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MAXIMAL FUNCTIONS WITH POLYNOMIAL DENSITIES IN LACUNARY DIRECTIONS

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ABSTRACT. Given a real polynomial $p(t)$ in one variable such that $p(0) = 0$, we consider the maximal operator in $\mathbb{R}^2$,

$$M_p f(x_1, x_2) = \sup_{h>0\,,\, i,j\in\mathbb{Z}} \frac{1}{h}\int_0^h \big|f\big(x_1 - 2^i p(t), x_2 - 2^j p(t)\big)\big|\, dt\ .$$

We prove that M_p is bounded on $L^q(\mathbb{R}^2)$ for $q > 1$ with bounds that only depend on the degree of p.

1. Introduction

Maximal operators on the real line of the form

$$f(x) \longmapsto \sup_{h>0} \frac{1}{h}\int_0^h \big|f\big(x - p(t)\big)\big|\, dt\ , \tag{1.1}$$

where p is a real polynomial with $p(0) = 0$, were considered in [CRW1], and it was shown that they satisfy weak-type 1-1 estimates that are uniform over all polynomials of fixed degree. Natural extensions of these operators to higher dimensions are discussed in [CRW2], in connection with $\mathbb{R}^n$-valued polynomials defined on $\mathbb{R}^m$.

We consider here a different kind of multi-dimensional analogue of (1.1), which is modelled on the maximal function in lacunary directions introduced in [NSW]. For simplicity, we restrict ourselves to two dimensions and to dyadic lacunary directions, i.e., determined by the vectors $v_k = (1, 2^k)$ with $k \in \mathbb{Z}$. In addition, we allow dyadic scaling along each of these directions.

To be precise, given a real polynomial $p(t)$ in one variable such that $p(0) = 0$, we define

$$\begin{aligned} M_p f(x_1, x_2) &= \sup_{h>0\,,\, i,k\in\mathbb{Z}} \frac{1}{h}\int_0^h \big|f\big(x - 2^i p(t) v_k\big)\big|\, dt \\ &= \sup_{h>0\,,\, i,j\in\mathbb{Z}} \frac{1}{h}\int_0^h \big|f\big(x_1 - 2^i p(t), x_2 - 2^j p(t)\big)\big|\, dt\ . \end{aligned} \tag{1.2}$$

We prove the following result.

Theorem 1. *M_p is bounded on $L^q(\mathbb{R}^2)$ for $q > 1$ with bounds that only depend on the degree of p.*

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It is easy to check that M_p cannot satisfy a weak-type 1-1 estimate.

The proof of Theorem 1 is based on the analysis of a general class of two-parameter maximal operators in the plane defined by compactly supported measures, subject to a decay assumption on their Fourier transforms. This result is in the spirit of [DR] and [RS], but here we consider the possibility that the Fourier transform of the measure has no decay within an angle that does not contain the coordinate axes.

Theorem 2. *For a probability measure μ supported on the unit square, let $\mu_{i,j}$ be the measure such that*

$$\int f\,d\mu_{i,j} = \int f(2^i x_1, 2^j x_2)\,d\mu(x_1,x_2)\ .$$

Assume that

(i) *there are constants $C,\delta > 0$ and $s > 1$ such that*

$$\big|\hat\mu(\xi)\big| \le C\big(1+|\xi|\big)^{-\delta} \tag{1.3}$$

away from the set where $s^{-1} < \frac{|\xi_1|}{|\xi_2|} < s$;

(ii) *the one-parameter maximal operator*

$$M^0_\mu f(x) = \sup_{i\in\mathbb{Z}} |f * \mu_{i,i}(x)| \tag{1.4}$$

is bounded on $L^q(\mathbb{R}^2)$ for $q > 1$.

Then also, the two-parameter maximal operator,

$$M_\mu f(x) = \sup_{i,j\in\mathbb{Z}} |f * \mu_{i,j}(x)|\ , \tag{1.5}$$

is bounded on $L^q(\mathbb{R}^2)$ for $q > 1$, with bounds that only depend on s, the constants C,δ in (1.3) and the norm of M^0_μ.

