

Fig. 41. The HLbL contribution depicted graphically. The bottom line is the muon. The blob indicates all possible hadronic interactions. The q_4 leg indicates the external magnetic field.

Source: Figure taken from Ref. [43].

covariance matrix in lattice-QCD calculations. Finally, in Ref. [500] it is proposed to calculate a version of the R -ratio convolved with Gaussian smearing kernels in lattice QCD using the method from Ref. [240] to extract smeared spectral densities from Euclidean correlators, including a proof-of-concept calculation with resolutions between 0.44 and 0.63 GeV.

With the new and upcoming lattice results available for the SD, intermediate, and LD windows and for other HVP related quantities those studies will have to be significantly updated. This is all the more important given the tensions now observed among the different $e^+e^- \rightarrow$ hadrons measurements, as well as with those of hadronic τ decays. Such studies should offer interesting perspectives in the comparison of lattice and data-driven results for $a_\mu^{\text{HVP, LO}}$. It is important that those studies account for all correlations in lattice and R -ratio observables, as well as for uncertainties on these [498]. Double-blinding, for observables not yet computed, is also important.

Finally, Fig. 40 presents a detailed overview of the current situation for HVP, comparing the data-driven, lattice, and hybrid results discussed in Sections 2 and 3, as well as Section 4.2. We note that the good agreement between the lattice and CMD-3-based results does not imply that the CMD-3 result is to be preferred. The discrepancy between the CMD-3 and other experimental measurements remains an independent puzzle. This issue, along with the well-known BABAR–KLOE discrepancy, will require further experimental investigations and detailed studies of the MC generators and other tools used by the different experiments in their data analyses.

In contrast, the complete lattice HVP results from Refs. [14,16,22,26,27] show good agreement, including with the hybrid result from Ref. [24] and the consolidated lattice averages. These averages are based on different combinations of a large number of results from well-defined sub-contributions, obtained independently by various lattice-QCD collaborations, as described in Section 3.

5. Data-driven and analytic approaches to HLbL

J. Bijnens, L. Cappiello, G. Eichmann, C. S. Fischer, S. González-Solís, N. Hermansson-Truedsson, M. Hoferichter, B.-L. Hoid, S. Holz, B. Kubis, A. Kupść, P. Masjuan, M. Procura, A. Rebhan, C. F. Redmer, A. Rodríguez-Sánchez, P. Roig, P. Sánchez-Puertas, P. Stoffer, M. Zillinger

5.1. Introduction

The HLbL contribution to the muon anomalous magnetic moment is depicted in Fig. 41.²³ For a very long time it was considered impossible to estimate it with any reliability. This changed during the 90s when a number of theoretically reasonably well argued models were used, final numbers can be found in Refs. [501,502]. However, it was realized much more recently that a proper well-defined separation of different hadronic contributions was possible [34]. This provided together with a proper definition of short-distance contributions (SDCs) from QCD [39,503] the basis for a full evaluation of the HLbL contribution in the previous white paper of the muon $g - 2$ theory initiative [1]. A proper description of the large amount of work done earlier can also be found there. The Fermilab experiment expects to release a new result for their total data set in the near future. We therefore need to update the HLbL result from Ref. [1]. The main improvements are that the dispersive framework has been improved to deal with spin-1 hadrons in a singularity-free fashion and first steps towards dealing with higher spin resonances have been done. The short-distance results have also been significantly improved. The model and phenomenological work has also been updated. This is described in the following sections.

The overall framework is discussed in Section 5.2. Improvements in the experimental inputs that are needed in the various approaches are discussed in Section 5.3. The short-distance part is improved in various ways, by including higher orders in the operator product expansion (OPE) as well as gluonic corrections, both for the general case and for the case with only two internal photons far off-shell. This is described in Section 5.4. The improved dispersive framework is used for determining the η , η' and axial-vector contributions as well as an update on scalars and a first estimate for tensors.

²³ We have kept the q_4 sign different in different sections to keep with the notation in the original references.

This is described in Sections 5.5.1–5.5.3. The method to perform a dedicated matching to the short-distance QCD results is explained in Section 5.5.4. Holographic QCD (hQCD) is a model for QCD that incorporates a lot of restrictions from full QCD. The HLbL contribution can be calculated in various versions of this class of models and the results are discussed in detail in Section 5.6. In particular we have used this to estimate errors on tensor and missing resonances from the results in Section 5.6.5. In WP20 the rational approximants were used as a cross-check for the dispersive result for the π^0 pole and as input for the η , η' pole. These together with the newer analyses using the resonance-chiral-theory ($R_\chi T$) approach for these contributions are described in Section 5.7.1. An improved phenomenological treatment of axial vectors and how they are used together with the short-distance results can be found in Section 5.7.2. Another approach to QCD are the functional methods using Dyson–Schwinger (DSE) and Bethe–Salpeter (BSE) equations. The approximations needed to be able to solve the equations make this approach a QCD-based model, but it can capture different aspects than the phenomenological or holographic approaches. The most recent results relevant for HLbL are given in Section 5.8. A few quantities can be easily compared between the various approaches. This comparison is performed in Section 5.9. The combination of all the results into a new number for the HLbL is presented in Section 5.10. Finally, prospects for further improvements on the theory side are given in Section 5.11.1 and a wish list and expected future experimental results in Section 5.11.2.

5.2. Framework

The HLbL tensor is defined as the hadronic Green's function of four EM currents in pure QCD

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int d^4x d^4y d^4z e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \langle 0 | T \{ j_{\text{em}}^\mu(x) j_{\text{em}}^\nu(y) j_{\text{em}}^\lambda(z) j_{\text{em}}^\sigma(0) \} | 0 \rangle, \quad (5.1)$$

which includes the contribution from the three lightest quarks:

$$j_{\text{em}}^\mu := \bar{q} Q \gamma^\mu q, \quad q = (u, d, s)^T, \quad Q = \text{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right). \quad (5.2)$$

Contracting $\Pi^{\mu\nu\lambda\sigma}$ with polarization vectors yields the hadronic contribution to the helicity amplitudes for off-shell photon–photon scattering, $\gamma^*(q_1, \mu) \gamma^*(q_2, \nu) \rightarrow \gamma^*(-q_3, \lambda) \gamma^*(q_4, \sigma)$. The starting point of a dispersive approach to HLbL is the decomposition of the tensor $\Pi^{\mu\nu\lambda\sigma}$ into a generating set of Lorentz structures $T_i^{\mu\nu\lambda\sigma}$ that are individually gauge invariant, with scalar coefficient functions Π_i free of kinematic singularities. Such a decomposition was constructed in [34] following the recipe by Bardeen, Tung, and Tarrach (BTT) [504,505], which leads to a redundant set of 54 structures with manifest crossing symmetry,

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i. \quad (5.3)$$

Although the scalar functions Π_i are free of kinematic singularities, which is a prerequisite for a dispersive representation, the redundancy in the decomposition needs to be addressed: a fully off-shell basis contains only 41 elements, given by the number of independent helicity amplitudes.

The HLbL contribution to a_μ can be derived from $\Pi^{\mu\nu\lambda\sigma}$ using Dirac-space projection operator techniques [506] in the limit $q_4 \rightarrow 0$. This leads to the following master formula, which is well-suited for a numerical evaluation:²⁴

$$a_\mu^{\text{HLbL}} = \frac{\alpha^3}{432\pi^2} \int_0^\infty d\Sigma \Sigma^3 \int_0^1 dr r \sqrt{1-r^2} \int_0^{2\pi} d\phi \sum_{i=1}^{12} T_i(\Sigma, r, \phi) \bar{\Pi}_i(Q_1^2, Q_2^2, Q_3^2), \quad (5.4)$$

with known kernel functions $T_i(\Sigma, r, \phi)$ [36] and Euclidean photon virtualities $Q_i^2 \equiv -q_i^2$ given by

$$\begin{aligned} Q_1^2 &= \frac{\Sigma}{3} \left(1 - \frac{r}{2} \cos \phi - \frac{r}{2} \sqrt{3} \sin \phi \right), \\ Q_2^2 &= \frac{\Sigma}{3} \left(1 - \frac{r}{2} \cos \phi + \frac{r}{2} \sqrt{3} \sin \phi \right), \\ Q_3^2 &= \frac{\Sigma}{3} (1 + r \cos \phi). \end{aligned} \quad (5.5)$$

The functions $\bar{\Pi}_i$ are linear combinations of the Π_i introduced above and fully parameterize the hadronic content in a_μ^{HLbL} . The ambiguities in the redundant set of functions Π_i cancel in the linear combinations $\bar{\Pi}_i$ in the limit $q_4 \rightarrow 0$, hence each of the 12 terms in the master formula Eq. (5.4) is a well-defined quantity.

Unitarity rigorously determines the absorptive part of the HLbL tensor, i.e., the residues of dynamical single-particle poles and the discontinuities of multi-particle branch cuts of the HLbL tensor in the different kinematic channels and

²⁴ Different variables for the integration can be used, some more examples are given in Ref. [54].

relates them to various sub-processes with on-shell intermediate states. In order to reconstruct the dispersive (real) part of the HLbL tensor, one makes use of analyticity to write dispersion relations for the BTT scalar functions in a given set of kinematic variables. This requires the scalar functions to be free of ambiguities, i.e., the redundancy in Eq. (5.3) needs to be resolved without introducing kinematic singularities in the dispersed variables. This has been achieved in the original approach of Refs. [34,36] in four-point kinematics in the limit of an external on-shell photon, $q_4^2 = 0$, i.e., dispersion relations have been derived in terms of the Mandelstam variables for $\gamma^*\gamma^* \rightarrow \gamma^*\gamma$ scattering with fixed photon virtualities, with the limit $q_4 \rightarrow 0$ relevant for a_μ^{HLbL} taken only afterwards. Although the derived scalar functions are free of singularities in the dispersed Mandelstam variable, they contain kinematic singularities in the fixed virtualities q_i^2 . In the static limit $q_4 \rightarrow 0$, their residues vanish due to a set of sum rules, hence a_μ^{HLbL} remains unaffected. These sum rules ensure that a_μ^{HLbL} is independent of the choice of the tensor basis. However, the fulfillment of the sum rules typically involves a cancellation between different hadronic intermediate states, which leads to complications in practice. In Ref. [49], an optimized basis has been constructed that limits the appearance of spurious kinematic singularities to intermediate states of spin ≥ 2 .

In Ref. [507], an alternative dispersive framework has been introduced, formulated directly for the functions $\bar{T}_i$ in the static limit $q_4 \rightarrow 0$. In this approach, all redundancies and kinematic singularities are manifestly absent, but the dispersion relations in the photon virtualities require new sub-processes as input, see Section 5.11.1.

In the dispersive framework in four-point kinematics, the contributions from one- and two-meson intermediate states have been explicitly accounted for as discussed in the next sections: pseudoscalar-pole and -box contributions, scalars, axial-vector contributions, as well as partial contributions from tensor mesons. These intermediate-state contributions are expressed in terms of physical transition form factors (TFFs) and helicity amplitudes, which can either be reconstructed dispersively, obtained from lattice QCD, or taken as input from hadronic models. Since the dispersive approach relates all contributions to on-shell intermediate states, these observables serve as the experimental input for the dispersive data-driven determination of a_μ^{HLbL} .

The dispersive treatment of exclusive intermediate states is feasible only at sufficiently low energies. Therefore, this description has to be complemented by matching to SDCs, which stem from the OPE and pQCD, see Section 5.4.

5.3. Experimental results

Experimental results can serve as input to the calculation of the contributions to HLbL in a direct and a more indirect way. Direct input to the pseudoscalar-meson pole contributions are TFFs $F_{P\gamma^*\gamma^*}(q_1^2, q_2^2)$, which can be measured in different regions of momentum transfer q_i^2 . The momentum transfer dependence of the relevant cross sections and decay rates due to the TFFs makes the experimental access to information at arbitrary values of q_i^2 challenging. Similar limitations hold for experimental information on multi-meson production in two-photon collisions, which can provide corresponding input for the $\pi\pi$ contribution or contributions of heavier scalar, axial-vector, and tensor resonances. In a more indirect way, experimental information on other hadronic and radiative processes is needed to determine missing direct information on TFFs from dispersion relations, as successfully demonstrated for the π^0 pole contribution in WP20.

With respect to the available experimental information and the possible approaches, priorities for new experimental input have been formulated in WP20. In the following an overview on new experimental results beyond those listed there is provided. An outlook on planned measurements and data to become available in the near future is given in Section 5.11.2.

5.3.1. Meson transition form factors

Experimental methods to obtain information on meson TFFs comprise Primakoff production of mesons P to address the normalization at $F_{P\gamma^*\gamma^*}(0, 0)$, meson production in two-photon scattering at e^+e^- machines to address the spacelike momentum dependence of $F_{P\gamma^*\gamma^*}(q_1^2, q_2^2)$, and radiative meson production at e^+e^- colliders as well as meson Dalitz decays to test the timelike momentum dependence of the respective TFFs. Furthermore, rare decays of pseudoscalar mesons into lepton pairs can be related to TFFs.

New measurements of Dalitz decays of η and η' involving a single off-shell photon are provided by the BESIII collaboration [508] using the radiative decays of a total of 10 billion inclusively recorded J/ψ decays as source of the mesons. The invariant mass distribution $M_{e\bar{e}}^2 = q^2$ of the lepton pairs is investigated to determine the TFFs, which are parameterized with respect to the slope at $q^2 = 0$. The slope parameter Λ^2 is obtained by fitting the data with the model parameterization²⁵

$$|F_{P\gamma^*\gamma^*}(q_1^2, 0)|^2 = \frac{\Lambda^2(\Lambda^2 + \gamma^2)}{(\Lambda^2 - q^2)^2 + \Lambda^2\gamma^2}, \quad (5.6)$$

where Λ and γ can be interpreted as the mass and width of a contributing effective vector meson, in the context of a vector-meson-dominance (VMD) picture. In case of the η' , the full parameterization is needed to properly describe the data with $\Lambda_{\eta'}^2 = 0.6432(56)(64)\text{GeV}^2$ and $\gamma_{\eta'}^2 = 0.0128(11)(2)\text{GeV}^2$, which is a four-times improved accuracy compared

²⁵ The pseudoscalar TFF $F_{P\gamma^*\gamma^*}$ is formally defined in Eq. (5.11).

to the previously reported first observation [509]. The slope parameter of the η TFF is obtained by setting $\gamma_\eta = 0$ as $\Lambda_\eta^2 = 0.561(20)(5) \text{ GeV}^2$. Compared to the previously reported results of the A2 [510] and NA60 [511] collaborations, the BESIII result tends towards smaller values, but it is compatible within errors. Currently, a new measurement of the η TFF is prepared at BESIII using η' decays as source of η mesons, which are more copiously produced. The different environment allows for a cross-check with similar statistical, but different systematic uncertainties. It should be mentioned that the timelike TFF has a cusp at the opening of the $\pi^+\pi^-$ channel corresponding to $q^2 = 4M_\pi^2$. The effect of the cusp is not included in Eq. (5.6), which is a source of bias in the determination of the slopes. The same data set of inclusive J/ψ decays also allowed one to observe the double Dalitz decay of the η' at BESIII with a significance of 5.7σ and to determine the branching fraction for the first time [512].

The CMS collaboration reported the branching ratio of the double Dalitz decay into muon pairs normalized to the decay into a single muon pair $\Gamma(\eta \rightarrow 2\mu^+2\mu^-)/\Gamma(\eta \rightarrow \mu^+\mu^-) = 0.86(14)(12) \times 10^{-3}$ [513]. The result can be related to the double off-shell η TFF [514].

Additional information on doubly-virtual TFFs through the investigation of dilepton decays of mesons is provided by the BESIII collaboration in the measurement of the direct production of the axial-vector state χ_{c1} in e^+e^- annihilation [515]. The meson production is established by exploiting the interference pattern of the decay particles and the radiative dimuon continuum predicted in Ref. [516] with a 5.1σ significance. The interference method allowed for the first measurement of nonvector meson production at e^+e^- colliders with an electronic width of χ_{c1} of $\Gamma_{ee} = (0.12_{-0.08}^{+0.13}) \text{ eV}$, and is currently tested to provide equivalent information on further mesons.

5.3.2. Further results from meson decays

The decay of η' into four pions was prioritized as one of the required inputs for the development of a dispersive framework of η and η' TFFs in WP20. The BESIII collaboration reported not only an upper limit of $\eta' \rightarrow 4\pi^0$, but also differential information of the decay $\eta' \rightarrow 2\pi^+2\pi^-$ [517]. An amplitude analysis is performed based on the work of Ref. [518] and finds the coupling constants to be in agreement with their assumptions.

Similarly relevant to the construction of a dispersive framework for the η' TFF is the decay into a pion pair and a lepton pair. First studies reported by the BESIII collaboration [519,520] were updated to the fully available data set of J/ψ decays [521]. Using different VMD models, the slope parameter of the TFF is determined from the differential decay distributions as $\Lambda_{\eta'}^2 = 0.77(11) \text{ GeV}^2$. Furthermore, a test for unconventional CP violation is performed based on the angular distributions of the pion and lepton decay planes.

5.3.3. Hadronic cross sections

Apart from information on meson decays also hadronic cross sections in e^+e^- collisions are relevant for the development of dispersive frameworks of the pseudoscalar meson TFFs. Predominantly in the context of the HVP contribution new measurements of the total cross section of $e^+e^- \rightarrow \pi^+\pi^-\pi^0$ using initial state radiation are reported by the BABAR and Belle-II collaborations [149,185]. Additional information on intermediate states are reported from a partial-wave analysis performed by the BESIII collaboration on high-statistics data points of an energy scan measurement at CM energies between 2.0 GeV and 3.08 GeV [522]. The three-pion system is found to be dominated by $\rho\pi$, $\omega\pi$, $\rho(1450)\pi$, $\rho(1700)\pi$, and $\rho_3(1690)\pi$.

An equivalent study of the $e^+e^- \rightarrow \pi^+\pi^-\eta$ cross section is performed on the same data [523]. The intermediate states are described by dominant contributions of $\rho\eta$ and $a_2(1320)\pi$. Further significant contributions of higher resonant states are identified. Additional results on the total cross sections of the related processes $e^+e^- \rightarrow \omega\pi^0/\eta^{(\prime)}$ and $e^+e^- \rightarrow \pi^+\pi^-\eta'$ are also reported based on the same data [524–526].

5.4. Short-distance constraints

SDCs can be derived to address the HLBL contributions to a_μ in kinematical regions where two or three photon virtualities Q_1^2 , Q_2^2 , and Q_3^2 become large. These constraints are obtained by applying OPEs to the EM quark currents that define the HLBL tensor [39,43,44,54,503,527]. In this section we review these constraints, in particular what has been done since WP20.

In the a_μ kinematics, the external momentum q_4 is always soft, but the other three vary in the integration over virtualities. SDCs can be obtained in two regions

1. Pure short-distance region: $Q_i^2 \gg \Lambda_{\text{QCD}}^2$.
2. Mixed, corner, or Melnikov–Vainshtein (MV), region: $Q_i^2, Q_j^2 \gg Q_k^2, \Lambda_{\text{QCD}}^2$.

In the underlying correlation function with four EM currents, the above limits respectively correspond to the cases where three and two currents are close in position space, which is how the OPE is applied. Here it should be noted that there is no inherent hierarchy on Q_k^2 and Λ_{QCD}^2 in the mixed region, but if one imposes $Q_k^2 \gg \Lambda_{\text{QCD}}^2$, there is partial overlap between the two regions. Below we discuss them separately. Additionally, other kinds of SDCs also exist, such as the Brodsky–Lepage constraint on form factors [528,529]. These are not further discussed in this section.

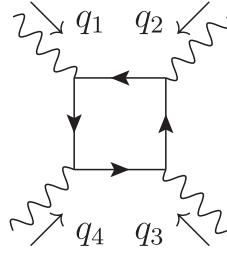


Fig. 42. The perturbative quark loop.
Source: Figure taken from Ref. [527].

5.4.1. Pure short-distance region

Whenever the rest of Euclidean momenta entering into the problem are large, the external soft photon at $q_4 \rightarrow 0$ can be treated as a background field for the OPE [530,531]. This was used for the electroweak contribution to a_μ in Ref. [71]. In Refs. [39,43] it was shown that the HLbL tensor in $g - 2$ kinematics can be directly related to a three-point function with the external field in the background captured in an external state,

$$\Pi^{\mu_1\mu_2\mu_3}(q_1, q_2) = -\frac{1}{e} \int \frac{d^4 q_3}{(2\pi)^4} \left(\prod_{i=1}^3 \int d^4 x_i e^{-iq_i x_i} \right) \langle 0 | T \left(\prod_{j=1}^3 J^{\mu_j}(x_j) \right) | \gamma(q_4) \rangle. \quad (5.7)$$

One may perform a well-defined and systematic OPE at this stage [39]. The leading operator is $F_{\mu\nu}$. Its contribution corresponds to the massless quark loop depicted in Fig. 42, where q_4 corresponds to the soft photon, $q_4 \rightarrow 0$.

The leading power correction, suppressed by the quark mass, was addressed in Ref. [39]. In Ref. [43] the OPE was further developed to study the subleading contributions beyond the perturbative quark loop, through orders $g_s^2 \frac{\Lambda_{\text{QCD}}^4}{Q^4}$, $m_q^2 \frac{\Lambda_{\text{QCD}}^4}{Q^4}$, $m_q \frac{\Lambda_{\text{QCD}}^3}{Q^4}$, and $m_q^3 \frac{\Lambda_{\text{QCD}}^3}{Q^4}$. The appearing operators are

$$\begin{aligned} Q_{1,\mu\nu} &= e e_q F_{\mu\nu}, & Q_{2,\mu\nu} &= \bar{q} \sigma_{\mu\nu} q & Q_{3,\mu\nu} &= i \bar{q} G_{\mu\nu} q, & Q_{4,\mu\nu} &= i \bar{q} \bar{G}_{\mu\nu} \gamma_5 q, \\ Q_{5,\mu\nu} &= \bar{q} q e e_q F_{\mu\nu}, & Q_{6,\mu\nu} &= \frac{\alpha_s}{\pi} G_a^{\alpha\beta} G_{\alpha\beta}^a e e_q F_{\mu\nu}, \\ Q_{7,\mu\nu} &= \bar{q} (G_{\mu\lambda} D_\nu + D_\nu G_{\mu\lambda}) \gamma^\lambda q - (\mu \leftrightarrow \nu), & Q_{[8],\mu\nu} &= \alpha_s (\bar{q} \Gamma q \bar{q} \Gamma q)_{\mu\nu}. \end{aligned} \quad (5.8)$$

Here $\sigma_{\mu\nu} = i/2 [\gamma_\mu, \gamma_\nu]$, $G_{\mu\nu}$ is the gluon field strength tensor, $\bar{G}_{\mu\nu}$ its dual and the $Q_{[8],\mu\nu}$ is a class of four-quark operators with Dirac matrix structure Γ . While the massless quark loop provides the leading contribution through the $F_{\mu\nu}$ operator, the leading quark-mass correction is not driven by the (massive) quark loop. Instead, it is linear in m_q and mediated by the $\bar{q} \sigma^{\mu\nu} q$ operator. The operators in Eq. (5.8) are defined in the $\overline{\text{MS}}$ scheme. The numerical estimates for their nonperturbative condensates labeled X_i are based on various methods and approximations as discussed in Ref. [43] and given in Table 19.

With an OPE in place for the HLbL tensor in the corresponding kinematic regime, the scalar functions in its Lorentz decomposition, $\hat{T}_i$, can be determined using projection techniques [43]. One may then use them to compute the short-distance regions of the a_μ^{HLbL} integral, $Q_{1,2,3} > Q_{\text{min}}$. The numerical impact on a_μ^{HLbL} of the associated terms in the OPE for two values of Q_{min} is shown in Table 19. The massless perturbative quark loop corresponds to contribution $X_{1,0}$, and is clearly dominating over all other corrections. It should be emphasized that also those contributions that are not suppressed by the light-quark masses are negligible at the current level of precision.

Another possibility for large corrections are the gluonic corrections to the massless quark loop, i.e., the leading term in the OPE. These have been worked out fully analytically in Ref. [44]. This perturbative two-loop calculation consists in adding all possible gluonic corrections to the massless perturbative quark loop as in Fig. 43, and reducing them to a set of six known master integrals [44,532,533]. The main conclusion is that the gluonic corrections are of the order of $\sim 10\%$ depending on the value of α_s used. In Fig. 44 we show the massless quark loop contribution and the gluonic corrections as a function of the lower cutoff Q_{min} on the three virtualities, $Q_i > Q_{\text{min}}$. The uncertainty indicated is from varying α_s [44].

The gluonic corrections have been partially included in model implementations. Especially in the holographic method as discussed in Section 5.6 it allowed for a significantly improved matching. In WP20 it was only known that the perturbative quark loop was the leading contribution and the directly subleading term was negligible, as had been shown in Ref. [39]. From Refs. [43,44] it has now been established that the perturbative quark loop is a good representation at the current level of precision up to 10% corrections from perturbative gluonic corrections.

5.4.2. Mixed region (Melnikov–Vainshtein)

In the mixed kinematical region there are two currents close in the underlying HLbL function. This momentum configuration was first studied in Ref. [503], and after WP20 this work was extended in Refs. [54,527]. Following the

Table 19

Numerical contribution to a_μ^{HLbL} for the indicated inputs. The X_i correspond to the operators in Eq. (5.8) and the order of quark masses is indicated as well. The input values for the condensates are described in detail in Ref. [43]. Table from Ref. [43].

Contribution	Inputs [GeV]	$Q_{\min} = 1 \text{ GeV}$	$Q_{\min} = 2 \text{ GeV}$
$X_{1,0}$		1.73×10^{-10}	4.35×10^{-11}
X_{1,m^2}		-5.7×10^{-14}	-3.6×10^{-15}
$X_{2,m}$	$X_2 = -4 \times 10^{-2}$	-1.2×10^{-12}	-7.3×10^{-14}
X_{2,m^3}	$X_2 = -4 \times 10^{-2}$	6.4×10^{-15}	1.0×10^{-16}
X_3	$X_3 = 3.51 \times 10^{-3}$	-3.0×10^{-14}	-4.7×10^{-16}
X_4	$X_4 = 3.51 \times 10^{-3}$	3.3×10^{-14}	5.3×10^{-16}
X_5	$X_5 = -1.56 \times 10^{-2}$	-1.8×10^{-13}	-2.8×10^{-15}
X_6	$X_6 = 2 \times 10^{-2}$	1.3×10^{-13}	2.0×10^{-15}
X_7	$X_7 = 3.33 \times 10^{-3}$	9.2×10^{-13}	1.5×10^{-14}
$X_{8,1}$	$\bar{X}_{8,1} = -1.44 \times 10^{-4}$	3.0×10^{-13}	4.7×10^{-15}
$X_{8,2}$	$\bar{X}_{8,2} = -1.44 \times 10^{-4}$	-1.3×10^{-13}	-2.0×10^{-15}

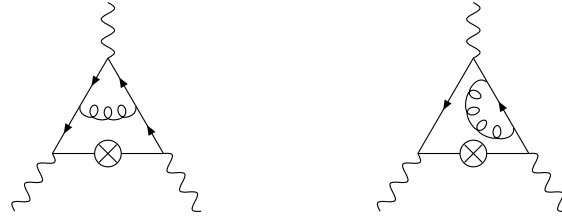


Fig. 43. Gluonic corrections to the perturbative quark loop, with the soft external field denoted by a crossed vertex. Source: Figures taken from Ref. [44].

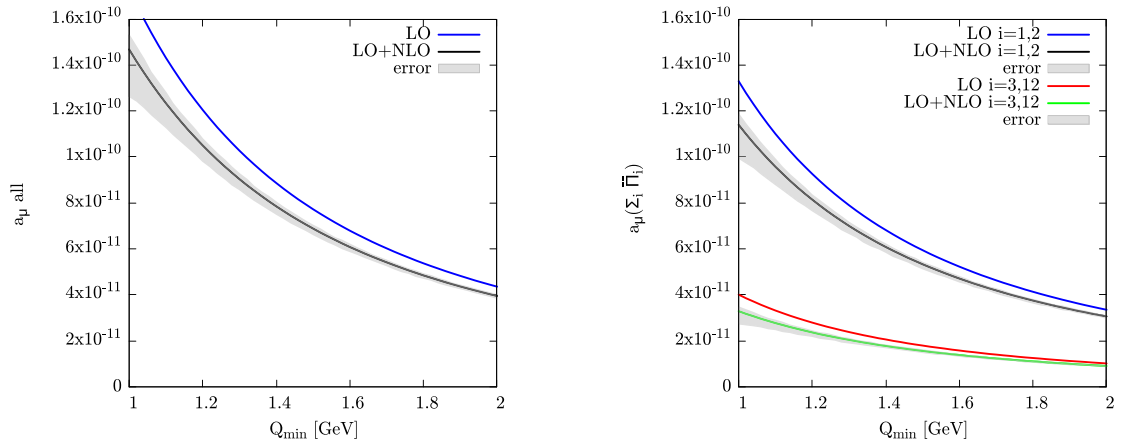


Fig. 44. Short-distance contribution to HLbL as a function of the cutoff on the three virtualities. Left: the full contribution. Right: separated in the longitudinal $\bar{T}_{i=1,2}$ and transverse $\bar{T}_{i=3,12}$ amplitudes. The blue line is the massless quark loop (LO). The full line is the quark loop including gluonic corrections (LO+NLO) and the shaded region the uncertainty due to varying the scale of evaluating α_s . We used the value $\alpha_s(1.5 \text{ GeV}) = 0.3501$ and varied the scale of α_s from $Q_{\min}/\sqrt{2}$ to $\sqrt{2}Q_{\min}$. The corresponding error becomes large below about 1.2 GeV and may be slightly overoptimistic for low $Q_{\min}$ values depending on the size of higher-order corrections.

notation of the latter references, for the specific case $Q_1^2, Q_2^2 \gg Q_3^2$, Λ_{QCD}^2 the HLbL tensor can be written as

$$\Pi^{\mu_1\mu_2\mu_3\mu_4} = \sum_{j,k} \frac{ie_{q_j}e_{q_k}}{e^2} \int \frac{d^4q_4}{(2\pi)^4} \int d^4x_1 \int d^4x_2 e^{-i(q_1x_1+q_2x_2)} \langle 0 | T \{ j^{\mu_1}(x_1) j^{\mu_2}(x_2) \} | \gamma^{\mu_3}(q_3) \gamma^{\mu_4}(q_4) \rangle. \quad (5.9)$$

The OPE between the two above currents at tree level can be diagrammatically depicted as in Fig. 45(a), and including gluonic corrections as in Fig. 45(b)–(e). Additional topologies corresponding to operators made out of photons and gluons start at $D = 4$. The large Euclidean scale in the expansion is $\hat{Q} = \sqrt{-\hat{q}^2}$, where $\hat{q} = (q_1 - q_2)/2$. The leading term (dimension $D = 3$) studied in Ref. [503] enters as $1/\hat{Q}^2$, and is directly related to the axial current with two nonperturbative form factors, ω_L and ω_T . It was also pointed out that the nonrenormalization theorems for the axial anomaly have significant consequences for $g - 2$. The implementation was discussed extensively in WP20 and also more

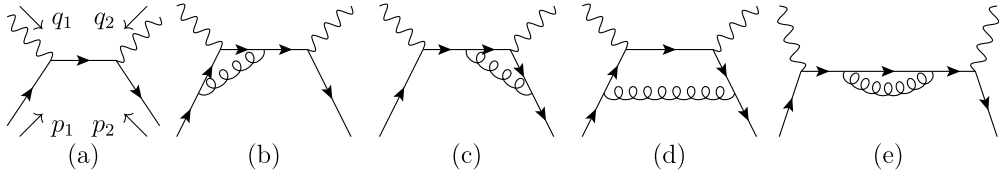


Fig. 45. Diagrammatic depiction of the two-current OPE in the mixed MV kinematical region. Here the particular case $Q_1^2, Q_2^2 \gg Q_3^2, \Lambda_{\text{QCD}}^2$ was chosen.

Source: Figures taken from Ref. [54].

Table 20

The leading power n of $1/\bar{Q}_k$, where the OPE through $D = 4$ lacks prediction. Table from Ref. [54].

	$\bar{Q}_1$	$\bar{Q}_2$	$\bar{Q}_3$
$\hat{H}_1$	5	5	4
$\hat{H}_4$	3	3	5
$\hat{H}_7$	4	5	6
$\hat{H}_{17}$	5	5	5
$\hat{H}_{39}$	5	5	5
$\hat{H}_{54}$	5	5	5

recently, see Refs. [40,42,45,47,534–543]. An update on the implementation in the different approaches can be found in the following, see Sections 5.5, 5.6, and 5.7.

The extension of the leading-order result to dimension $D = 4$ in Refs. [54,527] leads to a better understanding of the power counting and of the structure of perturbative and nonperturbative corrections to the HLbL tensor in the studied kinematic regime. Additionally, new constraints on the scalar functions determining the HLbL tensor were derived. Through dimension $D = 4$ the independent operators are [54,527]

$$\begin{aligned}
 D = 3 : \quad \mathcal{O}_{q3}^{\alpha\beta\gamma} &= \bar{q} \left[\gamma^\alpha \gamma^\beta \gamma^\gamma - \gamma^\gamma \gamma^\beta \gamma^\alpha \right] q, \\
 D = 4 : \quad \mathcal{O}_{q4}^{\alpha\beta} &= \bar{q} \gamma^\beta \left[\vec{D}^\alpha - \overleftarrow{D}^\alpha \right] q, \\
 \mathcal{O}_{\text{FF},1}^{\alpha\beta} &= F^{\alpha\gamma} F_\gamma^\beta, \quad \mathcal{O}_{\text{FF},2}^{\alpha\beta} = F^{\gamma\delta} F_{\gamma\delta} g^{\alpha\beta}, \quad \mathcal{O}_F = F \times F, \\
 \mathcal{O}_{\text{GG},1}^{\alpha\beta} &= G^{\alpha\gamma} G_\gamma^\beta, \quad \mathcal{O}_{\text{GG},2}^{\alpha\beta} = G^{\gamma\delta} G_{\gamma\delta} g^{\alpha\beta}, \quad \mathcal{O}_G = G \times G.
 \end{aligned} \tag{5.10}$$

The operators $\mathcal{O}_F$ and $\mathcal{O}_G$ represent classes of operators with Lorentz structure different from $\mathcal{O}_{\text{FF},i}^{\alpha\beta}$ and $\mathcal{O}_{\text{GG},i}^{\alpha\beta}$ [54]. In Ref. [54] also gluonic corrections to the OPE were included as well as the corresponding renormalization through dimension $D = 4$. This in particular proved the conjecture of Ref. [537] that at $D = 3$ the gluonic corrections enter as $-\alpha_s/\pi$ times the original result of Ref. [503]. It was also explicitly checked that in the overlap region between OPEs, where $Q_3^2 \gg \Lambda_{\text{QCD}}^2$, there is analytic agreement between derived scalar functions $\hat{H}_i$ through the relevant orders. The current limits of predictivity of the OPE in the three mixed regions for the $\hat{H}_i$ are shown in Table 20 (rewritten in terms of the corner variable $\bar{Q}_k = Q_i + Q_j$).

Although there in principle is a plethora of different nonperturbative form factors at dimension $D = 4$, it was found in Ref. [54] that, when incorporating the different components of the HLbL tensor into the $g - 2$ integrand, a very strong cancellation occurs for all computed contributions. This suggests that, up to chiral corrections, the leading term of the integrand in the corresponding kinematic regime may be determined by the axial current form factors ω_L and ω_T .

To assess how well the leading OPE contribution works, Ref. [54] compared the derived expansion through NLO for the sum $\sum T_i \bar{H}_i$ in the $g - 2$ integral with the LO one given by the quark loop. Since the NLO result depends on the nonperturbative form factors ω_L and ω_T , the perturbative limit was taken where $2\omega_T = \omega_L = -2/Q_i^2$ in the region where Q_i^2 is the small scale. At fixed large scale $\bar{Q}_i = 10 \text{ GeV}$ and corner variable $y_{jk} = (Q_j - Q_k)/Q_i = 0$ one can study $\sum T_i \bar{H}_i$ by varying the small scale Q_i in units of $\bar{Q}_i/2$. This is shown in Fig. 46 for $y_{jk} = 0$. Overall, there is good agreement up to regions relatively close to the symmetric point, where Q_i tends towards $Q_i/2$.

5.5. Dispersive approach

Within the dispersive framework [34,36,544–547], the different contributions to HLbL can be ordered according to the mass threshold in the unitarity relation. The lightest intermediate state is a single neutral pion, followed by two charged

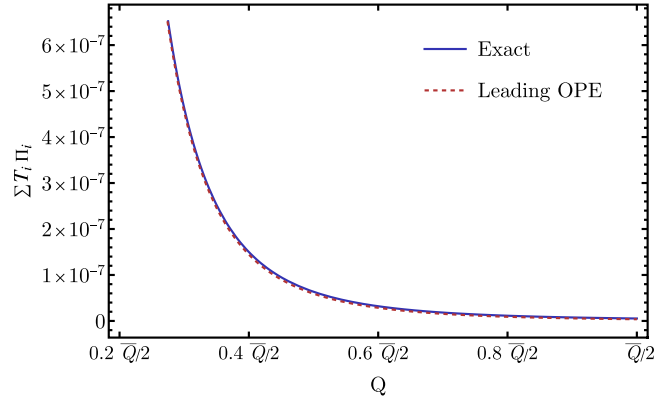


Fig. 46. Numerical comparison between the quark loop and OPE in the MV region.
Source: Figure adapted from Ref. [54].

pions, etc. This ordering turns out to be roughly reflected in the numerical dominance of the individual contributions to a_μ^{HLbL} . Single-pseudoscalar contributions are discussed in Section 5.5.1 and two-particle intermediate states in Section 5.5.2. A rigorous dispersive description of three-particle intermediate states would be challenging—this contribution is assumed to be negligible unless resonantly enhanced. It is therefore described in terms of single-resonance contributions in the narrow-width limit, see Section 5.5.3. The infinite sum over hadronic intermediate states needs to saturate the SDCs discussed in Section 5.4. In practice, only a finite number of intermediate states can be related to data input, hence the remainder typically involves some modeling, which is however controlled by the matching to the SDCs, see Section 5.5.4.

