



Review article

The anomalous magnetic moment of the muon in the Standard Model: an update



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ABSTRACT

We present the current Standard Model (SM) prediction for the muon anomalous magnetic moment, a_μ , updating the first White Paper (WP20) [1]. The pure QED and electroweak contributions have been further consolidated, while hadronic contributions continue to be responsible for the bulk of the uncertainty of the SM prediction. Significant progress has been achieved in the hadronic light-by-light scattering contribution using both the data-driven dispersive approach as well as lattice-QCD calculations, leading to a reduction of the uncertainty by almost a factor of two. The most important development since WP20 is the change in the estimate of the leading-order hadronic-vacuum-polarization (LO HVP) contribution. A new measurement of the $e^+e^- \rightarrow \pi^+\pi^-$ cross section by CMD-3 has increased the tensions among data-driven dispersive evaluations of the LO HVP contribution to a level that makes it impossible to combine the results in a meaningful way. At the same time, the attainable precision of lattice-QCD calculations has increased substantially and allows for a consolidated lattice-QCD average of the LO HVP contribution with a precision of about 0.9%. Adopting the latter in this update has resulted in a major upward shift of the total SM prediction, which now reads $a_\mu^{\text{SM}} = 116\,592\,033(62) \times 10^{-11}$ (530 ppb). When compared against the current experimental average based on the E821 experiment and runs 1–6 of E989 at Fermilab, one finds $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 38(63) \times 10^{-11}$, which implies that there is no tension between the SM and experiment at the current level of precision. The final precision of E989 (127 ppb) is the target of future efforts by the Theory Initiative. The resolution of the tensions among data-driven dispersive evaluations of the LO HVP contribution will be a key element in this endeavor.

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0. Executive summary

The experimental program to study the muon’s anomalous magnetic moment, $a_{\mu} \equiv (g - 2)_{\mu}/2$, which started over sixty years ago at CERN, continues to inspire theoretical efforts to obtain a Standard-Model (SM) prediction with matching precision. In the SM, the dominant contributions to a_{μ} are due to QED corrections, followed by hadronic contributions (governed by QCD) and electroweak (EW) corrections. The CERN experiments provided tests of the $(\alpha/\pi)^2$ [2] and $(\alpha/\pi)^3$ [3] QED contributions and sensitivity to the leading-order (LO) hadronic corrections [4]. The BNL E821 experiment [5] reached a precision of 540 parts-per-billion (ppb), yielding sensitivity also to EW and higher-order hadronic contributions and intriguing tensions with the SM predictions available at that time. This provided motivation to launch the Fermilab E989 experiment, which started taking data in 2017, and has obtained measurements with increasing precision, from 460 ppb in 2021 [6] to 200 ppb in 2023 [7], all consistent with each other as well as with E821. The Fermilab experiment has just announced its final measurement result, with a precision of 127 ppb, and hence will enable, in principle, an exquisitely precise test of the SM.¹ Meanwhile, a new experiment that will employ a largely different experimental method is currently under preparation at J-PARC [9], allowing an independent cross-check of these results.

On the SM side, the QED and EW corrections are at less than 5 ppb already well quantified, while the notoriously difficult-to-calculate hadronic contributions are the dominant sources of theory error, rendering SM predictions of a_{μ} still considerably less precise than experiment. They, therefore, continue to be the main focus of the Muon $g - 2$ Theory Initiative. A lot has happened regarding these corrections since the publication of the first White Paper (WP20) [1].

The hadronic-vacuum-polarization (HVP) contribution was estimated in WP20 from e^+e^- hadronic cross-section data in the dispersive, data-driven method, while lattice-QCD methods were not mature enough to yield reliable results at the

¹ This paper was posted on arXiv on May 28, 2025. Sections 2–8 and all numbers pertaining to the SM prediction have remained unchanged, but the experimental world average has been updated according to the E989 announcement on June 3, 2025 [8], and the description in Sections 0, 1 and 9 has been adapted accordingly. In particular, the experimental results in abstract, Table 1, Figs. 27, 40 and 83, and Section 9 have been updated.

Table 1

Summary of the contributions to a_μ^{SM} . The experimental number refers to the world average completely dominated by E989. The subsequent block summarizes the pertinent hadronic contributions from Sections 2, 3, 5 and 6 as well as the combination of data-driven and lattice-QCD evaluations of HLbL scattering from Section 9. The second block summarizes the quantities entering our recommended SM value, in particular, the total HVP contribution (using lattice QCD for LO and e^+e^- for higher-order iterations) and the total HLbL number. The construction of the total HLbL contribution takes into account correlations among the terms at different orders, and the final rounding includes subleading digits at intermediate stages. The experimental world average has been updated including the final results from the Fermilab experiment.

Contribution	Section	Equation	Value $\times 10^{11}$	References
Experiment (E989, E821)		Eq. (9.5)	116 592 071.5(14.5)	Refs. [5–8,10–13]
HVP LO (lattice)	Section 3.6.1	Eq. (3.37)	7132(61)	Refs. [14–30]
HVP LO (e^+e^- , τ)	Section 2	Table 5	Estimates not provided at this point	
HVP NLO (e^+e^-)	Section 2.9	Eq. (2.47)	−99.6(1.3)	Refs. [31,32]
HVP NNLO (e^+e^-)	Section 2.9	Eq. (2.48)	12.4(1)	Ref. [33]
HLbL (phenomenology)	Section 5.10	Eq. (5.69)	103.3(8.8)	Refs. [34–57]
HLbL NLO (phenomenology)	Section 5.10	Eq. (5.70)	2.6(6)	Ref. [58]
HLbL (lattice)	Section 6.2.8	Eq. (6.34)	122.5(9.0)	Refs. [59–63]
HLbL (phenomenology + lattice)	Section 9	Eq. (9.2)	112.6(9.6)	Refs. [34–57],[59–63]
QED	Section 7.5	Eq. (7.27)	116 584 718.8(2)	Refs. [64–70]
EW	Section 8	Eq. (8.12)	154.4(4)	Refs. [51,71–73]
HVP LO (lattice) + HVP N(N)LO (e^+e^-)	Section 9	Eq. (9.1)	7045(61)	Refs. [14–33]
HLbL (phenomenology + lattice + NLO)	Section 9	Eq. (9.3)	115.5(9.9)	Refs. [34–63]
Total SM Value	Section 9	Eq. (9.4)	116 592 033(62)	Refs. [14–73]
Difference: $\Delta a_\mu \equiv a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	Section 9	Eq. (9.6)	38(63)	

required precision. In the present review, we find that this situation is reversed. In the data-driven method, some tensions in the dominant $e^+e^- \rightarrow \pi^+\pi^-$ channel were present in WP20 and were accounted for by inflating uncertainties. New cross-section measurements of the same channel that became available after WP20 have increased these tensions to a level that prevents forming any meaningful average for use in obtaining a precise and reliable evaluation of the HVP contribution. Moreover, at present no scientific grounds have been identified that would allow one to disregard any of the data sets relevant for the HVP evaluation.

Resolving the tensions in the e^+e^- hadronic cross-section data will require new analyses and measurements as well as an improved understanding of higher-order radiative corrections. Both are currently being pursued by several experiments and theorists. Despite the present difficult situation, a data-driven estimate of HVP remains an important goal of the Theory Initiative and an essential cornerstone on which to build future precise SM predictions. In this regard, also the τ -based HVP determination, which relies on consistent data sets, is being reconsidered in view of recent progress towards a model-independent evaluation of the required isospin-breaking corrections. While promising, this is still work in progress: the current status of a τ -based HVP estimate is reviewed here, but is not yet included in the overall evaluation.

On the other hand, thanks to dedicated efforts by the world-wide lattice field theory community, lattice-QCD calculations of HVP have matured considerably. The present review is informed by more than a dozen new papers (since WP20), which enable consolidated averages of (almost) all the components that are computed separately to construct the total LO HVP contribution in lattice QCD. The result is a reliable determination of the LO HVP contribution at 0.9% precision that enters the SM prediction of a_μ . The change from a data-driven to a lattice-based estimate for the HVP contribution has resulted in a significant shift in the central value.

For the hadronic light-by-light (HLbL) contribution, significant progress since WP20 has been made in both the dispersive method and lattice QCD. On the data-driven side, improved calculations of short-distance constraints and a number of subleading contributions have become available, leading to a reduction of the uncertainty by about a factor of two compared to WP20. At the same time, new lattice-QCD calculations have reached a similar level of precision. The two averages, from phenomenology and lattice QCD, are combined into a final average with a precision below 10%.

Table 1 gives a summary of the contributions to the SM prediction, along with the locations in this White Paper (WP25) where the results are discussed.

1. Introduction

The anomalous magnetic moment of the muon a_μ has been an enduring testbed of the SM and of its possible extensions (generically defined as beyond the SM, or BSM, physics). Both experiment and SM theory provide superb precision beyond parts-per-million (ppm), allowing one to put constraints on BSM physics through the comparison between the two. The *Muon $g - 2$ Theory Initiative* (TI) was started in 2017 with the goal of bringing together the community of theorists and experimentalists working on this topic and of coordinating their efforts to provide a single, consensus number for the theoretical estimate of a_μ within the SM. The basis and the methodology behind such a number would be detailed in an extensive review, a so-called White Paper. The first edition was published in 2020 (WP20) [1]. Since then, new measurements have been announced by the Fermilab $g - 2$ experiment, the first result in 2021 [6] with a precision of

460 ppb, and the second result in 2023 [7] with 200 ppb. These new measurements are consistent with, but more precise than, the ones of the BNL experiment, and provide the basis of the current world average. The agreement of the two most precise experiments lends a high degree of reliability to this world average. The Fermilab E989 experiment has just announced its final results with the full set of all experimental data collected. Another experiment with a largely different experimental method is currently under preparation at J-PARC [9], allowing an independent cross-check of these results.

For the physical interpretation of the muon $g - 2$ experimental measurement, it is crucial to have a reliable SM prediction. Ideally, uncertainties on the SM prediction would be reduced to the same level as the experimental ones to maximize the BSM sensitivity. In the SM, a_μ is calculated from a perturbative expansion in the fine-structure constant α . The most important contributions to this quantity arise from pure QED diagrams, but starting from order α^2 also hadronic contributions arise, in particular HVP and HLbL. Corrections due to EW bosons are small because of their large mass. They have been calculated at next-to-leading order (NLO) and are relevant at the present level of precision. The uncertainties are currently dominated by hadronic contributions due to the nonperturbative nature thereof, which make the corresponding theoretical calculations highly nontrivial. The goal of this second edition of the White Paper (WP25), as for the previous one, is to provide a detailed snapshot of the present situation and a new consensus SM number, before the final Fermilab result is announced.²

In recent years, the theoretical and experimental work aimed at reducing the uncertainties of the SM prediction has been particularly intense and relied on higher-order calculations, dispersive methods, lattice QCD, effective field theories, as well as new data inputs from experiments. For HLbL these efforts have indeed been successful and the change from WP20 to the present review is that the uncertainty has been reduced by about a factor of two. Concerning HVP the situation is more complicated as reflected by the fact that the present review does not present a number for a data-driven estimate of this contribution, since not yet understood discrepancies among different experiments have emerged. On the other hand, there has been significant progress in lattice-QCD calculations of this quantity, and the consensus number for the HVP contribution presented in this review is based on them. The change from a data-driven to a lattice estimate of this contribution also means that the central value has shifted, unfortunately, by more than the uncertainty quoted in WP20.

Activities of the initiative are coordinated by a Steering Committee that consists of theorists, experimentalists, and representatives from the Fermilab and J-PARC experiments. This committee also functions as the Advisory Committee for the workshops organized by the TI. Since the release of WP20, we organized four plenary workshops as well as focused mini-workshops. The format of the recent plenary workshops follows the previous editions with plenary sessions consisting of updates from $g - 2$ experiments, updates from HVP and HLbL evaluations from dispersive approaches as well as lattice QCD. The plenary workshop in 2021 was held virtually during the period of the pandemic and was hosted by KEK [74]. After 2022, the three plenary workshops were held annually as in-person events in Edinburgh [75], Bern [76], and Tsukuba [77], respectively. The focused mini-workshops, held virtually, were devoted to high-precision calculations of HVP in lattice QCD [78], the CMD-3 result for $e^+e^- \rightarrow \pi^+\pi^-$ [79,80], WP25 preparations prior to the Tsukuba workshop [81], and the HVP evaluation from hadronic τ decays [82]. At the next plenary workshop, planned to take place in Paris [83], the TI will discuss the next steps after WP25 toward reducing the theory errors to match the experimental precision. A summary of the prospects for such future improvements was compiled as contribution to the US Community Study on the Future of Particle Physics (Snowmass 2021) [84].

The Steering Committee is chaired by Aida El-Khadra, supported by Christoph Lehner and Michel Davier as co-chairs. Other committee members are³ Gilberto Colangelo, Martin Hoferichter, Laurent Lellouch, Tsutomu Mibe, Lee Roberts, Thomas Teubner, and Hartmut Wittig. The Steering Committee's roles are the long-term planning of the Theory Initiative, the organization of the workshops, and the coordination of writing the White Paper and its updates. This review (WP25) has kept the same structure as the previous one and is organized in four main chapters, namely data-driven HVP, lattice HVP, analytic HLbL, and lattice HLbL, the writing of which was delegated to four corresponding working groups. The coordinators of the working groups are:

- Data-driven HVP: Vincenzo Cirigliano, Achim Denig, Fedor Ignatov, Bogdan Malaescu
- Lattice HVP: Steven Gottlieb, Antonin Portelli
- Analytic HLbL: Johan (Hans) Bijnens, Anton Rebhan
- Lattice HLbL: Luchang Jin, Harvey Meyer

In addition, sections on the QED and EW contributions are coordinated by Makiko Nio and Dominik Stöckinger, respectively.

This paper was written under the following agreements within the initiative. All participants of the past workshops, members of working groups, and their collaborators were invited to become co-authors of WP25. Essential input papers that were accepted for publication by May 22, 2025 were fully considered and included in the averages, while brief descriptions of unpublished papers available on arXiv before Apr 22, 2025 are also included.

² This paper was posted on arXiv on May 28, 2025. Sections 2–8 and all numbers pertaining to the SM prediction have remained unchanged, but the experimental world average has been updated according to the E989 announcement on June 3, 2025 [8], and the description in Sections 0, 1 and 9 has been adapted accordingly. In particular, the experimental results in abstract, Table 1, Figs. 27, 40 and 83, and Section 9 have been updated.

³ Simon Eidelman, a founding member of the Theory Initiative, served on the Steering Committee until he passed away in June 2021.

The rest of the review is organized as follows: we first discuss the evaluations of data-driven HVP calculations in Section 2 and lattice-QCD calculations in Section 3, followed by comparisons between them in Section 4. Then, we describe evaluations of HLbL by using data-driven and analytic approaches in Section 5, and lattice QCD in Section 6. Lastly, QED and EW contributions are given in Sections 7 and 8, respectively. We summarize the conclusions and outlook for the current SM prediction in Section 9.

2. Data-driven calculations of HVP

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2.1. Introduction

The calculation of the LO HVP contribution in terms of hadronic cross sections proceeds via the master formula [85–88]

$$a_\mu^{\text{HVP, LO}} = \left(\frac{\alpha m_\mu}{3\pi} \right)^2 \int_{s_{\text{thr}}}^\infty ds \frac{\hat{K}(s)}{s^2} R_{\text{had}}(s), \quad (2.1)$$

with muon mass m_μ , fine-structure constant $\alpha = e^2/(4\pi)$, and kernel function

$$\begin{aligned} \hat{K}(s) &= \frac{3s}{m_\mu^2} K(s), \quad x = \frac{1 - \beta_\mu}{1 + \beta_\mu}, \quad \beta_\mu = \sqrt{1 - \frac{4m_\mu^2}{s}}, \\ K(s) &= \frac{x^2}{2}(2 - x^2) + \frac{(1 + x^2)(1 + x)^2}{x^2} \left(\log(1 + x) - x + \frac{x^2}{2} \right) + \frac{1 + x}{1 - x} x^2 \log x. \end{aligned} \quad (2.2)$$

Here, the hadronic R-ratio,

$$R_{\text{had}}(s) = \frac{3s}{4\pi\alpha^2} \sigma[e^+e^- \rightarrow \text{hadrons}(+\gamma)], \quad (2.3)$$

is defined photon-inclusively, so that the integration starts at the threshold $s_{\text{thr}} = M_{\pi^0}^2$ due to the $e^+e^- \rightarrow \pi^0\gamma$ channel. The challenge in evaluating $a_\mu^{\text{HVP, LO}}$ at sub-percent level thus lies in the extraordinary precision requirements for the measurement of $e^+e^- \rightarrow \text{hadrons}$ cross sections, especially, for the crucial $e^+e^- \rightarrow \pi^+\pi^-$ channel. In this section, various aspects of this program are discussed, including reports from the e^+e^- experiments (Section 2.2), indirect cross-section measurements via τ decays (Section 2.3), Monte-Carlo (MC) tools (Section 2.4), global data combinations (Section 2.5), theory developments (Sections 2.6–2.8), higher orders (Section 2.9), and the MUonE project (Section 2.10). A summary of the current situation together with an outlook to future prospects is provided in Sections 2.11 and 2.12.

Throughout, in addition to the total LO HVP contribution as defined in Eq. (2.1), also so-called Euclidean-window observables will be considered. First introduced by RBC/UKQCD [14], weight functions in Euclidean time are introduced that separate the entire integral into a short-distance (SD), intermediate (W), and long-distance (LD) component, see Section 3.4 for the precise definitions. Importantly, as detailed in Section 4.1, these weight functions can be translated into center-of-mass (CM) energy $\sqrt{s}$, to be inserted into Eq. (2.1), so that data-driven evaluations of the same quantities become possible, a connection that is addressed in more detail in Section 4.

2.2. Status and perspectives of e^+e^- experiments

2.2.1. CMD-3

The CMD-3 [89,90] and SND [91] experiments have been in operation at the electron-positron collider VEPP-2000 [92,93] since 2010. The collider provides an instantaneous luminosity of up to $10^{32} \text{ cm}^{-2} \text{ s}^{-1}$ at the maximum CM energy $\sqrt{s} = 2 \text{ GeV}$, a world record for single-bunch luminosity. Overall, more than 1 fb^{-1} of data has already been collected by each experiment across the entire available CM energy range from 0.32 to 2.007 GeV. Today, VEPP-2000 is the only collider operating at these energies.

The new generation CMD-3 detector was designed and constructed with a major upgrade of all subsystems compared to its predecessor CMD-2 experiment. In particular, a new drift chamber provided higher efficiency and more than twice better momentum resolution, and a new liquid-xenon (LXe) calorimeter added multi-layer tracking capabilities and shower profile measurement. The detector also has completely new up-to-date electronics and an elaborate trigger system. The main goals of experiments at VEPP-2000 include the high-precision measurement of cross sections of various modes of $e^+e^- \rightarrow \text{hadrons}$ at low energies. All major channels are under analysis with final states of up to 7 pions, or

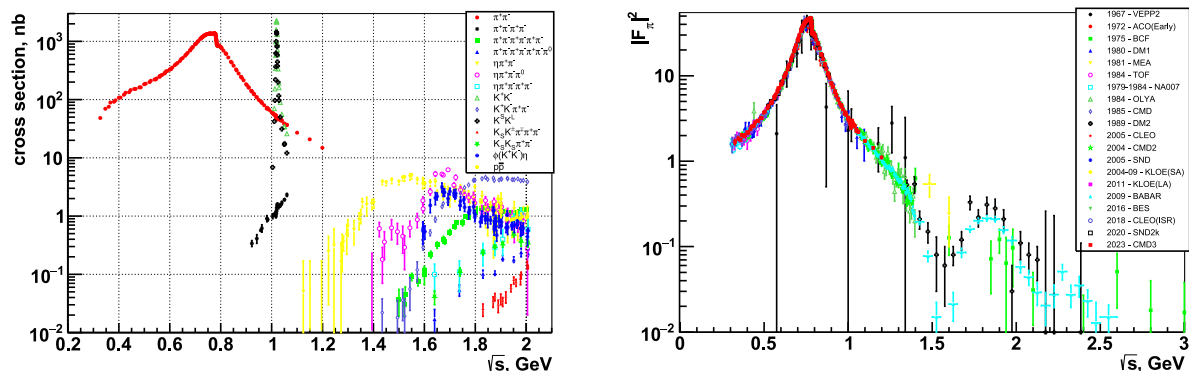


Fig. 1. Left: Published measurements of $e^+e^- \rightarrow \text{hadrons}$ cross sections by the CMD-3 experiment [95–108]. Right: Status of the $e^+e^- \rightarrow \pi^+\pi^-$ cross-section measurement with different experiments contributing over the years [95,96,109–140].

two kaons and three pions [94]. Many results have already been published by CMD-3 as shown in Fig. 1(left), where $e^+e^- \rightarrow K_S K_S \pi^+ \pi^-$ [106], $K_S K^\pm \pi^\mp \pi^+ \pi^-$ [105], and $\pi^+ \pi^-$ [95,96] have appeared since WP20, and many more channels are still being analyzed.

The most crucial channel for evaluation of $a_{\mu}^{\text{HVP, LO}}$ is the simplest $e^+e^- \rightarrow \pi^+\pi^-$ production. The study of this process has a long history since the first e^+e^- colliders, with numerous experiments that have contributed over the years, as shown in Fig. 1(right). This enormous effort is still not sufficient; to match the expected precision of the $g-2$ experiments, this channel should be known with a precision better than 0.2%. The latest, and one of the most precise measurements, was provided by the CMD-3 experiment [95,96]. This new measurement was intensely scrutinized and reviewed by the community within a specially organized series of seminars [79,80], where a comprehensive list of questions covering all aspects of the analysis was asked by a panel of experts nominated by the $g-2$ π Steering Committee. No pitfalls were found.

The $e^+e^- \rightarrow \pi^+\pi^-$ CMD-3 analysis is based on the largest ever data set of 34×10^6 selected $\pi^+\pi^-$ events at $\sqrt{s} < 1$ GeV, which is an order of magnitude larger statistics compared to most of the previous measurements. The large statistics were crucial to study various systematic effects in detail and to obtain a sharper view of possible detector effects. It would be impossible to apply the analysis procedure, developed for the CMD-3 measurement, to the data collected by the predecessor CMD-2 experiment, as many of the systematic effects would be hidden under statistical fluctuations. The estimation of the possible systematic effects in the CMD-3 analysis is done in the most conservative way, with the goal to achieve a high confidence in the final declared precision even at the expense of possible increase in error.

One of the key features of the analysis, in contrast to other measurements, is the use of three independent sources of information for measuring the number of detected $\pi^+\pi^-$ events: using momentum distributions of two particles measured in the tracking system, using detected energy depositions in the LXe calorimeter, or using the polar angular distribution. All three methods are consistent within 0.2%.

Another important aspect of the performed analysis by CMD-3 was a comprehensive study of the detector acceptance systematic uncertainty due to the determination of the polar angle. As a by-product of this investigation, the forward-backward charge asymmetry of $\pi^+\pi^-$ was measured with an integrated statistical precision of about 0.025%. A strong 1% deviation was observed from the prediction based on the conventional scalar QED (sQED) approach used for the calculation of radiative corrections, as shown in Fig. 2(left), which was never noted before. The improved generalized-vector-meson-dominance (GVMD) model was proposed in Ref. [141], which gives remarkable agreement with the experimental data. This was later confirmed by a calculation in the dispersive formalism [142], see Section 2.7.4. At first the dispersive approach prediction still displayed some visible deviation from the data, but a subsequent identification of an incorrect treatment of endpoint singularities in the dispersive calculation integrals led to an astonishing consistency with the experimental $\pi^+\pi^-$ asymmetry and the GVMD prediction, see Refs. [142,143] and Figs. 2 and 23. Independent beyond-sQED calculations within the GVMD and the dispersive formalism were also performed in Ref. [143], and some of the cross-checks were facilitated by the *RadioMonteCarLow 2* effort, see Section 2.4. The observed limitation of the sQED approach also suggests to revisit the radiative corrections used in the initial-state-radiation (ISR) measurements, where this can contribute to the C-even two-photon-exchange corrections and consequently affect the experimental extraction of the pion vector form factor (VFF) F_π^V , see Refs. [144,145]. The dispersive approach offers a more comprehensive description of the pion VFF, as it incorporates the correct analytic properties by construction. However, the GVMD model can provide a much simpler framework for more complicated calculations involving additional hard photons, and it also allows for iterative refinement of the VFF through the inclusion of additional terms. Ideally, having both calculations available would be beneficial for cross-checks.

The dominant systematic contribution in the CMD-3 $\pi^+\pi^-$ measurement comes from the fiducial-volume determination and was conservatively estimated as 0.5%/0.8% (RHO2018/2013) as shown in the summary systematic Table 2 of

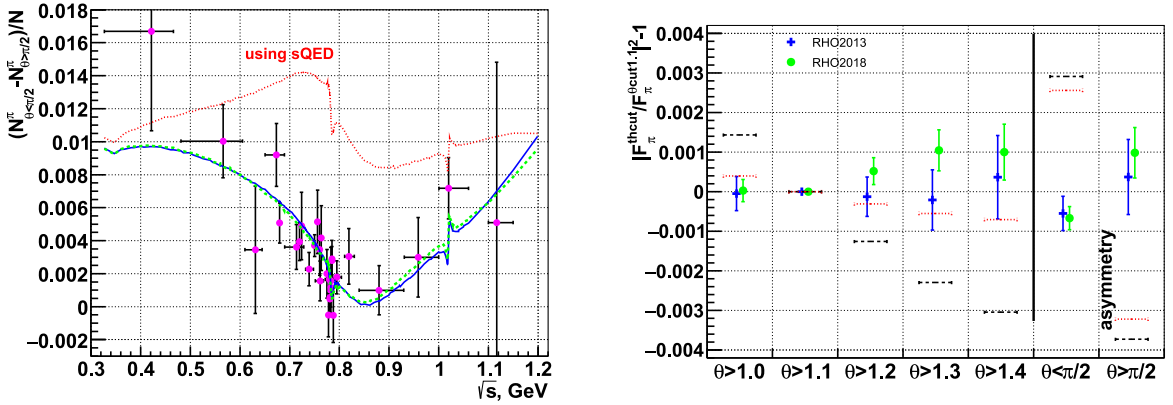


Fig. 2. Left: The measured asymmetry in the $\pi^+\pi^-$ process at CMD-3 in comparison with the prediction based on the conventional sQED approach (red dotted line), the GVMD model [141] (blue line), and the dispersive calculation [142] (green dashed line). Right: The relative differences between the pion form factors in the cases of the various θ^{cut} cut values. Different data taking seasons are shown by crosses (RHO2013) and circles (RHO2018). Also shown is the expected behavior in the case of the presence of a 0.5% systematic uncertainty in $|F^V_{\pi}|^2$ from the polar angle determination, either due to a Z-scale detector miscalibration (black dashed lines) or due to a constant shift in the reconstructed angle (red dotted lines). Source: Figures adapted from Ref. [96].

Ref. [96]. The achieved consistency at the permil level in the measured asymmetries of $e^+e^- \rightarrow \mu^+\mu^-$, $\pi^+\pi^-$ processes in CMD-3 together with overall stability of the result over different detector regions gives a quite confident level of control of this source of systematics. Such a cross-check with the extracted pion VFF within the different polar angle selections is shown in Fig. 2(right). For comparison, dashed lines show how the experimental points should look like in case of the presence of a 0.5% systematic effect from the fiducial volume in the measured cross section due to the most natural scenarios, either by a miscalibrated Z-scale of the detector or by a constant bias of the reconstructed polar angle for tracks of one of the charges. It is seen that the data is about 5 times more consistent than the declared assigned systematics. It should also be noted that some other previous measurements, which showed the dependence of results versus the detection polar angle, demonstrated variations an order of magnitude larger than those achieved by CMD-3.

Another important test was the measurement of $e^+e^- \rightarrow \mu^+\mu^-$ cross section, predicted by QED. It was done for the momentum-based analysis for $\sqrt{s} < 0.7$ GeV only, where the momentum resolution of the tracking system allowed to separate muons from other particles. The observed average ratio of the measured cross section to the QED prediction 1.0017(16)(61) proves the consistency of most parts of the analysis procedure, including the separation procedure, detector effects, evaluation of radiative corrections, etc.

Reaching improved precision requires support by MC tools. One of the drawbacks of the CMD-3 analysis was that only one generator, MCGPJ, with the required precision was available for the $e^+e^- \rightarrow \pi^+\pi^-$ process. Since the CMD-3 publication, the Pavia group presented a calculation of the $\pi^+\pi^-(\gamma)$ hadronic channel at NLO matched to a parton shower algorithm, with implementation in the new version of the BABAYAGA@NLO MC event generator [143]. The values of integrated radiative corrections and the forward-backward charge asymmetry in $\pi^+\pi^-$ were confirmed. The strong *RadioMonteCarLow* 2 community effort, going towards next-to-next-to-leading-order (NNLO) precision and above, will help further clarify and strengthen the theoretical side of the experimental $\pi^+\pi^-(\gamma)$ measurements, as discussed in Section 2.4. This effort will be mandatory if there is a wish to reach permil-level experimental precision in this channel. This aspect is even more critical for the ISR measurements right now, as the LO of the ISR processes starts at NLO precision of MC tools for the energy-scan experiments.

The total systematic precision of the CMD-3 $e^+e^- \rightarrow \pi^+\pi^-$ measurement was conservatively estimated as 0.7% in the central region near the ρ resonance. It is important to give a dedicated clarification regarding the treatment of the provided experimental systematic uncertainties. Being dominated mainly by a single source, the CMD-3 publication provides only the total systematic error for each point and suggests using a single-source full systematic covariance matrix accordingly, for simplicity of further external use. This yields an even more conservative evaluation of the uncertainty of the $a_{\mu}^{\text{HVP, LO}}$ integral compared to splitting over different statistically independent systematic contributions. Such a prescription in the case of direct point-by-point integration conventionally assumes using 100% correlated systematic uncertainty. However, strictly speaking, the exact energy dependence of the required systematic corrections are unknown. Therefore, it is essential to consider the uncertainties of the uncertainty, as argued in Sec. 2.3.6 of WP20. This is especially important given the reached advancements in statistical precision and the entirely dominant systematic uncertainties of the latest CMD-3 measurement. A strict interpretation of the provided systematic error values is that they accommodate an underlying systematic effect with possible internal correlations up to 100%. Furthermore, these values allow for any energy-dependent functional form of the required systematic correction, provided it remains within the specified systematic band. This prescription was commonly assumed by experimentalists in most of the $e^+e^- \rightarrow$ hadrons measurements and for most of the separate systematics sources (even when a correlation matrix was provided). Applications that exploit

correlations between data and consider only 100% strict correlation from the first to the last energy point of different split systematic contributions can produce a significant underestimate of the uncertainty on the quantity being evaluated. This, for example, applies to the all analyticity-constrained fits described in Section 2.6, where using only “100% correlated systematic” of each separate source strongly reduces the degrees of freedom of the fit and consequently shrinks the systematic part of errors from that. Such an underestimation of the systematic error of the evaluated $a_\mu^{\text{HVP, LO}}$ integrals may in some cases be greater than a factor 1.5, as shown in Ref. [146] and noted in Section 2.6.1 for the CMD-3 data treatment case.

The published CMD-3 pion VFF result was based on the full data sample collected at energies below 1 GeV before 2024 (62 pb⁻¹) and on a small subset of data, about 24 pb⁻¹, collected at energies above 1 GeV. The full data sample above 1 GeV is under analysis. The preliminary results on the pion VFF were obtained for data collected in 2019–2021 (about 170 pb⁻¹). The major data sample of 580 pb⁻¹ was collected in 2022 and 2023; these data are yet to be incorporated into the ongoing analysis. Data analysis at these CM energies differs from the one implemented in Ref. [96]:

- The momentum resolution of the CMD-3 tracking system does not allow one to distinguish momenta of e , μ , and π in $e^+e^- \rightarrow e^+e^-$, $\mu^+\mu^-$, and $\pi^+\pi^-$, correspondingly. Thus, the final-state identification is based on the analysis of energy deposition in the calorimeter. In order to cross-check the identification procedures, two independent methods are being developed—one is based on data from the LXe calorimeter only (similar to the method described in Ref. [96]) and another is based on the data from the full calorimeter and machine learning. Muons and pions cannot be robustly distinguished by the CMD-3 calorimeter response, thus the number of $\mu^+\mu^-$ pairs is evaluated from the number of detected e^+e^- . The identification procedure based on the analysis of angular distributions is statistically limited and is applied only as an overall cross-check, not point-by-point.
- At energies above 1 GeV, the cross section for the 2π final state drops, but it rises for many other hadronic final states, most notably for K^+K^- . In order to keep background low, strict selection cuts are used, which introduces larger efficiency corrections, which differ for e^+e^- , $\mu^+\mu^-$, and $\pi^+\pi^-$.

New data taking at energies below 1 GeV started in 2024 and is expected to continue through 2025 with the goal to scan the whole CM energy range from 0.32 to 1 GeV and to collect 2–3 times more data compared to previous scans. The CMD-3 result [96] is systematically limited. Nevertheless, the gain in statistics will be beneficial for the pion VFF measurement as well, as it will open additional possibilities for the systematics studies. It should be noted that a significant bulk of the data collected (and planned to be collected) in 2024/2025 is taken with the directions of e^+ and e^- beams reversed; that allows one to cross-check various detector-specific efficiency corrections.

The luminosity provided by VEPP-2000 at $\rho(770)$ energies allows for statistical accuracy of (0.1–0.2)% for the pion VFF. Such precision is required in order to match the accuracy of the $a_\mu^{\text{HVP, LO}}$ evaluation in the dispersive approach to the precision of the final FNAL result. Reduction of the systematic uncertainty to the same level requires a significant upgrade of the CMD-3 detector. The key directions of the upgrade are: a new tracking system with improved momentum resolution; the possibility for the precise and robust measurement of the polar angles of the tracks and the position of the vertex along the beam axis; a dedicated system to study nuclear interactions of pions with the material of the beam pipe and the tracking system. The upgrades are being designed. It is estimated that the data taking with the upgraded detector can start around 2030. It should be mentioned that 0.1% precision for the pion VFF also sets very high requirements for the theoretical models and corresponding MC codes used to evaluate the radiative corrections.

The same data sets discussed here are used for the measurement of cross sections of many exclusive channels of $e^+e^- \rightarrow \text{hadrons}$ beyond $e^+e^- \rightarrow \pi^+\pi^-$. The second most important channel after 2π for the $a_\mu^{\text{HVP, LO}}$ evaluation is $e^+e^- \rightarrow \pi^+\pi^-\pi^0$. The first result for this channel was published as a by-product of the 2π measurement [96], based on the small subset of $\pi^+\pi^-\pi^0$ identified as a background to $\pi^+\pi^-$. A dedicated analysis of the full $\pi^+\pi^-\pi^0$ data set is in progress and is expected to be released in 2025.

2.2.2. SND

Since 2020 several hadronic cross sections were measured at the SND experiment at the VEPP-2000 e^+e^- collider. A precise measurement of the $e^+e^- \rightarrow K_S K_L$ cross section was carried out in the vicinity of the ϕ meson resonance [147]. The systematic uncertainty in the measured cross section at the maximum of the ϕ resonance is 0.9%. The $e^+e^- \rightarrow \pi^+\pi^-\pi^0$ cross section was measured in the energy range (1.075–1.975) GeV [148]. The SND result is consistent with the previous BABAR measurement [149], but is more accurate. The SND data on the $e^+e^- \rightarrow K^+K^-\pi^0$ cross section below 2 GeV [150] are also in reasonable agreement with the BABAR measurement [151]. The $e^+e^- \rightarrow n\bar{n}$ process was studied in the energy range from threshold up to 2 GeV. The measured cross section [152–154] is presented in Fig. 3(left). New data on the processes with multiphoton final state $e^+e^- \rightarrow \eta\pi^0\gamma$ [155], $e^+e^- \rightarrow \eta\eta\gamma$ [156], $e^+e^- \rightarrow \eta\gamma$ [157], $e^+e^- \rightarrow \eta'\gamma$ [158] were obtained.

Currently, the SND collaboration works with $\pi^+\pi^-$ and $\pi^+\pi^-\pi^0$ channels in the $0.548 \text{ GeV} < \sqrt{s} < 1 \text{ GeV}$ and $0.548 \text{ GeV} < \sqrt{s} < 1.1 \text{ GeV}$ energy regions, respectively, using 69 pb⁻¹ data. They have faced serious challenges due to the fact that detector performance was very inconsistent during the 2017–2018 experiment. Despite these problems, it is possible to perform high-precision measurements of the $e^+e^- \rightarrow \pi^+\pi^-$ cross section, using a modified version of the reconstruction software. A new algorithm of e/π separation allows one to suppress $e^+e^- \rightarrow e^+e^-$ background by a factor

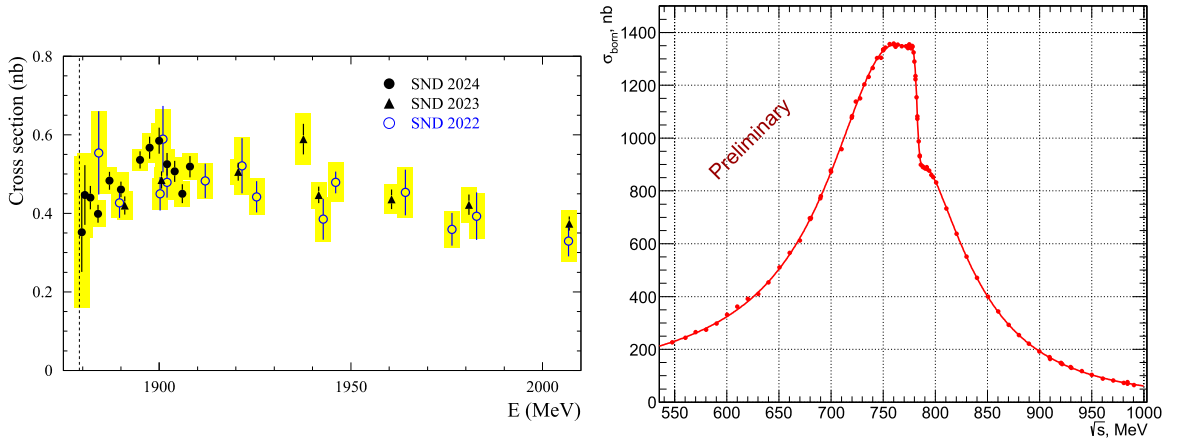


Fig. 3. Left: The $e^+e^- \rightarrow n\bar{n}$ cross section measured by SND [152–154]. The error bars and shaded boxes represent the statistical and systematic uncertainties, respectively. Right: The $e^+e^- \rightarrow \pi^+\pi^-$ born cross section measured by SND. The red line is a fit curve.

of 10^3 in the whole energy region, thus limiting the growth of the systematic error with decrease of $\sqrt{s}$. An improved reconstruction algorithm allows one to reduce the number of poorly reconstructed tracks. Simulated $e^+e^- \rightarrow \pi^+\pi^-$ events now better describe experimental inefficiencies of the cosmic veto and energy-deposition cut, after introduction of the light quenching effects in the plastic scintillator of the muon system and NaI(Tl) of the electromagnetic (EM) calorimeter. The current number of the selected $e^+e^- \rightarrow \pi^+\pi^-$ events makes the statistical uncertainty negligible compared to the systematic one. Due to the increased tensions between the most precise measurements of the $e^+e^- \rightarrow \pi^+\pi^-$ cross section, a blinding procedure was introduced for the earlier stages of the SND analysis. The efficiency of $e^+e^- \rightarrow \pi^+\pi^-$ was modified via multiplying it by $1 + a_0 + a_1 \times (\sqrt{s} - 775 \text{ MeV})$, where a_0 and a_1 are randomly generated parameters, unknown to the researchers. The preliminary results of the unblinded $e^+e^- \rightarrow \pi^+\pi^-$ cross-section measurement are shown in Fig. 3(right). Current estimates of the systematic uncertainty are (0.6–0.8)% in the whole energy range, due to much higher statistics and improvements in the simulation and reconstruction algorithms, compared to the 2020 SND analysis. Progress with the $\pi^+\pi^-\pi^0$ channel was delayed, since this analysis relies on the high-precision measurements of the integrated luminosity from the $\pi^+\pi^-$ analysis. Preliminary estimates of the systematic uncertainty of $e^+e^- \rightarrow \pi^+\pi^-\pi^0$ cross-section measurements show it to be no greater than 1.4%.

2.2.3. KLOE

The KLOE experiment took data during 2001–2006, with a total integrated luminosity of 2.5 fb^{-1} at $\sqrt{s} = 1.0194 \text{ GeV}$ (the ϕ peak) and 250 pb^{-1} off-peak at $\sqrt{s} = 1 \text{ GeV}$ (in 2006). Peak luminosity was reached during the 2005 run with a luminosity of $8.5 \text{ pb}^{-1}/\text{day}$. KLOE has published three precise cross-section measurements of $\sigma(e^+e^- \rightarrow \pi^+\pi^-\gamma(\gamma))$: KLOE08 [132], KLOE10 [133], and KLOE12 [134], which have been used for the evaluation of the 2π contribution to $a_\mu^{\text{HVP, LO}}$ [1]. The KLOE08 and KLOE12 analyses were performed with a small-angle (SA) selection where the photon is required to be within $\theta_\gamma < 15^\circ$ or $\theta_\gamma > 165^\circ$ (i.e., missing momentum) resulting in 240 pb^{-1} ($\simeq 3.5$ million $\pi\pi\gamma$ events) of data taken in 2002. The KLOE10 analysis was performed on 232 pb^{-1} from 2006 (corresponding to 0.6 million events), after the photon polar angle is selected at large angle (LA), i.e., $50^\circ < \theta_\gamma < 130^\circ$. Further details of the analyses can be found in Refs. [1,145,159], and details on the agreement and combination of the three results are found in Ref. [135].

In the KLOE08, KLOE10, and KLOE12 analyses, the MC generator PHOKHARA (version 5) was used, including NLO ISR, final-state-radiation (FSR) corrections, and simultaneous emission of one ISR and one FSR photon [160,161]. PHOKHARA was interfaced with the standard KLOE MC GEANFI to compute MC efficiencies as a function of the reconstructed $M_{\pi\pi}^2$, and to compute the radiator function H used to relate the measured differential cross section for $e^+e^- \rightarrow \pi^+\pi^-\gamma$ to the cross section $\sigma_{\pi\pi}$. In addition, PHOKHARA was used to correct for the shift between the measured $M_{\pi\pi}^2$ and the invariant mass of the intermediate photons for FSR events. The FSR treatment was done assuming point-like pions (F $\times$ sQED, see Ref. [145] for definitions).

The KLOE12 analysis normalized the cross section using a $\mu^+\mu^-\gamma$ cross-section measurement with the photon at SA. The error quoted by the authors of PHOKHARA on the ISR part of the generator is 0.5%, this error cancels out in the extraction of the pion VFF $|F_\pi^V|^2$ from the $\pi^+\pi^-\gamma/\mu^+\mu^-\gamma$ ratio. In the KLOE12 analysis, a global uncertainty of 0.2% was estimated for the factorization approximation when subtracting FSR for muons, and for the correction due to the FSR shift for pions. The effects of additional diagrams not present in PHOKHARA (v5) for muons was estimated to be 0.1% at most. The SA selection greatly suppresses events with only one hard photon from FSR and no ISR, as well as a large part of events with one photon from ISR and one from FSR. For KLOE08, the uncertainty due to the treatment of FSR events is estimated to be 0.3%. For KLOE10, the uncertainty on the treatment of FSR has been estimated to be 0.5% due to the

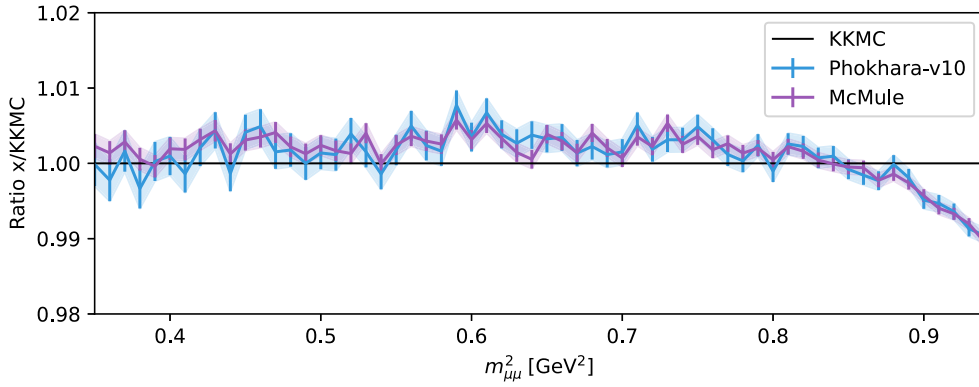


Fig. 4. Cross section of $e^+e^- \rightarrow \mu^+\mu^-\gamma$ vs. the invariant mass of two muons, in the KLOE SA acceptance selection scenario, with ISR and FSR. The results from PHOKHARA and McMULE were compared to KKMC (black line), showing an agreement better than 0.5% for most of the spectrum.

uncertainty of the pion VFF at $\sqrt{s} = 1$ GeV and 0.6% due to a comparison of FSR modeled with SU(3) chiral perturbation theory (ChPT) [145,162] with respect to the treatment used in PHOKHARA.

Within the *RadioMonteCarLow* 2 efforts [145], “tuned comparisons” (see Ref. [163] for the definition) between PHOKHARA 10.0⁴ (which has a full NLO matrix element for $\pi^+\pi^-\gamma$ and $\mu^+\mu^-\gamma$ final state) and other MC generators – namely, AFKQED, BABAYAGA@NLO, KKMC, and McMULE, all described in Ref. [145] – were performed [145,165], in order to investigate radiative correction effects both due to missing NNLO contributions in PHOKHARA and also to test the validity of the generator at NLO, as questioned in Ref. [166]. The following configurations have been studied:

- $e^+e^- \rightarrow \mu^+\mu^-\gamma(\gamma)$ before and after SA angular acceptance, considering both ISR and FSR emission;
- $e^+e^- \rightarrow \pi^+\pi^-\gamma(\gamma)$ before and after SA and LA angular acceptance for only ISR emission—due to limitations of other MC generators used in this comparison [165]. In the case of LA acceptance, it was also possible to compare AFKQED vs. PHOKHARA after the trackmass cut;⁵ this was not possible in the case of SA due to the AFKQED limitations for the KLOE configuration.

In this study, the $e^+e^- \rightarrow \pi^+\pi^-\gamma$ and $e^+e^- \rightarrow \mu^+\mu^-\gamma$ cross sections as a function of the invariant mass of the two pions and muons from PHOKHARA were compared with the predictions from other generators [145,159]. Vacuum polarization (VP) was switched off and detector effects (involving smearing of momenta) were not included. The selection cuts used for the SA and LA comparisons are described in Ref. [145], in the sections “KLOE-like small-angle scenario” and “KLOE-like large-angle scenario”, respectively.

The studies on muons showed good agreement with the SA acceptance between PHOKHARA, McMULE, and KKMC, with a difference below 0.5% between PHOKHARA and KKMC for most of the $M_{\mu\mu}^2$ spectrum, as shown in Fig. 4. We note, see Fig. 32 of Ref. [145], that this is not the case for the so-called “B-scenario” where differences between generators (particularly KKMC and AFKQED) reach a few percent at $M_{\mu\mu} < 1$ GeV. In the inclusive selection (i.e., without angular acceptance cuts), there are known differences for $M_{\mu\mu}^2 > 0.85$ GeV² between PHOKHARA and KKMC [167], up to $\simeq 1\%$, due to missing exponentiation in PHOKHARA, but with negligible effects on $a_\mu^{\text{HVP, LO}}$ (in the pion case)—which is dominated by the low-energy region.

The studies on pions showed excellent agreement between PHOKHARA and McMULE in the inclusive and SA [165] and LA acceptance selections, see Fig. 5(left). This is good confirmation of the correctness of the implementation of full NLO matrix elements in PHOKHARA. The LA acceptance selection (KLOE10) was studied also after the trackmass cut, see Fig. 5(right), which is sensitive to (N)NLO effects. With acceptance cuts only, an agreement below 0.5% was found around the ρ peak ($M_{\pi\pi}^2 \simeq 0.6$ GeV²) between PHOKHARA and AFKQED. With an additional cut on the trackmass variable on top of acceptance cuts, the agreement at the ρ peak was found to be below 1%, as shown in Fig. 5(right). The larger differences observed below 0.4 GeV² are expected to originate from the different treatment of radiative corrections. While the size of the differences is still within the systematic uncertainty of KLOE10 [133], see Fig. 5(right), for the future analyses this difference is a further indication of the need for a NNLO MC generator, on which work is in progress [145].

Preliminary results, which do not yet account for realistic detector effects, indicate the following:

- PHOKHARA shows excellent agreement with McMULE for $\mu\mu\gamma$ and $\pi\pi\gamma$. These results confirm the correct implementation of NLO matrix elements in PHOKHARA and exclude possible issues associated with PHOKHARA as questioned in scenario 1 of Ref. [166].

⁴ Differences between PHOKHARA5 (which was used by KLOE) and PHOKHARA10 have been investigated in Ref. [164].

⁵ The trackmass is a kinematic variable defined in Ref. [132]. The tails in its distribution are mainly due to radiative corrections, therefore, it is sensitive to (N)NLO effects.

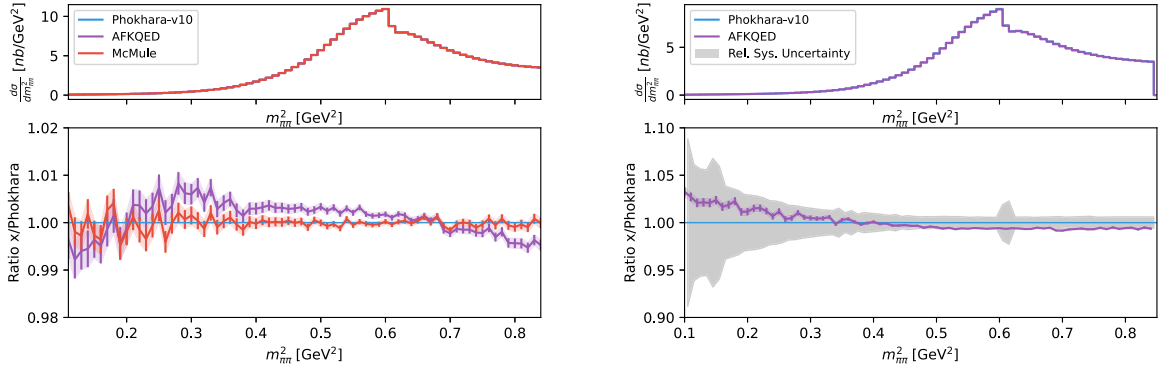


Fig. 5. Left: Cross section of $e^+e^- \rightarrow \pi^+\pi^-\gamma$ vs. the invariant mass of two pions, in the KLOE LA acceptance selection scenario, with ISR only, no trackmass cut is applied. The results from AFKQED and McMULE are compared to PHOKHARA (blue line), showing an agreement better than 0.5% at the ρ peak. Right: Cross section of $e^+e^- \rightarrow \pi^+\pi^-\gamma$ vs. the invariant mass of two pions, in the KLOE LA acceptance selection scenario, with ISR only, with the trackmass cut applied. The results from AFKQED are compared to PHOKHARA (blue line), showing an agreement better than 1% around the ρ peak (0.6 GeV²). The systematic uncertainty for KLOE10 LA analysis is shown as a function of the invariant mass of the two pions (gray band).

- For $\mu\mu\gamma$, the agreement between PHOKHARA and KLMC is mostly within $\leq 0.5\%$ for ISR, ISR+FSR in both inclusive and SA selections, before the trackmass selection is applied. Differences above 0.85 GeV² (in $m_{\mu\mu}^2$), reaching up to approximately 1% are due to missing exponentiation in PHOKHARA (and have negligible effects on $a_{\mu}^{\text{HVP, LO}}$ in the pion case). Work is currently in progress to estimate the effects of radiative corrections due to the trackmass cut on muons, which can be sensitive to the presence of additional photon emission.
- For $\pi\pi\gamma$ a comparison with AFKQED in the LA selection reveals differences below 0.5% when considering acceptance cuts, and below 1% when including also the trackmass cut, at the ρ peak. These findings limit the impact of NNLO effects on the LA analysis (KLOE10) as possible explanation for the discrepancy with BABAR and CMD-3 experiments.

More studies are in progress, specifically: enlarging the statistics; including realistic detector effects (GEANFI); including more refined hadron–photon models for FSR (beyond F×sQED) [145] and ISR NNLO radiative corrections; and applying the trackmass selection for SA $\pi\pi\gamma$ (KLOE08) and $\mu\mu\gamma$ (KLOE12) analyses.

A new SA KLOE 2π analysis on 2004–2005 data, normalized to $\mu\mu\gamma$, has started with the aim of clarifying the current tension with respect to other HVP evaluations. It is based on $\simeq 25$ million $\pi\pi\gamma$ events, corresponding to 7 times the statistics of the KLOE published analyses, which have not yet been analyzed. The 2006 off-peak data will also be used for additional cross-checks and systematic studies. This new analysis, called KLOE-nxt, will employ new software tools and additional validation tests aiming to a precision of 0.4% on the 2π contribution to $a_{\mu}^{\text{HVP, LO}}$, representing a two-fold improvement in accuracy compared to the previous KLOE12 result [168]. It will also benefit from the ongoing work towards a NNLO implementation of PHOKHARA [145]. KLOE-nxt will provide an important cross-check of the published results and will be conducted in a blinded manner [169].

In addition, KLOE is analyzing the three-pion cross section using the radiative-return method with 1.7 fb^{−1} of data collected at the ϕ meson mass [170].