We start with the proof of Theorem 2, which combines methods from [NSW], [C] and [RS]. This is done in Section 2. Theorem 1 is proved in Section 3.

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2. Proof of Theorem 2

Let σ_1 and σ_2 be the measures on the line defined by

$$\int_{\mathbb{R}} f(t)\,d\sigma_j(t) = \int_{\mathbb{R}^2} f(x_j)\,d\mu(x)\ .$$

Then $\hat\sigma_1(\tau) = \hat\mu(\tau,0)$ and $\hat\sigma_2(\tau) = \hat\mu(0,\tau)$, so that

$$|\hat\sigma_j(\tau)| \le C\big(1+|\tau|\big)^{-\delta}\ . \tag{2.1}$$

Let φ be a nonnegative smooth function on the line, supported on $[-1,1]$ and with integral equal to 1. Define

$$\nu = \mu - \sigma_1\otimes\varphi - \varphi\otimes\sigma_2 + \varphi\otimes\varphi\ .$$

Clearly, $\hat\nu$ satisfies (1.3), is supported on the unit square and

$$\hat\nu(\xi_1,0) = \hat\nu(0,\xi_2) = 0\ . \tag{2.2}$$

Since

$$M_\mu f \le M_\nu f + M_{\sigma_1\otimes\varphi} f + M_{\varphi\otimes\sigma_2} f + M_{\varphi\otimes\varphi}\ ,$$

we can discuss each of the maximal functions on the right-hand side separately.

The last term is controlled by the two-parameter strong maximal operator of Jessen, Marcinkiewicz and Zygmund. The L^q-boundedness of the two intermediate terms follows from Theorem 3.2 in [RS], once we observe that, by (2.1),

$$\left|\widehat{\sigma_1 \otimes \varphi}(\xi)\right| \leq C'\big(1+|\xi|\big)^{-\delta} \ ,$$

and similarly for $\varphi \otimes \sigma_2$. (Alternatively, one can argue that $M_{\sigma_1 \otimes \varphi}$ is controlled by the composition of the Hardy-Littlewood maximal operator in the x_2-variable with the one-parameter operator M_{σ_1} in the x_1-variable; to this operator one can apply Theorem A in [DR].)

Thus it remains to estimate $M_\nu f$. Due to the cancellations of ν that are implicit in (2.2), it is convenient to introduce appropriate square functions. Given a measure σ, we shall need two types of such functions:

$$S_\sigma f(x) = \bigg(\sum_{i,j\in\mathbb{Z}} \big|f * \sigma_{i,j}(x)\big|^2\bigg)^{\frac{1}{2}} \ , \tag{2.3.a}$$

$$\tilde{S}_\sigma f(x) = \bigg(\sum_{k\in\mathbb{Z}} \Big(\sup_{i\in\mathbb{Z}} \big|f * \sigma_{i,i+k}(x)\big|\Big)^2\bigg)^{\frac{1}{2}} \ . \tag{2.3.b}$$

Clearly, $M_\sigma f \leq \tilde{S}_\sigma f \leq S_\sigma f$. We shall also assume that q is finite, because there is nothing to prove for $q = \infty$.

Let $\eta_\ell(x) = 2^{2\ell}\eta(2^\ell x)$, $\ell \geq 0$, be a smooth approximate identity in $\mathbb{R}^2$, with η supported on the unit disk. We set $\psi_0 = \eta_0$, and $\psi_\ell = \eta_\ell - \eta_{\ell-1}$ for $\ell \geq 1$. Then

$$\nu = \sum_{\ell=0}^{\infty} \nu * \psi_\ell$$

and

$$S_\nu f \leq \sum_{\ell=0}^{\infty} S_{\nu*\psi_\ell} f \ .$$

Lemma 2.1. *For every $\varepsilon > 0$ and $1 < q < \infty$, $\|S_{\nu*\psi_\ell} f\|_q \leq A 2^{2\ell\varepsilon}\|f\|_q$, where the constant A depends only on ε and q.*