5.5.1. Pseudoscalars

In WP20 [1], the largest individual contribution to HLbL, the π^0 pole, was determined from three independent approaches: dispersively [37,548], using Canterbury approximants (CA) [35], and on the lattice [549]. For the subleading η - and η' -pole contributions, however, the consensus relied only on the determination from Canterbury approximants [35], as dispersion-theoretical or lattice-QCD calculations had not been completed. Since then, progress has been made on both latter approaches, see Section 6.3 for that on the lattice. Meanwhile, a dedicated effort to evaluate the $\eta^{(\prime)}$ -pole contributions with dispersion relations led to their first determination [56,550], which requires the dispersive reconstruction of the pertinent doubly-virtual TFFs [287,551–555] as input, extending a similar program for the π^0 TFF [37,282–284,548,556]. The pseudoscalar TFFs describe the decay $P(q_1 + q_2) \rightarrow \gamma^*(q_1, \mu)\gamma^*(q_2, \nu)$, formally defined by the matrix elements

$$i \int d^4x e^{iq_1 \cdot x} \langle 0 | T \{ j_\mu(x) j_\nu(0) \} | P(q_1 + q_2) \rangle = \epsilon_{\mu\nu\rho\sigma} q_1^\rho q_2^\sigma F_{P\gamma^*\gamma^*}(q_1^2, q_2^2), \quad (5.11)$$

where the normalization $F_{P\gamma^*\gamma^*}(0, 0) \equiv F_{P\gamma\gamma}$ is determined by the $P \rightarrow \gamma\gamma$ decay rate.

Such a reconstruction, based on the low-lying singularities, data input for $\eta^{(\prime)} \rightarrow \gamma\gamma$, $\eta^{(\prime)} \rightarrow \pi^+\pi^-\gamma$, $\eta^{(\prime)} \rightarrow 2(\pi^+\pi^-)$, and $e^+e^- \rightarrow e^+e^-\eta^{(\prime)}$, as well as an implementation of asymptotic constraints in the dispersive representations of η and η' transition form factors comparable to Refs. [37,548] was recently concluded [56,550]. This form factor representation reads

$$F_{\eta^{(\prime)}\gamma^*\gamma^*} = F_{\eta^{(\prime)}}^{(I=1)} + F_{\eta^{(\prime)}}^{(I=0)} + F_{\eta^{(\prime)}}^{\text{eff}} + F_{\eta^{(\prime)}}^{\text{asym}}. \quad (5.12)$$

In contradistinction to the π^0 case, the final-state photons here carry either both isovector ($I = 1$) or both isoscalar ($I = 0$) quantum numbers, leading to the two independent terms in Eq. (5.12). The first of these, the isovector dispersive piece, dominates the TFFs at low energies and can be cast in the form of a double-spectral representation

$$F_{\eta^{(\prime)}}^{(I=1)}(-Q_1^2, -Q_2^2) = \frac{1}{\pi^2} \int_{4M_\pi^2}^{\Lambda^2} dx dy \frac{\rho_{\eta^{(\prime)}}(x, y)}{(x + Q_1^2)(y + Q_2^2)} + (Q_1 \leftrightarrow Q_2), \quad (5.13)$$

with the double-spectral density

$$\rho_{\eta^{(\prime)}}(x, y) = \frac{x\sigma_\pi^3(x)}{192\pi} \text{Im} \left\{ [F_\pi^V(x)]^* \mathcal{F}_{\eta^{(\prime)}\pi\pi\gamma^*}(x, y) \right\}, \quad (5.14)$$

where $\sigma_\pi(s) = \sqrt{1 - 4M_\pi^2/s}$, and F_π^V is the pion vector form factor (VFF). $\mathcal{F}_{\eta^{(\prime)}\pi\pi\gamma^*}(t, k^2)$ is the P -wave amplitude for $\eta^{(\prime)}\gamma^*(k^2) \rightarrow \pi^+\pi^-$. It is constructed based on dispersive representations of $\eta^{(\prime)} \rightarrow 2(\pi^+\pi^-)$ amplitudes (cf. Ref. [518]) with the approximation of taking only pairwise rescattering of $\pi^+\pi^-$ into account. In addition, factorization-breaking

effects are included by considering the exchange of $a_2(1320)$ tensor mesons as a left-hand-cut contribution, which leads to a description by means of an inhomogeneous Omnès representation. The subtraction constants of these representations, which are subject to constraints from ChPT, are eventually fixed by fits to $\eta^{(\prime)} \rightarrow \pi^+\pi^-\gamma$ decay data [557,558]. For the description of F_π^V an Omnès representation [269] is utilized. Due to technical reasons [559], in the solution of the inhomogeneous Omnès problem, a representation of the $\pi\pi$ P -wave phase shift based on unitarized ChPT [337,560–562] is employed. The resulting representation of F_π^V is fixed by fitting to data of $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$ [192].

The isoscalar channel, on the other hand, is dominated by the narrow ω and ϕ resonances in the low-energy regime. Their (small) contribution is parameterized in a VMD ansatz,

$$F_{\eta^{(\prime)}}^{(I=0)}(-Q_1^2, -Q_2^2) = \sum_{V \in \{\omega, \phi\}} \frac{w_{\eta^{(\prime)}V\gamma} F_{\eta^{(\prime)}\gamma\gamma} M_V^4}{(M_V^2 + Q_1^2)(M_V^2 + Q_2^2)}, \quad (5.15)$$

with M_V as mass parameters and weight factors $w_{\eta^{(\prime)}V\gamma}$ [552,563] determined from the respective decays widths of $\omega \rightarrow \eta\gamma$, $\eta' \rightarrow \omega\gamma$, $\phi \rightarrow \eta^{(\prime)}\gamma$, and $\omega, \phi \rightarrow e^+e^-$ [225].

The third term in Eq. (5.12), the effective-pole piece, parameterizes higher intermediate states. It serves two purposes: it ensures that a sum rule for the form factor normalization is exactly fulfilled, and additionally, helps to describe high-energy, singly-virtual data on $e^+e^- \rightarrow e^+e^-\eta^{(\prime)}$. Two distinct parameterizations have been implemented,

$$\begin{aligned} F_{\eta^{(\prime)}}^{\text{eff}(A)}(-Q_1^2, -Q_2^2) &= \frac{g_{\text{eff}} F_{\eta^{(\prime)}\gamma\gamma} M_{\text{eff}}^4}{(M_{\text{eff}}^2 + Q_1^2)(M_{\text{eff}}^2 + Q_2^2)}, \\ F_{\eta^{(\prime)}}^{\text{eff}(B)}(-Q_1^2, -Q_2^2) &= \sum_{V \in \{\rho', \rho''\}} \frac{g_V F_{\eta^{(\prime)}\gamma\gamma} M_V^4}{(M_V^2 + Q_1^2)(M_V^2 + Q_2^2)}, \end{aligned} \quad (5.16)$$

where in the variant (A) the effective coupling g_{eff} is used to restore the normalization. For both η and η' , $|g_{\text{eff}}|$ is found in a range up to about 10%. In this variant, the effective mass parameter is fit to singly-virtual TFF data [564–567] and is found in the range (1.3–2.2) GeV. On the other hand, variant (B) describes the two poles of $\rho(1450) \equiv \rho'$ and $\rho(1700) \equiv \rho''$ resonances, with their mass parameters taken from Ref. [225]. One resonance coupling g_V is used to fulfill the normalization sum rule, while the other is fit to the singly-virtual TFF data.

The last term in Eq. (5.12) is used to implement constraints from pQCD [528,529,568]. As η and η' are much more massive than the π^0 , the masses are explicitly accounted for in this representation. Starting from an asymptotic piece in the massless limit [37,548] based on a dispersive reformulation [569] of the leading term of the light-cone sum rule expansion evaluated with asymptotic distribution amplitudes [528,529,568,570–572], a form including mass effects can be written as [55,56,543,550]

$$\begin{aligned} F_{\eta^{(\prime)}}^{\text{asym}}(q_1^2, q_2^2) &= \frac{-\bar{F}_{\text{asym}}^{\eta^{(\prime)}}}{M_{\eta^{(\prime)}}^4} \int_{2s_m}^{\infty} dv \left[\frac{q_2^2}{v - q_1^2} \left(\frac{1}{v - q_1^2 - q_2^2} - \frac{1}{q_1^2 - q_2^2} \right) f_{\eta^{(\prime)}}^{\text{asym}}(v, q_1^2) + (q_1^2 \leftrightarrow q_2^2) \right], \\ f_{\eta^{(\prime)}}^{\text{asym}}(v, q^2) &= \frac{(v - 2q^2)^2 - M_{\eta^{(\prime)}}^2 v}{\sqrt{(v - 2q^2)^2 - 2M_{\eta^{(\prime)}}^2 v + M_{\eta^{(\prime)}}^4}} + 2q^2 - v, \end{aligned} \quad (5.17)$$

which does not contribute to the singly-virtual direction, but maintains the correct doubly-virtual behavior, especially the limit

$$\lim_{Q^2 \rightarrow \infty} Q^2 F_{\eta^{(\prime)}\gamma^*\gamma^*}(-Q^2, -Q^2) = \frac{1}{3} \bar{F}_{\text{asym}}^{\eta^{(\prime)}}. \quad (5.18)$$

The asymptotic coefficient $\bar{F}_{\text{asym}}^{\eta^{(\prime)}}$ is extracted from data utilizing the singly-virtual limit

$$\lim_{Q^2 \rightarrow \infty} Q^2 F_{\eta^{(\prime)}\gamma^*\gamma^*}(-Q^2, 0) = \bar{F}_{\text{asym}}^{\eta^{(\prime)}}, \quad (5.19)$$

of the isovector, isoscalar, and effective-pole pieces in the TFF representation.

The pole contributions to HLbL can be evaluated according to the well-known expression

$$\begin{aligned} a_\mu^{\text{P-pole}} &= \left(\frac{\alpha}{\pi} \right)^3 \int dQ_1 dQ_2 d\tau \left[w_{1/2}(Q_1, Q_2, \tau) F_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) F_{P\gamma^*\gamma^*}(-Q_2^2, 0) \right. \\ &\quad \left. + w_2(Q_1, Q_2, \tau) F_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) F_{P\gamma^*\gamma^*}(-Q_3^2, 0) \right], \end{aligned} \quad (5.20)$$

where $Q_3^2 \equiv Q_1^2 + Q_2^2 + 2\tau Q_1 Q_2$, and the explicit form of the weight functions $w_{1/2}(Q_1, Q_2, \tau)$ is given, e.g., in Refs. [35,37,573,574]. Obviously, Eq. (5.20) requires the spacelike singly- and doubly-virtual TFFs as input. In Fig. 47, we therefore compare the dispersively reconstructed transition form factors $F_{\eta^{(\prime)}\gamma^*\gamma^*}(-Q^2, 0)$ and $F_{\eta^{(\prime)}\gamma^*\gamma^*}(-Q^2, -Q^2)$ to the previous approach based on CA [35] adopted in WP20 [1], as well as to lattice-QCD calculations by ETM [575]

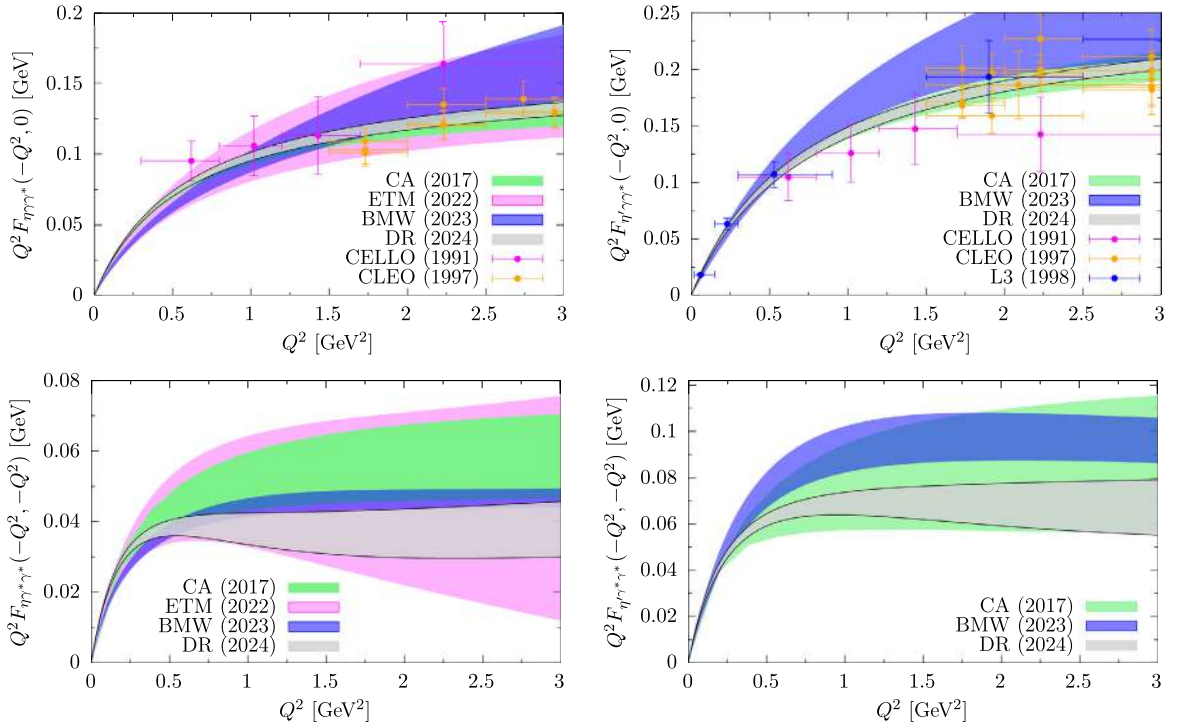


Fig. 47. Comparison of the singly-virtual (top) and doubly-virtual (bottom) dispersive η (left) and η' (right) TFFs (DR [56,550] with the results of CA [35], the lattice-QCD approach by ETM [575] and BMW [576], and singly-virtual experimental data from CELLO [564], CLEO [565], and L3 [566].

and BMW [576]. In general, we find very good agreement between the different approaches, as well as with the singly-virtual data [564–566] at low energies that the dispersive representation has not included in the fit. The narrowness of the uncertainty band, in particular in the singly-virtual direction, demonstrates the usefulness of the many constraints combined in the dispersive reconstruction of the TFFs. In the doubly-virtual direction, there is a tendency to approach the asymptotic behavior dictated by Eq. (5.18) slightly more slowly than found in Ref. [35].

Numerically, the η - and η' -pole contributions are determined as

$$\begin{aligned} a_{\mu}^{\eta\text{-pole}} &= 14.72(56)_{\text{norm}}(32)_{\text{disp}}(23)_{\text{BL}}(54)_{\text{asym}} \times 10^{-11} = 14.72(87) \times 10^{-11}, \\ a_{\mu}^{\eta'\text{-pole}} &= 13.50(48)_{\text{norm}}(15)_{\text{disp}}(20)_{\text{BL}}(48)_{\text{asym}} \times 10^{-11} = 13.50(72) \times 10^{-11}, \end{aligned} \quad (5.21)$$

where the different uncertainties arise from the following sources: (i) “norm”: the normalization uncertainty reflecting the uncertainty on the average of experimental determinations on the $\Gamma(\eta^{(\prime)} \rightarrow \gamma\gamma)$ decay widths; (ii) “disp”: the dispersive uncertainty, where different variants on the underlying representations of the isovector piece as well as different cutoff values for the dispersive integrals have been taken into account; (iii) “BL”: the fit to high-energy singly-virtual TFF data approaching the Brodsky–Lepage limit, as well as the difference of the two variants in Eq. (5.16); (iv) “asym”: the variation of the onset of the asymptotic piece through variation of s_m in Eq. (5.17) as well as variation of the asymptotic coefficient between the data-driven determination, by means of the ($I = 1$), ($I = 0$), and effective pieces, and lattice-QCD determinations [577,578]. Furthermore, the total uncertainty refers to the quadratic sum.

We mention two more data-driven consistency checks on the dispersive reconstruction of the η and η' transition form factors. The doubly-virtual η transition form factor is closely linked to the $\gamma^{(*)} \rightarrow \eta\pi^+\pi^-$ amplitude. In Ref. [554], $\pi^+\pi^-$ spectra in $e^+e^- \rightarrow \eta\pi^+\pi^-$ data [579,580] have been analyzed for consistency with those measured in the corresponding real-photon decay $\eta \rightarrow \pi^+\pi^-\gamma$ [557]. It was found that the dependence on the $\pi^+\pi^-$ invariant mass factorizes to a larger degree from that on the photon virtuality than what a data-driven model of singularities in the crossed channel induces [553], already suggesting rather mild breaking of factorization in the doubly-virtual η TFF at low-to-medium energies. For the η' singly-virtual TFF, a dispersive analysis [287] similarly relies on input for $\eta' \rightarrow \pi^+\pi^-\gamma$ [558] and the pion VFF (see Refs. [581,582] for related works). As both display a prominent isospin-breaking signal due to ρ - ω mixing [583], in Ref. [287], these were combined in a consistent coupled-channel framework. While such effects are of relevance for high-precision data on $\eta' \rightarrow \ell^+\ell^-\gamma$ in the future, the impact of these isospin-breaking corrections in the spacelike region is rather negligible.

5.5.2. Two-meson contributions: boxes and S -wave rescattering

The next-lightest intermediate state after a single pion is given by the two-pion state. It can be split further according to intermediate states in the crossed channels, resulting in the dispersively defined pion-box contribution [36,584]

$$a_{\mu}^{\pi\text{-box}} = -15.9(2) \times 10^{-11} \quad (5.22)$$

and two-pion rescattering contributions. The first only depends on the pion VFF, whereas the rescattering contribution can be expanded into partial waves and related to $\gamma^*\gamma^* \rightarrow \pi\pi$ helicity partial waves [36]. Both contributions have been discussed in detail in WP20.

Similarly to two-pion intermediate states, the dispersive analysis of two-meson intermediate states can be extended to $K\bar{K}$ and $\pi\eta$ intermediate states. Due to the heavier mass, the contribution of the kaon box is small.²⁶ It was previously evaluated based on model input for the kaon VFF [1,586]. In the meantime, a dispersive analysis of the kaon VFFs has become available [46], leading to

$$a_{\mu}^{K^{\pm}\text{-box}} = -0.48(1) \times 10^{-11}, \quad a_{\mu}^{K^0\text{-box}} = -0.5(4) \times 10^{-15}, \quad (5.23)$$

and confirming the evaluations based on model input.

The rescattering contributions are expanded into partial waves, with the S -waves providing a model-independent description of scalar resonances. While the low-energy contribution of the scalar $f_0(500)$ resonance was already treated dispersively in terms of $\pi\pi$ rescattering [36,584], WP20 combined additional scalar and tensor contributions together into $a_{\mu}^{\text{scalars+tensor}} = -1(3) \times 10^{-11}$ [1]. In Ref. [45], the S -wave rescattering contribution was extended by using as input the helicity partial waves for $\gamma^*\gamma^* \rightarrow \pi\pi/K\bar{K}_{I=0}$ [249,351–355], derived from a modified coupled-channel Muskhelishvili–Omnès analysis [355]. The input for the coupled-channel hadronic rescattering $\pi\pi/K\bar{K}_{I=0}$ was taken from the dispersive analysis of Ref. [587]. This covered the region of the $f_0(980)$ resonance within the dispersive rescattering formalism, updating the isospin $I = 0$ contribution. Together with the $I = 2$ estimates of Refs. [36,584], an S -wave rescattering contribution of

$$a_{\mu}^{\text{HLbL}}[\pi\pi/K\bar{K}_{I=0}, S\text{-waves}] = -8.7(1.0) \times 10^{-11} \quad (5.24)$$

was obtained. By investigating the line shape of the resonance, an $f_0(980)$ contribution of $a_{\mu}^{\text{HLbL}}[f_0(980)]|_{\text{rescattering}} = -0.2(1) \times 10^{-11}$ was isolated. In addition, Ref. [45] performed a comparison of the dispersive resonance contribution in terms of two-particle S -wave rescattering with a narrow-width approximation, using the same tensor decomposition of the dispersive framework. The dispersive definition of narrow-width contributions eliminates model-dependent nonpole contributions, but the use of a consistent framework, in particular the choice of the tensor basis, is crucial for a meaningful combination of different hadronic contributions: the independence of the dispersive result for HLbL on the choice of the tensor basis only follows from the requirement that a set of sum rules be fulfilled by the scalar functions [36]. While the pseudoscalar poles and the pion box fulfill the sum rules individually, this is no longer the case for rescattering contributions or narrow resonances. This implies that these individual contributions depend on the choice of basis, with only the sum over all hadronic states being unambiguous.

In Ref. [45], it was observed that similarly to the single-channel solution of [36,584] the coupled-channel solution for $\gamma^*\gamma^* \rightarrow \pi\pi/K\bar{K}_{I=0}$ only leads to small violations of the sum rule relevant for S -waves. Therefore, the obtained rescattering contribution is essentially basis independent. Differences in the narrow-width estimate of the $f_0(980)$ contribution between Ref. [45] and the earlier estimate of Ref. [588] were attributed mainly to the propagator definition of the scalar resonance, which only corresponds to a dispersive pole in a different tensor basis and therefore indicates sum-rule violations of the narrow resonance. To a lesser degree, the differences are related to the input for the transition form factors [589,590]. The comparison to Ref. [591] was complicated by the fact that this estimate was based on a single helicity amplitude, which suffers from kinematic singularities that had to be removed by hand via angular averages.

Based on the narrow-width approximation, Ref. [45] also evaluated the contribution of the isospin $I = 1$ scalar $a_0(980)$. In Ref. [52], the S -wave contribution of $\pi\eta/K\bar{K}_{I=1}$ intermediate states has been evaluated dispersively in terms of helicity partial waves. By incorporating precise experimental data from two-photon processes [592,593] and utilizing the modified Muskhelishvili–Omnès formalism, the dispersive analysis provides

$$a_{\mu}^{\text{HLbL}}[\pi\eta/K\bar{K}_{I=1}, S\text{-waves}] = -0.44(5) \times 10^{-11}. \quad (5.25)$$

This result represents an order of magnitude improvement in precision compared to the previous narrow-width estimate and is of the same order as the charged kaon box contribution in HLbL scattering. In total, the $I = 0, 1, 2$ rescatterings (including the effects of the $f_0(980)$ and $a_0(980)$ resonances) amount to a scalar contribution of

$$a_{\mu}^{\text{HLbL}}[\text{scalars}] = -9.1(1.0) \times 10^{-11}. \quad (5.26)$$

In Ref. [45], the contribution of the even heavier scalars $f_0(1370)$ and $a_0(1450)$ was estimated to be around -1×10^{-11} , but it was pointed out that the two-photon couplings and TFFs of these states are highly uncertain. In Ref. [55,594], these heavy-scalar contributions below a matching scale $Q_0 = 1.5$ GeV were estimated as

$$a_{\mu}^{\text{HLbL}}[\text{heavy scalars}, Q_i < 1.5 \text{ GeV}] = -0.7(3) \times 10^{-11}, \quad (5.27)$$

²⁶ Similarly, the proton box contribution (F_1 form factor) is very small, $0.18(1) \times 10^{-11}$ [585].

using a simple quark model for the TFFs [589]

$$\frac{\mathcal{F}_1^S(q_1^2, q_2^2)}{\mathcal{F}_1^S(0, 0)} = \frac{\Lambda_S^2(3\Lambda_S^2 - q_1^2 - q_2^2)}{3(\Lambda_S^2 - q_1^2 - q_2^2)^2}, \quad \frac{\mathcal{F}_2^S(q_1^2, q_2^2)}{\mathcal{F}_1^S(0, 0)} = -\frac{2\Lambda_S^4}{3(\Lambda_S^2 - q_1^2 - q_2^2)^2}, \quad (5.28)$$

with the normalization $\mathcal{F}_1^S(0, 0)$ determined from the two-photon widths $\Gamma_{\gamma\gamma}$ as discussed in Ref. [45]. The scale is set to $\Lambda_S = M_\rho$, based on the observation that for the scalar mesons $f_0(980)$ and $a_0(980)$, adopting the expected VMD scale results in better agreement with the explicit calculations using helicity partial waves [45,52].

5.5.3. Axial-vector and tensor contributions

As already discussed in WP20, the issue of sum-rule violations is more severe for the contribution of resonances beyond spin zero. At the time of WP20, the inclusion of axial-vector states or tensor resonances was not possible due to the presence of kinematic singularities. In the case of axial-vector states, this problem has been solved in Ref. [49] with the construction of an optimized basis that allows for the inclusion of axial-vector states while leaving previously evaluated contributions like scalar resonances in terms of S -wave rescattering unchanged. The contribution of pseudoscalar poles and the scalar-QED pion box are left unchanged, as they do not depend on the choice of basis: while pseudoscalar poles do not contribute to the set of sum rules, the box contributions fulfill them individually. Rescattering contributions or narrow resonances do not fulfill the sum rules exactly, which implies that these individual contributions depend on the choice of basis, with only the sum over all hadronic states being unambiguous.

The new basis optimized for axial-vector states has been tested by analyzing the convergence behavior of the partial-wave expanded pion box, observing an even faster convergence when summing up higher partial waves than in the original basis [49]. The new basis does not fully solve the issue of kinematic singularities, but significantly simplifies the remaining singularity structure even for tensor resonances. The tensor-meson contribution can in general be described in a narrow-width approximation in terms of five TFFs [590]. In the new basis, no kinematic singularities show up in the tensor-resonance contribution if only the TFF $\mathcal{F}_1^T$ is retained, as well as in the special cases when only $\mathcal{F}_{1,3}^T$ or $\mathcal{F}_{2,3}^T$ are present. In Refs. [55,594], the tensor contributions are estimated using the simple quark model of Ref. [589], which indeed only gives a contribution to $\mathcal{F}_1^T$ of the form

$$\frac{\mathcal{F}_1^T(q_1^2, q_2^2)}{\mathcal{F}_1^T(0, 0)} = \left(\frac{\Lambda_T^2}{\Lambda_T^2 - q_1^2 - q_2^2} \right)^2, \quad \mathcal{F}_{2-5}^T(q_1^2, q_2^2) = 0. \quad (5.29)$$

For a more realistic description at low energies, the contribution of the tensor mesons $f_2(1270)$, $a_2(1320)$, and $f'_2(1525)$ below the matching scale $Q_0 = 1.5$ GeV are calculated for $\Lambda_T = M_\rho$

$$a_\mu^{\text{HbL}}[\text{tensors}, Q_i < 1.5 \text{ GeV}] = -2.5(3) \times 10^{-11}, \quad (5.30)$$

where the uncertainty is propagated from the two-photon widths only. This reflects the expectation that the vector-meson scale should be important at low energies both because the $f_2(1270)$ arises primarily from the unitarization of vector-meson left-hand cuts and because the momentum dependence of the respective TFFs is again largely determined by vector-meson physics [249,351–355,546]. Further arguments to consider only $\mathcal{F}_1^T$ as a first estimate originate from the high-energy limit, since $\mathcal{F}_1^T$ gives the dominant contribution in the light-cone expansion [590], and from the low-energy behavior, as the leading term in a chiral-Lagrangian approach only generates $\mathcal{F}_1^T$ [595]. Finally, up to a small correction from $\mathcal{F}_2^T$, it is $\mathcal{F}_1^T$ that determines the on-shell two-photon decay width.²⁷

The tensor-resonance estimate of Ref. [597] included in WP20, $a_\mu^{\text{tensors}} = 0.9(1) \times 10^{-11}$, was based on a model-dependent propagator definition that resulted in the need to average over kinematic singularities [591], see Ref. [45] for further discussion. It can now be replaced by an estimate within the same basis as other contributions. However, we emphasize that even in the new basis, tensor-meson contributions beyond the effect of $\mathcal{F}_1^T$ still suffer from kinematic singularities. This issue has been solved within a new dispersive framework in triangle kinematics [507], see Section 5.11.1. The new framework requires further sub-processes as input, but opens a path towards a model-independent evaluation of tensor contributions in terms of two-meson rescattering [354,355]. As in the case of heavy scalars, the final contribution of tensor states should be studied in combination with the SDCs, see below.

In contrast to the rather small tensor contributions, model estimates point to a substantial contribution of axial-vector mesons as discussed in Sections 5.6 and 5.7. These contributions are tightly connected to the saturation of SDCs, see Sections 5.4, 5.5.4, and 5.10. With the solution of the problem of kinematic singularities for axial contributions in Ref. [49], the problem is reduced to the determination of the input for the axial-vector TFFs, which at present are poorly constrained from data. This is related to the fact that the Landau–Yang theorem forbids the decay of an axial-vector meson into two on-shell photons, hence constraining the TFFs requires data from processes involving at least one off-shell photon. For the three physical TFFs that enter the dispersive representation, the asymptotic behavior has been worked out in Ref. [590]

²⁷ The hQCD scenario described in Section 5.6.5, featuring $\mathcal{F}_{1,3}^T$, can also be evaluated in the optimized basis from Ref. [49], but the appearance of $\mathcal{F}_3^T$ is rather unexpected from the arguments given here, e.g., in contrast to hQCD the Lagrangian realization in Ref. [596] violates gauge invariance. However, given the limited information on tensor TFFs presently available, for the final compilation in Section 5.10 both the $\mathcal{F}_1^T$ and $\mathcal{F}_{1,3}^T$ scenarios will be taken into account.

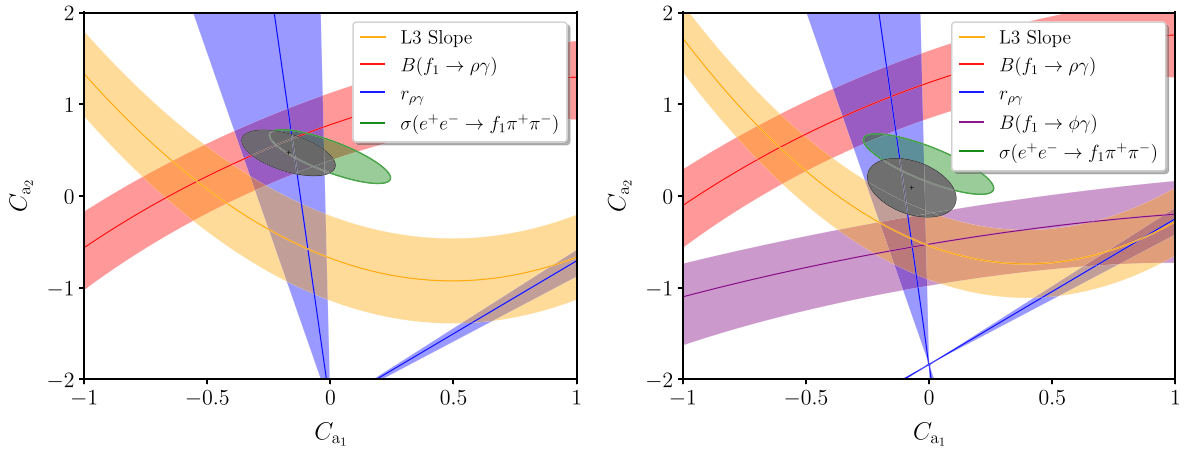


Fig. 48. Constraints on the normalizations of the two antisymmetric f_1 TFFs, C_{a_1} and C_{a_2} , excluding (left) and including (right) a possible constraint from $f_1 \rightarrow \phi\gamma$. The bands come from the L3 normalization and slope, branching and helicity fraction for $f_1 \rightarrow \rho\gamma$, and $\sigma(e^+e^- \rightarrow f_1\pi^+\pi^-)$, while the gray ellipse indicates the global fit. Source: Figures taken from Ref. [48].

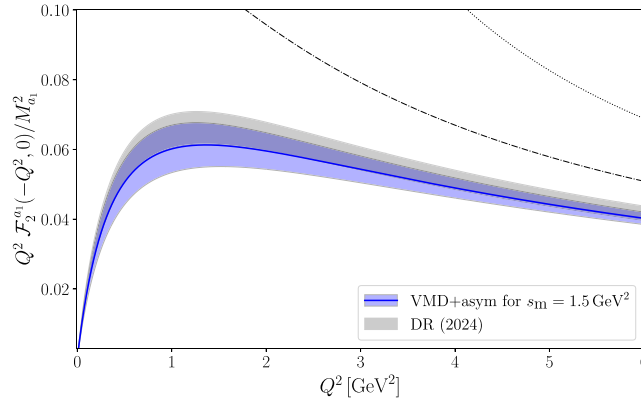


Fig. 49. Comparison of the a_1 singly-virtual TFF $\mathcal{F}_2^{a_1}(-Q^2, 0)$ from the VMD representation and U(3) symmetry (blue band) [48,55], in comparison with the dispersive result [51] (gray band).

using a light-cone expansion. In Ref. [48,543], a VMD representation was used to combine all available experimental constraints on the TFFs of the $f_1(1285)$. The minimal particle content that is necessary to fulfill asymptotic constraints as well as reproducing data from $e^+e^- \rightarrow f_1\pi^+\pi^-$ requires the inclusion of at least three multiplets of vector mesons. In Ref. [48], the $\rho = \rho(770)$, $\rho' = \rho(1450)$, and $\rho'' = \rho(1700)$ were used for the dominant isovector piece and $\omega = \omega(782)$, $\omega' = \omega(1420)$, $\omega'' = \omega(1650)$ as well as $\phi = \phi(1020)$, $\phi' = \phi(1680)$, $\phi'' = \phi(2170)$ for the isoscalar contributions.

Experimental input concerns the tree-level processes $e^+e^- \rightarrow e^+e^-f_1$ [598], which mainly determines the equivalent two-photon decay width and constrains the symmetric TFF rather well; the decay $f_1 \rightarrow \rho\gamma$; and, for the subleading isoscalar part, $f_1 \rightarrow \phi\gamma$. The decay $f_1 \rightarrow 4\pi$ does not provide information on the TFFs, as it is likely dominated by the decay chain $f_1 \rightarrow a_1\pi \rightarrow 4\pi$. In addition, the loop induced decay $f_1 \rightarrow e^+e^-$, measured for the first time in inverse kinematics by SND [599], could provide valuable information on all three TFFs. Unfortunately, the data from this decay are not yet at the required level of precision to have a meaningful impact on the global fit. Instead, the scattering process $e^+e^- \rightarrow f_1\pi^+\pi^-$ [173,579] was considered in Ref. [48], as it provides further sensitivity also to the antisymmetric TFFs, see Fig. 48. However, even combining all currently available data, at best the normalizations of the f_1 can be determined, while f_1' and a_1 are largely unconstrained. In this situation, estimates for their TFFs can be obtained assuming U(3) symmetry, since the corresponding mixing angle θ_A between f_1 and f_1' was measured by L3 via the ratio of the respective equivalent two-photon decay widths $\tilde{\Gamma}_{\gamma\gamma}$ yielding $\theta_A = 62(5)^\circ$ [598,600].

While full determinations of all three doubly-virtual TFFs are thus difficult to obtain, as reflected by the challenges to extract even the three couplings arising in a VMD-motivated representation for the f_1 from data, in the special case of the singly-virtual a_1 TFF $\mathcal{F}_2^{a_1}$ a cross-check is possible against a recent dispersive description derived in the context of the VVA correlator [51], see Fig. 49. This comparison shows that the assumption of U(3) symmetry leads to a result for the a_1 in surprisingly good agreement. Moreover, both results for $\mathcal{F}_2^{a_1}$ agree well with hQCD models, see Section 5.6.2.

Table 21

Summary of previously evaluated contributions in dispersion theory. The S -wave rescattering subsumes effects that correspond to the light scalar resonances $f_0(500)$, $f_0(980)$, and $a_0(980)$. Table from Ref. [594].