2.2.4. BABAR

A number of hadronic processes have been studied by BABAR since WP20, including $e^+e^- \rightarrow 2(\pi^+\pi^-)\pi^0\pi^0 + \pi^0/\eta$ [171] and $\pi^+\pi^-\pi^0\pi^0\pi^0 + \pi^0/\eta$ [172] in 2021, as well as $K^+K^-\pi^0\pi^0\pi^0$, $K_S^0K^\pm\pi^\mp\pi^0\pi^0$, $K_S^0K^\pm\pi^\mp\pi^+\pi^-$ [173] in 2022. In particular, the $\pi^+\pi^-\pi^0$ cross section, which represents the second largest input of the HVP contribution to a_{μ} , was measured in 2021 [149] using 469 fb^{−1} of data collected by the experiment near the $\Upsilon(4S)$ resonance, with CM energies ranging from 0.62 to 3.5 GeV via the ISR method. The analysis requires all final-state particles to be detected, namely two good-quality opposite-sign charged tracks and at least three photons, one being the ISR candidate with a CM energy larger than 3 GeV while the remaining photons must form one or more π^0 candidates with invariant masses $0.1 < M_{\pi^0} < 0.17$ GeV. A kinematic fit is performed on selected events and provides a χ^2 quality value used to reject background processes, estimated with MC simulation. The cross section is studied separately below and above 1.1 GeV because of the detector resolution which significantly distorts the 3π mass spectrum in the lower region. From threshold to 2.0 GeV, the measured HVP contribution of this channel is $45.86(14)(58) \times 10^{-10}$, with a systematic uncertainty of 1.3% near the ω and ϕ resonances, dominated by the detection efficiency, radiative correction, and luminosity. This result improves the precision by a factor of about 2 compared to calculations based on previous $\pi^+\pi^-\pi^0$ analyses.

The last BABAR analysis to measure the $\pi^+\pi^-(\gamma)$ cross section was published in 2009 and 2012 [139,174] with around half the BABAR statistics. To cancel out the ISR photon efficiency and the VP, the measured $\pi^+\pi^-(\gamma)$ mass spectrum is divided by the $\mu^+\mu^-(\gamma)$ spectrum, equivalent to the ratio of each final state's bare cross section. The separation between pions and muons was based on particle identification (PID) which required both charged tracks to have momenta larger

than 1 GeV in order to make the muon identification more reliable. PID was found to be the dominant source of systematic uncertainties on the a_μ prediction, with a total relative systematic error of 0.5% for energies from 0.5 to 1 GeV.

An upcoming BABAR analysis, intended to be published in 2025, aims at improving the precision on the HVP contribution to a_μ from $\pi^+\pi^-(\gamma)$. For that purpose, the entirety of BABAR data will be studied, while PID requirements on the tracks will be removed. An angular fit is considered as a new method [175] to distinguish the main signal ($\pi^+\pi^-(\gamma)$, $\mu^+\mu^-(\gamma)$) and remaining background ($K^+K^-(\gamma)$, $e^+e^-(\gamma)$) processes, based on the cosine of the angle between the negative charge track and the ISR photon in the 2-track CM frame. To further differentiate the shapes of the dipion and dimuon distributions, the track momentum selection required in the previous analysis is released, increasing the statistics at the same time. Overall, this new study will result in an independent measurement of the $\pi^+\pi^-(\gamma)$ cross section.

A first step towards this objective was taken with a study of additional radiation in ISR processes, published in 2023 [176] and relying on all the data collected by BABAR. The intent was to evaluate how accurate the PHOKHARA and AFKQED generators are in simulating data and to measure the relative proportions of additional radiations in $\mu^+\mu^-(\gamma)$ and $\pi^+\pi^-(\gamma)$ processes with ISR, at “LO” (no additional photon), “NLO” (+1 real photon), and “NNLO” (+2 real photons, simulated only by AFKQED). Tracks are assumed to have pion masses, while dipion and dimuon events are identified according to tight PID selections. Kinematic fits are performed on events according to the different configurations in which additional photons are emitted. In the case of “NLO” events, two fits are applied depending on whether the single additional photon is detected in the EM calorimeter at a large angle (LA) from the beams, between 0.35 and 2.4 radians, or it is emitted at a small angle (SA), assumed to be collinear with the beams. For “NNLO” events, three fits are devised considering the two additional photons can both make large angles (2LA), small angles (2SA), or one of each configuration (LA+SA).

Inputs to the fits are the measured energy and direction of the ISR photon, the momenta and angles of both charged tracks, as well as the measured energy and angles of detected additional photons in the LA fit. Returned quantities are χ^2 values and fit kinematic variables. Events are classified depending on the smallest χ^2 , with the additional requirement that fit energies are larger than 200 MeV either in the laboratory frame for LA photons or in the e^+e^- CM frame for SA photons. Events below these thresholds are classified as “LO”.

The distinction between “NLO” LA radiations from ISR and FSR is performed by fitting the distribution of the minimum angle between the additional photon and the charged tracks, using templates from AFKQED (FSR, no LA ISR simulated) and PHOKHARA (ISR after subtraction of FSR). The separation is fixed at 20 degrees in both $\pi^+\pi^-(\gamma)$ and $\mu^+\mu^-(\gamma)$ samples, with FSR peaking below 10 degrees and LA ISR forming a wide bump centered around 60 degrees. Distribution shapes of LA photon energy simulated by both generators are observed to be in good agreement with data.

In contrast, an excess of “NLO” SA photons, all ISR, is observed in PHOKHARA samples compared to data, evolving with a positive slope as a function of the additional photon’s energy in the CM frame as seen in Fig. 6. This excess has been shown not to be due to the collinear assumption in the “NLO” SA fits with the help of an alternative zero-constraint calculation that does not assume any collinearity. It is however affected by feed-through of “NNLO” 2SA photons, originating from a same beam, into the “NLO” SA category. After correction, the slope disappears and an almost constant excess of simulated events is observed with an overall data/MC ratio of 0.750(8) in $\mu^+\mu^-(\gamma)$ and 0.763(19) in $\pi^+\pi^-(\gamma)$.

Significant “NNLO” contributions are found at the above-mentioned photon energy thresholds, 3.47(38)% and 3.36(39)% of events in the muon and pion channels, respectively, more than 3% being due to two additional ISR photons. AFKQED shows overall good performance in describing the rates and energy distributions of data at “NNLO”. A slightly high data/MC ratio of 1.061(15) for muons and 1.043(10) for pions is found up to the maximum generated energy.

The consequences for past $\pi^+\pi^-(\gamma)$ cross-section measurements vary depending on the experiment. The previous BABAR result is unaffected as “NLO” and higher orders were already included in the analysis. The event acceptance determined with PHOKHARA must be corrected by a factor $0.3(1) \times 10^{-3}$, negligible compared to the total 0.5% systematic uncertainty. This independence from the treatment of additional radiation in PHOKHARA confirms the robustness of the BABAR analysis with respect to radiative corrections. On the contrary, unaccounted shortcomings from this generator could potentially affect the results of other experiments which apply more stringent “LO” selections and rely on PHOKHARA for additional radiations.

2.2.5. BESIII

The BESIII Collaboration is committed to providing experimental inputs to the dispersive evaluation of HVP by measuring the most relevant hadronic channels in electron–positron annihilation. The timelike pion VFF, and related cross section, $\sigma(e^+e^- \rightarrow \pi^+\pi^-)$, which represents the most important contribution to $a_\mu^{\text{HVP, LO}}$, was measured in 2015 in the $\rho(770)$ region with a systematic accuracy of 0.9% [140]. The analysis makes use of a data sample of 2.9fb^{-1} collected at a CM energy of 3.77 GeV and the ISR technique is employed to access the relevant energy range. The event selection requires the detection of the $\pi^+\pi^-$ pair as well as the ISR photon in the fiducial volume of the apparatus. The four-momenta of these particles are subject to a four-constraint kinematic fit (4C), which enforces energy–momentum conservation under the assumption of the $e^+e^- \rightarrow \pi^+\pi^-\gamma$ final state. Conventional particle-identification methods are applied to reject background contributions from Bhabha scattering events, while for the separation of signal events from the QED process $e^+e^- \rightarrow \mu^+\mu^-\gamma$ a neural network method has been worked out. The latter is retained as control sample for various cross-checks and for a QED test. The PHOKHARA [164] event generator (version 7) is employed to determine selection efficiencies and corrections (e.g., VP, FSR), as well as to extract the radiator function at NLO. A normalization is obtained by means of

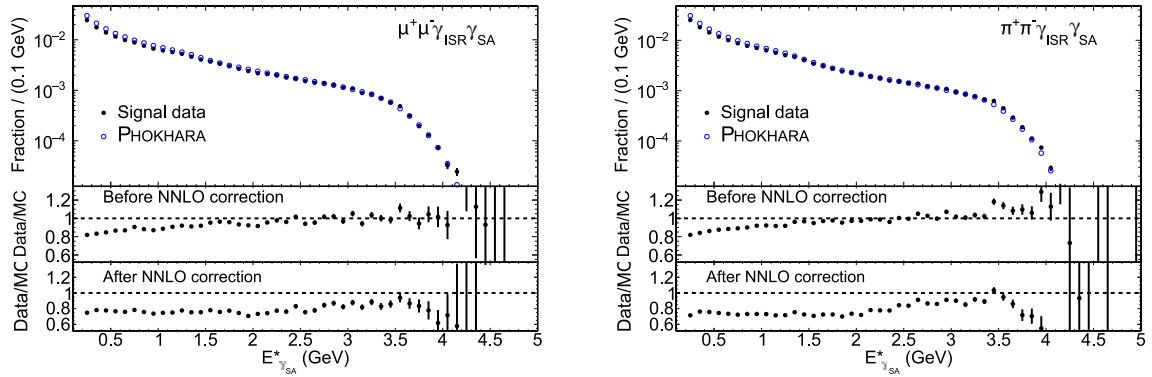


Fig. 6. Comparison of the fit energy $E_{\gamma SA}^*$ in the CM frame of the additional “NLO” SA photon, before and after “NNLO” correction, in the $\mu^+\mu^-(\gamma)$ (left) and $\pi^+\pi^-(\gamma)$ (right) data and PHOKHARA samples.

Source: Figures taken from Ref. [176].

the integrated luminosity, which is determined by measuring large-angle Bhabha events using the BABAYAGA@NLO [177] event generator.

Recent observations of the BABAR collaboration [176] have identified a limitation of the PHOKHARA event generator in the description of those NLO events in which two ISR photons are present (one at small angles, one at large angles). Such a limitation can introduce a mismatch in the χ^2 distribution of the 4C kinematic fit between the PHOKHARA simulation and real data, thus leading to an incorrect evaluation of the event-selection efficiency. As claimed in Ref. [166], the impact of this PHOKHARA limitation on the BESIII analysis might be as large as 3.2% in the case of the first of the two scenarios discussed in the paper, while the published BESIII result will be unaffected in the second scenario. The estimate of scenario 1, however, does not consider the effects of the photon efficiency calibration, which is based on a 1C fit. It has indeed been demonstrated that selections based on a 1C kinematic fit are much less affected by an imprecise description of NLO photons as these selections are more inclusive in NLO photon radiation. Effectively, the photon efficiency calibration procedure therefore mitigates the possible PHOKHARA issues found by the BABAR collaboration. A quantitative investigation has demonstrated that the impact on the BESIII pion VFF measurement can be constrained to $\leq 1\%$. Such a result is also confirmed independently by detailed comparisons of the selection efficiencies calculated using the PHOKHARA, AFKQED, and KKMC [178] event generators in the dimuon control sample. Fig. 7(left) shows the selection efficiencies for the three generators (upper plot) and the relative difference with respect to PHOKHARA (lower plot) for realistic selection cuts including the 4C kinematic fit and the photon efficiency calibration. Also in this case an agreement at the 1% level is observed among the three generators. Fig. 7(right) shows the same comparison for a selection in which the 1C kinematic fit is employed. Here, an even better agreement between the three event generators can be observed, which is due to the much more inclusive nature of the 1C kinematic fit. From these studies the scenario 1 of Ref. [166] with a big impact on the BESIII measurement cannot be confirmed. The development of additional event generators suited for ISR measurements, as well as the further development of those presently available, will be crucial for pinning down the actual uncertainty due to the employed version of PHOKHARA.

An improved measurement of the pion VFF at BESIII is foreseen in the upcoming years and aims to achieve $\mathcal{O}(0.5\%)$ accuracy, taking advantage of the additional 17 fb^{-1} collected in the past years at the CM energy of 3.77 GeV, increasing thereby the statistics by a factor of seven with respect to the 2015 result. The BESIII collaboration intends to improve the accuracy in a staged approach with the first result to be expected in 2025. New analysis techniques are presently being worked out, most importantly a selection without the explicit detection of the ISR photon. This is possible by constraining the event kinematics to the production of a $\pi^+\pi^-$ pair and one (undetected) photon in a 1C kinematic fit. As mentioned above, this method is largely insensitive to the possible limitations of the PHOKHARA event generator thanks to an increased acceptance of events with additional (soft) photons emitted. The updated analysis strategy is applied to the 2.9 fb^{-1} at 3.77 GeV considered in the 2015 analysis and to a data set of 3.1 fb^{-1} at 4.18 GeV, to produce an intermediate result with normalization to luminosity and a target precision of 0.7%. This measurement represents an important cross-check of the published measurement in terms of both analysis strategy and background contributions, thanks to the different CM energies of the analyzed data sets.

The ultimate precision of $\mathcal{O}(0.5\%)$ will be achieved by analyzing the full 20 fb^{-1} data sample, which opens the avenue to a normalization to $e^+e^- \rightarrow \mu^+\mu^-\gamma$ events with the additional advantage that the systematic uncertainties related to the radiator function and the luminosity drop in such an approach. In the intermediate future, corresponding high-precision analyses for the 3π , 4π , and K^+K^- channels are planned.

Recently, the BESIII Collaboration has published an inclusive R -value measurement in the energy region between 2.2 and 3.7 GeV, based on the energy-scan technique, with unprecedented precision at most of the considered energy points [179]. The analysis employs, as first in this kind of measurements, two independent event generators for simulating

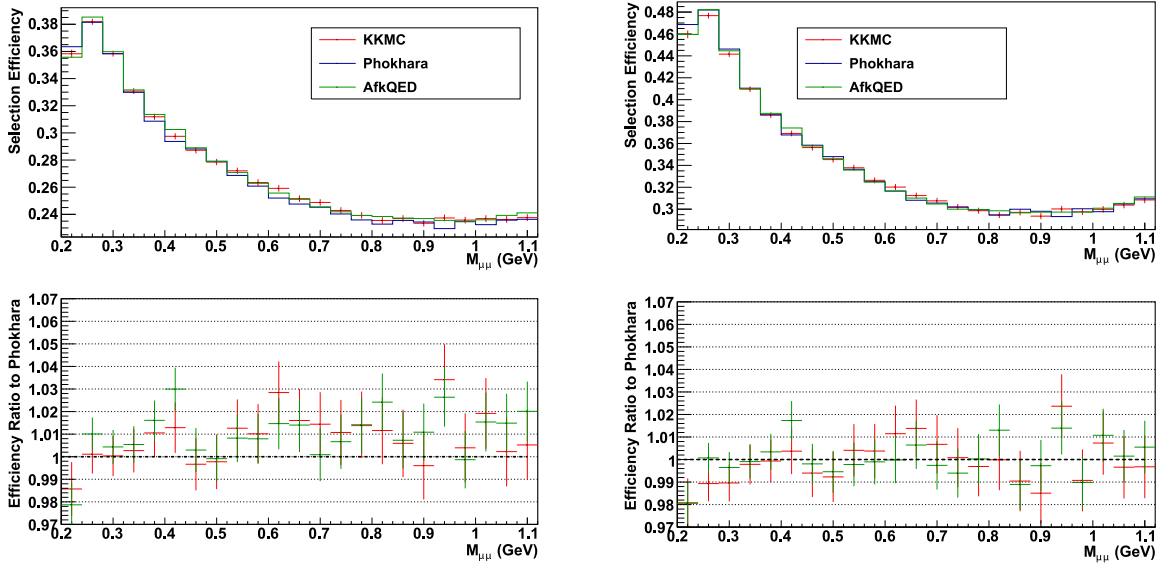


Fig. 7. Comparison of the pion VFF selection efficiency, after a kinematic fit with four (left) and one constraints (right), as evaluated with different MC generators.

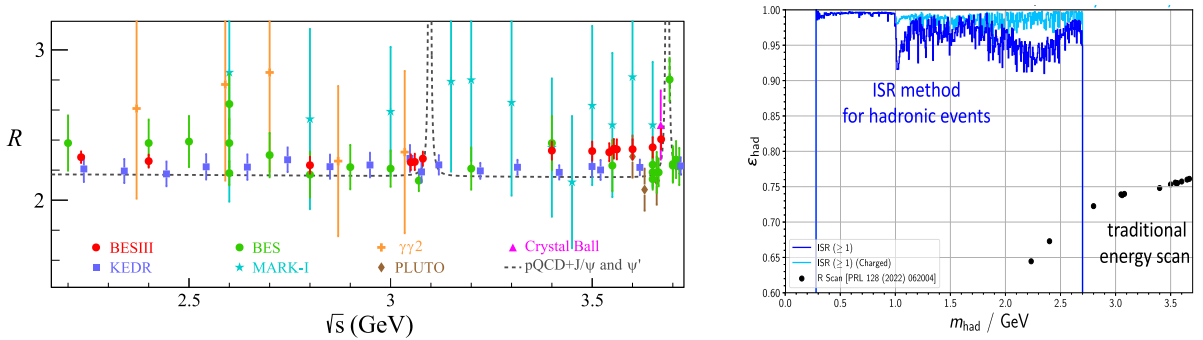


Fig. 8. Left: Result from the BESIII R -value measurement (figure adapted from Ref. [179]). Right: Preliminary selection efficiency for an inclusive measurement of R below 2 GeV with the ISR method.

the hadron production: a fully theoretical model based on the Lund model [180], called LUARLW [181], and the hybrid generator [182], which makes use of the well-established PHOKHARA [164] and the data-driven ConExc [183] event generators in combination with LUARLW used only for the remaining channels. The measured R -value shows some tensions with perturbative-QCD (pQCD) predictions, reaching a significance of 2σ above 3 GeV, see Fig. 8(left). The data sets analyzed in the published measurement represent a small fraction of the scan-data energy points collected by BESIII between 1.8 and 4 GeV, thus, further results are to be expected in the near future.

The energy region below approximately 2 GeV is inaccessible for inclusive R -value measurements using the traditional energy-scan method at e^+e^- colliders. This is mostly due to the dominant production of two-prong final states, e.g., $\pi^+\pi^-(\pi^0)$ or $K^+K^-(\pi^0)$, which can hardly be distinguished from the overwhelming QED background (mainly Bhabha events). Currently, an alternative approach based on the ISR technique is under development at BESIII for an inclusive R measurement below 2 GeV. It is based on the detection of the ISR photon in the central region of the detector and the reconstruction of (at least) one charged particle. The emission of a hard ISR photon boosts the hadronic system within the detector acceptance, such that the selection efficiency ($>95\%$) is found to be significantly higher than in traditional energy-scan measurements, see Fig. 8(right). As a consequence, the final result has a minimal dependence on fully inclusive event generators (e.g., LUARLW), which is currently the largest contribution to the systematic uncertainty in traditional measurements. Moreover, in large-angle radiative events, the background contribution from Bhabha scattering significantly reduces and can be effectively suppressed by simple particle-identification conditions. Preliminary studies suggest an accuracy of $\mathcal{O}(1\%)$ to be in reach, which will imply a valuable input to the dispersive evaluation of $a_\mu^{\text{HVP, LO}}$ as this inclusive method is a novel and independent approach for determining the HVP contribution.

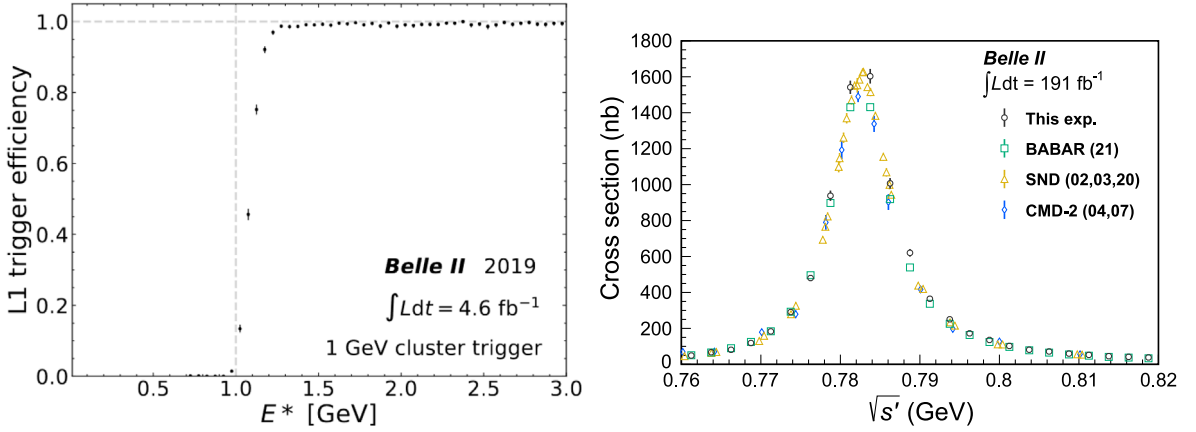


Fig. 9. Left: L1 trigger efficiency for high-energy photons as a function of the ISR photon energy in the CM frame. Right: Observed $e^+e^- \rightarrow \pi^+\pi^-\pi^0$ cross section as a function of energy compared with previous results at the ω resonance.
Source: Figure taken from Ref. [185].

2.2.6. Belle II

The $e^+e^- \rightarrow \pi^+\pi^-(\gamma)$ measurement at Belle II has adopted the same approach as the original BABAR analysis [139] and will use a data set of 427 fb⁻¹. ISR events are used to measure the cross section for $e^+e^- \rightarrow X\gamma$ at the reduced collision energy

$$\sqrt{s'} = m_X (X = \pi^+\pi^-, \mu^+\mu^-), \quad (2.4)$$

where s' is given by

$$s' = s \left(1 - \frac{2E_\gamma^*}{\sqrt{s}} \right), \quad (2.5)$$

and E_γ^* is the ISR photon energy in the CM frame. Muons and pions will be identified using the Belle-II particle-identification (PID) detectors and the ratio of cross sections of the pion pair to the muon pair will be measured. NLO QED effects and other backgrounds are validated or rejected using kinematic fits with multiple-photon hypotheses. Control-sample-based efficiency corrections $(\epsilon^{\text{data}}/\epsilon^{\text{MC}})_X$ related to different sources X are determined: trigger, PID, tracking, and fit χ^2 . Thus, the data efficiency ϵ^{data} is estimated from the MC efficiency ϵ^{MC} as

$$\epsilon^{\text{data}} = \epsilon^{\text{MC}} \left(\frac{\epsilon^{\text{data}}}{\epsilon^{\text{MC}}} \right)_{\text{trigger}} \left(\frac{\epsilon^{\text{data}}}{\epsilon^{\text{MC}}} \right)_{\text{PID}} \left(\frac{\epsilon^{\text{data}}}{\epsilon^{\text{MC}}} \right)_{\text{tracking}} \left(\frac{\epsilon^{\text{data}}}{\epsilon^{\text{MC}}} \right)_{\chi^2}. \quad (2.6)$$

To validate the analysis, the $e^+e^- \rightarrow \mu^+\mu^-(\gamma)$ cross section will be measured based on the luminosity derived from other QED processes [184].

The ISR-trigger efficiency is determined using a sample of $e^+e^- \rightarrow \mu^+\mu^-(\gamma)$ events selected by an independent track trigger. Fig. 9(left) shows the trigger efficiency as a function of E_γ^* in the barrel region of the EM calorimeter; the efficiency is 99.9% for $E_\gamma^* > 2$ GeV. The kinematic fits and background simulations were validated with a 1.8 fb⁻¹ subsample of data. Regarding tracking efficiency, a data-driven method tested with signal MC was developed. Studies of background, PID, and trigger are ongoing.

An $e^+e^- \rightarrow \pi^+\pi^-\pi^0$ measurement using 191 fb⁻¹ Belle II data has been published recently [185], with the main result shown in Fig. 9(right). The LO HVP contribution of this analysis to the muon magnetic anomaly is $a_\mu^{3\pi} = 48.91(0.23)(1.07) \times 10^{-10}$ in the (0.62–1.8) GeV energy range. This is 2.5 σ larger than the current most precise measurement from BABAR [149] and the global fit [186], see Section 2.6.2.

2.3. Tau data and isospin-breaking corrections

Since the hadronic weak charged current and the isovector component of the EM current belong to the same triplet of the isospin group, semi-leptonic τ decays can provide independent input for the computation of HVP, as first pointed out in Ref. [187]. In the limit of exact isospin symmetry, a simple kinematic factor relates the hadronic invariant mass distributions in $e^+e^- \rightarrow h$ and in $\tau \rightarrow \nu_\tau h'$, where h and h' denote hadronic states related by an isospin rotation. However, an analysis of HVP to sub-percent level requires an appropriate set of isospin-breaking (IB) corrections. In what follows, we first briefly review the experimental data sets (Section 2.3.1), then discuss the theoretical status and prospects for the IB corrections (Sections 2.3.2–2.3.6), and finally present the current HVP evaluation based on τ data (Section 2.3.6).

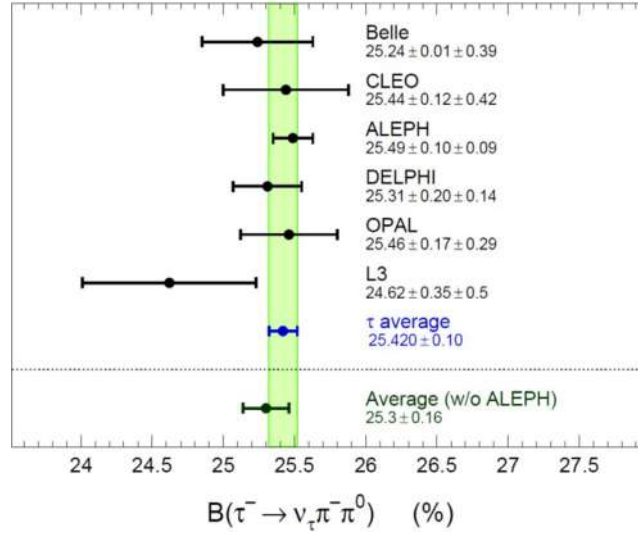


Fig. 10. Measured values of the branching ratio for $\tau \rightarrow \pi\pi^0(\gamma)\nu_\tau$. Good consistency is observed among the different experiments. Source: Figure adapted from Ref. [194].

2.3.1. Use of hadronic data from τ decays

Several high-quality data sets for semi-leptonic τ decays into vector states were collected more than 20 years ago at LEP and also at B factories, first by CLEO, later by Belle and BABAR. The available experimental information was very recently reconsidered [166], in view of the poor consistency among e^+e^- results. The spectral function in the dominant τ decay mode $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$, defined as

$$v_{\pi\pi^0}(s) = \frac{m_\tau^2}{6|V_{ud}|^2} \frac{\mathcal{B}_{\pi\pi^0}}{\mathcal{B}_e} \frac{dN_{\pi\pi^0}}{N_{\pi\pi^0}ds} \times \left[\left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \right]^{-1}, \quad (2.7)$$

has been precisely measured by several experiments [188–192] under very different conditions at LEP and the B factories. Here m_τ is the τ lepton mass, $|V_{ud}|$ the CKM matrix element, $\mathcal{B}_{\pi\pi^0}$ and $\mathcal{B}_e$ are the branching fractions of $\tau^- \rightarrow \pi^-\pi^0\nu_\tau(\gamma)$ (FSR is implied) and of $\tau^- \rightarrow e^-\bar{\nu}_e\nu_\tau$, and $dN_{\pi\pi^0}/N_{\pi\pi^0}ds$ is the normalized invariant mass spectrum of the hadronic final state. The precision achieved in the experiments for the branching fractions (0.4%) and the agreement between the different results, as seen in Fig. 10, provide a highly precise normalization of the spectral functions, even superior to that obtained in e^+e^- data. There is also good agreement between the spectral function results as shown in Ref. [189]. These measured spectral functions have been widely used (see, e.g., Ref. [193]) for a number of applications including in particular the evaluation of $a_\mu^{\text{HVP, LO}}$ and $\Delta\alpha_{\text{had}}^{(5)}$ as originally proposed in Ref. [187]. The evaluation of $a_\mu^{\text{HVP, LO}}$ using the τ hadronic decay has been valuable in earlier years when the e^+e^- data were not yet precise enough and in recent years given the large discrepancy among the most precise measurements from BABAR [139,174], CMD-3 [95,96], and KLOE [132–135]. In order to achieve the required precision in the τ -based evaluation of $a_\mu^{\text{HVP, LO}}$, IB corrections have to be understood and applied—a topic that we discuss in Sections 2.3.2–2.3.6.

2.3.2. Theoretical input for the HVP analysis based on τ data: generalities

We focus on the dominant $\tau \rightarrow \pi\pi\nu_\tau(\gamma)$ channel and denote with s the $\pi\pi$ invariant mass squared. The photon-inclusive differential decay spectrum $d\Gamma_{\pi\pi(\gamma)}/ds$ can be used to evaluate $a_\mu^{\text{HVP, LO}}[\pi\pi]$ according to the following dispersive formula [194–196] (with threshold $s_{\text{thr}} = 4M_{\pi^\pm}^2$)

$$a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] = \frac{1}{4\pi^3} \int_{s_{\text{thr}}}^\infty ds K(s) \left[\frac{K_\sigma(s)}{K_\Gamma(s)} \frac{d\Gamma_{\pi\pi(\gamma)}}{ds} \right] \times \frac{R_{\text{IB}}(s)}{S_{\text{EW}}^{\pi\pi}}, \quad (2.8)$$

where $K(s)$ is the QED kernel [85–88], see Eq. (2.2) for the explicit expression,

$$K_\sigma(s) = \frac{\pi\alpha^2}{3s}, \quad K_\Gamma(s) = \frac{\Gamma_e|V_{ud}|^2}{2m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right), \quad (2.9)$$

and the IB corrections are encoded in the product of several s -dependent factors

$$R_{\text{IB}}(s) = \frac{\text{FSR}(s) \beta_{\pi^+\pi^-(s)}^3}{G_{\text{EM}}(s) \beta_{\pi^\pm\pi^0(s)}^3} \left| \frac{F_\pi^V(s)}{f_+(s)} \right|^2. \quad (2.10)$$

The physical origin and meaning of the various IB corrections can be summarized as follows:

- The factor $S_{EW}^{\pi\pi} = S_{EW}/S_{EW}^{\text{sub, lept}}$ [197] is the s -independent short-distance EM correction to the weak semi-leptonic decay $\tau \rightarrow \pi\pi\nu_\tau$. The numerator

$$S_{EW} = 1 + \frac{2\alpha}{\pi} \log \frac{M_Z}{m_\tau} + \dots \quad (2.11)$$

encodes the universal leading logarithmic correction that affects all semi-leptonic decays [198–204] and is now known to next-to-leading logarithmic (NLL) accuracy [203,204]. The denominator

$$S_{EW}^{\text{sub, lept}} = 1 + \frac{\alpha(m_\tau)}{2\pi} \left(\frac{25}{4} - \pi^2 \right) \quad (2.12)$$

encodes the radiative corrections to the purely leptonic decay width

$$\Gamma_e \equiv \Gamma(\tau \rightarrow e\nu_\tau\bar{\nu}_e) = \frac{G_F^2 m_\tau^5}{192\pi^3} \times \left[1 + \mathcal{O}\left(\frac{m_e^2}{m_\tau^2}\right) \right] \times S_{EW}^{\text{sub, lept}}, \quad (2.13)$$

and needs to be included because the measurements provide $\tau \rightarrow \pi\pi\nu_\tau$ normalized to the leptonic mode. This is made explicit by expressing $K_I(s)$ in terms of Γ_e in Eq. (2.9).

- The $G_{EM}(s)$ factor [195,196,205,206] includes the long-distance QED corrections to the $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$ decay with virtual- plus real-photon radiation. The FSR(s) factor, first introduced in R_{IB} in Ref. [194], originates from FSR corrections to the $\pi^+\pi^-$ channel [207,208]. Both FSR(s) and $G_{EM}(s)$ are so far computed at $\mathcal{O}(\alpha)$ and with the help of hadronic models as specified below. However, the scheme-dependent nonlogarithmic term in $G_{EM}(s)$ is not currently known. In both cases, Eq. (2.10) implies a factorization assumption, separating long-range radiative effects subsumed in FSR(s) and $G_{EM}(s)$ from radiative corrections to the matrix elements, see Section 2.7.4 for the case of $F_\pi^V(s)$.
- The $\pi^\pm\pi^0$ mass difference generates a mismatch in the phase-space factors resulting in the correction term $\beta_{\pi^-\pi^+}^3(s)/\beta_{\pi^-\pi^0}^3(s)$, with $\beta_{pp'}(s) = 2p_{\text{cms}}(s)/\sqrt{s}$, where p_{cm} is the magnitude of the pion momentum in the dipion CM frame. This kinematic correction is large for s near threshold and carries negligible uncertainty.
- The last factor in Eq. (2.10) amounts to the ratio between the EM and the weak pion form factors $F_\pi^V(s)$ and $f_+(s)$, respectively, with IB only due to quark mass difference and to radiatively induced hadronic mass differences. Currently, this is the correction that carries the largest uncertainty, in particular because it is difficult to characterize in a model-independent way.

The shift in a_μ caused by the IB corrections, denoted by $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau]$, is obtained by replacing $R_{IB}(s)/S_{EW} \rightarrow R_{IB}(s)/S_{EW} - 1$ in Eq. (2.8).

We begin by discussing the impact of the s -independent short-distance corrections, which are not sensitive to hadronic structure. We use $S_{EW}^{\pi\pi} = 1.0233(3)$ [209], obtained by including the NLL correction from Ref. [204] and estimating the uncertainty to be of the order of missing nonlogarithmic terms of $\mathcal{O}(\alpha\alpha_s/\pi^2)$. The corresponding shift to HVP induced by $S_{EW}^{\pi\pi}$ is $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] = -12.16(15) \times 10^{-10}$ [209], consistent with the result of Ref. [166]. The above results do not account for the uncertainty in $S_{EW}^{\pi\pi}$ associated to the scheme dependence of the short-distance Wilson coefficient at NLL. We discuss this in Section 2.3.6. Next, the phase-space correction generates the shift $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] = -7.52 \times 10^{-10}$, with negligible intrinsic uncertainty, but the result does depend on the input for the spectral function, thus the small difference to Ref. [166].

In the remainder of this section, we discuss the status and prospects for computing the remaining hadron-structure-dependent corrections in various approaches: first, we summarize the current results based on a combination of ChPT and resonance hadronic models. This approach produces the corrections used to arrive at the current τ -based results for the HVP contribution in Section 2.3.6. We then discuss ongoing efforts based on dispersive methods and lattice QCD.

2.3.3. Chiral perturbation theory and phenomenological models

We summarize here the status of IB corrections obtained through a combination of ChPT, resonance chiral theory (R χ T), and resonance models for radiative corrections and for the pion VFF. This approach has been refined over the years, starting from the early work found in Refs. [187,194–197,205,210,211]. The state-of-the-art results of this approach have been recently presented in Refs. [166,209]. While differing in some details, these two references reach mutually consistent results, as shown below in Table 2. For the sake of simplicity, in this section we closely follow the analysis of Ref. [209], pointing out differences from Ref. [166] as needed (a comparison with lattice-QCD results in Euclidean-time windows, see Section 3.4, was reported in Ref. [212]).

The correction induced by the function $G_{EM}(s)$ is $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] = (-1.67)_{-1.39}^{+0.60} \times 10^{-10}$ [209], which corresponds to the reference value reported in Ref. [206] and is in good agreement with results obtained in Ref. [194] using Refs. [196,205]. This result updates the computation in Ref. [196], which used R χ T [213,214] and included only those resonance operators that – upon integrating out the resonances – contribute to the $\mathcal{O}(p^4)$ chiral low-energy constants (LECs). The uncertainty of the updated result is estimated by assessing the spread of results obtained by including a subset of resonance couplings that contribute to the chiral LECs at $\mathcal{O}(p^6)$ [215,216], precisely those couplings that are determined by short-distance QCD constraints (ensuring an appropriate asymptotic behavior of suitable two- and three-point Green functions).

This analysis concerns entirely the effects of structure-dependent real-photon emission ($\tau \rightarrow \nu_\tau \pi \pi \gamma$) and lacks the structure-dependent virtual-photon contribution. This missing piece, estimated as in Ref. [217] based on the results for the one-meson modes [218,219], is covered by the quoted uncertainty. Measurements of the $\tau \rightarrow \pi \pi \nu_\tau \gamma$ spectrum would greatly help reduce the model dependence of the corrections induced by $G_{\text{EM}}(s)$. The FSR(s) correction is based on sQED, producing a shift of $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] = 4.62(46) \times 10^{-10}$, where missing structure-dependent virtual- and real-photon corrections are assigned a 10% uncertainty [194,209].

The remaining IB corrections enter the ratio of form factors in Eq. (2.10). The modifications affecting $F_\pi^V(s)/f_+(s)$ can be split into the following sources:

- The ρ - ω and $-\omega$ to a much smaller extent – the ρ - ϕ mixing (leading to $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] = 2.87(8) \times 10^{-10}$ [209]).
- The $M_{\pi^\pm} - M_{\pi^0}$ difference, affecting the $\rho^{\pm,0}$ widths. An analogous mass difference for the kaons is considered in the Guerrero–Pich (GP) form factor [220] (and other chiral-based descriptions [221,222]), but not in the Gounaris–Sakurai (GS) [223] or Kühn–Santamaría (KS) [224] models (leading to $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] = 3.37 \times 10^{-10}$ [209]).
- The $M_{\rho^\pm} - M_{\rho^0}$ difference. For given $\rho^{\pm,0}$ mass values, the dominant on-shell $\rho^{\pm,0} \rightarrow \pi\pi$ widths are predictions in GP and related parameterizations, once short-distance QCD constraints [213,214] have been accounted for (leading to $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] = 1.95_{-1.55}^{+1.56} \times 10^{-10}$ [209]).
- The contributions to the $\rho^\pm - \rho^0$ width difference dominated by their $\pi\pi(\gamma)$ decay channels [211], where a 10% error is assigned to structure-dependent uncertainties [194,209,211] (leading to $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] = -6.66(73) \times 10^{-10}$ [209]). We note that the quoted references did not include IB in the $\rho\pi\pi$ coupling. The origin and possible size of this correction is discussed in Section 2.3.6.

These corrections were evaluated using a number of different parameterizations in Ref. [209], taking inputs for IB in the ρ masses and widths consistent with Ref. [194] and the PDG [225]. Based on analyticity tests, the dispersive form factor based on Ref. [226] is taken as the reference result. The central value is obtained from the dispersive result in the case where it has a conformal polynomial of fourth degree (constrained to comply with P -wave behavior) accounting for inelasticities. This yields $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] = 1.53_{-1.73}^{+1.74} \times 10^{-10}$ for the form factor correction [209]. The sum of all IB corrections entering (2.8) gives $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] = (-15.20)_{-2.27}^{+1.90} \times 10^{-10}$. The associated systematic uncertainties are estimated from the differences among the various dispersive results, and adding linearly a 2% uncertainty to account for the small difference with respect to the GS value, leading to $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau] = (-15.20)_{-2.63}^{+2.26} \times 10^{-10}$ [209].

Finally, we comment on ρ - γ mixing. First, we note that for a proper treatment of the ρ one just needs to correctly describe the resonant behavior of pion form factors, either of the weak or the EM currents. Gauge invariance, leading to the constraint $F_\pi^V(0) = 1$, as is respected in any theoretically sound approach, ultimately settles the issue of a possible ρ - γ mixing, since the ρ is never needed as an asymptotic state. This is explicit in a dispersive approach, since the ρ does not even enter as a degree of freedom, but also holds in any Lagrangian framework that couples hadrons to external sources representing the currents (this is the case in ChPT, R χ T, and any resonance model made consistent with chiral symmetry). The ρ - γ mixing issue may arise in intermediate steps of calculations within a given Lagrangian model. For example, in Ref. [227] the effect of such a mixing was studied within the hidden-local-symmetry model. In QFT the mixing of two fields can be addressed through a field redefinition. Since physical results should not depend on the field parameterization and given that the ρ meson is not an asymptotic state, performing the field redefinition that diagonalizes the ρ - γ quadratic terms is a possible, but far from necessary choice. In most approaches it turned out to be more convenient not to perform this diagonalization. However, if one performs the redefinition, the re-defined “ ρ ” field couples to all charged particles, and the resulting ρee and $\rho\mu\mu$ couplings should be taken into account, given that they affect the leptonic QED cross section used to normalize $\sigma(e^+e^- \rightarrow \pi^+\pi^-)$. This effect and its interplay with higher-order corrections was not discussed in Ref. [227]. Moreover, the large mixing corrections at $\sqrt{s} \gg M_\rho$ and the large slope at $\sqrt{s} \ll M_\rho$, leading to sizable numerical effects in Ref. [227], seem to arise from the fact that the pion VFF adopted does not incorporate the basic short-distance asymptotic conditions imposed by QCD (i.e., vanishing at high momentum transfer), a constraint whose relevance even at low energy has been thoroughly discussed in Ref. [214].

2.3.4. The dispersive approach

To first order in IB the photon-inclusive differential decay rate for $\tau^-(l_1) \rightarrow \pi^-(q_1)\pi^0(q_2)\nu_\tau(l_2)[\gamma(k)]$ is given by [195,196]

$$\frac{d\Gamma_{\pi\pi[\gamma]}}{ds} = S_{\text{EW}}^{\pi\pi} K_{\text{F}}(s) \beta_{\pi^\pm\pi^0}^3(s) |f_+(s)|^2 G_{\text{EM}}(s), \quad (2.14)$$

where $s = (q_1 + q_2)^2$ and $f_+(s)$ is the $\pi^-\pi^0$ form factor. $G_{\text{EM}}(s)$ the EM correction factor defined as:

$$G_{\text{EM}}(s) = \frac{\int_{t_{\min}(s)}^{t_{\max}(s)} dt D(s, t) [1 + 2f_{\text{elm}}^{\text{loop}}(s, t) + g_{\text{rad}}(s, t)]}{\int_{t_{\min}(s)}^{t_{\max}(s)} dt D(s, t)}, \quad (2.15)$$

with Mandelstam variable $t = (l_1 - q_1)^2$, $t_{\min, \max}(s)$ its lower, upper phase-space borders, and

$$D(s, t) = \frac{m_\tau^2}{2} (m_\tau^2 - s) + 2M_\pi^2 M_{\pi^0}^2 - 2t(m_\tau^2 - s + M_\pi^2 + M_{\pi^0}^2) + 2t^2. \quad (2.16)$$

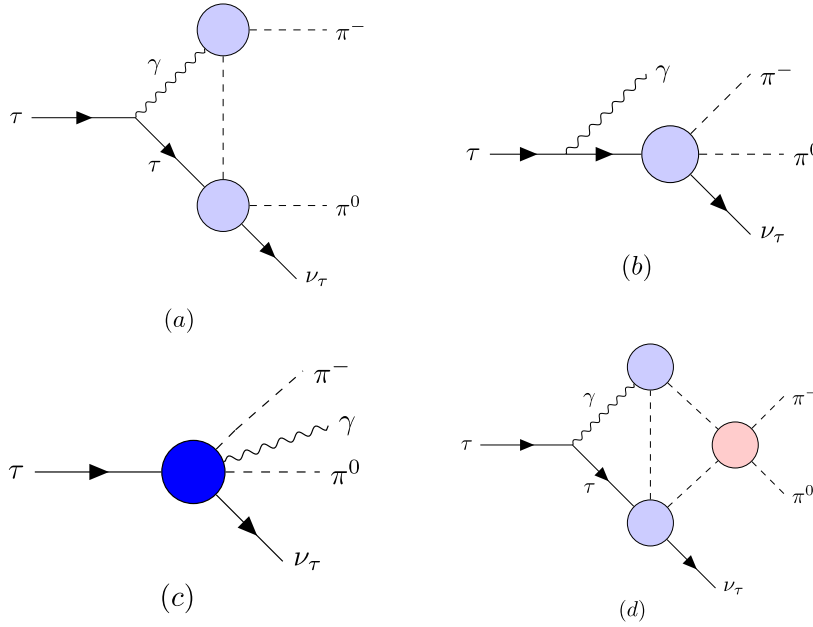


Fig. 11. Virtual (a) and real (b) + (c) contributions to the radiative corrections to $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ in the dispersive framework at first order in IB. Light-blue blobs indicate the pion VFF. Diagram (c) includes bremsstrahlung off the charged pion, but the blue blob represents the general matrix element [229]. A rescattering correction is shown in (d), where the light-red blob refers to the $\pi\pi$ scattering amplitude.

The functions $f_{\text{elm}}^{\text{loop}}(s, t)$ and $g_{\text{rad}}(s, t)$ in Eq. (2.15) account for virtual and real corrections, respectively. While previously these functions were calculated in ChPT or $R\chi T$ [195,196,206,209], in this section we outline progress in a dispersive approach [228]. First, the approximation of only taking into account intermediate hadronic states up to two pions is justified phenomenologically: in the energy region and in the channel of interest inelastic effects are known to be small. Radiative corrections to them are therefore considered to be negligible. In this approximation the contribution to $f_{\text{elm}}^{\text{loop}}$ in the dispersive framework is depicted in Fig. 11(a), where the light-blue blobs denote the pion VFF.

In analogous fashion to how radiative corrections to $e^+e^- \rightarrow \pi^+\pi^-$ are handled in a dispersive approach in Ref. [142], a dispersive representation is used for the pion VFF in the unitarity diagram of Fig. 11(a),

$$F_\pi^V(s) = 1 + \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\text{Im} F_\pi^V(s')}{s'(s' - s - i\epsilon)} = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\text{Im} F_\pi^V(s')}{s' - s - i\epsilon}. \quad (2.17)$$

In contrast to the approach in Ref. [142], an unsubtracted dispersion relation for the VFF (with an implicit sum rule for the value at $s = 0$) is chosen to ensure that the resulting representation of the triangle diagram is manifestly UV finite. However, IR singularities remain: these are handled through dimensional regularization and cancel with the IR-divergent interference of real contributions in Fig. 11(b) and (c). In this approach the effects of the VFF inside the loop integral are rigorously accounted for, an effect that was shown to be essential to describe the charge asymmetry in $e^+e^- \rightarrow \pi^+\pi^-$ due to resonance enhancement of the IR-finite remainder [141,142]. In previous model-based approaches such effects have only been crudely approximated.

Since the use of an unsubtracted dispersion relation for the pion VFF introduces sensitivity to its high-energy behavior, a matching procedure to the low-energy behavior calculable in ChPT is essential to mitigate this effect. Taking the virtual corrections in ChPT including virtual photons and leptons [162,230–233] at $\mathcal{O}(e^2 p^2)$ [195,196], denoted by $f_{\text{elm}}^{\text{loop, ChPT}}$, the matched virtual contribution can be expressed as

$$f_{\text{elm}}^{\text{loop, match}}(s, t) = f_{\text{elm}}^{\text{loop, VFF}}(s, t) - f_{\text{elm}}^{\text{loop, VFF}}(0, 0) + f_{\text{elm}}^{\text{loop, ChPT}}(0, 0), \quad (2.18)$$

where $f_{\text{elm}}^{\text{loop, VFF}}(s, t)$ refers to the dispersive result based on the unsubtracted dispersion relation. While $f_{\text{elm}}^{\text{loop, VFF}}(s, t)$ carries an IR divergence canceled by the interference of ISR and FSR, the term $f_{\text{elm}}^{\text{loop, ChPT}}(0, 0)$ covers the IR divergences of the remaining contributions, e.g., self-energy corrections, which are canceled by real-photon emission of either the initial or final state. The matching to ChPT in Eq. (2.18) entails several further advantages: first, the chiral logarithms only partially captured by the low-energy limit of the dispersive calculation of Fig. 11(a) are fully restored. Second, the chiral LECs define a convenient bridge to the short-distance amplitude, as their scale dependence needs to cancel in physical observables. While previously only resonance estimates of their finite parts were available, the matching to lattice-QCD calculation [234–236] and incorporation of RG corrections [202,204] now presents the opportunity to define a consistent

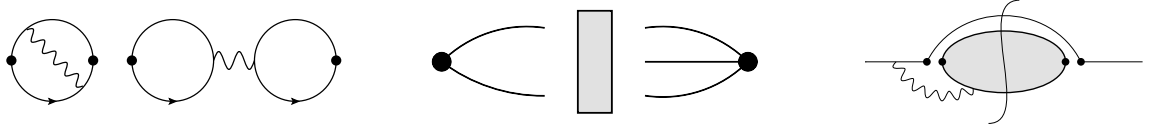


Fig. 12. Left: Diagrams contributing to the ρ mass shift (figure adapted from Ref. [14]). The black dots represent the external isovector currents, and only quark lines are drawn. The exchange of gluons between the two disconnected bubbles is implicitly assumed for the second diagram. Center: Phenomenological interpretation of the two-point isoscalar–isovector correlator, namely G_{01}^γ , with 2- and 3-pion states coupled by an IB operator. Right: An example of a mixed contribution where the photon is exchanged between the leptonic and hadronic sectors. The two horizontal external lines represent the τ lepton, the upper line the neutrino, and the gray blob the hadronic system. The vertical curly line denotes a final-state cut, while the black dots represent the four-Fermi interaction vertices.

set of correction factors S_{EW} and $G_{EM}(s)$, such that the scale dependence disappears up to a given order in the perturbative expansion.

Work along these lines is ongoing and should provide improved results for the combination of S_{EW} and $G_{EM}(s)$ with controlled uncertainties. Corrections include higher intermediate states and $\pi\pi$ rescattering corrections as shown in Fig. 11(d), which will also be addressed in a dispersive approach. For a complete estimate of IB corrections two main classes of additional corrections need to be considered:

1. FSR in $e^+e^- \rightarrow \pi^+\pi^-$ and ρ – ω mixing: these corrections can be determined from dispersive analyses of $e^+e^- \rightarrow \pi^+\pi^-$ and are therefore already under reasonable control [226,237];
2. 2π matrix element in charged and neutral current: these corrections can be expressed as a ratio of form factors $|f_+(s)/F_\pi^V(s)|$, where $f_+(s)$ refers to $\pi^-\pi^0$ system and $F_\pi^V(s)$ to the $\pi^+\pi^-$ one.

While frequently expressed in terms of ρ Breit–Wigner parameters and couplings, leading to model errors that are difficult to control, from a dispersive perspective the second class of corrections should be interpreted as IB in the respective QCD matrix elements. Accordingly, a better understanding of IB in the pion VFF is ultimately required, including pion-mass-difference effects and radiative corrections. Work along these lines was started in Ref. [238] and is in progress, see also Section 2.7.4.

2.3.5. Lattice-QCD approach

As discussed above, addressing IB effects in the $\pi^-\pi^0$ decay mode of the τ lepton from first principles is highly desirable. In this context lattice QCD+QED simulations can provide useful information, as we briefly review below, see also Section 3, in particular Section 3.7.1. The first difficulty that one encounters is in the exclusivity of the mode. In fact, even in the isosymmetric limit of QCD, providing the two-pion contribution to the spectral density is a challenging problem for a lattice calculation, due to the rotation to the Euclidean metric. The latter is by-passed in the limited elastic region (below the four-pion threshold) via the finite-volume formalism, while it requires inverse Laplace methods in the inelastic regime, methods that are currently being intensively developed in the community (see, e.g., Refs. [239–244]). IB effects constitute another layer of difficulty, so for these reasons in Ref. [245] it was first proposed to study the problem in a fully inclusive manner.

More specifically, the difference of the inclusive isovector spectral densities, charged and neutral, was examined. Since their relation to Euclidean correlators proceeds via a Laplace transform, e.g., for the HVP contribution to a_μ one starts from

$$C(x_0) = \int d\omega e^{-\omega x_0} \rho(\omega^2) \omega^2, \quad (2.19)$$

with $\rho(\omega^2)$ trivially related to the experimental R -ratio, the corresponding difference of two-point Euclidean correlators, denoted by $G_{11}^W - G_{11}^\gamma$, is examined in Section 3.7.1. Thanks to large cancellations it was shown that it is of purely EM nature and only the diagrams in Fig. 12(left) contribute. As pointed out in Ref. [246], this difference could be used as an intermediate quantity to address some open questions on the model dependence of some effects in $R_{IB}(s)$, as defined in Eq. (2.10). In simple terms, starting from a model of the charged and neutral pion form factor, after a Laplace transform, one could fit their (squared) difference to the lattice correlators to fix a few of their free parameters. More specifically, despite receiving EM corrections originating also from four-pion states (due to the inclusive nature of the Euclidean correlators considered), in analogy to the charged-neutral pion mass correction, the aforementioned difference may be used to fix the shift of the ρ mass parameter induced by EM interactions. An attractive feature is that being a property of the spectrum, this would not require the renormalization of the charged current.

While the latter does not constitute per se a conceptual obstacle for a lattice calculation, its matching from a scheme suitable for lattice calculations to the typical schemes used to match the EFT without the weak bosons to the SM, e.g., $\overline{MS}$, is still missing. As a consequence this still prevents the usage of lattice data to fix other model-dependent quantities such as the EM shift of the ρ decay width, a rather important effect as outlined in Table 2, which should be prioritized.