Proof. By the standard randomization argument, we can estimate the L^q-operator norm of the singular integral operators

$$f \longmapsto \sum_{i,j} \pm(\nu * \psi_\ell)_{i,j} * f \ .$$

We apply Lemma 2.3 in [RS]. Thus, it is necessary to prove that

$$\sup_{0<|h_2|<2} |h_2|^{-\varepsilon} \int \bigg(\sup_{0<|h_1|<2} |h_1|^{-\varepsilon} \int \big|\Delta^1_{h_1}\Delta^2_{h_2}(\nu * \psi_\ell)(x)\big|\, dx_1\bigg)\, dx_2 \leq C 2^{2\ell\varepsilon} \ ,$$

where

$$\begin{aligned} \Delta^1_{h_1} f(x_1,x_2) &= f(x_1+h_1,x_2) - f(x_1,x_2) \ , \\ \Delta^2_{h_2} f(x_1,x_2) &= f(x_1,x_2+h_2) - f(x_1,x_2) \ . \end{aligned}$$

We observe that

$$\Delta^1_{h_1}\Delta^2_{h_2}(\nu * \psi_\ell) = \nu * (\Delta^1_{h_1}\Delta^2_{h_2}\psi_\ell)$$

and that $\Delta^1_{h_1}\Delta^2_{h_2}\psi_\ell(x)$ is smaller than a constant times $2^{(2+2\varepsilon)\ell}|h_1|^\varepsilon|h_2|^\varepsilon$, and it is supported, for each $x,\ h_1,\ h_2$, on a set that is the union of four disks of radius $2^{-\ell}$. Therefore,

$$\begin{aligned}\int \big|\Delta^1_{h_1}\Delta^2_{h_2}(\nu*\psi_\ell)(x)\big|\,dx_1 &\le \int_{\mathbb{R}^2}\left(\int \big|\Delta^1_{h_1}\Delta^2_{h_2}\psi_\ell(x-y)\big|\,dx_1\right)d|\nu|(y)\\ &\le C2^{(1+2\varepsilon)\ell}|h_1|^\varepsilon|h_2|^\varepsilon\int_{\mathbb{R}^2}\chi_{y,h_2}(x_2)\,d|\nu|(y)\ ,\end{aligned}$$

where χ_{y,h_2} is the characteristic function of a set of measure $2^{-\ell}$ depending on y and h_2.

This concludes the proof. ∎

In order to obtain better estimates, we introduce a spectral decomposition of ν. Let $\Phi(\xi)$ be homogeneous of degree 0, smooth away from the origin, identically equal to 1 inside the angle $\Gamma_1=\{\xi: s^{-1}<|\xi_1|/|\xi_2|<s\}$, and identically equal to 0 outside of the angle $\Gamma_2=\{\xi:(2s)^{-1}<|\xi_1|/|\xi_2|<2s\}$.

We then define the "bad part" ν_b of ν as the distribution such that

$$\hat\nu_b(\xi)=\hat\nu(\xi)\Phi(\xi)\ ,$$

and the "good part" ν_g as $\nu_g=\nu-\nu_b$.

The square functions $S_{\nu_b}f$, $S_{\nu_b*\psi_\ell}f$, etc. are defined as in (2.3.a) and (2.3.b) for Schwartz functions f.

We show first that each part of ν shares the good properties of ν given by Lemma 2.1.

Lemma 2.2. *The conclusion of Lemma 2.1 remains valid if we replace ν by ν_b or ν_g.*

Proof. For $k\in\mathbb{Z}$, let $P_kf=\mathcal{F}^{-1}\big(\Phi(\xi_1,2^{-k}\xi_2)\hat f(\xi)\big)$. Because of the finite overlapping of the supports of the multipliers $\Phi(\xi_1,2^{-k}\xi_2)$, we have the Littlewood-Paley estimate

$$\left\|\left(\sum_{k\in\mathbb{Z}}|P_kf|^2\right)^{\frac12}\right\|_q\sim\|f\|_q\ , \tag{2.4}$$

for $1<q<\infty$. Also, observe that

$$(\nu_b)_{i,j}*f=\nu_{i,j}*(P_{i-j}f)\ ,\qquad(\nu_b*\psi_\ell)_{i,j}*f=(\nu*\psi_\ell)_{i,j}*(P_{i-j}f)\ .$$