Contribution	$a_\mu [10^{-11}]$	References
π^0, η, η' poles	$91.2^{+2.9}_{-2.4}$	[37,56,548,550]
$\pi^\pm$ box	$-15.9(2)$	[36,584]
$K^\pm$ box	$-0.5(0)$	[46]
S -wave rescattering	$-9.1(1.0)$	[36,45,52,584]
Sum	$65.7^{+3.1}_{-2.6}$	

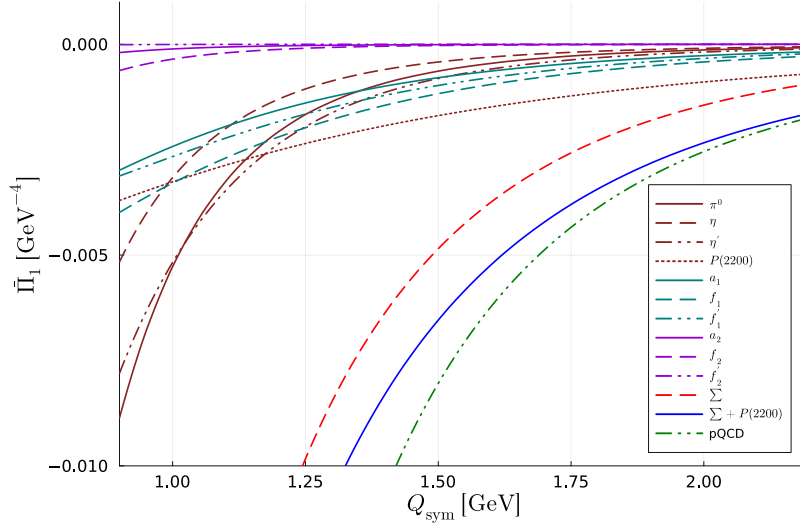


Fig. 50. Matching between the sum of hadronic states, $\Sigma = \pi^0 + \eta + \eta' + a_1 + f_1 + f_1' + a_2 + f_2 + f_2'$, in $\tilde{\Pi}_1$ as a function of $Q_{\text{sym}} \equiv Q_1 = Q_2 = Q_3$. $P(2200)$ denotes the contribution of the effective pseudoscalar pole. The maroon solid, dashed, and dot-dashed lines refer to the lowest-lying pseudoscalar triplet, the turquoise ones to the lowest axial-vector triplet, and the violet lines to the lowest tensor triplet. The red dashed line represents the sum of all these contributions, the solid blue line the sum including, in addition, the effective pseudoscalar pole, to be compared to the dot-dashed green curve representing pQCD.

Source: Figure taken from Ref. [594].

Finally, the VMD representation taking into account three multiplets of vector mesons does not reproduce the correct asymptotic scaling in the doubly-virtual direction, and the coefficient of the singly-virtual limit is also not reproduced exactly, making it necessary to supplement the VMD representation with an asymptotic piece. The asymptotic contribution takes a form that is an appropriate extension of the one established for pseudoscalars, see Section 5.5.1. In Ref. [55,594], a suitable representation was found that incorporates relevant mass effects, but at the same time leaves unaltered the low-energy properties such as normalization and slope, which are already determined by the fit of the VMD representation to data. The additional asymptotic form introduces a parameter s_m that describes the transition to the asymptotic regime of the corresponding form factor. Good agreement with the dispersive evaluation of the form factor $\mathcal{F}_2^{a_1}(q_1^2, q_2^2)$ of Ref. [51] was found for $s_m = 1.5 \text{ GeV}^2$, the typical matching scale expected from light-cone sum-rule calculations [569,601].

As the inclusion of axial-vector states is still affected by a basis dependence, their contribution needs to be studied in combination with the matching to SDCs, see Section 5.5.4. In this context, the tails of the pseudoscalar-pole contributions need to be subtracted, while the other contributions summarized in Table 21 have negligible overlap with the mixed and short-distance regions.

5.5.4. Matching to short-distance constraints

Matching the sum of hadronic intermediate states to the SDCs of the HLbL tensor is most conveniently organized in terms of a matching scale Q_0 that separates the low-energy regime from the high-energy part. Moreover, a parameter $r \in [0, 1]$ has to be introduced in order to keep track of the applicability of the OPE constraints in the mixed region. In the following, we describe the matching strategy from Refs. [55,594].

With a finite number of hadronic states, e.g., including the pseudoscalar poles, the heavy scalars $f_0(1370)$ and $a_0(1450)$, the first axial-vector multiplet $f_1(1285)$, $f_1'(1420)$, and $a_1(1260)$, and the first tensor multiplet $f_2(1270)$, $f_2'(1525)$, and $a_2(1320)$, the matching cannot be expected to be smooth in all possible directions of the HLbL tensor. In Fig. 50, the scalar function $\tilde{\Pi}_1$ for the symmetric limit $Q_{\text{sym}} = Q_1 = Q_2 = Q_3$ is shown. It displays the remaining discrepancy between

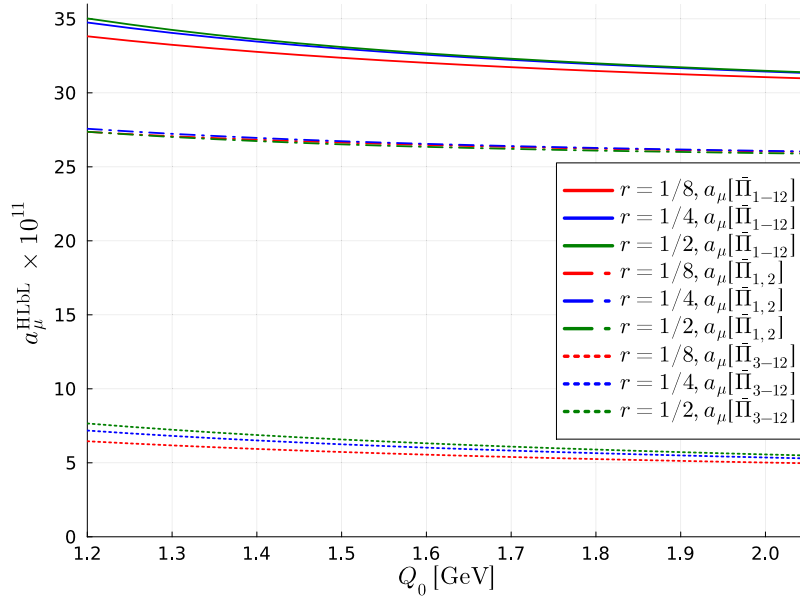


Fig. 51. Stability of the a_μ integral under variation of Q_0 , r .
Source: Figure taken from Ref. [594].

the sum of hadronic states and pQCD in the regime above 1 GeV. Therefore, in order to estimate the remaining uncertainty from the matching procedure, the relevant scales Q_0 and r are varied in suitable intervals.

In the region in which all $Q_i < Q_0$ are small, the sum of hadronic intermediate states is used, and in the opposite case, when all Q_i are larger than Q_0 , the pQCD result including α_s corrections evaluated at scale $\mu = Q_0$. The central value is chosen for $Q_0 = 1.5$ GeV, with scale varied between 1.2 GeV, the lower boundary at which α_s corrections can still be controlled, and 2 GeV, the upper limit at which a hadronic description should still be meaningful. The mixed region can be further divided into the part with $Q_1 > Q_0$ and $Q_2, Q_3 < Q_0$ and crossed versions thereof. In this regime, no OPE constraint applies and hence the sum of hadronic intermediate states is used. Instead, for

$$Q_3^2 \leq r \frac{Q_1^2 + Q_2^2}{2}, \quad Q_1^2 \geq Q_0^2, \quad Q_2^2 \geq Q_0^2, \quad Q_3^2 \leq Q_0^2, \quad (5.31)$$

and crossed versions thereof, the HLbL tensor is related to the VVA correlator, leading to the identification

$$\hat{\Pi}_1 = -\frac{1}{\pi^2 \hat{q}^2} \sum_{a=0,3,8} C_a^2 w_L^{(a)}(q_3^2),$$

$$\hat{\Pi}_5 = \hat{\Pi}_6 = -\hat{q}^2 \hat{\Pi}_{10} = -\hat{q}^2 \hat{\Pi}_{14} = \hat{q}^2 \hat{\Pi}_{17} = \hat{q}^2 \hat{\Pi}_{39} = 2\hat{q}^2 \hat{\Pi}_{50} = 2\hat{q}^2 \hat{\Pi}_{51} = -\frac{2}{3\pi^2 \hat{q}^2} \sum_{a=0,3,8} C_a^2 w_T^{(a)}(q_3^2), \quad (5.32)$$

in terms of longitudinal and transverse form factors $w_{L,T}(q^2)$.²⁸ In particular, it was shown in Ref. [54] that certain corrections at higher orders in the OPE accidentally cancel at the level of the a_μ integration, making the constraint Eq. (5.32) more robust. For the numerical analysis, the dedicated dispersive analysis of the hadronic VVA correlator from Ref. [51] is used for the triplet component $a = 3$, reproduced in Ref. [49] in a simplified set-up that can be readily generalized to $a = 0, 8$. For the central value $r = 1/4$ is used, varied by a factor of 2 in both directions in order to assess the uncertainty introduced by this specific matching parameter. The stability of a_μ by varying the scales Q_0 and r can be seen in Fig. 51.

The mismatch between the sum of hadronic intermediate states and pQCD in the region between 1.2 GeV and 2.0 GeV could have an impact on the low-energy region and the parts of the mixed region for which no OPE constraint applies. When including more states or even an infinite tower of states in order to saturate the asymptotic behavior and OPE constraints of the HLbL tensor, a potential nonnegligible contribution might arise at low energies as well, and this uncertainty would not be captured by the variation of the matching scales Q_0 and r . Therefore, effective poles are introduced that saturate the prescribed asymptotic behavior of the HLbL tensor. As a single state or a finite number of states in four-point kinematics cannot exactly reproduce the required behavior, the dispersive framework in triangle

²⁸ These form factors are identical to $\omega_{L,T}(q^2)$ introduced in Section 5.4, up to an overall factor of $N_c = 3$, $w_{L,T}(q^2) = N_c \omega_{L,T}(q^2)$.

Table 22

Summary of the various subleading contributions considered in Refs. [55,594], at the matching scale $Q_0 = 1.5$ GeV and with the OPE applied for $Q_3^2 < r(Q_1^2 + Q_2^2)/2$, $r = 1/4$. In the regions in which OPE and pQCD are used, the tails of the pseudoscalar poles are subtracted to avoid double counting. The effective poles are determined from the matching in the symmetric asymptotic limit, with TFF scale 1.5 GeV. The errors are labeled as in Eq. (5.34), see main text for details, while the matching uncertainties from the variation of Q_0 , r are not yet included. The errors for the pQCD contribution are propagated from α_s . Table from Ref. [55].

Region		$a_\mu[\tilde{T}_{1,2}] [10^{-11}]$	$a_\mu[\tilde{T}_{3-12}] [10^{-11}]$	Sum $[10^{-11}]$
$Q_i < Q_0$	$A = f_1, f'_1, a_1$	$7.2(1.4)_{\text{exp}}$	$5.0(1.0)_{\text{exp}}$	$12.2(2.3)_{\text{exp}}$
	$S = f_0(1370), a_0(1450)$	–	$-0.7(3)_{\text{exp}}$	$-0.7(3)_{\text{exp}}$
	$T = f_2, a_2, f'_2$	$2.6(3)_{\text{exp}}$	$-5.1(7)_{\text{exp}}$	$-2.5(3)_{\text{exp}}$
	Effective poles	2.5	–0.4	2.0
Mixed	A, S, T	$2.5(7)_{\text{exp}}$	$1.3(3)_{\text{exp}}$	$3.8(1.0)_{\text{exp}}$
	OPE	6.3	4.7	10.9
	Effective poles	1.1	0.1	1.2
$Q_i > Q_0$	pQCD	$4.8^{+0.1}_{-0.2}$	$1.6^{+0.0}_{-0.1}$	$6.3^{+0.2}_{-0.3}$
Sum		$26.9(2.1)_{\text{exp}}(3.7)_{\text{sys}}(3.2)_{\text{eff}}$	$6.3(1.5)_{\text{exp}}(0.2)_{\text{sys}}(2.2)_{\text{eff}}$	$33.2(3.3)_{\text{exp}}(4.6)_{\text{sys}}(3.9)_{\text{eff}}$

kinematics proves beneficial [507], introducing a corresponding pseudoscalar pole for the longitudinal components and an axial-vector state for the transverse ones, e.g.,

$$\begin{aligned}\hat{\Pi}_1^{\text{eff}} &= \frac{F_{P\gamma^*\gamma^*}(q_1^2, q_2^2)F_{P\gamma^*\gamma^*}(M_P^2, 0)}{q_3^2 - M_P^2}, \\ \hat{\Pi}_4^{\text{eff}} &= \frac{(q_1^2 + q_3^2 - M_A^2)\mathcal{F}_2^A(M_A^2, 0)[2\mathcal{F}_1^A(q_1^2, q_3^2) + \mathcal{F}_3^A(q_1^2, q_3^2)]}{2M_A^4(q_2^2 - M_A^2)} + (q_1^2 \leftrightarrow q_2^2).\end{aligned}\quad (5.33)$$

Note that the axial-vector state in triangle kinematics does not contribute to the longitudinal part in contrast to the dispersive framework in four-point kinematics. The overall couplings are determined from the short-distance matching in the symmetric limit $Q_{\text{sym}} = Q_1 = Q_2 = Q_3$, while the asymmetric matching is included in the error estimate. The mass scales are taken as $M_P^{\text{eff}} = 2.2$ GeV and $M_A^{\text{eff}} = 1.7$ GeV, motivated by the phenomenology of excited pseudoscalar and axial-vector states [55], and the TFF scale is varied in the same range as Q_0 , with central value at 1.5 GeV. The various contributions in each region are summarized in Table 22, displaying results for the central values and those errors directly derived from experiment (and α_s).

In total, the following results for the subleading HLBL contributions are obtained

$$\begin{aligned}a_\mu[\tilde{T}_{1,2}] &= 26.9(2.1)_{\text{exp}}(1.0)_{\text{match}}(3.7)_{\text{sys}}(3.2)_{\text{eff}}[5.4]_{\text{total}} \times 10^{-11}, \\ a_\mu[\tilde{T}_{3-12}] &= 6.3(1.5)_{\text{exp}}(1.4)_{\text{match}}(0.2)_{\text{sys}}(2.2)_{\text{eff}}[3.0]_{\text{total}} \times 10^{-11}, \\ a_\mu[\tilde{T}_{1-12}] &= 33.2(3.3)_{\text{exp}}(2.2)_{\text{match}}(4.6)_{\text{sys}}(3.9)_{\text{eff}}[7.2]_{\text{total}} \times 10^{-11},\end{aligned}\quad (5.34)$$

where the “exp” error refers to the two-photon couplings of the heavy scalars and tensor mesons as well as the axial-vector TFFs; the “match” error indicates the maximal variation under $Q_0 \in [1.2, 2.0]$ GeV and $r \in [1/8, 1/2]$; the “sys” error subsumes various systematic effects (a 30% uncertainty on the hadronic contributions to reflect U(3) assumptions for the axial-vector TFFs [48,543] and the simplified form of the tensor TFFs; and a 100% uncertainty for the total tensor contribution to $a_\mu[\tilde{T}_{1-12}]$ to reflect the strong cancellations observed among the different scalar functions); the “eff” error estimates the uncertainties in the effective-pole contributions (variation of TFF scale and symmetric vs. asymmetric matching), where the uncertainties for the pseudoscalar and axial-vector poles are considered separately and added in quadrature in the end, again due to a cancellation observed between them. Combined with the dispersively evaluated contributions from Table 21 one finds a total HLBL contribution $a_\mu^{\text{HLBL}} = 98.9(7.9) \times 10^{-11}$ (excluding the charm loop).

5.6. Holographic approach

In WP20, results from simple (chiral) holographic models, which had been shown to naturally implement the MV SDC through their infinite towers of axial-vector mesons [40,41], were observed to be roughly consistent with the combined estimate of the contributions of axials and SDCs, however, with numerical results around the upper value of the estimated error bar $a_\mu^{\text{axials+SDC}} = 21(16) \times 10^{-11}$. In the meantime, more realistic models including finite quark masses as well as the anomalous mass of the η' have been worked out that make a numerical comparison with the results from the updated data-driven approach as well as with lattice results even more interesting. Before doing so, it is useful to recall the theoretical basis of the holographic approach.

Holographic QCD (hQCD) refers to hadronic models of QCD in the large- N_c limit constructed in analogy to the conjectural but well-established AdS/CFT correspondence [602,603], where a conformally invariant supersymmetric Yang–Mills theory in the limit of infinite 't Hooft coupling can be mapped to a five-dimensional classical gravity theory on anti-de

Sitter (AdS) space. The extra dimension turns out to correspond to the energy scale of the holographically dual quantum field theory living on the conformal boundary of AdS space.

A “top-down” construction of a holographic dual to low-energy large- N_c QCD with chiral quarks has been achieved by Sakai and Sugimoto [604,605], building upon earlier work by Witten [606], where supersymmetry and conformal invariance is broken by an additional Kaluza–Klein compactification. While remarkably successful for a number of low-energy observables, this model does not make contact to QCD at large energies and momenta. However, there are by now a number of phenomenologically interesting “bottom-up” models of hadron physics that combine salient features of the top-down construction with a simpler geometry that is asymptotically AdS₅ and therefore can be matched to the asymptotic conformal symmetry of QCD in the UV. Confinement and chiral symmetry breaking is implemented either by a hard-wall (HW) cutoff [607–609] with appropriate boundary conditions or by a soft wall (SW) provided by a nontrivial dilaton [610–618], which has similarities with light-front hQCD [619].

Already the simplest hQCD models have proved to provide good qualitative and oftentimes quantitative predictions of observables in hadron physics with typical errors of (sometimes less than) 10 to 30% with a minimal set of free parameters [607–609]. While this is clearly much too crude to be of help with the HVP contribution to a_μ , bottom-up hQCD models provide interesting predictions for the HLbL amplitude, where they naturally satisfy the longitudinal SDC after matching leading-order OPE results for the vector correlation function [40,41,542]. In this context, it is useful to recall that hQCD relies on the so-called holographic dictionary underlying the original conjectured AdS/CFT duality, where for each gauge-invariant quantum operator $\mathcal{O}_\Delta(x)$ of the 4D gauge theory with scaling dimension Δ one has a corresponding 5D field $\phi(x, z)$, whose value on the UV boundary at $z = 0$ is identified (modulo a certain power of z) with the 4D source of the operator, $J(x) \propto \phi(x, 0)$, and whose 5D mass is determined by Δ and form degree p through $m_\phi^2 = (\Delta - p)(\Delta + p - 4)$.

The generating functional of the 4D theory is computed from the 5D action evaluated on-shell,

$$\exp(iW[J]) \equiv \langle 0|T \exp \left(i \int d^4x J(x) \mathcal{O}_\Delta(x) \right) |0\rangle_{4D} = \exp \left(iS_{5D}^{\text{on-shell}}[J] \right), \quad (5.35)$$

so that n -point functions of the 4D theory $\langle 0|T \{ \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) \cdots \mathcal{O}_\Delta(x_n) \} |0\rangle$ can be obtained from variational derivatives $\delta^n S_{5D}^{\text{on-shell}} / \delta J(x_1) \delta J(x_2) \cdots \delta J(x_n)$. The calculation of correlation functions of quantum operators thus amounts to computing tree-level (Witten) diagrams on the gravity side, where the external sources reside on the conformal boundary of AdS space, which connect with bulk-to-boundary propagators to interactions in the bulk of the 5D theory, involving also bulk-to-bulk propagators when $n \geq 4$ as in the case of the HLbL amplitude. In this case $\mathcal{O}_\Delta$ involves the vector current $\bar{q} \gamma^\mu t^a q$ with $\Delta = 3$, corresponding to a massless gauge field in the bulk. The global chiral symmetry $U(N_f)_L \times U(N_f)_R$ thus turns into a (flavor) gauge theory in the bulk with 5D action

$$S_{\text{YM}} = -\frac{1}{4g_5^2} \text{tr} \int d^4x \int_0^{z_0} dz e^{-\Phi(z)} \sqrt{-g} g^{PR} g^{QS} \left(\mathcal{F}_{PQ}^{(L)} \mathcal{F}_{RS}^{(L)} + \mathcal{F}_{PQ}^{(R)} \mathcal{F}_{RS}^{(R)} \right), \quad (5.36)$$

where $P, Q, R, S = 0, \dots, 3, z$ and $\mathcal{F}_{MN} = \partial_M \mathcal{B}_N - \partial_N \mathcal{B}_M - i[\mathcal{B}_M, \mathcal{B}_N]$. Here z_0 can either be a finite upper value, when confinement is implemented by a hard wall, or $z_0 = \infty$ when confinement is due to a nontrivial dilaton field $\Phi(z)$.

It is a striking feature of hQCD models that the correlation functions are saturated by infinite sums of narrow 4D resonances of increasing mass, as is expected in the large- N_c limit of QCD. In particular, the normalizable modes of the 5D Yang–Mills fields $\mathcal{B}_M^{L,R} = \mathcal{B}_M^V \mp \mathcal{B}_M^A$ correspond to an infinite tower of massive vector and axial-vector mesons. VMD is naturally built in through the bulk-to-boundary propagators for EM currents, which can be expressed in terms of mode functions of the massive vector mesons. Typically, hQCD models provide closed-form analytic or semi-analytic expressions summing up the contributions of those infinite towers of resonances. Moreover, hQCD models with (asymptotically) AdS metric reproduce the approximately conformal behavior of QCD at short distances, permitting in many cases a matching of SDCs.

Chiral symmetry is broken either by introducing an extra bifundamental scalar field X dual to quark bilinears $\bar{q}_L q_R$ as in the original HW model [607,608] (denoted by HW1 in the following) and in the SW model [611], or through different IR boundary conditions for vector and axial vector fields at z_0 as in the inherently chiral Hirn–Sanz (HW2) model [609] (as is also the case in the top-down Sakai–Sugimoto model [604,605]). Versions of the HW1 model with the latter boundary conditions have been advocated in Ref. [620] and further studied in Ref. [542] (denoted by HW3 therein). With the exception of the HW2 model, these bottom-up models have just enough parameters to match F_π , the ρ meson mass, and the leading OPE term of the vector–vector correlation function; the simpler HW2 model can match only two of those ingredients at once.

In all these models, flavor anomalies follow uniquely from 5D Chern–Simons terms $S_{\text{CS}}^L - S_{\text{CS}}^R$, where (in differential-form notation)

$$S_{\text{CS}} = \frac{N_c}{24\pi^2} \int \text{tr} \left(\mathcal{B} \mathcal{F}^2 - \frac{i}{2} \mathcal{B}^3 \mathcal{F} - \frac{1}{10} \mathcal{B}^5 \right). \quad (5.37)$$

In particular, this determines the anomalous interactions of photons with pseudoscalars and axial-vectors which appear together in the mode expansion of $\mathcal{B}_M^A = (\mathcal{B}_M^R - \mathcal{B}_M^L)/2$.

Although suppressed at large N_c , a Witten–Veneziano mechanism for the η_0 mass through the $U(1)_A$ anomaly is included in the Sakai–Sugimoto model [604] and yields satisfactory numerical results at $N_c = 3$ [621]. Extensions of the bottom-up models that implement the $U(1)_A$ anomaly have been proposed in Refs. [622,623], and have been employed in the context of HLbL scattering in Refs. [47,624].

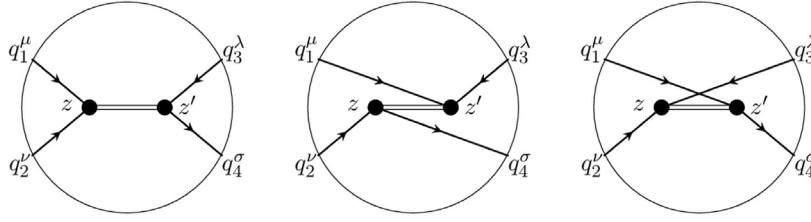


Fig. 52. 5D Witten diagrams contributing to the HLbL tensor in hQCD models, generalizing the one-particle exchange diagrams in 4D. The AdS_5 space is represented as the disk, with z as the holographic radial coordinate and ordinary 4D spacetime as its boundary. Solid lines represent the vector bulk-to-boundary propagators, depending on the 4D external momenta. Double lines denote the Green function of a 5D field, containing the whole tower of massive resonances in the given channel (pseudoscalars, axial vectors etc.) in the form $G(q, z, z') = \sum_n \varphi_n(z) \varphi_n(z') / (q^2 - m_n^2)$. Dots represent interaction vertices derived from the 5D action.

5.6.1. Pseudoscalar TFFs in hQCD

In a mode decomposition of the bulk-to-bulk propagator in the 5D Witten diagrams for the HLbL tensor shown in Fig. 52, integration over the holographic coordinate of an internal vertex yields the TFF of the exchanged meson. The internal vertex is provided by the Chern-Simons action Eq. (5.37), yielding

$$F_{\pi^0 \gamma^* \gamma^*}(Q_1^2, Q_2^2) = \frac{N_c}{12\pi^2 F_\pi} \int_0^{z_0} dz \mathcal{J}(Q_1, z) \mathcal{J}(Q_2, z) \Psi(z), \quad (5.38)$$

where $\mathcal{J}(Q, z)$ is the bulk-to-boundary propagator of a photon with virtuality $Q^2 = -q^2$ and $\Psi(s)$ a holographic pion profile function. The authors of Refs. [625–627] were the first to notice that in hQCD models with asymptotic AdS_5 geometry the expression in Eq. (5.38) reproduces the asymptotic momentum dependence obtained by BL in QCD [528,529,568],

$$F_{\pi^0 \gamma^* \gamma^*}(Q_1^2, Q_2^2) \rightarrow C \frac{2F_\pi}{Q^2} \left[\frac{1}{w^2} - \frac{1-w^2}{2w^3} \log \frac{1+w}{1-w} \right], \quad (5.39)$$

with $C = g_5^2 N_c / (12\pi^2)$, $Q^2 = (Q_1^2 + Q_2^2)/2 \rightarrow \infty$, $w = (Q_1^2 - Q_2^2)/(Q_1^2 + Q_2^2)$. The correct functional dependence on w is due to the infinite sum of vector resonances contained in $\mathcal{J}$, and cannot be obtained in VMD models with a finite number of resonances. When the vector–vector correlator is matched to the leading-order OPE result, one has $C = 1$, in exact agreement with the BL limit. This can be achieved in the bottom-up hQCD models with asymptotic AdS geometry except for the simple HW2 model, when the latter matches both F_π and M_ρ , which reduces C to 0.616 (fitting the asymptotic limit would instead lead to an overweight ρ meson with mass of 987 MeV).

Pion-pole contribution. First estimates of the corresponding holographic results for the pion pole contribution $a_\mu^{\pi^0}$ were obtained in Ref. [628,629] and more fully in Ref. [630]. The experimental data on the pion TFF turned out to be bracketed by the HW1 and HW2 results when these models are fit to F_π and M_ρ . Since the HW2 model then undershoots the asymptotic limit of Eq. (5.39) by 38%, its prediction of $a_\mu^{\pi^0} = 56.9 \times 10^{-11}$ is correspondingly on the low side. The HW1 model, which exactly satisfies the asymptotic limit Eq. (5.39) and which can also accommodate finite quark masses [542], predicts $a_\mu^{\pi^0} = (66.0\text{--}66.6) \times 10^{-11}$ depending on the precise boundary conditions [542]. This is consistent with, but slightly larger than the dispersive result $63.0^{+2.7}_{-2.1} \times 10^{-11}$ [1,37,548].

At high but still phenomenologically relevant energy scales, gluonic corrections modify the large- Q^2 behavior of the TFF by about 10% [44,631]. Holographic QCD models with a simple AdS_5 background have no running coupling constant and thus approach the asymptotic limit too quickly. Similar corrections, but with opposite sign, appear in the vector correlator determining also HVP. Indeed, the HW1 model with a complete UV fit underestimates the HVP contribution. Reducing g_5^2 by 10% (incidentally corresponding to an exact fit of the decay constant of the ρ meson) brings HVP in line with the dispersive result (to within 5%) [632]. It also makes the HW1 result with $a_\mu^{\pi^0} = 63.4 \times 10^{-11}$ almost completely coincident with the dispersive result [47,542]. In Fig. 53 the HW1 result (with finite quark masses) is compared with experimental data and the dispersive result of Ref. [37] for the two choices of g_5 .

More recently, the pseudoscalar TFFs and their contribution to a_μ have been evaluated also in the SW model [624]. In this model, the pseudoscalar TFFs turn out to approach the BL limit from above, and the good agreement with data obtained in the HW1 model is lost. The result for the pion contribution is $a_\mu^{\pi^0} = 75.2 \times 10^{-11}$, significantly larger than the results from data-driven and lattice approaches. A drawback of the SW model as set up in Ref. [624] is that the background of the bifundamental scalar, which implements chiral symmetry breaking, is taken over from the HW models but is not a solution of the field equations in the modified geometry. On the other hand, self-consistent solutions would make the chiral condensate proportional to quark masses [611], so that some modifications are needed which presumably would change the behavior at larger momenta. Improved hQCD models with nontrivial dilaton background exist in the literature [613,614,633], but they involve a rather large parameter space; so far they have not been worked out with regard to the HLbL amplitude.

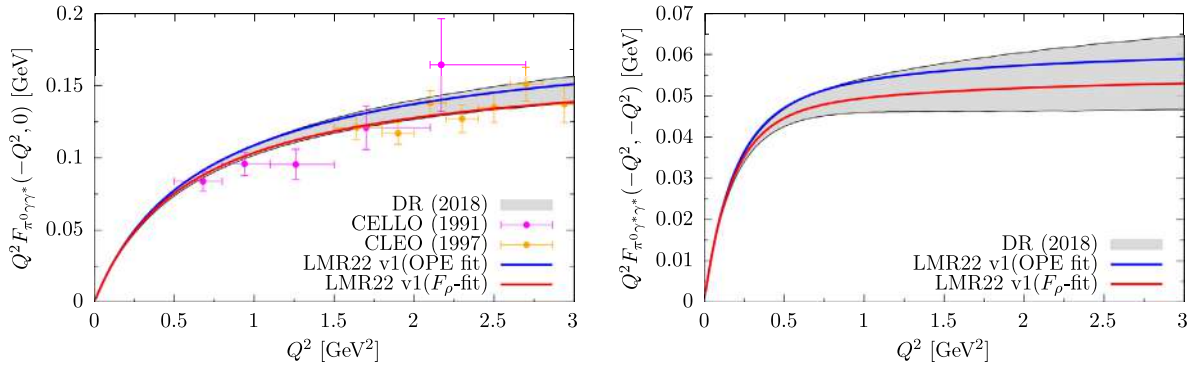


Fig. 53. Comparison of the hQCD results for the pion TFF of Ref. [47] (LMR22) with OPE fit of g_5^2 (blue lines) and a 10% reduced coupling g_5^2 (red lines) from fitting F_ρ instead, compared to the dispersive results of Ref. [37] and experimental data.

Excited pions. With the exception of the inherently chiral HW2 model, the hQCD models also have excited pseudoscalars. Those decouple from the anomaly in the chiral limit, where $F_{\pi_n} \rightarrow 0$ for $n > 1$, but even then have nonvanishing two-photon decay amplitudes. Away from the chiral limit, one has the sum rule [542]

$$\sum_{n=1}^{\infty} F_{\pi_n} F_{\pi_n \gamma \gamma} = \frac{N_c}{12\pi^2} \quad (5.40)$$

and asymptotic behavior according to the BL result given in Eq. (5.39).

In the massive HW models, the first excited pion has a mass of 1.9 GeV (HW1) or 1.7 GeV (HW3), contributing $a_{\mu}^{0*} = 0.7 \times 10^{-11}$ or 0.8×10^{-11} , respectively [542]. In Ref. [620] it was noted that with a different scaling dimension for the bifundamental operator one can adjust the HW3 model such that the mass of the $\pi(1300)$ resonance is matched. This increases the contribution to $a_{\mu}^{0*} = 1.5 \times 10^{-11}$; however, the predicted decay amplitude $|F_{\pi^{0*} \gamma \gamma}| = 0.206 \text{ GeV}^{-1}$ is in conflict with the estimated experimental upper bound [535] $|F_{\pi(1300) \gamma \gamma}| < 0.0544(71) \text{ GeV}^{-1}$. Without this modification, the contribution from the collection of excited pions with $n \geq 2$ is $\sum_{n \geq 2} a_{\mu}^{\pi_n} \simeq 0.8 \times 10^{-11}$ in the HW models. In the SW model of Ref. [624], where the first excited pion has a mass of about 2.1 GeV, already this mode would contribute 1.68×10^{-11} . In contrast to the HW model, the contributions from higher excited pions do not fall off quickly, despite their more rapidly growing masses [634].

η and η' . In order to describe the η and η' pseudoscalars consistently, the larger strange quark mass as well as the anomaly induced mass of the η_0 singlet has to be taken into account. In the chiral models studied initially in Refs. [40,41,629,630], the decay constants and masses were adjusted manually, whereas in the early HW1 study of Ref. [628] with finite quark masses the Chern–Simons action was implemented incorrectly [542]. From the chiral HW2 model, Ref. [41] obtained the ranges $a_{\mu}^{\eta} = (14 \dots 21) \times 10^{-11}$ and $a_{\mu}^{\eta'} = (10 \dots 16) \times 10^{-11}$, while from the chiral HW1 model Ref. [630] estimated $a_{\mu}^{\eta} = 18.2 \times 10^{-11}$ and $a_{\mu}^{\eta'} = 15.6 \times 10^{-11}$.

In the hQCD models with bifundamental scalar field, quark masses can be introduced by the nonnormalizable modes of the latter, while the anomalous mass of the η_0 singlet can be obtained by a Witten–Veneziano mechanism. This permits to predict masses, decay constants, and TFFs of η and η' when the quark masses are fixed to reproduce π and K masses. The best result so far was obtained in Ref. [47] by using the Katz–Schwartz model [622] for implementing the $U(1)_A$ anomaly in a HW AdS background, but allowing for a gluon condensate (referred to as model v1 in LMR22 [47]). This reproduced η and η' masses and $F_{\eta(\prime) \gamma \gamma}(0, 0)$ to within a few percent.²⁹

The HW results for the singly and doubly virtual TFFs are displayed in Fig. 54. While they are fully compatible with the CA results used in WP20, they are significantly above the new dispersive results [56,550]. Correspondingly the results for a_{μ}^{η} and $a_{\mu}^{\eta'}$ listed in Table 23, which would agree with WP20 within errors, are larger than those of Refs. [56,550], by 3σ and 2σ , respectively.

5.6.2. Axial-vector TFFs in hQCD

All hQCD models involve an infinite tower of axial vector mesons. Their coupling to two photons is determined by the same action Eq. (5.37) that gives rise to the anomalous TFFs of the pseudoscalars. It has the form [40,41]

$$\mathcal{M}_{A_n \gamma^* \gamma}^* = i \frac{N_c}{4\pi^2} \text{tr}(\mathcal{Q}^2 t^a) \epsilon_{(1)}^{\mu} \epsilon_{(2)}^{\nu} \epsilon_A^{*\rho} \epsilon_{\mu\nu\rho\sigma} \left[q_{(2)}^{\sigma} Q_1^2 A_n^a(Q_1^2, Q_2^2) - q_{(1)}^{\sigma} Q_2^2 A_n^a(Q_2^2, Q_1^2) \right], \quad (5.41)$$

²⁹ The SW model of Ref. [624] with $U(1)_A$ anomaly also achieves this, but the TFFs of η and η' approach the BL limit from above like the SW result for π^0 , leading to rather poor agreement with data, in particular in the case of η .

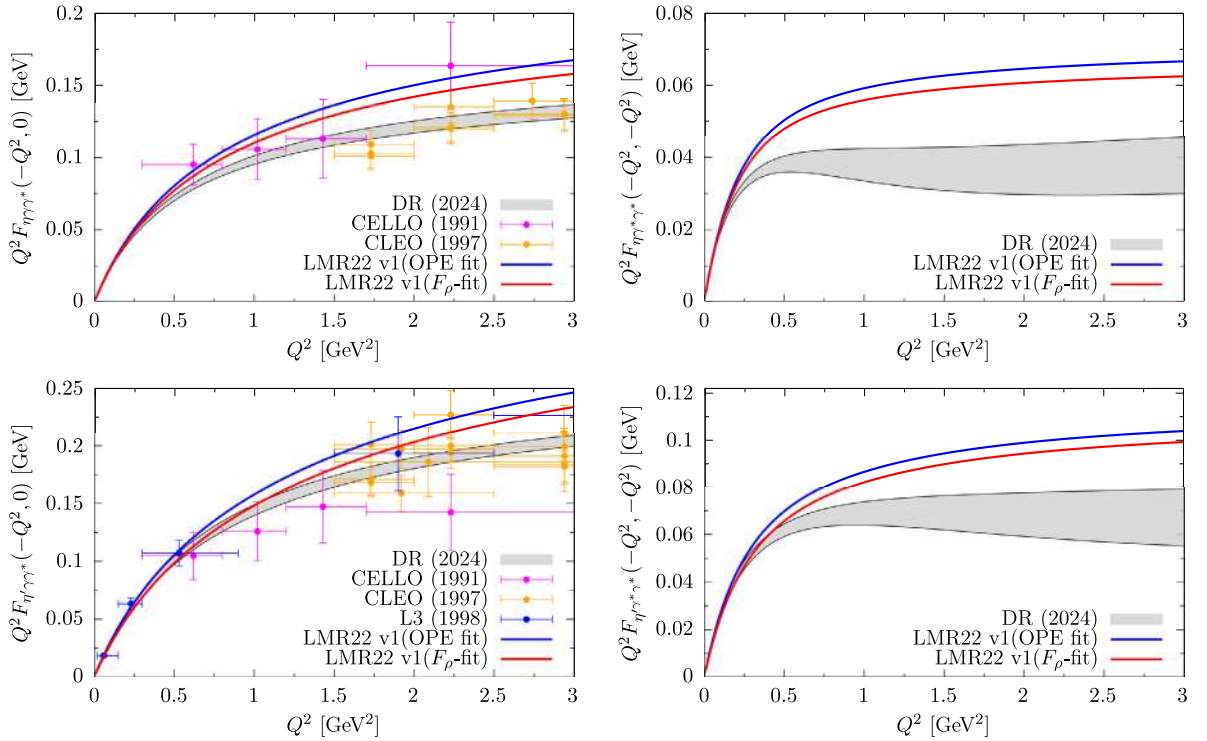


Fig. 54. Comparison of the hQCD results for the η and η' TFFs of Ref. [47] (LMR22) with OPE fit of g_s^2 (blue lines) and a 10% reduced coupling g_s^2 (red lines) from fitting F_ρ instead, compared to the dispersive results of Refs. [56,550] and experimental data.