Another interesting object is the two-point correlator involving the isoscalar and isovector neutral currents, denoted as G_{01}^γ in Section 3.7.1. From a phenomenological point of view, it contains the mixing between two- and three-pion

Table 2

Summary of the different classes of IB corrections contributing to $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau]$ (in units of 10^{-10}). For Refs. [166,194], the second errors due to the difference between the GS and KS models are added linearly to the quadratic sum of all other uncertainties. For Ref. [209], the total uncertainty includes an estimate of the systematic uncertainty arising from using different dispersive parameterizations. An additional 2% uncertainty is added linearly to account for the difference between the result based on the dispersive and the GS parameterizations. The entries in the last column are discussed in the main text.

	Refs. [166,194]	Ref. [209]	Refs. [237,247]	Our estimate
Phase space	−7.88	−7.52	–	−7.7(2)
S_{EW}	−12.21(15)	−12.16(15)	–	−12.2(1.3)
G_{EM}	−1.92(90)	(−1.67) $^{+0.60}_{-1.39}$	–	−2.0(1.4)
FSR	4.67(47)	4.62(46)	4.42(4)	4.5(3)
ρ – ω mixing	4.0(4)	2.87(8)	3.79(19)	3.9(3)
ΔM_ρ	0.20($^{+27}_{-19}$)(9)	1.95 $^{+1.56}_{-1.55}$	–	
$\Delta\Gamma_\rho(\Delta M_\pi)$	4.09(0)(7)	3.37	–	
$\frac{F_+^V}{F_+}(w/o \rho$ – $\omega)$	$\Delta\Gamma_\rho(\pi\pi\gamma)$ −5.91(59)(48)	−6.66(73)	–	
	$\Delta\Gamma_\rho(g_{\rho\pi\pi})$ –	–	–	
Total	−1.62(65)(63)	(−1.34) $^{+1.72}_{-1.71}$	–	−1.5(4.7)
Sum	−14.9(1.9)	(−15.20) $^{+2.26}_{-2.63}$	–	−15.0(5.1)

states and therefore information on the ρ – ω mixing parameters, as sketched in Fig. 12(center). While on the one hand this could be used again to fix a few model-dependent parameters, its inclusive nature shows the difficulty in using it for a direct estimate of the 2π IB effects alone, since in practice one would have to account and remove the IB effects of the three-pion channel, and neglect the effects of even higher multiplicity channels.

The remaining term that could be ideally addressed in a lattice calculation is the G_{EM} function. The current strategy so far pursued in Ref. [245] consists of borrowing the available knowledge on G_{EM} from ChPT [195,196], since a complete lattice calculation would require a study of the necessary triangle diagrams, see Fig. 12(right), from Euclidean space–time, where problems with analytic continuation are present. Developments in this direction are currently being pursued for simpler quantities, but progress is certainly to be expected as these methods become more and more mature. From the short-distance perspective performing the entire calculation using the lattice regulator would presumably simplify the renormalization pattern, leaving only the matching with S_{EW} as the open question.

In summary, while an exclusive study of IB effects for the 2π channel remains a challenging problem for a lattice calculation, an inclusive approach is currently being developed and intermediate quantities, such as the difference of isovector charged and neutral correlators may turn out to be useful, in the short term, to constrain model-dependent parameters.

2.3.6. Summary on isospin-breaking corrections and τ -based HVP result

In this section we summarize the theoretical IB corrections needed for a τ -based analysis of HVP, discuss the robustness of the associated uncertainties, and provide recommended values. Table 2 compiles results for various IB effects from recent state-of-the-art analyses, as discussed in previous subsections, and includes in the last column “our estimate”, based on our assessment of the uncertainties.

Before discussing each line in Table 2, we observe that estimating uncertainties in this mostly nonperturbative regime is not a simple exercise, due to the ensuing model-dependence of most results. The current main analyses [166,209] address this issue by assigning systematic uncertainties associated with the spread of results obtained with different models. We will not repeat this exercise. Rather, we adopt this approach: (i) for most IB corrections, we combine the results from Refs. [166,194,209,237,247]; (ii) where appropriate, we identify uncertainties not included in current analyses and provide recommended numerical values for them.

- **Phase space:** The small differences reflect the use of different form-factor parameterizations. We adopt the midpoint of the results shown in the second and third column of Table 2, assigning an uncertainty to cover the full range. Also note that higher-order IB corrections could play a role here, depending on whether one uses the spectra from τ decay or e^+e^- data.
- **Scheme dependence in S_{EW} :** As anticipated earlier, current analyses do not account for the $\mathcal{O}(\alpha/\pi)$ scheme dependence of the Wilson coefficient whose square gives the factor S_{EW} . As is well known, to NLL, in general the Wilson coefficients acquire a scheme dependence of $\mathcal{O}(\alpha/\pi)$, to be absorbed by corresponding long-distance matrix elements ($G_{\text{EM}}(s)$ in this case). As discussed in Ref. [204] in the context of β decays, the current result $S_{\text{EW}}^{\pi\pi} = 1.0233(3)$ is based on a particular definition of “scheme-independent” NLL Wilson coefficient [248] that effectively factors out the scheme dependence of $\mathcal{O}(\alpha)$ and assumes it to be reabsorbed in the LECs entering the long-distance correction $G_{\text{EM}}(s)$. Since currently we have no control over the scheme dependence in $G_{\text{EM}}(s)$, we should add this perturbative scheme-dependence uncertainty to S_{EW} . Taking the ambiguity in the decay rate to be $\alpha(m_\tau)/\pi \simeq 0.24\%$ leads to $S_{\text{EW}}^{\pi\pi} = 1.0233(3)(24)$, which entails an additional uncertainty of 1.3×10^{-10} in $\Delta a_\mu^{\text{HVP, LO}}[\pi\pi, \tau]$, adopted in Table 2.

Note that if we had assumed an $\mathcal{O}(\alpha/\pi)$ uncertainty at the level of the Wilson coefficient, the impact on S_{EW} would double, leading to an ambiguity in $\Delta a_{\mu}^{HVP, LO}[\pi\pi, \tau]$ of 2.5×10^{-10} .

- **Structure-dependent virtual corrections in G_{EM} :** Combining the results from Refs. [166,194] and Ref. [209] leads to the G_{EM} -induced correction $\Delta a_{\mu}^{HVP, LO}[\pi\pi, \tau] = -2.0(1.0) \times 10^{-10}$. As discussed in Ref. [209], existing results for $G_{EM}(s)$ do not include the effect of hadronic structure dependence in the virtual-photon corrections to $\tau \rightarrow \pi\pi\nu_{\tau}$ —pions are treated as point-like objects. In Ref. [217] it was argued that the effect of the missing structure-dependent loops can be estimated by analogy with the single-meson decay modes $\tau \rightarrow P\nu_{\tau}$ ($P = \pi, K$), for which structure effects have been studied [218,219]. There is, however, one class of diagrams that may lead to a substantial difference between the corrections to $\tau \rightarrow \pi\nu_{\tau}$ and $\tau \rightarrow \pi\pi\nu_{\tau}$. Before integrating out the W , these take the form of box diagrams, in which two gauge bosons (γ and W) are exchanged between the lepton and pion lines (after integrating out the W boson this reduces to the diagram in Fig. 11(a)). In the case of $\tau \rightarrow \pi\nu_{\tau}$, the weak current involves just the pion decay constant, while in the case of $\tau \rightarrow \pi\pi\nu_{\tau}$ the weak $\pi\pi$ form factor appears and the box diagram involves the interplay of weak and EM form factors. In Refs. [141,142] it was shown that such diagrams are sensitive to structure-dependent corrections, as the remainder after the cancellation of IR divergences can be strongly enhanced by the ρ resonance. In fact, the measurement of the forward–backward asymmetry by CMD-3 [95,96] showed that these effects were critical to explain the data, substantially increasing the size of the asymmetry in the ρ peak, see Sections 2.2.1 and 2.7.4. A similar effect could also be present in the virtual corrections to the τ decay, whose size depends on the strength of the IR enhancement and the interplay with the kernel function in the a_{μ} integral, as currently under investigation (see Section 2.3.4). Given the large numerical impact observed in the e^+e^- case, it is prudent to assign to G_{EM} an additional uncertainty, which is currently hard to estimate. As a reference point, we assign to the structure-dependent virtual-photon corrections a similar uncertainty as the one emerging from the structure-dependent real-photon corrections, leading to the G_{EM} -induced correction $\Delta a_{\mu}^{HVP, LO}[\pi\pi, \tau] = -2.0(1.0)_{\text{real}}(1.0)_{\text{virtual}} \times 10^{-10} = -2.0(1.4) \times 10^{-10}$.
- **FSR effects in $e^+e^- \rightarrow \pi^+\pi^-$:** The results reported in Table 2 are quite consistent and the small differences are likely due to the form-factor input. All calculations use sQED as theoretical input. The result of Ref. [247] includes the estimate from Ref. [249] for the non-IR-enhanced contributions, which amounts to an effect of $\simeq 0.2 \times 10^{-10}$ in $\Delta a_{\mu}^{HVP, LO}[\pi\pi, \tau]$ and shows that in this case sQED indeed captures the dominant effect. We take as our estimate the average of the sQED results and assign an uncertainty to cover the spread of central values, which is large enough to cover possible effects beyond sQED [249]. This leads to $\Delta a_{\mu}^{HVP, LO}[\pi\pi, \tau] = 4.5(3) \times 10^{-10}$.
- **ρ – ω mixing:** For this contribution the result of Ref. [209] differs considerably from the results of Ref. [166] and Refs. [237,247]. The source of this difference can be traced back to the phase of the ρ – ω mixing parameter, which is $\simeq 10^\circ$ in the reference dispersive analysis of Ref. [209], while it is $\simeq 4^\circ$ in the other analyses. In Ref. [237] this phase has been bound fairly robustly and therefore in our combination we give more weight to the results from Refs. [166,237,247], leading to $\Delta a_{\mu}^{HVP, LO}[\pi\pi, \tau] = 3.9(3) \times 10^{-10}$.⁶
- **$F_{\pi}^V(s)/f_{\pi}(s)$ and impact of IB in the $\rho\pi\pi$ coupling:** The ratio of form factors leads to a number of corrections that largely compensate in the total, as shown in Table 2. The results of Refs. [166,194] and Ref. [209] are fairly consistent. However, they both omit a potentially large source of IB in the width of the ρ resonance, as discussed below. IB effects in the width of the ρ resonance are usually discussed in the context of resonance models, assuming the form $\Gamma_{\rho}(s) \propto g_{\rho\pi\pi}^2 \sqrt{s} \beta_{\pi\pi}^3(s) (1 + \delta_{\rho\pi\pi})$ for the off-shell $\rho \rightarrow \pi\pi$ width [194,211]. Here, $\delta_{\rho\pi\pi}$ encodes long-distance radiative corrections, including the effect of the $\pi\pi\gamma$ decay channels. In this context there are four effects that can generate the splitting $\Delta\Gamma_{\rho} \equiv \Gamma_{\rho^0} - \Gamma_{\rho^+}$: the coupling $g_{\rho\pi\pi}$, the ρ mass, the kinematic factor $\beta_{\pi\pi}^3(s)$, and the radiative corrections $\delta_{\rho\pi\pi}$. In the literature only the last three effects are discussed and it is found [194,211] that they partially cancel. Here we point out that the effect of $g_{\rho^0\pi^+\pi^-} \neq g_{\rho^{\pm}\pi^{\mp}\pi^0}$ is of similar size to the ones included, and therefore, absent a detailed analysis, its impact should be reflected in the total uncertainty. In a purely phenomenological parameterization, one should adopt $g_{\rho^0\pi^+\pi^-} \neq g_{\rho^{\pm}\pi^{\mp}\pi^0}$. In the context of Lagrangian models such as $R\chi T$, for example, this kind of IB can be induced by operators involving two EM spurions, following common practice in ChPT [232] and resonance chiral Lagrangians [250]. The impact of $\delta_g \equiv (g_{\rho^0\pi^+\pi^-} - g_{\rho^{\pm}\pi^{\mp}\pi^0})/g_{\rho^0\pi^+\pi^-}$ on $\Delta\Gamma_{\rho}$ can be estimated by observing that

$$\Delta\Gamma_{\rho} \simeq 2\Gamma_{\rho} \delta_g \simeq 300 \delta_g \text{ MeV}. \quad (2.20)$$

δ_g could range anywhere from $\delta_g \simeq \alpha/\pi \simeq 0.23\%$ to $\delta_g \simeq 1\%$ (the typical size of form-factor-related IB effects, e.g., FSR and ρ – ω mixing), corresponding to $\Delta\Gamma_{\rho} \simeq 0.7 \text{ MeV}$ and $\Delta\Gamma_{\rho} \simeq 3 \text{ MeV}$, respectively. The choice $\delta_g \simeq \alpha/\pi$ is likely too aggressive, because a similar estimate would not capture the actual size of the long-distance radiative corrections to $\Gamma(\rho \rightarrow \pi\pi[\gamma])$ (these induce a shift $\Delta\Gamma_{\rho} = 1.8(2) \text{ MeV}$ [194,211]). On the other hand, the choice $\delta_g \simeq 1\%$ would lead to $\Delta\Gamma_{\rho} \simeq 3 \text{ MeV}$, which exceeds the current PDG average $\Delta\Gamma_{\rho} = 0.3(1.3) \text{ MeV}$ [225].⁷ Taking this into account, we adopt as a compromise value $|\delta_g| \leq 2\alpha/\pi \simeq 0.46\%$, which leads to $|\Delta\Gamma_{\rho}| \leq 1.4 \text{ MeV}$,

⁶ Ref. [209] obtains 3.87(8) for the ρ – ω mixing correction using the GS form factor parameterization. Again, the difference can be traced back to the phase of the ρ – ω mixing parameter. Other (partially compensating) differences are found between dispersive and GS parameterizations in Ref. [209], and they are accounted for by increasing the overall uncertainty, as discussed in the text.

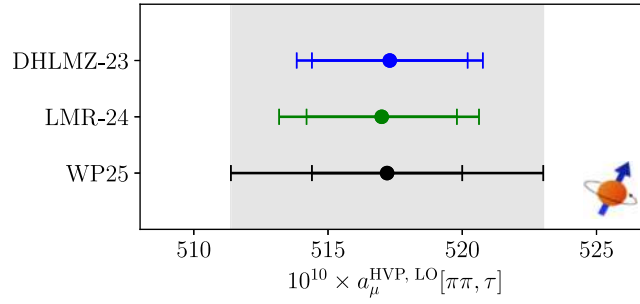


Fig. 13. Summary of evaluation of $a_{\mu}^{\text{HVP, LO}}[\pi\pi]$ from τ decays. The points refer to DHLMZ-23 [166], LMR-24 [209], and Eq. (2.22), respectively. The inner error bar reflects the current experimental uncertainties, while the outer error represents the combination of experimental and theoretical uncertainties.

reflecting the current PDG uncertainty. Scaling the effect from the pion mass difference $\Delta\Gamma_{\rho}(\Delta M_{\pi}) = 1.1$ MeV from Refs. [166,194,209], this corresponds to an additional uncertainty of $\simeq 4.7 \times 10^{-10}$ in $\Delta a_{\mu}^{\text{HVP, LO}}[\pi\pi, \tau]$, motivating the corresponding entry in Table 2 for F_{π}^V/f_{+} .

As an alternative to using theoretical predictions for IB in the ρ parameters, a data-driven approach using the two-pion τ and e^+e^- spectral functions has very recently been proposed in Ref. [251]. Given the direct relevance of this work for the topic discussed here, we briefly summarize it, also noting that there has not been enough time for detailed scrutiny. This approach proposes to decouple the normalization of the measured spectral function from its shape, the latter characterized mainly by the mass and width of the ρ resonance, with some small residual contribution from higher resonances. For the correction to the dispersive integral from the pion form factors Ref. [251] finds $\Delta a_{\mu}[\pi\pi, \tau](F_{\pi}^V/f_{+}) = +1.68(2.92)(1.39) \times 10^{-10}$, where the first uncertainty is dominated by the τ data and the second is due to the line-shape dependence. This data-based estimate agrees with the previous results from two different groups as quoted in Table 2, $-1.62(65)(63) \times 10^{-10}$ [166,194], and $-1.34(1.71) \times 10^{-10}$ [209], respectively at the 1.0 and 0.8 σ level. It also agrees with the new theoretical estimate in Table 2, $-1.5(4.7) \times 10^{-10}$. These figures emphasize the need for model-independent input for the ratio of matrix elements $F_{\pi}^V(s)/f_{+}(s)$.

Putting all the corrections together, the individual analyses arrive at $\Delta a_{\mu}^{\text{HVP, LO}}[\pi\pi, \tau] = -14.9(1.9) \times 10^{-10}$ [166,194], $\Delta a_{\mu}^{\text{HVP, LO}}[\pi\pi, \tau] = (-15.20)_{-2.63}^{+2.26} \times 10^{-10}$ [209], and $\Delta a_{\mu}^{\text{HVP, LO}}[\pi\pi, \tau] = -12.2(3.4) \times 10^{-10}$ [251]. The latter data-based determination is not used in the following because it appeared late in the WP25 schedule and it is not published (at the time of writing). Altogether, based on the above discussion and Table 2, our current best estimate for the IB corrections to be used in a τ -based evaluation of HVP is

$$\Delta a_{\mu}^{\text{HVP, LO}}[\pi\pi, \tau] = -15.0(5.1) \times 10^{-10} \quad (\text{our estimate}). \quad (2.21)$$

While the central value fully reflects the state-of-the-art published analyses, the uncertainty attempts to account for sources of IB that are not yet fully addressed in the literature. Work to tackle the issues summarized above is ongoing and there are good prospects of integrating in the near future new information from the dispersive approach (see Section 2.3.4), the lattice-QCD approach (see Section 2.3.5), and data-driven constraints (see Ref. [251]).

In summary, there are two consistent evaluations of HVP based on τ data [166,209], corresponding to the first two columns in Table 2. They lead to $a_{\mu}^{\text{HVP, LO}}[\pi\pi, \tau] = 517.3(1.9)_{\text{stat}}(2.2)_{\text{syst}}(1.9)_{\text{th}} \times 10^{-10}$ [166] and $a_{\mu}^{\text{HVP, LO}}[\pi\pi, \tau] = 517.0(2.8)_{\text{exp}}(^{+2.3}_{-2.6})_{\text{th}} \times 10^{-10}$ [209]. Based on our conservative estimate of the IB uncertainties, larger than the ones quoted in the individual analyses, we arrive at

$$a_{\mu}^{\text{HVP, LO}}[\pi\pi, \tau] = 517.2(2.8)_{\text{exp}}(5.1)_{\text{th}} \times 10^{-10} \quad (\text{our estimate}), \quad (2.22)$$

see Fig. 13 for a summary. Adding the offset $187.3(2.2) \times 10^{-10}$ from WP20 for the remainder of the LO HVP contribution, one arrives at a total LO HVP value of

$$a_{\mu}^{\text{HVP, LO}}[(\pi\pi, \tau) + \text{WP20}] = 704.5(6.2) \times 10^{-10}. \quad (2.23)$$

The above offset from WP20 is not updated in this work, we instead focus on the major tensions in the 2π channel. However, we emphasize that relevant new information on the subleading channels has become available, e.g., new measurements for $e^+e^- \rightarrow 3\pi$ [149,185], $e^+e^- \rightarrow K_S K_L$ [147], and the inclusive R -ratio [179]. As described in Sections 2.2.6 and 2.6.2, tensions between the Belle-II 3π data and previous measurements are now visible, other tensions are present in the K^+K^- channel and in the comparison of the BESIII inclusive R -ratio measurement with pQCD. While

⁷ Alternatively, Ref. [225] also quotes separate values $\Gamma_{\rho^0} = 147.4(8)$ MeV, $\Gamma_{\rho^+} = 149.1(8)$ MeV, and thus $\Delta\Gamma_{\rho} = -1.7(1.1)$ MeV, suggesting a somewhat larger uncertainty.

the overall effect is minor compared to the tension seen in the 2π channel, these effects will need to be carefully studied for a new, complete data-driven evaluation of $a_\mu^{\text{HVP, LO}}$, for which continued progress in the overall $R_{\text{had}}(s)$ measurement program is essential.

2.4. Monte-Carlo generators and radiative corrections

The *RadioMonteCarLow 2* community effort, which started in 2022 [144,252–254], is dedicated to improving the theoretical predictions for hadron and lepton production at low-energy e^+e^- colliders. In Phase I, whose results are documented in Ref. [145], the focus was on $2 \rightarrow 2$ and $2 \rightarrow 3$ processes $e^+e^- \rightarrow X^+X^- (+\gamma)$ for $\sqrt{s} = (1\text{--}10)\text{ GeV}$ with $X \in \{e, \mu, \pi\}$. This concerns scan-based measurements (no hard photon required) and radiative return measurements (hard photon required).

The long-term goal of the project is to preserve the well-established codes; to provide canonical versions of codes; and to further develop and improve codes. An important aspect is the inclusion of new ideas and approaches. This includes exploiting the technical progress since the last systematic review of low-energy MC tools in 2010 [163], which is leading to new codes getting developed or extended to cover the processes of interest.

RadioMonteCarLow 2 is committed to open science and has released all codes, input data, and output together with a report as part of a living review at <https://radiomontecarlow2.gitlab.io>. Phase I was mostly a theoretical exercise in which seven codes (AFKQED, BABAYAGA@NLO, KKMCMC, MCGPJ, McMULE, PHOKHARA, and SHERPA) were used in five scenarios inspired by former and current experiments. It should be stressed that the goal was *not* to prove any one code wrong or obtain a theory error for the prediction. The main purpose was to ascertain the importance of various effects, exploiting the wide variation of approaches used in the codes. As documented in the review, the differences among the results of the codes can be understood, giving valuable information for further progress. Beyond Phase I, this community effort is expected to continue for many years yet during which the living review will be continuously updated as new results become available.

2.4.1. Terminology

When discussing the features of the different codes and the effects they include it is important to use a systematic terminology. An often used one is the fixed-order counting in α where LO, NLO, and NNLO refer (almost) universally to the full contribution at α^2 , α^3 , and α^4 for $ee \rightarrow XX$ (or α^3 , α^4 , α^5 for $ee \rightarrow XX\gamma$). This includes *per definition* both real and virtual corrections as either would not be finite or physical on their own. Beyond fixed order, a MC might utilize some form of approximations of soft and/or collinear effects – potentially resummed – to capture dominant effects that are related to additional photon emission. However, it is important to stress that even if a given code can generate configurations with two or more extra photons this does not mean it is NNLO exact. The proper use of the term (N)NLO, indicating that *all* terms of a given order in α are taken into account, is encouraged.

Resummation of logarithms in the electron mass involves modeling the splitting $\ell \rightarrow \ell\gamma$, which can be done (semi-) analytically using structure and fragmentation functions or fully numerically using a parton shower. This leads to a good description of observables involving the lepton but does not model the transverse momentum $p_\perp$ of electrons and photons unless special care is taken. Soft logarithms can be resummed using the Yennie–Frautschi–Suura (YFS) formalism [255], leaving an IR finite perturbative expansion. The calculation of these finite residuals allows for the matching of the YFS resummation with higher-order corrections. This can be achieved at the amplitude squared level using the exclusive exponentiation (EEX) [256,257] or directly at the amplitude level using coherent exponentiation (CEEX) [178].

Ultimately the goal should always be to combine a fixed-order calculation with some form of approximate higher-order corrections as both effects contribute at the percent level. The current state of the art is NNLO (NLO) or NLO (LO) matched to resummation/additional approximate emission for $2 \rightarrow 2$ ($2 \rightarrow 3$) scattering.

To further classify the QED aspects, initial-state corrections (ISC), final-state corrections (FSC), mixed corrections (interferences between ISC and FSC), and VP corrections (VPC) are distinguished. This is done by assigning formally different charges to the initial electron and outgoing muon or pion and all parts are gauge invariant and IR finite. The reason for such a distinction is of a purely technical nature. It allows one to use adapted techniques for the various parts. It is quantum mechanically invalid to interpret any photon as being emitted from either the initial or final state. Hence, Ref. [145] refrains from using the term “ISR process” for $e^+e^- \rightarrow X^+X^-\gamma$. Furthermore, it is stressed that a factorized treatment of the corrections into ISC and FSC is always an approximation, the validity of which depends strongly on process and observable.

Beyond the purely perturbative QED aspects, the nonperturbative nature of the hadronic states needs to be considered. First, this concerns the hadronic VPC. While there are different compilations available, in the *RadioMonteCarLow 2* report all results were produced with the same HVP, based on Refs. [163,258,259]. However, there were different implementations in that some codes resum the HVP corrections, others follow an approach closer to the fixed-order approach. Second, final-state pions require a nonperturbative treatment. In principle the hadronic matrix elements of $\pi\pi \rightarrow n\gamma^{(*)}$ is well-defined. However, especially if some or all of the photons involved are hard, there is not always sufficient information available for practical calculations.

n = 1 The matrix element reduces to the pion VFF $F_\pi^V(Q^2)$ which is known sufficiently well. All results in the report were produced with the same version of the VFF.

$n = 2$ The two-pion pole contribution (with two insertions of F_π^V , either using GVMD or dispersively) can be an acceptable approximation, which is called FsQED. On the other hand, a simple multiplication of the sQED result with a form factor evaluated at some scale to ensure finiteness – a method referred to as F×sQED – is known to fail in producing the correct asymmetries, see Sections 2.2.1 and 2.7.4.

$n = 3$ This contribution is not yet systematically studied beyond F×sQED and will require an improved understanding of the hadronic matrix element. Unfortunately, this contribution is especially important for the full NLO corrections to $ee \rightarrow \pi\pi\gamma$.

beyond $n = 3$ These matrix elements, or approximations of them, will be required for a full NNLO description of $ee \rightarrow \pi\pi\gamma$. Needless to say, this is still some time away.

It should once again be stressed that despite these shortcomings it is possible to generate approximate higher-order effects using resummation tools. However, these results are not NLO or even NNLO for $ee \rightarrow \pi\pi\gamma$. It is vital that investigations beyond F×sQED be carried out to estimate the impact of this approximation.

2.4.2. Monte-Carlo comparison

A major part of the *RadioMonteCarLow 2* report for Phase I was the comparison of seven MC codes in five scenarios to loosely approximate the experiments on which they are based: CMD, KLOE small-angle, KLOE large-angle, BES, and a generic B factory. Designing these was a trade-off between complexity, computing requirements, and realism. A major goal of Phase II is streamlining this process and improving the realism of these scenarios.

In the following, some broad conclusions from the comparison of different codes that were used as proxies for the different effects and approximations will be discussed. However, the reader is strongly recommended to refer to the review for a more detailed and nuanced discussion. Specifically, the spread of results for the different codes (“code spread”) cannot be used as an estimate for the theory error.

Starting with $2 \rightarrow 2$ processes, for simple, well-behaved observables the code spread is as low as 0.2%. The notion of “well-behaved” refers to the fact that the result is nonvanishing at LO for the $2 \rightarrow 2$ kinematics. Outside this region, the spread gets significantly larger. This is not surprising, since the technical accuracy is reduced by one order. In this region multiple emission becomes the main effect. In most cases that were studied, the resummation methods were dominated by the emission of one extra photon. This is not to say that these effects are small; indeed they can become substantial (several percent) depending on the specific observable under consideration.

The importance of the interference effects between ISC and FSC can be most plainly seen in asymmetries. This is especially important since the ISC are conceptually the simplest and here improvements are expected first. Further, unless special care is taken to match a resummation (completely or approximately) to an NLO-or-better fixed-order result, these effects will not be captured. The treatment of the $\pi^+\pi^-$ final state for the $2 \rightarrow 2$ process is not too severe an issue. FsQED and GVMD appear to be reasonable approximations.

Turning to $2 \rightarrow 3$ processes, the situation is more delicate. Most codes are either NLO without resummation or LO with resummation, allowing one to estimate both the importance of fixed-order and multi-photon topologies. In the case of $ee \rightarrow \mu\mu\gamma$, the fixed-order result is unambiguous and extensions to NNLO are underway, making it a valuable testing ground for various approximations. The importance of supplementing a parton shower with $p_\perp$ -effects as is done, e.g., in BABAYAGA@NLO, both for capturing the correct shape and the correct cross section, is noted. Similarly, the interference terms that are included in CEEX have proved to be very important and are capable of capturing most of the NLO dynamics though the effect is less pronounced when only considering the shape of the distribution rather than its absolute value. To achieve a precision below the percent level effects beyond NLO need to be incorporated, though the exact details of course depend on scenario and cuts.

For $ee \rightarrow \pi\pi\gamma$, often small FSC at LO were witnessed, (0.1–0.5)%, due to suppression from the VFF since this leads to a larger Q^2 in the form factor. However, this trend does not continue to NLO and large FSC at NLO around 1% were observed, even if FSC were negligible at LO. It is, therefore, important to realize that extrapolating to the next uncomputed order is fraught with difficulty. Even at LO, there are notable exceptions such as the KLOE-like large-angle scenario where the cuts prefer the photon closer to pion, leading to FSC effects of more than 10%.

The importance of FSC in $ee \rightarrow \pi\pi\gamma$ is a major issue, especially at NLO. This $n = 2$ case was not treated really satisfactorily in any of the codes at the time of the comparison. If it is considered at all, it is done using F×sQED, the validity of which has been questioned, at least in some cases. Especially worrying are topologies that involve real or virtual corrections to the pion VFF. These are C -even, i.e., contributing to the total cross section rather than merely an asymmetry, and at least partially FSC. Without a calculation beyond F×sQED for $ee \rightarrow \pi\pi\gamma$, it is difficult to estimate how large this effect really is.

When it comes to resummation for $ee \rightarrow \pi\pi\gamma$ it is possible to use similar technology as for $ee \rightarrow \mu\mu\gamma$. For soft logarithms nothing changes because the soft limit is independent of the spin of the emitter, so YFS can be directly used at LO, and can be extended to higher orders by the inclusion of the corresponding amplitudes (even if changes to the code may be required). For collinear logarithms the $\pi \rightarrow \pi\gamma$ splitting function needs to be used instead. Both methods inherently assume that the emitting pion is nearly on-shell as part of their constructions.

Finally, many scenarios were seen in which VPC were small at NLO but not at NNLO, where their interplay with the non-VP corrections can be complicated. The size of the VPC depends strongly on the scenario but their careful treatment is very important, especially where narrow resonances such as the J/ψ become relevant. This again shows the danger of extrapolating to the next uncomputed order.

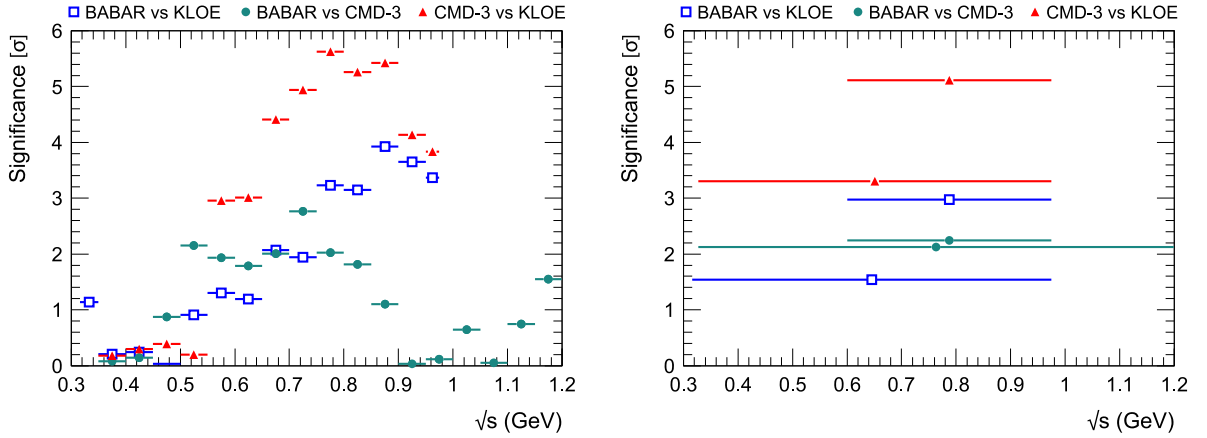


Fig. 14. Significance of the difference between pairs of the three most precise $e^+e^- \rightarrow \pi^+\pi^-$ cross section measurements (BABAR [139,174], CMD-3 [96], KLOE [132–135]) for narrow energy intervals of 50 MeV or less (left) and larger energy intervals (right) indicated by the horizontal lines.

Source: Figures taken from Ref. [166].

2.4.3. Next steps

So far, only the first phase of *RadioMonteCarLow* 2 has concluded, most of which was a review of the existing state-of-the-art. Actual improvements to the codes are expected to take place during Phase II (see, e.g., BABAYAGA@NLO [143]). The *RadioMonteCarLow* 2 community strives to update their living review to reflect these developments as they get released.

A major improvement will be the development of NNLO results for $2 \rightarrow 3$ where possible. Even for $ee \rightarrow \mu\mu\gamma$, a full calculation without any approximation would be challenging. However, there are well-established procedures to obtain sufficiently precise results in the near future. These tools could also be used for $ee \rightarrow \pi\pi\gamma$ in F $\times$ sQED but it is unclear how good an approximation this would be.

Another point that is of high priority for all the developers of MC tools is the inclusion of structure-dependent corrections for pion production. In the case of $ee \rightarrow \pi\pi$, this was already done by a few codes (most recently in BABAYAGA@NLO [143] using GVMD or FsQED). For $ee \rightarrow \pi\pi\gamma$, these effects were identified to be problematic as they could enter the cross section rather than just asymmetries (cf. Figs. 14 and 15 of Ref. [145]). First investigatory steps are being taken in cases where GVMD or FsQED are assumed to be reliable approximations. The development of a full model of the $n = 2$ case is also underway.

The existing NLO calculations and future NNLO calculations will have to be matched with resummation techniques, either based on parton shower or YFS. This will improve the reliability of the prediction and estimate of the truncation error. Further, a strict NLO (NNLO) calculation for $2 \rightarrow 3$ processes will never generate events that have more than four (five) photons. If such events turn out to be experimentally important, the exclusive generation of the resummation techniques will be able to at least approximate these events.

Finally, only a few fairly simple scenarios have been considered and only for $ee \rightarrow ee, \mu\mu, \pi\pi(+\gamma)$. As a first step it is planned to consider more complicated experimental scenarios by streamlining the production pipelines. It is also planned to consider more types of hadronic final states with $ee \rightarrow \pi^+\pi^-\pi^0$ as a first likely candidate. Phase II will also hopefully see estimates of theory errors for some of the processes and scenarios under consideration.

2.5. New developments for the evaluation of HVP

2.5.1. The DHLMZ studies

Data combination and comparisons of mass spectra in the $\pi^+\pi^-$ channel. The DHLMZ studies employ since 2009 a procedure for combining cross-section data with arbitrary point spacing or binning, redistributed in a fine common binning using spline-based interpolations, as implemented in the HVPTools software [166,194,260–263]. For each narrow final bin, a χ^2 is minimized to get the average weights and locally test the level of agreement among the input measurements. The average weights also account for the different bin sizes and point-spacing of measurements, in order to compare their precisions on the same footing. This procedure has been validated through a closure test. It features full and realistic (i.e., not too optimistic) treatment of uncertainties and correlations, between the measurements (data points or bins) of a given experiment, between experiments and between different channels. It also fully accounts for systematic tensions between experiments.

The DHLMZ average is dominated by the most precise experiments (BABAR [139,174], CMD-3 [96], KLOE [132–134], SND20 [109], followed by CMD-2 [124–128], BESIII [140], and SND06 [129]), BABAR covering the full energy range of interest. Taking the ratio between various measurements and this average, it is found that in the [0.5, 1] GeV range the

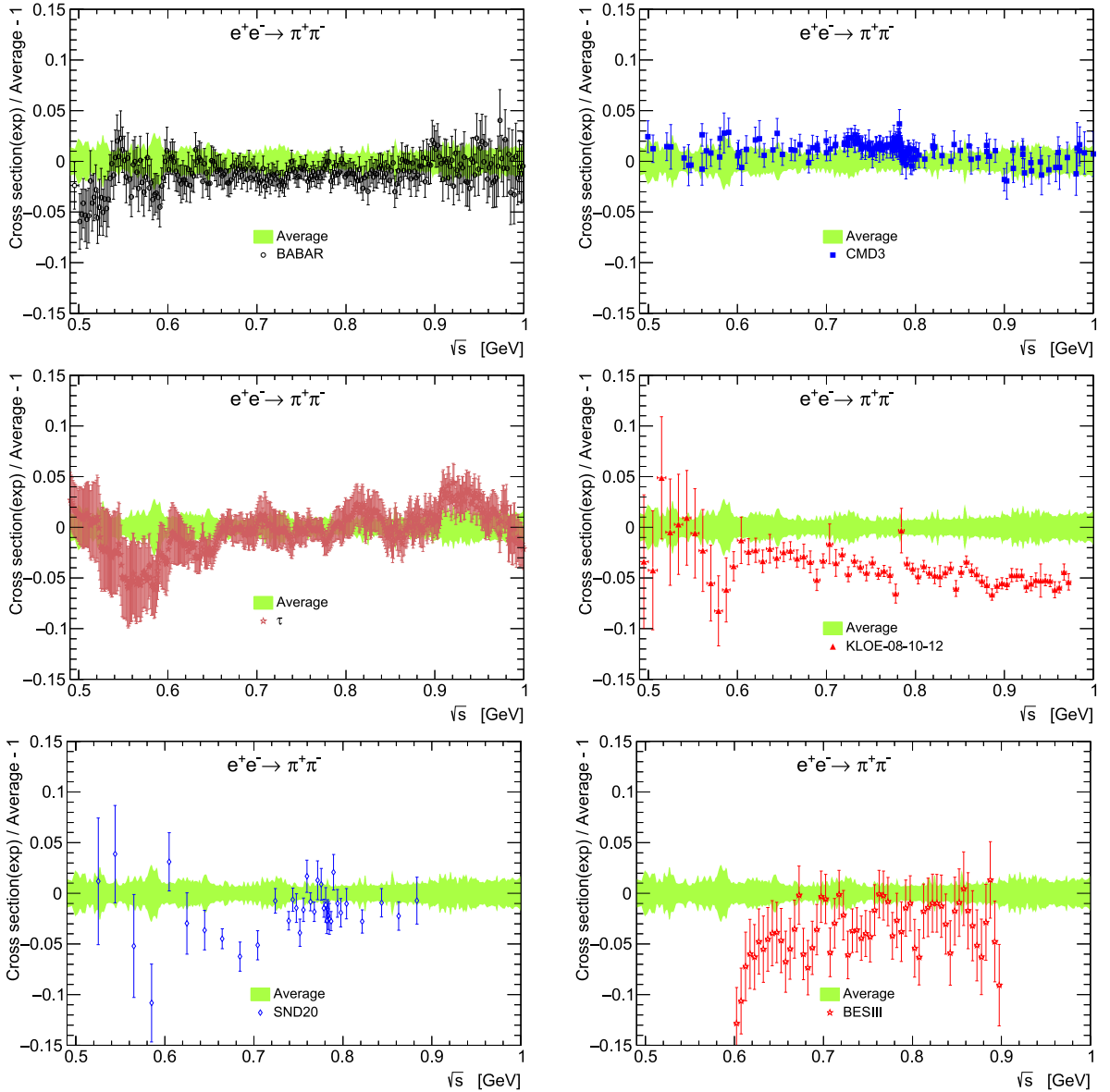


Fig. 15. Relative difference between the $\pi\pi$ cross sections from BABAR (top left) [139,174], CMD-3 (top right) [96], the IB-corrected τ data (middle left) [189], and their average. The IB corrections and uncertainties employed here correspond to the ones from Ref. [166]. Relative difference with respect to the same average for the measurements from KLOE [135] (middle right), SND20 [109] (bottom left), and BESIII [140] (bottom right). Source: Figures taken from Ref. [264].

BABAR and SND20 data overlap rather well with the average, KLOE is systematically below it, while CMD-3 is above [166]. These tensions, especially between KLOE and CMD-3, which provide the smallest and respectively largest cross sections in the ρ region, are also reflected by the enhanced values of χ^2/dof . For KLOE slopes are observed when comparing the 2010 and 2012 data with the 2008 ones. These tensions were quantified through fits and found to be at the $(2.5\text{--}3)\sigma$ level. No significant slope is present between 2010 and 2012 data (see Ref. [1] and references therein).

In order to further quantify these tensions, the integrals for individual experiments were computed and compared in various restricted energy ranges. The significance of the difference between pairs of experiments is determined, taking into account the correlations of the uncertainties, see Fig. 14. The largest observed tensions are between CMD-3 and KLOE, going beyond 5σ on the ρ peak [166]. The impact of these tensions on the comparison of the experimental result to the SM expectation for a_μ is displayed in the Fig. 5 from Ref. [166] (see more detailed discussion below).

The presence of these tensions among experimental measurements represents a clear indication of underestimated uncertainties. This calls for a conservative uncertainty treatment in combination fits and in the determination of the

averaging weights, as implemented in the DHMZ approach [1,263]. These systematic tensions go well beyond the effects accounted through the local χ^2/dof rescaling. This had already motivated the inclusion of the dominant BABAR–KLOE systematic by DHMZ, since the studies reported in Ref. [263]. However, the tensions are larger now and therefore require one to understand their actual source.

Impact of higher-order photon emissions: the implications of a unique “(N)NLO” BABAR study. As discussed in Section 2.2.4, the higher-order photon emissions (i.e., in addition to the hard ISR photon), were studied in-situ with BABAR data [176], for the first time at NLO and NNLO, in the $e^+e^- \rightarrow \mu^+\mu^-\gamma$ and $e^+e^- \rightarrow \pi^+\pi^-\gamma$ channels. This allows one to test the most frequently used MC generators, PHOKHARA and AFKQED. The “(N)NLO” order counting in data and simulations is performed based on the number of additional photons in the final state, having the energy above some given threshold.

It is found that the rate of “NLO” small-angle ISR in PHOKHARA is higher than in data, while the data/MC ratios for large-angle photon emissions are consistent with unity [176]. An independent confirmation of the PHOKHARA problem has been provided by the measurement of the $\pi^+\pi^-\pi^0$ channel performed by the Belle-II Collaboration [185]. The “NNLO” contributions are also clearly observed in data, while they are missing in PHOKHARA. At the same time, AFKQED provides a reasonable description of the rate and energy distributions for “(N)NLO” data.

The discrepancies observed between BABAR and PHOKHARA in the energy distributions of additional ISR photons reveal PHOKHARA shortcomings that cannot be fully interpreted due to the absence of complete NNLO MC generators. A full range of scenarios with extremes labeled 1 and 2, has been considered according to presently unknown NNLO calculations [166]. Scenario 1 questions the validity of PHOKHARA at the NLO level in addition to missing NNLO, while in scenario 2 the discrepancy observed by BABAR would originate solely from missing NNLO. The realistic situation is expected to lie between these two extremes. These findings have triggered further studies performed by BESIII and KLOE. Tests of MC generators regarding the $\mu\mu/\pi\pi$ mass distributions are now available (see Sections 2.4, 2.2.3 and 2.2.5) which confirm the validity of NLO PHOKHARA at the 1% level (where the uncertainties of the other generators employed in the comparisons are relevant too). This is to be compared with a systematic uncertainty of 0.5% for PHOKHARA as quoted in the KLOE/BESIII publications. However, similar comparisons should be done for the energy distribution of additional ISR photons, playing an important role in the event selection. For the moment, the question of re-evaluating PHOKHARA-related systematic uncertainties for the published KLOE/BESIII results has not yet been considered, particularly for KLOE, which has competitive precision. It is worth noting that the sensitivity to these effects does depend on analysis strategy choices that are somewhat less critical for BESIII. It is important to note that the BABAR measurements [139,174] employ a loose selection, incorporating “NLO” and higher-order radiation, minimizing hence the dependence on MC simulations.

A new perspective on $a_\mu^{\text{HVP, LO}}$. The a_μ and central a_μ^W calculations (see Sections 3.4 and 4.1) were performed employing the dispersive approach, based on the most precise measurements available in the $\pi^+\pi^-(\gamma)$ channel (see pages 7, 8, 18, and 19 in Ref. [264]), from BABAR [139,174], CMD-3 [96], and KLOE [135], as well as from hadronic τ decays [189] as discussed in Section 2.3.1, see the Fig. 5 from Ref. [166]. For KLOE, both the full available range (KLOE_{wide}) and a restricted range of (0.6–0.975) GeV (KLOE_{peak}) were considered. For the latter, the data are most precise and KLOE’s weight in the combination is largest. These various $a_\mu^{\text{HVP, LO}}$ integrals are completed with the combination of all the available measurements in the $\pi\pi$ channel [166], in order to cover the full mass range of interest, as well as with contributions from other hadronic channels (as evaluated in Ref. [1]), in view of the comparisons with the BMW lattice QCD result [16] and with the experimental measurement [7].

The τ -based HVP contribution is close to the values provided by BABAR and CMD-3, see the Fig. 5 from Ref. [166]. Their combination is compatible with BMW for a_μ , but a 2.9σ tension persists for a_μ^W . Combining BABAR, CMD-3, τ (and BMW), a difference of 2.5σ (2.8σ) is found with respect to the experiment. When including KLOE in the dispersive calculation, the difference to the experiment becomes larger than 5σ . This situation concerning the most precise experiments can be illustrated at the level of the cross sections by comparison with the average of BABAR [139,174], CMD-3 [96], and the IB-corrected τ data [189], showing their consistency, while KLOE [135] disagrees, both in magnitude and shape, see Fig. 15.⁸ The SND20 [109] and BESIII [140] measurements show also some tensions with this same average, although within somewhat larger uncertainties and hence less statistically significant. To further investigate these problems, tests of MC generators must be performed for experiments which rely on them to correct for missing higher-order contributions. These studies should focus on the rate and angular distributions of additional photons, which impact the selection efficiency [166].

These findings provide additional insight into the longstanding deviation among the muon $g - 2$ measurement, the SM prediction using the data-driven dispersive approach for calculation of HVP, and the comparison with lattice-QCD calculations.

⁸ The average of BABAR, CMD-3, and the IB-corrected τ data is employed here just as a reference in the comparisons. It enables a clear visualization of agreement/differences in terms of shape and normalization among various measurements, which would not be the case if, e.g., a full combination were used instead.

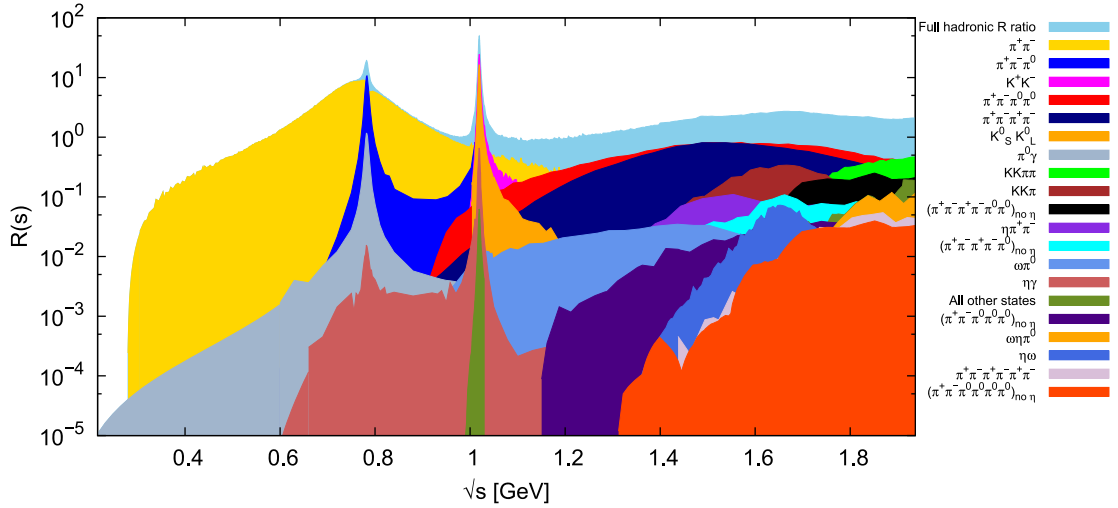


Fig. 16. Contributions to the KNT data compilation of the total hadronic R -ratio from the different hadronic final states below 1.937 GeV [31,265]. The full R -ratio is shown in light blue. Each final state is included as a new layer on top in decreasing order of the size of its contribution to $a_\mu^{\text{HVP, LO}}$. Source: Figure adapted from Ref. [265].

2.5.2. The KNTW studies

For the first edition of WP20 [1], KNT (now KNTW) provided their data-driven HVP compilation [31,265]. Their data for the full hadronic cross section, see Fig. 16, result in the most precise evaluation of $a_\mu^{\text{HVP, LO}}$ to date of $a_\mu^{\text{HVP, LO}}[\text{KNT19}] = 692.8(2.4) \times 10^{-10}$, with overall good agreement observed when compared to other groups. The results were used in the merging procedure between data-driven analyses, which determined the adopted consistent value $a_\mu^{\text{HVP, LO}}[\text{WP20}] = 693.1(4.0) \times 10^{-10}$ in Ref. [1]. KNT have also provided the prediction for the NLO HVP contribution, $a_\mu^{\text{HVP, NLO}}[\text{WP20}] = -9.83(7) \times 10^{-10}$, the data input for the NNLO HVP contribution, $a_\mu^{\text{HVP, NNLO}}[\text{WP20}] = 1.24(1) \times 10^{-10}$ [33], predictions for the HVP contributions to the electron and tau $g - 2$ (both at LO and NLO), for the running of the EM coupling, $\alpha(Q^2)$, and for the hyperfine structure of muonium. Their publicly available, open-access data compilation includes: (1) the full hadronic cross section ranging from the production threshold to 1 TeV, complete with a full covariance matrix for all data, (2) cross-section data and covariance matrices for all individual hadronic modes, and (3) a software package (`knt_vp`) that produces $\alpha(Q^2)$ for both spacelike and timelike Q^2 .⁹ These data have been used by the wider particle physics community including in many studies related to the muon $g - 2$, notably for predictions of window-based quantities.

The philosophy of the KNTW compilation is summarized by the main features: (i) Prioritize, wherever possible, direct input from data without modeling, parameterization, or other constraints. (ii) Include all published data unless they are known to be defective. (iii) Understand and minimize/avoid all possible bias that could be present in the analysis. (iv) Incorporate full correlation information between different data points due to systematic uncertainties in the combination to constrain the fits by using all available experimental data. (v) Evaluate and incorporate additional theoretical systematic uncertainties arising from the analysis. (vi) Be fully open-access with regard to the sharing of studies, results, and resulting combination data.

At the time of Ref. [1], the KNT data combination could accommodate tensions between the data sets, still achieving an acceptable fit quality. This was reflected in a global $\chi^2/\text{dof} = 1.26$ for the most important two-pion channel. For the evaluation of the a_μ integral, an energy-dependent chi-square inflation was used, which amounted to a 14% error inflation for $a_\mu^{\pi^+\pi^-}$ in the range $0.305 < \sqrt{s} < 1.937$ GeV. This situation changed dramatically with the publication of the CMD-3 two-pion data [95,96]. Despite sustained efforts by the community, coordinated by the Muon $g - 2$ Theory and the *RadioMonteCarLow* 2 initiatives, see Sections 2.2.1 and 2.4, no explanations for the discrepancy have been found so far. With further dedicated efforts on the calculation of higher-order radiative corrections and their implementation in MC generators, and new data analyses in the two-pion channel underway, the picture is not yet settled. Therefore, the persistent strong tension with the previous data across the entire energy range in the dominant $\pi^+\pi^-$ channel means that the most precise data sets and their combination are highly inconsistent, and extra caution should be taken before trying to proceed with the direct combination as before. KNTW have consequently refrained from providing an updated compilation.

⁹ All data are available upon request by contacting the authors directly.

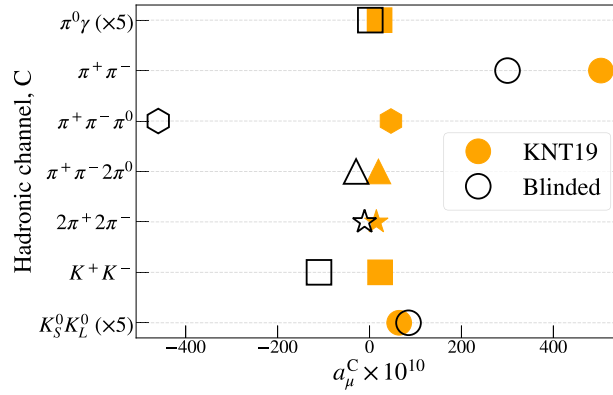


Fig. 17. Values for $a_\mu^{\text{HVP, LO}}$ for different hadronic channels from the original unblinded KNT19 analysis [31] (filled), and what the blinded results of that analysis would have looked like prior to unblinding had the new KNTW blinding procedure [266] been employed (unfilled). For the offsets of this specific example, the blinded results for $\pi^0 \gamma$ and $K_S^0 K_L^0$ are randomly similar to their unblinded counterparts so both have been scaled by $\times 5$ to make the differences visible.

Source: Figure taken from Ref. [266].

Instead, KNTW are taking the opportunity to fully scrutinize, overhaul, and modernize their analysis and data combination procedure in preparation for crucial, new hadronic cross-section data expected in the future, particularly for the $\pi^+ \pi^-$ channel, see Section 2.2. The aim is to improve all aspects of the data treatment, data combination, and error estimation, and to provide a versatile and modern database and software tool for wider use. This is particularly important given that current tensions in different evaluations of $a_\mu^{\text{HVP, LO}}$ and in the $e^+ e^- \rightarrow \pi^+ \pi^-$ cross-section data indicate either a discovery of new physics or a multi-method confirmation of the SM. Crucially, different analysis choices in $e^+ e^- \rightarrow$ hadrons data combinations by different groups can lead to different results and, in Ref. [1], have been shown to differ at the level of the uncertainty on the combined cross section. It follows that future data-driven determinations of $a_\mu^{\text{HVP, LO}}$ must attempt to avoid analysis bias wherever possible, including any past or future analysis choices on how to combine the available data. Consequently, implementing analysis blinding in data-driven determinations of $a_\mu^{\text{HVP, LO}}$ is paramount and critical before including new data whose impact on the resulting $a_\mu^{\text{HVP, LO}}$ will be influenced by such analysis choices. The first blinding scheme for data-driven evaluations of HVP has been developed by KNTW and is in place as part of their new, ongoing KNTW analysis, see Fig. 17. The full blinding procedure is described in Ref. [266]. Unblinding will occur when the analysis is complete and at an appropriate time with respect to the release of new or updated hadronic cross-section data.