Therefore,

$$\begin{aligned}S_{\nu_b*\psi_\ell}f&=\left(\sum_{i,j\in\mathbb{Z}}\big|(\nu*\psi_\ell)_{i,j}*(P_{i-j}f)(x)\big|^2\right)^{\frac12}\\ &\le\left(\sum_{i,j,k\in\mathbb{Z}}\big|(\nu*\psi_\ell)_{i,j}*(P_kf)(x)\big|^2\right)^{\frac12}.\end{aligned}$$

The last quantity equals the L^2-norm on $[0,1]^3$ of the function

$$(t,u,v)\longmapsto\sum_{i,j,k\in\mathbb{Z}}(\nu*\psi_\ell)_{i,j}*(P_kf)(x)r_i(t)r_j(u)r_k(v)\ ,$$

where r_n is the nth Rademacher function. By the properties of Rademacher functions, the L^2-norm is equivalent to the L^q-norm. Therefore,

$$\|S_{\nu_b*\psi_\ell} f\|_q^q \leq C \int_{\mathbb{R}^2} \int_{[0,1]^3} \Big| \sum_{i,j,k\in\mathbb{Z}} (\nu * \psi_\ell)_{i,j} * (P_k f)(x) r_i(t) r_j(u) r_k(v) \Big|^q \, dt\, du\, dv\, dx \ .$$

We denote

$$K_{t,u} = \sum_{i,j} r_i(t) r_j(u) (\nu * \psi_\ell)_{i,j} \ , \qquad f_v = \sum_k r_k(v) P_k f \ .$$

Changing the order of integration, we have

$$\|S_{\nu_b*\psi_\ell} f\|_q^q \leq C \int_{[0,1]^3} \|K_{t,u} * f_v\|_q^q \, dt\, du\, dv \ .$$

The proof of Lemma 2.1 shows that the L^q-operator norms of the $K_{t,u}$ are uniformly bounded by a constant times $2^{2\ell\varepsilon}$. Hence,

$$\|S_{\nu_b*\psi_\ell} f\|_q^q \leq C 2^{2\ell\varepsilon} \int_{[0,1]} \|f_v\|_q^q \, dv \ .$$

Changing the order of integration again, replacing the L^q-norm on $[0,1]$ with the L^2-norm, and using (2.4), we obtain the conclusion for ν_b.

For ν_g it is sufficient to observe that $S_{\nu_g*\psi_\ell} f \leq S_{\nu*\psi_\ell} f + S_{\nu_b*\psi_\ell} f$. □

We shall now improve the estimate on $S_{\nu_g*\psi_\ell}$, using the uniform decay of $\hat{\nu}_g(\xi)$ as ξ goes to infinity. In fact, as we already observed, $\hat{\nu}$ satisfies (1.3); hence,

$$|\hat{\nu}_g(\xi)| \leq C\big(1 + |\xi|\big)^{-\delta} \ . \tag{2.5}$$

We shall assume, w.l.o.g., that $\delta < 1$.

Lemma 2.3. $\|S_{\nu_g*\psi_\ell} f\|_2 \leq A 2^{-\ell\delta/4} \|f\|_2$, *with A depending only on δ and C.*

Proof. By the Plancherel formula, we have to prove that

$$\sum_{i,j\in\mathbb{Z}} \big|\hat{\nu}_g(2^i\xi_1, 2^j\xi_2)\big|^2 \big|\hat{\psi}_\ell(2^i\xi_1, 2^j\xi_2)\big|^2 \leq A 2^{-\ell\delta/2} \ . \tag{2.6}$$

By (2.2),

$$\hat{\nu}(\xi) = \int (e^{-ix_1\xi_1} - 1)(e^{-ix_2\xi_2} - 1)\, d\nu(\xi) \ .$$