Table 23

Breakdown of the axial-sector contributions to a_μ^{HLbL} in $N_f = 2 + 1$ hQCD models, where the HLbL contributions of pseudoscalars and axial-vector mesons have been evaluated completely. LMR22 v1(F_ρ -fit) represents the best-guess model of Ref. [47]. The table gives a partial summary of the results from Refs. [41,47,635] for the different contributions to a_μ^{HLbL} .

$a_\mu \times 10^{11}$	CCDGI(set1-set2)	LMR22 v1(OPE fit)	LMR22 v1(F_ρ -fit)	CS'(F_ρ -fit)
π^0	57–75	66.1	63.4	68.8
η	14–21	19.3	17.6	17.2
η'	10–16	16.9	14.9	12.2
G/η''	–	0.2	0.2	2.6
$\sum_{PS^*}$	–	1.6	1.4	3.2
$\pi^0 + \eta + \eta'$	81–112	102.3	95.9	98.2
PS poles	81–112	104	97.5	104
a_1	–	7.8	7.1	4.9
$f_1 + f_1'$	–	20.0	17.9	12.0
$\sum_{a_1^*}$	–	2.5	2.6	2.3
$\sum_{f_1^{(\prime)*}}$	–	4.0	3.5	7.9
AV+LSDC	28(2)	34.3	31.1	27.0
AV+P*+LSDC	28(2)	36.0	32.7	32.8
Total	110–140	138	129	131

involving an asymmetric structure function

$$A_n(Q_1^2, Q_2^2) = \frac{2g_5}{Q_1^2} \int_0^{z_0} dz \left[\frac{d}{dz} \mathcal{J}(Q_1, z) \right] \mathcal{J}(Q_2, z) \psi_n^A(z), \quad (5.42)$$

where $\psi_n^A(z)$ is the holographic profile function of the n th axial vector meson. The Landau–Yang theorem [636,637], which forbids the decay of an axial vector meson into two real photons, is satisfied because $\partial_z \mathcal{J}(0, z) \equiv 0$.

In the notation of Refs. [48,543,590], $A_n(Q_1^2, Q_2^2) \propto \mathcal{F}_2(q_1^2, q_2^2) \equiv -\mathcal{F}_3(q_2^2, q_1^2)$. The most general decay amplitude of axial-vector mesons would permit one further structure function [543,638,639], denoted as $\mathcal{F}_1 \equiv \mathcal{F}_{a_1}$ in Refs. [48,543,590].

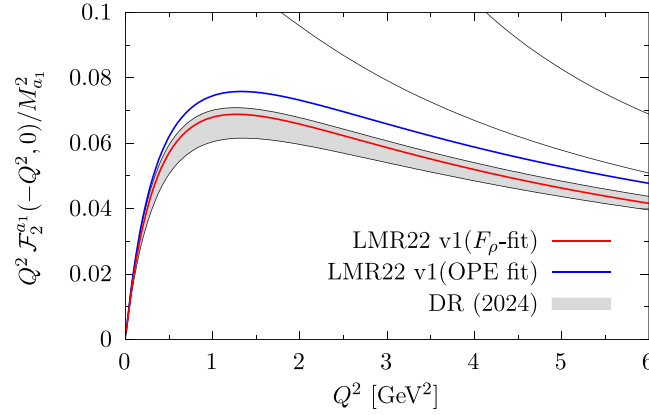


Fig. 55. Comparison of the hQCD results for the a_1 TFF of Ref. [47] (LMR22) with OPE fit of g_5^2 (blue lines) and a 10% reduced coupling g_5^2 (red lines) from fitting F_ρ instead, compared to the dispersive result of Ref. [51]. The BL limit [590] is shown as a dotted (dash-dotted) gray line without (with) mass corrections.

The holographic prediction that this vanishes is consistent with the results of the study of all available experimental data on $f_1(1285)$ in Ref. [48].

In all hQCD models that are asymptotically AdS, the asymptotic form of $A_n(Q_1^2, Q_2^2)$ is given by [40]

$$A_n(Q_1^2, Q_2^2) \rightarrow C \frac{12\pi^2 F_n^A}{N_c Q^4} \frac{1}{w^4} \left[w(3-2w) + \frac{1}{2}(w+3)(1-w) \log \frac{1-w}{1+w} \right], \quad (5.43)$$

for $Q^2 = (Q_1^2 + Q_2^2)/2 \rightarrow \infty$, $w = (Q_1^2 - Q_2^2)/(Q_1^2 + Q_2^2)$, which agrees with the result obtained by light-cone expansions in Ref. [590], where in fact $\mathcal{F}_1$ does not contribute terms of order $1/Q^4$. Again, $C = g_5^2 N_c / (12\pi^2) = 1$ when g_5 is fixed such that the asymptotic behavior of the vector correlator matches the leading-order OPE result in QCD.

In Fig. 55 the result obtained for the singly-virtual a_1 TFF in the HW model of Ref. [47] (LMR22) is compared to the recent dispersive result of Ref. [51]. With g_5 fixed by fitting F_ρ instead of the OPE result, the BL limit is reached asymptotically only at the level of 89%, but for virtualities at moderate values of Q^2 a perfect agreement with the dispersive result is obtained.

5.6.3. Short-distance constraints in hQCD

In the BTT basis of the HLbL four-point function [34], the longitudinal SDC of MV [503] in the region $Q_1^2 \simeq Q_2^2 \gg Q_3^2 \gg M_\rho^2$ and $Q_4 = 0$, which is governed by the chiral anomaly and protected by its nonrenormalization theorem, reads [39,503,534,535]

$$\lim_{Q_3 \rightarrow \infty} \lim_{Q \rightarrow \infty} Q^2 Q_3^2 \bar{\Pi}_1(Q, Q, Q_3) = -\frac{2}{3\pi^2} \quad (5.44)$$

for $N_c = N_f = 3$, while the symmetric limit is 2/3 of this,

$$\lim_{Q \rightarrow \infty} \lim_{Q \rightarrow \infty} Q^4 \bar{\Pi}_1(Q, Q, Q) = -\frac{4}{9\pi^2}. \quad (5.45)$$

The short-distance behavior of the form factors of both pseudoscalars and axial-vector mesons implies that each individual meson gives a pole contribution with $\bar{\Pi}_1(Q, Q, Q_3) \simeq Q^{-2} Q_3^{-4}$. However, in Ref. [40,41] it was shown that in hQCD a summation over the infinite tower of axial-vector mesons changes this. The infinite sum yields

$$\bar{\Pi}_1^{AV} = -\frac{2C}{\pi^2 Q_3^2} \int_0^{z_0} dz \int_0^{z_0} dz' \mathcal{J}'(Q, z) \mathcal{J}(Q, z) \mathcal{J}'(Q_3, z') G_A(0; z, z'), \quad (5.46)$$

where G_A is the Green function for the axial-vector mode equation at $q^2 = 0$. For large $Q, Q_3 \gg M_\rho$, Eq. (5.46) is dominated by $z, z' \ll z_0$, where all hQCD models with (at least asymptotic) AdS geometry have $\mathcal{J}(Q, z) \rightarrow QzK_1(Qz)$, and

$$G_A(0, z, z') = \frac{1}{2} (\min(z, z'))^2 (1 + \mathcal{O}(Q^{-n}) + \mathcal{O}(Q_3^{-n})), \quad n > 0, \quad (5.47)$$

when $z = \xi/Q$ and $z' = \xi'/Q_3$. For $Q \gg Q_3 \gg M_\rho$ this leads to

$$\lim_{Q_3 \rightarrow \infty} \lim_{Q \rightarrow \infty} Q^2 Q_3^2 \bar{\Pi}_1^{AV}(Q, Q, Q_3) = \frac{C}{2\pi^2} \int_0^\infty d\xi \xi^2 \frac{d}{d\xi} [\xi K_1(\xi)]^2 = -\frac{2}{3\pi^2} C, \quad (5.48)$$

reproducing Eq. (5.44) for $C = 1$, i.e., whenever the leading-order OPE behavior of the vector correlator is matched. As shown in Ref. [542], an infinite tower of excited pseudoscalars, which is present in the hQCD models apart from the HW2 model, does not contribute to Eq. (5.48). An analogous analysis of the symmetric limit $Q_1^2 \simeq Q_2^2 \simeq Q_3^2 \gg M_\rho^2$ yields

$$\lim_{Q \rightarrow \infty} Q^4 \bar{\Pi}_1^{\text{AV}}(Q, Q, Q) = -\frac{C}{\pi^2} \int_0^\infty d\xi \xi [\xi K_1(\xi)]^3 = -0.812 \frac{4}{9\pi^2} C, \quad (5.49)$$

which reproduces the correct power behavior, but for $C = 1$ is only 81% of the result Eq. (5.45).

5.6.4. Complete axial-sector HLbL contributions to a_μ in hQCD

The virtue of bottom-up hQCD models is to provide simple, self-contained hadronic models for the HLbL tensor covering all virtualities, including the consequences of the axial anomaly for the longitudinal SDC in full, while also capturing 81% of the transverse SDC. Because the pseudoscalar and axial-vector contributions are coupled, complete results for the HLbL contribution in a given hQCD model require that both be fully evaluated. So far this has been done in the chiral HW models studied in Ref. [40,41], flavor-symmetric HW models with a quark mass that matches M_π in Ref. [542], and more recently in an $N_f = 2 + 1$ model including the anomalous η' mass in Ref. [47]. The resulting values for a_μ^{HLbL} are summarized in Table 23. There CCDGI(set1-set2) refers to the results for the HW2 model from Ref. [41], which made an estimate for the $N_f = 2 + 1$ case by using the chiral TFFs with physical pseudoscalar masses and by introducing a different decay constant for η' , with two sets of parameters which either undershoot the BL limit by a factor of 0.616, or overshoot the ρ mass, but give pseudoscalar TFFs that bracket the experimental results. LMR22 refers to the HW model Ref. [47] with quark masses and an η_0 mass contribution from the $U(1)_A$ anomaly. The latter entails a pseudoscalar glue ball (G/η'') that mixes with η and η' and also couples to two photons. Most recently, variations of such hQCD models have been considered in Ref. [635] with scalar-extended Chern–Simons actions, motivated by string-theoretic constructions [623]. One of those named $\text{CS}''(F_\rho\text{-fit})$ is included in Table 23, which has the virtue of reproducing quite well the experimentally observed [598,600] equivalent photon rates and mixing of $f_1^{(\prime)}$ axial-vector mesons, albeit at the expense of a less good fit of η' data. Moreover, the π^0 TFF appears to be overestimated at nonzero virtualities. However, the total sum of all contributions turns out to be similar to the LMR22 result due to larger contributions from the tower of excited pseudoscalars and axials, which compensate for the smaller result from ground-state axial-vector mesons. Ref. [635] still considers the LMR22 v1($F_\rho\text{-fit}$) as the best-guess hQCD result, with the variation obtained in the CS'' (and the partially scalar-extended CS') models providing estimates for inherent errors.

A comparison of these results with other approaches that determine the pole contributions from single meson exchanges is made in Sections 5.9 and 5.10 below, and reasonable agreement is obtained within estimated systematic errors.³⁰

5.6.5. Scalar and tensor contributions in hQCD

In WP20 the broad resonance $f_0(500)$ was covered by the S -wave $\pi\pi$ scattering contribution and largely dominates those of heavier scalars, which have been estimated in the narrow-width approximation. The contributions of the lightest tensor mesons $f_2(1270)$, $f_2(1565)$, $a_2(1320)$ and $a_2(1700)$, were evaluated in the narrow-resonance approximation in Refs. [591,597]. The total contribution from scalar and tensor resonances with masses greater than 1 GeV was estimated to be $a_\mu^{\text{scalar+tensor}} = -1(3) \times 10^{-11}$. There were no dispersive estimate for the single tensor resonance exchanges due to some spurious dependence on the result on the tensor basis. Recently, this problem has been overcome in Ref. [49] allowing for the dispersive evaluation reported in Ref [594].

The hQCD models which involve a bifundamental scalar field for implementing chiral symmetry breaking also feature nonets of scalar mesons and their excitations. However, in the minimal setup of Refs. [607,608,611], they do not give rise to a coupling to photons. Such a coupling is only naturally present for the flavor-singlet dilaton, which corresponds to a scalar glue ball, and metric fluctuations, corresponding to a tensor glue ball. In Ref. [643], this was considered within the Witten–Sakai–Sugimoto model with two-photon decay rates in the keV range, but the resulting a_μ contributions have been found to be negligible.

In Ref. [539], the minimal HW1 model of Ref. [607,608,611] was extended by adding, in the 5D Lagrangian, two new interaction terms quadratic in both the 5D scalar field and tensor gauge fields in Eq. (5.36). Chiral symmetry breaking generated scalar–photon–photon interaction vertices and hence a nonvanishing scalar TFF. Only the chiral limit was considered in that model. The contribution of the full tower of scalar resonances in each channel was obtained. The final result for the total contribution of $\sigma(500)$, $a(980)$, and $f_0(980)$ was $a_\mu^S = -9(2) \times 10^{-11}$, with roughly 90% of it coming from $\sigma(500)$, being compatible with the estimates in WP20. A shortcoming of the model is that the asymptotic behavior of the scalar TFF does not match the one expected from a BL analysis [590] in QCD for large photon virtualities. In Ref. [635] it has been shown that this mismatch persists also beyond the chiral limit, although consistency with the OPE limit is achieved.

³⁰ An exception in recent work concerns Refs. [640–642], where significantly smaller results, $a_\mu^{\pi^0+\eta+\eta'} = 58.5(8.7) \times 10^{-11}$ and $a_\mu^{a_1+f_1+f_1'} = 3.6(1.8) \times 10^{-11}$, are obtained in a nonlocal quark model. However, these results and the other contributions in those papers do not correspond to pure pole contributions, and it is thus not clear how to compare to the results discussed here.

Table 24

a_μ results for the tensor meson pole contributions, in units of 10^{-11} , obtained by inserting the hQCD results for $\mathcal{F}_{1,3}^T$ in the formulae obtained in Ref. [49], where the first line corresponds to the lowest HW tensor mode normalized as in Ref. [57,644] (F_ρ -fit) and the remaining lines to a refit to experimental data. The IR region is defined by $Q_i \leq Q_0 = 1.5$ GeV for all $i = 1, 2, 3$, the mixed region by having only one or two below Q_0 . The first term of each sum is from $\overline{T}_{1,2}$, while the second one is from the other 10 structure functions. Table adapted from Ref. [57].

M_T [GeV]	$\Gamma_{\gamma\gamma}$ [keV]	IR	Mixed	a_μ [10^{-11}]
1.235	$2.3 + 0.8 + 0.2$	$9.61 - 6.68 = 2.93$	$0.55 - 0.32 = 0.23$	3.17
1.2754(8)	2.65(45)	$6.6 - 4.3 = 2.3$	$0.4 - 0.2 = 0.2$	2.4(4)
1.3182(6)	1.01(9)	$2.2 - 1.3 = 0.9$	< 0.1	0.9(1)
1.5173(24)	0.08(2)	$0.1 - 0.04 = 0.06$	< 0.01	0.06(2)
$f_2 + a_2 + f_2'$		$8.9 - 5.7 = 3.2$	$0.5 - 0.3 = 0.2$	3.4(4)
$f_2 + a_2 + f_2'$ w/o $\mathcal{F}_3^T$		$4.6 - 9.1 = -4.5$	$0.2 - 0.3 = -0.1$	-4.6(6)

In Ref. [624] a holographic description of tensor mesons was considered which postulated two-photon vertices as obtained from metric fluctuations, analogously to tensor glue balls, but with phenomenologically determined coupling strengths. In this paper a result of $a_\mu^{f_2(1270)} \simeq +0.6 \times 10^{-11}$ was reported for both HW and SW models, when the experimental decay width is matched, however, in a simplified and thus incomplete evaluation. Recently, in Ref. [57], a full evaluation of the tensor meson pole contribution has been performed for the HW models.³¹ This involves two structure functions, $\mathcal{F}_1^T$ and $\mathcal{F}_3^T$ in the notation of Ref. [49], which have the correct Q^2 scaling expected from the analysis of Ref. [590], but not the same asymmetry functions. The normalized TFF of singly-virtual tensor decays involves only $\mathcal{F}_1^T$. This turns out to agree well with available Belle data, see Fig. 56, in particular at the lowest available Q^2 values. Using this TFF together with $\mathcal{F}_3^T$, which only contributes in the doubly-virtual case, the pole contribution as defined by the dispersive approach in the optimized basis of Ref. [49] yields a positive a_μ result of $3.4(4) \times 10^{-11}$ when refit to the lowest f_2, a_2, f_2' multiplet ($3.2(4) \times 10^{-11}$ from the integration region $Q_i < 1.5$ GeV). This is in stark contrast to the quark model, where similarly large but negative results are obtained [55]. In fact, setting $\mathcal{F}_3^T$ to zero in the holographic result would reverse the sign and increase the absolute value to $-4.6(6) \times 10^{-11}$ for $a_\mu^{f_2+a_2+f_2'}$ (of which $-4.5(6) \times 10^{-11}$ is due to $Q_i < 1.5$ GeV), see Table 24.

In Ref. [644] it was moreover shown that summing over the infinite tower of tensor mesons gives a contribution to the symmetric longitudinal SDC, where axial vector mesons give only 81% of the OPE result, see Eq. (5.49), without modifying the result for the MV constraint Eq. (5.48). Fixing the tensor normalization by the condition that tensor and axial-vector contributions together match both longitudinal SDCs leads to a somewhat larger result than the original prescription of Ref. [645] and it also restores the correct N_c dependence of tensor meson couplings.

In contrast to the case of pseudoscalar and axial-vector contributions, evaluating the complete contribution of tensor mesons to the HLbL amplitude in the HW model does give a different result than the pole contribution as defined by the dispersive approach. Instead of 3.17×10^{-11} (2.93×10^{-11} from the IR region), the lowest tensor mode as given by the model contributes [57,644]

$$a_\mu^{T,n=1} = 8.3 \times 10^{-11}, \quad a_\mu^{T,n=1}|_{\text{IR}} = 7.4 \times 10^{-11}, \quad (F_\rho\text{-fit}) \quad (5.50)$$

which increases to a value of

$$a_\mu^T = 11.1 \times 10^{-11}, \quad a_\mu^T|_{\text{IR}} = 8.5 \times 10^{-11}, \quad (F_\rho\text{-fit}) \quad (5.51)$$

when the whole tower of tensor meson modes is included. From the perspective of the holographic model, which is self-contained and does not need matching with pQCD results except for the parameters, it is more natural to take the complete contribution into account, which is independent of the choice of basis. Remarkably, the sum over individual pole contributions from the tensor tower converges much more slowly than the sum over the complete contributions, while both sums tend to equally large totals [57].

In Table 29 below, the contribution in hQCD that is attributed to the ground-state tensor mesons, however, refers only to the $n = 1$ pole contribution as defined in the dispersive approach; the remaining contribution from the tower of tensor modes is included under “other” contributions, together with the contributions from the axial sector as given by the LMR22 v1(F_ρ -fit) result of Table 23. However, a larger value is obtained than in all previous estimates, with a positive sign due to the presence of $\mathcal{F}_3^T$. Unfortunately, at present there are no experimental data available to test the hQCD prediction of a nonzero $\mathcal{F}_3^T$ as this requires double virtuality.³²

³¹ This contribution is the same in all the HW models considered above, depending only on z_0 , which is matched to M_ρ .

³² A nonzero $\mathcal{F}_3^T$ also appears in so-called minimal models of tensor mesons [596], which contributes even at vanishing virtualities due to poles at $Q^2 = 0$, thereby violating gauge invariance, in contrast to the hQCD case; see Section 5.5.3 for a related discussion of the tensor TFFs.

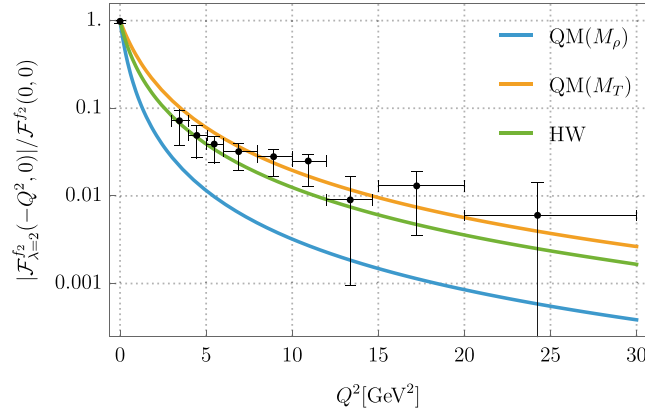


Fig. 56. Comparison of hQCD result (HW) and quark model ansatz with $\Lambda = M_\rho$ [55] and M_T [589,590] for the singly-virtual tensor TFF for helicity $\lambda = 2$ with Belle data [646] for the $f_2(1270)$, normalized to $\mathcal{F}^{f_2}(0, 0) \equiv \sqrt{5}\Gamma_{\gamma\gamma}/(\pi\alpha^2 M_T)$. Source: Figure adapted from Ref. [57].

5.6.6. Holographic QCD summary

The strongest point of hQCD models so far has been clearly the ability to match anomalies in the hadron sector in a way that is consistent with asymptotic conformal invariance and corresponding SDCs. Comparatively minimal models succeed in reproducing the masses and photon couplings of π^0 , η , and η' rather well, which makes their predictions for the experimentally less well-known axial-vector sector that is responsible for the saturation of the MV SDC also quantitatively interesting.

In principle, such models would also permit the study of $\pi^\pm$ and $K^\pm$ box contributions, by evaluating one-loop Witten diagrams, but this has not been carried out so far. However, such contributions are anyway determined with negligible errors by the dispersive approach. On the other hand, the similarly important S -wave rescattering contribution, corresponding to light scalar resonances, depends on phenomenological data for which hQCD models do not provide particular clues. However, using available data on scalars as input, the hQCD model calculation of Ref. [539] obtained a result that completely agrees with the dispersive approach, with somewhat enlarged error.

It thus seems more pertinent to compare only the available hQCD results for the axial sector and the tensor sector with corresponding results from other approaches, or to combine it with the contribution from pseudoscalar boxes, which in the dispersive approach [36,46,584] amounts to $-16.4(2) \times 10^{-11}$ together with the S -wave rescattering contribution from either the dispersive approach or the hQCD model calculation [539], the latter amounting to $-9(2) \times 10^{-11}$. The sum total given in Table 23 for the best-fitting LMR22 v1(F_ρ -fit) model, combined with the relatively large positive contribution from the tower of tensor modes [57,644], $11.1^{+1.3}_{-3.0} \times 10^{-11}$, would then correspond to

$$a_\mu^{\text{HLbL}} \Big|_{\text{hQCD completed}} = 114^{+10}_{-4} \times 10^{-11}, \quad (5.52)$$

where the estimate for an (hQCD-systematic) error accommodates the range of the (F_ρ -fit) models considered in Ref. [635] (including also the OPE-fit models would enlarge the upwards error to +17 points).

5.7. Rational approximants, resonance chiral theory, and Regge evaluations

5.7.1. π^0 , η , η' poles

Until recently, the estimate for the η , η' poles was obtained from CA alone, which provides us with a mathematical framework to approximate the TFFs in a systematically improvable way, while other approaches available at the time were deemed insufficient to meet the quality criteria outlined in WP20 [1]. As such, new alternative evaluations are necessary to better establish their current estimate. In particular, while for the singly-virtual case a wealth of experimental data confirms the idea that TFFs are reasonably described by VMD, the doubly-virtual case stands on a different footing. More specifically, while the singly-virtual behavior is dominated by the physics of the light vector resonances (and any approach faithfully describing them should provide an excellent parameterization), for the doubly-virtual case the lightest vector mesons cannot fulfill pQCD asymptotics. Instead, an infinite number of them, or some alternative mechanism supplying the missing physics, is required. While guidance can be found from pQCD at asymptotically large energies, the fact that fundamental physics is missing at intermediate energies calls for a critical inspection of this region and potential systematic effects therein. Advances along these lines have been undertaken in different theoretical approaches, see Sections 5.8, 5.5.1 and 5.6.1.

Concerning data, the very useful singly-virtual (spacelike) measurements from the BABAR [567,647], Belle [648], CELLO [564], CLEO [565], and LEP [566] collaborations are complemented, for the doubly-virtual case, only by five data

points published by BABAR in the η' channel [649]. Therefore, it is natural to wonder if η' data could shed light on the π^0, η cases based on $U(3)_F$ symmetry (and its breaking) arguments. In this context, Ref. [50] explores this possibility incorporating, also, lattice QCD results for double virtuality [575,576,650], that improves notably the control on this difficult dependence of the lightest pseudoscalar TFFs and has a nonnegligible impact on $a_\mu^{\text{PS-poles}}$ [50]. To such endeavor, Ref. [50] uses the framework of $R_\chi T$, incorporating resonances along similar lines as developed in Refs. [213,214] in a large-number of colors expansion [651] and accounting for flavor-breaking³³ and includes all operators with resonance fields that – after being integrated out – saturate the $\mathcal{O}(p^6)$ chiral low-energy constants [216]. In this approach, the effect of the infinite tower of vector and pseudoscalar resonances is accounted for with a second (effective) meson multiplet and contact terms – of ChPT origin – that, together with the first multiplet, comply with the singly- and doubly-virtual leading asymptotic behaviors predicted by pQCD [568,570,655,656]³⁴ and order by order with the chiral-symmetry-breaking expansion. The relations found among the Lagrangian couplings agree with the consistent SDCs in the odd-intrinsic parity sector [657]. In addition to the singly- and doubly-virtual experimental and lattice data quoted above,³⁵ Ref. [50] also fits the $P \rightarrow \gamma\gamma$ decay widths [306] and uses the η – η' mixing parameters [658] as stabilization points. It is instructive [50] to relate this $R_\chi T$ approach to the CA: the resulting π^0 TFF corresponds to a C_2^2 and the $\eta^{(\prime)}$ to C_4^4 CA³⁶ (for which clearly there is not enough data to determine all their unknown coefficients), although in the chiral limit they correspond as well to a C_2^2 . Even though Ref. [50] considers data fits using independent C_2^2 for the three P channels in which the OPE is constrained, it is concluded that these would require more doubly-virtual data (either from experimental measurements or from lattice QCD) to prevent overfitting. Interestingly, $R_\chi T$ allows one to predict from BABAR η' doubly-virtual measurements the π^0 and η corresponding dependence due to the built-in $U(3)_F$ relations, though lattice results (as future measurements on any of the P channels would also do) improve this knowledge. This is not possible in CAs that, in addition, provide limited ability to extract information from lattice QCD data due to the absence of $U(3)_F$ relations [50]. Concluding, the $R_\chi T$ model predicts

$$\begin{aligned} a_\mu^{\{\pi^0, \eta, \eta'\}} &= \left\{ 61.9(0.6)(^{+2.4}_{-1.5}), 15.2(0.5)(^{+1.1}_{-0.8}), 14.2(0.7)(^{+1.4}_{-0.9}) \right\} \times 10^{-11}, \\ a_\mu^{\text{PS-poles}} &= 91.3(1.0)(^{+3.0}_{-1.9}) \times 10^{-11}, \end{aligned} \quad (5.53)$$

where the first uncertainty is statistical and the second one is the systematic theory error. In order of importance, the sources of the main systematic uncertainties are: the effect of modeling the $n = 2, \dots, \infty$ multiplets by a second effective one, subleading corrections in $1/N_c$, and the combination of experimental and lattice data. In turn, they contribute by $\{+1.8, \pm 1.5, +0.4\}$, $\{+1.0, \pm 0.5, -0.6\}$, $\{+1.4, \pm 0.3, -0.8\}$ to the error for the $\{\pi^0, \eta, \eta'\}$ channels (see Ref. [50] for a detailed discussion, including smaller uncertainties as well). This result is fully compatible for the π^0 , compared to the estimate based on DR or CA, and the η, η' ones agree well with previous estimates based on CA ($a_\mu^\eta = 16.3(1.0)(0.5)(0.9) \times 10^{-11}$, $a_\mu^{\eta'} = 14.5(0.7)(0.4)(1.7) \times 10^{-11}$).

The results are compared to CAs for the singly and doubly-virtual TFFs in Fig. 57 together with a z -expansion fit to lattice data from Ref. [576]. Good agreement is found – even within statistical uncertainties – for the singly-virtual TFFs between $R_\chi T$ and CA. In turn, differences appear in doubly-virtual kinematics for the π^0 and η case, yet results are fully consistent if systematic uncertainties are accounted for (for CA this is estimated as the difference among the C_2^1 and C_1^0 approximants). In this respect, it would be interesting to better constrain the doubly-virtual behavior in the region relevant for the HLbL contribution. In particular, in the last row from Fig. 57 we show the difference of the contributions among CA and $R_\chi T$ to a_μ^{HLbL} if the TFFs are cut off at an upper scale Λ (the band stands for the central one obtained with CA, but lacks any kind of uncertainties). As can be observed, the main differences arise in the energy region up to $Q_i^2 \simeq 2 \text{ GeV}^2$, well below BABAR's data for the doubly-virtual η' case. This region could be accessed at BESIII [659] and could help improve the uncertainties above provided the experimental errors are below the plotted bands.

As a final remark, the size of doubly-virtual systematics emphasizes the relevance of less-controlled intermediate physics effects in both approaches. This can be better appreciated in CA, where no data was employed to constrain the doubly-virtual TFFs. There, one observes that singly-virtual data is vastly dominated by a single hadronic scale in the spacelike region, that relates to the success of VMD. In turn, the slower convergence for the doubly-virtual one is related to the existence of further scales that require one to go beyond the simplest C_1^0 approximant or naive VMD approaches, requiring further input to determine the additional parameters in CA and $R_\chi T$ approaches.

5.7.2. Other mesons and short-distance constraints

Previous results on axial-vector-meson contributions scattered over a large range [1] due to differences in their implementation. While some of them may be ascribed to an incorrect high-energy doubly-virtual behavior [591],

³³ It improves over and/or extends earlier descriptions in this framework [216,652–654].

³⁴ The subleading correction to the doubly-virtual result, computed within the OPE, is also fit, with its coefficient determined from QCD sum rules [570].

³⁵ Ref. [50] explains that only three lattice data points per channel can be used, to represent faithfully the lattice information.

³⁶ Note that coefficients must be chosen to reproduce the appropriate high-energy behavior. Further, it must be noted that, by contrast to the most general CA (such as in Ref. [35]) the denominator reduces to a product of poles, which would relate to what is known in the literature as Padé-type approximants, with different convergence properties.

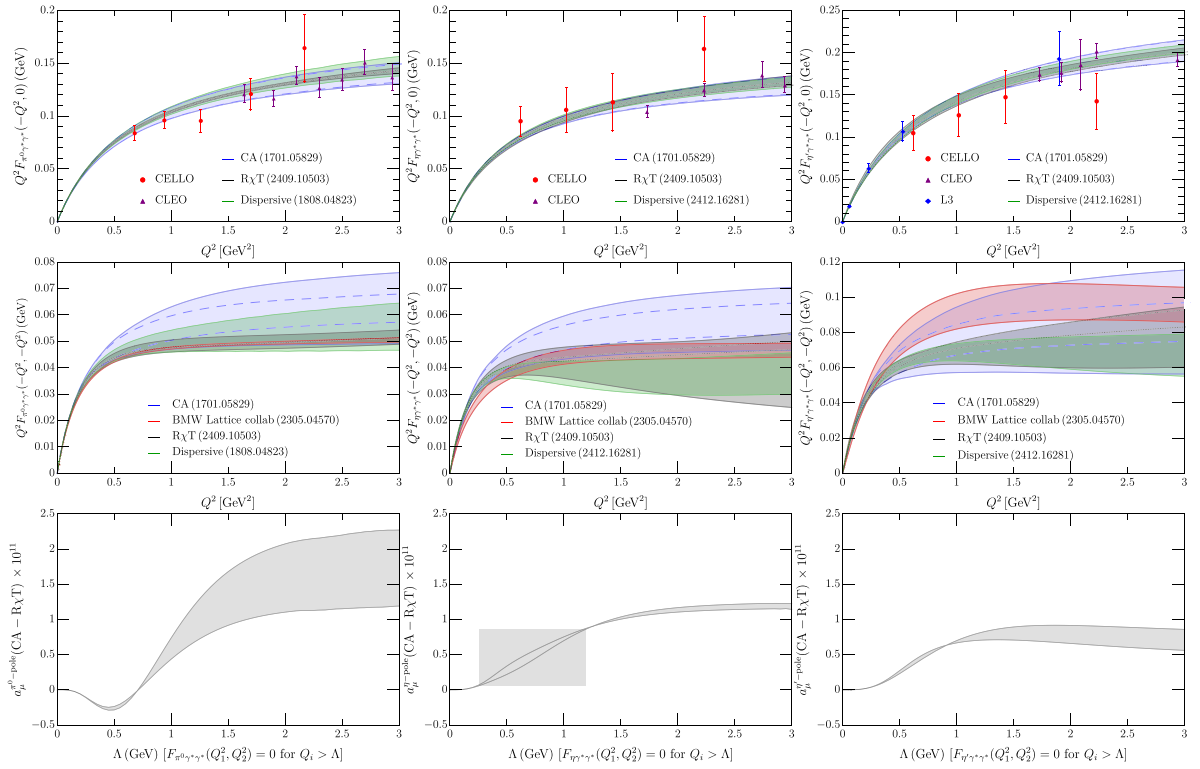


Fig. 57. First row: the singly-virtual TFFs. The inner dashed-blue (dotted-black) band is the CA (R χ T) result with statistical uncertainties, whereas the full ones include a combination in quadrature of statistics and systematic uncertainties. The green band is the dispersive results from Refs. [56,550] (cf. Section 5.5.1). The red band stands for the z-expansion fit to lattice data [576] (no systematics). Second row: the results analogous to the first row for doubly-virtual TFFs. Third row: the difference of the two approaches to the partial contribution to $a_{\mu}^{\text{HLbL},P}$ up to a given scale.

additional ambiguities arise from incorrect claims on the Landau–Yang theorem, kinematic zeros [660], or the treatment of the axial-vector meson propagator [639,660], that clearly required an improved understanding in Lagrangian/resonance-exchange models. Along these lines, recent evidence has accumulated that axial-vector mesons play a leading role in fulfilling SDCs [40–42].

In particular, Ref. [42] makes use of the important connection of SDCs in the MV limit [503] of the HLbL tensor to the $\langle VVA \rangle$ Green’s function and the anomaly. The central point is to advocate for a tensor basis free of kinematic singularities that allows for a transparent identification of the role of intermediate states. This contrasts to the standard $\langle VVA \rangle$ basis employed in earlier discussions of the MV limit, which basis naturally splits longitudinal, w_L , from transverse contributions at the cost of introducing kinematic singularities. More specifically, borrowing the notation from Ref. [1], Eq. (4.76), one can show that $(q_{12} = q_1 + q_2)$ [42,538]

$$\frac{w_L(q_1^2, q_2^2, q_{12}^2)}{8\pi^2} = \frac{\tilde{w}_0(q_1^2, q_2^2, q_{12}^2)}{8\pi^2} - \frac{(q_1^2 + q_2^2)}{q_{12}^2} \frac{w_T^+(q_1^2, q_2^2, q_{12}^2)}{8\pi^2} + \frac{(q_1^2 - q_2^2)}{q_{12}^2} \frac{w_T^-(q_1^2, q_2^2, q_{12}^2)}{8\pi^2}. \quad (5.54)$$

The form factors on the RHS ($\tilde{w}_0, w_T^\pm$) contain only physical singularities, but w_L clearly contains, in general, kinematic ones. More specifically, pseudoscalar poles exclusively appear in $w_L, \tilde{w}_0$, whereas $w_T^\pm$ contain axial-vector meson poles. Clearly, the only physical pole at $q_{12}^2 \rightarrow 0$ (in the chiral limit) identifies with the π^0 one, and its residue relates to its TFF, that only fulfills the anomaly ($w_L = 2N_c/q_{12}^2$) for real photons. For virtual photons, the anomaly can only be fulfilled if transverse contributions in Eq. (5.54) are included, showing that transverse physics cannot be excluded when accounting for the anomaly. It is also along these lines that Ref. [538] discusses the $(g-2)_\mu$ soft-photon limit. Identifying the form factors there [538], $-8\pi^2\{w_0 + w_2, w_1, w_2, w_3\} = \{\tilde{w}_0, w_T^-, w_T^+, \tilde{w}_T^-\}$, one obtains $-w_L(q_1^2, 0, q_1^2)/8\pi^2 = w_0(q_1^2, 0, q_1^2) + w_1(q_2^2, 0, q_1^2)$. Only the full tensor is free from unphysical poles. This is the principal line of thought in Ref. [42], which investigates axial-vector meson contributions to the HLbL tensor and $\langle VVA \rangle$. There it is shown how (an infinite number of) axial-vector mesons, that introduce pole-free contributions to w_L in Eq. (5.54), together with the π^0 pole, can fulfill the anomaly. In particular, this fixes all the ambiguities with axial-vector-meson propagators previously discussed that, together with high-energy constraints, fix the way in which Lagrangian models/resonance-exchange axial contributions should be accounted for. Note also that similar findings have been obtained recently albeit in a dispersive approach for the $\langle VVA \rangle$ function [51]. In addition, the discussion explains why R χ T calculations of axial exchanges – that

are necessarily transverse — cannot reproduce SDCs with axial-vector mesons alone, and contact terms would be required. This is nevertheless standard in R χ T, but discourages its use in the context of axial contributions to HLBL scattering.