2.6. Dispersive representations for exclusive channels

For the dominant exclusive channels in a data-based evaluation of HVP, the form of the required hadronic matrix elements is sufficiently simple that strong constraints from general principles of QCD can be derived, including analyticity, unitarity, crossing symmetry, and chiral low-energy theorems. In Ref. [1] such constraints were already described for the 2π [226,263,267] and 3π [268] channels, here, we review the new developments in dispersive representations in these cases, as well as new applications to $\pi^0 \gamma$ and $\bar{K} K$.

2.6.1. 2π

The central object that defines the 2π contribution to a_μ is the pion VFF $F_\pi^V(s)$, related to the cross section via

$$\sigma(e^+ e^- \rightarrow \pi^+ \pi^-)(s) = \frac{\pi \alpha^2}{3s} \sigma_\pi^3(s) |F_\pi^V(s)|^2, \quad \sigma_\pi(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}. \quad (2.24)$$

To derive dispersive constraints, it is most useful to separate the contributions from different cuts according to

$$F_\pi^V(s) = \Omega_1^1(s) G_\omega(s) G_{\text{in}}^N(s), \quad (2.25)$$

where the Omnès factor $\Omega_1^1(s)$ [269] accounts for 2π intermediate states, $G_\omega(s)$ includes the IB 3π cut, and further inelastic intermediate states such as 4π are expanded into a conformal polynomial $G_{\text{in}}^N(s)$ (with $N - 1$ degrees of freedom). In the formalism from Ref. [226], the strength of $G_\omega(s)$ is parameterized by the ρ - ω mixing parameter ϵ_ω , which can be interpreted as the residue at the ω pole, and apart from a tiny phase from the analytic continuation into the complex plane has to be a real quantity. In Ref. [237], this formalism was extended to account for radiative intermediate states

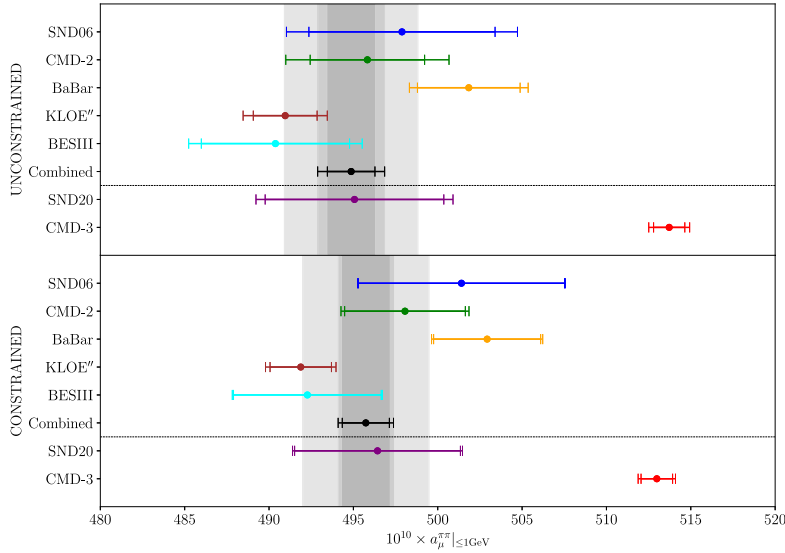


Fig. 18. 2π contribution to a_μ for $\sqrt{s} \leq 1$ GeV, in the unconstrained dispersive representation from Refs. [237,270] (upper panel) and imposing the absence of zeros in the VFF [146] (lower panel). The data sets are SND06 [129,130], CMD-2 [124,125,128], BABAR [139,174], KLOE [132–135], BESIII [140], SND20 [109], and CMD-3 [95,96]. The combined fit includes all data sets apart from SND20 and CMD-3. Inner error bars are derived from the experimental uncertainties, the outer ones include theory uncertainties from the dispersive representation. In the case of the combined fit, the outmost gray error band includes, in addition, an estimate of the BABAR–KLOE tension as in Ref. [1].
Source: Figure taken from Ref. [146].

mediating the ρ – ω transition, such as $\rho \rightarrow \pi^0 \gamma \rightarrow \omega$, which can produce a small but relevant phase in ϵ_ω . The improved parameterization of $G_\omega(s)$ takes the form [237]

$$G_\omega(s) = 1 + \frac{s}{\pi} \int_{9M_\pi^2}^{\infty} ds' \frac{\text{Re } \epsilon_\omega}{s'(s' - s)} \text{Im} \left[\frac{s'}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s'} \right] \left(\frac{1 - \frac{9M_\pi^2}{s'}}{1 - \frac{9M_\pi^2}{M_\omega^2}} \right)^4 + \frac{s}{\pi} \int_{M_{\pi^0}^2}^{\infty} ds' \frac{\text{Im } \epsilon_\omega}{s'(s' - s)} \text{Re} \left[\frac{s'}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s'} \right] \left(\frac{1 - \frac{M_{\pi^0}^2}{s'}}{1 - \frac{M_{\pi^0}^2}{M_\omega^2}} \right)^3, \quad (2.26)$$

ensuring the absence of imaginary parts below the respective thresholds and their correct threshold behavior above. Moreover, the size of the expected phase of ϵ_ω could be estimated to $\delta_\epsilon = 3.5(1.0)^\circ$, based on a narrow-width approximation for the intermediate-state vector mesons in $\pi^0 \gamma$, $\pi \pi \gamma$, and $\eta \gamma$. A fit to the different $e^+e^- \rightarrow \pi^+ \pi^-$ data sets was performed in Refs. [237,270], leading to the picture in the upper panel of Fig. 18.¹⁰ As key result, it was found that all but the SND20 data set could be described in a statistically satisfactory manner, which was therefore excluded from the combined fit of experiments prior to CMD-3. Already in this unconstrained fit, the tension between CMD-3 and the combination evaluates to 7.3σ (excluding the additional BABAR–KLOE tension represented by the outmost gray band in Fig. 18).

The largest uncertainty in the dispersive representation of the VFF from Refs. [226,237,270] is a systematic (theory) uncertainty related to variations with the order N of the conformal polynomial $G_{\text{in}}^N(s)$ describing the inelastic effects. In Ref. [146], it was observed that these large variations coincide with the appearance of zeros in the VFF fits for $N \geq 5$. Although the assumption that the VFF is free of complex zeros is not proved, there are theoretical arguments against the presence of such zeros [275]. In Refs. [276,277], zeros have been excluded in a large region of the complex s -plane. In Ref. [146] it was shown that imposing the absence of zeros as a constraint in the fit removes the largest systematic uncertainty without a significant impact on the fit quality. The results of these constrained fits are shown in the lower panel in Fig. 18. While qualitatively leading to similar results, the additional constraint exacerbates the discrepancies between dispersive fits to different experiments. Notably, the discrepancy between the fits to KLOE and CMD-3 reaches 8.9σ [146].

In case that the VFF has no zeros (or the position of zeros is known), its phase can be reconstructed dispersively from the modulus [275,278,279]. In Ref. [277], such a modulus dispersion relation was used to determine the phase of the

¹⁰ The figure shows the implementation from Ref. [146], which also involves improved error estimates for the P -wave elasticity parameter in the solution of $\pi\pi$ Roy equations [271–274] and the truncation of the conformal expansion.

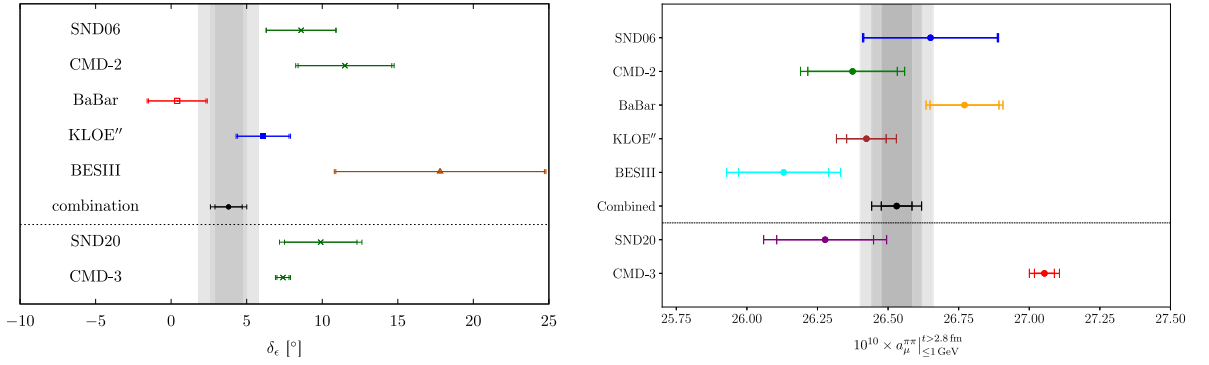


Fig. 19. Left: Phase δ_ϵ of the ρ - ω mixing parameter, according to the dispersive representation Eq. (2.26). Right: Two-pion contribution to the $t > 2.8$ fm Euclidean time window (Eq. (3.19), with $t_1 = 2.8$ fm), obtained from low-energy fits of the VFF constrained to be free of zeros. Legends as in Fig. 18.

Source: Figures taken from Refs. [146,270].

VFF up to $s = (2.5 \text{ GeV})^2$ from BABAR data, by parameterizing the data on the modulus with a GS [223] function. This approach can also be combined with an Omnès representation for the elastic $\pi\pi$ rescattering contribution, resulting in a hybrid representation that requires input for the modulus only above the inelastic threshold [276,280]. In Ref. [146], such a hybrid representation was fit to the different experiments at low energies and to BABAR data [139,174] above 1.4 GeV. The results are compatible with the unconstrained or constrained low-energy Omnès fits, but lead to even smaller uncertainties, at the price of introducing a parameterization dependence for the modulus above the inelastic threshold. While the representation of inelastic contributions in terms of a conformal polynomial $G_{\text{in}}^N(s)$ in Eq. (2.25) is restricted to energies below $\simeq 1$ GeV, the hybrid representation has the advantage that it can be used also above 1 GeV. An assessment of the parameterization dependence and a further reduction of the uncertainties would be possible with more high-statistics data on the modulus of the VFF above the inelastic threshold.

The dispersive representation of the VFF has the feature that it provides a global fit function: it thus allows one to evaluate each experiment in the entire region below 1 GeV, not only in the range in which data were taken, and to sharpen the tensions among the experiments by taking into account that the allowed form of the cross section is highly constrained by analyticity and unitarity. Moreover, the fit parameters can be compared as well, implying interesting correlations with other low-energy observables, in particular the pion charge radius, see Section 2.7.1. An example is provided in Fig. 19(left), where δ_ϵ is shown for the fits to the different experiments. Again, considerable spread is observed among the experiments, but in a different order than for the cross section, comparing to Fig. 18, with some experiments preferring phases much bigger than the narrow-width estimate $\delta_\epsilon = 3.5(1.0)^\circ$. Another example is given by Euclidean window quantities: as shown in Ref. [146], the dispersive constraints correlate the discrepancies in the ρ -peak region with tensions even at very long Euclidean distance (corresponding to very low energies), see Fig. 19(right) for the spread of results for the two-pion contribution below 1 GeV to the Euclidean window $t > 2.8$ fm.

In contrast to some other experiments, CMD-3 [95,96] does not provide detailed information on the bin-to-bin correlations of the systematic uncertainties, see Section 2.2.1. The fits of Refs. [146,270] were performed assuming fully correlated systematic uncertainties. Since this assumption might not apply to all systematic effects, Ref. [146] also investigated a toy scenario of a de-correlation of the systematic uncertainties, ranging from fully correlated uncertainties to only diagonal errors, which shows some shifts in the central fit values: in the considered de-correlation scenario for CMD-3, the maximal shift of the central value for $a_\mu^{\pi\pi}|_{\leq 1 \text{ GeV}}$ amounts to 1.8 times the fit uncertainty obtained with fully correlated systematics, whereas for all other experiments, the shifts in the central values induced by neglecting correlations remain within the fit uncertainties. Since no detailed information is available on the systematic correlations, for a most conservative treatment of systematic errors the CMD-3 fit uncertainties should be somewhat enlarged. Since the observed shifts for CMD-3 are towards larger values of $a_\mu^{\pi\pi}|_{\leq 1 \text{ GeV}}$, these studies indicate that reduced correlations would lead to even larger differences between CMD-3 and the other experiments than the treatment with fully correlated systematic uncertainties. In either case, these studies emphasize the need for improved information on the correlation of systematic uncertainties, especially in cases such as CMD-3 in which both statistical and truncation uncertainties in the conformal expansion are small.

2.6.2. 3π and $\pi^0\gamma$

The $e^+e^- \rightarrow 3\pi$ cross section can be described in a dispersive approach by its underlying $\gamma^* \rightarrow 3\pi$ decay amplitude $\mathcal{F}(s, t, u; q^2)$ defined as

$$\langle 0 | j_\mu(0) | \pi^+(p_+) \pi^-(p_-) \pi^0(p_0) \rangle = -\epsilon_{\mu\nu\alpha\beta} p_+^\nu p_-^\alpha p_0^\beta \mathcal{F}(s, t, u; q^2), \quad (2.27)$$

Table 3

VP-subtracted resonance parameters of ω and ϕ from $e^+e^- \rightarrow 3\pi$ [186,268], $e^+e^- \rightarrow \pi^0\gamma$ [300], and $e^+e^- \rightarrow \bar{K}K$ [46]. The last column gives the PDG values [306], with VP removed using the corrections from Ref. [287]. Table from Ref. [186]. The values from Ref. [268] are superseded by Ref. [186], illustrating the impact of the BABAR data [149]. At this level of precision, tensions among the different channels start to emerge, most notably in Γ_ϕ between $e^+e^- \rightarrow \bar{K}K$ and $e^+e^- \rightarrow 3\pi$.

	$e^+e^- \rightarrow \pi^0\gamma$ Ref. [300]	$e^+e^- \rightarrow \bar{K}K$ Ref. [46]	$e^+e^- \rightarrow 3\pi$ Ref. [268]	Ref. [186]	PDG [306]
M_ω [MeV]	782.584(28)	–	782.631(28)	782.697(32)	782.53(13)
Γ_ω [MeV]	8.65(6)	–	8.71(6)	8.711(26)	8.74(13)
M_ϕ [MeV]	1019.205(55)	1019.219(4)	1019.196(21)	1019.211(17)	1019.201(16)
Γ_ϕ [MeV]	4.07(13)	4.207(8)	4.23(4)	4.270(13)	4.249(13)

where s , t , and u are Mandelstam variables and q^2 is the photon virtuality. A dispersive representation of this amplitude was constructed in Refs. [268,281–285] in terms of a Khuri–Treiman equation [286], and was used to evaluate the 3π contribution to a_μ . The new developments since then mainly concern IB effects, with ρ – ω mixing and radiative corrections being incorporated into the dispersive representation in Ref. [186]. The first effect, ρ – ω mixing, can be predicted by the coupled-channel formalism from Ref. [287], from which the correction factor

$$g_\pi(q^2) = 1 - \frac{g_{\omega\gamma}^2 \epsilon_\omega}{e^2} \Pi_\pi(q^2), \quad (2.28)$$

depending on the 2π VP function $\Pi_\pi(q^2)$, emerges. In particular, the ρ – ω -mixing parameter ϵ_ω is introduced in a way consistent with that of $e^+e^- \rightarrow 2\pi$, see Eq. (2.26). The value of the mixing parameter obtained from the 3π global fit, including the data sets of SND [288–291], CMD-2 [125,292–294], and BABAR [149], is in agreement with ϵ_ω extracted from the 2π channel at the level of 1.9σ . An estimate of FSR was derived by focusing on IR-enhanced corrections, and a correction factor $\eta_{3\pi}$ analogous to $\eta_{2\pi}$ in 2π channel was obtained for the first time, see Ref. [186] for details. The total HVP contribution amounts to $a_\mu^{3\pi}|_{\leq 1.8 \text{ GeV}} = 45.91(53) \times 10^{-10}$, of which $a_\mu^{\rho-\omega}[3\pi] = -2.68(70) \times 10^{-10}$ and $a_\mu^{\text{FSR}}[3\pi] = 0.51(1) \times 10^{-10}$ arise from IB.

The dispersive representation for $\mathcal{F}(s, t, u; q^2)$ is constrained by the Wess–Zumino–Witten (WZW) anomaly for $3\pi\gamma$ [295,296], which, in the chiral limit, predicts $\mathcal{F}(0, 0, 0; 0) = F_{3\pi}^{\text{WZW}} = 1/(4\pi^2 F_\pi^3)$ in terms of the pion decay constant F_π . Including quark-mass corrections [283,297,298], this constraint anchors the dispersive representation directly at low energies and indirectly via a sum rule. In Ref. [299] an extended data fit was performed to test this input, obtaining $F_{3\pi}/F_{3\pi}^{\text{WZW}} = 1.028(53)$ and thus validating the anomaly constraint at the 5% level. In addition to the global fit, also the recent Belle II data [185] were considered, see Section 2.2.6, observing tensions with the dispersive constraints, the width parameters of ω and ϕ , and the chiral anomaly; the latter reflecting the tension in the 3π contribution to a_μ .

Similarly, the $\pi^0\gamma$ channel is based on a dispersive representation of the singly-virtual $\pi^0 \rightarrow \gamma\gamma^*$ transition form factor. A once-subtracted representation was implemented as [300]

$$F_{\pi^0\gamma^*\gamma^*}(q^2, 0) = F_{\pi\gamma\gamma} + \frac{1}{12\pi^2} \int_{4M_\pi^2}^\infty ds' \frac{q_\pi^3(s')(F_\pi^V(s'))^*}{s'^{3/2}} \times \left\{ f_1(s', q^2) - f_1(s', 0) + \frac{q^2}{s' - q^2} f_1(s', 0) \right\}. \quad (2.29)$$

This process is closely related to the 3π channel since the $\gamma^* \rightarrow 3\pi$ P -wave amplitude $f_1(s, q^2)$ turns out to be a building block of its unitarity relation. The resonance parameters were then fit to the existing data sets [301–305] below 1.4 GeV. The final combined fit produced the HVP contribution to a_μ from this lowest radiative channel, $a_\mu^{\pi^0\gamma}|_{\leq 1.35 \text{ GeV}} = 43.8(6) \times 10^{-11}$. As a by-product, the resonance parameters of ω and ϕ can be compared to the determinations from other channels, as shown in Table 3.¹¹ In particular, good agreement of the ω mass was found between 3π and $\pi^0\gamma$ in the dispersive approach, evading unphysical phases that afflict previous conflicting determinations in the same channel [303].

2.6.3. $\bar{K}K$

In Ref. [46], a dispersive analysis of the charged- and neutral-kaon EM form factors was performed, based on their lowest-lying singularities, and thus allowing one to correlate both time- and spacelike data. For this purpose, the physical form factors are decomposed into isovector (v) and isoscalar (s) components according to

$$F_{K^\pm}(s) = F_K^s(s) + F_K^v(s), \quad F_{K^0}(s) = F_K^s(s) - F_K^v(s). \quad (2.30)$$

The unitarity relation for the isovector part is, at low energies, entirely dominated by two-pion intermediate states, resulting in [311]

$$\text{Im} F_K^v(s) = \frac{s}{4\sqrt{2}} \sigma_\pi^3(s) (g_1^1(s))^* F_\pi^V(s). \quad (2.31)$$

¹¹ The resonance parameters also play a crucial role in R χ T and related approaches, see Refs. [307–310] for recent applications to evaluations of the HVP contribution.

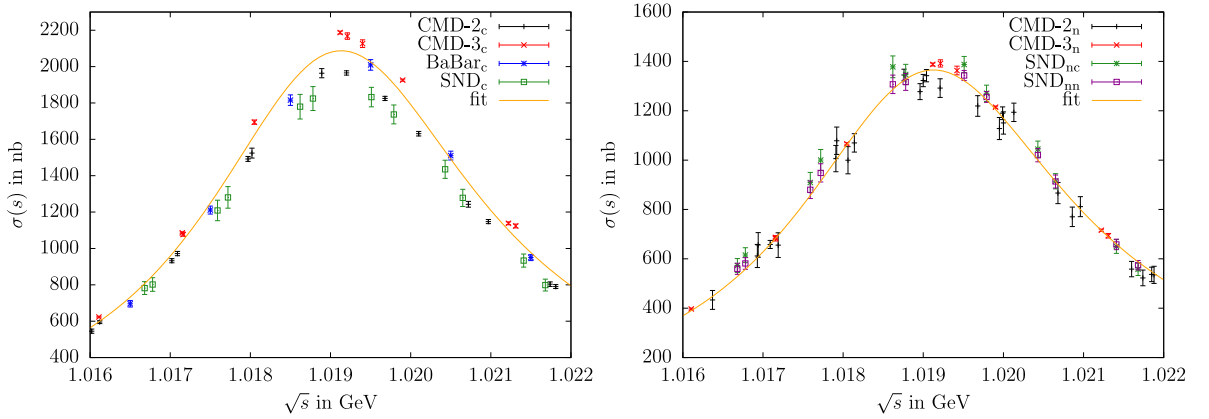


Fig. 20. Cross-section data and combined fits to all data sets for $e^+e^- \rightarrow K^+K^-$ (data from Refs. [102,288,317,318], left) and $e^+e^- \rightarrow K_S K_L$ (data from Refs. [104,125,288,288], right), respectively. Both are shown focused on the ϕ peak region. Source: Figure adapted from Ref. [46].

Here, $g_1^1(s)$ denotes the $\pi\pi \rightarrow \bar{K}K$ P -wave amplitude known from corresponding Roy–Steiner analyses [312–314]. The isovector kaon form factor itself is calculated from Eq. (2.31) using an unsubtracted dispersion relation. Higher intermediate states are modeled effectively by a ρ' -pole contribution, whose strength is adjusted to fulfill the normalization sum rule $F_K^v(0) = 1/2$. The isoscalar part, on the other hand, is dominated by the narrow $\omega(782)$ and $\phi(1020)$ resonances. While the ϕ dominates the $e^+e^- \rightarrow \bar{K}K$ cross sections above threshold, and its spectral form needs to be described carefully using energy-dependent widths [284], the ω lies in the unphysical region and can be constrained only to some extent as a background effect, from spacelike data [315,316], or using SU(3) symmetry. Once more, a heavier, effective, ω' pole is added to ensure the correct isoscalar normalization.

An interesting observation is that combined fits to the $e^+e^- \rightarrow K^+K^-$ [102,288,317,318] and $e^+e^- \rightarrow K_S K_L$ [104, 125,288] data sets suggest an IB difference in the ϕ residues at the level of 2.6(9)%. This is found despite the kaon mass difference, universal FSR for the charged kaons, and the isovector background in the form factors being taken into account. Mass and width of the ϕ come out consistently with determinations from the 3π and $\pi^0\gamma$ channels, cf. Table 3. The combined HVP contributions in the ϕ -resonance region are found to be

$$a_\mu^{\text{HVP}}[K^+K^-, \leq 1.05 \text{ GeV}] = 184.5(2.0) \times 10^{-11}, \quad a_\mu^{\text{HVP}}[K_S K_L, \leq 1.05 \text{ GeV}] = 118.3(1.5) \times 10^{-11}, \quad (2.32)$$

while results based on individual data sets reflect the tensions between the data, in particular BABAR [318] and CMD-3 [102] for the charged kaons; see Fig. 20 and Ref. [46].

The resulting spacelike form factors have also been used to determine the kaon-box contributions to HLbL, see Section 5.5.2. Besides, radii for both neutral and charged kaons can be extracted, yielding

$$\langle r^2 \rangle_n = -0.060(4) \text{ fm}^2, \quad \langle r^2 \rangle_c = 0.359(3) \text{ fm}^2, \quad (2.33)$$

which both present a significant gain in precision compared to the values cited by the PDG [225]. Finally, knowledge of the form factors allows one to evaluate the elastic contributions to the EM mass shifts from the Cottingham formula [319],

$$(M_K^2)_{\text{EM}} = \frac{\alpha}{8\pi} \int_0^\infty ds [F_K(-s)]^2 \left(4W + \frac{s}{M_K^2} (W - 1) \right), \quad W = \sqrt{1 + \frac{4M_K^2}{s}}, \quad (2.34)$$

which results in an estimate of the EM charged-to-neutral kaon mass difference $(\Delta M_K^2)_{\text{EM}} = 2.12(18) \times 10^{-3} \text{ GeV}^2$ and allows one to disentangle EM from quark-mass-induced effects in $\Delta M_K^2 = M_{K^\pm}^2 - M_{K^0}^2$. This separation serves as important input to the discussion of IB in Section 2.7.3. In particular, the scheme implicitly defined by Eq. (2.34), and the resulting mass decomposition in Eq. (2.37), agree well with the FLAG-recommended convention in lattice QCD, see Section 3.2.

2.7. Applications of dispersive representations

2.7.1. Correlations with other observables

Having a dispersive representation for the 2π cross section allows one to study interesting correlations with other low-energy observables, see, e.g., Ref. [320]. First, the resulting values of the P -wave $\pi\pi$ phase shift at $s_0 = (0.8 \text{ GeV})^2$ and $s_1 = (1.15 \text{ GeV})^2$, which enter as fit parameters in $\Omega_1^1(s)$ via the Roy-equation solution, can be compared to partial-wave analyses [321,322]. The result of the global VFF fit proves consistent with these partial-wave solutions,

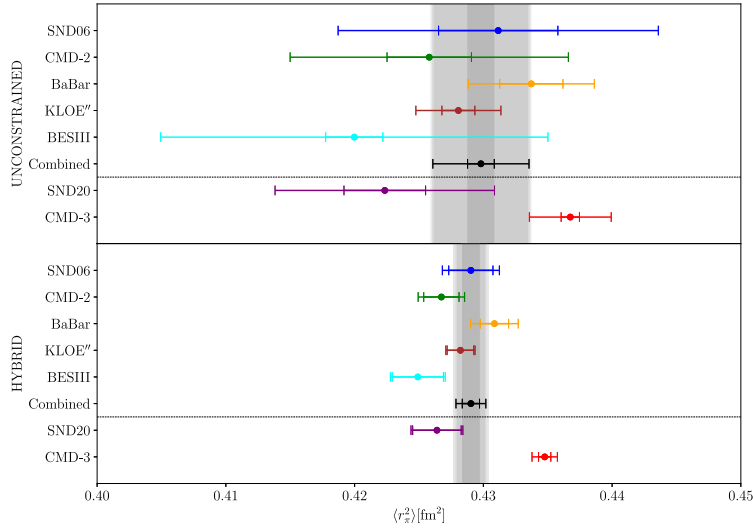


Fig. 21. Values of the pion charge radius $\langle r_\pi^2 \rangle$ (in fm²) from the (unconstrained) Omnès representation Eq. (2.25) and from the hybrid representation of Ref. [146], fit to single experiments. Figure taken from Ref. [146], legend as in Fig. 18.

but appreciably more precise, to the extent that the corresponding phase shift enters as input for global analyses of $\pi\pi$ scattering [323]. Surprisingly, even for the fits to CMD-3 the change in these phase-shift values is small [146,270], essentially realizing scenario (2) from Ref. [320] in which all changes occur in the conformal polynomial $G_{\text{in}}^N(s)$, see Section 2.6.1. This observation could potentially allow one to discriminate among the 2π data sets using additional input from $e^+e^- \rightarrow 4\pi, \pi\omega$ data, and work in this direction is ongoing [324,325].

Next, in addition to studying δ_ϵ , the results for the ω pole parameters in VFF fits can be contrasted to determinations from $e^+e^- \rightarrow 3\pi$ and $e^+e^- \rightarrow \pi^0\gamma$, see Section 2.6.2. While the sensitivity cannot compete with the 3π channel, some deficit in M_ω tends to remain, correlated with δ_ϵ (in line with a similar observation in Ref. [139] in the context of GS fits).

Finally, once the pion VFF is determined, the pion charge radius follows via the sum rule

$$\langle r_\pi^2 \rangle = 6 \frac{dF_\pi^V(s)}{ds} \Big|_{s=0} = \frac{6}{\pi} \int_{4M_\pi^2}^{\infty} ds \frac{\text{Im} F_\pi^V(s)}{s^2}, \quad (2.35)$$

defining another low-energy observable that could help discriminate among the 2π data sets in case lattice-QCD calculations [326–329] at the required level of precision became available. For such a comparison, the improvements in the evaluation of the inelastic contributions in Ref. [146] become critical, since, with the imaginary part entering in Eq. (2.35) in principle up to arbitrarily high energies, $\langle r_\pi^2 \rangle$ displays an increased sensitivity to G_{in}^N . In particular, the hybrid representation of Ref. [146] leads to significantly smaller uncertainties in $\langle r_\pi^2 \rangle$ than the unconstrained Omnès representation Eq. (2.25), see Fig. 21, but in all fits the hybrid representation relies on BABAR data above 1.4 GeV. Therefore, the main results of Ref. [146] for the pion charge radius are based on the hybrid representation, fit to a combination of all data sets apart from SND20 and CMD-3, as well as the constrained Omnès representation, fit to CMD-3, leading to

$$\langle r_\pi^2 \rangle^{\text{comb}} = 0.4290(17) \text{ fm}^2, \quad \langle r_\pi^2 \rangle^{\text{CMD-3}} = 0.4367(24) \text{ fm}^2. \quad (2.36)$$

Apart from these low-energy correlations, connections also exist with Z-pole observables via the hadronic running of the fine-structure constant, see Refs. [330–333] and Section 3.7.2.

2.7.2. Chiral extrapolation

A dispersive representation of the 2π cross section further allows one to improve chiral extrapolations. That is, the pion-mass dependence of Eq. (2.25), and thus the two-pion contribution to a_μ , can be evaluated once chiral extrapolations of the phase shift $\delta_1^1(s)$ and the conformal polynomial are available. For the former, one may use the inverse-amplitude method (IAM) [334,335], relying on input for the chiral amplitudes up to two loops [336,337] to be able to assess the convergence by comparing NLO and NNLO results. For the conformal polynomial, the pion-mass dependence can be estimated by matching to ChPT for the pion VFF, e.g., the charge radius, whose M_π dependence is also known up to two loops [338], allows one to include the first nontrivial term in the conformal expansion. In particular, resonance saturation for the required two-loop low-energy constant $r_{V_1}^2$ is validated by lattice-QCD calculations of $\langle r_\pi^2 \rangle$ at heavier-than-physical pion masses [327,335].

A detailed analysis of the pion-mass dependence of the two-pion contribution along these lines was performed in Ref. [335]. The results inform the chiral extrapolation (or interpolation) of lattice-QCD calculations, e.g., suggesting the

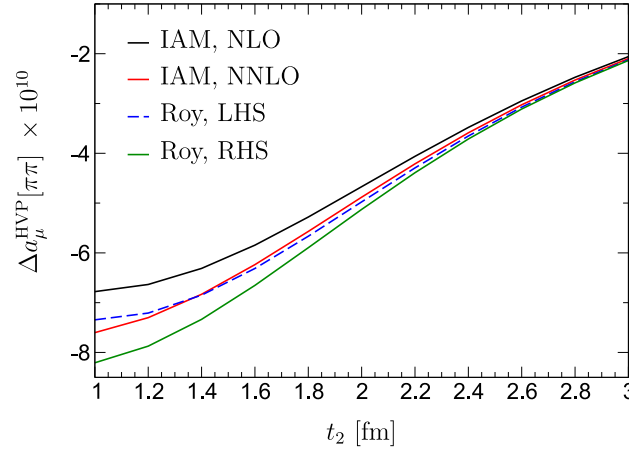


Fig. 22. Projection of the pion-mass-difference correction to the two-pion contribution $\Delta a_{\mu}^{\text{HVP, LO}}[\pi\pi]$ onto the Euclidean window $[t_2, \infty)$. The curves refer to the IAM at NLO and NNLO as well as the LHS and RHS of the Roy-equation solution, respectively. Based on Refs. [247,335].

presence of an $1/M_{\pi}^2$ term in empirical fits in the range $M_{\pi} \in [0.14, 0.25]$ GeV. Moreover, also the important IB effect when performing iso-symmetric calculations at the mass of the neutral pion can be estimated, see Fig. 22 for the projection of this correction onto the Euclidean window $[t_2, \infty)$. As a cross-check, the same effect was evaluated in Ref. [247] from a Roy-equation analysis, determining the $\pi\pi$ scattering lengths at the mass of the neutral pion from two-loop ChPT [336,337,339], where the difference between left-hand side (LHS) and right-hand side (RHS) of the Roy equations provides an estimate of the systematic uncertainty. Fig. 22 shows that the different variants agree very well especially in the long-distance tail, while uncertainties increase towards $t_2 = 1$ fm. This result enters the phenomenological estimate of IB corrections discussed in Section 2.7.3.

2.7.3. Isospin breaking

The detailed understanding of the dominant exclusive channels summarized in Section 2.6 presents an opportunity to estimate a number of IB effects from phenomenology. First, the radiative channels $\pi^0\gamma$, $\eta\gamma$, and $\pi^0\pi^0\gamma$ are naturally booked as QED contributions $\mathcal{O}(e^2)$. They are large compared to the naive expectation $\frac{\alpha}{\pi} a_{\mu}^{\text{HVP, LO}} \simeq 1 \times 10^{-10}$, especially considering the energy suppression of the latter two channels. Their size can be explained by resonance enhancement, due to $V = \omega, \rho, \phi$ for the $P\gamma$ channels and a double enhancement due to $\rho(1450) \rightarrow \omega\pi^0 \rightarrow \pi^0\pi^0\gamma$. Next, FSR effects contribute to the 2π , 3π , and K^+K^- channels, and can again be extracted from dispersive fits to data, as can the contribution due to ρ - ω mixing thanks to the improved representations discussed in Sections 2.6.1 and 2.6.2. For the pion-mass correction when defining the isospin limit by the mass of the neutral pion the treatment according to Section 2.7.2 applies. Finally, for the kaon channels one needs to further specify the isospin conventions, for which Refs. [247,340] suggest the decomposition

$$M_{K^{\pm}} = (494.58 - 3.05_{\delta} + 2.14_{e^2}) \text{ MeV}, \quad M_{K^0} = (494.58 + 3.03_{\delta}) \text{ MeV}, \quad (2.37)$$

into $\mathcal{O}(e^2)$ and $\mathcal{O}(\delta)$, $\delta = m_u - m_d$, contributions, as motivated by the kaon self energies determined via the Cottingham approach [46], see Section 2.6.3. Eq. (2.37) agrees well with the FLAG prescription in lattice QCD, see Section 3.2.

The complete summary of all IB effects collected in Ref. [247] is reproduced in Table 4, separated into the benchmark RBC/UKQCD SD, intermediate, and LD Euclidean windows as defined in Section 3.4, as well as $\mathcal{O}(e^2)$ and $\mathcal{O}(\delta)$ contributions. For each channel and effect, the uncertainties are well defined and reasonably small, so that the final uncertainties are dominated by potential IB corrections that cannot be fully quantified. The first of these concerns the residue of the ϕ in the $\bar{K}K$ decay, since it is not clear with which channel the isospin limit should be identified. Accordingly, the difference between the possible choices is added as an additional source of uncertainty.¹² Finally, higher multiplicity channels such as 4π may also involve IB contributions, but in those cases no strong enhancements from resonances, threshold effects, or IR cancellations are expected, so that a generic 1% error is assigned in Table 4.

From the sum of exclusive channels one observes that individually large contributions tend to cancel in the sum, leaving remarkably small overall corrections. Moreover, for the SD window a null effect is predicted, with a large uncertainty dominated, as expected, by the 1% estimate of missing channels. In the intermediate window, such effects still play an important role, while evidence for a nonvanishing effect is found largely driven by the radiative channels. The most robust

¹² A similar effect also occurs for the ω residue in the 3π channel, which could subsume IB effects, but in this case no phenomenological way to estimate the potential impact is available. Assigning an additional 1% uncertainty for the 3π channel would amount to $\{0.03, 0.19, 0.25, 0.46\}$, for SD, intermediate, LD, and total.

Table 4

IB contributions to $a_\mu^{\text{HVP, LO}}$ from the various radiative channels, FSR, ρ – ω mixing, and threshold effects, separated into $\mathcal{O}(e^2)$ and $\mathcal{O}(\delta)$ contributions as well as the RBC/UKQCD SD, intermediate, and LD windows. ρ – ω mixing is booked as $\mathcal{O}(\delta)$ following the LO R χ T argument from Ref. [341], see Ref. [237] for details. The penultimate panel reflects the uncertainties from the quadratic sum of the individual-channel errors, the ambiguity in the ϕ residues, and a generic 1% error for the channels not explicitly included, while the last line gives the total quadratic sum. Table adapted from Ref. [247], all entries in units of 10^{-10} .

	SD		W		LD		Full	
	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$
$\pi^0\gamma$	0.16(0)	–	1.52(2)	–	2.70(4)	–	4.38(6)	–
$\eta\gamma$	0.05(0)	–	0.34(1)	–	0.31(1)	–	0.70(2)	–
$\omega(\rightarrow \pi^0\gamma)\pi^0$	0.15(0)	–	0.54(1)	–	0.19(0)	–	0.88(2)	–
FSR (2π)	0.12(0)	–	1.17(1)	–	3.13(3)	–	4.42(4)	–
FSR (3π)	0.03(0)	–	0.20(0)	–	0.28(1)	–	0.51(1)	–
FSR (K^+K^-)	0.07(0)	–	0.39(2)	–	0.29(2)	–	0.75(4)	–
ρ – ω mixing (2π)	–	0.06(1)	–	0.86(6)	–	2.87(12)	–	3.79(19)
ρ – ω mixing (3π)	–	–0.13(3)	–	–1.03(27)	–	–1.52(40)	–	–2.68(70)
Pion mass (2π)	0.04(8)	–	–0.09(56)	–	–7.62(63)	–	–7.67(94)	–
Kaon mass (K^+K^-)	–0.29(1)	0.44(2)	–1.71(9)	2.63(14)	–1.24(6)	1.91(10)	–3.24(17)	4.98(26)
Kaon mass ($\bar{K}^0K^0$)	0.00(0)	–0.41(2)	–0.01(0)	–2.44(12)	–0.01(0)	–1.78(9)	–0.02(0)	–4.62(23)
Sum of channels	0.33(8)	–0.04(4)	2.34(57)	0.02(33)	–1.97(63)	1.48(44)	0.71(95)	1.47(80)
ϕ residue	0.00(8)	0.00(8)	0.00(47)	0.00(47)	0.00(36)	0.00(36)	0.00(90)	0.00(90)
Missing channels	0.00(49)	0.00(49)	0.00(55)	0.00(55)	0.00(12)	0.00(12)	0.00(1.16)	0.00(1.16)
Sum	0.33(51)	–0.04(50)	2.34(92)	0.02(79)	–1.97(74)	1.48(58)	0.71(1.75)	1.47(1.67)

prediction concerns the LD window, in which case the contribution from higher-multiplicity channels is very small and the dominant effects well under control. As illustrated in Fig. 22 for the case of the pion-mass correction, the uncertainties will reduce further if the lower boundary t_2 is increased. The complementarity to lattice-QCD results is discussed in Section 3.6, especially for the LD window. Within uncertainties, the full $\mathcal{O}(\delta)$ result given in Table 4 agrees with the ChPT analysis of Ref. [342].

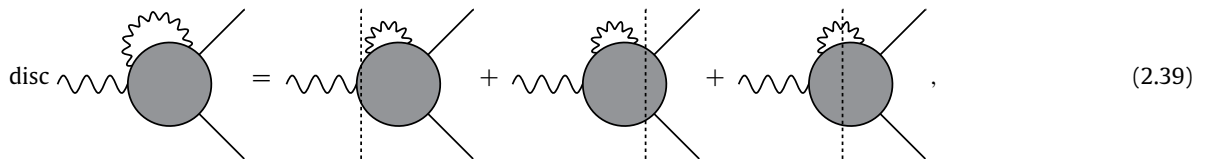
2.7.4. Radiative corrections

At the required level of precision, radiative corrections for the main $\pi^+\pi^-$ channel are important and need to be evaluated carefully. In particular, since the 2π final state is produced in an e^+e^- collision, one needs to remove all effects produced by photons emitted by the electron–positron pair before the collision, i.e., all ISR effects, see Ref. [145] and Section 2.4 for the required MC framework. In this section, we address the application of dispersive techniques for the calculation of radiative corrections, starting from FSR and interference effects. For both contributions, it is necessary to perform calculations in QCD+QED and to rely on a perturbative expansion only for the latter. At leading order in α , one needs hadronic matrix elements involving one, two, or three EM currents in the nonperturbative regime, making model-independent calculations particularly challenging.

In principle, all experiments measuring $e^+e^- \rightarrow \pi^+\pi^-$ aim to determine the inclusive $2\pi(\gamma)$ final state, so that a calculation of FSR effects would not be strictly necessary if a measurement with complete angular coverage were possible. In practice, corrections are so far evaluated using sQED multiplied by the pion VFF (F $\times$ sQED in the classification of Ref. [145]), leading to a universal correction factor $\eta_{2\pi}(s)$ that captures the leading, IR-enhanced effects [343–346], see also Section 2.6.2. Removing this effect is not only important for the application of dispersive techniques to the QCD matrix elements, see Section 2.6.1, but also allows one to estimate one class of isospin corrections for τ decays, see Section 2.3.4. A first step towards a dispersive calculation of FSR corrections was performed in Ref. [249], supporting the assumption that the IR-enhanced contributions are numerically dominant. Here, we discuss the recent approach from Refs. [238,347,348]. Since only the two-pion final state is considered, the quantity of interest becomes the VFF of the pion in QCD+QED, for which an expansion in α is employed:

$$F_{\pi, \text{QED}}^V(s) = F_{\pi}^V(s) + F_{\pi}^{V, \alpha}(s) + F_{\pi}^{V, \alpha^2}(s) + \dots \quad (2.38)$$

The first term is the VFF in pure QCD, while the goal of the dispersive analysis is a complete calculation of the second term, the correction at $\mathcal{O}(\alpha)$, by reconstructing this matrix element from its unitarity cuts. For $F_{\pi}^{V, \alpha}(s)$ there are three possible cuts that determine its discontinuity:



where the vertical dashed lines represent the unitarity cuts, the gray blob the hadronic matrix element, and when cutting through it, in principle, all possible hadronic intermediate states have to be considered. However, since we are interested only in the energy region around 1 GeV and below – for which inelastic contributions to $F_\pi^V(s)$ are known to be small – the analysis can be restricted to two-pion intermediate states in a first step. This allows one to translate this schematic representation into a sum of products of well-defined sub-amplitudes by replacing the cut hadronic blob with a two-pion state. These contributions are integrated over a two- or three-body phase space, so that the discontinuity becomes

$$\frac{\text{disc } F_\pi^{V,\alpha}(s)}{2i} = \frac{(2\pi)^4}{2} \left[\int d\Phi_2 F_\pi^V(s) T_{\pi\pi}^{\alpha*}(s, t) + \int d\Phi_2 F_\pi^{V,\alpha}(s) T_{\pi\pi}^*(s, t) + \int d\Phi_3 F_\pi^{V,\gamma}(s, t) T_{\pi\pi}^{\gamma*}(s, \{t_i\}) \right], \quad (2.40)$$

where each term corresponds to a cut in Eq. (2.40). The first term is the product of the purely hadronic VFF with the $\mathcal{O}(\alpha)$ correction to the $\pi\pi$ scattering amplitude $T_{\pi\pi}^\alpha(s, t)$. The second term is the product of $F_\pi^{V,\alpha}(s)$ with the purely hadronic $\pi\pi$ scattering amplitude. In the last term, $F_\pi^{V,\gamma}(s, t)$ and $T_{\pi\pi}^\gamma(s, \{t_i\})$ are the transition amplitudes for the processes $\gamma^* \rightarrow \pi^+\pi^-\gamma$ and $\pi^+\pi^- \rightarrow \pi^+\pi^-\gamma$, respectively, where the latter involves five external particles, introducing a dependence on five Mandelstam variables (s and t_i , $i \in \{1, 2, 3, 4\}$). The three-body phase-space integral of the last term is due to the presence of an additional photon in the intermediate state, contrary to the two first terms.

If one is able to determine the discontinuity of $F_\pi^{V,\alpha}(s)$ from this expression, a dispersive integral will allow one to reconstruct the whole function. Among all the sub-amplitudes appearing on the RHS of Eq. (2.40), only the pion VFF in pure QCD can be considered to be known, while all other amplitudes need to be determined. A striking feature of this equation is that the initial amplitude $F_\pi^{V,\alpha}(s)$ also appears as a sub-amplitude in the two-body phase-space integral of the second term. Therefore, determining $F_\pi^{V,\alpha}(s)$ not only requires additional input, such as $T_{\pi\pi}^\alpha(s, t)$, but also involves solving an implicit integral equation once all sub-amplitudes are known. The building blocks $T_{\pi\pi}^\alpha(s, t)$, $F_\pi^{V,\gamma}(s, t)$, and $T_{\pi\pi}^\gamma(s, \{t_i\})$ can also be determined using a dispersive approach, following similar lines to those discussed here. A preliminary account of this determination, along with a numerical estimate of these effects, is provided in Ref. [238]. Based on these findings, no significant shifts with respect to the F $\times$ sQED treatment are expected. Ongoing studies of these effects are addressing, in particular, the following important aspects:

1. Self-energy photonic corrections to the pions are responsible for the mass difference between the charged and the neutral pions. These effects can be analyzed by solving Roy equations in the presence of a nonzero pion mass difference, see Ref. [347].
2. An important step in the determination of $F_\pi^{V,\alpha}(s)$ within a dispersive approach is represented by the matching to the chiral representation of the same quantity (which is available in the literature [349,350]). This matching is a necessary step also for the sub-amplitudes occurring in the discontinuity of $F_\pi^{V,\alpha}(s)$. While a global matching was performed in Ref. [238], an improved treatment should consider the matching for the individual sub-amplitudes, which also rigorously takes into account the chiral counting, see Ref. [348].

Finally, an important application of the study of FSR effects concerns the calculation of differences between the VFF in the $\pi^+\pi^-$ and $\pi^-\pi^0$ channels, presenting an avenue for a model-independent determination of this class of IB corrections in the analysis of τ decays, see Section 2.3.6.

Beyond FSR corrections, dispersive techniques have been applied to the charge asymmetry in $e^+e^- \rightarrow \pi^+\pi^-$. Based on a GVMD model, it was observed in Ref. [141] that the F $\times$ sQED description is inadequate for this quantity, see Section 2.2.1. In a dispersive approach, these findings can be put onto more solid grounds, by avoiding the systematic uncertainties due to unphysical imaginary parts below the 2π threshold. Keeping the dominant 2π cuts (the F $\times$ sQED approximation in the classification of Ref. [145]), the results shown in Fig. 23 were obtained in Ref. [142], demonstrating that in this case the difference between GVMD and the dispersive calculation is astonishingly small. Both calculations have already been implemented in MC generators [143]. In particular, the full dispersive calculation involves subtle effects related to the treatment of an end-point divergence in the imaginary part, and collaboration within the *RadioMonteCarLow* 2 project, see Sections 2.4 and 2.2.1, was critical to establish the correct finite terms.

For energy-scan experiments, the charge asymmetry is C odd and therefore does not directly contribute to the 2π cross section that enters the HVP integral. Accordingly, in this case the above studies mainly serve as a consistency check for the CMD-3 data [95,96]. However, as first pointed out in Ref. [144], for the ISR process a similar effect is possible for the C-even cross section: this implies that resonance-enhanced corrections that are not captured by the current F $\times$ sQED treatment [164] could affect such measurements, see Ref. [145] for a more detailed discussion of these classes of diagrams. The calculation of these effects for ISR processes becomes significantly more challenging due to the appearance of $\pi\pi \rightarrow \gamma\gamma^*\gamma^*$ matrix elements (in addition to the well-studied pion Compton scattering $\pi\pi \rightarrow \gamma^*\gamma^*$ [249,351–355]), but work in this direction is ongoing and could be critical to help resolve the current situation in the data-driven evaluation of the HVP contribution.

2.8. Data-driven results for comparisons to lattice QCD

With the advent of precise calculations of $a_\mu^{\text{HVP, LO}}$ from lattice QCD, the quantitative comparison of data-driven results with lattice-QCD-based counterparts is, obviously, necessary. A detailed comparison should go beyond simply comparing

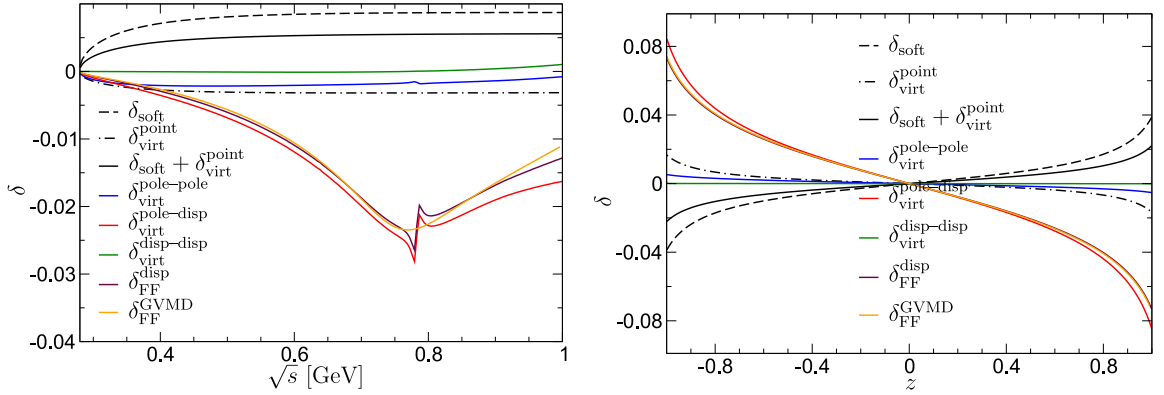


Fig. 23. Correction factors δ for the charge asymmetry in $e^+e^- \rightarrow \pi^+\pi^-$ as a function of $\sqrt{s}$ for fixed $z = \cos(1)$ (left) and as a function of z for fixed $\sqrt{s} = 0.75$ GeV (right), evaluated using a dispersive representation of the pion VFF [142]. The different lines refer to the point-like result (black dashed: real, dot-dashed: virtual, solid: sum of real and virtual), the pole–pole (blue), pole–dispersive (red), and dispersive–dispersive (green) contributions in the dispersive approach, and the GVMD model of Ref. [141] for comparison (orange). In all cases, the logarithmic term of the IR divergence is subtracted in the conventions of Ref. [141].

Source: Figure taken from Ref. [142].

results for $a_\mu^{\text{HVP, LO}}$ itself since, in both the dispersive and lattice approaches, the full result is the sum of components that are physical quantities in their own right. However, the very nature of the lattice approach, which is quark based and inclusive, makes a comparison of intermediate physical quantities obtained from the lattice with data-driven ones built up by summing experimental data for exclusive-mode hadronic cross sections not straightforward. The lattice computation of $a_\mu^{\text{HVP, LO}}$, as reviewed in Section 3, is commonly split into a number of separately computed components. The dominant contributions arise from the isospin-limit light- and strange-quark connected and disconnected diagrams, with the light-quark connected accounting for about 90% of $a_\mu^{\text{HVP, LO}}$ and the sum of all strange- and light-quark disconnected diagrams for about 6%. Additional, smaller, contributions from charm and bottom quarks have also been computed. IB is accounted for by including corrections from EM (QED) and strong (SIB) origin, to first order in an expansion in α and the up-down quark-mass difference $\delta = m_u - m_d$. In order to scrutinize potential conflicts between the data-driven and lattice approaches to $a_\mu^{\text{HVP, LO}}$, it is highly desirable to obtain reliable data-driven estimates for different components of the lattice-QCD result.

In a series of recent papers [356–360], it has been shown that it is possible to reliably determine, in an almost purely data-driven manner, the light-quark connected (ud) and the sum of the full strange- and light-quark disconnected ($s + \text{disc}$) contributions to $a_\mu^{\text{HVP, LO}}$. Subtracting from the latter an averaged lattice result for the strange-quark connected contribution also gives access to an estimate of the total light- and strange-quark disconnected components. The strategy starts from the isospin ($\text{SU}(3)_F$) decomposition of the EM current into its $I = 1$ (flavor 3) and $I = 0$ (flavor 8) parts, and, from this, the decomposition of the EM VP into pure $I = 1$ (flavor 33), pure $I = 0$ (flavor 88), and mixed-isospin (MI) (flavor 38) parts. It is well known that, in the isospin limit, the MI part vanishes, and the $I = 1$ part is purely light-quark connected (see, for example, Refs. [361,362]). One has then

$$a_\mu^{\text{HVP, LO}}(ud) = \frac{10}{9} a_\mu^{I=1}, \quad a_\mu^{\text{HVP, LO}}(s + \text{disc}) = a_\mu^{I=0} - \frac{1}{9} a_\mu^{I=1}. \quad (2.41)$$

The main task to be accomplished is then the identification, with sufficient precision, of the $I = 1$ and $I = 0$ components of the EM spectral function ρ_{EM} . Once a procedure is established to accomplish this goal, it is straightforward to obtain the ud and $s + \text{disc}$ parts of the total HVP contribution, or of any windowed quantity, including, in particular, the three RBC/UKQCD windows [14], introduced in Section 3.4. The decomposition into $I = 1/0$ parts can be achieved on a mode-by-mode basis in the exclusive-mode region of the $R_{\text{had}}(s)$ data, with this region giving the largest contribution to $a_\mu^{\text{HVP, LO}}$. The smaller contributions from the inclusive region are obtained from pQCD supplemented with a conservative error estimate for potential residual quark–hadron duality violations. Finally, EM IB contributions have to be subtracted from the data-based pure $I = 1$ and $I = 0$ components to obtain the isospin-limit results to be compared with lattice-QCD determinations (as shown in Ref. [247] they are not large enough to explain discrepancies between data-driven and lattice results). Below we outline how this is done in practice; see the original publications [356–360] for further details. Data-driven estimates for the RBC/UKQCD window quantities were considered before in Ref. [363], but not the breakdown in ud and $s + \text{disc}$ parts.