Since ν is supported on the unit square,

$$|\hat{\nu}(\xi)| \leq C |\xi_1| |\xi_2| \ .$$

Combining this with (2.5), we obtain that, if $0 < \varepsilon < 1$,

$$|\hat{\nu}_g(\xi)| \leq C \frac{|\xi_1|^\varepsilon |\xi_2|^\varepsilon}{(1 + |\xi|)^{\delta(1-\varepsilon)}} \ .$$

If $\ell \geq 1$, then

$$|\hat{\psi}_\ell(\xi)| = |\hat{\psi}_1(2^{-(\ell-1)}\xi)| \leq C 2^{-\ell\varepsilon} |\xi|^\varepsilon \ ,$$

because $\hat{\psi}_1(0) = 0$. Hence,

$$|\hat{\nu}_g(\xi) \hat{\psi}_\ell(\xi)| \leq C 2^{-\ell\varepsilon} \frac{|\xi_1|^\varepsilon |\xi_2|^\varepsilon}{(1 + |\xi|)^{\delta(1-\varepsilon)-\varepsilon}} \ .$$

We can assume that $|\xi_1| \sim |\xi_2| \sim 1$ in (2.6). Then we simply have to observe that, taking $\varepsilon = \delta/4$, the exponent in the denominator is bigger than $\delta/2 = 2\varepsilon$, and that the series

$$\sum_{i,j\in\mathbb{Z}} \frac{2^{2\varepsilon i}2^{2\varepsilon j}}{(1+2^i+2^j)^{2\alpha}}$$

is convergent for $\alpha > 2\varepsilon$. □

Interpolating between the L^2-estimate in Lemma 2.3 and the L^q-estimate in Lemma 2.2 for $S_{\nu_g * \psi_\ell}$, we obtain that for every $q \in (1,\infty)$ there is an $\varepsilon_q > 0$ such that $\|S_{\nu_g*\psi_\ell} f\|_q \le A 2^{-\ell\varepsilon_q}\|f\|_q$. Therefore,

Proposition 2.4. *S_{ν_g} is bounded on L^q for $1 < q \le 2$.*

In order to complete the proof of Theorem 2, we may just observe that we are in the hypotheses of Theorem B in [C] (attributed to M. Christ). We give, however, an independent proof, based on the extrapolation argument in [NSW], adapted to $\tilde S_{\nu_b}$.

End of the proof of Theorem 2. The starting point is that $\tilde S_{\nu_b}$ is bounded on L^2. In fact, assumption (ii) implies that $M^0_{\nu_{0,k}}$ is uniformly bounded on L^q independently of k. Therefore,

$$\begin{aligned}\int \tilde S_{\nu_b} f(x)^2\,dx &= \sum_{k\in\mathbb{Z}} \int \sup_{i\in\mathbb{Z}} \left|\nu_{i,i+k} * P_k f(x)\right|^2 dx \\ &= \sum_{k\in\mathbb{Z}} \int \big(M^0_{\nu_{0,k}} P_k f(x)\big)^2\,dx \\ &\le C\sum_{k\in\mathbb{Z}} \int \big(P_k f(x)\big)^2\,dx \\ &= C\|f\|_2^2\,.\end{aligned}$$

In general, the boundedness of $\tilde S_{\nu_b}$ on some L^q implies, by Proposition 2.4, the boundedness of M_ν on the same L^q, and hence that of M_μ.

Assume now that M_μ is bounded on some L^q, and consider the inequality

$$\left\|\Big(\sum_{k\in\mathbb{Z}} M^0_{\mu_{0,k}} f_k(x)^r\Big)^{1/r}\right\|_s \le C\left\|\Big(\sum_{k\in\mathbb{Z}} |f_k(x)|^r\Big)^{1/r}\right\|_s\,. \tag{2.7}$$

This is equivalent to saying that the linear operator

$$T : \{f_k\} \longmapsto \{\mu_{i,i+k} * f_k\}$$

is bounded from $L^s(\ell^r)$ to $L^s\big(\ell^r(\ell^\infty)\big)$.