Accounting for all these subtleties and focusing on the symmetric form factor exclusively, Ref. [42] finds, for the ground state multiplet,

$$a_\mu^{a_1} + a_\mu^{f_1} + a_\mu^{f'_1} = [5.4(^{+3.7}_{-3.3}) + 8.3(^{+3.4}_{-2.9}) + 2.3(^{+1.1}_{-0.9})] \times 10^{-11} = 16.0(^{+5.1}_{-4.5}) \times 10^{-11}, \quad (5.55)$$

where the symmetric form factor is analogous in functional form to that in Eq. (5.56). Note, however, that for a complete evaluation also the two asymmetric TFFs need to be taken into account, as their numerical contribution might not be negligible and only $\mathcal{F}_1$ is suppressed asymptotically [590], see Sections 5.5.3 and 5.6.2. Still, in this framework, this suffices to fulfill the SDCs for the HLBL tensor in the MV regime when an infinite number of axial-vector mesons are considered (see further comments in this respect below). The a_1 input used for Eq. (5.55) was obtained from the R χ T estimate in Ref. [639] under U(3)_F relations and the assumption that f_1, f'_1 are, analogously to the vector case, mostly light and strange mesons, respectively. This seems reasonable from BR[$f_1 \rightarrow \gamma \rho$]/BR[$f_1 \rightarrow \gamma \phi$] decays, and the equivalent 2γ decay widths for f_1, f'_1 are in line with experiment due to their R χ T scaling $\propto M_A/\Lambda_H^3$ differing from the standard assumption $\propto M_A^{-2}$ [600] (in holographic models $\propto \Lambda_H^{-2}$), outlining once more the need for experimental input for the axials.³⁷ The estimates $B_{25}^{a_1}(0, 0) = 0.245(63) \text{ GeV}^{-2}$ (in agreement with $0.252(30) \text{ GeV}^{-2}$ [51] or 0.230 GeV^{-2} [635]) and $\Lambda_{a_1} = 1.0(1) \text{ GeV}$ lead in addition to a doubly-virtual high-energy behavior in agreement with the OPE, $\lim_{Q^2 \rightarrow \infty} Q^4 B_{25}(-Q^2, -Q^2) = \sum_a M_A F_A^a \text{Tr}(\mathcal{Q}^2 \lambda^a)$ [42]. Further, in order to fulfill the MV SDCs, Ref. [42] introduces a Regge-like model for an infinite tower of axial-vector meson resonances and their form factors where

$$B_{25}^{A_n^a} = \frac{4F_A \text{Tr}(\mathcal{Q}^2 \lambda^a) M_{A_n^a}}{[q_1^2 + q_2^2 - (M_{A_n^a}^2 + n\Lambda_V^2)]^2}, \quad M_{A_n^a}^2 = M_{A_0^a}^2 + n\Lambda_A^2, \quad a = \{3, q, s\}, \quad (5.56)$$

with A_n^a the n th axial-vector resonance with isospin index a and decay constant $F_A = 140(10) \text{ MeV}$. MV SDCs, the anomaly, and the asymptotic behavior of the doubly-virtual π^0 TFF completely fix the parameter for the form factors to $\Lambda_V = 4\pi F_A/\sqrt{N_c}$ and $M_3^2 = (8\pi^2/3)[F_A^2 + 2F_\pi^2]$, predicting $B_{25}^{a_1}(0, 0) = 0.246 \text{ GeV}^{-2}$ and a slope for the π^0 TFF in agreement with recent estimates [35,37], that warrants then the appropriate low-energy behavior for the transverse part of the (VVA) Green's function (find similar discussions in Ref. [51]). This reflects the connection between the pseudoscalar and axial-vector sectors in the model. To bypass a detailed analysis of η - η' mixing, the parameters $M_{q,s}$ in Eq. (5.56) were fixed to reproduce the experimental 2γ equivalent decay widths for the ground state, and Λ_A^2 was taken to reproduce the Regge trajectories. As a result, the estimate for the tower of axial-vector mesons was found to be

$$a_\mu^{\text{HLbL}; \{a_1 f_1 f'_1\}} = \{5.89, 10.52, 1.97\} \times 10^{-11}, \quad \sum_{n=1}^{99} a_\mu^{\text{HLbL}; \{a_1 f_1 f'_1\}} = \{3.67, 8.46, 0.90\} \times 10^{-11}. \quad (5.57)$$

While not discussed in Ref. [42], by choosing $\Lambda_A = \Lambda_V$, it is possible to satisfy the MV SDC for the transverse part as well ($w_L(Q^2) = 2w_T(Q^2)$ asymptotically [503]). In the following, we provide an update for the tower of excited axial-vector meson states if $\Lambda_A = \Lambda_V$ is taken, including as well uncertainties from the half-width rule [661], F_A [662], and experimental uncertainties on $B_{25}^{f_1 f'_1}(0, 0)$ ³⁸

$$a_\mu^{\{a_1 f_1 f'_1\}} = \{6.0(0.2), 10.5(1.5), 2.0(0.5)\} \times 10^{-11}, \quad \sum_{n=1}^{99} a_\mu^{\{a_1 f_1 f'_1\}} = \{3.6(0.4), 8.6(0.7), 0.91(0.09)\} \times 10^{-11}, \quad (5.58)$$

but still lacking a genuine systematic uncertainty from the model itself. In order to compare against other results, it is interesting to provide their contributions divided in the different regions introduced in Refs. [55,594] for $Q_0 = 1.5 \text{ GeV}$ (cf. Section 5.5.3). In particular, $a_\mu^{a_1 f_1 f'_1}|_{Q_i < Q_0} = 10.9(1.0) \times 10^{-11}$, $a_\mu^{\sum A^*}|_{Q_i < Q_0} = 3.2(6) \times 10^{-11}$, $a_\mu^{\sum A_1}|_{\text{Mixed}} = 12.8(5) \times 10^{-11}$, $a_\mu^{\sum A}|_{Q_i > Q_0} = 4.8(1) \times 10^{-11}$, $a_\mu^{\sum A} = 31.7(1.6) \times 10^{-11}$, where $\sum A^*$ refers to the sum of excited axial-vector multiplets and $\sum A$ to the full tower (including the ground state), cf. Table 29. In addition, we show the prediction of the model above for the ground state a_1 meson $M_{a_1}^{-2} \mathcal{F}_2^{a_1}(-Q^2, 0)$ TFF compared to the dispersive result in Fig. 58. Remarkably, the plot shows a nice agreement at low energies. Note, in particular, that the model above predicts $M_{a_1}^{-2} \mathcal{F}_2^{a_1}(0, 0) = 0.25(5) \text{ GeV}^{-2}$, to be compared with the dispersive result $0.25(2) \text{ GeV}^{-2}$. At high energies some differences appear, that are expected from the simplistic model and the omission of the antisymmetric form factors. More specifically, the latter seem necessary, in connection with the anomaly, to correctly match the singly-virtual π^0 TFF asymptotic behavior. Note that, still, the pQCD relation $w_L(Q^2) = 2w_T(Q^2)$ holds.

³⁷ The mixing angle of f_1, f'_1 can be estimated as well from the equivalent two-photon decay widths [598,600] under U(3) assumptions. Apart from data on axial-vector TFFs directly, these flavor assumptions can also be tested in radiative decays to vector mesons. For a more complete discussion, see Refs. [48,543] and Table 30.

³⁸ We note that changes are minor, and differences are mostly due to the updated numerical integration recipe Divonne from CUBA library [663].

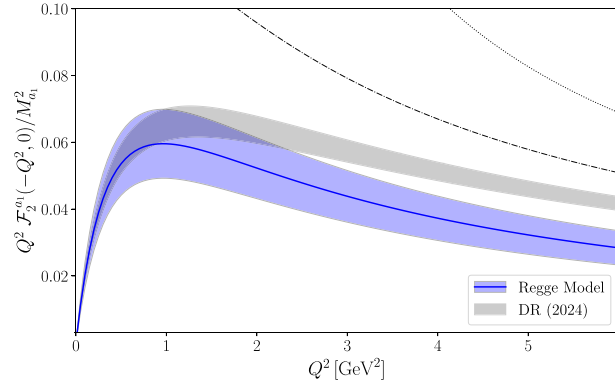


Fig. 58. The ground state a_1 meson singly-virtual TFF $M_{a_1}^{-2} \mathcal{F}_{a_1}^{a_1}(-Q^2, 0)$. The results from the Regge model [42] identify with Eq. (5.56) and are shown in blue, with uncertainties from F_{a_1} and the half-width rule applied to M_{a_1} . The dispersive result [51] is shown as a gray band and the BL limit [590] is shown as a dotted (dash-dotted) gray line without (with) mass corrections.

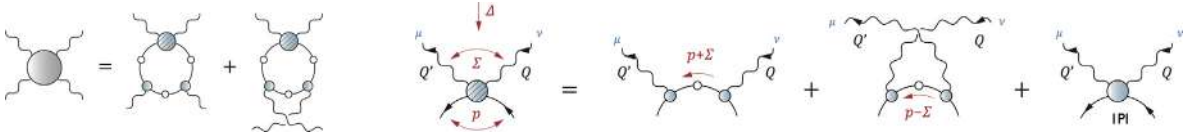


Fig. 59. HLbL in terms of a quark Compton vertex including contributions from the dressed quark loop and meson exchange and box diagrams, as well as other contributions subsumed in the IPI part of the Compton vertex.
Source: Figures taken from Ref. [671].

Finally, also the tensor contributions $a_\mu^{(a_2+f_2+f_2')\text{-pole}}$ have recently been computed in $R_\chi T$ [664]. Using a large- N_c expansion, working in the chiral limit, and including two vector resonance multiplets to fulfill the SDCs, this contribution reads $(-4.5_{-0.3}^{+0.5}) \times 10^{-11}$ (cutting off the integrals above $Q_0 = 1.5$ GeV). This coincides with the result given in the last line of Table 24, corresponding to the fact that in this approach the only nonvanishing TFF is $\mathcal{F}_1^T$, in line with the discussion in Section 5.5.3, and its shape turns out similar to hQCD, see Section 5.6.5. The numerical effect is therefore covered by the uncertainty estimates discussed in Section 5.10.

5.8. Functional methods

Another approach to the muon $g - 2$ problem employs functional methods, in particular a formalism based on coupled Dyson–Schwinger equations (DSEs) and Bethe–Salpeter equations (BSEs) in QCD. The DSEs are the quantum equations of motion which relate QCD's n -point correlation functions with each other, including the dressed quark and gluon propagators, the quark–gluon and three-gluon vertices, etc. These are exact equations in QCD, but in practice they require truncations to obtain numerical solutions. The status quo in the DSE sector is by now quite advanced and arriving at a point where ab-initio solutions, whose only approximations amount to neglecting higher n -point functions, have become feasible [665–667]. On the other hand, its implementation in hadronic BSEs is technically challenging and has so far generally been limited to the rainbow-ladder (RL) truncation, where the quark–(anti-)quark interaction is modeled by an effective gluon exchange [668]. This is the leading order in a systematic expansion of the kernel, which satisfies chiral symmetry constraints by construction, including its spontaneous breaking which is responsible for hadron mass generation. Over the past decades this approach has been employed in studying a wide range of hadron properties from light- and heavy-meson spectroscopy to baryon spectroscopy, four-quark states, EM elastic and TFFs, parton distributions and various scattering amplitudes; see Refs. [668–670] for reviews.

In principle, the HLbL contribution within the DSE/BSE formalism can be evaluated through two different frameworks: (i) directly and (ii) indirectly via the dispersive framework (see below). First, in the direct approach, the HLbL amplitude can be systematically expressed in terms of the underlying correlation functions like the quark propagator, the quark–photon vertex and quark–two-photon vertex, which are then computed through their DSEs and BSEs [668]. This yields a decomposition of the amplitude in terms of a dressed quark loop plus diagrams with intermediate $q\bar{q} \rightarrow q\bar{q}$ four-point functions, which automatically contain all possible meson poles. The latter can be rearranged such that the central ingredient is the quark–two-photon vertex, see Fig. 59, which has been calculated in the context of nucleon Compton scattering [671]. Therefore, this establishes an independent approach to compute HLbL scattering that is complementary to lattice QCD and dispersion relations.

Table 25

DSE/BSE results for various contributions to a_μ^{HLbL} in units of 10^{-11} compared to the values quoted in WP20 [1]. The results for pseudoscalar exchange and pseudoscalar-box contributions are from Refs. [38,586] and those from axial-vector and scalar exchange from Ref. [53].

	DSE/BSE	WP20
π^0 exchange	62.6(1)(1.3)	63.0 $^{+2.7}_{-2.1}$
η exchange	15.8(2)(3)(1.0)	16.3(1.4)
η' exchange	13.3(4)(3)(6)	14.5(1.9)
π^0, η, η' exchange	91.6(1.9)	93.8 $^{+4.0}_{-3.6}$
π box	−15.7(2)(3)	−15.9(2)
K box	−0.48(2)(4)	−0.46(2)
π, K boxes/loops	−16.2(5)	−16.4(5)
S -wave $\pi\pi$ rescattering	–	−8.0(1.0)
higher scalar exchange	−1.6(5)	−2.0(2.0)
AV exchange (single)	17.4(6.0)	6.0(6.0)
AV exchange (tower) + SDC	24.8(6.1)	21.0(16.0)

The full HLbL scattering amplitude is electromagnetically gauge invariant and thus transverse. This is preserved by the RL truncation but does not hold for the individual diagrams contributing to the amplitude like the dressed quark loop. The main obstacle so far is the lack of a complete basis for the HLbL amplitude that is “minimal”, i.e., free of kinematic constraints. The amplitude depends on 136 linearly independent tensors, of which 41 are purely transverse and 95 nontransverse [672,673]. A minimal basis for the transverse part alone is known [544,673] but its analog for the nontransverse part is still missing. However, the latter is needed for a projection onto all 136 tensors. This allows one to remove gauge artifacts because any further approximation (in addition to the RL) would break gauge invariance and induce kinematic singularities, which impedes an extraction of a_μ^{HLbL} . An analogous but simpler example was recently discussed in the context of axial-vector-meson TFFs contributing to HLbL scattering: In that case, three tensors are transverse and three are nontransverse; the full RL calculation is gauge invariant, but any further approximation would introduce gauge artifacts that can only be removed by projecting onto the complete basis [53]. The construction of a complete basis for HLbL scattering is ongoing work.

The second approach to HLbL scattering with functional methods has been advocated in Refs. [38,53,586]. Here the meson EM and transition form factors entering in the HLbL amplitude are calculated separately in the DSE/BSE framework and then inserted into the dispersive formalism described in Section 5.5. In this case the systematics of meson-exchange and meson-box diagrams is identical to the dispersive approach, while the effects from the quark loop must be implemented through SDCs, see Section 5.4. Once again, all ingredients in the form factor calculations (quark propagators, quark–photon vertices and meson amplitudes) are determined in the RL truncation without further approximations. In particular, the quark–photon vertex dynamically develops vector-meson poles so that the form factors in the timelike region automatically capture the physics of VMD.

The results obtained in the DSE/BSE framework are collected in Table 25 and compared to the values quoted in WP20.³⁹ The sum of intermediate π^0 , η , and η' single-meson pole contributions amounts to $a_\mu^{\text{PS-poles}} = 91.6(1.9) \times 10^{-11}$, where the error contains the variation of the parameter in the RL effective interaction, the numerical error, and the uncertainties in the η and η' mixing parameters [38]. The corresponding $\pi^0, \eta, \eta' \rightarrow \gamma^{(*)}\gamma^{(*)}$ TFFs, displayed in Fig. 60, are calculated in the spacelike domain including the doubly- and singly-virtual kinematic limits [38,676,677]. Similar results in a slightly different framework have been found in Ref. [674]. The sum of the pion and kaon box diagrams, which depend on the pion and kaon EM form factors, yields $a_\mu^{\text{PS-box}} = -16.2(5) \times 10^{-11}$ [38,586]. All these results are in good agreement with results from other approaches and the WP20 estimate.

Since WP20, three new issues have been addressed in the DSE/BSE approach. First, the authors of Ref. [678] have determined corrections to the pseudoscalar pion and kaon box contributions beyond RL (BRL), see also Ref. [679] for a summary. To this end they determined the (RL and BRL) EM form factors of the pion and the kaon in the space- and timelike region [680–682]. A major result is the remarkable robustness of the RL truncation in the spacelike region, leading to the observation that BRL pion and kaon box contributions to HLbL are safely contained within the error bars of previous RL results. Second, after calculating the EM form factor of the first radial excitation of the pion, π_1 , its box contribution to the HLbL component of the a_μ has been computed in Ref. [683], producing, respectively, $a_\mu^{\pi_1\text{-box}}(\text{RL}) = -2.03(12) \times 10^{-13}$ and $a_\mu^{\pi_1\text{-box}}(\text{BRL}) = -2.02(10) \times 10^{-13}$ in the RL and BRL truncation schemes. Third, the contributions from intermediate axial-vector and scalar mesons have been calculated [53]. Their corresponding TFFs have been determined in the very same setup used previously to determine pseudoscalar meson exchange (π^0, η and η') as well as meson (π and K) box contributions. The corresponding results are also shown in Table 25. In the scalar meson sector, contributions from $f_0(980)$,

³⁹ The DSE/BSE framework also delivers results for the HVP contribution in the ballpark of dispersive and lattice results, although the error of roughly 3% is much too large to make any claims for this quantity [675].

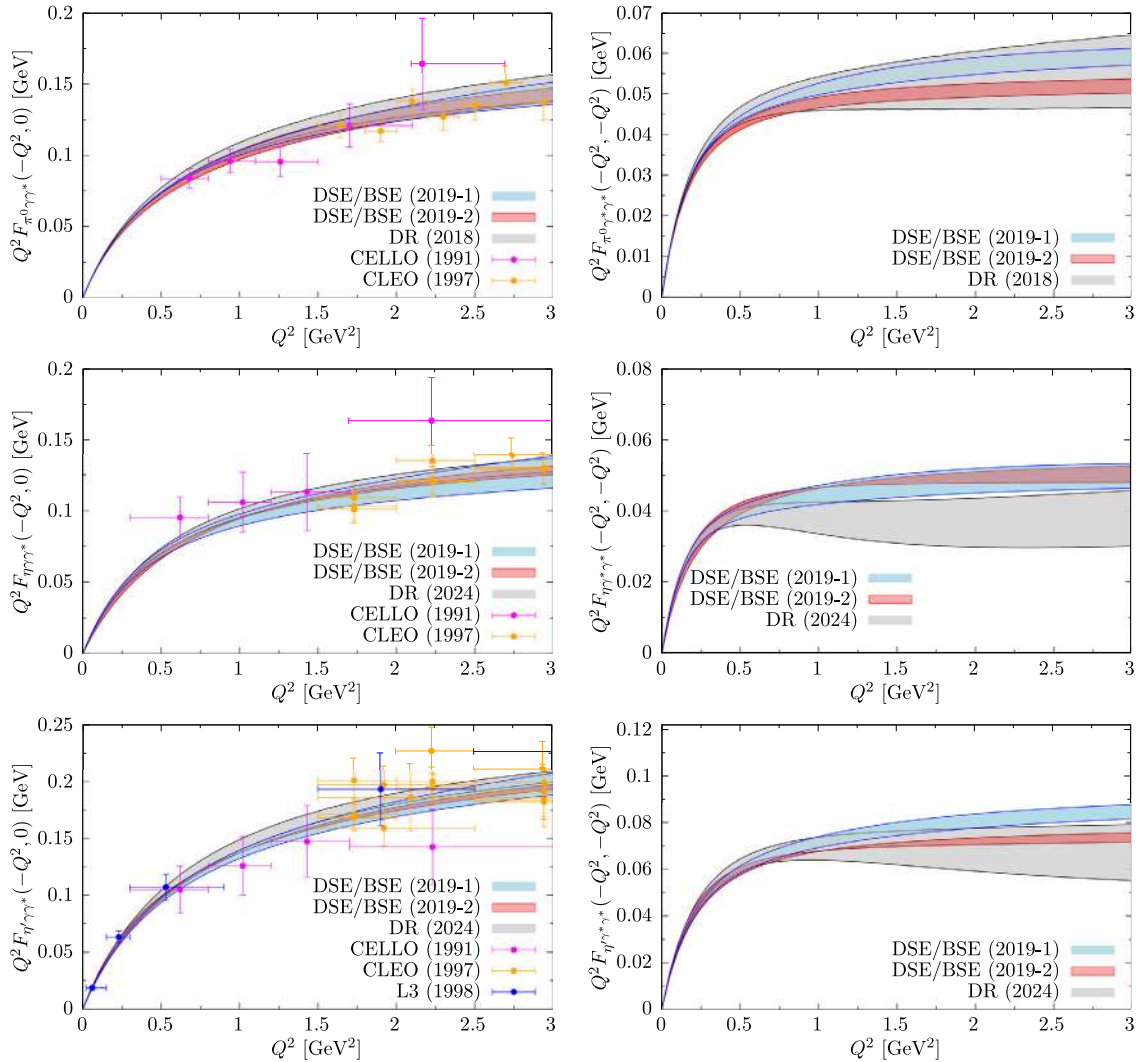


Fig. 60. Results for the singly- (left) and doubly-virtual (right) TFFs of π^0 (first row), η (second row), and η' (third row) from the DSE/BSE approach of Ref. [38] (2019-1) and Ref. [674] (2019-2) compared to dispersive results [37,56,548,550] and experimental data [564–566].

$a_0(980)$, $f_0(1370)$, and $a_0(1450)$ have been combined into a single number, $a_\mu^{\text{HLbL}}[\text{scalar}] = -1.6(5) \times 10^{-11}$. In the axial-vector sector, the result $a_\mu^{\text{HLbL}}[\text{AV exchange (single)}] = 17.4(6.0) \times 10^{-11}$ combines contributions from the lowest lying multiplet of a_1 , f_1 , and f_1' states, whereas the result $a_\mu^{\text{HLbL}}[\text{AV-tower+SDC}] = 24.8(6.1) \times 10^{-11}$ combines contributions from a whole tower of a_1 , f_1 , and f_1' multiplets including short-distance contributions in the form of a quark loop using the matching procedure outlined above and in Refs. [535,540]. The latter is necessary because, even though a whole tower of axial-vector contributions has been taken into account, one cannot show analytically whether an infinite resummation of such a tower would satisfy the SDCs as it does in hQCD [40,41,47,542]. In contrast to hQCD, the TFFs in the DSE/BSE framework are given numerically, and much more refined numerical methods would be needed to extract the necessary information. A comparison of the $\mathcal{F}_2$ TFF to the dispersive result is shown in Fig. 61.

In summary, systematic cross-checks and comparisons between the dispersive approach, lattice QCD, hQCD, and the approach to QCD via functional methods certainly has the potential to further decrease the spread of results and consequently the error bar of the axial-vector and other contributions to HLbL in the future.

5.9. Form factor comparisons

For the pseudoscalar TFFs, several cross-checks between the various approaches have already been discussed and illustrated throughout the various subsections. Here, we add a broader comparison of all phenomenological theoretical

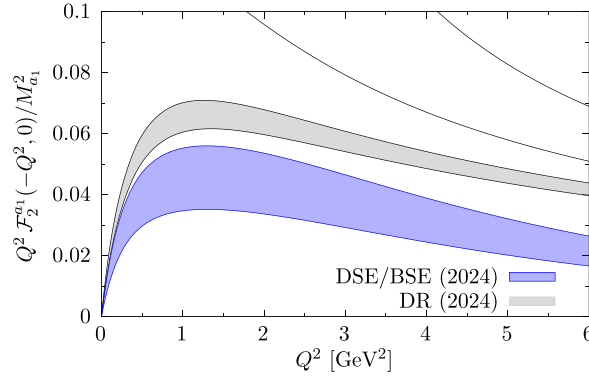


Fig. 61. Results for the axial-vector TFF $\mathcal{F}_2$ from the DSE/BSE approach of Ref. [53] (2024) compared to the dispersive result of Ref. [51]. The momentum dependency of the form factors agrees nicely; differences in an overall factor can be mostly attributed to the assumption of ideal mixing in the DSE/BSE result. The BL limits [590] are shown as dash-dotted/dotted lines similar to previous plots.

Table 26

The slope and the asymptotic value of the pseudoscalar singly-virtual TFFs from the different approaches.

	Dispersive [37,56,548,550]	CA [35]	R χ T [50]	hQCD [47]	DSE/BSE [38]
b_{π^0} [GeV $^{-2}$]	1.73(5)	1.76(10)	1.74(1)	1.68(8)	1.71(1)
b_η [GeV $^{-2}$]	1.83(4)	1.91(3)	2.00(3)	1.53(3)	1.87(1)
$b_{\eta'}$ [GeV $^{-2}$]	1.49(3)	1.43(3)	1.37(2)	1.31(2)	1.54(3)
$\bar{F}_{\text{asym}}^{\pi^0}$	$2F_\pi$	$2F_\pi$	$2F_\pi$	$2F_\pi [\times 0.9]$	$2.6(4)F_\pi$
$\bar{F}_{\text{asym}}^\eta$ [GeV]	0.186(13)	0.180(12)	0.174(3)	0.194(14)	0.21(2)
$\bar{F}_{\text{asym}}^{\eta'}$ [GeV]	0.264(13)	0.255(4)	0.260(4)	0.32(4)	0.36(4)

Table 27

The normalization of the a_1 TFF $\mathcal{F}_2(0, 0)/M_{a_1}^2$ and M_{a_1} used from the different approaches in units of GeV $^{-2}$ and GeV, respectively.

	Dispersive [51]	VMD, U(3) [48,590]	Regge [42]	R χ T [42,639]	hQCD [47]	DSE/BSE [38,586]
$\mathcal{F}_2(0, 0)/M_{a_1}^2$	0.25(2)	0.23(3)	0.25(5)	0.25(6)	0.265(20)	0.18(3)
M_{a_1}	1.23(4)	1.23(4)	1.23(21)	1.23(21)	1.28(8)	1.23(4)

calculations, specifically for the TFF slopes and asymptotic behavior. We define the slope parameters for the pseudoscalar TFFs according to

$$b_P \equiv \frac{1}{F_{P\gamma\gamma}} \frac{\partial}{\partial q^2} F_{P\gamma^*\gamma^*}(q^2, 0) \Big|_{q^2=0}. \quad (5.59)$$

In the simplest VMD model, the slopes are expected to be given approximately by $b_P \simeq M_\rho^{-2} \simeq 1.66 \text{ GeV}^{-2}$ for all three pseudoscalars. These slope parameters, as well as the singly-virtual asymptotic values

$$\lim_{Q^2 \rightarrow \infty} Q^2 F_{P\gamma^*\gamma^*}(-Q^2, 0) = \bar{F}_{\text{asym}}^P, \quad (5.60)$$

are collected, for various theoretical approaches, in Table 26. We find that the slopes b_{π^0} of the π^0 TFF agree within uncertainties for all calculations. The hQCD results for b_η and $b_{\eta'}$ are smaller than all others, with the most significant discrepancy for b_η , while the R χ T number for b_η deviates somewhat towards a larger value. Dispersive, CA, and DSE/BSE slopes are well consistent with each other also for η and η' . The results for the singly-virtual asymptotic behavior are largely consistent with each other within uncertainties for all TFFs, with discrepancies at most at the 2σ level.

In Table 27, we compare the different models for the TFF normalization $\mathcal{F}_2(0, 0)/M_{a_1}^2$ for the $a_1(1260)$ axial-vector meson, again finding excellent agreement among all of them. Note that this only constrains one out of the in total three axial-vector TFFs, cf. the discussion in Section 5.5.3.

5.10. A new final number for analytic HLbL

We now have a fully dispersive evaluation of the π^0 , η , and η' contributions, which agrees well with the other approaches based on phenomenological analyses, CA and R χ T modeling as well as the more QCD modeling approaches

Table 28The pseudoscalar pole contributions to a_μ in units of 10^{-11} from the different approaches.

	Dispersive [37,56,548,550]	CA [35]	R χ T [50]	hQCD [47]	DSE/BSE [38]
π^0	$63.0^{+2.7}_{-2.1}$	63.6(2.7)	$61.3^{+2.5}_{-1.6}$	63.4(2.7)	62.6(1.3)
η	14.7(9)	16.3(1.4)	$15.2^{+1.2}_{-0.9}$	17.6(1.7)	15.8(1.1)
η'	13.5(7)	14.5(1.9)	$14.2^{+1.6}_{-1.1}$	14.9(2.0)	13.3(8)
Sum	$91.2^{+2.9}_{-2.4}$	94.4(3.6)	$91.3^{+3.2}_{-2.1}$	95.9(3.8)	91.6(1.9)

hQCD and the functional approaches. The comparison of these is shown in Table 28. This contribution is improved over WP20 [1] of $93.8^{+4.0}_{-3.6} \times 10^{-11}$ to

$$a_\mu^{\text{PS-poles}} = 91.2^{+2.9}_{-2.4} \times 10^{-11}. \quad (5.61)$$

The next contributions are those from the pion and kaon box and rescattering in the S -wave channel. The dispersive box contributions for the pion have not changed compared to WP20. The kaon box is now evaluated with dispersive methods with a very small change compared to WP20. Both contributions are now evaluated as well by functional methods in good agreement with the dispersive results. The sum is unchanged giving

$$a_\mu^{\pi,K\text{-box}} = -16.4(2) \times 10^{-11}. \quad (5.62)$$

The S -wave rescattering has been updated and now includes $I = 0, 1, 2 \pi\pi, \bar{K}K, \pi\eta$ rescattering, leading to

$$a_\mu^{S\text{-waves}} = -9.1(1.0) \times 10^{-11}. \quad (5.63)$$

The above contributions are well established and agree well among the different approaches. They sum to

$$a_\mu^{\text{disp-low}} = 65.7^{+3.1}_{-2.6} \times 10^{-11}. \quad (5.64)$$

This is slightly lower than the WP20 result of $69.4(4.1) \times 10^{-11}$ and a somewhat smaller error. This is mainly due to the improved estimates of the η, η' contributions from dispersive methods compared to the CA results used previously.

The main source of the error in WP20 came from the short- and intermediate-distance domain with both missing resonances and issues of double counting. The SDCs have been improved very much as described in Section 5.4. The remaining contributions are split in different integration regions which we treat differently. This allows one to deal with the double counting issue. The integration regimes are split by a scale $Q_0 = 1.5 \text{ GeV}$ in a purely short-distance regime with all three $Q_i > Q_0$, a purely long-distance regime with all three $Q_i < Q_0$, and a mixed regime.

The purely short-distance regime can be calculated by using the QCD results discussed in Section 5.4 with contributions already included in Eq. (5.64) subtracted. Similarly, in the phenomenological approaches all other contributions can be included. This is shown in Table 29 in the row $Q_i > Q_0$. The agreement from the QCD result with the other approaches is reasonable with the possible exception of the functional approach. For this region we will use the QCD result with

$$a_\mu^{\text{SD-QCD}} = 6.2^{+0.2}_{-0.3} \times 10^{-11}. \quad (5.65)$$

The error is mainly from the uncertainty in α_s .

The next part is the mixed region where at least one of the Q_i is larger than Q_0 and one smaller. In this regime there are known short-distance results [54,503,527,538]. These constraints have been included to a large extent in all the modeling as well for direct evaluation of some parts of this region. For the modeling approaches the results have been calculated for the sum. The dispersive results follow from Table 22. The uncertainty on the hadronic part has a 30% error added quadratically to the input uncertainties added. The results from the four approaches are in reasonable agreement but the dispersive is somewhat larger due to a large contribution from the OPE domain. The error on the OPE domain has been determined from varying r , defined in Eq. (5.31), and shows that there is some remaining mismatch even after adding the effective poles. For this part we will use

$$a_\mu^{\text{Mixed}} = 15.9(1.7) \times 10^{-11}. \quad (5.66)$$

The remaining error we expect to be covered by the Q_0 variation discussed below.

The last remaining contribution is the low-energy one with all three $Q_i < Q_0$. The contribution from the lowest axial-vector nonet has been much clarified compared to WP20 and all approaches are now in acceptable agreement. The heavy scalars are estimated both in the functional and dispersive approach and are in good agreement with each other as well as with estimates for the scalars used in WP20, but now with a smaller error. For the tensor contributions the error estimate used in WP20 was shown to not be reliable. Here the dispersive approach uses a phenomenological estimate including only one of the five TFFs, as argued from the quark model. The hQCD estimate includes one more and finds a different sign due to this; with only the same single TFF as in the dispersive calculation the result for hQCD for the tensors is somewhat more negative than the dispersive estimate. We will use as the tensor contribution $-1(4)$ covering all available results. With this and the dispersive results agreeing well with the others we will use for this contribution

$$a_\mu^{\text{Low}} = 12.5(5.9) \times 10^{-11}. \quad (5.67)$$

Table 29

The contributions in units of 10^{-11} from the different regions with the parts included in $a_\mu^{\text{disp-low}}$ removed. Dispersive results from Table 22, with a 30% error added for the hadronic contributions; hQCD, Regge, and DSE/BSE results are calculated from the models described in Section 5.7.2, Section 5.6, and Section 5.8 with this value of Q_0 and numbers can thus be slightly different from those quoted there. The row “other” includes all contributions not specified explicitly, in particular from excited axial-vector mesons, and for hQCD also from excited tensors and tensor nonpole contributions (the latter two give 5.6 out of the 8.0). In the row labeled “Sum” the errors for hQCD have been added linearly.

Region		Dispersive	hQCD	Regge	DSE/BSE
$Q_i > Q_0$		$6.2^{+0.2}_{-0.3}$	6.3(7)	4.8(1)	2.3(1.5)
Mixed	A, S, T	3.8(1.5)			
	OPE	10.9(0.8)			
	Effective pole	1.2			
	Sum	15.9(1.7)	13.5(2.4)	12.8(5)	10.1(3.0)
$Q_i < Q_0$					
	$A = f_1, f'_1, a_1$	12.2(4.3)	13.1(1.5)	10.9(1.0)	8.6(2.6)
	$S = f_0(1370), a_0(1450)$	−0.7(4)			−0.8(3)
	$T = f_2, a_2$	−2.5(8)	2.9(4)		
	Other	2.0	8.0(9)	3.2(6)	2.8(6)
	Sum	11.0(4.4)	24.0(2.8)	14.1(1.2)	10.6(2.7)
Sum		33.2(4.7)	43.8(5.9)	31.7(1.6)	23.0(7.4)

When putting all together there are still some remaining uncertainties. These we estimate by varying Q_0 as described in Section 5.5.4 and Ref. [55] to be an error of 1.9 and the estimate of missing contributions by varying how matching to the SDCs is done. This is estimated by varying how the matching with the effective poles is done and adds an error of 3.9. However, while the estimate of “Other” from the dispersive estimates agrees well with the Regge and DSE/BSE results, the hQCD estimate of the remaining contributions is higher, we therefore increase the error for this uncertainty to 5.0.

Finally we use the same estimates for the NLO HLbL charm contribution as in WP20:

$$a_\mu^{\text{charm}} = 3(1) \times 10^{-11}. \quad (5.68)$$

Adding all the errors quadratically and the contributions gives a total (LO) HLbL contribution of

$$a_\mu^{\text{HLbL}} = 103.3(8.8) \times 10^{-11}. \quad (5.69)$$

This should be compared with the WP20 result of $92(19) \times 10^{-11}$. Our new number is perfectly compatible with the previous number within errors and has a significantly improved precision due to improvements of many hadronic contributions as well as improvements in short-distance calculations and matching.

Finally, there is the estimate of the NLO HLbL contribution from Ref. [58], which we update along the lines of Ref. [1]:

$$a_\mu^{\text{HLbL, NLO}} = 2.6(6) \times 10^{-11}. \quad (5.70)$$

5.11. Prospects for future improvements

5.11.1. Theory

The original approach of Refs. [34,36] is based on dispersion relations for the HLbL tensor in general four-point kinematics, which can be derived from the Mandelstam double-spectral representation. The photon virtualities are treated as fixed external variables, while dispersion relations are written in terms of the Mandelstam variables for off-shell photon–photon scattering. A disadvantage of this approach is that, even in the optimized BTT decomposition of Ref. [49], the cancellation of kinematic singularities from the contribution of intermediate states of spin equal to 2 or larger is not manifest. This is due to the fact that in this approach – based on the decomposition of the HLbL tensor in terms of a redundant set of BTT structure – kinematic singularities in the photon virtualities are eliminated by sum rules that involve a tower of intermediate states in the unitarity relation. This issue is solved within a novel approach in triangle kinematics [507], in which the external photon is taken soft prior to setting up the dispersion relations for the scalar functions entering the master formula in Eq. (5.4). In this framework the original cuts in the Mandelstam variables and in the photon virtualities are no longer separated, which leads to more complicated unitarity relations and the fact that the dependence of pole contributions on the photon virtualities is not automatically resummed as in the original approach. In the new framework, a reshuffling of intermediate-state contributions takes place and further sub-processes enter the two-pion unitarity relations, e.g., $\gamma^* \gamma^* \gamma \rightarrow \pi\pi$. These can be dispersively reconstructed without introducing kinematic singularities nor ambiguities [507]. A comparison and a suitable combination of the two dispersive approaches to HLbL [507] will, in the future, provide optimal control over residual uncertainties in the matching between hadronic and asymptotic continuum contributions. A detailed study of the relationship between the two types of dispersion relations and the reshuffling of different hadronic intermediate-state contributions has been recently performed in Ref. [51] for the simpler case of the VVA correlator.