In the exclusive-mode region of the $R_{\text{had}}(s)$ data, there are two classes of contributions. The first, which gives the dominant contribution to the ud results, arises from modes with well-defined G -parity: modes with positive/negative G -parity have isospin $I = 1/0$. We refer to these as “unambiguous modes”. The large contributions from $n\pi$ modes, for example, fall into this category. The numerically most important $\pi^+\pi^-$ mode, in particular, is $I = 1$, and therefore linked directly with the ud contribution. The second class consists of contributions from modes without well-defined isospin,

referred to as “ambiguous modes”. These are of two types: those from modes for which external information can be used to separate the isospin components and those for which no such information is available.

For ambiguous modes for which no external information is available, which turn out to give small contributions, we rely on a maximally conservative separation, based on the observation that, in the isospin limit, due to the positivity of the $I = 0, 1$ spectral functions, the $I = 0$ and $I = 1$ parts of a given mode contribution must lie between zero and the full experimental $I = 1 + 0$ total. The $I = 1$ and $I = 0$ components thus lie in the range $(50 \pm 50)\%$ of that total, with $I = 1$ and 0 errors 100% anticorrelated. Fortunately, for the ambiguous modes that give large contributions, leading to unacceptably large errors with this maximally conservative split, external information can be used to achieve a sufficiently precise isospin separation. This is the case for the $K\bar{K}$, $K\bar{K}\pi$ and $\pi^0/\eta + \gamma$ modes. In the case of the $K\bar{K}$ mode, BABAR’s results for the differential decay distribution of $\tau \rightarrow K^- K^0 \nu_\tau$ [364] provide a measurement of the charged-current $I = 1$ vector spectral function. Using the conservation of the vector current, one then obtains an estimate of the $I = 1$ $K\bar{K}$ contribution to ρ_{EM} , and hence an estimate of the $I = 1$ contribution from this channel to integrated quantities up to near the end-point of BABAR’s spectrum ($s = 2.7556 \text{ GeV}^2$). The decomposition of the remaining small contribution from higher s is obtained via the maximally conservative separation treatment of the integrated $K\bar{K}$ -mode $R_{\text{had}}(s)$ contribution from this region. This approach is crucial to achieving good control of the $K\bar{K}$ contributions, dramatically reducing the final uncertainty for the $I = 1/0$ components. A similar treatment of the $K\bar{K}\pi$ contribution is possible thanks to BABAR’s Dalitz plot separation of $I = 1/0$ parts of the $e^+e^- \rightarrow K\bar{K}\pi$ cross sections [151]. Finally, for the radiative channels $\pi/\eta + \gamma$ a reliable decomposition into pure $I = 1/0$ and MI components is possible due to the strong dominance of the observed cross sections by intermediate vector-meson contributions. A detailed description of this separation is given in App. B of Ref. [359].

The remaining ingredients for the data-driven determination of the ud and $s + \text{disc}$ contributions to $a_\mu^{\text{HVP, LO}}$ are: (i) the treatment of the inclusive $R_{\text{had}}(s)$ data region, and (ii) the corrections for IB (both EM and SIB effects). In the inclusive $R_{\text{had}}(s)$ region the type of separation that was outlined above cannot be performed and one has to turn to pQCD, in which the $I = 0/1$ split is straightforward. The value of s at which the exclusive-mode region ends and the inclusive region begins depends on the data compilation used [31,262,263,265], but is always close to $s = (2 \text{ GeV})^2$. The pQCD spectral function for massless quarks is exactly known to α_s^4 [365] and Refs. [356–360] use these results supplemented with an estimate for the α_s^5 coefficient. Given, however, that recent experimental results from BESIII [179] show some tension with pQCD below the charm threshold, Refs. [356–360] include a conservative error estimate based on a model for potential residual duality violations, relying on previous work [366,367]. The pQCD contribution turns out to be small in all quantities of interest, except, as expected, the SD RBC/UKQCD window.

To correct for EM and SIB effects, it is sufficient to define the isospin-symmetric world as that in which all pions have mass M_{π^0} . SIB effects, which to order $m_d - m_u$ are MI only, are then split into those associated with the 2π and 3π exclusive modes, where SIB effects are expected to be dominated by $\rho - \omega$ mixing, and those associated with the remaining exclusive modes and perturbation theory above the inclusive threshold. For the 2π and 3π modes, rather precise SIB results are available from Refs. [186,237,247], see also Section 2.7.3 and Table 4, based on dispersive representations of the 2π and 3π channels, both for $a_\mu^{\text{HVP, LO}}$ and the RBC/UKQCD window quantities. For the remaining modes, Refs. [356–360] allow for an SIB-induced uncertainty equal to 1% of the total contribution. For the SD window, the only quantity for which the perturbative contribution is not very small, an additional 1% SIB uncertainty is assigned to that contribution.

EM corrections are the only building blocks for which no complete data-driven estimates are available (see Ref. [357] for a discussion). Therefore, inclusive lattice results for the EM contributions to $a_\mu^{\text{HVP, LO}}$ and the intermediate window from Ref. [16] (with updates from Ref. [24]) were used to estimate these corrections, with conservative estimates for the SD and LD windows for which such lattice results are not available. For all quantities considered, these corrections are very small (the largest being the 0.4% for the LD window [360]), and therefore, although relying on this small lattice-QCD input, the results of Refs. [356–360] are almost entirely data driven.

2.8.1. Light-quark connected contributions

The isospin-limit data-driven ud contribution $a_\mu^{\text{HVP, LO}}(ud)$ to $a_\mu^{\text{HVP, LO}}$ was first discussed in Ref. [357] and slightly updated in Refs. [359,360]. In this case, the required exclusive-mode $a_\mu^{\text{HVP, LO}}$ contributions can be found in tables in the publications of the two main $R_{\text{had}}(s)$ data compilations by DHMZ [262,263] and KNT [31,265]. Supplementing this information with the external results required for the separation of ambiguous-mode contributions, the pQCD contribution in the inclusive region, and estimates of IB corrections that partially rely on BMW results [16,24], as explained above, one obtains the following pre-CMD-3 results for $a_\mu^{\text{HVP, LO}}(ud)$, based on either the DHMZ or the KNT data compilation [357,360]

$$\begin{aligned} a_\mu^{\text{HVP, LO}}(ud) &= 638.9(4.1) \times 10^{-10} \quad (\text{DHMZ based}), \\ a_\mu^{\text{HVP, LO}}(ud) &= 635.8(2.6) \times 10^{-10} \quad (\text{KNT based}). \end{aligned} \quad (2.42)$$

These results are smaller than, and show an important tension with, the most precise pure lattice determinations of the same quantity by the BMW [16], the RBC/UKQCD [26], the Mainz [27], and the Fermilab/HPQCD/MILC [30] collaborations.

Access to the individual exclusive-mode spectra from the KNT data compilation allowed the authors of Refs. [356–360] to obtain analogous results for the three RBC/UKQCD and other windows proposed in the literature [19,368]. Results for

the three RBC/UKQCD windows can be found in Ref. [360]. The pre-CMD-3, purely KNT-based versions of these results are

$$\begin{aligned} a_\mu^{\text{SD}}(ud) &= 46.96(48) \times 10^{-10}, \\ a_\mu^{\text{W}}(ud) &= 199.0(1.1) \times 10^{-10}, \\ a_\mu^{\text{LD}}(ud) &= 389.9(1.7) \times 10^{-10}. \end{aligned} \quad (2.43)$$

An interesting exercise, performed to explore the potential impact of the new CMD-3 $\pi\pi$ data on the data-driven determinations of the various isospin-limit quantities, was the replacement of the combined $\pi^+\pi^-$ data of the KNT compilation with CMD-3 data alone, in the region between 0.33 and 1.2 GeV covered by the CMD-3 data. Of course, as this exercise ignores all other $\pi^+\pi^-$ data in this region, these “CMD-3” results are purely exploratory. The results of this exploration are shown in Fig. 38, together with recent lattice results for the same quantities.

In Ref. [359] also a KNT-based data-driven result for the 1.5 to 1.9 fm window “W2” of Ref. [19] was obtained. The value, $93.75(36) \times 10^{-10}$, is again significantly lower than recent lattice values obtained in Refs. [19,24,369]. This discrepancy also vanishes if one replaces the KNT $\pi\pi$ combination with CMD-3 results in the CMD-3 data region.

2.8.2. Strange + light-quark disconnected contributions

The isospin-limit data-driven $s + \text{disc}$ contribution $a_\mu^{\text{HVP, LO}}(s + \text{disc})$ to $a_\mu^{\text{HVP, LO}}$ was first discussed in Ref. [356] and again slightly updated in Refs. [359,360]. The methodology employed is much the same as for the ud contribution, with the exception of the very small EM corrections which can only be bounded based on the results of Ref. [16]. The DHMZ and KNT data-compilation based results are [359,360]

$$\begin{aligned} a_\mu^{\text{HVP, LO}}(s + \text{disc}) &= 39.8(2.4) \times 10^{-10} \quad (\text{DHMZ based}), \\ a_\mu^{\text{HVP, LO}}(s + \text{disc}) &= 40.7(1.7) \times 10^{-10} \quad (\text{KNT based}). \end{aligned} \quad (2.44)$$

For the three RBC/UKQCD window quantities only results based on the KNT compilation have been obtained [359,360]; they read [360]

$$\begin{aligned} a_\mu^{\text{SD}}(s + \text{disc}) &= 9.21(36) \times 10^{-10}, \\ a_\mu^{\text{W}}(s + \text{disc}) &= 26.98(84) \times 10^{-10}, \\ a_\mu^{\text{LD}}(s + \text{disc}) &= 4.53(73) \times 10^{-10}. \end{aligned} \quad (2.45)$$

Results for the SD and intermediate windows, as well as the $s + \text{disc}$ part of the total HVP, are shown in Fig. 39, together with recent lattice results for the same quantities. The only lattice determination of the $s + \text{disc}$ contribution to the LD window to date, by the Mainz collaboration [27], is $a_\mu^{\text{HVP, LO}}(s + \text{disc}) = 1.3(2.4) \times 10^{-10}$ and is compatible with Eq. (2.45) within 1.3σ .

2.9. Higher-order iterations of HVP

At the required level of precision, also higher-order HVP iterations, including both NLO [370] and NNLO [33] effects, need to be considered. While traditionally, in the timelike method, the contributions are expressed in terms of the R -ratio via the corresponding kernel functions, also spacelike representations are possible, in which case the integrands are related to the HVP function $\Pi(Q^2)$ and the Adler function $D(Q^2)$. Specifically, the relation between the spacelike kernel $K_\Pi(Q^2)$ in terms of the timelike kernel $K_R(s)$ has independently been obtained in Refs. [371,372] and the corresponding explicit expressions calculated at NLO [371,372] and NNLO [372]. Additionally, the complete set of six relations that mutually express all three kernels $K_\Pi(Q^2)$, $K_D(Q^2)$, and $K_R(s)$ in terms of each other has been obtained in Ref. [371], which also provides the results for $K_D(Q^2)$ at NLO. These results are important for calculations in lattice QCD, see Section 3, and the HVP measurement in MUonE, see Section 2.10, as in both cases spacelike representations are needed. In particular, the time-momentum representation of the NLO spacelike kernel was provided in Ref. [373].

Based on these developments, improved higher-order HVP calculations will become available in lattice QCD over the next years, while for existing calculations [374] the precision is not yet competitive with the data-driven approach despite the tensions among the e^+e^- measurements. To define a conservative estimate, we consider two evaluations obtained in the context of Ref. [32]

$$a_\mu^{\text{HVP, NLO}} = \begin{cases} -9.83(4) \times 10^{-10} & \text{KNT19,} \\ -10.08(6) \times 10^{-10} & \text{KNT19/CMD-3,} \end{cases} \quad (2.46)$$

where the second variant corresponds to the compilation from Ref. [31] with the low-energy $\pi^+\pi^-$ contribution replaced by the CMD-3 measurement [95,96] where applicable, and assign the mean and spread as our value¹³

$$a_\mu^{\text{HVP, NLO}} = -9.96(13) \times 10^{-10}. \quad (2.47)$$

¹³ The same procedure for $a_\mu^{\text{HVP, LO}}$ would produce an uncertainty of $\pm 10.9 \times 10^{-10}$, which roughly reflects the spread observed in Fig. 26, so that the uncertainty assignment in Eq. (2.47) should reasonably cover all presently observed tensions in the e^+e^- data.

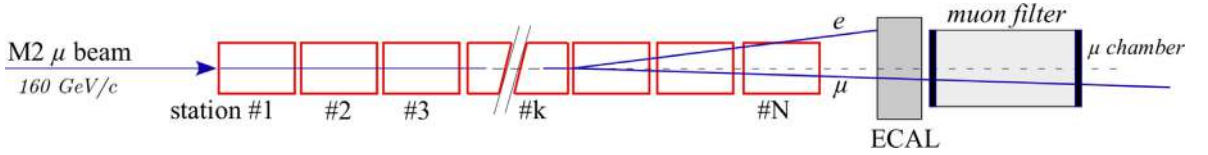


Fig. 24. Schematic view of the MUonE experimental apparatus (not to scale).
Source: Figure taken from Ref. [378].

For the NNLO contribution, we continue to use [33]

$$a_{\mu}^{\text{HVP, NNLO}} = 1.24(1) \times 10^{-10}, \quad (2.48)$$

even though the uncertainty is likely somewhat underestimated. The impact of mixed leptonic and hadronic corrections at NNLO was estimated as $\lesssim 0.1 \times 10^{-10}$ in Ref. [375], while NNLO hadronic corrections due to virtual photons emitted and reabsorbed by the HVP insertion were estimated to be of size $\mathcal{O}(10^{-12})$ [372].

2.10. An alternative approach: MUonE

The MUonE experiment aims to provide an independent calculation of $a_{\mu}^{\text{HVP, LO}}$ using an innovative method based on the measurement of the hadronic contribution to the running of the EM coupling constant, $\Delta\alpha_{\text{had}}(t)$, for negative squared four-momentum transfers t [376]. The hadronic running will be extracted from the shape of the μ - e elastic scattering differential cross section [377], which will be measured by colliding the 160 GeV muon beam available at CERN's M2 beam line with a thin low- Z target. The detector will be segmented into 40 identical stations, each consisting of a target followed by a tracking system equipped with silicon strip sensors. The apparatus is completed by an EM calorimeter and a muon ID system placed downstream of the tracking stations, to improve event selection, and a spectrometer placed upstream to measure the beam momentum. A sketch of the experimental apparatus is shown in Fig. 24.

The goal of MUonE is to determine $a_{\mu}^{\text{HVP, LO}}$ with a $\simeq 0.3\%$ statistical accuracy and comparable systematics. This poses several challenges not only on the experimental side [378,379], but also requires a huge effort to determine the higher-order radiative corrections to the μ - e scattering. The complete set of NLO electroweak, NNLO QED, and hadronic corrections have been calculated in Refs. [380–391]. The first steps towards an N3LO computation of the QED corrections were taken recently in Refs. [392–398], and atomic binding corrections to μ - e scattering were recently studied in Refs. [399,400]. The dominant NNLO QED and hadronic corrections are implemented in two different independent MC codes, MESMER [385] and McMULE [387]. While the former includes also the main background processes [389,390,401], and is already integrated in the full detector simulations, the latter contains the complete NNLO photonic calculation [391]. In addition to this, analytic expressions have been provided in Refs. [371,372] to compute also $a_{\mu}^{\text{HVP, NLO}}$ and $a_{\mu}^{\text{HVP, NNLO}}$ in the spacelike region, see Section 2.9, thus extending the capability of MUonE to determine the HVP contribution to a_{μ} to higher orders. Finally, a different method has been proposed in Ref. [402] to determine $a_{\mu}^{\text{HVP, LO}}$ from MUonE data in the spacelike region through the derivatives of $\Delta\alpha_{\text{had}}(t)$ at zero momentum transfer. Possible fit functions for $\Delta\alpha_{\text{had}}(t)$ are also discussed in Refs. [403,404].

In the last few years, MUonE carried out a series of short beam tests at the M2 beam line with a first tracking station instrumented with prototype 2S modules, silicon strip sensors foreseen for the CMS Phase-2 Upgrade [405]. This allowed one to commission the system mechanics and the data acquisition chain in view of a longer test performed in Summer 2023, whose goal was to demonstrate the ability to identify and reconstruct μ - e elastic events. The apparatus consisted of two tracking stations equipped with prototype 2S modules, followed by a prototype calorimeter. Graphite targets 2 or 3 cm thick were installed between the two tracking stations, in order to evaluate multiple scattering effects and study background processes in different configurations. Tracker data are currently being analyzed to optimize reconstruction algorithms, software alignment procedures and event selection. A sample of candidate elastic events is shown in Fig. 25(left). The outgoing tracks are labeled according to the magnitude of their angles, denoted as θ_{max} and θ_{min} , and a large fraction of background events is clearly visible at low θ_{min} . Fig. 25(right) shows instead the effect of a preliminary event selection, including cuts on the total number of hits in the downstream station, on the acoplanarity of the event and on the vertex position, which is capable of rejecting most of the background. Residual background can be removed by cutting events with $\theta_{\text{min}} \leq 0.2$ mrad. Further preliminary results on Test Run data analysis can be found in Refs. [406–408].

The MUonE Collaboration has recently submitted a proposal to the CERN SPS Committee [406] to run four weeks in 2025 with a small-scale version of the final apparatus, comprising three tracking stations, a calorimeter, a muon filter, and the beam spectrometer, obtaining positive recommendations. The Phase-1 MUonE Run will occur in the first half of 2025, allowing one to study signal and background under realistic conditions with a possible sensitivity to $\Delta\alpha_{\text{had}}(t)$. The 2025 data will also serve as a basis for a full-scale experiment proposal to be prepared during the CERN Long Shutdown 3 (2027–2029), aimed at taking data after 2030.

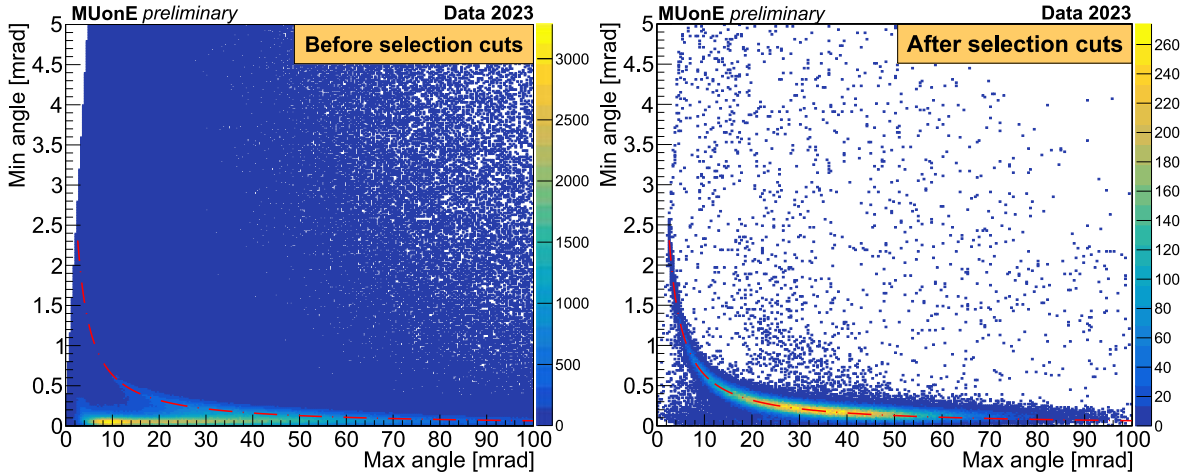


Fig. 25. $(\theta_{\max}, \theta_{\min})$ distribution of elastic scattering candidates before (left) and after (right) the preliminary event selection described in the text. The red dashed line represents the expected elasticity curve for a 160 GeV muon beam.

Source: Figure adapted from Ref. [406].

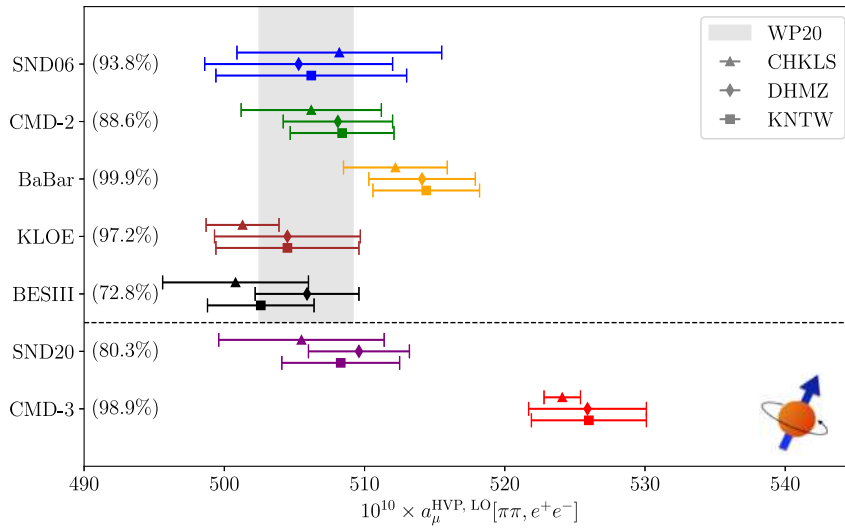


Fig. 26. Dispersive theoretical predictions for $a_{\mu}^{\text{HVP, LO}}[\pi\pi]$, based on various measurements of $e^+e^- \rightarrow \pi^+\pi^-$, fit/interpolated and complemented for the uncovered mass ranges (percentages of the integral covered by each measurement are shown), for the three approaches “CHKLS”, “DHMZ”, and “KNTW” as detailed in the main text. The gray band indicates the result from WP20, including the error inflation due to the BaBar–KLOE tension. The experiments above the dashed line entered the result for WP20, whilst those below are new measurements since then. The numerical values shown are reproduced in Table 5.

2.11. HVP dispersive evaluation: comparison of experimental inputs and methods

Fig. 26 presents a summary of the dispersive theoretical predictions for $a_{\mu}^{\text{HVP, LO}}[\pi\pi]$, based on various measurements of $e^+e^- \rightarrow \pi^+\pi^-$, treated using either of the “CHKLS” (Section 2.6.1), “DHMZ” (Section 2.5.1), and “KNTW” (Section 2.5.2) approaches. The various measurements cover the range from threshold up to 1.8 GeV according to the percentages indicated in the figure.

Contributions from other channels and from the high mass region above 1.8 GeV are based on WP20. It is, however, worth mentioning that significant progress has been made in these other channels since WP20 (see Section 2.2), e.g., next by uncertainty amplitude, the contribution from the 3π process was improved by more than a factor of two. However, tensions between the Belle II data and previous measurements are now visible. Other discrepancies are observed in a few other channels as well – see the corresponding discussion in Section 2.3.6 in the context of Eq. (2.23) – although their overall impact remains minor compared to the tension seen in the 2π channel. Continued progress in the overall $R_{\text{had}}(s)$ measurement program is essential for a future complete data-driven evaluation of $a_{\mu}^{\text{HVP, LO}}$.

Table 5

$a_{\mu}^{\text{HVP, LO}}[\pi\pi, e^+e^-]$ up to 1.8 GeV (in units of 10^{-10}) for the different $e^+e^- \rightarrow \pi^+\pi^-$ experiments, using the “CHKLS,” “DHMZ,” and “KNTW” approaches as detailed in the main text and illustrated in Fig. 26, where the second column indicates the percentage of the integral covered by each experiment. Depending on the method, a given experiment is either complemented by an average of others outside the covered mass range (DHMZ and KNTW) or extrapolated from the range in which data are taken (CHKLS), and also the treatment of the region above 1 GeV differs, as explained in the main text. We emphasize that these numbers are meant to illustrate the spread among the different experiments and analysis methods, and since the discrepancies are currently not understood, we do not attempt to derive a global $a_{\mu}^{\text{HVP, LO}}$ number based on e^+e^- data. For comparison, the table also includes our estimate for $a_{\mu}^{\text{SM}}[\pi\pi, \tau]$ from Eq. (2.22). To obtain the corresponding ranges for a_{μ}^{SM} shown in Fig. 27, first the offset $187.3(2.2) \times 10^{-10}$ from WP20 is added to arrive at $a_{\mu}^{\text{HVP, LO}}$ (see main text). All other contributions are taken from WP25.

Experiment	Percentage covered			
e^+e^-		CHKLS	DHMZ	KNTW
SND06	93.8%	508.2(7.3)	505.3(6.7)	506.2(6.8)
CMD-2	88.6%	506.2(5.0)	508.1(3.9)	508.4(3.7)
BaBar	99.9%	512.2(3.7)	514.1(3.8)	514.4(3.8)
KLOE	97.2%	501.3(2.6)	504.5(5.2)	504.5(5.1)
BESIII	72.8%	500.8(5.2)	505.9(3.7)	502.6(3.8)
SND20	80.3%	505.5(5.9)	509.6(3.6)	508.3(4.2)
CMD-3	98.9%	524.1(1.3)	525.9(4.2)	526.0(4.1)
τ		DHLHZ-23+LMR-24+WP25		
Belle+CLEO+ALEPH+OPAL	100%		517.2(5.8)	

The most important feature of Fig. 26 concerns the strong tensions observed among the available 2π measurements, regardless of the various methodologies employed for the data integration and extrapolation/completion to the full mass range. The differences among the various methodologies are nevertheless also visible for the resulting central values and uncertainties, the latter differing up to a factor of two in some cases. This comes from different treatment of correlations for fits/weight derivation, strategies to complete the data sets, and usage of additional constraints. The increasing precision of data and the larger tensions among experiments highlight the importance of a conservative treatment of experimental systematic uncertainties. The conventionally used practice to treat the provided experimental systematic errors, whose knowledge is inherently limited, is generally incomplete compared to what was assumed by the measured e^+e^- data as discussed in Secs. 2.3.1–2.3.6 of WP20 and Sections 2.2.1 and 2.6.1 of WP25. In some cases, this leads to an underestimation of the evaluated integral errors due to assumptions made for the correlations of systematic uncertainties or the energy-dependent amplitude of uncertainties.

The “CHKLS” [146,226,237,270] points are obtained using a dispersive representation to implement constraints from analyticity, unitarity, and crossing symmetry, so that the $\pi^+\pi^-$ cross sections can be analyzed in a global fit (with input for covariance matrices as provided by experiment), and thus the entire low-energy contribution can be evaluated for each experiment individually. In the figure, the variant of the dispersive approach without imposing the absence of zeros in the form factor is shown, see Section 2.6.1 for details, evaluated up to $\sqrt{s} = 1$ GeV, with the remainder taken from WP20.

“DHMZ” perform a direct integration in the full range where a given measurement is available, complemented with data combinations for the rest, while “KNTW” limit their use of individual $\pi^+\pi^-$ data to the upper range of 1 GeV, using data combinations for the remaining values. It is important to note that the latter combination-based integrals are by construction more precise than the integrals derived from one measurement alone (i.e., a measurement covering the low-and/or high-mass range can be misleadingly “penalized” when comparing the uncertainties). A local χ^2 -based uncertainty rescaling is performed when necessary, although there are relatively little tensions in the relevant mass regions.

The “KNTW” numbers are solely based on the KNT19 combination [31,265] and are, in particular, deliberately not including the CMD-3 data (see Section 2.5.2 for further details). The combination is fully model-independent, incorporates experimental correlation information in data fits for the full available energy range whilst simultaneously avoiding procedural systematic biases (such as the d’Agostini bias in correlated χ^2 fits [409,410]), and determines additional systematic uncertainties.

The “DHMZ” result employs a spline-based relatively-local averaging (minimization of χ^2 with correlations), with weight derivation accounting for different point-spacing/binning (see Section 2.5.1 for further details). All existing data are used on the full mass ranges where combinations are employed. The combination procedure is also validated through a closure test.

The “DHMZ” and “KNTW” direct integration methods yield similar values in most cases (differences not exceeding much more than 1 unit), except for the BESIII integral, where some difference (of about 3 units) arises due to the larger range on which the combined data are employed and the differences of methodologies discussed above and in WP20. The “CHKLS” points show lower uncertainties in a few cases, since the additional theoretical constraints in a global fit reduce the uncertainties in the resulting integral. This reduction is most effective in cases in which the data are relatively

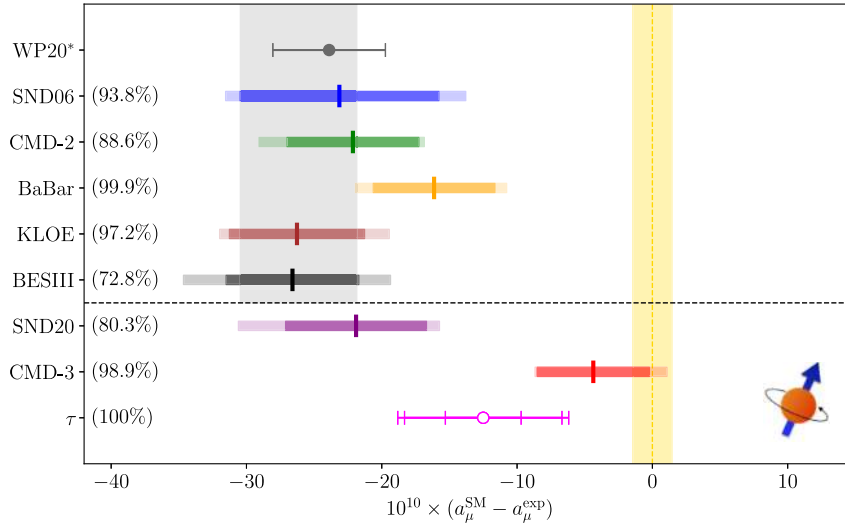


Fig. 27. Summary of current data-driven evaluations of HVP, propagated to a_μ^{SM} (the yellow band indicates a_μ^{exp} , the gray band the WP20 SM prediction based on the e^+e^- data sets above the dashed line and the remainder from WP20, in particular, the WP20 HLBL value; the data point labeled WP20* indicates the shift upon using WP25 input for the other contributions besides LO HVP). The τ point corresponds to WP25 in Fig. 13, with the third, outmost error including the additional uncertainties beyond the 2π channel (the remainder of HVP is taken from WP20, the other contributions from WP25). The other points use input from the various $e^+e^- \rightarrow \pi^+\pi^-$ experiments according to Fig. 26 (again with HVP remainder from WP20 and the other contributions from WP25), where for each experiment the central values are obtained as simple average of the three combination methods, the inner ranges as simple average of the uncertainties obtained in each method, and the outer ranges reflect the maximal range covered by all methods (the percentages indicate how much of the 2π contribution to the HVP integral is covered by each measurement). We emphasize that these ranges are merely meant to illustrate the current spread, they cannot be interpreted as uncertainties with a proper statistical meaning. The numerical values follow from Tables 1 and 5. The experimental world average has been updated including the final results from the Fermilab experiment.

scarce in some energy region – the extreme case concerning the ability to extrapolate in a robust manner into regions in which no data were taken – and in which the systematic uncertainties, most prominently from the truncation of the conformal expansion, are small. In the latter case, as for CMD-3, the limited knowledge of exact correlations of the systematic uncertainties in experiment becomes more critical, suggesting to somewhat enlarge the fit uncertainties, see Sections 2.2.1 and 2.6.1.

2.12. Summary and outlook

The unsatisfactory situation regarding the knowledge of the cross section for the process $e^+e^- \rightarrow \pi^+\pi^-$, which is known to contribute more than 70% to the total HVP dispersion integral, presents a significant limitation to the data-driven evaluation of the HVP contribution and, consequently, to the SM prediction of the anomalous magnetic moment of the muon. This is illustrated in Fig. 27. Since WP20, the introduction of the new CMD-3 measurement, which has a systematic uncertainty of 0.7%, has changed the landscape of hadronic cross-section measurements. The CMD-3 result provides a significantly higher value for the HVP contribution compared to older evaluations. Previously, for WP20, an averaging procedure with inflated uncertainties was adopted to accommodate the two most precise measurements from KLOE and BABAR, which exhibited already some discrepancies. However, this approach is no longer appropriate given the current situation with the new CMD-3 data. Due to the significant spread of experimental results, with relative differences exceeding the claimed systematic uncertainties by substantial factors, it has been decided not to perform a new average for the two-pion channel and, consequently, the HVP contribution within WP25. Clarifying the current situation is of utmost importance, and the following developments from the recent past and the near future will be essential for achieving progress:

- The detailed understanding and control of radiative corrections in experimental analyses have become a focal point of research in recent years. The availability of high-precision event generators, along with a well-defined understanding of their uncertainties, is fundamentally important for the field of hadronic cross-section measurements, as well as for the QED normalization processes utilized by experiments. All ISR experiments (BABAR BESIII, KLOE) use the NLO PHOKHARA event generator, which simulates hadronic and muonic events with one high-energy ISR (or FSR) photon representing the LO configuration. The dependence on PHOKHARA, however, strongly depends on the event analysis. The BABAR collaboration has conducted a detailed study on the fraction of (N)LO photons and identified significant limitations of PHOKHARA in simulating these higher-order corrections [176]. They also found

that the BABAR measurement, due to its highly inclusive selection of (N)NNLO photons, remains unaffected. A subsequent publication by the DHMZ group [166] explored the potential impact of these shortcomings on existing ISR publications for the process $e^+e^- \rightarrow \pi^+\pi^-(\gamma)$. Fast simulation by the DHMZ group suggested that the KLOE and BESIII measurements may have significantly underestimated the uncertainties attributed to radiative corrections. This latter claim holds especially for scenario 1 of the two scenarios considered in Ref. [166]. Further analysis by BESIII indicated that the large effect predicted for the BESIII measurement did not account for specific corrections implemented in the BESIII analysis [411]. Moreover, both KLOE and BESIII demonstrated agreement at the 1% level for the mass spectra for various event generators, which differ in their simulation of higher-order corrections [165,411]. As a result of these studies, it was concluded that scenario 1 in Ref. [166] seems unlikely, and the shortcomings of PHOKHARA most likely do not explain the seen differences between different measurements.

- The *RadioMonteCarLow 2* initiative is committed to improving the theoretical predictions for hadron and lepton production at low-energy e^+e^- colliders by bringing the available MC generators to NNLO+ precision. In the first phase, a review of the existing state-of-the-art available generators has been concluded [145]. The comparison was performed by using approximated experimental selections as benchmarks. The study will continue by incorporating additional important selection variables. Moreover, Ref. [145] presents a detailed discussion of the different classes of higher-order radiative corrections for both direct-scan and ISR processes, identifying a critical class of virtual ISR corrections that, due to resonance enhancement, could contribute to the cross section at the relevant level (as first pointed out in Ref. [144]). A similar effect in the C-odd asymmetry in direct-scan experiments was observed by CMD-3, demonstrating that such structure-dependent radiative corrections can indeed far exceed estimates in a form-factor-times-sQED prescription [141,142]. Actual improvements to the codes are expected to take place during the next Phase II. To ultimately investigate the accuracy of event generators, high-quality codes with different approaches in simulating the radiative corrections are needed for cross-checks (both for energy scan and ISR experiments). The ongoing extension of the BABAYAGA@NLO, PHOKHARA, McMULE, and additional codes are important steps in that context.
- Fortunately, new precision measurements of the process $e^+e^- \rightarrow \pi^+\pi^-(\gamma)$ are being prepared with significant effort by several experimental collaborations. The CMD-3 and SND experiments will continue their campaigns to analyze the energy scans conducted at the VEPP-2000 collider, while the BABAR, BESIII, and KLOE experiments will produce updated results using the ISR technique. For this purpose, considerably larger data sets will be utilized, and the event analyses will be designed to address possible limitations arising from event generators. In the case of BABAR a new analysis strategy based on angular fits will be employed. The Belle-II collaboration has already presented a preliminary precision ISR analysis for the 3π channel [185] and will soon contribute to the leading two-pion channel as well.
- Theory developments since WP20 include the derivation and improvement of unitarity and analyticity constraints for the 2π , 3π , $\pi^0\gamma$, and $\bar{K}K$ channels, with applications to chiral extrapolations, the determination of IB contributions, and structure-dependent radiative corrections. In addition, methods were developed to extract quark-flavor-specific results from the data. These efforts provide consistency checks on the data sets, establish correlations with other (low-energy) observables, and facilitate detailed comparisons to lattice-QCD calculations of HVP. Examples include the observed tensions with the dispersive constraints in the case of SND20 (2π) – while all other 2π measurements pass the consistency check – and Belle II (3π); spread in the phase of the ρ - ω mixing parameter and the charge radius among the 2π experiments; predictions for IB in the long-distance tail of the QED correction due to the pion mass difference (as well as other IB effects); the derivation of theoretically robust amplitudes for structure-dependent radiative corrections; and the determination of data-driven comparison values for light-quark-connected and strange + disconnected Euclidean windows.
- The phenomenological analysis groups have made significant progress since the release of WP20. The DHMZ group has conducted detailed investigations to quantify the tensions among the various data sets, while the KNTW group has implemented the first fully blind data-driven HVP analysis and is in the process of scrutinizing, improving, and modernizing their data combination procedure. All these efforts over different groups are expected to lead to improved averaging procedures in the near future when the new data sets become available.
- The hadronic τ decay $\tau \rightarrow \pi\pi\nu_\tau$ provides an alternative path to the dominant two-pion contribution to HVP. The data sets from the LEP experiments, CLEO, and Belle are consistent and provide a competitive uncertainty, with excellent prospects for further precision improvements with Belle-II measurements. The current uncertainty in the τ -based determination of HVP is dominated by challenging IB corrections. The two analyses by the DHLMZ and LMR groups have been combined to obtain the WP25 estimate: while the central value fully reflects the published results, the uncertainty has been reassessed to account for sources of IB that are not yet fully addressed in the literature. Work to tackle structure-dependent radiative corrections and IB effects in the ratio of EM and weak form factors is ongoing and new information is expected in the near future from dispersive methods, lattice-QCD, and data-driven constraints.

As an outlook, it can be stated that the coming years will bring a series of new experimental results for the dominant hadronic channels used to evaluate the HVP contribution to the anomalous magnetic moment of the muon. Both the BESIII and BABAR collaborations anticipate presenting new results for the dominant $e^+e^- \rightarrow \pi^+\pi^-(\gamma)$ channel already in 2025.

The increased sensitivity to the treatment of higher-order radiative corrections in the experimental analyses, along with the development of new event generators to support these analyses, marks a significant improvement and is expected to help clarify the current situation. Furthermore, upcoming new measurements of hadronic τ decays with unprecedented statistics will be performed at Belle II and will also be crucial to further improve the precision of the SM prediction of a_μ . In the more distant future, the MUonE experiment aims to provide an independent and inclusive approach to determining the HVP contribution by measuring the spacelike running of HVP. Additionally, new analyses and highly complementary analysis techniques will be developed, such as BESIII's attempt to measure the inclusive hadronic cross section at low energies via ISR.

3. Lattice-QCD calculations of HVP

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3.1. Introduction

This section reviews lattice calculations of the HVP contributions to the muon $g - 2$. We start with an introduction summarizing standard definitions and the outcome of WP20. We adopt a Euclidean metric, which is the appropriate setup for lattice formulations. The position-space HVP tensor is given by the QCD and QED expectation value

$$C_{\mu\nu}(x) = \langle j_\mu(x) j_\nu(0) \rangle, \quad (3.1)$$

where j_μ is the quark EM current

$$j_\mu(x) = \sum_{f=1}^{N_f} Q_f \bar{q}_f(x) \gamma_\mu q_f(x), \quad (3.2)$$

where, for a flavor f , q_f is the quark field and Q_f is the electric charge in units of the elementary charge. The current above is conserved in continuous space-time, and its implementation on a discrete space-time might differ in lattice-QCD calculations. The momentum-space HVP tensor is then given by the Fourier transform

$$\Pi_{\mu\nu}(Q) = \int d^4x C_{\mu\nu}(x) e^{-iQ \cdot x}. \quad (3.3)$$

The tensor above is known to be transverse as a consequence of gauge invariance and Ward–Takahashi identities, and can be described through a single form factor $\Pi(Q^2)$:

$$\Pi_{\mu\nu}(Q) = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi(Q^2). \quad (3.4)$$

The function $\Pi(Q^2)$ is known to be UV divergent, and is conventionally regularized by using the subtracted function $\hat{\Pi}(Q^2) = 4\pi^2[\Pi(Q^2) - \Pi(0)]$, which fixes the residue of the photon propagator to one. Finally, with the definitions above, the LO HVP contribution to the muon $g - 2$ is given by

$$a_\mu^{\text{HVP, LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 f(Q^2) \hat{\Pi}(Q^2), \quad (3.5)$$

where the kernel function $f(s)$ is defined by

$$f(s) = \frac{r[Z(r)]^3}{m_\mu^2} \frac{1 - rZ(r)}{1 + r[Z(r)]^2}, \quad \text{with} \quad Z(r) = -\frac{r - \sqrt{r^2 + 4r}}{2r} \quad \text{and} \quad r = \frac{s}{m_\mu^2}. \quad (3.6)$$

Eq. (3.5) gives the spacelike analog of the timelike master formula Eq. (2.1), see also Section 4.1. In practice, it has become more standard in lattice calculations to evaluate the integral above on the time variable, for zero spatial momentum, as summarized below in Section 3.3. Furthermore, we define the LO HVP contribution, denoted by $a_\mu^{\text{HVP, LO}}$ in Eq. (3.5), in terms of the current–current correlator $C_{\mu\nu}$ restricted to one-photon-irreducible contributions.

We conclude this introduction by reviewing the status of lattice-QCD calculations of the LO HVP contribution to the anomalous magnetic moment of the muon at the time of WP20. Further details can be found in WP20 and references therein. Additionally, the definitions of the various components of $a_\mu^{\text{HVP, LO}}$ are summarized in the next section. In WP20, six groups quoted complete results for the total LO HVP and were included in a world lattice average. Notice, however, that the disconnected contribution was computed on the lattice by only three groups, and some groups used phenomenological inputs in order to estimate the strong and QED isospin-breaking (IB) contributions. There was a relatively large spread among lattice results, but all calculations were in agreement, at the 1.5σ level. The final estimate was

$$a_\mu^{\text{HVP, LO}}(\text{WP20}) = 711.6(18.4) \times 10^{-10}, \quad (3.7)$$

with an uncertainty of 2.6%. As a consequence, the lattice world average was consistent with both the dispersive data-driven estimate and the “no new physics” scenario. The uncertainty was a factor of 4.5 larger than that of the data-driven estimate, and the lattice average was not included in the final SM value for $a_\mu^{\text{HVP, LO}}$. We also note that the BMW-20 [16] lattice calculation, which reported a precision of 0.8%, was posted on arXiv after the WP20 deadline and published in April 2021. Consequently, it was not included in the WP20 lattice world average of Eq. (3.7).

Since none of the results retained in WP20 for the lattice HVP average of Eq. (3.7) included complete nonperturbative calculations of all components, it was decided to perform averages on individual contributions. In a second step, the errors of the individual contributions were added conservatively, assuming 100% correlation on the uncertainties, leading to the final result given by Eq. (3.7). The total uncertainty was dominated by the isospin-symmetric connected light-quark contribution, followed by strong IB (SIB) and QED corrections and the quark disconnected contribution. For the dominant light-quark contribution itself, the total error was dominated by the uncertainties associated with finite-volume (FV) corrections, statistics, and the continuum extrapolation. The uncertainties on the charm and strange quark contributions were already at or below the permil level relative to $a_\mu^{\text{HVP, LO}}$.

In order to achieve the permil-level precision needed to match the final Fermilab precision, the community identified a number of challenges early on. Within this section, we present in detail the progress made on them since WP20. Briefly, the main challenges and the progress made to address them so far are:

- FV corrections are significant, even on 6 fm lattices, and are among the largest contributions to the total error quoted in Eq. (3.7). They can be estimated using EFTs or EFT-inspired models, including NNLO ChPT [412–415], the Chiral Model [416], the Meyer–Lellouch–Lüscher (MLL) formalism with a Gounaris–Sakurai parameterization of the timelike pion form factor (MLLGS) [223,417–419], and the Hansen and Patella formalism [420,421]. However, there are now several direct FV studies also at the physical point [16,26,27] that guide these extrapolations and provide well-quantified control over the associated uncertainties.
- The lattice scale uncertainty is amplified in $a_\mu^{\text{HVP, LO}}$ by close to a factor of two [419], due to its dependence on the muon mass, and, hence must itself be determined with permil-level precision. This is part of the program of every lattice collaboration engaged in an effort to compute HVP at permil-level precision.
- A permil determination of $a_\mu^{\text{HVP, LO}}$ requires the full inclusion of IB effects even for quark-disconnected contributions. In principle, these effects also enter the determination of the lattice spacing. In WP20 only partial calculations of these effects were available, while now results for almost all IB effects are available from several groups [14,16,27,29,422], except for a small fraction associated with sea–sea and sea–valence QED corrections, for which only one calculation exists [16]. The long-distance (LD) QED contributions, which amount to about 0.3% of the total HVP, are, however, still a challenge.
- Many of these systematics (e.g., cutoff effects) are discretization dependent, so it is imperative to have results from different lattice actions. The results presented in this review employ gauge field ensembles generated independently by several lattice groups with improved gauge actions and a variety of different fermion actions, including Wilson-Clover, twisted-mass Wilson, Domain Wall, HISQ, and stout-smear staggered fermion actions.

The rest of this section is organized as follows: Section 3.2 details the standardized lattice breakdown of $a_\mu^{\text{HVP, LO}}$ into the isospin-symmetric flavor components and IB corrections. Section 3.3 is devoted to the discussion of some special features in the analysis of lattice calculations of HVP that are common to all calculations (e.g., blinding). Section 3.4 describes the calculation of time windows, which allow sharper and more detailed comparisons between lattice QCD calculations. Section 3.5 discusses the isospin-symmetric flavor contributions to the full Euclidean-time integration region. In Section 3.6, the results from the different collaborations are combined in order to produce a final estimate. The rest of the section describes the IB correction to the data from τ decays and the hadronic contribution to the running of α . Those are presented in Section 3.7. Finally, possible cross-checks resulting from the use of alternative approaches such as the covariant coordinate-space representation are introduced in Section 3.8.

3.2. Breakdown of the total HVP contribution to $a_\mu^{\text{HVP, LO}}$ into its various contributions

IB corrections to $a_\mu^{\text{HVP, LO}}$ due to the mass difference between up and down quarks, as well as EM effects, are expected to be an $\mathcal{O}(1\%)$ correction relative to the total $a_\mu^{\text{HVP, LO}}$. Therefore, it is desirable to write

$$a_\mu^{\text{HVP, LO}} = a_\mu^{\text{HVP, LO}}(\text{iso}) + \delta a_\mu^{\text{HVP, LO}}, \quad (3.8)$$

where $a_\mu^{\text{HVP, LO}}(\text{iso})$ denotes the isospin-symmetric component and $\delta a_\mu^{\text{HVP, LO}}$ the first-order IB corrections, which are sufficient in the present context. However, the separation above is not prescription independent. Indeed, since all quarks interact strongly and electromagnetically, any experimental input used to renormalize QCD and QED will only define the sum of the two terms above. Defining the individual terms requires additional conditions which define what physics is kept constant while taking α to zero. This issue was discussed in the most recent FLAG review (cf. Sec. 3 in Ref. [423]), where a prescription referred to as the *Edinburgh Consensus* was agreed upon after consultation with the community. This scheme is the recommended prescription for future calculations. However, most calculations in the present review were initiated before the Edinburgh Consensus was agreed upon, and we will use a slightly modified prescription. We

define *isospin-symmetric QCD* (*isoQCD*) to be the $\alpha = 0$ theory where quark masses are tuned to reproduce the following complete set of inputs

$$M_{\pi^+} = 135.0 \text{ MeV}, \quad M_{K^0} = M_{K^+} = 494.6 \text{ MeV}, \quad M_{D_s^+} = 1967 \text{ MeV}, \quad \text{and} \quad w_0 = 0.17236 \text{ fm}, \quad (3.9)$$

where w_0 is the Wilson flow scale introduced in Ref. [424]. We call this prescription the *WP25 scheme*. The only difference between the WP25 scheme and the Edinburgh Consensus is the use of w_0 to set the QCD scale instead of the pion decay constant. Recent results estimated that this change does not generate significant deviations within the current level of uncertainty (cf. comparison in Section 3.4.4), although the matching between the two prescriptions will require more scrutiny in the future. Additionally, the variables used in the prescription above can be changed to other hadronic inputs while keeping the scheme fixed, although this change will generate an additional matching uncertainty in actual numerical calculations. Another set of variables introduced originally by the BMW collaboration [16] are the connected meson masses M_{qq} , where for a quark flavor q , M_{qq} is the mass of the pseudoscalar meson $\bar{q}q$ considering only quark-connected contributions. These masses, although purely theoretical, are well-defined quantities that can be matched to the physical inputs above. The scheme employed in BMW-20 [16] resulted in *isoQCD* defined by

$$M_{uu} = M_{dd} = M_{\pi^0}^{\text{PDG}}, \quad M_{ss} = 689.89(49) \text{ MeV}, \quad M_{D_s^+} = 1967 \text{ MeV}, \quad \text{and} \quad w_0 = 0.17236(70) \text{ fm}, \quad (3.10)$$

where $M_{\pi^0}^{\text{PDG}} = 134.9768(5) \text{ MeV}$ is the current world average of the π^0 mass experimental measurements [225]. It was shown in BMW/DMZ-24 [24] that Eq. (3.10) matches Eq. (3.9) up to uncertainties small enough to be discarded in the context of the present review. Because of this, Eq. (3.9), or Eq. (3.10) with discarded uncertainties, may be referred to in the literature as *BMW-20* or *BMW scheme*. In summary, we consider that it is well motivated to average results produced with either prescription, and that propagation of matching uncertainties is unnecessary at the current level of precision.

Furthermore, it is convenient to decompose $a_\mu^{\text{HVP, LO}}(\text{iso})$ into quark-disconnected and single-flavor quark connected contributions:

$$a_\mu^{\text{HVP, LO}}(\text{iso}) = a_\mu^{\text{HVP, LO}}(ud) + a_\mu^{\text{HVP, LO}}(s) + a_\mu^{\text{HVP, LO}}(c) + a_\mu^{\text{HVP, LO}}(b) + a_\mu^{\text{HVP, LO}}(\text{disc}). \quad (3.11)$$

For a quark flavor q , $a_\mu^{\text{HVP, LO}}(q)$ in the equation above is obtained from Eq. (3.5) by considering only the quark-connected contribution for flavor q in the current–current correlator $C_{\mu\nu}$, and $a_\mu^{\text{HVP, LO}}(\text{disc})$ is the sum for all flavors of the remaining quark-disconnected contributions. We now discuss standard methodologies to predict the quantities defined in this introduction from lattice simulations.

3.3. Methodology

Here, we briefly discuss some of the methodology that is common to the lattice-HVP calculations presented in more detail below.

- *Time–momentum representation (TMR)*: The TMR has become the standard method for estimating the LO HVP contribution using lattice methods [425]. In this approach, Eq. (3.5) is re-written

$$a_\mu^{\text{HVP, LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0 C(x_0) \tilde{f}(x_0), \quad (3.12)$$

with the zero-momentum time correlator

$$C(x_0) = -\frac{1}{3} \sum_{k=1}^3 \int d^3x C_{kk}(x_0, \mathbf{x}), \quad (3.13)$$

and the kernel function

$$\tilde{f}(x_0) = 8\pi^2 \int_0^\infty \frac{d\omega}{\omega} f(\omega^2) \left[\omega^2 x_0^2 - 4 \sin^2 \left(\frac{\omega x_0}{2} \right) \right]. \quad (3.14)$$

In the equations above, $C_{\mu\nu}$ and f are defined in Section 3.1. The function $\tilde{f}$ can be expressed in terms of a modified Bessel function of the second kind and Meijer's G function [419] as

$$\tilde{f}(x_0) = \frac{2\pi^2}{m_\mu^2} \left[-2 + 8\gamma_E + \frac{4}{\hat{t}^2} + \hat{t}^2 - \frac{8}{\hat{t}} K_1(2\hat{t}) + 8 \log \hat{t} + G_{1,3}^{2,1} \left(\begin{matrix} \frac{3}{2} \\ 0, 1, \frac{1}{2} \end{matrix} \middle| \hat{t}^2 \right) \right], \quad (3.15)$$

where $\hat{t} = m_\mu x_0$. The nonperturbative input computed on the lattice is the time correlator C . Other approaches, based on a direct evaluation of the HVP tensor, $\Pi_{\mu\nu}(Q)$ (see, for example, Ref. [426]), or on the method of time moments introduced in Ref. [427] have been superseded by the TMR.