Since μ is a positive measure and we are assuming that M_μ is bounded on L^q, (2.7) is verified for $r=\infty$ and $s=q$. In addition, it is verified for $r=s>1$ by the uniform boundedness of $M^0_{\mu_{0,k}}$. Hence, T is bounded from $L^q(\ell^\infty)$ to $L^q\big(\ell^\infty(\ell^\infty)\big)$ and from $L^r(\ell^r)$ to $L^r\big(\ell^r(\ell^\infty)\big)$ for $r>1$. By interpolation, (2.7) holds for $r=2$ and $\frac{1}{q} < \frac{1}{s} < \frac{1}{2}\big(1+\frac{1}{q}\big)$.

The same inequality holds with μ replaced by $\sigma\otimes\varphi$, $\varphi\otimes\sigma$ and $\varphi\otimes\varphi$, and hence with μ replaced by ν.

Taking $f_k = P_k f$, this implies that $\tilde{S}_{\nu_b}$ is bounded on the same spaces L^s. Since each $q\in(1,2)$ can be reached by iteration in a finite number of steps, we conclude that M_μ is bounded on L^q for every $q>1$. □

3. Proof of Theorem 1

The starting point for the proof of Theorem 1 is Lemma 2.5 in [CRW1]. We give a slightly different (and less complete) formulation of it.

Lemma 3.1. *For every n there are constants $A(n)\ge 1$ and $B=B(n)$ with the following property: if $p(t)$ is a monic real polynomial of degree n such that $p(0)=0$, $A\ge A(n)$, and $m\in\mathbb{Z}$ is such that no complex zero of p lies in the strip*

$$\{z : A^{m-1}\le |z|\le A^{m+2}\},$$

then the following properties hold:

(i) *p has constant sign and is strictly monotonic on $I_m=[A^m,A^{m+1}]$;*
(ii) *$|p(t)|\le Bt|p'(t)|$ for $t\in I_m$;*
(iii) *$\max_{t\in I_m}|p(t)|\le B\min_{t\in I_m}|p(t)|$.*

Observe that we are allowed to replace the polynomial $p(t)$ in (1.2), when convenient, by $\tilde p(t)=bp(at)$, with $a,b>0$. In fact, the identity

$$M_{\tilde p}f(x)=M_p f_b\left(\frac{x}{b}\right)\ ,$$

where $f_b(x)=f(bx)$, implies that M_p and $M_{\tilde p}$ have the same operator norm. In particular, we can assume that p is monic.

Also, the maximal function M_p can be replaced by

$$\tilde M_p f(x_1,x_2)=\sup_{m\in\mathbb{Z}}\tilde M_{p,m}f(x_1,x_2)$$

where

$$\tilde M_{p,m}f(x_1,x_2)=\sup_{i,j\in\mathbb{Z}}A^{-m}\int_{I_m}\big|f\big(x_1-2^ip(t),x_2-2^jp(t)\big)\big|\,dt\ . \tag{3.1}$$

Let I_m be one of the "good" dyadic intervals satisfying properties (i)–(iii) in Lemma 3.1. Making the change of variable $u=p(t)$, we have

$$\begin{aligned}A^{-m}\int_{I_m}&\big|f\big(x_1-2^ip(t),x_2-2^jp(t)\big)\big|\,dt\\
&\le A\int_{A^m}^{A^{m+1}}\big|f\big(x_1-2^ip(t),x_2-2^jp(t)\big)\big|\,\frac{dt}{t}\\
&\le AB\int_{I_m}\big|f\big(x_1-2^ip(t),x_2-2^jp(t)\big)\big|\frac{|p'(t)|}{|p(t)|}\,dt\\
&= AB\int_{p(I_m)}\big|f\big(x_1-2^iu,x_2-2^ju\big)\big|\,\frac{du}{|u|}\ .\end{aligned}$$