Ultimately, one would like to have in addition a data-driven approach given in terms of a single observable, as is done for the HVP contribution. Such an alternative approach is, in principle, provided by the Schwinger sum rule [684]. It expresses the anomalous magnetic moment via an integral of a polarized photoabsorption cross section as follows:

$$a_\mu = \frac{m_\mu^2}{\pi^2 \alpha} \int_{\nu_0}^{\infty} d\nu \left[\frac{\sigma_{LT}(\nu, Q^2)}{Q} \right]_{Q^2 \rightarrow 0}, \quad (5.71)$$

where Q^2 is the photon virtuality, ν is the photon lab frame energy, and ν_0 is the first inelastic particle-production threshold. The polarized photoabsorption cross section σ_{LT} is the observable quantity that corresponds to the sum of the muon spin structure functions, $g_1(x, Q^2) + g_2(x, Q^2)$. In this approach the HVP and HlBL contributions appear on the same footing; see Ref. [685] for more details. Work in this direction is ongoing [686–688].

5.11.2. Experiment

Experimental input for a data-driven evaluation of the HlBL contribution requires measurements of TFFs $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ of pseudoscalar mesons P at arbitrary virtualities Q_i^2 to improve on the pseudoscalar-pole contributions. The normalization of the TFF at $Q_i^2 = 0$ is given by the radiative width $\Gamma(P \rightarrow \gamma\gamma)$ of the respective pseudoscalar meson. A new measurement of the radiative width of η using Primakoff-type meson production is announced by the PrimEx- η collaboration at JLab [689]. A first, preliminary evaluation without considering systematic effects yields $\Gamma(\eta \rightarrow \gamma\gamma) = 0.499(36) \text{ keV}$ [690], which is in agreement with the current value $\Gamma(\eta \rightarrow \gamma\gamma) = 0.515(18) \text{ keV}$ [225] listed by the PDG. As the PDG only considered results from e^+e^- experiments in their average and fits, the agreement suggests that the previously observed discrepancy with results from Primakoff-type experiments is resolved.

With the proposed upgrade of JLab to 22 GeV beam energy significant improvements on the accuracy of the radiative width measurements of π^0 , η , and η' will be possible [691]. While the Primakoff-type measurements for η and, for the first time, also for η' , will be performed on nuclear targets with projected accuracies of 2% and 3.5%, respectively, the measurement for π^0 can be performed off atomic electrons, which will allow one to avoid uncertainties of nuclear effects and achieve sub-percent accuracy on the radiative width. The Primakoff program is to be extended to measure the momentum dependence of the meson TFFs in a range of 0.001 GeV^2 up to 0.3 GeV^2 , which is complementary to the ongoing investigations at e^+e^- colliders.

The preliminary result of the BESIII collaboration on the singly virtual spacelike TFF of the π^0 [692] has already demonstrated the potential of the e^+e^- experiment to contribute high-precision data in the most relevant region of momentum transfer around 1 GeV^2 . In spring 2024 the collaboration finalized the data taking of a 20 fb^{-1} data set at $\sqrt{s} = 3.773 \text{ GeV}$, which improves on the statistics of the preliminary result by a factor seven and will be the new standard sample for two-photon fusion reaction studies at BESIII. First results on singly-virtual TFFs of π^0 , η , and η' based on the complete data set are expected in 2025, providing information for virtualities from $Q^2 \geq 0.1 \text{ GeV}^2$ to 5 GeV^2 . A comparison with the results of BABAR and Belle at lower values Q^2 should be possible. The results will consider the full radiative corrections at NLO, as provided by the EKHARA 3.0 event generator [693] without applying the previously accepted approximations [694,695]. The size of the data set will also allow one to provide information on the doubly-virtual spacelike TFF of the lightest pseudoscalar mesons.

The data will also be used to provide information on the partial waves in $\gamma^*\gamma^{(*)} \rightarrow \pi^+\pi^-/\pi^0\pi^0$. Results with a single off-shell photon are reported to be provided for invariant masses from the two-pion threshold up to 2.0 GeV at $0.2 \text{ GeV}^2 \leq Q^2 \leq 3.0 \text{ GeV}^2$ and the full coverage of the pion helicity angle. Furthermore, a revised analysis scheme allows for doubly-virtual information in a limited region of the second virtuality [696].

The analysis of single-virtual production of higher meson multiplicities will allow to provide results on the TFFs of the axial-vector meson $f_1(1285)$ at virtualities in the region at $0.2 \text{ GeV}^2 \leq Q^2 \leq 3.0 \text{ GeV}^2$. Here, a partial-wave analysis is performed to separate the $a_0(980)\pi$ and $f_0\eta$ contributions in the $\pi^+\pi^-\eta$ final state. Further investigations of higher-multiplicity pion final states as well as final states involving kaons are being prepared to provide information on further axial and tensor states.

Complementary measurements at $Q_i^2 > 3 \text{ GeV}^2$ are to be expected from the Belle-II collaboration as a continuation of the successful activities at Belle [697].

In addition to the ongoing investigation Table 30 summarizes recommendations for intensified experimental activities to improve on HlBL.

5.11.3. Monte-Carlo event generators

The measurement of multi-particle final states produced in two-photon fusion reactions with virtual photons beyond the individual decay modes of η and η' mesons requires new event generators to evaluate detection efficiencies.

The EKHARA 3.0 generator [693] has been extended to provide a data-driven description of the resonance contributions in $\gamma^{(*)}\gamma^{(*)} \rightarrow \pi^+\pi^-/\pi^0\pi^0$ up to invariant pion pair masses of 2 GeV and virtualities $Q_{1,2}^2 \leq 4 \text{ GeV}^2$. The dominant resonances $f_0(500)$, $f_0(980)$, and $f_2(1270)$ are included based on Refs. [353,698].

The development of EKHARA will unfortunately not continue. To provide the MC simulations necessary in experimental investigations for efficiency studies and partial-wave analyses, a new event generator project has been started [699]. The focus is on multi-particle final states, which are relevant for measurements of axial-vector and tensor mesons, but the design allows for flexibility in the generated final states. The generator determines the cross section observable at

Table 30
Examples of useful experimental inputs related to the exclusive hadronic channels.

	Experimental input
Axial-vector TFFs	$e^+e^- \rightarrow e^+e^-A$, $A = f_1, f'_1, a_1$ Radiative decays $A \rightarrow V\gamma$, $V = \rho, \omega, \phi$ Dilepton decays $A \rightarrow e^+e^-$
Scalar and tensor TFFs	$\gamma^*\gamma^{(*)} \rightarrow \pi\pi, \pi\eta, \bar{K}K, \pi\pi\pi$
Pseudoscalar TFFs	$\gamma\gamma \rightarrow \eta, \eta'$ $e^+e^- \rightarrow e^+e^-(\gamma^*\gamma^{(*)} \rightarrow \pi^0, \eta, \eta')$ $e^+e^- \rightarrow e^+e^-(\gamma\gamma \rightarrow P)$, $P = \pi(1300), \eta(1295), \eta(1405)$

e^+e^- machines as the product of the luminosity function [638,700] and two-photon cross-sections. The latter depend on the polarization of the virtual photons and can be taken from theory calculations, e.g., for two-pion [353,355] or $f_1(1285) \rightarrow \pi^+\pi^-\eta$ [701] production in two-photon collisions, or can be modeled from existing experimental results. The ongoing measurements of $\gamma\gamma^* \rightarrow \pi^0\pi^0$ and $\gamma\gamma^* \rightarrow f_1(1285)$ at BESIII are the first applications of the new event generator.

6. Lattice approaches to HLbL

X. Feng, A. Gérardin, L. Jin, T. Lin, H. Meyer

6.1. Introduction

The idea to compute the HLbL contribution to the muon ($g - 2$) in lattice QCD was pioneered in Ref. [702]. The relevant diagram is illustrated on the RHS of Fig. 62. A first peer-reviewed publication on the subject based on these methods appeared in 2014 [703]. Substantial improvements to the methodology [704] led to a lattice calculation of a_μ^{HLbL} at physical pion mass [705]. An important feature of this methodology is the calculation of lepton and photon propagators on the same (periodic) lattice as the QCD degrees of freedom. By the time of WP20, one complete lattice calculation, by the RBC/UKQCD collaboration [59], had been published; it entered the WP20 average. Starting in 2015, in a series of conference proceedings [463,706–709] and papers [710], the Mainz/CLS group developed a method treating the QED parts of the diagram in continuum and infinite volume coordinate-space perturbation theory. This type of method, sometimes referred to as QED_∞ , was also investigated by the RBC/UKQCD collaboration starting with Ref. [711], providing some important refinements. Since WP20, three collaborations (Mainz/CLS, RBC/UKQCD, and BMW) have produced lattice results for a_μ^{HLbL} based on QED_∞ . Their methodology and results are reviewed in Section 6.2, which concludes with a lattice average for a_μ^{HLbL} .

A separate line of approach to HLbL scattering in the muon ($g - 2$) is to provide the relevant hadronic input to compute individual contributions as defined in a given dispersive framework. The most important case is that of the pseudoscalar meson exchanges. Starting with Ref. [712], lattice calculations have aimed at providing the $F_{P\gamma^*\gamma^*}(q_1^2, q_2^2)$ TFFs ($P = \pi^0, \eta, \eta'$), or else directly evaluating the relevant integral in coordinate space [713]. By now, several evaluations have appeared for the π^0 pole contribution and first corresponding calculations for the η are available as well. All these are reviewed in Section 6.3.

6.2. Direct lattice calculations of the hadronic light-by-light contribution

The starting point for computing a_μ^{HLbL} in lattice QCD using Euclidean position-space techniques can be written as [463]

$$a_\mu^{\text{HLbL}} = \frac{m_\mu e^6}{3} \int d^4x d^4y \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p, x, y) i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x, y), \quad (6.1)$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x, y) = - \int d^4z z_\rho \left\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \right\rangle_{\text{QCD}}. \quad (6.2)$$

The nonperturbative physics of the strong interaction enters via the QCD four-point function Eq. (6.2) of the EM current carried by the quarks, $j_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \dots$. The QED kernel

$$\begin{aligned} \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p, x, y) &= \frac{1}{16m_\mu^2} \int d^4u d^4v d^4w G(w-x) G(u-y) G(v) e^{-ip \cdot (w-v)} \\ &\quad \times \text{Tr} \left\{ [\gamma_\rho, \gamma_\sigma] (-i\not{p} + m_\mu) \gamma_\mu S(w-u) \gamma_\nu S(u-v) \gamma_\lambda (-i\not{p} + m_\mu) \right\} \end{aligned} \quad (6.3)$$

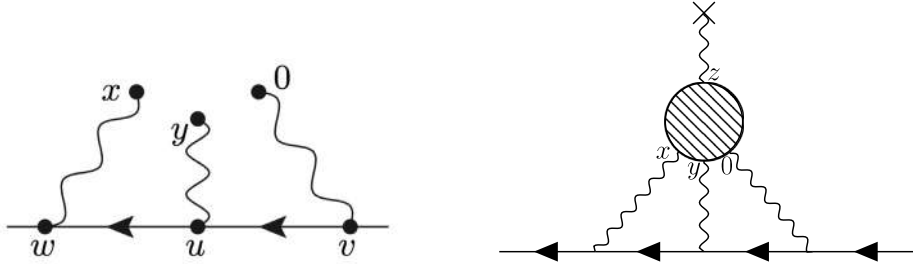


Fig. 62. Left: QED part of the HLbL Feynman diagram, with the muon in- and outgoing with a definite momentum p . The position-space QED kernel $\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p, x, y)$ results from integrating over the internal vertices w, u, v . Note that the represented amplitude is translationally invariant, hence the origin 0 can be chosen to coincide with one of the photon end-points without loss of generality. Figure taken from Ref. [715]. Right: the full amplitude yielding a_μ^{HLbL} , with the blob corresponding to the HLbL scattering amplitude. Figure taken from Ref. [710].

represents the photon and muon elements of the HLbL diagram, which are depicted in the left diagram of Fig. 62. In Eq. (6.3), $G(x) = 1/(4\pi^2 x^2)$ is the massless scalar propagator,

$$S(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{-i\not{p} + m_\mu}{p^2 + m_\mu^2} e^{ip \cdot x} \quad (6.4)$$

is the muon propagator, and the notation is fully Euclidean, in particular for the space-time scalar products, and the (Hermitian) Dirac matrices obey $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The kernel is dimensionless as well as infrared finite thanks to the trace of Eq. (6.3), which projects onto the anomalous magnetic moment, and obeys the properties

$$\begin{aligned} \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(-p, -x, -y) &= -\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p, x, y), \\ \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(-p^*, x, y) &= \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p, x, y)^*, \\ \mathcal{L}_{[\rho,\sigma];\lambda\nu\mu}(p, x, x-y) &= -\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(-p^*, x, y)^*. \end{aligned} \quad (6.5)$$

While in earlier approaches the QED kernel used to be “immersed” in the lattice formulation of a_μ^{HLbL} , recent calculations have worked with the kernel in the continuum and infinite-volume theory. A reliable calculation of Eq. (6.3), or its momentum-averaged version $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}$ defined below in Eq. (6.6) is a demanding task in itself [706,710,711]. A code with auxiliary look-up files providing $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}$ is publicly available [710].⁴⁰

As for the muon momentum, one option, adopted by the RBC/UKQCD collaboration [62,711], is to choose the rest frame of the muon, *de facto* averaging over the Euclidean momenta $p = (\pm im_\mu, \mathbf{0})$; see Section 6.2.2 for more details. The option followed by all other collaborations [60,63,714] so far is to employ a full average of the QED kernel over the direction of the muon momentum, thus making the following substitution [463] in Eq. (6.1),

$$\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p, x, y) \rightarrow \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y) \equiv \frac{1}{2\pi^2} \int d\Omega_{\hat{e}} \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p = im_\mu \hat{e}, x, y). \quad (6.6)$$

Here $\hat{e}$ is understood to be a real, four-component unit vector, and $d\Omega_{\hat{e}}$ the corresponding solid-angle differential. An advantage of using $\bar{\mathcal{L}}$ is that, after contraction of the Lorentz indices and integration (say) over x in Eq. (6.1), the remaining integral over y reduces to a one-dimensional integral over $|y|$ that can be performed reliably with $\mathcal{O}(20)$ points.

Irrespective of the choice made for handling the muon momentum, a significant amount of flexibility remains to optimize the QED kernel for numerical purposes. First, there is a freedom to add total-divergence terms – sufficiently well-behaved not to generate boundary terms upon partial integration – to the kernel, due to the conservation of the vector currents [709,711]. A proposal that has been adopted by several collaborations [60,63,714] is [709]

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}^{(\Lambda)}(x, y) = \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y) - \partial_\mu^{(x)}(x_\alpha e^{-\Lambda m_\mu^2 x^2/2}) \bar{\mathcal{L}}_{[\rho,\sigma];\alpha\nu\lambda}(0, y) - \partial_\nu^{(y)}(y_\alpha e^{-\Lambda m_\mu^2 y^2/2}) \bar{\mathcal{L}}_{[\rho,\sigma];\mu\alpha\lambda}(x, 0), \quad (6.7)$$

where the value 0.4 was chosen for the free parameter Λ . The rationale behind this choice is to make the integrand neither too long-range, nor too short-distance dominated, as this leads to smaller overall uncertainties in the lattice calculation. Second, there is the option of symmetrizing the kernel with respect to the Bose symmetries of $\hat{\Pi}$, for instance under the exchange of the pairs $(x, \mu) \leftrightarrow (y, \nu)$. Thirdly, for each individual Wick contraction contributing to $\hat{\Pi}$, translation invariance can be exploited [704,705,716] so as to minimize the overall computational cost.

The notation used in Eq. (6.1) is the one introduced by the Mainz/CLS group. The connection to the notation used by the RBC/UKQCD collaboration [62], provided here for convenience, is

$$\frac{i}{4} \text{Tr}[\{\gamma_\rho, \gamma_\sigma\} \mathfrak{G}_{\mu\nu\lambda}^{\text{RBC}}(x, y, z)] = \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p = (im_\mu, \mathbf{0}), x - z, y - z),$$

⁴⁰ <https://github.com/RJHudspith/KQED>

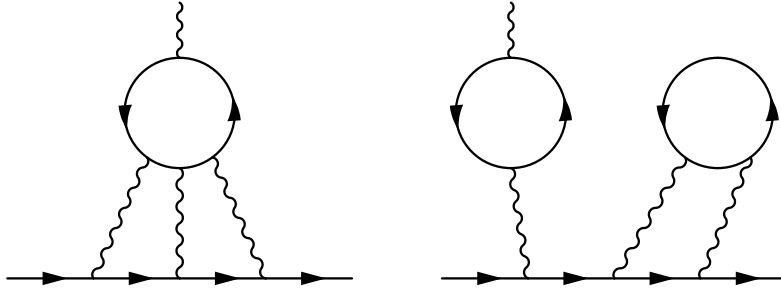


Fig. 63. Illustration of the connected and (2+2) disconnected Wick-contraction diagrams for the quarks. The latter are not $SU(3)_F$ suppressed and turn out to be the dominant class of disconnected diagrams.

Source: Figures taken from Ref. [62].

$$-6 \int d^4 x_{\text{op}} (x_{\text{op}} - x_{\text{ref}})_j \mathcal{H}_{k,\rho,\sigma,\lambda}^{\text{RBC}}(x_{\text{op}}, x, y, z) = i \widehat{\Pi}_{j;\rho\sigma\lambda k}(x - z, y - z). \quad (6.8)$$

A further redundancy in the master formula Eq. (6.1) for a_μ^{HLbL} exploited in Ref. [62] is that in the definition of $i\widehat{\Pi}$, Eq. (6.2), the factor z_ρ could be replaced by⁴¹ $(z_\rho - z_\rho^{\text{ref}})$ —as made explicit in Eq. (6.8), where in RBC/UKQCD notation the external vertex is written x_{op} instead of z .

The flavor structure of $i\widehat{\Pi}$ plays an important role. In terms of the isospin decomposition $j_\mu = j_\mu^3 + j_\mu^8/\sqrt{3}$ of the EM current in the (u, d, s) sector, $i\widehat{\Pi}$ can be decomposed into three terms, involving either zero, two, or four isovector currents j_μ^3 , which lead respectively to the contributions $a_\mu^{\text{HLbL}[j^8]}$, $a_\mu^{\text{HLbL}[j^3, j^8]}$, and $a_\mu^{\text{HLbL}[j^3]}$. On the other hand, for a given quark current there are five classes of Wick contractions, the connected (4) as well as the (2+2), the (3+1), the (2+1+1), and the (1+1+1+1) disconnected diagrams. For instance, we denote the (2+2) contribution to a_μ^{HLbL} from the u, d quarks $a_\mu^{\text{HLbL},(2\ell+2\ell)}$. In particular, the contribution involving four isovector currents j_μ^3 corresponds to a linear combination of the two leading quark-level contractions, $a_\mu^{\text{HLbL},(4)\ell}$ and $a_\mu^{\text{HLbL},(2+2)\ell}$. We write

$$a_\mu^{\text{HLbL}} = a_\mu^{\text{HLbL},\ell} + a_\mu^{\text{HLbL},s} + a_\mu^{\text{HLbL},c} + a_\mu^{\text{HLbL},\text{rest}}, \quad (6.9)$$

where

$$\begin{aligned} a_\mu^{\text{HLbL},\ell} &\equiv a_\mu^{\text{HLbL},(4\ell)} + a_\mu^{\text{HLbL},(2\ell+2\ell)}, \\ a_\mu^{\text{HLbL},s} &\equiv a_\mu^{\text{HLbL},(4s)} + a_\mu^{\text{HLbL},(2s+2\ell)} + a_\mu^{\text{HLbL},(2s+2s)}, \\ a_\mu^{\text{HLbL},c} &\equiv a_\mu^{\text{HLbL},(4c)} + a_\mu^{\text{HLbL},(2c+2\ell)} + a_\mu^{\text{HLbL},(2c+2c)}, \\ a_\mu^{\text{HLbL},\text{rest}} &\equiv a_\mu^{\text{HLbL},(3\ell+1\ell s)} + a_\mu^{\text{HLbL},(2\ell+1\ell s+1\ell s)} + a_\mu^{\text{HLbL},(1\ell s+1\ell s+1\ell s+1\ell s)} + \dots, \end{aligned} \quad (6.10)$$

where we will neglect the contributions represented by the ellipsis. The notation $1\ell s$ refers to the difference of a light- and a strange-quark loop with a single, vector-current insertion. We begin by reviewing the calculations by three different collaborations of the sum $a_\mu^{\text{HLbL},\ell}$ of the connected and leading disconnected light-quark contributions illustrated in Fig. 63. Subsequently, we treat the strange, as well as the charm-quark contribution, and finally we present the status of the subleading contributions in the (u, d, s) quark sector. The term $a_\mu^{\text{HLbL},\ell}$ by far dominates the total a_μ^{HLbL} , and the other contributions are found to be smaller than the current uncertainty of $a_\mu^{\text{HLbL},\ell}$.

6.2.1. $a_\mu^{\text{HLbL},\ell}$: the Mainz/CLS calculation

A calculation of a_μ^{HLbL} by the Mainz group [60] was published in 2021. In this calculation, the kernel employed is $\tilde{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}^{(\Lambda=0.4)}(x, y)$ as given in Eq. (6.7). The lattice formulation consists of a nonperturbatively $\mathcal{O}(a)$ -improved Wilson fermion action with an $\mathcal{O}(a^2)$ tree-level improved Lüscher–Weisz gauge action, and the calculations are performed on ensembles generated as part of the Coordinated Lattice Simulations (CLS) initiative [717]. The Mainz group follows the strategy of performing the x and z integrals as explicit four-dimensional sums over the lattice, whereas the final integral over y is performed in spherical coordinates, by sampling $\mathcal{O}(20)$ values of $|y|$, see Fig. 64.

A pilot study at $M_\pi = M_K \simeq 415 \text{ MeV}$. The Mainz group initially performed a pilot study [716] at the $SU(3)_F$ -symmetric point $M_\pi = M_K \simeq 415 \text{ MeV}$. The enlarged flavor symmetry leads to a simplification, namely only the fully-connected and the $(2 + 2)$ topologies contribute. For the connected contribution, two methods were explored: the first based on

⁴¹ A more general class of subtractions to the integrand $j_\sigma(z)z_\rho$ is $j_\sigma(z)\partial_\alpha^{(z)}f_{\rho\sigma}(x, y, z)$, where the special case is recovered for $f_{\rho\sigma}(x, y, z) \equiv z_\sigma z_\rho^{\text{ref}}(x, y)$.

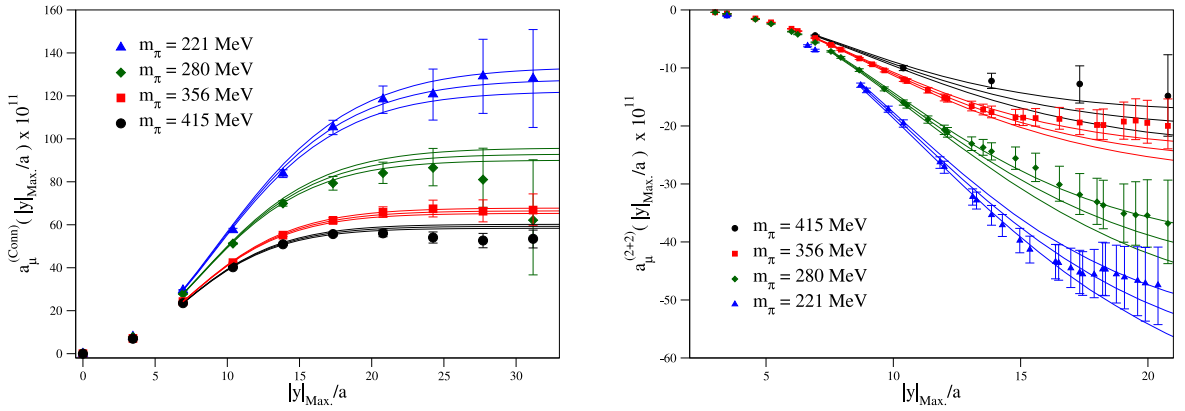


Fig. 64. Partially-integrated light-quark connected (left) and (2,2) disconnected contribution to a_μ^{HLbL} versus $|y|_{\text{Max.}}/a$ for four ensembles which have a broad range of pion masses but the same lattice spacing ($a = 0.086$ fm) and similar volumes ($4 \lesssim M_\pi L \lesssim 6$). The points are the numerically integrated lattice data and the curves result from switching the integrand to a fit above a certain distance.
Source: Figures taken from Ref. [60].

computing all three connected Wick-contractions of the HLbL amplitude called “Method 1”. This requires the calculation of a large number of (sequential) propagators and is therefore rather expensive. The second by contrast, “Method 2”, exploits the symmetries of the tensor $i\vec{T}$ to reduce the connected contribution to a_μ^{HLbL} to a single Wick contraction. This can then be computed with far fewer propagator calculations. “Method-2” was found to broaden the integrand somewhat, and the subtraction scheme of Eq. (6.7) was necessary to have it peak in a range suitable for the lattice. In the end, the two methods led to the same result for a_μ^{HLbL} within uncertainties. For the $(2+2)$ disconnected contribution, only one method was used: the contraction in which a quark loop connects the origin to point y was handled by interchanging the integration variables x and y in Eq. (6.1).

One challenge in the calculation is the control over long-distance effects. This includes finite-size effects on the final $|y|$ integrand, as well as the extension of this integrand to arbitrarily long distances, since the statistical noise is difficult to tame in this regime. Both issues were handled using a coordinate-space calculation of the π^0 exchange in finite volume, based on a VMD TFF with parameters determined from a direct lattice calculation of the TFF on the same ensemble [549]. A 25% uncertainty was assigned to the total long-distance correction applied to the data.

The other major challenge is to control cutoff effects. For the (positive) connected contribution at the $\text{SU}(3)_F$ -symmetric point, these were found to lead to a 15% (upward) correction between the lattice data at the coarsest lattice spacing and the continuum result. Discretization errors on the (negative) disconnected contribution turned out to have the same sign as for the connected. Thus, while a strong cancellation takes place when adding connected and disconnected contributions in the continuum, no such cancellation occurs for the corresponding discretization errors. At the $\text{SU}(3)_F$ -symmetric point [716], this means that the continuum result for a_μ^{HLbL} is a factor of about 1.6 larger than a_μ^{HLbL} at the coarsest lattice spacing used, $a = 0.086$ fm. The uncertainty of the continuum extrapolation is made larger by the expected presence of $\mathcal{O}(a)$, as opposed to $\mathcal{O}(a^2)$ discretization errors, due to contact terms in the vector four-point function and the use of unimproved vector currents. The relative size of the $\mathcal{O}(a)$ and the $\mathcal{O}(a^2)$ discretization errors is not known in the range of lattice spacings employed (0.049 to 0.086 fm) and is difficult to determine from the lattice data. Nevertheless, applying various fit ansätze reflecting this uncertainty, the result obtained in [716] was:

$$a_\mu^{\text{HLbL}} = 65.4(4.9)(6.6) \times 10^{-11} \quad \text{at } M_\pi = M_K \simeq 415 \text{ MeV}, \quad (6.11)$$

with the first error resulting from the uncertainties on the individual gauge ensembles, and the second from the systematic error of the continuum extrapolation.

Lattice result extrapolated to physical (u, d, s) quark masses. The calculation of Ref. [716] was extended away from the $\text{SU}(3)_F$ -symmetric point towards physical quark masses in Ref. [60]. The lattice ensembles used Ref. [717] keep the sum of the (u, d, s) quark masses constant, and the smallest pion mass reached was around 200 MeV. They found a strong increase in the value of the light-quark fully-connected contribution to a_μ^{HLbL} with decreasing pion mass. Similarly, the $(2+2)$ disconnected contribution was found to increase in magnitude in this limit. The absolute amount of cancellation between the two contributions thus strongly increases as the pion mass is lowered, and the authors concluded that adding the two contributions prior to the continuum extrapolation was beneficial. Furthermore, three more Wick-contraction topologies appear away from the $\text{SU}(3)_F$ -symmetric point. They have all been computed within Ref. [60] and are discussed in Section 6.2.7.

In Ref. [60], the tail of the integrand and the finite-size correction, though still based on the π^0 exchange, were handled differently: first, an ansatz describing the final integration in the variable $|y|$ faithfully for the π^0 exchange was fit to

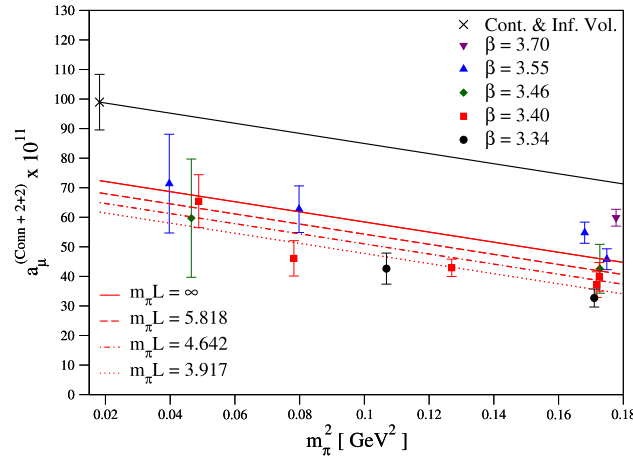


Fig. 65. Example of a chiral, continuum, and infinite-volume extrapolation of the sum of the light-quark, fully-connected, and (2 + 2) contributions to a_μ^{HLbL} . For this fit ansatz, the discretization errors are assumed to be linear in a^2 and M_π -independent, and the chiral dependence to be linear in M_π^2 . The red straight lines labeled by an $M_\pi L$ value correspond to the lattice spacing $a = 0.086$ fm. The infinite-volume, continuum behavior inferred from the fit is shown as the black, uppermost straight line.

Source: Figure taken from Ref. [60].

the lattice data in order to extend the integration up to $|y| = \infty$. Secondly, a finite-size correction term of the form $\exp(-M_\pi L/2)$ with a fit coefficient was included in the final extrapolation to the physical point. This alternative, more data-driven procedure was shown to yield results consistent with those of Ref. [716] at the $\text{SU}(3)_F$ symmetric point.

The final value of a_μ^{HLbL} at the physical point is the result of an extrapolation to infinite volume, zero lattice spacing, and physical pion mass, see Fig. 65. Various fit ansätze as well as cuts in the pion mass and lattice spacing were used to assess the systematic uncertainty. The final result reads

$$a_\mu^{\text{HLbL},\ell} = 107.4(15.8) \times 10^{-11}, \quad (6.12)$$

where the indicated total error of 15.8 consists of the statistical error (11.3), of the systematic error from varying the fit ansatz and data cuts (9.2), which is dominated by the difference in the results obtained from extrapolations linear in a and in a^2 ; and finally, of an error associated with the chiral extrapolation quantified as (6.0). The different components were then combined in quadrature. In Ref. [718], a modified strategy for the chiral extrapolation was investigated, namely by first subtracting the π^0 exchange contribution from a_μ^{HLbL} computed on each gauge ensemble, and adding back the π^0 contribution at the every end. This procedure leads to a very flat chiral extrapolation, with a result entirely consistent with Eq. (6.12).

6.2.2. $a_\mu^{\text{HLbL},\ell}$: the RBC/UKQCD calculation

By the time of WP20, the RBC/UKQCD collaboration had published a lattice calculation [59] with both QCD and QED in a finite-size box using the so-called QED_L [462] scheme. The finite volume error of HLbL with QED_L scales with $\mathcal{O}(1/L^2)$. Lattices with different sizes ranging from $L = 4.67$ fm up to $L = 9.33$ fm were used to extrapolate to the infinite-volume limit. Two different lattices $a = 0.114$ fm and $a = 0.084$ fm are used for extrapolating to the continuum limit. The results obtained are listed below:

$$\begin{aligned} a_\mu^{\text{HLbL},\ell} + a_\mu^{\text{HLbL},(2s+2\ell)} + a_\mu^{\text{HLbL},(2s+2s)} &= 78.7(30.6)_{\text{stat}}(17.7)_{\text{syst}} [35.4] \times 10^{-11}, \\ a_\mu^{\text{HLbL},(4\ell)} &= 241.6(23.0)_{\text{stat}}(51.1)_{\text{syst}} [56.0] \times 10^{-11}, \\ a_\mu^{\text{HLbL},(2\ell+2\ell)} + a_\mu^{\text{HLbL},(2s+2\ell)} + a_\mu^{\text{HLbL},(2s+2s)} &= -164.5(21.3)_{\text{stat}}(39.9)_{\text{syst}} [45.2] \times 10^{-11}. \end{aligned} \quad (6.13)$$

This publication does not quote $a_\mu^{\text{HLbL},\ell}$ or $a_\mu^{\text{HLbL},(2\ell+2\ell)}$ alone. For the purpose of comparison, we subtract the result $a_\mu^{\text{HLbL},(2s+2\ell)} + a_\mu^{\text{HLbL},(2s+2s)} = -3.6(2.2)_{\text{stat}}(0.3)_{\text{syst}} \times 10^{-11}$ calculated in RBC/UKQCD's new publication [62], and obtain the following results:

$$\begin{aligned} a_\mu^{\text{HLbL},\ell} &= 82.3(30.7)_{\text{stat}}(17.7)_{\text{syst}} [35.4] \times 10^{-11}, \\ a_\mu^{\text{HLbL},(2\ell+2\ell)} &= -160.9(21.4)_{\text{stat}}(39.9)_{\text{syst}} [45.3] \times 10^{-11}. \end{aligned} \quad (6.14)$$

The remaining part of this subsection is dedicated to the new RBC/UKQCD calculation [62] with QED_∞ . In this calculation, the muon momentum p is set to its rest frame value, $p^2 = -m_\mu^2$ with $\mathbf{p} = \mathbf{0}$. Note that the time direction is

identified with the axis of the hypercubic lattice that has the longest extent. A first observation is that due to the property $\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(-\mathbf{x}, -\mathbf{y}) = -\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(\mathbf{x}, \mathbf{y})$, the kernel $\mathcal{L}$ can be replaced by its odd part,

$$\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}^I(p, \mathbf{x}, \mathbf{y}) = \frac{1}{2} (\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p, \mathbf{x}, \mathbf{y}) - \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p, -\mathbf{x}, -\mathbf{y})) . \quad (6.15)$$

This modification, which by property Eq. (6.5) is equivalent to averaging the kernel over the two values $(\pm i\mathbf{m}_\mu, \mathbf{0})$ of the muon momentum, has the technical advantage that $\mathcal{L}^I$ computed using Eq. (6.3) is infrared finite prior to taking the Dirac trace. The kernel $\mathcal{L}^I$ is then fed into the linear combination

$$\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}^{II}(p, \mathbf{x}, \mathbf{y}) = \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}^I(p, \mathbf{x}, \mathbf{y}) - \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}^I(p, \mathbf{y}, \mathbf{y}) - \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}^I(p, \mathbf{x} - \mathbf{y}, \mathbf{0}) , \quad (6.16)$$

which has the property of vanishing for $\mathbf{y} = \mathbf{0}$ and for $\mathbf{x} = \mathbf{y}$. The second term in Eq. (6.16) does not contribute to a_μ^{HLbL} (in infinite volume) due to the property $\partial_\mu^{(x)} \widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(\mathbf{x}, \mathbf{y}) = 0$, and neither does the third term due to the conservation of the EM current at the origin, which implies $(\partial_\lambda^{(x)} + \partial_\lambda^{(y)}) \widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(\mathbf{x}, \mathbf{y}) = 0$. The main motivation for this subtraction is to reduce the discretization error. Compared with the scheme adopted by the Mainz group proposed in Ref. [709], this choice of the subtraction scheme introduced in Ref. [711] has more contributions from the long-distance region, and needs a dedicated study for this region as described below.

Subsequently, the kernel $\mathcal{L}^{II}$ is replaced by the version $\mathcal{L}^{II,\text{sym}}$, which is symmetrized under the permutation group of the three photon end-points $(\mathbf{x}, \mu)$, $(\mathbf{y}, \nu)$ and $(\mathbf{0}, \lambda)$. Exploiting the invariance of $\mathcal{L}^{II,\text{sym}}$ under permutation of the internal photon vertices, the calculation of the connected contribution is reduced (from three) to a single Wick contraction with quark flow in both directions. The selected Wick contraction is such that it would lead to an overall planar quark–photon–muon diagram if combined with (the unsymmetrized) $\mathcal{L}^{II}$. Similarly, using $\mathcal{L}^{II,\text{sym}}$ the three disconnected diagrams of topology (2+2) are traded for the single diagram in which the origin is connected by quark lines to the external vertex.

The subtraction vector $\mathbf{z}^{\text{ref}}$ (introduced below Eq. (6.8)) can depend on $\mathbf{x}, \mathbf{y}$. In the treatment of the (2+2)-type diagrams, $\mathbf{z}^{\text{ref}}$ is set to be located at the internal vertex on the quark loop that couples to the external photon. With the reduction of the Wick contractions due to the symmetrization in $\mathcal{L}^{II,\text{sym}}$, this choice of $\mathbf{z}^{\text{ref}}$ simply means $\mathbf{z}^{\text{ref}} = \mathbf{0}$. This choice of $\mathbf{z}^{\text{ref}}$ suppresses the (2+2)-type diagram contribution when the external photon vertex is close to the internal vertex belonging to the same quark loop, whereby one expects milder cutoff effects. The choice of $\mathbf{z}^{\text{ref}}$ is legitimate due to the classical current conservation condition being satisfied at every vertex in each individual (2+2)-type diagram.