- *Error treatment in HVP calculations*: Lattice-QCD calculations yield *ab initio* theoretical predictions that are systematically improvable. Since lattice QCD is a MC method, statistical analysis is a crucial part of producing physical results. In addition to statistical errors from the MC process, systematic errors can enter in a variety of forms, for example,

through truncating the functional form that guides continuum limit extrapolations of lattice data obtained at finite sets of lattice spacings. While the details may vary from group to group, all lattice analyses described below include estimates of systematic errors based on studies of variations of the functional forms of the underlying EFT expansions, e.g., consideration of various functional forms for continuum extrapolation, that allow the assessment of systematic errors associated with the respective extrapolation (or interpolation). More detailed discussions of specific types of systematic errors, for which the dominant uncertainties may be different for different windows, will be provided in context below.

- **Blinding:** Most of the HVP results presented below, including (for example) all three calculations discussed for the dominant LD light-quark connected contribution [26,27,30], are based on blinded analyses. Blinding has been used widely in lattice QCD for many years, to prevent the introduction of bias in calculations of quantities whose values are known either from experiment or from previous calculations. To conduct a blind analysis, a small but unknown blinding factor is introduced into the analysis, modifying the final HVP result. Different lattice groups introduce the blinding in different places, but most commonly the blinding is applied as part of the analysis code, which has the advantage that different blinding factors can be easily applied for different windows. As the precision of lattice calculations improves, the importance of blinding will only increase, and we encourage all groups to adopt blind analyses going forward.
- **Treatment of correlations using FLAG procedure:** Except where explicitly noted otherwise, to combine results from different lattice calculations, we adopt the procedure used by the FLAG group for averaging [423]. Central values in this procedure amount to uncorrelated weighted averages of individual central values, with statistical and systematic errors added in quadrature. Error bars for averages are constructed using a procedure by which 100% correlation can be assumed between specific contributions to individual error estimates, as described in Eqs. (7)–(10) of the reference. In cases where the overall χ^2 per degree of freedom implied by the average over individual results is poor (i.e., $\chi^2/\text{dof} > 1$), a rescaling to increase the error bars by $\sqrt{\chi^2/\text{dof}}$ is performed to compensate. In some cases, results for certain quantities from lattice groups have been superseded by updated calculations. Such superseded results are shown in tables and figures below, but are excluded from averages. Depending on the quantity considered, different sources of uncertainty are taken to conservatively be 100% correlated. This may include statistical uncertainties in cases where the two calculations share the same gauge configurations, or systematic uncertainties for shared FV correction schemes, valence quark discretization, or other sources of error as described below.

3.4. Window contributions

Dividing the Euclidean time integral into different ranges has proved to be a very valuable tool in comparing the results of various collaborations and also for comparison with calculations of HVP based on dispersion relations (where time ranges correspond to energy ranges). After defining the three windows most commonly used, we consider details of the results for each window.

3.4.1. Definitions

Breaking the time integral Eq. (3.12) into three components allows for a better separation of the different sources of uncertainty [14]. The window contributions are defined by

$$a_{\mu}^{\text{HVP, LO}} = a_{\mu}^{\text{SD}} + a_{\mu}^{\text{W}} + a_{\mu}^{\text{LD}}, \quad (3.16)$$

$$a_{\mu}^{\text{SD}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^{\infty} dx_0 C(x_0) \tilde{f}(x_0) [1 - \Theta(x_0, t_0, \Delta)], \quad (3.17)$$

$$a_{\mu}^{\text{W}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^{\infty} dx_0 C(x_0) \tilde{f}(x_0) [\Theta(x_0, t_0, \Delta) - \Theta(x_0, t_1, \Delta)], \quad (3.18)$$

$$a_{\mu}^{\text{LD}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^{\infty} dx_0 C(x_0) \tilde{f}(x_0) \Theta(x_0, t_1, \Delta), \quad (3.19)$$

where

$$\Theta(t, t', \Delta) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{t - t'}{\Delta}\right), \quad (3.20)$$

defines the window as a smooth function in Euclidean time. The width parameter Δ , in particular, smooths out the window on both sides of the interval. The community advocated the use of the widely studied choice $\Delta = 0.15 \text{ fm}$, $t_0 = 0.4 \text{ fm}$, and $t_1 = 1.0 \text{ fm}$ [14] as benchmark parameters. Each window can then be treated independently and extrapolated to the continuum and infinite-volume limits. With this choice, the *intermediate window* (W) a_{μ}^{W} is less sensitive to discretization effects than the *short-distance* (SD) window a_{μ}^{SD} . It is also designed to be less sensitive to LD effects than the *long-distance* window a_{μ}^{LD} and hence much less affected by the signal-to-noise problems at large x_0 .

Finally, it is understood that all the window quantities can be decomposed as per Eqs. (3.8) and (3.11), using the same definitions and conventions as in the case of $a_{\mu}^{\text{HVP, LO}}$. We now turn to results for each of the windows.

Table 6

Compilation of lattice results for flavor-specific contributions to a_μ^{SD} , in units of 10^{-10} . The $N_f = 2 + 1$ Mainz/CLS calculation includes charm contributions from first principles in the valence sector, but not in the sea. Values marked with an asterisk are older results superseded by more recent determinations by the same collaboration. When results are displayed with two errors, the first is the statistical uncertainty and the second the systematic one. With only one quoted error, the statistical and systematic uncertainties are combined.

Collaboration	N_f	$a_\mu^{\text{SD}}(ud)$	$a_\mu^{\text{SD}}(s)$	$a_\mu^{\text{SD}}(c)$	$a_\mu^{\text{SD}}(\text{disc})$
ETM-24 [28]	2+1+1	–	9.063(16)(22)	11.61(07)(11)	–
FNAL/HPQCD/MILC-24 [29]	2+1+1	48.139(11)(91)	9.111(03)(16)	11.46(00)(17)	–0.0019(04)(26)
BMW/DMZ-24 [24]	2+1+1	47.84(04)(15)	9.04(03)(06)	11.68(15)(21)	–0.0007(019)(102)
RBC/UKQCD-23 [22]	2+1(+1)	48.51(43)(53)	–	–	–
ETM-22 [21]	2+1+1	48.24(03)(20)	9.074(14)(62)*	11.61(09)(25)*	–0.006(5)
χ QCD-22 [18]	2+1(+1)	48.58(07)(1.2)	9.18(01)(25)	–	–
ETM-21 [428]	2+1+1	48.21(56)(57)*	–	–	–
SL-24 [25]	2+1/0	47.62(32)(60)	–	–	–
Mainz/CLS-24 [23]	2+1	47.84(04)(24)	9.072(10)(58)	11.53(13)(26)	0.0013(25)(41)

3.4.2. Short-distance window

We begin by reviewing the SD window observable, denoted as a_μ^{SD} and defined in Eq. (3.17). As in the case of the total LO HVP correction, also in the SD window region the most important hadronic contribution is the one due to the light u - and d -quark-connected contractions, $a_\mu^{\text{SD}}(ud)$, representing $\simeq 70\%$ of $a_\mu^{\text{SD}}(\text{iso})$, while the strange- and charm-quark-connected terms, $a_\mu^{\text{SD}}(s)$ and $a_\mu^{\text{SD}}(c)$, constitute $\simeq 13\%$ and $\simeq 17\%$ of the total SD contribution, respectively. In Table 6 and Fig. 28, we collect results for the flavor-specific contributions, namely, $a_\mu^{\text{SD}}(ud)$, $a_\mu^{\text{SD}}(s)$, and $a_\mu^{\text{SD}}(c)$, as well as for the subleading correction $a_\mu^{\text{SD}}(\text{disc})$ reported by the various lattice QCD groups.

The calculations in BMW/DMZ-24 [24], Fermilab/HPQCD/MILC-24 [29], and ETM-22/24 [21,28] were performed on $N_f = 2 + 1 + 1$ ensembles. Those in Mainz/CLS-24 [23] and SL-24 [25] are based on $N_f = 2 + 1$ ensembles. The former allow SD matching to the full SM at renormalization scales equal to m_b , up to $1/m_b^2$ corrections, while that scale is m_c and corrections are of order $1/m_c^2$ for calculations based on $N_f = 2 + 1$ ensembles. The Mainz/CLS-24 [23] determination of a_μ^{SD} , however, includes the leading correction due to the quenching of sea-charm quarks, estimating it to be at the level of 0.02×10^{-10} based on perturbation theory [429]. The RBC/UKQCD calculation (RBC/UKQCD-23 [22]) addresses that subleading systematic effect from first principles, by generating gauge ensembles with dynamical charm quarks matched to those with $N_f = 2 + 1$ sea quarks, and finds it to be small compared to the quoted uncertainties. The gauge- and fermion-action dependence of the SD window is studied by the χ QCD group (χ QCD-22 [18]) using the overlap valence fermion action on ensembles with either the domain-wall fermion (and the Iwasaki gauge action) $N_f = 2 + 1$ -flavor sea from the RBC/UKQCD collaboration or the highly-improved-staggered-quark (with the Symanzik gauge action) $N_f = 2 + 1 + 1$ -flavor sea from the MILC collaboration. Further details of the lattice formulations adopted for the QCD action by the various groups can be found in the corresponding references indicated in Table 6.

The SD window observable can be obtained very precisely in lattice-QCD calculations as far as statistical errors are concerned, not being affected by the signal-to-noise problems present at large Euclidean time distances. FV effects, which typically represent a source of uncertainty relevant for the calculation of a_μ^{LD} , play hardly any role in a_μ^{SD} . Quantitatively, direct lattice evaluations, estimates based on ChPT and data-driven determinations point to sub-permil FV corrections for $L \simeq 6$ fm and physical m_{ud} [24]. As observed in Refs. [22,23,29] the dependence of a_μ^{SD} on quark masses is mild as well. Also, the sensitivity to the lattice scale is found to be subdominant [21,23]. On the contrary, the SD window suffers from large discretization effects, associated to the behavior of the lattice vector correlator at small-time distances, which are expected to be logarithmically enhanced with respect to the naive a^2 scaling typical of $\mathcal{O}(a)$ -improved calculations (compared to the usual case with on-shell observables), as discussed in Refs. [21,430–432]. Even though this may present a significant challenge, the comparison among the results obtained with different lattice regularizations provides the opportunity to test the robustness of the continuum-limit extrapolation.

The QCD action and the vector correlator employed in the calculations listed in Table 6 are $\mathcal{O}(a)$ -improved and all groups use data with at least three lattice spacings (or two lattice spacings and two or more lattice regularizations of the observable of interest) to control the extrapolation to the continuum limit. Furthermore, different strategies are adopted to reduce the impact of ($\mathcal{O}(a^2 \log a)$) lattice artifacts from very short Euclidean distances: in Refs. [21–24,28], tree-level perturbative cutoff effects on lattice correlators are subtracted from the nonperturbative data; the use of a suitably subtracted SD TMR kernel function together with perturbative computations at very high momenta has been devised by the Mainz group [23]; SL scrutinize the continuum limit of the SD window using purely lattice QCD methods, without perturbative input, via a combination of a quenched ($N_f = 0$) continuum extrapolation with a separate one of the dynamical sea-quark corrections [25]; while the Fermilab/HPQCD/MILC collaboration adopts a highly-improved vector-current discretization [29]. In the case of the staggered formulations adopted in Refs. [24,29], additional lattice artifacts that generate the taste splittings among the pions are heavily suppressed in the SD window region and found to be negligible at the level of precision reached. The quality of the extrapolations to the continuum limit is shown in Fig. 29 through the examples of the computations of $a_\mu^{\text{SD}}(ud)$ by BMW/DMZ-24, who use the largest number (seven) of lattice spacings, and of $a_\mu^{\text{SD}}(c)$ by ETM-24, who quote the smallest relative uncertainty for that flavor-specific contribution (1.2%).

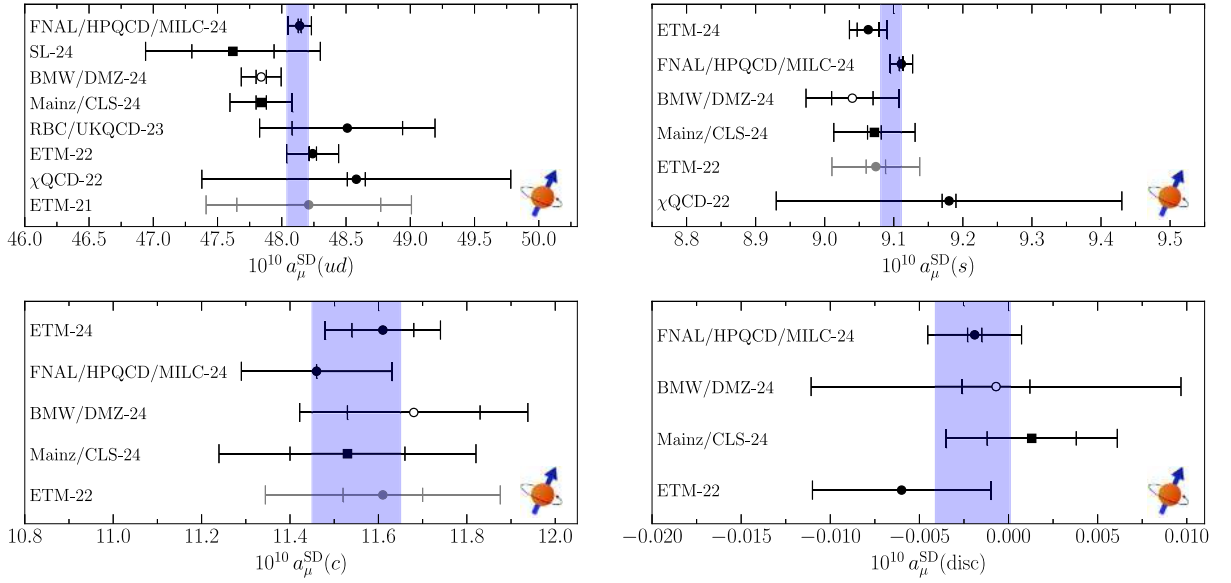


Fig. 28. Comparisons of lattice results for flavor-specific contributions to a_μ^{SD} . Upper-Left: Light-quark-connected contribution $a_\mu^{\text{SD}}(ud)$. Upper-Right: Strange-quark-connected contribution $a_\mu^{\text{SD}}(s)$. Lower-Left: Charm-quark-connected contribution $a_\mu^{\text{SD}}(c)$. Lower-Right: Quark-disconnected contribution $a_\mu^{\text{SD}}(\text{disc})$. We show in gray the results that have been superseded by new determinations. Since the BMW/DMZ-24 results have not yet been published, they are excluded from the averages as denoted by the unfilled symbols. The plotting symbol indicates the number of sea quarks in the gauge ensembles employed in the underlying calculation. Circles correspond to “ $N_f = 2 + 1 + 1$ ” ensembles with up, down, strange, and charm quarks in the sea, while squares denote “ $N_f = 2 + 1$ ” in which charm quarks are omitted in the sea. The inner error bars, when displayed, represent the statistical uncertainties, while the outer error bars show the total uncertainties given by adding the statistical and systematic errors in quadrature. The lattice averages of Table 7 are displayed here as light blue bands at the level of one standard deviation. See Table 6 for further details.

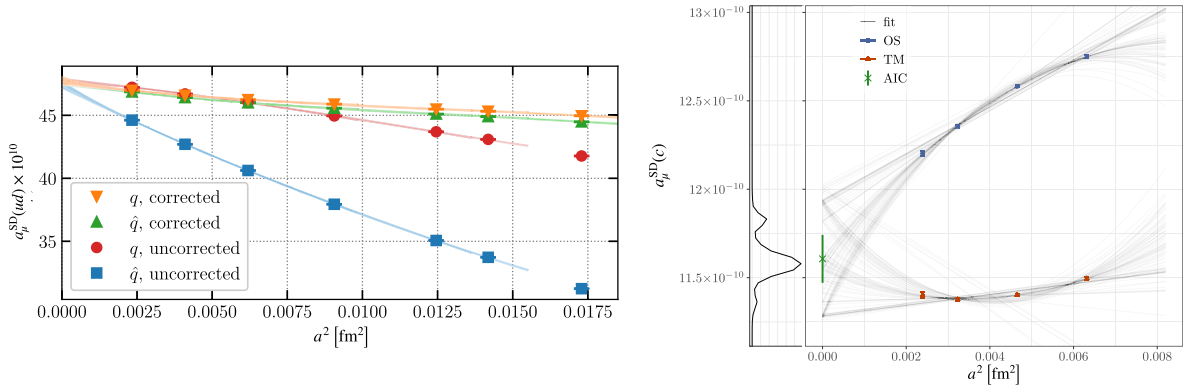


Fig. 29. Quality of the extrapolations to the continuum limit for the light- and charm-quark connected contributions to a_μ^{SD} performed in BMW/DMZ-24 (left panel) and ETM-24 (right panel), respectively. BMW/DMZ-24 uses data at seven lattice spacings and four variants of the observable of interest, while ETM-24 adopts two different valence-quark regularizations at four lattice spacings. Tree-level perturbative cutoff effects on lattice correlators are subtracted from the nonperturbative data in order to reduce the impact of dangerous $\mathcal{O}(a^2 \log a)$ artifacts. Source: Figures adapted from Refs. [24,28].

All results presented in Table 6 are extrapolated to the continuum and infinite-volume limits and interpolated or extrapolated to the physical point. The quoted errors in all lattice results include statistical and systematic uncertainties, where the latter estimates effects from scale setting, input parameters, continuum extrapolation (which represents the main source of uncertainty in the case of a_μ^{SD}), infinite-volume extrapolation, and chiral interpolations/extrapolations. Typically, these systematic errors are estimated by varying the chiral, continuum, or FV fit functions, including adding higher-order terms in the corresponding EFT expansions, or varying which lattice data are included, among other things. Overall, agreement is observed among the various (seven), currently available and more recent determinations of the ud -quark connected contribution, shown in the upper-left panel of Fig. 28, with the most pronounced differences occurring between the calculations in Mainz/CLS-24 [23] and BMW/DMZ-24 [24] on the one hand and those in ETM-22 [21] and Fermilab/HPQCD/MILC-24 [29] on the other, at most at the level of about 1.7σ . As far as the s -quark connected contribution

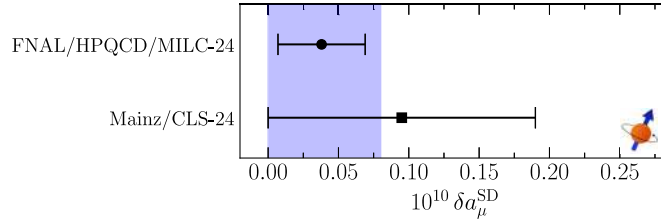


Fig. 30. Comparison of the available lattice results for the SIB+QED IB corrections to the SD window observable, δa_μ^{SD} , displayed along with the corresponding average given in Table 7. Error ticks and plotting symbols and colors have the same meaning as in Fig. 28.

Table 7

Summary results for the individual flavor contributions to $a_\mu^{\text{SD}}(\text{iso})$ (in the isospin limit, $m_u = m_d$ and $\alpha = 0$) and for the SIB and QED corrections, in units of 10^{-10} .

$a_\mu^{\text{SD}}(ud)$	$a_\mu^{\text{SD}}(s)$	$a_\mu^{\text{SD}}(c)$	$a_\mu^{\text{SD}}(b)$	$a_\mu^{\text{SD}}(\text{disc})$	δa_μ^{SD}
48.123(83)	9.096(15)	11.55(10)	0.30(02)	−0.0020(21)	0.04(04)

is concerned, except for a 1.5σ tension between the ETM-24 [28] and Fermilab/HPQCD/MILC-24 [29] determinations, the five most up-to-date results, listed in Table 6 and collected in the upper-right panel of Fig. 28, exhibit good compatibility. The four results for $a_\mu^{\text{SD}}(c)$ and for $a_\mu^{\text{SD}}(\text{disc})$ (lower-left and -right panels of Fig. 28, respectively) from Refs. [21,23,24,28,29] are nicely consistent. For the b -quark connected contribution the ETM, Mainz, and Fermilab/HPQCD/MILC groups provide the following estimates: 0.32×10^{-10} [21] (based on perturbation theory); $0.29(03) \times 10^{-10}$ [23] (given by a combination of phenomenological/perturbative estimates [433–436], and a nonrelativistic-QCD-based lattice calculation [437]); and $0.296(15) \times 10^{-10}$ [29] (from the direct lattice determination in Ref. [438] and pQCD inputs), respectively. Finally, the challenging nonperturbative calculation of the subleading IB contributions δa_μ^{SD} has been performed by only two collaborations so far. The Mainz group performs an explicit calculation of the IB corrections to the connected light- and strange-quark contributions due to unequal up- and down-quark masses (SIB) and electric charges [439,440], while the QED correction to the charm-quark contribution is estimated in perturbation theory. IB effects in the lattice scale and in the quark sea, as well as quark-disconnected contributions in the calculation of the relevant correlation functions are neglected. The total quoted correction is $0.095(95) \times 10^{-10}$ [23]. The Fermilab/HPQCD/MILC collaboration computes from first principles the LO SIB corrections for both the connected- and disconnected-quark contractions. QED corrections are estimated in perturbation theory. Higher-order contributions, including QED corrections to disconnected diagrams are neglected. The resulting $-0.0049(35) \times 10^{-10}$ SIB connected, $0.015(12) \times 10^{-10}$ SIB disconnected and $0.028(28) \times 10^{-10}$ [29] QED perturbative estimates yield a total SIB+QED contribution equal to $0.038(31) \times 10^{-10}$.

To average the lattice results for $a_\mu^{\text{SD}}(ud)$, $a_\mu^{\text{SD}}(s)$, $a_\mu^{\text{SD}}(c)$, and $a_\mu^{\text{SD}}(\text{disc})$ given in Table 6, we use the strategy of the FLAG group described in Ref. [423]. Any result that has been updated or superseded by a more recent determination by the same lattice group is excluded from the average. In Table 6 superseded results are marked with an asterisk. In addition, the results from BMW/DMZ-24 are not yet published, hence are also excluded from the following average. We treat the statistical and systematic errors in $\chi\text{QCD-22}$ [18] and RBC/UKQCD-23 [22], $\chi\text{QCD-22}$ [18] and Fermilab/HPQCD/MILC-24 [29], as well as in RBC/UKQCD-23 [22] and SL-24 [25] as 100% correlated since these calculations are based on overlapping MILC and/or RBC/UKQCD gauge configurations, (partially) on the same formulation of the QCD action, on common ensembles for the evaluation of systematic errors (such as FV corrections) and/or on common scale uncertainty. All the other determinations are treated as uncorrelated. Since sea-charm-quark effects on a_μ^{SD} are found to be below the permil level (see the above estimate in QCD perturbation theory [23]), we can combine $N_f = 2 + 1$ and $N_f = 2 + 1 + 1$ results in Table 6. Note that b -quark effects are even more subleading and, therefore, can be neglected here. For the valence-bottom-quark-connected contribution, $a_\mu^{\text{SD}}(b)$, we consider the uncertainties of the estimates mentioned above as 100% correlated, since the methods used by different teams to get them are similar. We provide an estimate for the SIB+QED IB corrections, δa_μ^{SD} , by combining the two previously described results given in Mainz/CLS-24 [23] and Fermilab/HPQCD/MILC-24 [29] (see Fig. 30). Since both determinations come from first-generation calculations of some of those effects, we choose the conservative approach of averaging Mainz/CLS-24 and Fermilab/HPQCD/MILC-24 under the assumption that the uncertainties of the two calculations are 100% correlated. IB effects in the lattice scale and in the quark sea, as well as quark-disconnected QED contributions neglected in those calculations are expected to be small and, thus, covered by the quoted conservative error. The averages carry a $\chi^2/\text{dof} < 1$ in all cases.

Our averages for the individual flavor contributions to a_μ^{SD} , in the isospin limit and for the leading SIB and QED corrections, are summarized in Table 7. It should be stressed that the separation into isospin-symmetric flavor terms ($m_u = m_d$ and $\alpha = 0$) and IB corrections is scheme dependent, while the sum of those contributions is unambiguous. We refer the reader to the various references given in Table 6 for details on the schemes adopted by the lattice groups. Here it is sufficient to note that ambiguities related to the different prescriptions used to define isoQCD with respect to the full QCD+QED theory are negligible for the SD window at the current permil precision level, as can be inferred from the very modest impact of variations in the lattice-scale setting [21,30] and quark-mass tuning [22,23,29] on a_μ^{SD} .

The contributions in Table 7 can now be combined to obtain average lattice numbers for the SD window contribution to LO HVP in the isospin limit, $a_\mu^{\text{SD}}(\text{iso})$. Adding to this result the SIB and QED corrections yields the LO HVP correction a_μ^{SD} , which can be compared with the phenomenological determinations of a_μ^{SD} (see also Figs. 38 and 39 below). A final choice has to be made on how to combine uncertainties in adding up the individual flavor and SIB+QED contributions. The uncertainty associated with the determination of the overall scale is 100% correlated among these contributions in a same calculation. Also, in a calculation of the various contributions performed on the same set of ensembles, statistical errors will be correlated. Thus, we choose the conservative approach of adding the errors of the individual contributions linearly in our final results. This procedure leads the following world average for the isospin-symmetric SD window contribution to the total LO HVP correction:

$$a_\mu^{\text{SD}}(\text{iso}) = 69.06(22) \times 10^{-10}. \quad (3.21)$$

Adding to this result the average SIB+QED correction, we obtain the following lattice world average for the total SD, LO HVP contribution to the muon ($g - 2$):

$$a_\mu^{\text{SD}} = 69.10(26) \times 10^{-10}, \quad (3.22)$$

where we include subleading digits at intermediate stages before rounding to the digits shown. This result, which is roughly 10% of the total LO HVP contribution, has a relative uncertainty of about 0.4%, and it is compatible with the estimate $68.4(5) \times 10^{-10}$ from R -ratio data given in Ref. [363] in the pre-CMD-3 scenario. The absence of significant tensions in comparing lattice and R -data-driven results for a_μ^{SD} is consistent with what is found for the HVP $\Pi(Q^2)$ at large Euclidean momentum Q^2 , as well as with the constraints from EW precision tests of the SM.

Another way of getting an average value for a_μ^{SD} , though with a less conservative error, is to use directly the results quoted by a few groups, specifically $69.26(25) \times 10^{-10}$ by the ETM collaboration [21,28], $68.85(45) \times 10^{-10}$ by Mainz [23], $69.05(21) \times 10^{-10}$ by Fermilab/HPQCD/MILC [29]. The average *à la* FLAG of those independent determinations, treating the errors as uncorrelated across calculations, amounts to $69.10(15) \times 10^{-10}$ which, remarkably, is in excellent agreement with our final value given in Eq. (3.22).

3.4.3. Intermediate window

The standard intermediate window observable a_μ^{W} contributing to the LO HVP term of a_μ is defined in Section 3.4.1, Eq. (3.18). Over the past few years the intermediate window has played an important role, since it allows for a high-precision comparison between SM predictions and experimental data for the e^+e^- annihilation cross section into hadrons in the CM energy range around 1 GeV. Indeed, from the perspective of lattice QCD, the computation of the intermediate window is significantly less challenging than that of the full HVP contribution. The LD tail of the vector correlator is exponentially suppressed by the integration kernel of a_μ^{W} , thereby mitigating both the signal-to-noise ratio issues that affect the correlator at large times and the large finite-size effects associated with two-pion states. Moreover, unlike the SD window, a_μ^{W} does not suffer from large cutoff effects. Altogether these features enable very precise lattice calculations of a_μ^{W} .

We first review results for a_μ^{W} that are given in isoQCD. Since tiny differences in the scheme defining isoQCD are present in the results obtained by the various groups we adopt the WP25 scheme as a reference scheme prior to building averages (see details below). Then we consider determinations of a_μ^{W} in QCD+QED as quoted by individual groups or coming from our average in the WP25-scheme isoQCD and from available estimates of IB effects (see Table 9).

We begin by separately discussing the connected light, strange, and charm quark contributions to a_μ^{W} in isospin-symmetric lattice QCD, denoted as $a_\mu^{\text{W}}(ud)$, $a_\mu^{\text{W}}(s)$, and $a_\mu^{\text{W}}(c)$, respectively. For the connected ud -quark term, $a_\mu^{\text{W}}(ud)$, which is by far the numerically dominant contribution to a_μ^{W} , we consider here all the lattice results obtained using data with at least one ensemble at the physical pion mass point, three lattice spacings (or two lattice spacings and two or more lattice regularizations of the observable), and $M_\pi L \geq 3$. This criterion is relaxed for the subdominant contributions, where results based on two lattice spacings are also included. Since the deadline for inclusion in WP20, several lattice results have appeared: LM-20 [17], BMW-20 [16], ABGP-22 [19], Mainz/CLS-22 [20], χ QCD-22 [18], ETM-22 [21], RBC/UKQCD-23 [22], Fermilab/HPQCD/MILC-23 [369], BMW/DMZ-24 [24], and Fermilab/HPQCD/MILC-24 [29]. Concerning the lattice formulation adopted, LM-20, ABGP-22, Fermilab/HPQCD/MILC-23, and Fermilab/HPQCD/MILC-24 employ the HISQ staggered fermion discretization, BMW/DMZ-24 (BMW-20) use a staggered quark action with stout smearing (4stout), RBC/UKQCD use domain-wall fermions, ETM-22 use Wilson-Clover twisted-mass fermions, χ QCD-22 use overlap valence fermions on both domain-wall and HISQ sea, and Mainz/CLS-22 use $\mathcal{O}(a)$ -improved Wilson fermions. These determinations of the light-connected term, $a_\mu^{\text{W}}(ud)$ are collected in Table 8, and graphically shown in the upper-left panel of Fig. 31. Within the quoted (statistical plus systematic) errors a striking agreement is observed among all of them. The somewhat older and since superseded result from RBC/UKQCD-18 [14] is also included in the table for completeness.

For the s - and c -quark connected contributions, $a_\mu^{\text{W}}(s)$ and $a_\mu^{\text{W}}(c)$, new results have recently been presented by χ QCD-22 (s -quark only), Mainz/CLS-22, ETM-22, BMW/DMZ-24, and Fermilab/HPQCD/MILC-24, with an additional update from the ETM collaboration (ETM-24 [28]). All currently available data for these contributions to the intermediate window are compiled in Table 8 and displayed in the upper-right and lower-left panels of Fig. 31. Overall, good agreement is observed among the various determinations of the s -quark connected contribution, with the Mainz/CLS-22 result lying slightly above the other ones. For the c -quark connected contribution, a somewhat larger spread is evident,

Table 8

Single-flavor and disconnected contributions to a_μ^W in isoQCD, in units of 10^{-10} . In many cases, the results of the various groups correspond to slightly different definitions of isoQCD. To account for the impact of this ambiguity (even if small), we have added, prior to averaging, an additional systematic error to the published results, as described in the text. Values marked with an asterisk have been superseded by more recent results by the same collaboration and are hence excluded from the average that is given in the last row. Since the BMW/DMZ-24 results, which are indicated with a hash in the table, have not yet been published, we use the values quoted in the BMW-20 paper for the averages. For the specific contributions $a_\mu^W(s)$ and $a_\mu^W(c)$, however, the BMW/DMZ-24 analysis reports substantially larger systematic uncertainties. To remain conservative, in our averages we keep the BMW-20 central values while replacing their systematic errors with those quoted by BMW/DMZ-24 and adding the absolute difference between the BMW-20 and BMW/DMZ-24 determinations as an additional systematic uncertainty. This procedure yields $a_\mu^W(s) = 27.175(28)(90)(95)[134]$ and $a_\mu^W(c) = 2.70(1)(17)(24)[29]$, where the error in square brackets is the total error.

	N_f	$a_\mu^W(ud)$	$a_\mu^W(s)$	$a_\mu^W(c)$	$a_\mu^W(\text{disc})$
FNAL/HPQCD/MILC-24 [29]	2+1+1	207.40(15)(39)	27.32(2)(5)	2.717(0)(42)	−0.85(6)(19)
ETM-24 [28]	2+1+1	–	27.16(15)(20)	2.920(43)(48)	–
BMW/DMZ-24 [24]	2+1+1	−206.57(25)(60) [#]	27.08(9)(9) [#]	2.94(11)(17) [#]	−1.084(22)(49) [#]
FNAL/HPQCD/MILC-23 [369]	2+1+1	206.6(1.0) [*]	–	–	–
RBC/UKQCD-23 [22]	2+1	206.36(44)(43)	–	–	–
ETM-22 [21]	2+1+1	206.5(7)(1.1)	27.28(13)(15) [*]	2.90(3)(12) [*]	−0.78(21)(0)
χ QCD-22 [18]	2+1(+1)	206.7(1.5)(1.0)	26.8(1)(3)	–	–
Mainz/CLS-22 [20]	2+1	207.0(8)(1.2)	27.68(18)(22)	2.89(3)(13)	−0.81(4)(8)
ABGP-22 [19]	2+1+1	206.75(81)(2.10)	–	–	–
BMW-20 [16]	2+1+1	207.3(4)(1.3)	27.175(28)(13)	2.70(1)(10)	−0.907(35)(54)
LM-20 [17]	2+1+1	205.97(79)(90)	27.06(8)(21)	–	–
RBC/UKQCD-18 [14]	2+1	202.9(1.4)(0.3) [*]	27.0(2)(1)	3.0(0)(1)	−1.0(1)(0)
Average		206.97(41)	27.274(62)	2.805(55)	−0.893(44)

Table 9

Results for a_μ^W , in units of 10^{-10} . We quote in different columns: the number of active quark flavors; the isoQCD values discussed in this subsection (see Eq. (3.23) and related text), $a_\mu^W(\text{iso})$, with the average in last line given in the WP25 scheme (see text about how we treat the ETM-22 and Mainz/CLS-22 results that are quoted in a slightly different scheme); the QCD+QED values based on the papers cited in each line, a_μ^W , with their average given in last line; the IB contribution, δa_μ^W , with respect to isoQCD in the WP25 scheme (see text for details), and its average in the last line. As in the case of the single-flavor and disconnected contributions, since the BMW/DMZ-24 results have not yet been published (they are indicated with a hash in the table), we use the values quoted in the BMW-20 paper for the averages.

	N_f	$a_\mu^W(\text{iso})$	a_μ^W	δa_μ^W
FNAL/HPQCD/MILC-24 [29]	2+1+1	236.60(16)(47)	236.45(17)(83)	−0.15(59)
BMW/DMZ-24 [24]	2+1+1	235.51(29)(63) [#]	235.94(29)(63) [#]	–
RBC/UKQCD-23/18 [14,22]	2+1	235.36(49)(46)	235.56(65)(50)	0.2(4)
ETM-22 [21]	2+1+1	235.90(74)(1.12)	236.30(74)(1.12)	–
Mainz/CLS-22 [20]	2+1	236.60(79)(1.13)	237.30(79)(1.22)	0.70(47)
BMW-20 [16]	2+1+1	236.3(4)(1.3)	236.7(4)(1.3)	0.43(08)
Average		236.18(36)	236.26(47)	0.42(07)

with the most pronounced differences occurring between Fermilab/HPQCD/MILC-24 and RBC/UKQCD-18, and between Fermilab/HPQCD/MILC-24 and ETM-24, each at the level of about $(2.6\text{--}2.7)\sigma$.

Turning to the quark-disconnected contribution to a_μ^W in isoQCD, which we denote as $a_\mu^W(\text{disc})$, six lattice results are available: RBC/UKQCD-18 [14], BMW-20 [16], Mainz/CLS-22 [20], ETM-22 [21], BMW/DMZ-24 [24], and Fermilab/HPQCD/MILC-24 [29]. These are listed in Table 8 and shown in the lower-right panel of Fig. 31. The disconnected term gives only a tiny contribution to the total a_μ^W , of the order of 0.05%, which is similar in size to the uncertainty on the dominant light-connected term. A moderate compatibility is observed overall among the different lattice results: the largest discrepancy, about 2.5σ , arises between the new BMW/DMZ-24 result and Mainz/CLS-22. We note that the BMW/DMZ-24 result is 20% larger in magnitude than BMW-20 (which had a relative uncertainty of about 7%).

In order to average the lattice results for $a_\mu^W(ud)$, $a_\mu^W(s)$, $a_\mu^W(c)$, $a_\mu^W(\text{disc})$, we follow the procedure established by the FLAG group [423]. We exclude from the average any result that has been superseded by a more recent determination by the same collaboration, with the only exception of the BMW/DMZ-24 results, which have not yet been published. Consequently in our averages we use the BMW-20 determinations, supplementing them, when necessary, as specified in the caption of Table 8, with an additional systematic uncertainty. In Fig. 31, superseded results appear in gray, while in Table 8 they are indicated by an asterisk. The final central values are determined via an uncorrelated weighted average of the results, as described in Eqs. (3)–(6) of Ref. [423]. To estimate the final uncertainties, we apply the method summarized in Eqs. (7)–(10) of Ref. [423], which takes into account correlations among the various results. As a conservative choice, in averaging we assume a 100% correlation in the statistical errors for groups that fully or partially share gauge configurations, and a 100% correlation in the systematic errors for groups using the same discretization in the valence sector.

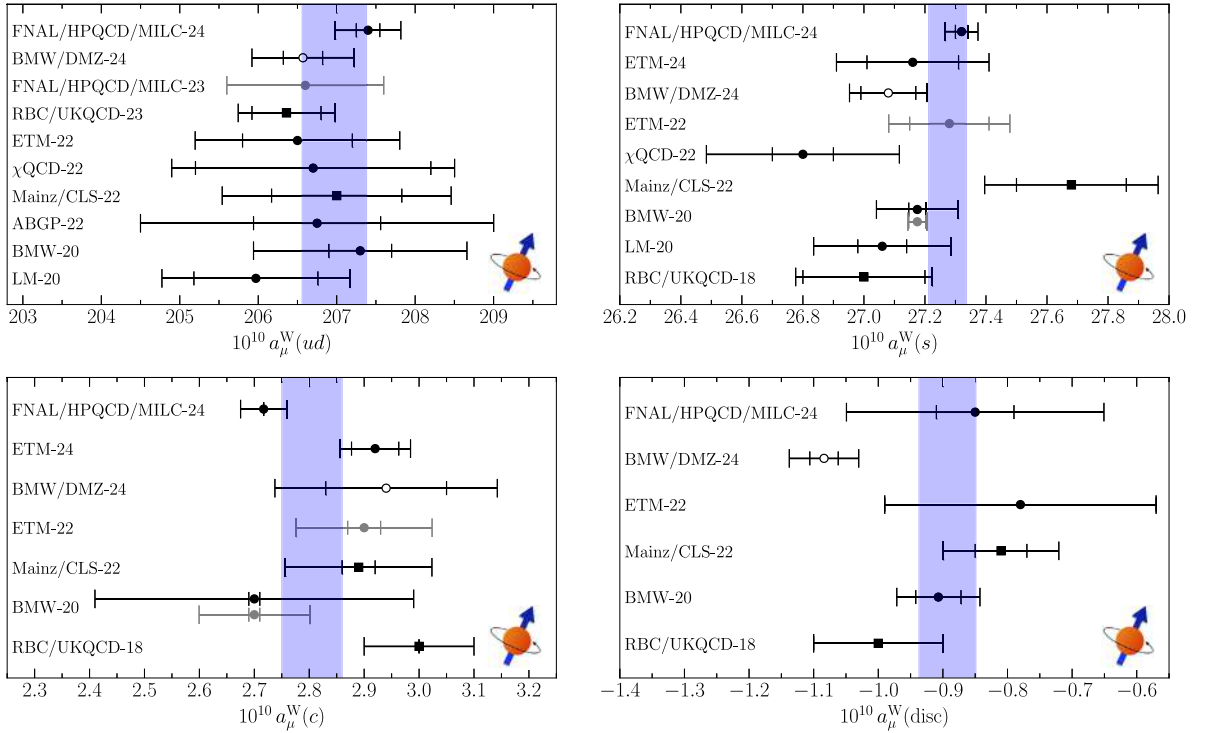


Fig. 31. Compilation of lattice results for the partial flavor contributions to a_μ^W . Upper-Left: ud -quark connected, $a_\mu^W(ud)$. Upper-Right: s -quark connected, $a_\mu^W(s)$. Lower-Left: c -quark connected, $a_\mu^W(c)$. Lower-Right: quark disconnected, $a_\mu^W(\text{disc})$. Error ticks and plotting symbols and colors have the same meaning as in Fig. 28. For the BMW-20 results for $a_\mu^W(s)$ and $a_\mu^W(c)$, both the original published values and those including the additional systematic uncertainty explained in the caption of Table 8 are shown in gray and black, respectively. The light blue band in each panel corresponds to the average $\pm 1\sigma$ obtained using the procedure described in the text.

As anticipated, the average values for a_μ^W partial and total results in isoQCD are given in the reference WP25 scheme (see Section 3.2). Besides the BMW-20 and BMW/DMZ-24 results, the one of RBC/UKQCD-23 on $a_\mu^W(ud)$, and the ones by Fermilab/HPQCD/MILC-24, the other intermediate-window results are given by the various authors in slightly different schemes: we do not alter their central values and errors in the tables and plots of this subsection, but in order to account for this systematic effect, prior to and only for the sake of averaging, we add in quadrature to the results not in the WP25 scheme a 0.2% systematic uncertainty for $a_\mu^W(ud)$, $a_\mu^W(c)$, $a_\mu^W(\text{disc})$, and a 0.3% systematic uncertainty for $a_\mu^W(s)$. In almost all cases, this leads to a very small increase of the total errors and thus to a correspondingly slight decrease of the weight of these results in the average.

The value of the systematic uncertainty added in quadrature to the results not in the WP25 scheme can be justified as follows. Among the four renormalization inputs (for scale, ud -, s -, c -quark mass) that specify an isoQCD scheme, all lattice groups employ $M_\pi \simeq 135.0$ MeV to fix the ud -quark mass and $M_{D_s} \simeq 1968.4$ MeV or a similar closely related input (e.g., $m_c/m_s \simeq 11.8$) to fix the c -quark mass (which moreover has a tiny impact on the total a_μ^W). One is thus only left with the ambiguities coming from use of different inputs as compared to the WP25 scheme about the scale and the s -quark mass.

The scale input in the WP25 scheme, which originally in the BMW-20 results is given by $w_0 \simeq 0.1724(7)$ fm (updated to 0.1725(5) fm by BMW/DMZ-24), is commonly replaced by $M_\Omega \simeq 1672.5$ MeV (used, e.g., by RBC/UKQCD-23 and Fermilab/HPQCD/MILC-24) or by the FLAG value of the pion decay constant $f_\pi = 130.5$ MeV (used, e.g., by ETM-22/24, Mainz/CLS-22, and Fermilab/HPQCD/MILC-24). From RBC/UKQCD-23 we know that using the WP25 or the M_Ω scale input leads to $\lesssim 0.05\%$ central value shift in $a_\mu^W(ud)$. From the FLAG24 review [423] one learns that using f_π as scale input leads to values of w_0 that are within 0.5% of the WP25 value. This can be seen in Table 77 of Ref. [423] by comparing the results for w_0 from ETM-21 [441], MILC-15 [442], HPQCD13A [443], and CLS-21 [444] (the last one inferred by comparing the value of $\sqrt{t_0}$ to that of ETM-21). Using the estimate $d \log a_\mu^W(ud)/d \log a \simeq 0.4$ (see, e.g., ETM-22, Eq. (A.20)) implies a systematic uncertainty on $a_\mu^W(ud)$ not larger than 0.2%. Using analogous arguments and information on the spectral density of the current–current correlator in each flavor channel, a similar relative uncertainty due to the w_0 -scale ambiguity is estimated or expected for $a_\mu^W(s)$ and $a_\mu^W(\text{disc})$, and a substantially smaller one for $a_\mu^W(c)$.

The systematic error we add to non-WP25 scheme results for $a_\mu^W(s)$ is 0.3% because it amounts to the sum in quadrature of 0.2% arising from the scale ambiguity and another 0.2% due to the difference in s -quark mass input for isoQCD as compared to the WP25 one. Indeed, from an unpublished analysis of ETM-24 data at four lattice spacings and physical

pion mass we infer that the WP25 scheme input, which is the fictitious strange-strange quark-connected meson mass $M_{ss} = 689.9$ MeV, leads to a value of the kaon mass M_K which is 0.7 MeV larger than $M_K(\text{FLAG}) = 494.6$ MeV (used, e.g., by several lattice groups) and only 0.4 MeV smaller than $M_K(\text{RBC/UKQCD}) = 495.7$ MeV. Combining the fact that the M_K input spread compared to WP25 is about 0.2% with the estimate of $d \log a_\mu^W(s)/d \log M_K^2 = -0.41(6)$, which is provided by Mainz/CLS-22 (see Eq. (B.26) of Ref. [20]) and fully confirmed by ETM-24 unpublished data, we estimate a systematic uncertainty of less than 0.2% on $a_\mu^W(s)$ due to ambiguity in the s -quark input for isoQCD.

The averages of the single-flavor and disconnected contributions obtained in this way are provided in the last row of Table 8 and illustrated by the colored vertical bands in Fig. 31. In the case of $a_\mu^W(c)$, the reduced χ^2 of the fit is approximately 3 and, due to the lower level of compatibility among the lattice results in this case, we inflate the final uncertainty by a factor of $\sqrt{\chi^2/\text{dof}}$.¹⁴

We now turn to the results for $a_\mu^W(\text{iso})$ in isoQCD. There are five groups who provided (or provided sufficient evidence to infer) results on $a_\mu^W(\text{iso})$: Fermilab/HPQCD/MILC, BMW, RBC/UKQCD, ETM, and Mainz. The results are summarized in Table 9 and shown in the left panel of Fig. 32. For BMW/DMZ-24, we sum the single-flavor and disconnected contributions, combining their errors in quadrature. Instead, ETM-22, Mainz/CLS-22, and Fermilab/HPQCD/MILC-24 directly quote the value of $a_\mu^W(\text{iso})$. For RBC/UKQCD-23/18, we begin with the value of $a_\mu^W(ud)$ quoted in the WP25 scheme [22] and then add the s - and c -quark connected contributions, as well as the disconnected contribution given in Ref. [14]. Because these three contributions are not originally provided in the WP25 isoQCD scheme, we assign systematic errors of 0.3%, 0.2%, and 0.2%, respectively, before summation. The resulting RBC/UKQCD value and error are then treated as if given in the WP25 scheme. As for BMW-20, we subtract an IB correction $\delta a_\mu^W = 0.43(8) \times 10^{-10}$ from the total a_μ^W computed in QCD+QED.

In analogy to the procedure above, the average of results for $a_\mu^W(\text{iso})$ is also given in the WP25 isoQCD scheme. In order to account for the slightly different definitions of the isoQCD entering the results by ETM and Mainz in Table 9, we add, prior to averaging, a systematic uncertainty of 0.2% (i.e., the same systematic error we attributed previously on $a_\mu^W(ud)$) to all results not quoted in the WP25 isoQCD scheme. The average, performed according to the FLAG procedure just as for the partial single-flavor and disconnected contributions, yields¹⁵

$$a_\mu^W(\text{iso}) = 236.18(36) \times 10^{-10}, \quad (3.23)$$

with an uncertainty of about 0.15%.

An alternative procedure to obtain $a_\mu^W(\text{iso})$ is to sum the (average of) single-flavor and disconnected contributions listed in the last row of Table 8. Noting that the quark-connected correlators for different flavors and the quark-disconnected correlators all have very different statistical fluctuations, reflecting the completely different physics of the corresponding intermediate-state channels, we expect their statistical correlations to be tiny already within the results of a single group and further diluted by the average leading to last line of Table 8, and hence safely negligible. For this reason we proceed by adding the uncertainty of these contributions in quadrature to obtain

$$a_\mu^W(\text{iso, single-flav-sum}) = 236.16(42) \times 10^{-10}. \quad (3.24)$$

This result is in excellent agreement with that of Eq. (3.23). If one were to treat the individual flavor contributions as fully correlated one would obtain a result with a 36% larger uncertainty, namely

$$a_\mu^W(\text{iso, single-flav-sum-correlated}) = 236.16(57) \times 10^{-10}. \quad (3.25)$$

For the reasons outlined above, we consider this latter value to be overly conservative and advocate the value given in Eq. (3.24) as our central result for the isospin-symmetric window contribution $a_\mu^W(\text{iso})$ in the WP25 scheme.

To provide a QCD+QED result with a conservative error for a_μ^W we add the average of IB effects δa_μ^W to the result Eq. (3.24) obtaining the value

$$a_\mu^W = 236.58(43) \times 10^{-10}. \quad (3.26)$$

The estimates of the IB effect δa_μ^W with respect to isoQCD in the WP25 scheme are given for each group in the last column of Table 9. For the cases where the value of δa_μ^W is not taken from the BMW-20 paper, we proceed as follows: for Fermilab/HPQCD/MILC-24 and RBC/UKQCD-23/18 we obtain it by combining the published information on the correlated difference in $a_\mu^W(\text{iso})$ between the used scheme and the WP25 scheme with the requirement that the sum $a_\mu^W(\text{iso}) + \delta a_\mu^W$ be scheme independent; for Mainz/CLS-22 we keep the published central value of δa_μ^W in the scheme of choice, assuming that the scheme dependence is accounted for in the estimated systematic error.

Another way of obtaining a final result for a_μ^W is to use directly the results quoted in QCD+QED by a few groups, see the corresponding column of Table 9. An average *à la* FLAG leads to $a_\mu^W = 236.26(47) \times 10^{-10}$, which is reassuringly in very good agreement with the value in Eq. (3.26).

Finally, we stress that the result of Eq. (3.26) is potentially of high relevance for the experimental collaborations measuring $e^+e^- \rightarrow$ hadrons and for precision tests of the SM. Indeed, the quantity a_μ^W can be accessed experimentally and lattice QCD+QED provides for it a pure SM prediction with an accuracy of about 0.18%.

¹⁴ For $a_\mu^W(s)$, the reduced χ^2 of the fit is slightly larger than one (about 1.3), too, and we inflated the final error by a factor $\sqrt{\chi^2/\text{dof}}$.

¹⁵ As in the case of the single-flavor and disconnected contributions, since the BMW/DMZ-24 results have not yet been published, we base our averages on the BMW-20 determinations (see also the caption of Table 9).

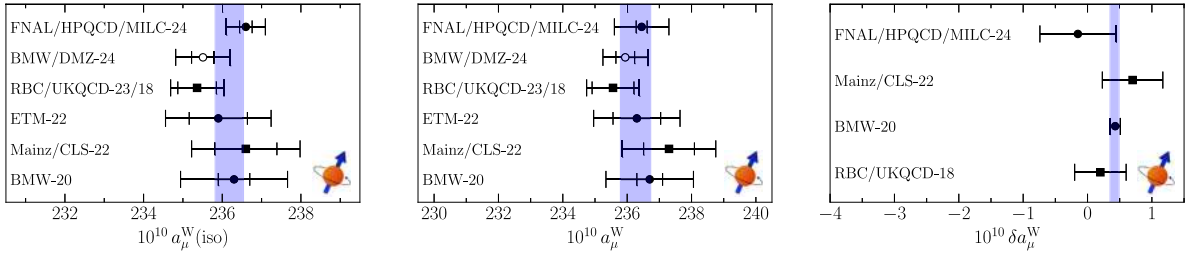


Fig. 32. Compilation of lattice results for the total a_μ^W in isospin-symmetric QCD (left), QCD+QED (center), and the available independently estimated IB corrections (right). Error ticks and plotting symbols and colors have the same meaning as in Fig. 28. The light blue band in each panel denotes the average $\pm 1\sigma$ using the procedure described in the text.

3.4.4. Long-distance window

A precise estimation of the LD contribution is crucial for achieving high precision in $a_\mu^{\text{HVP, LO}}$, as it dominates both the central value and the uncertainty in lattice-QCD calculations. The latter arises primarily from the exponential signal-to-noise problem, which results in larger statistical uncertainties compared to a_μ^{SD} and a_μ^W . Additionally, significant FV effects pose challenges for conventionally sized lattices with $M_\pi L \leq 5$ at the physical value of the pion mass. Moreover, when the staggered fermion formulation is used to simulate quark fields, substantial taste-breaking effects introduce deviations from the continuum theory, further complicating the continuum extrapolation.

Four groups have performed dedicated computations of contributions to the LD window a_μ^{LD} . RBC/UKQCD-24 [26], Mainz/CLS-24 [27], and Fermilab/HPQCD/MILC-24 [30] have all evaluated the dominant light-quark connected contribution to a_μ^{LD} . In addition, Mainz/CLS-24 provide results for the remaining flavor contributions, yielding a complete determination of the LD window in isoQCD. The ETM collaboration has computed the subleading strange- and charm-connected contribution to a_μ^{LD} in ETM-24 [28].

The RBC/UKQCD collaboration employs simulations with $2 + 1$ flavors of domain-wall fermions at three lattice spacings, down to 0.073 fm. Three ensembles at physical quark masses are complemented by six simulations with heavier-than-physical pion masses. A spectral reconstruction of the vector correlation function from two-pion states is used to reduce statistical noise in the LD regime. The Mainz calculation is based on $2 + 1$ flavors of $\mathcal{O}(a)$ -improved Wilson fermions at six lattice spacings, with the finest reaching 0.039 fm. The analysis includes 34 gauge ensembles with pion masses ranging from 420 MeV down to 132 MeV, including three ensembles with physical quark masses. Low-mode averaging is supplemented by a spectral reconstruction on selected ensembles to compute the vector correlation function. Fermilab/HPQCD/MILC use four ensembles with $2 + 1 + 1$ flavors of highly improved staggered quarks, all at physical quark masses. The finest ensemble corresponds to a lattice spacing of 0.06 fm. Low-mode averaging is used for the vector correlation function, and EFT-based taste-breaking corrections are applied alongside finite-size corrections to mitigate cutoff effects specific to the staggered formulation. The ETM collaboration computes s - and c -quark connected correlation functions on six physical-mass ensembles with $2 + 1 + 1$ flavors of twisted-mass fermions at four lattice spacings, with the finest being 0.049 fm.