By (i) and (iii), the interval $p(I_m)$ is contained in an interval of the form $\pm[\alpha_m,B\alpha_m]$, with $\alpha_m>0$. Therefore, assuming w.l.o.g. that p is positive on I_m,

$$\begin{aligned} A^{-m}\int_{I_m}\left|f\big(x_1-2^ip(t),x_2-2^jp(t)\big)\right|dt &\le AB\int_{\alpha_m}^{B\alpha_m}\left|f\big(x_1-2^iu,x_2-2^ju\big)\right|\frac{du}{u} \\ &\le \frac{AB}{\alpha_m}\int_{\alpha_m}^{B\alpha_m}\left|f\big(x_1-2^iu,x_2-2^ju\big)\right|du \\ &\le \frac{AB^2}{B\alpha_m}\int_0^{B\alpha_m}\left|f\big(x_1-2^iu,x_2-2^ju\big)\right|du\ . \end{aligned}$$

This shows that the contribution to $\tilde{M}_p f$ given by the "good" intervals is controlled by the maximal function in lacunary directions

$$\mathcal{M}f(x_1,x_2)=\sup_{h>0\,,\,k\in\mathbb{Z}}\frac{1}{h}\int_0^h\left|f\big(x_1-t,x_2-2^kt\big)\right|dt$$

of [NSW]. Since $\mathcal{M}$ is bounded on L^q for $q>1$ [NSW], it remains to consider the contribution from the "bad" intervals. Since there are at most $3n$ of these intervals, it is enough to prove that $\tilde{M}_{p,m}$ acts on L^q for $q>1$, with operator norm bounded independently of the polynomial p and integer m.

We claim it suffices to show that there exists a constant $C_{q,n}$ such that

$$\sup_{m\in\mathbb{Z}}\|\tilde{M}_{p,m}f\|_q\le C_{q,n}\|f\|_q$$

for every $f\in L^q$ and monic polynomial p of degree n satisfying $p(0)=0$ and

$$A^{-n}\le\max_{t\in I_m}|p(t)|\le 1. \tag{3.2}$$

To see this, suppose p is an arbitrary monic polynomial with $p(0)=0$, and choose $k\in\mathbb{Z}$ such that

$$A^{-n}\le\max_{t\in I_m}A^{-kn}|p(t)|\le 1.$$

Let $\underset{\sim}{p}(t)=A^{-kn}p(A^kt)$. Since

$$A^{-n}\le\max_{t\in I_{m-k}}|\underset{\sim}{p}(t)|\le 1,$$

the (L^q,L^q) operator norm of $\tilde{M}_{\underset{\sim}{p},m-k}$ is at most $C_{q,n}$. Since

$$\tilde{M}_{p,m}f(x)=\tilde{M}_{\underset{\sim}{p},m-k}f_{A^k}(A^{-k}x),$$

$\tilde{M}_{p,m}$ also acts on L^q with bounds that are independent of m and p.

Consequently, we need to investigate the measure μ given by

$$\int f\,d\mu=A^{-m}\int_{I_m}f(p(t),p(t))dt$$

where p satisfies (3.2). This measure is supported on the segment $\{(u,u):-1\le u\le 1\}$ and, up to a factor depending on A, is a probability measure.

The proof of Theorem 1 will be complete once we show that the operator M_μ is bounded on L^q for $q>1$ with bounds that depend only on n and q. We apply Theorem 2.

The Fourier transform of μ is

$$\hat{\mu}(\xi_1,\xi_2)=A^{-m}\int_{I_m}e^{-i(\xi_1+\xi_2)p(t)}\,dt\ . \tag{3.3}$$

Lemma 3.2. *There is an integer $k \in \{1, 2, \dots, n\}$ such that, if A is large enough (depending on n), then*

$$\left|\hat{\mu}(\xi_1, \xi_2)\right| \le CA^n \big(1 + |\xi_1 + \xi_2|\big)^{-1/k} \ ,$$

with C independent of p and m.

Proof. Let $t_1 = 0, t_2, \dots, t_n$ be the zeroes of p, ordered so that $0 \le |t_2| \le \dots \le |t_n|$. Let m' be the smallest integer greater than m such that $I_{m'}$ does not contain any of the $|t_j|$. Then $m' \le m + n$, so that $A^{m'}$ is comparable with A^m. Also let k be such that $|t_j| < A^{m'}$ for $j \le k$ and $|t_j| > A^{m'+1}$ for $j > k$.