For the connected diagrams, the same choice of $\mathbf{z}^{\text{ref}}$ as the (2+2)-type diagrams does not apply, since all vertices belong to the same quark loop. Therefore, a slightly different choice is made which (a) maintains the symmetry among the three internal vertices and (b) mimics the choice for the (2+2)-type diagrams in the long-distance region: $\mathbf{z}^{\text{ref}}$ is set to coincide with the internal vertex that is opposite to the shortest side of the triangle formed by the internal vertices $(\mathbf{0}, \mathbf{x}, \mathbf{y})$. Indeed, in the long-distance region, the π^0 -exchange contribution dominates, and the two vertices coupled to one end of the π^0 propagator are relatively close to each other, but far away from the other two vertices, which are coupled to the other end of the π^0 propagator. Furthermore, in the (2+2)-type diagram case, the two vertices on the same quark loop are coupled to the same end of the π^0 propagator. Therefore the integrand of the (2+2)-type diagram in which the external vertex and the vertex at $\mathbf{z}^{\text{ref}}$ are in the same quark loop is dominated precisely by the constellation where $\mathbf{z}^{\text{ref}}$ is far away from the other two internal vertices.

The position-space integral as a function of its upper boundary is illustrated in Fig. 66 for the connected and (2+2) disconnected diagrams, where R_{max} represents the longest side of the triangle formed by the internal vertices. Reading off the values from the left plot in Fig. 66 for contributions within $R_{\text{max}} < 4$ fm:

$$\begin{aligned} a_\mu^{\text{HLbL},\ell}(R_{\text{max}} < 4 \text{ fm}) &= 111.1(21.1)_{\text{stat}} \times 10^{-11} , \\ a_\mu^{\text{HLbL},(4\ell)}(R_{\text{max}} < 4 \text{ fm}) &= 186.1(12.2)_{\text{stat}} \times 10^{-11} , \\ a_\mu^{\text{HLbL},(2\ell+2\ell)}(R_{\text{max}} < 4 \text{ fm}) &= -74.9(18.2)_{\text{stat}} \times 10^{-11} . \end{aligned} \quad (6.17)$$

These results are obtained with the 481 MDWF ensemble from RBC/UKQCD collaborations. The lattice spacing is 0.114 fm. Due to the subtraction of the QED kernel described in Eq. (6.16), the discretization error is expected to be smaller than the 481 results obtained from the previous QED_L calculation. Due to the lack of light-quark data from a second lattice spacing, the discretization error of the above results were estimated based on the continuum extrapolation for the strange-quark-connected contribution, in which case the continuum limit is 8% larger than the 481 result. Therefore, RBC/UKQCD estimate 8% discretization systematic error for the total light quark contribution $(8.3)_{\text{syst}} \times 10^{-11}$.

The observed cancellation between connected and disconnected contributions motivates the partitioning of the dominant HLbL contribution into a term proportional to the contribution from the purely isovector part $j_\mu^3 = (\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)/2$ of the photon coupling, $a_\mu^{\text{HLbL}[j^3]}$, and a remaining fraction of the light-connected contribution,

$$a_\mu^{\text{HLbL},\ell} = \underbrace{\frac{100}{81} a_\mu^{\text{HLbL}[j^3]}}_{=a_\mu^{\text{no pion}}} + \frac{9}{34} a_\mu^{\text{HLbL},\text{ud conn}} . \quad (6.18)$$

The integrand for the first term, denoted $a_\mu^{\text{no pion}}$ in Ref. [62], being free of the pion-pole contribution [719], is expected to be of shorter range. On the other hand, $a_\mu^{\text{HLbL}[j^3]}$ is expected to contain the entire charged-pion-loop contribution to

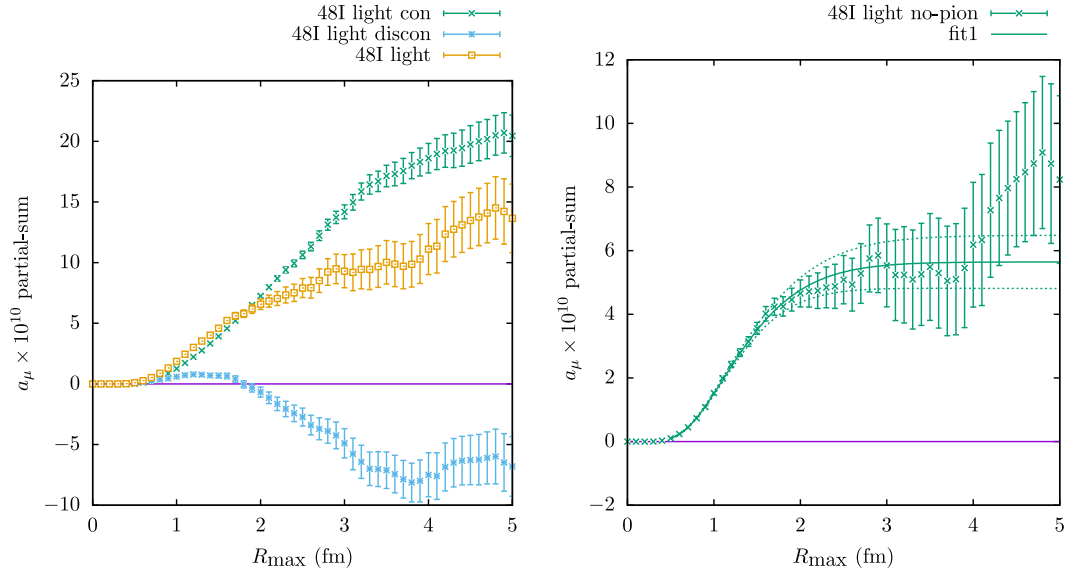


Fig. 66. Left: partially integrated light-quark contributions computed on a Möbius domain-wall fermion ensemble of size $48^3 \times 96$ with $M_\pi = 139$ MeV, $a = 0.114$ fm from the connected diagrams, the (2+2) disconnected diagrams, and the total. Right: linear combination of light-quark connected and (2+2) disconnected diagrams canceling the π^0 exchange contribution from long distances. The variable $R_{\max}$ is the longest side of the triangle formed by the internal vertices.

Source: Figures taken from Ref. [62].

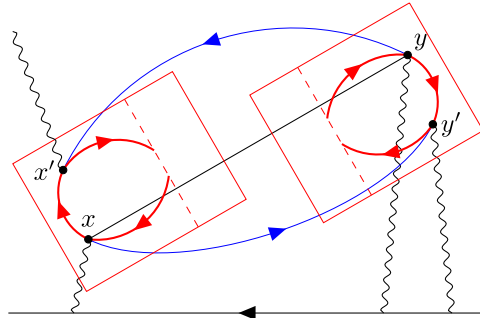


Fig. 67. The long-distance HLbL contribution to the muon $g - 2$ associated with π^0 exchange. The amplitudes inside the boxes are calculated in lattice QCD, while the pion propagator linking them (solid lines) is given by the analytic, infinite-volume, continuum expression.

Source: Figure taken from Ref. [62].

a_μ^{HLbL} , which is negative. Clearly, the positive result shown on the right panel of Fig. 66 indicates that other effects, such as the η and η' exchanges and shorter-distance contributions, dominate over the charged pion loop. To take advantage of the shorter-range property of $a_\mu^{\text{no pion}}$, this contribution is included directly from a lattice calculation up to $R_{\max} < 2.5$ fm, which gives $50.9(7.6)_{\text{stat}} \times 10^{-11}$. For the region where $R_{\max} > 2.5$ fm, a fit is performed

$$f(R_{\max}) = A/\text{fm}^4 \frac{R_{\max}^6}{R_{\max}^3 + (C \text{ fm})^3} e^{-BR_{\max}/(\text{fm} \times \text{GeV})}, \quad (6.19)$$

as illustrated in the right plot of Fig. 66. This tail contribution is $3.1(2.2)_{\text{stat}}(3.1)_{\text{syst}} \times 10^{-11}$. This is the only part of the entire calculation that is based on a fit. Many different fitting forms are studied and 100% systematic error is assigned to this contribution.

As mentioned above, the long-distance π^0 exchange contribution is calculated separately as illustrated in Fig. 67 based on the following long-distance approximation:

$$\langle T j_{\mu'}(x') j_{\mu}(x) j_{\nu'}(y') j_{\nu}(y) \rangle \simeq D_{\pi^0}(x-y) \mathcal{F}_{\mu',\mu} \left(x' - x, iM_\pi \frac{x-y}{|x-y|} \right) \mathcal{F}_{\nu',\nu} \left(y' - y, iM_\pi \frac{y-x}{|y-x|} \right), \quad (6.20)$$

where the π^0 TFF $\mathcal{F}_{\mu',\mu}(x, iM_\pi \hat{n})$ is directly calculated on the lattice with

$$\mathcal{F}_{\mu,\nu}(x, iM_\pi \hat{t}) = \langle 0 | T j_\mu(x) j_\nu(0) | \pi^0(\mathbf{p} = \mathbf{0}) \rangle \quad (6.21)$$

and a proper Euclidean space–time rotation. Here, the long-distance contribution includes regions with $R_{\max} > 4$ fm. This contribution is calculated to be $20.0(1.1)_{\text{stat}}(2.8)_{\text{syst}} \times 10^{-11}$.

The finite-volume correction for the contribution in the region $R_{\max} < 4$ fm is calculated based on the coordinate space formulation of the π^0 -pole contribution in finite volume, with the π^0 transition form factor given by the LMD model. The correction is $-4.7(1.1)_{\text{syst}} \times 10^{-11}$. This model can also be used to calculate the above long-distance contribution and is consistent with the above lattice results based on the long-distance approximation.

The last correction is from a small mismatch of the physical pion mass 135 MeV and the pion mass used in the above lattice calculation, 139 MeV. The correction is based on a separate lattice calculation at $M_\pi = 340$ MeV. The correction is calculated to be $3.5(0.7)_{\text{stat}}(1.7)_{\text{syst}} \times 10^{-11}$.

The final value of $a_\mu^{\text{HLbL},\ell}$ after applying the corrections from finite volume, small mismatch of pion mass is

$$a_\mu^{\text{HLbL},\ell} = 122.0(10.1)_{\text{stat}}(9.5)_{\text{syst}} [13.8] \times 10^{-11}. \quad (6.22)$$

The leading source of systematic error is from the estimation of the discretization effects, which is estimated to be $(8.3)_{\text{syst}} \times 10^{-11}$. A future calculation at a finer lattice spacing can help reduce this systematic error. Results for the individual contributions are also obtained

$$\begin{aligned} a_\mu^{\text{HLbL},(4\ell)} &= 257.0(13.3)_{\text{stat}}(19.9)_{\text{syst}} [23.9] \times 10^{-11}, \\ a_\mu^{\text{HLbL},(2\ell+2\ell)} &= -135.0(13.6)_{\text{stat}}(12.1)_{\text{syst}} [18.2] \times 10^{-11}, \\ a_\mu^{\text{no pion}} &= \frac{25}{34} a_\mu^{\text{HLbL},(4\ell)} + a_\mu^{\text{HLbL},(2\ell+2\ell)} = 54.0(9.4)_{\text{stat}}(5.3)_{\text{syst}} [10.8] \times 10^{-11}. \end{aligned} \quad (6.23)$$

6.2.3. $a_\mu^{\text{HLbL},\ell}$: the BMW calculation

In Ref. [63], the staggered-fermion formulation of QCD is used with $N_f = 2 + 1 + 1$ dynamical quarks, whose action involves four steps of stout smearing. The pion and kaon masses are very close to their physical values. The light-quark contribution is computed on seven ensembles at three values of the lattice spacing, 0.132, 0.112 and 0.095 fm. The linear extent of three ensembles is just above $L = 6$ fm, one has $L = 4.2$ fm, and the other three, which do not enter the final continuum extrapolation, have $L \simeq 3.1$ fm. The conserved vector current, which does not receive any multiplicative renormalization, is used throughout.

The starting point for the integral representation chosen of $a_\mu^{\text{HLbL},\ell}$ is the kernel $\tilde{\mathcal{L}}^{(A)}$ defined in Eq. (6.7), for which the same numerical implementation [710] is used as in the Mainz/CLS calculation. In contrast to the latter, however, the kernel $\tilde{\mathcal{L}}^{(A)}$ is then entirely symmetrized with respect to the three internal vertices. The x and z integrals are performed as sums over the lattice, while the remaining y integral is handled as a one-dimensional integral over $|y|$ thanks to the use of spherical coordinates. The position vector y of the vertex is chosen to be an integer multiple of $(1,1,1)$ or $(3,1,1)$. In fact, this vertex is spread over the eight nearest neighbors of its nominal position in order to suppress oscillating lattice artifacts.

For the connected contribution $a_\mu^{\text{HLbL},(4\ell)}$, the calculation is reduced to the quark-contraction diagram in which the vertex at the origin is connected to the internal vertex at x and to the external vertex at z . Due to the use of translation invariance to perform this reduction to one diagram, effectively a subtraction term $z_\rho^{\text{ref}} = -x_\rho/3$ appears (see the comment below Eq. (6.8)). The chosen representation is validated by reproducing the known contribution of a lepton loop to a_μ^{LbL} on the lattice, for a lepton twice as heavy as the muon.

For the disconnected contribution $a_\mu^{\text{HLbL},(2\ell,2\ell)}$, there are again three Wick contractions, which can all be expressed in terms of the diagram in which 0 is connected (by two quark lines) with x , or in terms of the diagram where 0 is connected with z . Each of these two possibilities leads to a valid expression for $a_\mu^{\text{HLbL},(2\ell,2\ell)}$, and the final estimator is the average of these two expressions. This choice has the effect of giving the pseudoscalar-pole contribution to the connected diagram and the two disconnected diagrams precisely the same weight function, so that the π^0 contribution can be made to cancel at the *integrand* level in the linear combination $a_\mu^{\text{no pion}}$. The final $|y|$ integral as a function of its upper bound is displayed in the left panel of Fig. 68 for $a_\mu^{\text{HLbL},(4\ell)}$, $a_\mu^{\text{HLbL},(2\ell,2\ell)}$ and for their sum, $a_\mu^{\text{HLbL},\ell}$.

The pion-pole contribution is computed in position space and in finite volume in order to extend the $|y|$ integrand at long distances and to correct for finite-size effects. The switch from lattice data to the pion-pole prediction is done at $y_{\text{cut}} \simeq 2.3$ fm. It is also used to assess how soon the x integral can be truncated once $|y| > 1.5$ fm, a technique that is used to reduce statistical fluctuations. The parameterizations used for the pion TFF are the VMD and LMD ones. The latter is found to provide a satisfactory description of finite-size effects, while the prediction using the VMD form factor yields corrections that are 20 to 25% smaller. Overall, the corrections applied to the raw lattice data summed up to y_{cut} are on the order of 4 to 6.5×10^{-11} for $a_\mu^{\text{HLbL},\ell}$.

The continuum extrapolation of $a_\mu^{\text{HLbL},\ell}$ is shown in the right panel of Fig. 68. The alternative route, which consists in working with the linear combination Eq. (6.18) of the shorter-range $a_\mu^{\text{no pion}}$ and the connected part, gives consistent results. The final results are collected in the rightmost column of Table 31. No continuum-extrapolated result is quoted

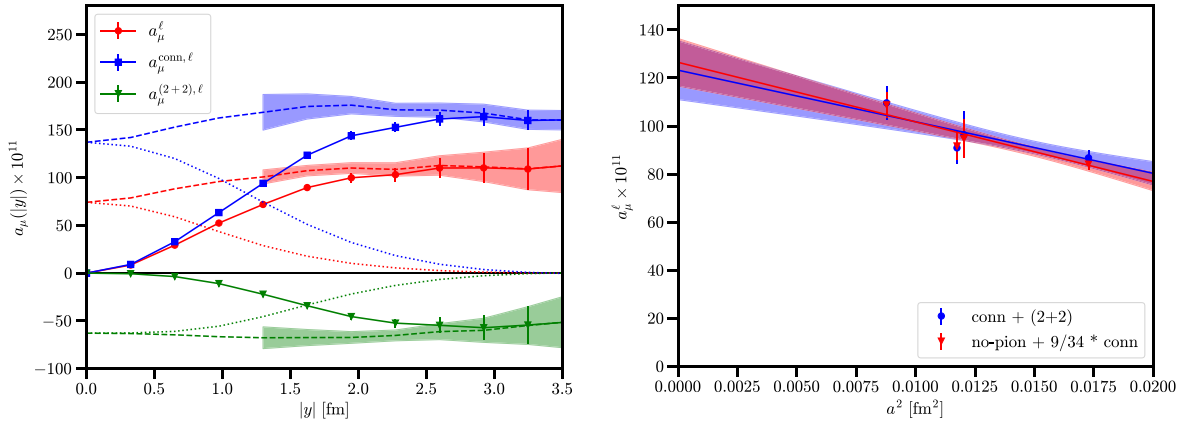


Fig. 68. Left: partially integrated light-quark contributions computed on a staggered-quark, physical-mass ensemble of size $64^3 \times 96$ with $a = 0.095$ fm from the connected diagrams, the (2+2) disconnected diagrams, and the total. Right: continuum extrapolation of $a_\mu^{\text{HLbL}, \ell}$, either based on the direct calculation of the sum of the connected and disconnected diagram, or based on combining the $a_\mu^{\text{no pion}}$ with the connected contribution. Source: Figures taken from Ref. [63].

Table 31

Results for the light-quark contributions in various linear combinations, as well as the total $a_\mu^{\text{HLbL}, \ell}$.

	[10 ⁻¹¹]	RBC/UKQCD QED _L [59]	
$a_\mu^{\text{HLbL}, (4\ell)}$	241.6(23.0) _{stat} (51.1) _{syst} [56.0]		
$a_\mu^{\text{HLbL}, (2\ell+2\ell)}$	−160.9(21.4) _{stat} (39.9) _{syst} [45.3]		
$a_\mu^{\text{HLbL}, \ell}$	82.3(30.7) _{stat} (17.7) _{syst} [35.4]		

	[10 ⁻¹¹]	Mainz/CLS [60]	RBC/UKQCD [62]	BMW [63]
$a_\mu^{\text{HLbL}, (4\ell)}$	see text		257.0(13.3) _{stat} (19.9) _{syst} [23.9]	220.1(13.0) _{stat} (3.8) _{syst}
$a_\mu^{\text{HLbL}, (2\ell+2\ell)}$	see text		−135.0(13.6) _{stat} (12.1) _{syst} [18.2]	−101.1(12.4) _{stat} (3.2) _{syst}
$a_\mu^{\text{no pion}}$	–		54.0(9.4) _{stat} (5.3) _{syst} [10.8]	see text
$a_\mu^{\text{HLbL}, \ell}$	107.4(11.3)(9.2)(6.0)[15.8]		122.0(10.1) _{stat} (9.5) _{syst} [13.8]	122.6(11.5) _{stat} (1.8) _{syst}

for $a_\mu^{\text{no pion}}$, however, the results at finite lattice spacing are in the range 49 to 59×10^{-11} , increasing toward the continuum. This contribution would end up being higher than the central value obtained by the RBC/UKQCD collaboration (penultimate column of Table 31), but still compatible given the uncertainties.

6.2.4. $a_\mu^{\text{HLbL}, \ell}$: our average

Before proceedings to average $a_\mu^{\text{HLbL}, \ell}$, we briefly compare the results for the connected and the disconnected contributions; see Table 31. The Mainz/CLS publication does not quote a final result for the two topologies separately, but provides the results of four different fit ansätze with regards to the chiral extrapolation. This results in a fairly large spread. Nevertheless, all three collaborations obtain central values in the range 200 to 260 for the connected, while the disconnected part is in the range −145 to −100. Clearly, the absolute size of systematic effects is larger when the connected and the disconnected contributions are handled separately, due to the enhanced weight with which the π^0 exchange contributes in these. For that reason, we refrain for now from presenting an average for $a_\mu^{\text{HLbL}, (4\ell)}$ and $a_\mu^{\text{HLbL}, (2\ell+2\ell)}$ separately. The decomposition into connected and $a_\mu^{\text{no pion}}$ is more favorable in that respect.

We now proceed to perform an average of the light-quark contribution $a_\mu^{\text{HLbL}, \ell}$. The four calculations were done with three qualitatively different quark actions on independent ensembles; therefore the statistical errors are fully uncorrelated (the two RBC/UKQCD calculations are performed with the same quark action but with two almost independent data sets). We adopt the approach of taking into account correlations among the systematic errors conservatively. In the Mainz/CLS case, the systematic uncertainty of 6.0×10^{-11} associated with the chiral extrapolation will be treated as being uncorrelated to the other two calculations, since the latter are performed directly at (near-)physical quark masses.

Now treating the systematic errors $\{17.7, 9.2, 9.5, 1.8\} \times 10^{-11}$ of the four calculations as being 100% correlated (that is the systematic errors from different results added linearly in the average), we construct the covariance matrix of the

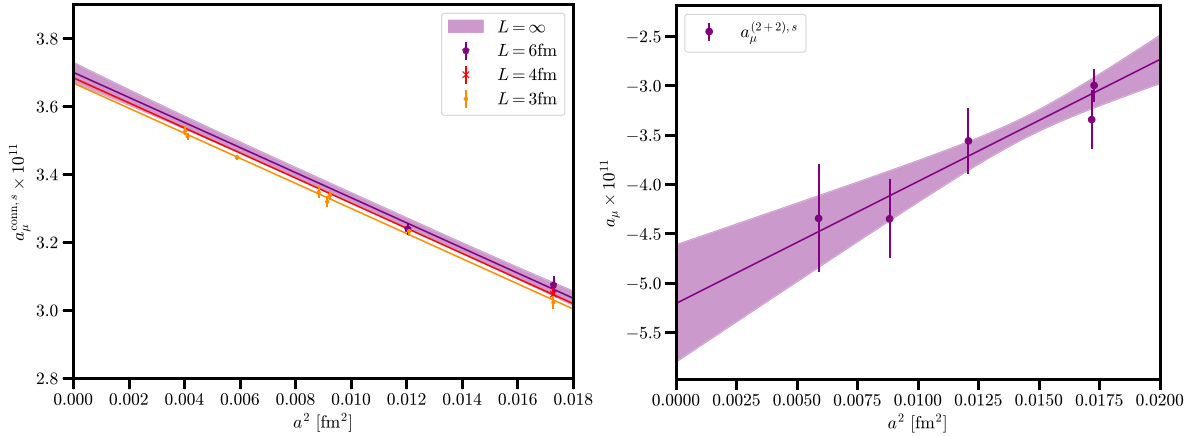


Fig. 69. Extrapolation of the strange connected (left, $a_\mu^{\text{HLbL},(4s)}$) and disconnected contribution (right, $a_\mu^{\text{HLbL},(2l+2s)} + a_\mu^{\text{HLbL},(2s+2s)}$) in the calculation by the BMW collaboration [63] based on ensembles with dynamical up/down, strange, and charm quarks tuned to their physical masses. Source: Figures taken from Ref. [63].

results and perform a fit to a constant.⁴² The χ^2/dof is 0.69 and the result is

$$a_\mu^{\text{HLbL},\ell} = 119.6(7.1)_{\text{stat}}(5.5)_{\text{syst}}[9.0]_{\text{tot}} \times 10^{-11} \quad (\text{our average}). \quad (6.24)$$

If we treated the systematic errors as being uncorrelated, the result would change to $117.1(6.5)_{\text{stat}}(4.0)_{\text{syst}}[7.6]_{\text{tot}} \times 10^{-11}$. We have checked that interpolating between these two results with a scale factor $\xi \in [0, 1]$ multiplying the off-diagonal components of the covariance matrix, the error of the average increases monotonously towards the fully correlated $\xi = 1$ case.

6.2.5. The strange contribution $a_\mu^{\text{HLbL},s}$

Three collaborations [60,62,63] have produced results for $a_\mu^{\text{HLbL},s}$. We begin by discussing the connected part alone. The strange connected contribution $a_\mu^{\text{HLbL},(4s)}$ is an excellent benchmark quantity to compare lattice calculations, since the signal is of good quality, not too long range, and finite-size effects are expected to be suppressed. We quote the results

$$a_\mu^{\text{HLbL},(4s)} = \begin{cases} 3.530(70)_{\text{stat}} \times 10^{-11} & \text{RBC/UKQCD} \\ 3.694(17)_{\text{stat}}(18)_a(8)_{\text{syst}} \times 10^{-11} & \text{BMW} \end{cases}. \quad (6.25)$$

Both calculations are performed directly at (practically) physical quark masses. The BMW calculation uses five lattice spacings in the range 0.064 to 0.132 fm, see Fig. 69, and the statistical precision was sufficiently high that finite-size effects could be resolved—and corrected for. The cutoff effect on the coarsest lattice spacing is about 16% of the continuum value. The RBC/UKQCD calculation uses two lattice spacings, 0.114 and 0.084 fm to obtain the continuum limit, relative to which the data on the coarser ensemble is 8% lower. Here we also mention the preliminary results of the ETM collaboration [714], that obtains continuum-extrapolated values somewhat lower than RBC/UKQCD or BMW, with an error still to be quantified.

The Mainz/CLS publication [60] does not quote a result for $a_\mu^{\text{HLbL},(4s)}$ alone. For the purpose of a comparison with the two results above, we produce an estimate from (a) the continuum-extrapolated result at the $M_\pi = M_K \simeq 415$ MeV point, Eq. (22) [716] with the charge factor adjust from 18/81 to 1/81 in order to isolate the strange contribution of $5.49(14) \times 10^{-11}$, and (b) by fitting the $a_\mu^{\text{HLbL},(4s)}$ values obtained on three ensembles at a fixed lattice spacing $a = 0.086$ fm and $M_K = 415, 461$ and 487 MeV in order to correct to the physical $M_K = 494.6$ MeV. Indeed, the data for $a_\mu^{\text{HLbL},(4\ell)}$ suggests that the cutoff effects on the valence-quark mass dependence is weak. We do so using a quadratic fit in M_K^2 and obtain $a_\mu^{\text{HLbL},(4s)} = 3.64(14)_{\text{SU}(3)_F}(9)_{M_K \text{ fit}} \times 10^{-11}$, where we have assigned a 5% uncertainty to the M_K dependence.

We conclude that the agreement between the three calculations of $a_\mu^{\text{HLbL},(4s)}$ is satisfactory. At the same time, there is evidence that the quoted errors are somewhat underestimated. A straightforward fit to a constant leads to $\chi^2/\text{dof} = 1.8$;

⁴² In the weighted average we always add linearly the absolute value of the weighted systematic errors. The fit is performed to find the weights w_i (the weights should satisfy $\sum_i w_i = 1$) that minimize the total error of the weighted average. The formula for the final error is $\sqrt{\sum_i w_i^2 e_{\text{stat},i}^2 + (\sum_i |w_i e_{\text{syst},i}|)^2 + (\sum_i |w_i e_{\text{chiral-extrap},i}|)^2}$, where $e_{\text{stat},i}$ and $e_{\text{syst},i}$ are the statistical and systematic errors for each lattice calculations, $e_{\text{chiral-extrap},i}$ are the errors for the chiral extrapolations, which are equal to zero for calculations performed at physical quark masses. Note that, with the above minimization procedure, the final weights should always be nonnegative.

the error inflated by the square-root of this quantity would be 0.034×10^{-11} . For simplicity, we then quote

$$a_{\mu}^{\text{HLbL},(4s)} = 3.68(4) \times 10^{-11} \quad (\text{our average}). \quad (6.26)$$

After this important consistency check, we quote the following results for the full contribution $a_{\mu}^{\text{HLbL},s}$,

$$a_{\mu}^{\text{HLbL},s} = \begin{cases} -0.6(2.0)_{\text{stat}} \times 10^{-11} & \text{Mainz/CLS} \\ -0.0(2.2)_{\text{stat}}(0.3)_{\text{syst}} \times 10^{-11} & \text{RBC/UKQCD} \\ -1.7(8)_{\text{stat}}(3)_{\text{syst}} \times 10^{-11} & \text{BMW} \end{cases}. \quad (6.27)$$

Due to, in part, the much larger statistical errors of the disconnected diagrams, these results are entirely consistent with each other. Performing a straightforward average, we again slightly inflate the error and obtain

$$a_{\mu}^{\text{HLbL},s} = -1.4(8) \times 10^{-11} \quad (\text{our average}). \quad (6.28)$$

6.2.6. The charm contribution $a_{\mu}^{\text{HLbL},c}$

The Mainz/CLS and the BMW collaboration have performed dedicated lattice calculations of the charm-quark contribution $a_{\mu}^{\text{HLbL},c}$ [61,63], see Fig. 70, and preliminary results by the ETM collaboration are available [714]. Due to the large mass scale involved, the main challenge in this case is to control the cutoff effects.

In the Mainz/CLS calculation, based on quarks with an $\mathcal{O}(a)$ -improved Wilson action, the adopted strategy consisted in using very fine lattice spacings (down to 0.039 fm) and lighter-than-physical charm-quark masses. These calculations were followed by a simultaneous extrapolation in both variables to the continuum and physical charm quark mass, respectively, to obtain the final result. To make this strategy computationally affordable, the calculation was performed on ensembles with dynamical (u, d, s) quarks at the $\text{SU}(3)_F$ -symmetric point with $M_{\pi} = M_K \simeq 415$ MeV. Indeed, neither the charm-quark loop, nor the η_c exchange is expected to depend strongly on the light-quark masses.

An important question is then what fit ansätze to employ in order to describe the charm mass and lattice spacing dependence of $a_{\mu}^{\text{HLbL},c}$. Being a heavy degree of freedom, general arguments [720,721] strongly suggests a leading $1/m_c^2$ dependence. As for the lattice-spacing dependence, both $\mathcal{O}(a)$ and $\mathcal{O}(a^2)$ effects are present in the Mainz setup. The (dominant) connected charm contribution and its uncertainty were estimated so as to cover the continuum results emerging from all fits with good χ^2/dof .

The correlation between a light-quark and a charm-quark two-vector-insertion loop (2+2 Wick-contraction topology), expected to be the next most important contribution, was computed as well and found to amount to a -10% correction to the connected piece. The corresponding (2+2) diagram involving two charm loops was found to be negligible. Altogether, the final result of Ref. [61] was

$$a_{\mu}^{\text{HLbL},c} = 2.8(5) \times 10^{-11} \quad (\text{Mainz/CLS}), \quad (6.29)$$

where the uncertainty is entirely systematics dominated. It also contains an estimate of the effect of not performing the calculation at physical (u, d, s) quark masses.

In the BMW calculation [63], which is performed directly at physical quark masses in the staggered-quark formulation, the continuum extrapolation also represented a major source of uncertainty. Additively implemented tree-level improvement of the observable $a_{\mu}^{\text{HLbL},c}$ was applied to data obtained from six lattice spacings in the range 0.048 to 0.132 fm. Two directions were used for the vertex y integrated over last, $(1, 1, 1, 1)$ and $(0, 1, 1, 1)$, leading to separate continuum extrapolations. The difference in results was taken as an additional systematic error. Additional checks were performed, by using lighter-than-physical charm quark masses as well as by reproducing the result for a lepton loop in the continuum. As a result of the spread obtained in the latter exercise, a sizable uncertainty (0.25×10^{-11}) was associated with the numerical implementation of the kernel.

The leading disconnected diagrams involving a charm-quark loop, namely those of topology (2+2), were computed at four values of the lattice spacing, extrapolated to the continuum and found to be negative. The final integrand is more extended than for the connected diagram, reaching its maximum size around $|y| = 0.6$ fm and is found to yield a contribution on the order of -0.2×10^{-11} after continuum extrapolation. All in all, the total charm contribution was found to be

$$a_{\mu}^{\text{HLbL},c} = 3.73(26) \times 10^{-11} \quad (\text{BMW}), \quad (6.30)$$

where the error is entirely dominated by the systematic uncertainty associated with the short-distance part of the kernel, see the previous paragraph, and the continuum extrapolation of the connected part.

The ETM collaboration has performed preliminary calculations of the connected charm contribution [714] at physical quark masses with twisted-mass Wilson sea quarks, using three lattice spacings in the range 0.057 to 0.080 fm. While an upward trend is observed in the $a_{\mu}^{\text{HLbL},c}$ results as the continuum is approached, pointing to a similar order of magnitude as found by Mainz/CLS and BMW, it is too early to derive a quantitative result.

We thus proceed to average the Mainz/CLS and BMW results. The BMW result Eq. (6.30) exhibits a mild 1.7σ tension⁴³ with the Mainz result Eq. (6.29). Therefore, after fitting the results of the two collaborations to a constant, we need to

⁴³ Recall that the statistically precise strange connected contribution agrees well between the two collaborations.

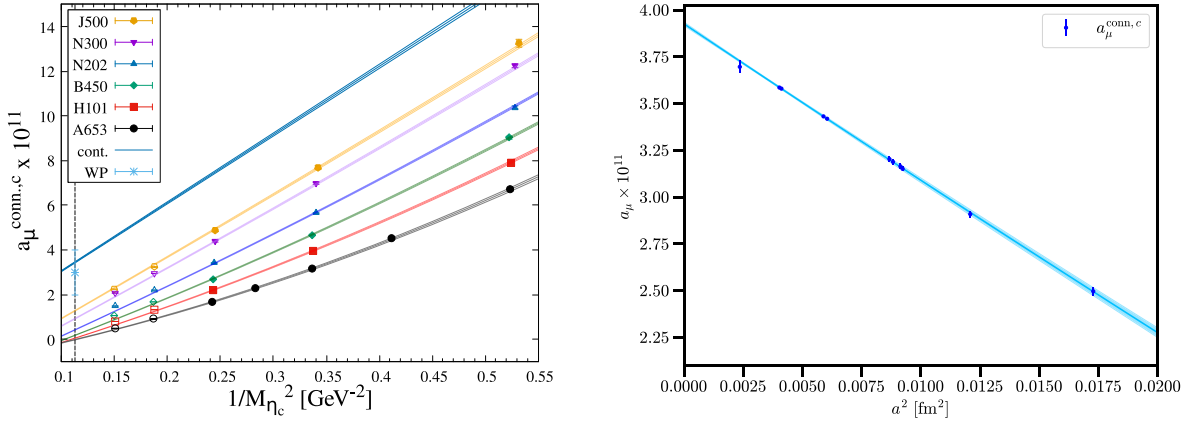


Fig. 70. Extrapolations of the connected charm contribution. Left: One representative fit to the lattice data of the Mainz/CLS collaboration [61] at different charm-quark masses and lattice spacings. The η_c meson mass is used as a proxy for the charm mass. The continuum limit of the lattice data is represented by the blue, uppermost curve. For comparison, the WP20 estimate of $a_\mu^{\text{HLbL},c} = 3(1) \times 10^{-11}$ is displayed at the physical η_c mass. Figure taken from Ref. [61]. Right: Representative continuum extrapolation at physical charm quark mass by the BMW collaboration [63]. Figure taken from Ref. [63].

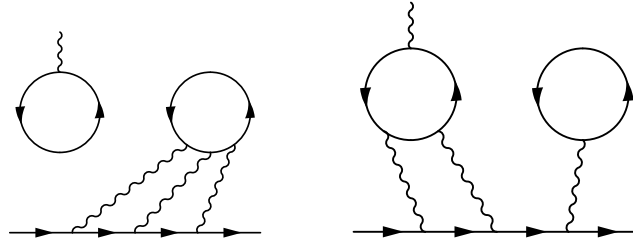


Fig. 71. The two quark-contraction diagrams of the (3+1) topology class, up to permutations of the vertices on the muon propagator. Source: Figures taken from Ref. [62].

inflate the error. The scale factor obtained from χ^2/dof would lead to an error of 0.4×10^{-11} . We make a slightly more conservative choice for the latter and thereby obtain for the charm light-by-light contribution

$$a_\mu^{\text{HLbL},c} = 3.5(5) \times 10^{-11} \quad (\text{our average}). \quad (6.31)$$

6.2.7. Subleading disconnected diagrams, $a_\mu^{\text{HLbL},\text{rest}}$

For those topologies (X+1) containing one or more quark loops consisting of a single vector insertion, $L_\alpha^f(x) = -\text{Tr}[\gamma_\alpha S_f(x, x)]$, where $S_f(y, x)$ denotes the quark propagator of flavor f , it is helpful to include all three flavors, $\sum_{f=u,d,s} Q_f L_\alpha^f(x)$, exploiting the fact that $\sum_{f=u,d,s} Q_f = 0$. The latter property implies the vanishing of these diagrams at the $\text{SU}(3)_F$ -symmetric point, $m_l = m_s$ and therefore a suppression at short distances.

The topology (3+1), see Fig. 71, consists of two quark loops and is therefore of the same order in $1/N_c$ counting as the (2+2) topology. Also, the pion loop contributes to this topology with a significant weight. Therefore, a direct lattice calculation is called for. The Mainz/CLS calculation could not resolve a signal and yielded $a_\mu^{\text{HLbL},3+1} = 0.0(6) \times 10^{-11}$. The more recent BMW calculation [63] did obtain a positive signal,

$$a_\mu^{\text{HLbL},3+1} = 0.82(18)_{\text{stat}}(17)_{\text{syst}} \times 10^{-11} \quad (\text{BMW}). \quad (6.32)$$

Since the Mainz/CLS result only amounts to an upper bound on the magnitude of this contribution, we will adopt the result Eq. (6.32) for our average.