Currently, no lattice calculation exists for IB effects in the pure LD regime, restricting the averaging of results to the isospin-symmetric case. Given the enhanced scale dependence of a_μ^{LD} compared to a_μ^{SD} and a_μ^W , and the fact that IB corrections primarily affect large Euclidean distances, a significant dependence on the scheme used for isoQCD cannot be excluded at present. To mitigate this and enable meaningful comparisons and averaging of lattice-QCD results, groups computing the light-connected contribution to a_μ^{LD} have performed their analyses in multiple schemes.

RBC/UKQCD-24 quote results in the RBC/UKQCD-18 scheme [14] as well as in the WP25 scheme, both of them using the Ω baryon mass M_Ω to set the scale. Mainz/CLS-24 provide results in a scheme defined in Ref. [20], which employs a combination of the pion and kaon decay constants, f_π and f_K , to set the scale, as well as in the WP25 scheme. Similarly, Fermilab/HPQCD/MILC-24 quote results in two schemes that differ only in the scale-setting quantity, either M_Ω or f_π , and also in the WP25 scheme. ETM-24 employ the scheme of the Edinburgh consensus based on f_π to set the scale.

While the values provided by the collaborations in the WP25 scheme are expected to match within the quoted uncertainties, the degree to which the collaboration-specific M_Ω - and f_π -based schemes differ has not yet been systematically studied. We recognize that the spread of the f_π -based schemes between Fermilab/HPQCD/MILC-24 and Mainz/CLS-24 is significant and needs to be scrutinized in future work. To account for this effect in WP25, we consider additional averages in the WP25 scheme with Fermilab/HPQCD/MILC-24 and Mainz/CLS-24 excluded, respectively. We then take half of the difference of the central value of those two averages as an additional uncertainty that is added in quadrature, following the approach adopted in WP20 for the BABAR–KLOE tension of the data-driven HVP evaluation. As a result, an average of the three results for the light-quark connected contribution can be robustly performed in the WP25 scheme. The results across all schemes are summarized in Table 10 and shown in Fig. 33 for the common WP25 scheme.

While all three results are statistically independent, they adopt similar approaches to correct for finite-size effects. RBC/UKQCD-24 apply the method developed by Hansen and Patella [420,421]. Mainz/CLS-24 employ the same method

Table 10

Results for the light-connected contribution $a_\mu^{\text{LD}}(ud)$ in units of 10^{-10} , obtained by three groups using different schemes for isoQCD. The third column lists results computed in the WP25 scheme. The remaining two columns show results in each collaboration's own scheme labeled by the quantity used for scale setting (see main text for details). We note that (except for WP25) schemes with the same scale setting used by different collaborations do not match exactly. Uncertainties are given in parentheses, where the first denotes statistical errors and the second systematic errors.

	N_f	WP25	Own scheme (M_Ω)	Own scheme (f_π)
FNAL/HPQCD/MILC-24 [30]	2+1+1	401.3(2.3)(3.1)	400.2(2.3)(3.7)	396.6(2.2)(3.3)
Mainz/CLS-24 [27]	2+1	411.4(4.8)(6.0)	–	420.8(4.1)(3.5)
RBC/UKQCD-24 [26]	2+1	411.4(4.3)(2.4)	413.6(6.0)(2.9)	–

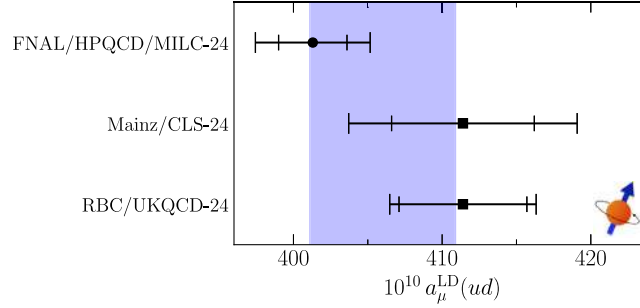


Fig. 33. Compilation of lattice results for $a_\mu^{\text{LD}}(ud)$ as listed in Table 10. The light blue band corresponds to the average given in Eq. (3.27). Error ticks and plotting symbols and colors have the same meaning as in Fig. 28.

for short Euclidean distances, complemented at larger distances by the Meyer–Lellouch–Lüscher (MLL) formalism, which uses a Gounaris–Sakurai parameterization of the timelike pion form factor [223,417,418].¹⁶ Both studies explicitly validate these corrections through measurements in multiple volumes with identical Lagrangian parameters. Fermilab/HPQCD/MILC-24, on the other hand, perform a model average over three different approaches to account for FV effects, including the MLL method. RBC/UKQCD-24, Mainz/CLS-24, and Fermilab/HPQCD/MILC-24 estimate the uncertainty for the light-connected contribution $a_\mu^{\text{LD}}(ud)$ arising from FV corrections to be 0.8, 1.7, and 1.4 units of 10^{-10} , respectively. To ensure a conservative assessment, we adopt these values as a correlated contribution to the systematic uncertainty across all three results.

When averaging according to the FLAG procedure, we apply a slight $\sqrt{\chi^2/\text{dof}}$ rescaling of 1.6 to account for the fact that the result from Fermilab/HPQCD/MILC-24 is significantly smaller than the other two. The p -value of the average is 0.2. The outcome of the partly correlated average, combined with the additional systematic uncertainty of 3.1×10^{-10} , is

$$a_\mu^{\text{LD}}(ud) = 406.0(4.9) \times 10^{-10}, \quad (3.27)$$

where the contribution of the uncertainty of w_0 that is part of the scheme definition is not included. This uncertainty originates from the determination of the physical value of the gradient flow scale $w_0 = 0.17236(70)$ fm, based on M_Ω in Ref. [16]. The average is displayed as a blue band alongside the three inputs in Fig. 33.

To compute the full LD window observable, the averaged result for the isovector contribution can be combined with the Mainz/CLS-24 determination of the isoscalar contribution within the WP25 scheme. The strange-connected contribution is included in the isoscalar part. Its determination in Mainz/CLS-24 is slightly larger than, but statistically compatible with, the result from ETM-24, where the two employed schemes are similar but not identical. The charm-connected contribution, computed in Mainz/CLS-24 and ETM-24, is numerically negligible at the current level of precision and has not been obtained in the WP25 scheme in either work. We note that this procedure uses the LD disconnected and strange contributions from only Mainz/CLS-24.

By adding the averaged isovector contribution, $a_\mu^{\text{LD}}(I1) = \frac{9}{10} a_\mu^{\text{LD}}(ud)$, to the isoscalar contribution, $a_\mu^{\text{LD}}(I0) = 42.5(1.8)_{\text{stat}}(1.5)_{\text{syst}}$ [27], one obtains

$$a_\mu^{\text{LD}}(\text{iso}) = 407.9(5.0) \times 10^{-10}, \quad (3.28)$$

for the LD window observable in isoQCD within the WP25 scheme, again without scale uncertainty.

¹⁶ A recent study finds that the MLL formalism is valid at all Euclidean distances and reproduces the result of Hansen–Patella by including sufficiently high-energy two-pion states [445].

Table 11

Single-flavor and disconnected contributions to $a_\mu^{\text{HVP, LO}}$, see also Fig. 34, in units of 10^{-10} . All results are given in the isospin-symmetric scheme of the individual groups, information on these schemes can be found in the associated references. Results which have been superseded by new calculations by the same collaboration are denoted with an asterisk.

	N_f	$a_\mu^{\text{HVP, LO}}(ud)$	$a_\mu^{\text{HVP, LO}}(s)$	$a_\mu^{\text{HVP, LO}}(c)$	$a_\mu^{\text{HVP, LO}}(\text{disc})$
FNAL/HPQCD/MILC-24 [30]	2+1+1	655.2(2.3)(3.9)	–	–	–
ETM-24 [28]	2+1+1	–	53.57(41)(48)	14.56(10)(9)	–
Mainz/CLS-24 [27]	2+1	675.7(4.1)(3.7)	54.5(3)(3)	14.4(2)(2)	–16.1(1.2)(1.2)
RBC/UKQCD-24 [26]	2+1	668.7(6.1)(2.9)	–	–	–
ABGP-22 [19]	2+1+1	646 (11.4)(7.8)	–	–	–
HPQCD-20 [446]	2+1+1	–	–	14.606(47)	–
LM-20 [17]	2+1+1	657(26)(12)	52.83(22)(65)	–	–
BMW-20 [16]	2+1+1	652.4(2.0)(5.3)	53.393(89)(68)	14.6(0)(1)	–15.4(1.2)(1.4)
ETM-18/19 [447,448]	2+1+1	629.1(13.7)	53.1(1.6)(2.0)*	14.75(42)(37)*	–
Mainz/CLS-19 [449]	2+1	674(12)(5)*	54.5(2.4)(0.6)*	14.66(45)(6)*	–23.2(2.2)(4.5)*
ABGP-19 [414]	2+1+1	659(22)*	–	–	–
FNAL/HPQCD/MILC-19 [450]	2+1+1	637.8(8.8)*	–	–	–
PACS-19 [451]	2+1	673(9)(11)	52.1(2)(5)	11.7(0.2)(1.6)	–
RBC/UKQCD-18 [14]	2+1	649.7(14.2)(4.9)*	53.2(4)(3)	14.3(0)(7)	–11.2(3.3)(2.3)
BMW-17 [452]	2+1+1	647.6(7.5)(17.7)*	53.73(4)(49)*	14.74(4)(16)*	–12.8(1.1)(1.6)*
HPQCD-14 [427]	2+1(+1)	–	53.41(0)(59)	14.42(0)(39)*	–

Table 12

Single-flavor and disconnected results of Table 11 in the WP25 isospin-symmetric scheme, see Eq. (3.9), in units of 10^{-10} . The BMW-20 results are duplicated for clarity.

	N_f	$a_\mu^{\text{HVP, LO}}(ud)$	$a_\mu^{\text{HVP, LO}}(s)$	$a_\mu^{\text{HVP, LO}}(c)$	$a_\mu^{\text{HVP, LO}}(\text{disc})$
FNAL/HPQCD/MILC-24 [30]	2+1+1	656.9(2.3)(3.2)	–	–	–
Mainz/CLS-24 [27]	2+1	666.2(4.9)(6.1)	53.56(21)(33)	–	–16.3(1.7)(1.7)
RBC/UKQCD-24 [26]	2+1	666.2(4.3)(2.5)	–	–	–
BMW-20 [16]	2+1+1	652.4(2.0)(5.3)	53.393(89)(68)	14.6(0)(1)	–15.4(1.2)(1.4)

3.5. Flavor decomposition

In this section, we summarize the results for flavor specific contributions to $a_\mu^{\text{HVP, LO}}$. Following the breakdown in Eq. (3.11), we consider the light-, strange-, and charm-quark connected contributions, followed by a discussion of the disconnected contribution. These results are obtained by considering the individual flavor contributions over the entire integration range. In principle, this corresponds to performing the integral of Eq. (3.12). However, in practice, the results are now more commonly obtained from summing the contributions to the three windows as in Eq. (3.16). This summed approach leverages the fact that each window contribution has a tailored analysis, resulting in a more robust determination, once inter-window correlations are taken into account [26,27,30]. Final averages are given in the WP25 isospin-symmetric scheme of Eq. (3.9). Where possible, results in figures are also presented in this scheme.

For the dominant light-quark connected component $a_\mu^{\text{HVP, LO}}(ud)$, which accounts for $\simeq 88\%$ of the total HVP contribution, there have been six new results since the release of WP20: BMW-20 [16], LM-20 [17], ABGP-22 [19], RBC/UKQCD-24 [26], Mainz/CLS-24 [27], and Fermilab/HPQCD/MILC-24 [30]. The first three are obtained from an analysis of the full integral, whereas the latter three come from summing the windows. These six new results are collected in the third column of Table 11, along with previous determinations that were included in WP20. All these new results (aside from LM-20) are updates on previous determinations, which we denote with an asterisk in the table. As discussed in Section 3.4.4, a central component of these newer calculations (aside from LM-20) is the use of exact low-modes of the Dirac operator, which is the primary reason for the dramatic improvement in statistical precision over older determinations. The four most precise determinations are all also available in the WP25 isospin-symmetric scheme, which we collect in Table 12.

For the strange and charm connected components, which are $\simeq 7\%$ and $\simeq 2\%$ of the total HVP contribution, respectively, there are four new determinations of $a_\mu^{\text{HVP, LO}}(s)$, from BMW-20 [16], LM-20 [17], Mainz/CLS-24 [27], and ETM-24 [28], and four new determinations of $a_\mu^{\text{HVP, LO}}(c)$, from BMW-20 [16], HPQCD-20 [17], Mainz/CLS-24 [27], and ETM-24 [28], since WP20. These are given in the fourth and fifth columns of Table 11. Again, aside from LM-20, these supersede previous determinations from the same groups. Finally, for the disconnected contribution $a_\mu^{\text{HVP, LO}}(\text{disc})$, given in the last column of Table 11, there have been two new results, BMW-20 [16] and Mainz/CLS-24 [27], both updates on previous determinations.

The results for each contribution are compared in Fig. 34, where we show superseded results with a gray marker.¹⁷ We stress here, as in Section 3.4.4, that these determinations, in particular $a_\mu^{\text{HVP, LO}}(ud)$ and $a_\mu^{\text{HVP, LO}}(s)$, are sensitive to how one defines the isospin-symmetric pure QCD scheme. Hence, when comparing results directly one should be aware of the particular scheme definitions, which are described in the caption and references of Tables 11 and 12.

¹⁷ The results of Refs. [416,419] are not included in the comparison, because they were already superseded in WP20 by newer results.

Table 13

Results for the individual flavor contributions to $a_\mu^{\text{HVP, LO}}$ (in the isospin-symmetric scheme of Eq. (3.9) in units of 10^{-10}). Also given is the total isospin-symmetric contribution to $a_\mu^{\text{HVP, LO}}$, which includes the $a_\mu^{\text{HVP, LO}}(b)$ contribution from Table 7. The first row (Avg. A) corresponds to directly averaging over the results of Tables 11 and 12 using the FLAG procedure. The second row (Avg. B) corresponds to combining the window averages of Sections 3.4.2–3.4.4.

	$a_\mu^{\text{HVP, LO}}(ud)$	$a_\mu^{\text{HVP, LO}}(s)$	$a_\mu^{\text{HVP, LO}}(c)$	$a_\mu^{\text{HVP, LO}}(\text{disc})$	$a_\mu^{\text{HVP, LO}}(\text{iso})$
Avg. A	659.5(4.7)	53.37(11)	14.576(68)	−15.7(1.5)	712.0(4.9)
Avg. B	661.1(5.0)	53.18(28)	14.37(11)	−16.4(2.3)	712.5(5.4)

We perform the world averages for each of the flavor contributions using two independent approaches. The first approach makes direct use of the results in Tables 11 and 12. For this, we follow the FLAG averaging procedure discussed at the end of Section 3.3. In particular, results that have been superseded are not included, as well as results based on less than three lattice spacings for dominant contributions and less than two for subdominant ones. Before performing the averages, the results that are not in the preferred isospin-symmetric scheme of Eq. (3.9) must be adjusted. To do this we follow the strategy laid out in Section 3.4.3.

For $a_\mu^{\text{HVP, LO}}(ud)$, the results from Fermilab/HPQCD/MILC-24 [30], RBC/UKQCD-24 [26], and BMW-20 [16] are available in the preferred scheme, which relies on M_Ω to determine w_0 in fm. The result from Mainz/CLS-24 [27] is not; however, the corresponding determination of $a_\mu^{\text{LD}}(ud)$ is available in the WP25 scheme, which can be combined with the relatively scale-insensitive short- and intermediate-distance contributions. All these WP25 determinations are given in Table 12. The remaining results, ABGP-22 [19], LM-20 [17], ETM-18/19 [447,448], and PACS-19 [451], all have significantly larger uncertainties and use f_π directly to set the scale or to determine w_0 in fm. Hence, for this average, we include only the four sub-percent results in the WP25 scheme. We account for correlations due to the shared approaches to correct for FV effects in the same fashion as in Section 3.4.4. In addition, we correlate the systematic uncertainty due to taste-breaking effects from BMW-20 and Fermilab/MILC/HPQCD-24, corresponding to 3.7 and 0.9 in units of 10^{-10} , respectively. The additional uncertainty of 3.1×10^{-10} from the spread of the f_π -based LD window results in Section 3.4.4 is also applied to $a_\mu^{\text{HVP, LO}}(ud)$ here.

For $a_\mu^{\text{HVP, LO}}(s)$, there are only two¹⁸ results available in the WP25 scheme (see Table 12), hence, following Section 3.4.3, for results not in this scheme we include an additional 0.9% and 0.2% systematic uncertainty (in quadrature) due to scale setting input and the spread in input values for M_K . These relative uncertainties are estimated using the derivatives $d \log a_\mu^{\text{HVP, LO}}(s)/d \log w_0 \simeq 1.8$ and $d \log a_\mu^{\text{HVP, LO}}(s)/d \log M_K^2 \simeq 0.6$ (Eq. (B.29) of Ref. [20]). As the charm contribution is dominated by $a_\mu^{\text{SD}}(c)$ and $a_\mu^{\text{W}}(c)$ we assign just the 0.2% systematic uncertainty obtained in Section 3.4.3.¹⁹ Finally, for the disconnected contribution, there are only two qualifying results obtained from a lattice calculation with more than one lattice spacing, BMW-20 [16] and Mainz/CLS-24 [27]. The BMW-20 result is already in the preferred scheme, whereas the Mainz/CLS-24 result can be obtained by the same procedure as described above for $a_\mu^{\text{HVP, LO}}(ud)$. This first set of flavor averages (Avg. A), as well as the corresponding result for $a_\mu^{\text{HVP, LO}}(\text{iso}) = a_\mu^{\text{HVP, LO}}(ud) + a_\mu^{\text{HVP, LO}}(s) + a_\mu^{\text{HVP, LO}}(c) + a_\mu^{\text{HVP, LO}}(b) + a_\mu^{\text{HVP, LO}}(\text{disc})$, is collected in the first row of Table 13. We note here that the $a_\mu^{\text{HVP, LO}}(s)$ average is largely dominated by the BMW-20 result, with a weight of 0.84. This is due to it being one of only two results already in the WP25 scheme, and already having smaller errors than all other determinations.

The second approach for obtaining world averages is to make use of the windowed averages obtained in Sections 3.4.2–3.4.4. For the short- and intermediate-distance windows, the separate flavor averages are all performed in the WP25 scheme. For the LD window, the light-quark contribution $a_\mu^{\text{LD}}(ud)$ of Eq. (3.27) is also obtained in the WP25 scheme. Due to the sensitivity of $a_\mu^{\text{LD}}(s)$ to the choice of scale-setting and strange-quark mass input, we take the sole determination in the WP25 scheme from Mainz/CLS-24 [27] as our window-“average” for this quantity.²⁰ The LD charm contribution, $a_\mu^{\text{LD}}(c)$, is almost negligible since it contributes less than 0.1% to the charm total. In addition, the sensitivity of $a_\mu^{\text{LD}}(c)$ to the choice of scale setting is suppressed by the large value of the ratio m_c/m_μ . To obtain $a_\mu^{\text{LD}}(c)$, we perform a FLAG average of the two determinations of this quantity from Mainz/CLS-24 [27] and ETM-24 [28]. For the LD disconnected contribution $a_\mu^{\text{LD}}(\text{disc})$, we take the only available determination of this quantity from Mainz/CLS-24 [27].

With all these results in hand, the sum in Eq. (3.16) can be performed for each flavor. Given the conservative choices made to account for correlations between groups in Sections 3.4.2 and 3.4.3, we perform an uncorrelated sum of the windows to obtain the window-summed averaged values for $a_\mu^{\text{HVP, LO}}(s)$ and $a_\mu^{\text{HVP, LO}}(c)$. For $a_\mu^{\text{HVP, LO}}(ud)$ and $a_\mu^{\text{HVP, LO}}(\text{disc})$,

¹⁸ The Mainz/CLS-24 $a_\mu^{\text{HVP, LO}}(s)$ result in the WP25 scheme is constructed in the same fashion as described for the $a_\mu^{\text{HVP, LO}}(ud)$ result, with an additional 0.3% relative uncertainty applied to the $a_\mu^{\text{W}}(s)$ component. This additional 0.3% uncertainty is estimated in Section 3.4.3.

¹⁹ For $a_\mu^{\text{HVP, LO}}(c)$ results obtained using $M_{J/\psi}$ as opposed to M_{D_s} as in Eq. (3.9), namely HPQCD-20, we include a 0.5% systematic uncertainty, estimated similarly as described for $a_\mu^{\text{HVP, LO}}(s)$.

²⁰ The other determination of this quantity from ETM-24 [28] uses f_π to set the scale as opposed to w_0 via M_Ω as in the WP25 scheme Eq. (3.9). The shift in $a_\mu^{\text{LD}}(s)$ from the two scale-setting choices can be estimated using the two results for $a_\mu^{\text{LD}}(s)$ in Ref. [27](Table 3 and Eq. (A.9)), giving $\Delta a_\mu^{\text{LD}}(s) \simeq 0.92 \times 10^{-10}$. Including this shift as an additional source of uncertainty on the result of Ref. [28] dramatically inflates the overall error and renders a FLAG average of the two results pointless.

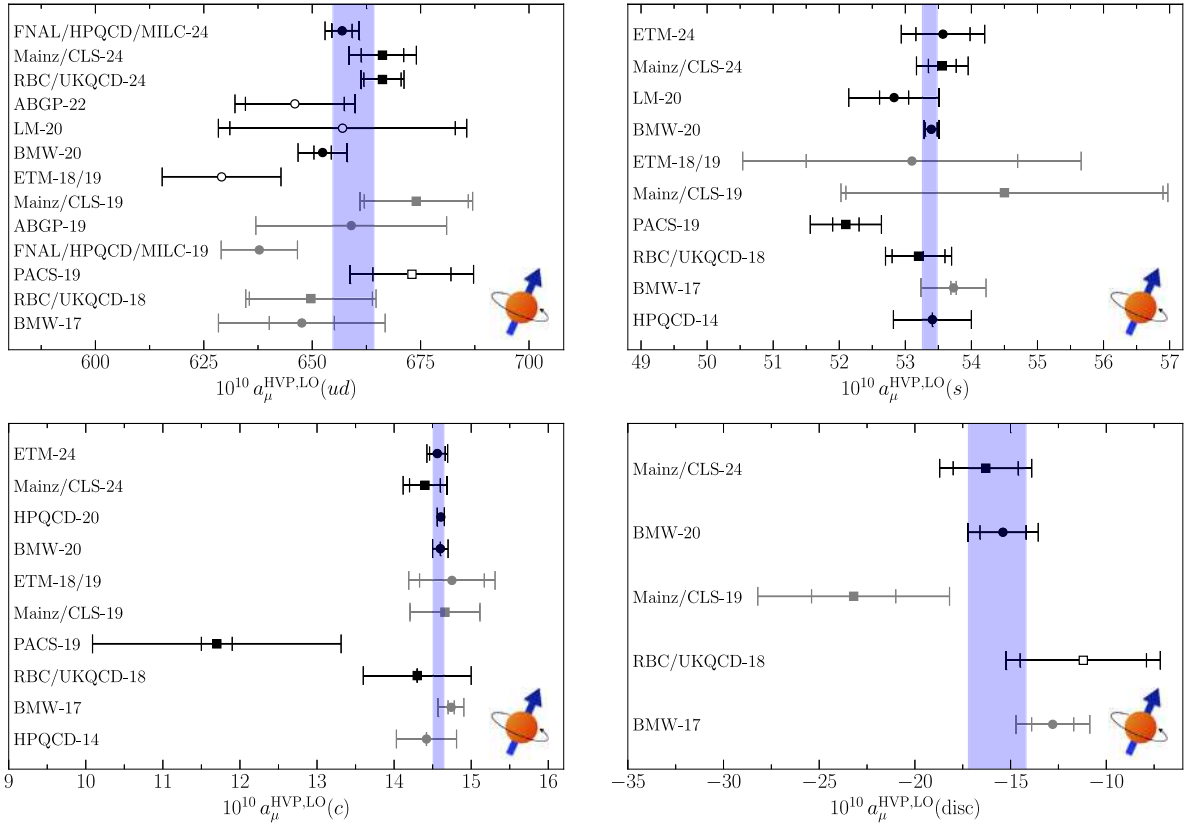


Fig. 34. Compilation of lattice results for the flavor contributions to $a_\mu^{\text{HVP, LO}}$. Upper-Left: light-quark connected $a_\mu^{\text{HVP, LO}}(ud)$. Upper-Right: strange-quark connected $a_\mu^{\text{HVP, LO}}(s)$. Lower-Left: charm-quark connected $a_\mu^{\text{HVP, LO}}(\text{disc})$. Lower-Right: quark disconnected $a_\mu^{\text{HVP, LO}}(\text{disc})$. Where possible, we display results in the WP25 isospin-symmetric scheme (Eq. (3.9)), corresponding to the results in Table 12. The light blue bands correspond to “Avg. A” in the first row of Table 13. Results not included in the average are denoted by unfilled symbols. Error ticks and plotting symbols and colors have the same meaning as in Fig. 28.

a systematic uncertainty that is common to all determinations of these quantities in the W and LD windows concerns the EFT-based approaches to correcting for FV effects. To account for this, we assume 100% correlation of FV uncertainty between the intermediate and LD windows. To obtain a correlation coefficient that corresponds to this assumption, we examine results from the three groups that have both $a_\mu^{\text{W}}(ud)$ and $a_\mu^{\text{LD}}(ud)$ determinations [20,22,26,27,29,30]. In particular, we compute the following correlation coefficient $\rho = \sigma_{\text{W}}^{\text{FV}} \sigma_{\text{LD}}^{\text{FV}} / (\sigma_{\text{W}} \sigma_{\text{LD}})$, where σ^{FV} is the uncertainty due to FV, for each case. A weighted average of $\rho = 0.1$ is obtained using the weights from the FLAG average in Eq. (3.27). This value is used for $a_\mu^{\text{HVP, LO}}(\text{disc})$ as well. The final window-summed averaged results (Avg. B) are given in the second row of Table 13. We note here that the Avg. B determination of $a_\mu^{\text{HVP, LO}}(\text{iso})$ neglects the nontrivial anti-correlations between the Mainz/CLS-24 results for $a_\mu^{\text{LD}}(ud)$ and $a_\mu^{\text{LD}}(\text{disc})$, which arise when obtaining these flavor contributions from linear combinations of $a_\mu^{\text{LD}}(10)$ and $a_\mu^{\text{LD}}(11)$ as in Ref. [27]. Therefore, the uncertainty on this determination of $a_\mu^{\text{HVP, LO}}(\text{iso})$ is less well estimated than that of Eq. (3.29) discussed in Section 3.6.

We observe that all values from this second average are broadly consistent with the averages from the first approach. The central value of $a_\mu^{\text{HVP, LO}}(ud)$ from Avg. A is slightly lower than Avg. B and its uncertainty is smaller. This is due to the LD contribution from BMW-20 playing a part in the first average but not in the second. For the same reason, the uncertainty in $a_\mu^{\text{HVP, LO}}(\text{disc})$ is significantly smaller in the first average.

Comparing the two averages further, we note that every flavor-specific average that enters Avg. A is based on three or more inputs, while Avg. B takes as inputs for $a_\mu^{\text{LD}}(s)$ and $a_\mu^{\text{LD}}(\text{disc})$ a result from a single lattice group. However, Avg. A excludes some of the most recent results for the a_μ^{SD} and a_μ^{W} contributions, particularly for the (ud) and (c) flavor contributions. We can consider a variation to Averages A and B, where (almost) all recent results are included and where all contributions are based on two or more inputs, by combining $a_\mu^{\text{HVP, LO}}(s)$ and $a_\mu^{\text{HVP, LO}}(\text{disc})$ from Avg. A with $a_\mu^{\text{HVP, LO}}(ud)$ and $a_\mu^{\text{HVP, LO}}(c)$ from Avg. B. This yields the value $a_\mu^{\text{HVP, LO}}(\text{iso}) = 713.5(5.2)$, which is virtually identical to Avg. B and to the average in Eq. (3.29) in Section 3.6.1. Other variations yield similar results.

3.6. Total HVP

In this section, we present our averages for the total HVP contribution to a_μ computed in lattice QCD. When combined with QED, EW, and the remaining hadronic contributions (also computed using lattice methods), these averages yield purely theoretical SM predictions for a_μ . We emphasize that most of the values from individual groups that comprise our averages have used blinding procedures, and we use the FLAG averaging procedure throughout.

Since the pioneering works of Blum [453] and Aubin and Blum [454], and the seminal paper of ETM in 2011 [455] in two-flavor QCD, many lattice calculations of $a_\mu^{\text{HVP, LO}}$ have been performed [14,16,19,22,24,26,27,416,419,447–449,451,452,456,457]. In addition to these, there are now many works giving the individual windows and other contributions as discussed in the previous two sections [17,18,20,21,23,25,28–30]. Hence, to obtain a final lattice prediction for $a_\mu^{\text{HVP, LO}}$, there are a number of ways to combine the lattice results from the various groups. These differ in the contributions to $a_\mu^{\text{HVP, LO}}$ that are averaged before being combined. Since groups do not necessarily compute all contributions, it matters in which order they are averaged and added up. Here we consider four different combinations:

1. *Total $a_\mu^{\text{HVP, LO}}$ from a combination of the averages of all available lattice results for the SD, W, and LD windows in the isospin-symmetric limit plus the average of the total IB corrections:* The advantage of this approach is that there are many results for some flavor components of the isospin-symmetric windows (see Table 8). This is particularly true for the ud contribution to the intermediate-distance window for which there are eleven results and, more generally, for all flavor contributions to that window. While this large number of computations allows a particularly robust estimate of the corresponding systematic errors, this intermediate-distance window contributes only about 35% to the total. There is also a fair number of computations of the SD window (see Table 6). For the LD window, which contributes 55%, the situation is less favorable. There are only three results [26,27,30], of which only the one from the Mainz group [27] includes all flavor and IB contributions, as shown in Table 10. More generally, the situation for the IB corrections to all windows is quite spotty. In addition, this averaging approach requires a common definition of the isoQCD limit for each of the windows and flavor contributions. Indeed, for the most important ud LD window, Mainz [27], RBC/UKQCD-[26], and Fermilab/HPQCD/MILC [30] have gone the extra mile to convert their results to the common WP25 scheme. We note that the shift required for results that are not available in the WP25 scheme includes an additional uncertainty, as discussed in Sections 3.4 and 3.5.
2. *Total $a_\mu^{\text{HVP, LO}}$ as a combination of lattice results for isospin-symmetric $a_\mu^{\text{HVP, LO}}(\text{iso})$, obtained via flavor decomposition, with an average of the total IB correction:* Here $a_\mu^{\text{HVP, LO}}(\text{iso})$ is obtained by summing the results of determinations of individual flavor contributions to that quantity. Section 3.5 considers two ways of obtaining those flavor contributions. In the first (approach 2A), the flavor components of $a_\mu^{\text{HVP, LO}}(\text{iso})$ are obtained from averages of the corresponding results by the different collaborations. That procedure may appear to be a rearrangement of approach 3 described below. However, for approach 3 there are only three computations of the total isospin-symmetric contribution to $a_\mu^{\text{HVP, LO}}$, while there are many more of the various flavor components of $a_\mu^{\text{HVP, LO}}(\text{iso})$ (see Table 11). The second way (approach 2B) of obtaining the flavor components of $a_\mu^{\text{HVP, LO}}(\text{iso})$ is by summing the averages of the individual flavor contributions to the SD, W, and LD windows. As explained in Section 3.5, this procedure, which uses the same inputs as approach 1, described above, and hence is very closely related to it, yields a slightly less-well quantified uncertainty.
From the two determinations of $a_\mu^{\text{HVP, LO}}(\text{iso})$ obtained in Section 3.5, two values of $a_\mu^{\text{HVP, LO}}$ are obtained by adding the average SIB and QED contributions to that quantity obtained in Section 3.6.1.
3. *Total $a_\mu^{\text{HVP, LO}}$ as a combination of an average of lattice results for isospin-symmetric $a_\mu^{\text{HVP, LO}}(\text{iso})$ and for the corresponding IB corrections:* This is similar to the first combination described in the preceding paragraph except that the isospin-symmetric average is taken over the totals from each group without averaging flavor-specific contributions as an intermediate step. Thus, one of the drawbacks compared to approach 2A is that there are fewer computations of the total isospin-symmetric contribution to $a_\mu^{\text{HVP, LO}}$ than there are of some individual flavor-specific contributions. However, there are no more computations of the ud contribution to the isospin-symmetric LD window [26,27,30], which overwhelms the uncertainty on $a_\mu^{\text{HVP, LO}}$, than of the total isospin-symmetric contribution [14,16,26,27]. Indeed, here the full isospin-symmetric result from BMW-20 [16] re-enters the average. Thus, the uncertainties in this averaging approach will not necessarily be larger than in approach 1. It is worth noting that the groups who have computed the total isospin-symmetric value are the same as those who have computed the complete $a_\mu^{\text{HVP, LO}}$, so that the overlap in input between this approach and approach 4 (below) is significant.
4. *Averaging lattice results for total $a_\mu^{\text{HVP, LO}}$:* The advantage of this approach is that all ambiguity surrounding different conventions used in the separation into strictly QCD results and QED and strong IB corrections is absent. Each collaboration can choose its preferred decomposition scheme or way of fixing the scale and tuning quark masses. The result obtained corresponds to the same quantity: $a_\mu^{\text{HVP, LO}}$ is a well-defined physical observable. This makes averaging results straightforward. The downside is that very few collaborations have computed all necessary contributions. The BMW [16] and Mainz [27] collaborations have done so. It is possible to obtain such a result from the RBC/UKQCD collaboration publications [14,22,26]. We note the latter two employed blinded analyses, at least for part of their calculation.

Table 14

IB contributions to $a_\mu^{\text{HVP, LO}}$ in units of 10^{-10} . The values in the second column are as published (in the collaboration's scheme), while the third is given in the WP25 scheme. They also include corrections for possibly omitted contributions but not the estimate, 2×10^{-10} , of the common uncertainty associated with the assumptions made about the LD behavior of the IB-correction correlators. As described in the main text, the latter is added linearly to the average IB contribution.

Collaboration	$\delta a_\mu^{\text{HVP, LO}}$	$\delta a_\mu^{\text{HVP, LO}}(\text{WP25}) _{\text{corr}}$	References
Mainz/CLS-24	−4.1(4.4)	5.2(4.4)	[27]
BMW-20	−0.2(1.3)	−0.2(1.4)	[16]
ETM-19	7.1(2.6)	−3.7(4.3)	[15]
RBC/UKQCD-18	9.5(10.4)	6.9(10.4)	[14]

In addition to the advantages and disadvantages of the averaging methods discussed above, other features can play an important role in the final uncertainties. In particular, because we perform the canonical PDG rescaling of uncertainties by $\sqrt{\chi^2/\text{dof}}$ in averages of results with slight tensions, for which $\chi^2/\text{dof} > 1$, the order in which different contributions are averaged and combined may significantly impact the final uncertainty. Such effects will be found below.

3.6.1. Result for $a_\mu^{\text{HVP, LO}}$ from the sum of averaged windows

Here we simply rely on Section 3.4, which provides the following averages for the SD, W, and LD windows in the isospin-symmetric limit: $a_\mu^{\text{SD}}(\text{iso}) = 69.06(22) \times 10^{-10}$, $a_\mu^{\text{W}}(\text{iso}) = 236.16(42) \times 10^{-10}$, and $a_\mu^{\text{LD}}(\text{iso}) = 407.9(5.0)$. These results are to be understood as given in the WP25 scheme for the isospin limit. Adding the contributions together without any correlations, we obtain the following result for $a_\mu^{\text{HVP, LO}}$ in the WP25 isospin limit:

$$a_\mu^{\text{HVP, LO}}(\text{iso}) \Big|_{\text{Avg. 1}}^{\text{lat}} = 713.1(5.0) \times 10^{-10}. \quad (3.29)$$

To that result, we have to add QED and SIB effects. We average the results of BMW-20 [16], RBC/UKQCD-18 [14], ETM-19 [15], and Mainz/CLS-24 [27]. We must first account for the following modifications to those effects and also consider that the IB corrections computed by Mainz/CLS-24, ETM-19, and RBC/UKQCD-18 are not given in the WP25 scheme.

To begin with, it should be noted that many of the correlation functions that enter the computation of those effects are very noisy at distances above $t \simeq 2$ fm, with details depending on the specific effect considered or on the approach used. Thus, all collaborations make assumptions about the behavior of those correlators at long distances. At such separations, the QED corrections are dominated by the $\pi^\pm\pi^0$ mass difference. There are also small contributions from $\pi\pi\gamma$ and $\pi^0\gamma$ states as well as from ρ – ω mixing. For Euclidean times $t \geq 2.8$ fm, those effects integrate to $-2.0(1) \times 10^{-10}$ according to the phenomenological evaluation of Ref. [247].²¹ That time range approximately corresponds to the region above which the QED and SIB correlators are set to zero in Ref. [16]. More generally, it is important to account for the uncertainties associated with the assumptions made by the various collaborations. To do so we will add the absolute value of this phenomenological estimate to the uncertainty on the average of the determinations, without shifting the central value. Since that correction is only an estimate, we choose to add it linearly. It should be noted that this effect is not relevant for the hybrid BMW/DMZ-24 result [24] discussed in Section 4.2, because in that calculation e^+e^- and τ spectral functions are used to compute the contribution for $a_\mu^{\text{HVP, LO}}$ for $t \geq 2.8$ fm.

In Ref. [24] a small mistake in the sea–sea QED contribution to $a_\mu^{\text{HVP, LO}}(ud)$ of BMW-20 was found, as described in Sec. 7.3 of that reference. The corresponding correction and its uncertainty are tiny compared to the total 5.5×10^{-10} uncertainty quoted for $a_\mu^{\text{HVP, LO}}$ in Ref. [16]. Once propagated in a correlated manner to the total QED contribution, its net effect is to reduce the central value of the BMW-20 result for that contribution by 0.64×10^{-10} . It would also reduce its squared uncertainty by 0.45×10^{-21} . However, because BMW/DMZ-24 is not yet published, we add to the uncertainty the absolute value of the central-value shift in quadrature. Taking all of those changes into account, the sum of the total QED and SIB corrections for BMW-20 that we use here is $-0.2(1.5) \times 10^{-10}$.

Turning to RBC/UKQCD-18, the total QED and SIB correction given in Ref. [14] is $9.5(10.4) \times 10^{-10}$ in their isospin-decomposition scheme. Many QED and SIB corrections are computed, except for the quark-disconnected SIB one and for a number of quark-disconnected and all sea QED ones. BMW [16] calculated these to be $-4.67(54)(69) \times 10^{-10}$ for the former and $-0.40(80) \times 10^{-10}$ for the latter (including the small correction and the error increase mentioned above). The quark-disconnected SIB result is also consistent with an estimate from partially quenched ChPT of $-6.9(3.5) \times 10^{-10}$ [17]. We note that the suppression of the disconnected SIB is not as pronounced as in the isospin-symmetric case. This can be understood from partially quenched ChPT [17], where the cancellation of two-pion connected and disconnected SIB contributions is manifest at LO. The partially quenched ChPT argument, together with the significant cancellation

²¹ The breakdown of the different sources of IB is provided in Table 4, in particular, the dependence of the dominant LD effect due to the pion mass difference (Δ_π) on the cut in Euclidean time t_2 is shown in Fig. 22. The full estimate for $t_2 = 2.8$ fm roughly decomposes as $-2.0 \simeq -2.5_{\Delta_\pi} + 0.1_{\pi^0\gamma} + 0.3_{\text{FSR } (2\pi)} + 0.1_{\rho-\omega}$.

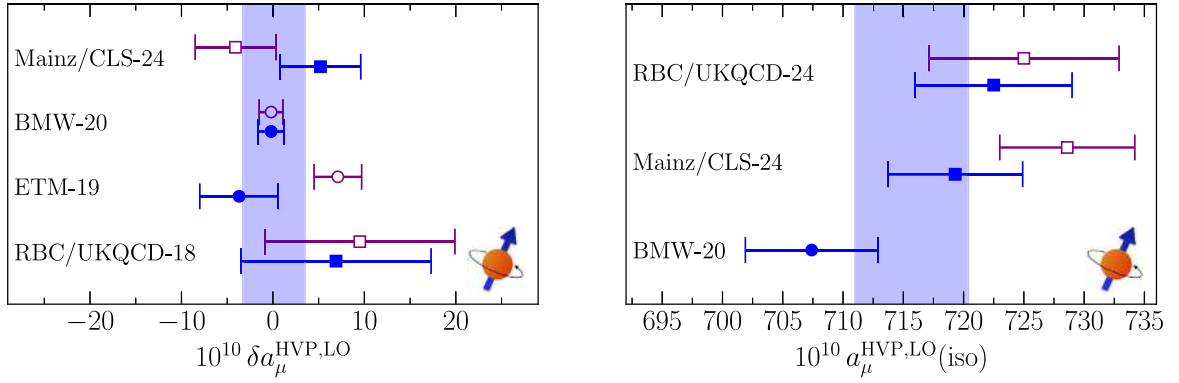


Fig. 35. Total IB (left) and isospin-symmetric (right) contributions to $a_\mu^{\text{HVP}, \text{LO}}$. On the left both corrected (blue, filled symbols) and uncorrected (purple, unfilled symbols) results, as described in the text, are shown. The blue band corresponds to the average of the corrected values in the WP25 scheme. On the right results are shown in original (purple, unfilled symbols) and WP25 schemes (blue, filled symbols). The light blue band is the average in the WP25 scheme. Plotting symbols have the same meaning as in Fig. 28.

of quark-connected and quark-disconnected SIB contributions calculated on the lattice [16,458], stresses the importance of quark-disconnected SIB for future computations of $a_\mu^{\text{HVP}, \text{LO}}$. In Ref. [14], an error of 1.1×10^{-10} is ascribed for the neglect of the SIB correction and of 0.3×10^{-10} for the QED ones. While the combination of these errors may appear quite generous, it does not cover the corresponding large shift in central value. Thus, we add to RBC/UKQCD's result the shift $-5.1(1.2) \times 10^{-10}$ as well as the difference between the central values of the ud contribution in the WP25 and RBC/UKQCD-18 schemes, i.e., 2.5×10^{-10} [26]. Finally, we arrive at $\delta a_\mu^{\text{HVP}, \text{LO}}(\text{RBC/UKQCD-18}) = 6.9(10.4) \times 10^{-10}$.

We include in the average the ETM-19 calculation of IB corrections [15], where quark connected SIB and QED contributions are computed for light, strange, and charm quarks in a quenched QED setup. ETM-19 includes an extra uncertainty of 1.2×10^{-10} for the missing disconnected SIB and QED contributions, as well as the sea QED contributions. According to the BMW-20-based determination discussed above, these neglected corrections amount to $-5.6(1.2) \times 10^{-10}$ and are not covered by ETM-19's extra uncertainty. Thus, after removing that uncertainty, we add to ETM-19's result, $7.1(2.6) \times 10^{-10}$, the determination of the neglected contribution and combine uncertainties in quadrature. While Refs. [24,459] provide sufficient evidence that the differences between the Gasser–Rusetsky–Scimemi (GRS) [460] and WP25 schemes are negligible as far as the hadronic inputs to extract quark masses are concerned, ETM-19's GRS scheme is implemented employing the pion decay constant to set the scale. To account for the scheme mismatch, we use the Fermilab/HPQCD/MILC-24 [30] computation of the correlated difference of the isospin-symmetric total HVP [29,30] between a scheme based on f_π , close to the ETM one, and the one used for our averages. This difference amounts to $-5.2(3.3) \times 10^{-10}$, which we also add to the ETM-19 estimate, combining uncertainties in quadrature. All the above corrections yield $\delta a_\mu^{\text{HVP}, \text{LO}}(\text{ETM-19}) = -3.7(4.4) \times 10^{-10}$.

For Mainz/CLS-24, we get the total QED plus SIB correction in their scheme directly from Ref. [27]. It is $-4.1(4.4) \times 10^{-10}$. Most QED and SIB corrections were computed, except for a number of quark-disconnected and sea QED ones, which were either taken into account using scalar QED [27,458] or included via an additional systematic error estimate. It is worth noting that Mainz does not account for the renormalization of the lattice spacing due to QED effects. This effect is believed to be negligible compared to their quoted total error. We must now convert this result to the WP25 scheme. We do the latter by adding the difference between the central values of the LD window in the WP25 and Mainz/CLS-24 schemes, i.e., 9.3×10^{-10} [27]. Thus, in the WP25 scheme we obtain $\delta a_\mu^{\text{HVP}, \text{LO}}(\text{Mainz/CLS-24}) = 5.2(4.4) \times 10^{-10}$.

Before averaging we have to account for correlations between BMW-20, ETM-19, and RBC/UKQCD-18 which arise through the corrections applied to the latter two for missing QED and SIB contributions. The uncertainty associated with the SIB contribution is 0.9×10^{-10} for BMW-20 and 1.1×10^{-10} for RBC/UKQCD-18. For the sea QED corrections, the value is 0.8×10^{-10} for the two results. For ETM-19 the uncertainty is 1.2×10^{-10} for both. We can now average the total QED plus SIB correction from Mainz/CLS-24, BMW-20, ETM-19, and RBC 18. These are summarized in Table 14 and shown in the left panel of Fig. 35. We find $0.1(1.4) \times 10^{-10}$ with a $\chi^2/\text{dof} = 2.6/3$. As discussed above, to account for the uncertainties associated with assumptions made concerning the LD behavior of lattice IB-correction correlators, we add a 2.0×10^{-10} uncertainty linearly to that average. Thus, we obtain

$$\delta a_\mu^{\text{HVP}, \text{LO}} \Big|_{\text{Avg.}}^{\text{lat}} = 0.1(3.4) \times 10^{-10}. \quad (3.30)$$

Adding these IB corrections to the isospin-symmetric results of Eq. (3.29), we obtain:

$$a_\mu^{\text{HVP}, \text{LO}} \Big|_{\text{Avg.}}^{\text{lat}} = 713.2(6.1) \times 10^{-10}. \quad (3.31)$$

Table 15

Values corresponding to the world averages in Tables 7, 8 and 13 and Eqs. (3.21), (3.22), (3.25), (3.26), (3.29), (3.31) and (3.32) with the inclusion of the as-of-yet-unpublished BMW/DMZ-24 [24] results.

Contribution	(<i>ud</i>)	(<i>s</i>)	(<i>c</i>)	(disc)	(iso)	Total
a_{μ}^{SD} [BMW-20→BMW/DMZ-24]	48.063(73)	9.094(14)	11.57(10)	−0.0020(21)	69.02(20)	69.06(24)
a_{μ}^{W} [BMW-20→BMW/DMZ-24]	206.89(37)	27.261(68)	2.810(56)	−0.995(59)	235.97(38)	236.39(39)
$a_{\mu}^{\text{HVP, LO}} \Big _{\text{Avg. 1}}^{\text{lat}}$ [BMW-20→BMW/DMZ-24]	—	—	—	—	712.9(5.0)	713.0(6.1)
$a_{\mu}^{\text{HVP, LO}} \Big _{\text{Avg. 2B}}^{\text{lat}}$ [BMW-20→BMW/DMZ-24]	661.0(5.0)	53.16(28)	14.39(11)	−16.5(2.3)	712.3(5.4)	712.4(6.4)

In addition to the IB estimates averaged here, we refer to an independent estimate of the SIB for the SD and W contributions recently provided by Fermilab/HPQCD/MILC-24 [29] and their ongoing calculation of the QED effects [461]. Alternative formulations beyond QED_L [462] and QED_{∞} [14,463,464] implemented in the estimates discussed in detail here, include a novel IR-improved QED action QED_r [465,466], simulating at nonzero photon mass m_{γ} and taking the limit $m_{\gamma} \rightarrow 0$ [467–469], and a local, gauge invariant formulation with C^* boundary conditions [470], which has recently been used to compute HVP in an unphysical setup [471].

3.6.2. Result for $a_{\mu}^{\text{HVP, LO}}$ from the sum of the individual flavor components of $a_{\mu}^{\text{HVP, LO}}(\text{iso})$ and the average isospin-breaking corrections

Here we rely on the results of Section 3.5 for the determination of $a_{\mu}^{\text{HVP, LO}}(\text{iso})$, in the WP25 scheme, from averages of its individual flavor components. These are obtained in two ways. In the first, the flavor components of $a_{\mu}^{\text{HVP, LO}}(\text{iso})$ are obtained from averages of the corresponding results by the different collaborations. The results that enter these averages are collected in Table 11 and plotted in Fig. 34. Their averages are summarized in the first row of Table 13. In that same row is given the result $a_{\mu}^{\text{HVP, LO}}(\text{iso}) = 712.0(4.9) \times 10^{-10}$, obtained by combining those averages.

The second way of obtaining the flavor components of $a_{\mu}^{\text{HVP, LO}}(\text{iso})$ is by summing averages of the individual flavor contributions to the SD, W, and LD windows computed in Sections 3.4.2–3.4.4. This procedure may appear to be merely a rearrangement of that of approach 1 above, where $a_{\mu}^{\text{HVP, LO}}(\text{iso})$ is obtained by first summing the flavor contributions window-by-window and then adding the three windows together. They are clearly closely related. However, there are some differences. These arise from taking as only result for $a_{\mu}^{\text{LD}}(s)$ the one from Ref. [27], from including a very small charm contribution to a_{μ}^{LD} obtained by averaging the results of Mainz/CLS-24 [27] and ETM-24 [28], and from considering the disconnected contribution to $a_{\mu}^{\text{LD}}(\text{iso})$ instead of the iso-singlet one. Nevertheless, because those differences are small, it is not surprising that the resulting value for $a_{\mu}^{\text{HVP, LO}}(\text{iso})$ given in the third row of Table 13, $712.5(5.4) \times 10^{-10}$, is close to the result in Eq. (3.29). We note that the central values of the two averages are very close to each other.

Now we combine each of these determinations of $a_{\mu}^{\text{HVP, LO}}(\text{iso})$ from averages of its individual flavor components with the average of the IB corrections computed in Section 3.6.1 and given in Eq. (3.30). Using the first determination we obtain

$$a_{\mu}^{\text{HVP, LO}} \Big|_{\text{Avg. 2A}}^{\text{lat}} = 712.1(6.0) \times 10^{-10}, \quad (3.32)$$

and for the second,

$$a_{\mu}^{\text{HVP, LO}} \Big|_{\text{Avg. 2B}}^{\text{lat}} = 712.6(6.4) \times 10^{-10}. \quad (3.33)$$

These two results are virtually identical to $a_{\mu}^{\text{HVP, LO}} \Big|_{\text{Avg. 1}}^{\text{lat}}$, which is also true of the alternate average considered in Section 3.5.

As noted in Sections 3.4 and 3.5, because Ref. [24] is, at the time of the writing of WP25, not yet published, the BMW/DMZ-24 results obtained therein are not included in any of the averages reported there, and hence also not included in the averages of this section and Section 3.6.1. For completeness, we list in Table 15 values for the averages from Sections 3.4, 3.5 and 3.6.1 as well as this section that include the corresponding BMW/DMZ-24 results. In particular, Table 15 lists BMW/DMZ-24-inclusive averages for the a_{μ}^{SD} and a_{μ}^{W} observables of Tables 7 and 8 and Eqs. (3.21), (3.22), (3.25) and (3.26) and the $a_{\mu}^{\text{HVP, LO}}$ results of Table 13 and Eqs. (3.29), (3.31) and (3.32). The corresponding BMW-20 [16] results are then excluded as inputs as they are then superseded by BMW/DMZ-24.

3.6.3. Result for $a_{\mu}^{\text{HVP, LO}}$ from the sum of the averages for the isospin-symmetric total and isospin-breaking contributions

Table 16 lists the results for the total isospin-symmetric contribution to $a_{\mu}^{\text{HVP, LO}}$ that we consider here. These are also shown in the right panel of Fig. 35. These are from the BMW, Mainz, and RBC/UKQCD collaborations whose results enter the average of approach 4 and whose calculations are discussed in more detail in Section 3.6.4. We perform a weighted average of those values in the WP25 scheme, while accounting for the correlations among the FV corrections in the Mainz/CLS-24 and RBC/UKQCD-24 computations. We obtain

$$a_{\mu}^{\text{HVP, LO}}(\text{iso}) \Big|_{\text{Avg. 3}}^{\text{lat}} = 715.7(4.7) \times 10^{-10}, \quad (3.34)$$

Table 16

Total isospin-symmetric contribution to $a_\mu^{\text{HVP, LO}}$ in units of 10^{-10} in the collaboration's original isospin-decomposition scheme (second column) and in the WP25 scheme (third column).

Collaboration	$a_\mu^{\text{HVP, LO}}(\text{iso})$	$a_\mu^{\text{HVP, LO}}(\text{iso, WP25})$	References
BMW-20	–	707.4 (5.5)	[16]
Mainz/CLS-24	728.6 (5.6)	719.3(5.6)	[20,23,27]
RBC/UKQCD-24	725.0 (7.9)	722.5 (6.5)	[14,22,26]

Table 17

Total HVP contribution to a_μ in units of 10^{-10} . The values in the second column are as published, while the third includes corrections for omitted contributions as described in the text, except for Ref. [27] that is left unchanged. Not included is the estimate, 2×10^{-10} , of the common uncertainty associated with the assumptions made about the LD behavior of the IB-correction correlators. As described in Section 3.6.1, the latter is added linearly to the average of corrected results.

Collaboration	$a_\mu^{\text{HVP, LO}}$	$a_\mu^{\text{HVP, LO}} _{\text{corr}}$	References
BMW-20	707.5 (5.5)	707.2 (5.5)	[16]
Mainz/CLS-24	724.5 (7.1)	–	[20,23,27]
RBC/UKQCD-24	734.5 (13.0)	729.4 (13.0)	[14,22,26]

with $\chi^2/\text{dof} = 3.8/2$, so the quoted error has been enlarged by a factor of 1.4. Taking the IB correction from Eq. (3.30) computed for Avg. 1, we arrive at

$$a_\mu^{\text{HVP, LO}} \Big|_{\text{Avg. 3}}^{\text{lat}} = 715.8(5.8) \times 10^{-10}, \quad (3.35)$$

which is slightly larger, but consistent, with the averages in the previous two subsections.