The kth derivative of p equals

$$p^{(k)}(t) = \prod_{j=k+1}^{n} (t - t_j) + r(t) \ ,$$

where $r(t)$ is a sum where each term is a product of $n - k$ factors $t - t_j$, with at least one of the j less than or equal to k.

If $t \in I_m$, $|t - t_j| < 2A^{m'}$ for $j \le k$, and $|t - t_j| > (1 - A^{-1})|t_j| > (A - 1)A^{m'}$ for $j > k$. Therefore, if A is large enough,

$$\left|p^{(k)}(t)\right| \ge C \prod_{j=k+1}^{n} |t_j| \ ,$$

for $t \in I_m$.

By van der Corput's lemma,

$$A^{-m}\left|\int_{I_m} e^{-i\lambda p(t)}\, dt\right| \le CA^{-m}\left(\prod_{j=k+1}^{n} |t_j|\right)^{-1/k} |\lambda|^{-1/k} \ .$$

If $\bar{t} \in I_m$ is such that $|p(\bar{t})| \ge A^{-n}$, we have

$$A^{-n} \le \left|p(\bar{t})\right| \le 2^n A^{km'} \prod_{j=k+1}^{n} |t_j| \ .$$

Therefore, $\prod_{j=k+1}^{n} |t_j| \ge CA^{-n}A^{-km}$, so that

$$A^{-m}\left|\int_{I_m} e^{-i\lambda p(t)}\, dt\right| \le C|\lambda|^{-1/k} A^n \ ,$$

with C independent of p and m. Since the left-hand side is trivially bounded by 1, this concludes the proof. □

Thus, $\hat{\mu}$ clearly satisfies hypothesis (i) of Theorem 2. It remains to prove that the one-parameter maximal operator M^0_μ in (1.4) is bounded on L^q for $q > 1$ with bounds that only depend on n and q. This follows from a transference argument: because μ is supported on a line, it is sufficient to consider the maximal operator on $\mathbb{R}$,

$$M_{\tilde{\mu}} g(x) = \sup_{i \in \mathbb{Z}} |g * \tilde{\mu}_i(x)| \ ,$$

where

$$\int_{\mathbb{R}} g\, d\tilde{\mu} = A^{-m} \int_{I_m} g\big(p(t)\big)\, dt \ .$$

By Lemma 3.2, $\left|\widehat{\tilde{\mu}}(\eta)\right| \le C\big(1 + |\eta|\big)^{-1/k}$, with $1 \le k \le n$ and C depending only on n. The conclusion follows from Theorem A in [DR].

Remark. In [CRW1] the authors show that the "supermaximal function" on the real line

$$f(x) \longmapsto \sup_{\substack{h>0 \\ \deg p \leq k\,,\, p(0)=0}} \frac{1}{h}\int_0^h \big|f\big(x-p(t)\big)\big|\,dt$$

is of restricted weak type $k-k$ and hence of strong type $q-q$ for $q>k$.

The proof can be adapted to show that the operator

$$\begin{aligned}\mathcal{M}_k f(x) &= \sup_{\deg p\leq k\,,\, p(0)=0} M_p f(x) \\ &= \sup_{\substack{h>0\,,\, j\in\mathbb{Z} \\ \deg p\leq k\,,\, p(0)=0}} \frac{1}{h}\int_0^h \big|f\big(x_1-p(t), x_2-2^j p(t)\big)\big|\,dt\end{aligned}$$

is bounded on $L^q(\mathbb{R}^2)$ for $q>k$.

In fact, if f is the characteristic function of a measurable set in the plane, the same proof as in [CRW1] gives the pointwise domination

$$\mathcal{M}_k f(x) \leq C\big(M^* f^k(x)\big)^{1/k}\ ,$$

where M^* is the maximal function in dyadic direction of [NSW]. This implies that $\mathcal{M}_k$ is of restricted weak type $q-q$ for $q>k$, and hence of strong type.

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