3.6.4. Averaging lattice results for total $a_\mu^{\text{HVP, LO}}$

Here we focus on the computations that include all flavor contributions and at least the dominant QED and SIB corrections (for details about the less recent ones see Section 3.5 and Ref. [1]). We also require that the precision of the isoQCD contribution be around one percent. This makes them suitable for a comparison with data-driven determinations of $a_\mu^{\text{HVP, LO}}$ and, as input to the SM prediction, with the measurement of a_μ [5–7]. We now discuss the three calculations that satisfy those criteria and that we use to obtain our final lattice average of $a_\mu^{\text{HVP, LO}}$. These are presented in Table 17 and shown in Fig. 36.

The BMW collaboration was the first to reach the precision goal in 2020, quoting a 0.8% total uncertainty [16]. The 31 gauge ensembles used are obtained with $N_f = 2 + 1 + 1$ flavors of stout-smear staggered fermions, quark masses closely bracketing their physical values, and six lattice spacings. The calculation includes all LO QED and SIB corrections. FV corrections are determined via dedicated simulations on $L = 11$ fm lattices. In addition to the modifications to the QED and SIB corrections computed in BMW-20 and discussed in Section 3.6.1, the subtraction of the very small one-photon-reducible contribution, $0.321(11) \times 10^{-10}$, performed in BMW-20 was not required and thus should be removed.

Recently, Mainz reported results for the LD window [27], which they combined with earlier results for the SD [23] and intermediate [20] windows to give a final prediction for $a_\mu^{\text{HVP, LO}}$. They quote a prediction for $a_\mu^{\text{HVP, LO}}$ with a total relative error of 1.0%. Their results are obtained from 34 ensembles with $N_f = 2 + 1$ flavors of $\mathcal{O}(a)$ -improved fermions, four of which have near-physical light quark masses. Their simulations provide them with six lattice spacings. QED and SIB corrections are explained in Section 3.6.1.

The third collaboration, whose work alone can provide a determination of $a_\mu^{\text{HVP, LO}}$, is RBC/UKQCD. They were the first collaboration to present a determination of the connected ud contribution to the LD window in the isospin-symmetric limit [26]. By combining that result with an earlier determination of the connected ud contribution to the SD and intermediate windows [22], they obtain a prediction for that contribution to the full $a_\mu^{\text{HVP, LO}}$ [26], denoted here $a_\mu^{\text{HVP, LO}}(ud)$. In turn, that prediction can be combined with RBC/UKQCD's 2018 computation of the quark-disconnected, strange, charm, QED, and SIB contributions [14] to obtain a result for $a_\mu^{\text{HVP, LO}}$. In that process, the QED and SIB corrections of RBC/UKQCD-18 must be supplemented by missing contributions, as discussed in Section 3.6.1. Their prediction for $a_\mu^{\text{HVP, LO}}(ud)$ is based on ten $N_f = 2 + 1$ domain-wall fermion ensembles at three lattice spacings, of which four are generated with masses of light quarks around their physical values [22,26]. Their results for the strange-quark connected contribution are obtained from two ensembles with different lattice spacings [14] and, for the charm, eight ensembles with three spacings. Both determinations involve two ensembles with physical pion masses. For the light- and strange-quark disconnected contributions [362], as well as for QED and SIB corrections, one ensemble with the larger lattice spacing and physical masses is used.

These three results are given in Table 17 and shown in Fig. 36, with and without the applied correction factors discussed in Section 3.6.1. We are now almost ready to combine the corrected results of BMW-20, Mainz/CLS-24, and RBC/UKQCD-24.

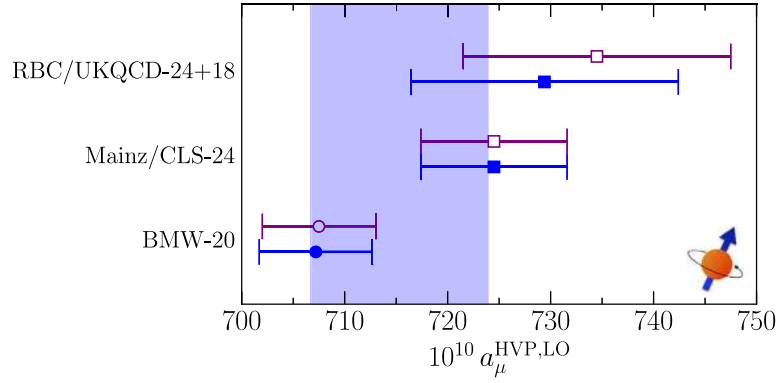


Fig. 36. LO HVP contribution to a_μ . Results shown are published (purple, unfilled symbols) and with corrections (blue, filled symbols) as explained in the text. The vertical band displays the weighted average of the corrected results as given in Eq. (3.36) (note that the Mainz/CLS-24 result has not been corrected). Plotting symbols have the same meaning as in Fig. 28.

First, we must discuss the correlations among them. Mainz/CLS-24 and RBC/UKQCD-24 employ very similar models for FV corrections. By far the largest contribution to those corrections concerns the $I = 1$ contribution to the LD window. The way in which the models are used differs in the two calculations, but both their physical-mass simulations are performed on $L \simeq 6$ fm lattices. Thus, we fully correlate the uncertainties on the collaborations' estimates of FV corrections for the LD window, i.e., 1.5×10^{-10} for Mainz and 0.8×10^{-10} for RBC/UKQCD. Correlations arising from IB corrections between BMW-20 and RBC/UKQCD-24 are also included, see Section 3.6.1.

Putting all of those ingredients together, we obtain the average $715.3(6.6) \times 10^{-10}$ for $a_\mu^{\text{HVP, LO}}$. This average carries a $\chi^2/\text{dof} = 5.0/2$, leading to a rescaling of its error by 1.6. While this rescaling factor is not excessively large, it applies to the total uncertainty on $a_\mu^{\text{HVP, LO}}$, instead of being applied to only the sub-components that are in tension, as in the three other combination procedures described above. Since the uncertainty on the total $a_\mu^{\text{HVP, LO}}$ is obviously larger than those on its sub-components, a similar rescaling factor has a larger impact, in absolute terms, on the final uncertainty of this particular combination of results than on the others. For example, the current average makes use of the same set of results as approach 3, except for the addition of ETM-19 [422] in the determination of $\delta a_\mu^{\text{HVP, LO}}|_{\text{Avg.}}^{\text{lat}}$ in Eq. (3.30). The lesser precision here comes from the fact that the tensions between different lattice inputs, which are strongest between respective isospin-symmetric results, also induce a rescaling of the large uncertainties associated with IB corrections, while they only induce a rescaling of the uncertainty on the isospin-symmetric averages in approach 3. This is an example that shows that the order in which averages are combined can have a significant impact on the final uncertainty.

Now, the uncertainty on the above average does not take into account the 2.0×10^{-10} uncertainty associated with the assumptions made on the LD behavior of the IB-correction correlators discussed in Section 3.6.1. Following that discussion, we add this uncertainty linearly to the one of the average, leading to:

$$a_\mu^{\text{HVP, LO}}|_{\text{Avg. 4}}^{\text{lat}} = 715.3(8.6) \times 10^{-10}. \quad (3.36)$$

This result also carries a significantly larger uncertainty than those of the first two approaches, described in Sections 3.6.1 and 3.6.2, which is due to the fact they include many more lattice inputs.

3.6.5. Summary of lattice world averages for $a_\mu^{\text{HVP, LO}}$

Table 18 collects the main results of this “Total HVP” section, and they are also shown in Fig. 37. While a few of the lattice inputs enter all five averages, the overlap among the inputs used in the different approaches varies somewhat. Averages 1 and 2B use the exact same lattice inputs, while the overlap among the inputs used in Avgs. 1, 2B, and 2A is considerable. Similarly, Avgs. 3 and 4 are based on almost the same lattice inputs. We note that the small tensions observed among results from different collaborations for some components of $a_\mu^{\text{HVP, LO}}$ contribute differently depending on the order in which they are combined. Nevertheless, these different averages and combinations are remarkably similar, well within quoted uncertainties, demonstrating the general consistency of the lattice results.

We note that Avgs. 1 and 2B include all of the most recent lattice inputs and thus information about $a_\mu^{\text{HVP, LO}}$, where the windowed sum of Avg. 1 yields a slightly more reliable uncertainty estimate. Average 2A, obtained from a flavor decomposition, includes some different inputs that yield better consolidation for some sub-components, see Section 3.5. We also note that Section 3.5 discusses an alternate average that mixes the inputs in Avgs. 2A and 2B to yield a virtually identical result to Avg. 1. Incidentally, the Avg. 1 central value and error are close to being medians for the five results listed in Table 18. In summary, we take $a_\mu^{\text{HVP, LO}}|_{\text{Avg. 1}}^{\text{lat}}$ as our final result for the WP25 consolidated lattice HVP average. Hence, quoting Eq. (3.31), we obtain

$$a_\mu^{\text{HVP, LO}}|_{\text{WP25}}^{\text{lat}} = 713.2(6.1) \times 10^{-10}, \quad (3.37)$$

Table 18

Summary of lattice weighted averages for $a_\mu^{\text{HVP, LO}}$ in units of 10^{-10} ; “1” from the sum of averages of the isospin-symmetric SD, intermediate, and LD windows from Sections 3.4.2–3.4.4 plus our average of IB corrections; “2A” using averages of flavor components of $a_\mu^{\text{HVP, LO(iso)}}$ obtained in Section 3.5 plus that same IB correction average; “2B” using a flavor decomposition of the windows implemented in Section 3.5 plus the same IB correction; “3” from our average of the total $a_\mu^{\text{HVP, LO(iso)}}$ values plus the IB correction average; and “4” is our average of total $a_\mu^{\text{HVP, LO}}$ values.

Avg. ID	$a_\mu^{\text{HVP, LO}} _{\text{Avg. ID}}^{\text{lat}}$	
1	713.2(6.1)	Eq. (3.31)
2A	712.1(6.0)	Eq. (3.32)
2B	712.6(6.4)	Eq. (3.33)
3	715.8(5.8)	Eq. (3.35)
4	715.3(8.6)	Eq. (3.36)

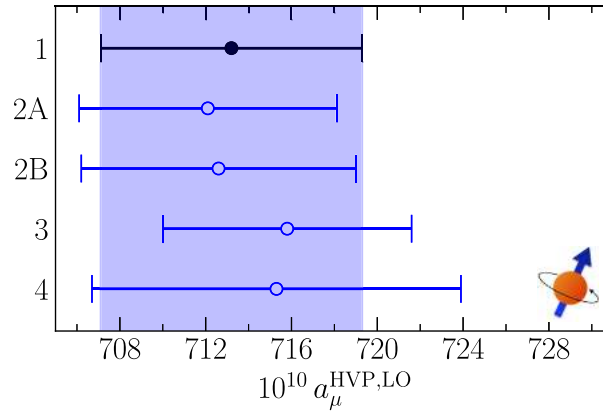


Fig. 37. Weighted averages of the total HVP contribution to a_μ for the various averaging procedures described in the text, see Table 18 for more details. The final quoted value for $a_\mu^{\text{HVP, LO}}$ is denoted by the black, filled circle; results for the other averaging procedures are shown as blue, unfilled symbols.

which has a precision of 0.9%. This result is a testimony to the significant progress accomplished by lattice computations of $a_\mu^{\text{HVP, LO}}$ since WP20 [1].

3.7. Other observables

3.7.1. Isospin-breaking corrections for τ data

Dispersive determinations of $a_\mu^{\text{HVP, LO}}$ based on τ data, specifically on vector-mediated decays of the τ lepton into final hadronic states, have been proposed long ago [187]. In the limit of exact isospin symmetry the underlying charged spectral density agrees with the corresponding neutral one, obtained from e^+e^- data, but given the high-precision target of $a_\mu^{\text{HVP, LO}}$, accurate experimental determinations of the differential rates together with a precise control of IB effects are necessary. These aspects are discussed, from a phenomenological point of view, in Section 2.3 with a specific focus on the dominant 2π channel. In this problem, lattice QCD+QED simulations can play an important role in providing solid first-principle results as proposed in Ref. [245], where the difference of the (fully) inclusive spectral densities in the charged and neutral channels has been first examined, see also Section 2.3.5.

Contrary to the calculation of IB effects for HVP, the presence of electric charge in initial and final states severely complicates the problem. Setting aside the technicalities on the definition of QED on a torus, an important challenge arising here is in the IR divergences. By studying the contributions to the decay rate, rather than the individual Feynman diagrams, it is possible to isolate three classes of IR safe contributions, as reported in Ref. [472], amounting to factorized initial- and final-state effects, and mixed nonfactorizable contributions (where a photon is exchanged between the lepton and the hadronic system). For the first, thanks to the perturbative nature of QED, pure analytic control is achieved [195,196], see also Section 2.3, and progress in defining a factorized correction has been recently reported [473].

The focus of Ref. [245] was instead on the final-state effects, with the intent to study them in a fully inclusive manner by re-using QED and SIB correlation functions, already generated for the study of IB effects of the (Euclidean lattice) HVP contribution. By splitting the ud EM current j_μ into its two isospin components, j_k^I (with $I = 0, 1$ and $I_3 = 0$), we introduce the Euclidean-time correlators $G_{II'}(x_0)$ involving two isospin-projected currents [245]

$$G_{II'}(x_0) \equiv -\frac{1}{3} \sum_{k=1}^3 \int d^3x \langle j_k^I(x_0, \mathbf{x}) j_k^{I'}(0) \rangle, \quad (3.38)$$

such that the two-point function used for the leading lattice HVP calculation obeys the relation

$$C(x_0) \equiv G_{00}^\gamma(x_0) + 2G_{01}^\gamma(x_0) + G_{11}^\gamma(x_0). \quad (3.39)$$

In the isospin limit, $G_{01}^\gamma(x_0)$ vanishes and $G_{11}^\gamma(x_0)$ reduces to the so-called Wick-connected diagram. A Laplace transform relates the latter to the isovector spectral density, which is dominated at low-energies by the two-pion channel. The same spectral density appears in hadronic decays of the τ lepton and the corresponding Euclidean correlator is given by

$$G_{11}^W(x_0) \equiv -\frac{1}{3} \sum_{k=1}^3 \int d^3x \langle 0 | j_k^{1,+}(x_0, \mathbf{x}) j_k^{1,-}(0) | 0 \rangle, \quad (3.40)$$

with $j_k^{1,+}$ the (Euclidean) charged ud vector current. By turning on QED and SIB effects, both G_{01}^γ and the difference $G_{11}^W - G_{11}^\gamma$ acquire a nonzero value. The correlator G_{11}^W , being charged, is evidently gauge dependent, and requires an appropriate scale- and scheme-dependent renormalization. It is therefore defined only as an intermediate quantity, which may nevertheless be used to extract some model-dependent parameters, such as the ρ mass shift induced by EM interactions, where details of the operators do not matter. While eventually a model-independent calculation of the matrix elements is required, at present the phenomenological estimate is informed by IB in the ρ parameters, see Section 2.3.6.

At present, a possible strategy to calculate the IB corrections required for a τ -based inclusive determination of the HVP contribution to $(g - 2)_\mu$ has been reported in several talks [472,473]. Some additional developments are, however, still required. One is the need to define the matching between a renormalization scheme suitable for a lattice calculation of G_{11}^W and the W -regularization or $\overline{\text{MS}}$ schemes used to determine the SD matching with the SM. The mixed terms where the photon is exchanged between the τ and the hadronic system constitute an additional problem: the strategy advocated so far in Ref. [472] consists of using the available knowledge on the LD effects of these terms from ChPT [195,196], which is, however, limited to the two-pion intermediate channel. To make further progress one should consider matching these terms to the other phenomenological descriptions of the LD effects (including their scheme dependence) discussed in Section 2.3. The restriction to the two-pion channel generates an additional systematic error if one is interested in a fully inclusive approach, and eventually one should consider the complete calculation of these mixed nonfactorizable effects from lattice QCD+QED simulations, possibly solving the corresponding inverse problem.

In conclusion, while further work is still required to properly address all systematic errors of a τ -based prediction of $a_\mu^{\text{HVP, LO}}$ using lattice QCD+QED inputs, it is nevertheless worth remarking that several of these systematic effects are expected to be mitigated in the intermediate window contribution a_μ^W , where high-multiplicity channels are suppressed, a quantity that should likely be the target of a first lattice QCD+QED prediction from τ data.

3.7.2. Running of the fine-structure constant

The dependence of the QED coupling α on the interaction energy can be parameterized in the on-shell scheme as $\alpha(q^2) = \alpha/[1 - \Delta\alpha(q^2)]$, where the $q^2 = 0$ value $\alpha = 1/137.035999\dots$ is the precisely-measured fine-structure constant. The uncertainty in the running $\Delta\alpha$ is dominated by the hadronic contribution $\Delta\alpha_{\text{had}}$. For spacelike momenta, $q^2 = -Q^2 < 0$, it is proportional to the subtracted HVP function defined in Section 3.1, i.e.,

$$\Delta\alpha_{\text{had}}(-Q^2) = \frac{\alpha}{\pi} \hat{\Pi}(Q^2), \quad \text{with} \quad \hat{\Pi}(Q^2) = 4\pi^2[\Pi(Q^2) - \Pi(0)]. \quad (3.41)$$

Thus, $\Delta\alpha_{\text{had}}(-Q^2)$ is intimately related to the HVP contribution $a_\mu^{\text{HVP, LO}}$ and can be computed from first principles on the lattice. First, we discuss results at a spacelike Q^2 of a few GeV^2 . These constitute yet another set of quantities that are directly accessible by lattice computations and data-driven methods alike (see also Sections 2.7.1 and 4) and can be used as an alternative to a_μ^{HVP} and the window observables to compare independent determinations. In particular, if $R_{\text{had}}(s)$ is split in three intervals of $\sqrt{s}$ below 600 MeV, above 900 MeV and between, the relative weight of the three contributions is similar in $\Delta\alpha_{\text{had}}(-1 \text{ GeV}^2)$ and a_μ^W [20]. Following pioneering works by ETM [474] and Mainz [475], the BMW collaboration provided values for $\Delta\alpha_{\text{had}}(-Q^2)$ at $Q^2 = 1.0, 2.0, \dots, 5.0 \text{ GeV}^2$ in the supplementary material of Ref. [452], albeit not including the FV corrections. The first complete lattice calculation of the five-flavor $\Delta\alpha_{\text{had}}^{(5)}(-Q^2)$ including contributions up to the bottom quark and all known systematic effects is given in Ref. [16] at $Q^2 = 1 \text{ GeV}$, $\Delta\alpha_{\text{had}}^{(5)}(-1 \text{ GeV}^2) = 0.003784(22)$, and in terms of the difference with $Q^2 = 10 \text{ GeV}$, $\Delta\alpha_{\text{had}}^{(5)}(-10 \text{ GeV}^2) - \Delta\alpha_{\text{had}}^{(5)}(-1 \text{ GeV}^2) = 0.004867(32)$. In Ref. [476] Mainz published their dedicated study of the hadronic contribution to the running of α up to a scale of 7 GeV^2 . The result at $Q^2 = 1 \text{ GeV}^2$ is $0.003864(32)$, larger than the corresponding BMW result. These results can be directly compared with the data-driven estimates of Refs. [263,265,477], which are in agreement among themselves. For instance, using the data

from Ref. [265] one obtains $0.003682(14)$ at $Q^2 = 1 \text{ GeV}^2$. One concludes that lattice results are in significant tension with data-driven estimates obtained prior to CMD-3's publication of their data in the $\pi^+\pi^-$ channel, even if only the lower BMW result is considered. This persists at larger values of Q^2 : Ref. [476] obtains $\Delta\alpha_{\text{had}}^{(5)}(-5 \text{ GeV}^2) = 0.00716(9)$, that is 3.5% larger and in a 2.6σ tension with the data-driven result of $0.006915(33)$ [265]. In general, the discrepancy observed in Refs. [16,452,476] between the lattice results and the data-driven estimates is consistent with the scenario suggested by the lattice a_μ^{EW} results.

In terms of phenomenological impact, the most relevant quantity is the running contribution to the Z pole, $\Delta\alpha_{\text{had}}(M_Z^2)$, which is an input to global EW fits. Current lattice simulations cannot access spacelike Q^2 much larger than a few GeV^2 , beyond which cutoff effects are not under control. A solution is writing the running to the Z pole using the so-called Euclidean split technique (also known as Adler function approach) [478,479] as the sum of three terms

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = \Delta\alpha_{\text{had}}^{(5)}(-Q_0^2) + [\Delta\alpha_{\text{had}}^{(5)}(-M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-Q_0^2)] + [\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) - \Delta\alpha_{\text{had}}^{(5)}(-M_Z^2)], \quad (3.42)$$

where the first term on the RHS at an intermediate spacelike scale Q_0^2 of the order of a few GeV^2 has been computed on the lattice. The second term on the RHS of Eq. (3.42) can either be estimated with data-driven methods, or, if one is after a result independent of R -ratio data, it can be obtained by integrating the Adler function $D(Q^2)$ that, for sufficiently large Q^2 , is calculable in pQCD plus minor nonperturbative corrections [478,480–482]. The third term on the RHS of Eq. (3.42), which provides the link between the spacelike and timelike regions at M_Z , has been computed in pQCD and has a negligible error compared to the other two terms [477].

In Ref. [483] the Euclidean split technique is applied combining the lattice results of Ref. [452] with the pQCD running at $Q_0^2 = 4 \text{ GeV}^2$, obtaining $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = 0.02766(10)$. This fit prior results in a small pull of 1.3σ (or 1.1σ in a conservative scenario), leading the authors of Ref. [483] to conclude that there is no inconsistency between this lattice evaluation and the EW fit. In Ref. [476], the result of the combination of the lattice calculation with the pQCD expansion of the Adler function is $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = 0.02773(15)$, substantially constant when varying Q_0^2 in the 3 to 7 GeV^2 range. While this result is larger than a pure data-driven determination, e.g., $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = 0.02761(11)$ from Ref. [31], the clear tension in the running up to Q_0^2 is hidden by the relatively large error of the higher-energy part of the running corresponding to the second term in Eq. (3.42). Indeed, the result of Ref. [476] is little more than 1σ larger than the indirect determinations from most EW global fits [332,333,483,484], except for the global fit in Ref. [331] (based on Ref. [485]) that obtains a slightly smaller value with a smaller error hinting at a larger tension. When interpreting these findings, one must keep in mind that differences among global fits can originate from the input values for m_t and M_H , as well as the treatment of M_W .

Similar connections exist between $\hat{\Pi}(q^2)$ and the running of the weak mixing angle, whose nonperturbative contributions arise from a different flavor combination of the same correlator. Ref. [476] provides the first lattice calculation thereof, including the SU(3) breaking correlator $\bar{\Pi}^{08}$, which removes the SU(3) assumptions required in previous phenomenological determinations [486–489]. In particular, the evolution of the weak mixing angle to $q^2 = M_Z^2$ can be better controlled [490], which reduces the uncertainty in the γ -Z correlator $\bar{\Pi}^{\gamma Z}(-M_Z^2)$, required for the prediction of a_μ^{EW} , see Section 8 and Ref. [73].

3.8. Further cross-checks

Given the high precision that is being aimed at, it is important to provide stringent cross-checks of the lattice calculations. Reaching consistency among the results of different lattice collaborations is a crucial validation step, in particular as a test of universality. In this subsection, we discuss further cross-checks aimed at eliminating potentially underestimated common sources of systematic error.

In recent years, all lattice collaborations have used the TMR, which means that the states created by the EM current are at rest. An alternative method, which is particularly natural on physically large lattices, is the covariant coordinate-space (CCS) representation [491],

$$a_\mu^{\text{HVP, LO}} = \int d^4z H_{\lambda\sigma}(z) \langle j_\sigma(z) j_\lambda(0) \rangle, \quad (3.43)$$

with $H_{\mu\nu}(x)$ a known tensorial kernel. The latter can be chosen to be, e.g., transverse, traceless, or proportional to $x_\mu x_\nu$, by exploiting the conservation of the vector current. A recipe to translate a quantity defined in the TMR representation to the CCS representation is provided in Ref. [492]. Consistency between results obtained from the TMR and the CCS representation are ultimately a test of the restoration of Lorentz symmetry in the continuum and infinite-volume limit. In particular, finite-size effects are different for the two representations, and therefore the comparison TMR vs. CCS provides a test of our understanding of these effects.

The CCS representation with a traceless kernel has been applied recently [492] to compute the intermediate window quantity, thereby providing a cross-check for the Mainz calculation [20] at the point $M_\pi = 350 \text{ MeV}$, $M_K = 450 \text{ MeV}$ with a precision of 1.1%. Good consistency was found in the continuum, even though the discretization errors are rather different for the lattice correlator based on one local and one conserved vector current.

The CCS representation can also be used to compute the bare EM correction to the HVP contribution [493]

$$a_{\mu}^{1\gamma^*}(\Lambda) = -\frac{e^2}{2}\delta_{\mu\nu}\int d^4x d^4y d^4z H_{\lambda\sigma}(z) G_0(\Lambda; y-x) \langle j_{\sigma}(z) j_{\nu}(y) j_{\mu}(x) j_{\lambda}(0) \rangle, \quad (3.44)$$

where G_0 is the massless scalar propagator UV-regulated at scale Λ , either using the same lattice as the QCD degrees of freedom, in which case $\Lambda = 1/a$, or at a scale Λ well below the inverse lattice spacing, for instance à la Pauli–Villars. In the latter case, the expression has a similar structure as the coordinate-space expression for a_{μ}^{HLbL} . The continuum limit of $a_{\mu}^{1\gamma^*}(\Lambda)$ can be taken at a fixed value of Λ . It is entirely determined by the forward light-by-light amplitude, and a Cottingham-like formula [493] allows for an evaluation of $a_{\mu}^{1\gamma^*}(\Lambda)$ in the continuum. Quark-contraction diagrams in which the photon propagator connects two distinct quark loops are UV-finite and may be evaluated without regularization [458].

If $\mathbf{M}$ denotes an $(N_f + 1)$ -component vector of stable hadron masses defining the parameters of QCD in the presence of dynamical photons, the master equation for the correction to HVP to be added to its value in isoQCD reads

$$\delta a_{\mu}^{\text{HVP, LO}} = \lim_{\Lambda \rightarrow \infty} \left[a_{\mu}^{1\gamma^*}(\Lambda) - \nabla_{\mathbf{M}} a_{\mu}^{\text{HVP, LO}} \cdot \mathbf{M}^{\text{self}}(\Lambda) \right] + \nabla_{\mathbf{M}} a_{\mu}^{\text{HVP, LO}} \cdot \delta \mathbf{M}. \quad (3.45)$$

The term containing the bare EM self-energies $\mathbf{M}^{\text{self}}(\Lambda)$ of the reference hadrons corresponds to subtracting from the bare correction $a_{\mu}^{1\gamma^*}(\Lambda)$ the effect that merely comes from an EM shift of the reference hadron masses. The last term corresponds to a shift in $a_{\mu}^{\text{HVP, LO}}$ due to the fact that at the isoQCD expansion point, not all reference hadron masses are at their physical values, $\delta \mathbf{M} = \mathbf{M}^{\text{phys}} - \mathbf{M}^{\text{isoQCD}}$. Eq. (3.45) is also a rigorous starting point for computing $\delta a_{\mu}^{\text{HVP, LO}}$ in the continuum, since $\mathbf{M}^{\text{self}}(\Lambda)$ can be evaluated using Cottingham formulae for each reference hadron (see, e.g., Ref. [494]). Thus, the correction $\delta a_{\mu}^{\text{HVP, LO}}$ can ultimately be cross-checked by continuum evaluations, for which one might expect a precision of 2×10^{-10} to be achievable, by analogy with a_{μ}^{HLbL} .

3.9. Conclusions and outlook

At the time of WP20 lattice-QCD calculations were not precise enough to impact the SM prediction, and, as a result, the value for HVP in the SM prediction of WP20 was entirely based on data-driven analyses of hadronic e^+e^- cross-section data. Since then a great deal of progress has been achieved, thanks to dedicated efforts by the world-wide lattice field theory community, allowing for a precise and robust first-principles calculation of the HVP contribution from a variety of lattice fermion actions including Wilson-Clover and twisted-mass Wilson, domain wall, highly improved staggered quark (HISQ), and other implementations of staggered quark actions.

A key ingredient for the observed consolidation of lattice results was the introduction of window observables and their adoption by the lattice community as a standard for cross-checking and benchmarking lattice calculations for sub-contributions to HVP. The SD, intermediate, and LD windows amplify or suppress technical challenges related to discretization effects, FV corrections, and statistical noise, allowing for tailored approaches for each of them. The second important development since WP20 is the widespread adoption of blinding procedures by the lattice community to avoid confirmation bias. This has proved instrumental for establishing the reliability of the observed consolidation among independent lattice-QCD results and, in turn, for producing a robust and precise average for LO HVP.

The intermediate-distance window was the first quantity for which a consolidated set of lattice results became available around 2022, hinting at a significant tension with the corresponding data-driven estimates based on e^+e^- data prior to CMD-3's 2023 result. The currently available set of results for the light-quark, isospin-symmetric short- and intermediate-distance window observables shows a remarkable degree of consistency and precision among different lattice collaborations. Fewer results are available for the LD window or for unwrapped contributions, but the results for the latter are in good agreement with sums of windows.

As the precision of lattice results improves, the focus is gradually shifting towards the calculation of IB corrections. Only a small fraction associated with sea–sea and sea–valence QED corrections, which amounts to approximately 0.06% of the total HVP contribution, has been calculated by a single lattice collaboration so far. For all other contributions, at least three independent lattice results exist. The LD QED contributions, which amount to about 0.3% of the total HVP correction, are particularly challenging. For these contributions additional cross-checks against phenomenology have been performed.

The review of lattice-QCD results in WP25 is based on seventeen different papers from eight independent lattice-QCD collaborations [14–30], including three lattice calculations of the entire LO HVP contribution [16,26,27]. We use five approaches for obtaining averages for $a_{\mu}^{\text{HVP, LO}}$, which are in excellent agreement. As our final SM prediction of $a_{\mu}^{\text{HVP, LO}}$ we take Avg. 1 from Eq. (3.31), which includes the maximum number of independent lattice results from Refs. [14–30]. This consolidated average of lattice-QCD results provides a reliable determination of the LO HVP contribution to the SM prediction of a_{μ} . The total error of $\pm 6.1 \times 10^{-10}$ is larger compared to that of the data-driven estimate quoted in WP20. Compared with WP20, the relative precision for $a_{\mu}^{\text{HVP, LO}}$ obtained from lattice QCD has improved by a factor of approximately three.

This section also describes how lattice QED+QCD can provide model-independent results for the isospin rotation needed to extract the spectral function from hadronic τ decays. In addition to considering the LO HVP contribution, we have also reviewed lattice calculations of the closely related hadronic running of the EM coupling α , which, unsurprisingly, shows

similar tensions between lattice results and dispersive analyses based on e^+e^- cross-section data prior to the results published by CMD-3. Finally, a valuable consistency check of the entire lattice approach to HVP can be performed by employing the CCS representation as an alternative to the standard TMR.

Continuing efforts by the world-wide lattice community are expected to yield further significant improvements in precision and, hopefully, even better consolidation thanks to a diversity of methods. We expect, in particular, more precise evaluations of IB effects and the noisy contributions at long distances.

4. Interplay of lattice-QCD and data-driven evaluations of HVP

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4.1. Data-driven evaluations of Euclidean windows

For the full HVP contribution, timelike [85–88] and spacelike [425,453,495] representations

$$a_\mu^{\text{HVP, LO}} = \left(\frac{\alpha m_\mu}{3\pi} \right)^2 \int_{s_{\text{thr}}}^\infty ds \frac{\hat{K}(s)}{s^2} R_{\text{had}}(s) = \left(\frac{\alpha}{\pi} \right)^2 \int_0^\infty dx_0 C(x_0) \tilde{f}(x_0), \quad (4.1)$$

are connected via change of the corresponding kernel functions $\hat{K}(s)$ and $\tilde{f}(x_0)$, where $\hat{K}(s)$ and $R_{\text{had}}(s)$ are defined in Section 2.1, $\tilde{f}(x_0)$ and $C(x_0)$ in Section 3.3. Ultimately, the two formulations are related by a change of kernel functions, $\hat{K}(s)$ vs. $\tilde{f}(x_0)$. In particular, this implies that additional weight functions introduced in the integrals in Eq. (4.1) in one approach can be translated to the other accordingly,²² e.g., the spacelike weight functions $\Theta(x_0)$ introduced in Ref. [14], see Eqs. (3.16), (3.17) and (3.17)–(3.20), translate to timelike ones $\tilde{\Theta}(s)$ via

$$\tilde{\Theta}(s) = \frac{3s^{5/2}}{m_\mu^2 \hat{K}(s)} \int_0^\infty dx_0 \Theta(x_0) e^{-x_0 \sqrt{s}} \int_0^\infty \frac{d\omega}{\omega} f(\omega^2) \left[\omega^2 x_0^2 - 4 \sin^2 \left(\frac{\omega x_0}{2} \right) \right], \quad (4.2)$$

with $f(s)$ defined in Eq. (3.6). The possibility of evaluating the Euclidean window quantities defined by $\Theta(x_0)$ also with data-driven techniques renders them an invaluable diagnostic tool to compare lattice-QCD and data-driven evaluations of HVP in more detail, and, ultimately, to combine the two approaches via hybrid evaluations, using either method in the parameter space where it is most precise [14].

In practice, the comparison is currently performed at the level of the spacelike windows, as the inversion to the timelike domain amounts to an ill-posed inverse Laplace transform. Optimizing the resolution in $\sqrt{s}$ that can be achieved in such an inversion therefore requires precise knowledge of the covariances within a given lattice-QCD calculation, and several studies already explored the consequences for detailed comparisons between phenomenological and lattice-QCD evaluations [17,363,368,497,498], including the development of strategies to extract the maximum amount of information on the timelike domain from a given set of lattice calculations for $a_\mu^{\text{HVP, LO}}$, Euclidean windows, and $\Delta\alpha_{\text{had}}(-Q^2)$. In addition to comparing the full HVP contribution in a given window [16,363,498], also comparison quantities for specific parts of a complete lattice calculation have been extracted with data-driven techniques, such as IB contributions, see Section 2.7.3, the light-quark-connected contribution, see Section 2.8.1, and the strange+disconnected contribution, see Section 2.8.2.

4.2. Hybrid calculations

The Euclidean windows defined in Ref. [14] are well-defined individually and are independent of the regulator. They can therefore also be computed with different methods and then combined to obtain the total HVP value. This was already recognized in Ref. [14], where the SD window and the LD window were obtained from a data-driven approach and the intermediate-distance window was obtained using the lattice method. This requires that the lattice calculation be complete, i.e., include all quark flavors and also QED and SIB corrections. Since Ref. [14] appeared, tensions in the data-driven sector have been identified that make the original hybrid result of RBC/UKQCD-18 no longer appropriate to use since it included the LD contribution from $t_1 = 1.0$ fm from the data-driven method. Variations of this idea can, however, be useful with well-consolidated results.

BMW/DMZ-24 builds on this idea, strongly limiting the use of the data-driven component to a LD window beyond $t_1 = 2.8$ fm, see Eq. (3.19) [24]. Thus, more than 95% of their result is obtained with a lattice calculation and only the remaining $\lesssim 5\%$ utilizes a subset of the measured $e^+e^- \rightarrow \text{hadrons}$ (and $\tau \rightarrow \nu_\tau + \pi^- \pi^0$) spectral function discussed in Section 2. More importantly, this LD contribution is predominantly determined by the low-mass region of the spectrum where all measurements agree within uncertainties, below the ρ -peak region that displays the tensions discussed in Section 2. Even such a small replacement brings a significant reduction in overall uncertainty. As discussed in Section 3, the statistical noise of light-quark contributions increases rapidly with distance when present-day lattice methods are utilized. The same is true for effects associated with the finite volume of the lattice. However, those challenges are circumvented when the measured spectra are used to determine LD contributions instead. The authors find that for the

²² In a perturbative approach, also the effect of quark-flavor thresholds needs to be taken into account [496].

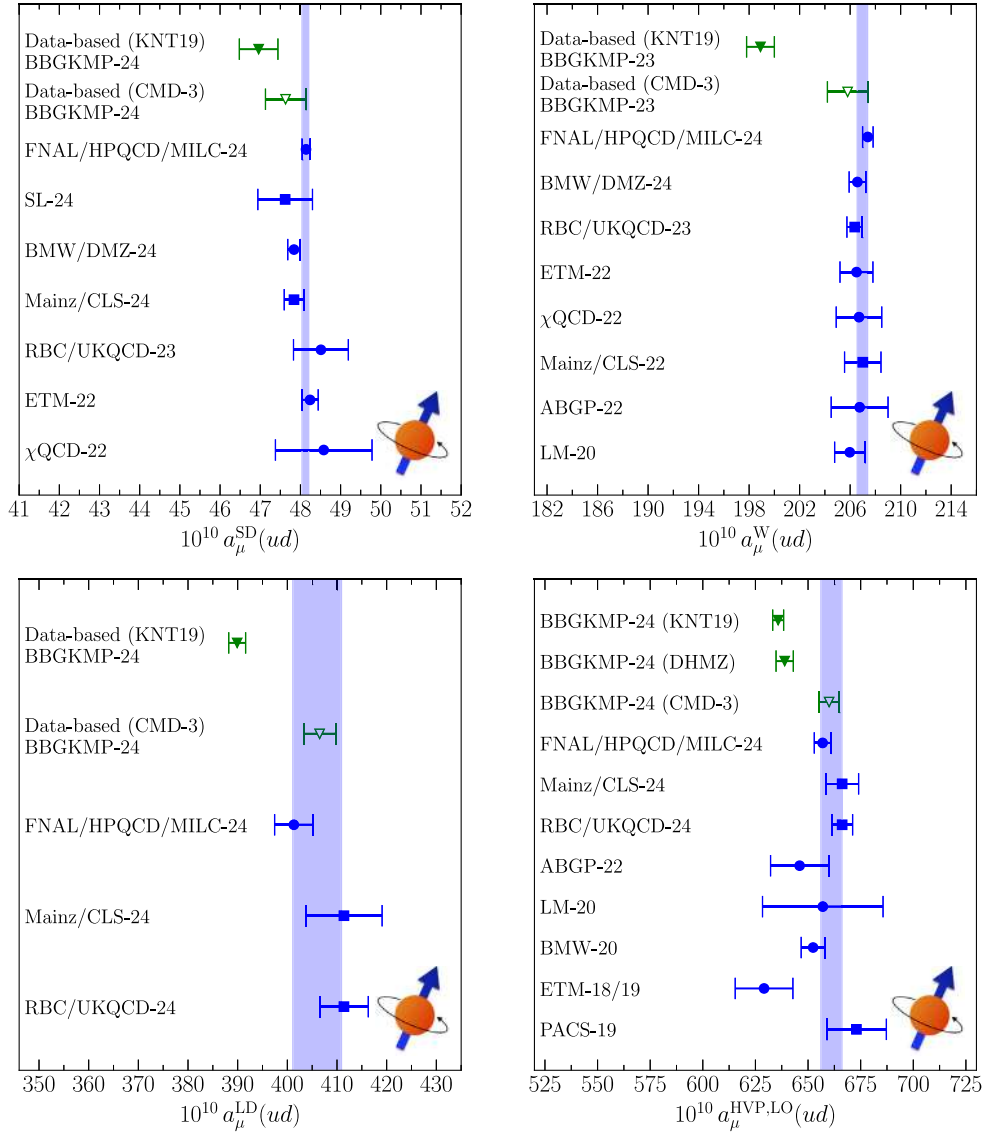


Fig. 38. Data-driven results for the light-quark connected (ud) components of the RBC/UKQCD windows compared with isospin-symmetric lattice-QCD determinations of the same quantities. The label “BBGKMP-24(23)” refers to Refs. [357–360]. The bands show averages for these quantities taken from Section 3. Upper-Left: Data-driven $a_\mu^{\text{SD}}(ud)$ compared with results from Refs. [18,21–25,29]. Upper-Right: Data-driven $a_\mu^{\text{W}}(ud)$ compared with results from Refs. [17–22,24,29]. Lower-Left: Data-driven $a_\mu^{\text{LD}}(ud)$ compared with the recent results of Refs. [26,27,30]. Lower-Right: Data-driven $a_\mu^{\text{HVP, LO}}(ud)$ compared with results from Refs. [16,17,19,26,27,30,447,448,451]. Entries marked with “CMD-3” in each figure show the exploration of the impact of the replacement of KNT 2π data for energies between 0.33 and 1.2 GeV with CMD-3 data (see main text).

choice of t_1 considered in Ref. [24], the resulting uncertainty on this LD part becomes a negligible contribution to the one on the final result. It is worth noting that investigations of the dependence of $a_\mu^{\text{HVP, LO}}$ on t_1 have been performed in Refs. [14,499]. The result of Ref. [24] for $a_\mu^{\text{HVP, LO}}$ is not yet published and the authors plan to complete the documentation of the lattice-QCD calculations that enter the hybrid evaluation in the published version. Moreover, it is important that the different components, which enter that result, be computed independently by other groups in view of providing a consolidated combination with similar or even improved precision in the future. We note that once the discrepancies among experimental e^+e^- data are resolved and if data-driven and lattice results are found to be in good agreement, the hybrid method has the potential to yield valuable additional insights, including evaluations of HVP with improved precision, to fully leverage the Fermilab experiment’s measurement of the muon $g - 2$.

4.3. Comparison

The results of the analysis in Section 2.8 are shown in Fig. 38 for isospin-symmetric light-quark connected quantities, and in Fig. 39 for isospin-symmetric s -disc quantities, comparing the data-driven results with those obtained from

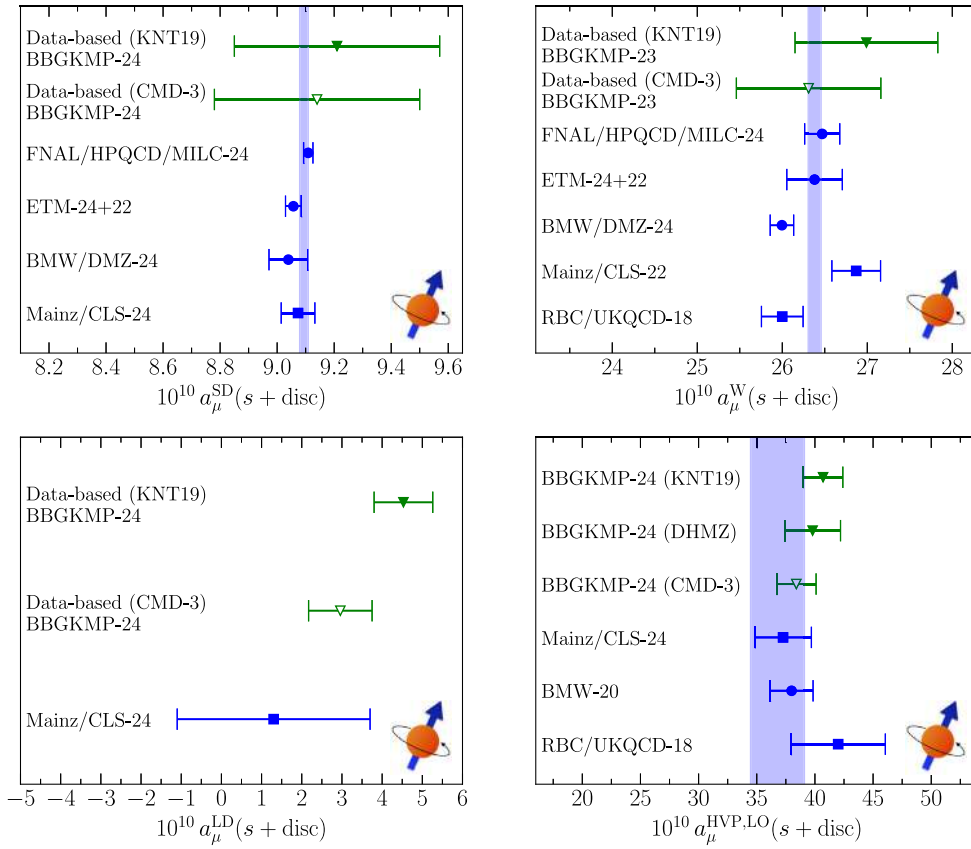


Fig. 39. Data-driven results for the s +disc components of the RBC/UKQCD windows compared with isospin-symmetric lattice determinations of the same quantities obtained by adding strange-connected and disconnected contributions given in Section 3. The label “BBGKMP 24(23)” refers to Refs. [356,358–360]. The bands show averages for these quantities obtained from Section 3 by adding the averages of s and disc contributions, see main text. Upper-Left: Data-driven results for $a_\mu^{\text{SD}}(s + \text{disc})$ compared with results from Refs. [21,23,24,28,29]. Upper-Right: Data-driven results for $a_\mu^{\text{W}}(s + \text{disc})$ compared with results from Refs. [14,20,21,24,28,29]. Lower-Left: Data-driven results for $a_\mu^{\text{LD}}(s + \text{disc})$ compared with the result from Ref. [27] (see also Ref. [360]). Lower-Right: Data-driven results for $a_\mu^{\text{HVP, LO}}(s + \text{disc})$ compared with results from Refs. [14,16,27]. Entries marked with “CMD-3” in each figure show the exploration of the impact of the replacement of KNT 2π data for energies between 0.33 and 1.2 GeV with CMD-3 data (see main text).

the lattice. Since the analyses in question [356–360] use phenomenological input for IB corrections in the 2π and 3π channels [186,237,247], which to good approximation is scheme independent once the pion mass is identified with the mass of the neutral pion, and BMW-20 [16] input for small additional QED IB corrections, it is justified to compare with lattice-QCD results in the BMW/WP25 scheme, see also Section 2.7.3. The lattice averages shown in Fig. 38 are taken from Section 3, while those in Fig. 39 are obtained by adding strange-connected and disconnected averages from Section 3 (adding errors in quadrature, as suggested therein), using the fact that charm-disconnected contributions are very small, and can thus be neglected. The three lattice points shown in the lower-right panel of Fig. 39 average to a value larger than the band shown because the band includes more strange-connected results from Tables 7 and 8. For the single lattice point in the lower-left panel, see Ref. [360]. For the ud and s +disc HVP lattice averages, the line “Avg. B” in Table 13 has been used.

Figs. 38 and 39 show that for the LD and intermediate light-quark connected RBC/UKQCD window quantities, there are significant discrepancies between the KNT-compilation-based data-driven and the lattice-based estimates, which lead to a significant discrepancy in the total $a_\mu^{\text{HVP, LO}}(ud)$ when comparing with the most precise lattice determinations. In contrast, for $a_\mu^{\text{SD}}(ud)$ and for all the s +disc window quantities, there are no discrepancies (though the data-driven errors for the latter are relatively large). Moreover, the exploratory exercise of replacing the $\pi^+\pi^-$ KNT-compilation data in the interval between 0.33 and 1.2 GeV with the CMD-3 $\pi^+\pi^-$ data suggests that these discrepancies could be due to discrepancies in the experimental data for the $\pi^+\pi^-$ component of $R_{\text{had}}(s)$ in the region around the ρ peak. With this replacement, the discrepancies in the light-quark connected results are eliminated without disturbing the good agreement for the s +disc and the light-quark connected SD parts. The $\pi^+\pi^-$ channel is responsible for 72% of the data-driven $a_\mu^{\text{W}}(ud)$ result and for 88% of the data-driven $a_\mu^{\text{LD}}(ud)$, but only 32% of the $a_\mu^{\text{SD}}(ud)$ value and only very small fractions of the s +disc results. These conclusions for the quark-flavor-specific contributions agree with the findings from Ref. [498], obtained for

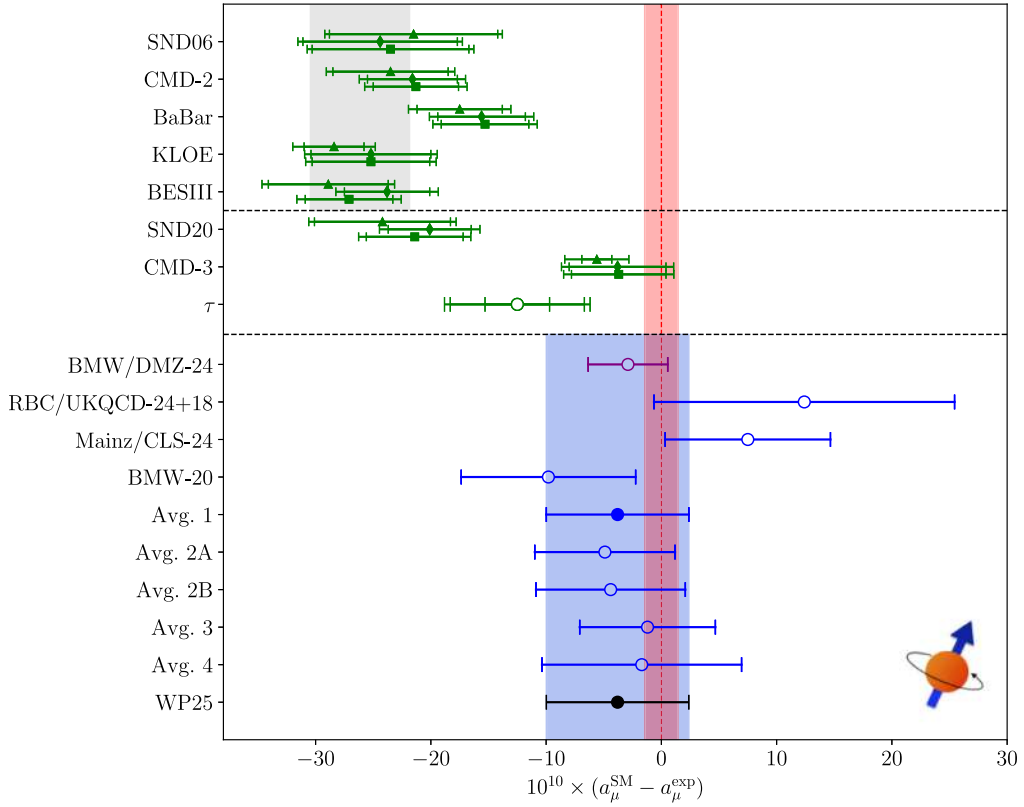


Fig. 40. Final summary of various determinations of $a_\mu^{\text{HVP, LO}}$ discussed in Sections 2 and 3, propagated to a_μ^{SM} . The first two panels refer to data-driven determinations, where the three points for each e^+e^- experiment reflect the “CHKLS”, “DHMZ”, and “KNTW” methods, see Figs. 26 and 27 for more details. The gray band indicates the WP20 result, based on the e^+e^- experiments above the first dashed line. The τ point corresponds to Eq. (2.23). The last panel summarizes lattice-QCD determinations, including the hybrid evaluation [24], the three individual lattice-QCD calculations shown in Fig. 36, and the five lattice HVP averages from Fig. 37. The blue band refers to the final WP25 result, which coincides with “Avg. 1”. In all cases, except for the gray WP20 band, the remaining contributions to a_μ^{SM} beyond $a_\mu^{\text{HVP, LO}}$ are taken from WP25, as given in Table 1. The red band denotes the experimental world average, which has been updated including the final results from the Fermilab experiment.

the full contributions, including all flavors and all IB corrections. Confirming the original findings of Ref. [16], the authors observe a significant discrepancy between the pre-CMD-3 data-driven and lattice results for the full intermediate-window contribution and a large, but less statistically significant one, for the total HVP contribution to a_μ . In addition they show that there is relatively good agreement for $\Delta\alpha_{\text{had}}(-10 \text{ GeV}^2) - \Delta\alpha_{\text{had}}(-1 \text{ GeV}^2)$. Using a new approach that allows one to investigate how the experimentally measured spectral functions would have to be modified to reconcile the data-driven results with the lattice ones, they show that an enhancement of that function in any interval of CM energy that includes the ρ peak could explain the observed disagreement pattern. In particular, this pattern can be explained by a rescaling by 5% of the contribution to each of those observables from the ρ -peak region, defined via the interval $\sqrt{s} \in [0.63, 0.92]$. Of course, such a rescaling is significantly larger than the uncertainties quoted in pre-CMD-3 combinations of the e^+e^- spectra. This study included statistical-and-systematic-uncertainty correlations among the observables computed on the lattice and found the conclusions to be stable with respect to them. It was also performed with an initial blinding on the data-driven results.

A priori, it is not evident that significant modifications to the 2π spectral function can be introduced without violating QCD constraints from analyticity and unitarity, but it was shown in Ref. [320] that this is possible. One option is a modification of the $\pi\pi$ phase shift, which would, however, induce rather large changes in the cross section concentrated around the ρ resonance. The second possibility concerns changes in the inelastic contributions, which lead to largely uniform relative shifts of the cross section in the whole low-energy region; the CMD-3 measurement amounts to a realization of this latter scenario. Indeed, the CMD-3 data pass the tests of unitarity and analyticity [146,270], see Section 2.7.1. However, inverting the Laplace transform to translate from Euclidean windows back to CM energies is inherently ill-posed, where one possible avenue concerns optimizing linear combinations of a number of Euclidean windows [17,363,497], e.g., Ref. [363] explicitly constructs such linear combinations for localization in CM energy, leading to strong oscillations in Euclidean time, and also proposes to isolate a given hadronic channel via suitable linear combinations of Euclidean windows. It is emphasized that such optimizations require a precise knowledge of the full