Two collaborations [61,63] have found the (2+1+1) and the (1+1+1+1) contributions (see Fig. 72) to be entirely negligible, hence we will not go into any details. No statistically significant signal is obtained for these contributions. However, the lattice data allows one to bound them. The Mainz/CLS calculation arrives at

$$\begin{aligned} a_\mu^{\text{HLbL},(2+1+1)} &= 0.0(3) \times 10^{-11}, \\ a_\mu^{\text{HLbL},(1+1+1+1)} &= 0.0(1) \times 10^{-11}, \end{aligned} \quad (6.33)$$

while the BMW collaboration quotes even tighter bounds, 0.04×10^{-11} and 0.0005×10^{-11} , respectively. Therefore, we will neglect these contributions altogether.

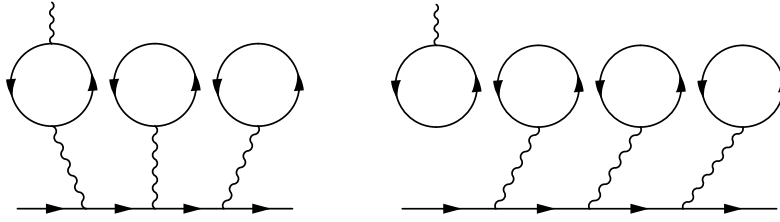


Fig. 72. Representatives of the (2+1+1, left) and (1+1+1+1, right) class of diagrams.
Source: Figures taken from Ref. [62].

6.2.8. Final average for a_μ^{HLbL}

Adding the four contributions of Eqs. (6.24), (6.28), (6.31) and (6.32), treating their uncertainties as being uncorrelated, we arrive at our final average of lattice QCD results for a_μ^{HLbL} ,

$$a_\mu^{\text{HLbL}} = 122.5(7.1)_{\text{stat}}(5.6)_{\text{syst}} \times 10^{-11} = 122.5(9.0) \times 10^{-11} \quad (\text{lattice QCD average}). \quad (6.34)$$

Clearly, the uncertainty is entirely dominated by that of the light-quark contribution $a_\mu^{\text{HLbL},\ell}$. The entire remaining contribution is

$$a_\mu^{\text{HLbL},s} + a_\mu^{\text{HLbL},c} + a_\mu^{\text{HLbL},\text{rest}} = 2.92(98) \times 10^{-11} \quad (\text{lattice QCD average}). \quad (6.35)$$

A summary of the status of the HLbL contribution is shown in Fig. 82 in Section 9.

6.3. Exclusive state contributions

In the dispersive framework, the dominant contribution to the HLbL diagram comes from the light pseudoscalar-poles according to Eq. (5.20). The hadronic inputs are the TFFs $F_{P\gamma^*\gamma^*}$ at spacelike virtualities with $P = \pi^0, \eta, \eta'$. The integrand in Eq. (5.20) is peaked at low virtualities and the dominant contribution comes from the region $Q_{1,2}^2 < 3 \text{ GeV}^2$ [574] that can be reached by lattice calculations.

Regarding the determination of TFFs, two distinct approaches are currently employed. The first approach adopts the time-momentum representation, as proposed by Ref. [722], where the momenta for the pseudoscalar meson and photons are constrained to discrete values. This necessitates the use of parameterizations like the z -expansion for momentum interpolation. This method was used in early calculations to evaluate the TFFs and the lifetime of the pion [723] and has recently been adopted by three groups to determine the pseudoscalar-pole contributions to HLbL, extending from the pion pole to the η - and η' -poles [549,575,576,650,712,724]. The second approach, proposed very recently by the RBC/UKQCD collaboration [713], calculates TFFs with arbitrary photon momenta by introducing an appropriate coordinate-space weight function (coordinate-space representation) [725,726]. To date, this method has only been applied to the pion-pole contribution, where the significant signal-to-noise problem in computing TFFs at large virtualities is mitigated by introducing a structure function. Below, we present the results obtained from both approaches.

6.3.1. Time-momentum representation

By the time of WP20, a single complete lattice calculation of the pion TFF, by the Mainz group [549,712], had been published. The TFF had been computed in the kinematic range relevant to the $(g-2)_\mu$ using $N_f = 2 + 1$ Wilson-Clover quarks. Four lattice spacings in the range [0.050–0.086] fm and pion masses down to 200 MeV were used to extrapolate the result to the physical point using a systematically improvable parameterization, inspired by the analysis of other hadronic form factors. This parameterization satisfies the short-distance constraints [528,529,568,570,571] and can be further constrained by experimental inputs, such as the two-photon decay width. In the first publication [712] the calculation was limited to the pion rest-frame. In Ref. [549] it was realized that including data in a moving frame, where the pion carries one unit of momentum, allows a better coverage of the (Q_1^2, Q_2^2) -plane, especially when one of the virtualities tends to zero (single-virtual regime). Using the parameterization of the form factor, the Mainz group was able to provide the first lattice calculation of the pion-pole contribution to the HLbL. The result is

$$a_\mu^{\pi^0\text{-pole}} = 59.7(3.4)(0.9)(0.5) \times 10^{-11} = 59.7(3.6) \times 10^{-11}, \quad (6.36)$$

where the first error is statistical, the second one is the systematic error associated with the parameterization of the TFF, and the third one from the disconnected contribution. The disconnected contribution is $-1.0(5) \times 10^{-11}$. Using, in addition, the experimental measurement of the two-photon decay width [727,728] as a constraint in the fit of the TFF, the authors quote $a_\mu^{\pi^0\text{-pole}} = 62.3(2.3) \times 10^{-11}$.

Since the publication of WP20, two additional groups have presented lattice calculations of the pion TFF in the time-momentum representation. The ETM collaboration [650] uses $N_f = 2 + 1 + 1$ flavors of Wilson Clover twisted mass

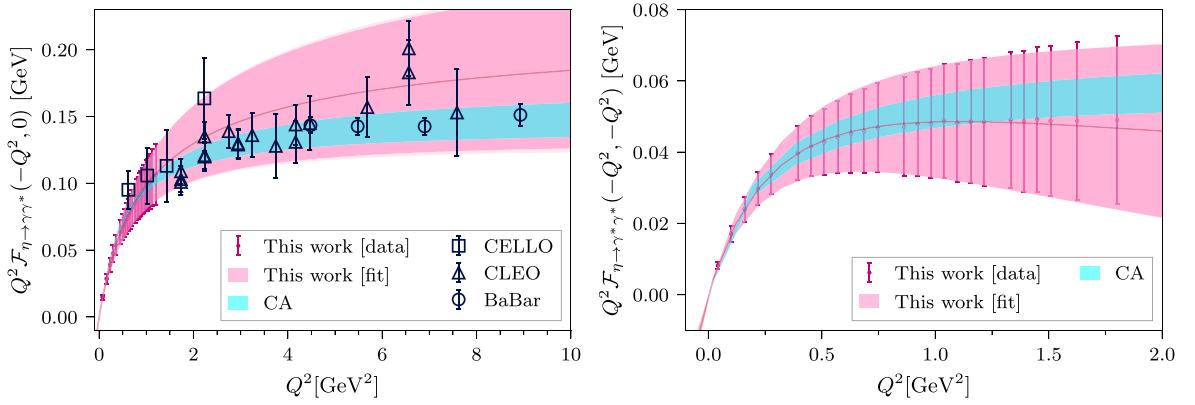


Fig. 73. The η TFF from the ETM collaboration in the single-virtual (left) and double-virtual (right) kinematics [575]. The result, obtained at a single lattice spacing, is compared with experimental results and the Canterbury estimate (cyan bands).
Source: Figures taken from Ref. [575].

quarks. The simulations are performed at the physical pion mass and at maximal twist [729,730]. This ensures automatic $\mathcal{O}(a)$ -improvement of the three-point function. The latter is evaluated in the pion rest frame for three values of the lattice spacing in the range [0.057–0.080] fm using lattices of spatial extents $L \simeq 5.5$ fm. The calculation includes both connected and disconnected contributions. The form factor is extrapolated to the continuum limit using the parameterization introduced by the Mainz group [549]. The authors quote

$$a_{\mu}^{\pi^0\text{-pole}} = 56.7(3.1)(1.0) \times 10^{-11} = 56.7(3.2) \times 10^{-11}, \quad (6.37)$$

where the first error is statistical and the second is the systematic uncertainty.

The BMW collaboration [576] uses $N_f = 2 + 1 + 1$ flavors of staggered quark fermions with four steps of stout smearing. The calculation is also performed at the physical pion mass and includes five values of the lattice spacing in the range [0.065–0.132] fm. The spatial extents of the lattices are in the range [6.1–6.7] fm. In addition to the pion rest frame, the pion frame with one unit of momentum is also considered. As for the other groups, both connected and disconnected contributions are included and the parameterization introduced in Ref. [549] is used to extrapolate the TFF in the continuum limit. The result reads

$$a_{\mu}^{\pi^0\text{-pole}} = 57.8(1.8)(0.9) \times 10^{-11} = 57.8(2.0) \times 10^{-11}. \quad (6.38)$$

where the disconnected contribution is $-1.33(19) \times 10^{-11}$.

Besides the pion, the η and η' -poles are the largest pseudoscalar-pole contributions. Compared to the pion, the TFFs are much more challenging to compute on the lattice. The calculation requires additional quark-disconnected diagrams that involve a single pseudoscalar loop. This noisy disconnected contribution is large and of opposite sign as compared to the fully connected part. Thus, large statistics is needed to control this delicate cancellation. Because of the heavier pseudoscalar masses, the integrand in Eq. (5.20) is also peaked at larger virtualities and the pole contribution is less sensitive to the normalization of the TFF [574]. The η meson is the lowest-lying eigenstate with quantum numbers $I^G(J^{PC}) = 0^+(0^{-+})$ and the associated form factor can be extracted using the same strategy as the one used for the pion. In this case, the η' contribution is seen as an excited-state contribution that vanishes for sufficiently large time separations between the pseudoscalar insertion and the two vector currents. This is the strategy followed by the ETM collaboration in Ref. [575,731]. In principle, any pseudoscalar interpolating operator can be used as long as it overlaps with the physical η meson. For the η' meson, a proper treatment of the mixing between the flavor-octet and flavor-singlet states is required to isolate the eigenstate of the Hamiltonian.

In Ref. [575], the ETM collaboration has presented the first lattice calculation of the η TFF, at the physical pion mass but at a single lattice spacing. The result for the double-virtual and single-virtual kinematics is shown in Fig. 73 and compared with experimental data and the CA estimate [35]. Using this TFF and the master formula Eq. (5.20), they quote

$$a_{\mu}^{\eta\text{-pole}} = 13.2(5.2)_{\text{stat}}(1.3)_{\text{syst}}[5.3]_{\text{tot}} \times 10^{-11}, \quad (6.39)$$

where the error is dominated by statistics. The main systematic error comes from varying the order of their z -expansions used to parameterize the virtuality dependence of the form factor. The diagrams that contain single vector loops are suppressed in the SU(3) flavor limit and have been neglected in this work.

The BMW collaboration has presented a first calculation including both the η and η' contributions. The results for the single-virtual regime are depicted in Fig. 74. All connected and disconnected contributions have been estimated. It has been shown that the disconnected diagrams that are suppressed in the SU(3) flavor limit are small and negligible at the

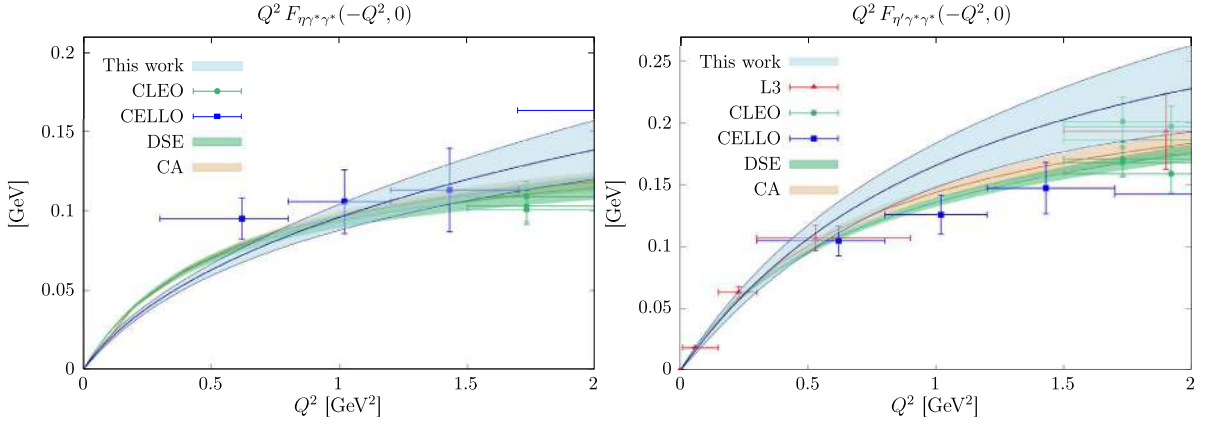


Fig. 74. The η (left) and η' (right) single-virtual TFFs from the BMW collaboration at the physical point and in the continuum limit. The Canterbury approximant (CA) result is extracted from Ref. [35] and the Dyson–Schwinger equation (DSE) result comes from Ref. [38]. Measurements from CELLO [564], CLEO [565], and L3 [566] are shown for comparison. Source: Figures taken from Ref. [576].

current statistical precision. As for the pion, two pseudoscalar frames have been included for cross-checks. This calculation includes a continuum extrapolation based on five lattice spacings. The corresponding pseudoscalar-pole contributions read

$$\begin{aligned} a_{\mu}^{\eta\text{-pole}} &= 11.6(1.6)_{\text{stat}}(0.5)_{\text{syst}}(1.1)_{\text{FSE}} \times 10^{-11}, \\ a_{\mu}^{\eta'\text{-pole}} &= 15.7(3.9)_{\text{stat}}(1.1)_{\text{syst}}(1.3)_{\text{FSE}} \times 10^{-11}. \end{aligned} \quad (6.40)$$

The uncertainty is dominated by statistics, followed by the finite-volume effects. The η and η' contributions are comparable in size and their sum is about half of the pion-pole contribution.

The η -pole contribution is compatible within 1.4 standard deviations with the data-driven dispersive result presented in Ref. [56]. Good agreement is found for the η' -pole contribution, although with larger uncertainties on the lattice side.

6.3.2. Coordinate-space representation

In the coordinate-space representation, the methodology is designed within the framework of infinite volume, where spatial momentum can take arbitrary values. The TFFs can be extracted via the Fourier transform

$$\mathcal{F}_{\mu\nu}(Q, Q') = \int d^4x e^{-i(Q - \frac{Q'}{2}) \cdot x} \mathcal{H}_{\mu\nu}(x) = i\varepsilon_{\mu\nu\alpha\beta} Q_{\alpha} Q'_{\beta} F_{\pi^0 \gamma^* \gamma^*}^*(-Q^2, -Q'^2), \quad (6.41)$$

with $Q' = Q_{\pi} - Q$ and $Q = (iE, \mathbf{q})$ parameterized in terms of the real-valued kinematic variables E and $\mathbf{q}$. The hadronic matrix element $\mathcal{H}_{\mu\nu}(x) = \langle 0 | T \{ j_{\mu}(\frac{x}{2}) j_{\nu}(-\frac{x}{2}) \} | \pi(Q_{\pi}) \rangle$ is defined in Euclidean spacetime with $j_{\mu}(x)$ representing the EM current and $Q_{\pi} = (iE_{\pi}, \mathbf{q}_{\pi})$ denoting the four-momentum of the on-shell pion. This matrix element can be decomposed as $\mathcal{H}_{\mu\nu}(x) = -\varepsilon_{\mu\nu\alpha\beta} x_{\alpha} Q_{\pi\beta} H(x^2, Q_{\pi} \cdot x)$. Consequently, the TFFs can be determined through an integral [713]

$$F_{\pi^0 \gamma^* \gamma^*}(-Q^2, -Q'^2) = \int d^4x \omega(K, Q_{\pi}, x) H(x^2, Q_{\pi} \cdot x), \quad (6.42)$$

where the weight function $\omega(K, Q_{\pi}, x)$ is defined as

$$\omega(K, Q_{\pi}, x) \equiv i e^{-iK \cdot x} \frac{(K \cdot x) Q_{\pi}^2 - (K \cdot Q_{\pi})(Q_{\pi} \cdot x)}{K^2 Q_{\pi}^2 - (K \cdot Q_{\pi})^2}, \quad (6.43)$$

with $K = Q - Q_{\pi}/2$. Unfortunately, the factor $e^{-iK \cdot x} = e^{(E - E_{\pi}/2)t} e^{-i(\mathbf{q} - \mathbf{q}_{\pi}/2) \cdot \mathbf{x}}$ grows rapidly when E becomes large, leading to a severe signal-to-noise problem, which is also common in the time-momentum representation. The challenge is addressed by introducing a pion structure function $\phi_{\pi}(x^2, u)$ and expressing $H(x^2, Q_{\pi} \cdot x)$ as

$$H(x^2, Q_{\pi} \cdot x) = \int_0^1 du e^{i(u - \frac{1}{2})Q_{\pi} \cdot x} \phi_{\pi}(x^2, u) H(x^2, 0). \quad (6.44)$$

In this way, the TFFs can be obtained through

$$F_{\pi^0 \gamma^* \gamma^*}(-Q^2, -Q'^2) = \int_0^1 du \int d^4x \omega(\bar{K}, Q_{\pi}, x) \phi_{\pi}(x^2, u) H(x^2, 0), \quad (6.45)$$

with $\bar{K} = Q - uQ_{\pi}$. Since $\phi_{\pi}(x^2, u) H(x^2, 0)$ is symmetric under $\text{SO}(4)$ spacetime rotations, an $\text{SO}(4)$ average can be performed for the weight function $\omega(\bar{K}, Q_{\pi}, x)$, yielding $\langle \omega(\bar{K}, P, x) \rangle_{\text{SO}(4)} = 2 \frac{J_2(|\bar{K}| |x|)}{\bar{K}^2}$, with $J_2(x)$ being the Bessel function.

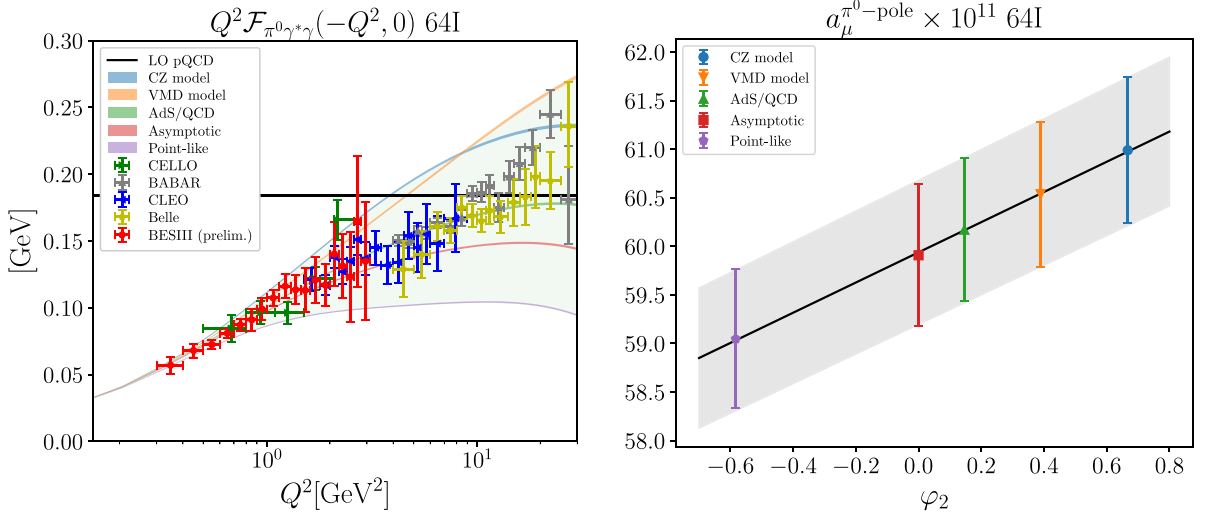


Fig. 75. Left: Comparison between experimental measurements of TFFs with a single virtual photon and lattice QCD calculations using different parameterizations of $\phi_{\pi}(x^2, u)$ as inputs. Right: $a_{\mu}^{\pi^0-pole}$ as a function of $\varphi_2(|x|)$. The results align closely with a linear form, indicating that higher-order effects are negligible.

Source: Figures taken from Ref. [713].

As $J_2(|\vec{K}||x|)$ does not exhibit exponential growth at large $|\vec{K}||x|$, there is no signal-to-noise problem in computing the TFFs. The integral can be further improved by replacing $H(x^2, 0)$ with $H(x^2, Q_{\pi} \cdot x)$ based on the relation Eq. (6.44).

The methodology involves a Gegenbauer expansion to express the pion structure function $\phi_{\pi}(x^2, u)$. The size of different Gegenbauer terms is evaluated by incorporating lattice data, and the LO Gegenbauer term is demonstrated to be model independent. Five parameterizations for $\phi_{\pi}(x^2, u)$, which span a wide range of large-momentum behaviors for the TFFs and cover all experimental data as shown in Fig. 75, are used to estimate the systematic effects induced by $\phi_{\pi}(x^2, u)$. These parameterizations yield the coefficients for different Gegenbauer terms. The results of $a_{\mu}^{\pi^0-pole}$ align closely with a linear form of the NLO coefficient $\varphi_2(|x|)$ as shown in Fig. 75, indicating that the higher-order effects are negligible. More details are presented in Ref. [713].

Using eight gauge ensembles generated with 2+1 flavor domain wall fermions, incorporating multiple pion masses, lattice spacings, and volumes, the RBC/UKQCD calculation provides the result⁴⁴

$$a_{\mu}^{\pi^0-pole} = 59.6(1.6)_{\text{stat}}(1.0)_{\phi}(1.0)_a(0.4)_{\text{FV}} \times 10^{-11} = 59.6(2.2) \times 10^{-11}. \quad (6.46)$$

In this result, the first error is statistical, the second reflects the dependence on the structure function, the third represents lattice artifacts, and the fourth accounts for residual finite-volume effects. It is noteworthy that the finite-volume effects are estimated by assuming that the hadronic function $\mathcal{H}_{\mu\nu}(x)$ at long distances is dominated by the ρ meson and $\pi\rho$ states. These finite-volume effects can reach 0.8×10^{-11} for a lattice size of $L \simeq 4.6$ fm and are quickly suppressed to 0.2×10^{-11} at $L \simeq 5.5$ fm. In Eq. (6.46), the uncertainty $(0.4)_{\text{FV}} \times 10^{-11}$ indicates potential residual effects after the finite-volume correction is applied. When using small lattice volumes, the finite-volume effects are not negligible, making corrections essential, particularly as lattice precision improves in future studies. In the total contribution to $a_{\mu}^{\pi^0-pole}$, the connected diagrams account for $57.8(2.2) \times 10^{-11}$, while the disconnected pieces contribute $1.81(15) \times 10^{-11}$, based on the results from the finest ensemble.

6.3.3. Conclusion

A summary of the lattice calculations of pseudoscalar-pole contributions is given in Fig. 76. Four groups, using different lattice regularizations, have presented results on the pion-pole contribution, with precisions between 3.5 and 6%. They agree within one standard deviation. However, the sign of the small disconnected contribution $a_{\mu}^{\pi^0-pole, \text{disc}}$ is found to be different among groups and needs further investigations. At present, Mainz, BMW, and ETM find the disconnected contribution to be negative, and the more recent RBC/UKQCD calculation finds it to be positive. Also, Mainz, BMW, and RBC/UKQCD provided the results for the disconnected contribution $a_{\mu}^{\pi^0-pole, \text{disc}}$, in which case, we can subtract it from the total and obtain the connected-only contribution $a_{\mu}^{\pi^0-pole, \text{conn}}$. The ETM group has only obtained the total contribution,

⁴⁴ While this paper was under review, the final version of Ref. [713] appeared, $a_{\mu}^{\pi^0-pole} = 61.2(1.1)_{\text{stat}}(1.0)_{\phi}(0.7)_a(0.4)_{\text{FV}} \times 10^{-11} = 61.2(1.7) \times 10^{-11}$, with changes to Eq. (6.46) due to increased statistics. The average in Eq. (6.47) is based on Eq. (6.46).

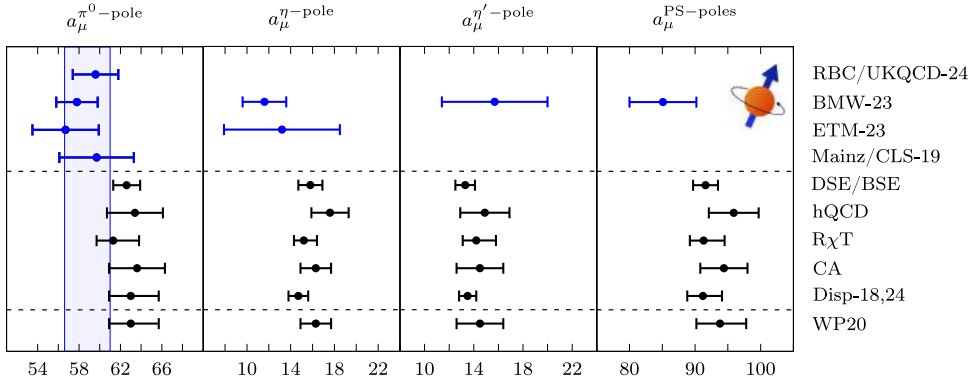


Fig. 76. Summary of the lattice calculations (in blue) and of the approaches presented in Section 5, and comparison with the WP20 average. The blue band represents the lattice average for the pion-pole contribution.

Table 32

Current lattice results for the direct calculation of a_μ^{HLbL} , where statistical and systematic errors have been added in quadrature.

Contribution	Average [10^{-11}]	Sources
$a_\mu^{\text{HLbL},\ell}$	119.6(7.1) _{stat} (5.5) _{syst} [9.0] _{tot}	Eq. (6.24), Table 31
$a_\mu^{\text{HLbL},s}$	−1.4(8)	Eqs. (6.27) and (6.28)
$a_\mu^{\text{HLbL},c}$	3.5(5)	Eqs. (6.29) to (6.31)
$a_\mu^{\text{HLbL},3+1}$	0.82(25)	Eq. (6.32)
a_μ^{HLbL}	122.5(7.1) _{stat} (5.6) _{syst} [9.0] _{tot}	Eq. (6.34)

which should equal to $a_\mu^{\pi^0\text{-pole,conn}} - |a_\mu^{\pi^0\text{-pole,disc}}|$ in their calculation. We can perform a global fit and obtain both the averaged connected-only contribution and the magnitude of the disconnected contribution. We find good agreements in the connected contribution with $\chi^2/\text{dof} = 0.24$. For the disconnected diagram, in addition to the sign disagreement, there is also about 2σ tension in the magnitude between the BMW and RBC/UKQCD results. Therefore, we use the connected-only contribution as the central value and the disconnected contribution is treated as an additional uncertainty. The final lattice QCD average is⁴⁵

$$a_\mu^{\pi^0\text{-pole}} = a_\mu^{\pi^0\text{-pole,conn}} \pm |a_\mu^{\pi^0\text{-pole,disc}}| = 58.8(1.1)_{\text{stat}}(1.1)_{\text{syst}}(1.6)_{\text{disc}}[2.2]_{\text{tot}} \times 10^{-11} \quad (\text{lattice QCD average}). \quad (6.47)$$

The lattice average is shown in Fig. 76, where it is compared to the data-driven approaches presented in Section 5. In particular, the value Eq. (6.47) is compatible with the data-driven result of Ref. [548] within 1.3σ . A more detailed comparison between the latter and lattice calculations, at the level of the TFF, is made in Fig. 77. Both at single- and at double-virtual kinematics, one observes broad agreement between the different determinations. We note that the number of parameters used in parameterizing the TFF has a significant impact on the uncertainty bands.

Since WP20, first lattice calculations for the η and η' have appeared. They suffer from the noisy disconnected contribution that largely cancels the connected contribution. The ETM collaboration has published results for the η -pole contribution at the physical point using a single lattice spacing and the BMW collaboration has presented results for both the η - and η' -pole contributions at the physical point and in the continuum limit. Finally, it is worth noting that the light pseudoscalar TFFs can be valuable inputs in reducing the statistical and systematic errors in the direct HLbL calculation.

6.4. Summary and future prospects

6.4.1. Summary of current knowledge from the lattice

Great strides have been made in the lattice computation of the HLbL contribution to $(g-2)_\mu$ compared to WP20. In WP20, the only determination was that of the RBC/UKQCD collaboration using finite-volume QED_L, but now an infinite-volume QED result has also been determined and the two are consistent. Chiral-continuum determinations of all contributing diagrams using lattice QCD and infinite-volume QED have now been measured in the direct calculation [60]. It is shown that only the fully-connected and $(2+2)$ -disconnected light quark contributions are significant to the overall value, and there are three independent lattice calculations of these contributions: those of the RBC/UKQCD collaboration [59,62], the Mainz group [60], and the BMW collaboration [63].

Table 32 summarizes various contributions to a_μ^{HLbL} based on the averages of different lattice studies currently available.

⁴⁵ Using the final result of Ref. [713], this average would change to $a_\mu^{\pi^0\text{-pole}} = 59.4(0.9)_{\text{stat}}(1.1)_{\text{syst}}(1.5)_{\text{disc}}[2.1]_{\text{tot}} \times 10^{-11}$.

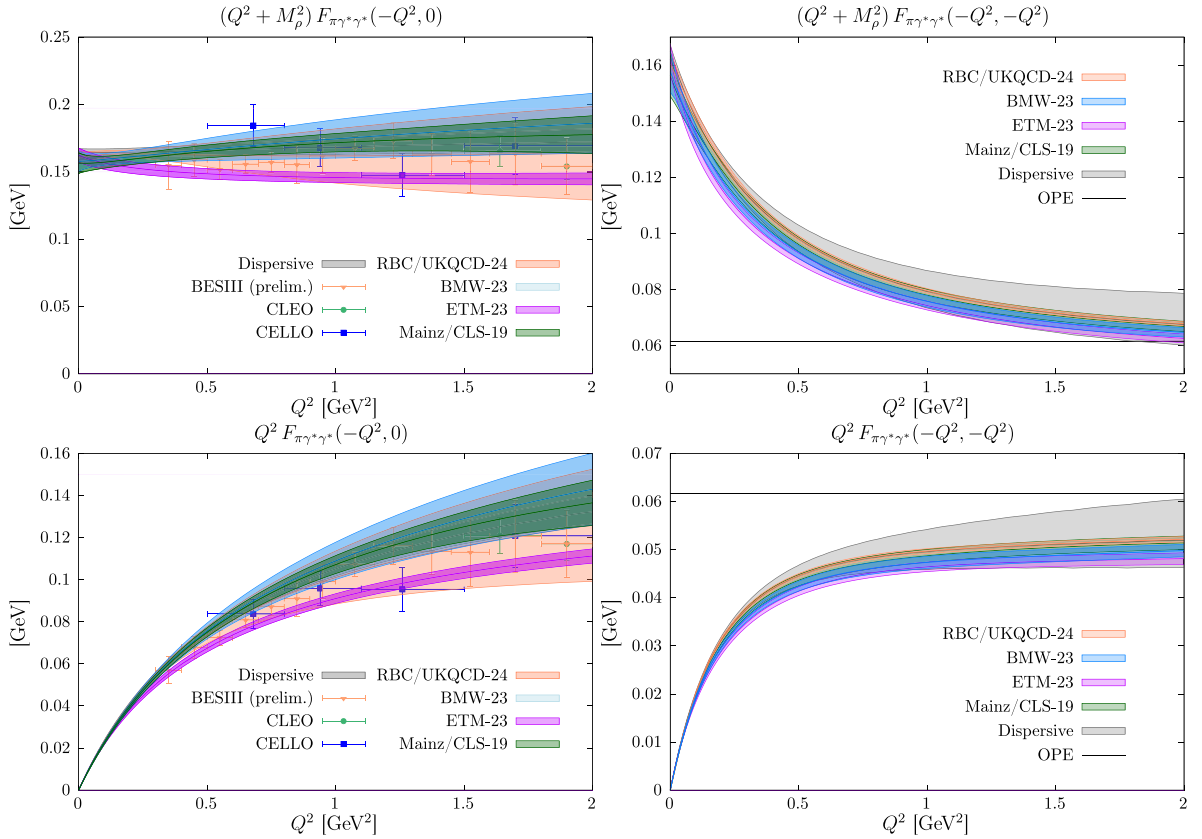


Fig. 77. Comparison of lattice calculations [549,576,650,713], the dispersive determination of Ref. [548], and experimental data (Refs. [564,565] and preliminary BESIII data [1,732]) for the single-virtual and the double-virtual π^0 TFF.

6.4.2. Final estimate and outlook

Our estimate for the lattice determination of the HLbL contribution reads as follows:

$$a_\mu^{\text{HLbL}} = 122.5(7.1)_{\text{stat}}(5.6)_{\text{syst}}[9.0]_{\text{tot}} \times 10^{-11} \quad (\text{lattice QCD average}). \quad (6.48)$$

The individual contributions of the determinations are listed in Table 32. It has been shown that the current techniques and methods are sufficient for the expected precision on this quantity that are required. It is now evident that only two contributions are really relevant, the light-quark (u, d) fully-connected, and $(2+2)$ disconnected contributions, thus facilitating cheaper future studies of a_μ^{HLbL} . Future improvements in the lattice determination are possible with higher statistics and more gauge ensembles with different lattice spacings and volumes.

In addition, lattice QCD calculations have obtained the π^0 -pole contribution in HLbL. The lattice average is

$$a_\mu^{\pi^0\text{-pole}} = 58.8(1.1)_{\text{stat}}(1.1)_{\text{syst}}(1.6)_{\text{disc}}[2.2]_{\text{tot}} \times 10^{-11} \quad (\text{lattice QCD average}). \quad (6.49)$$

7. The QED contributions to a_μ

T. Aoyama, M. Hayakawa, M. Nio, S. Volkov

No significant changes have arisen in the QED contribution to a_μ since WP20 [1].⁴⁶ However, several remarkable new results were obtained in related studies. They are summarized in this section, and their influences on the theoretical value of a_μ are discussed.

7.1. The fine-structure constant α from atom-interferometer experiments

The EM coupling constant known as the fine-structure constant α is a fundamental and essential input for predicting the theoretical value of the lepton $g-2$. The latest and most accurate values of α are provided in two methods. One is the

⁴⁶ Slight changes are due to the updated mass-independent tenth-order coefficient $A_1^{(10)}$ as detailed in Section 7.3 as well as the updated sixth-order coefficient $A_2^{(6)}(m_\mu/m_\tau) = 0.000\,360\,601(83)$. The latter change does not affect any digits reported in this section.

atom-mass measurement using an atom interferometer, and the other is the measurement of the electron's anomalous magnetic moment a_e together with its theoretical prediction.

The atom-interferometer experiment determines the quotient of the Planck constant h and the mass of an atom X m_X . Since May 2019, the Planck constant h is a defined constant, so that it becomes a direct measurement of the mass of a neutral atom. The cesium (^{133}Cs) atom mass [68] and the rubidium (^{87}Rb) atom mass [69] are available for the determination of α :

$$\begin{aligned} h/m_{\text{Cs}} &= 3.002\,369\,4721(12) \times 10^{-9} \text{ m}^2\text{s}^{-1} & [0.40 \text{ ppb}], \\ h/m_{\text{Rb}} &= 4.591\,359\,258\,90(65) \times 10^{-9} \text{ m}^2\text{s}^{-1} & [0.14 \text{ ppb}]. \end{aligned} \quad (7.1)$$

Both experiments are ongoing, aiming for an order-of-magnitude improvement. Another atom-interferometer experiment with strontium (Sr) or ytterbium (Yb) has started also, aiming at a relative precision 0.01 ppb [733]. The mass m_X is converted to α with the help of more precisely determined physical constants:

$$\alpha(X) = \left[\frac{2R_\infty}{c} \frac{A_r(X)}{A_r(e)} \frac{h}{m_X} \right]^{1/2}, \quad (7.2)$$

where R_∞ is the Rydberg constant, c is the speed of light in vacuum, and $A_r(X)$ and $A_r(e)$ are the relative atomic mass of the atom X and that of the electron, respectively.

The Rydberg constant R_∞ has been determined by combining many measurements on various transition frequencies of the hydrogen, deuterium, and muonic atoms from 1979 to 2022 and the theoretical QED predictions of these spectra. The latest CODATA2022 recommended value of R_∞ is [734]

$$R_\infty = 10\,973\,731.568\,157(12) \text{ m}^{-1} \quad [1.1 \text{ ppt}]. \quad (7.3)$$

The relative atomic masses of Cs and Rb atoms were also updated in 2020 [735,736]. The relative atomic masses of the ions of these nuclei were measured and then converted to the neutral atom masses by adding the appropriate number of electron masses and subtracting the ionization energies. The latest values are

$$\begin{aligned} A_r(^{133}\text{Cs}) &= 132.905\,451\,9585(86) & [0.065 \text{ ppb}], \\ A_r(^{87}\text{Rb}) &= 86.909\,180\,5291(65) & [0.075 \text{ ppb}]. \end{aligned} \quad (7.4)$$

The relative atomic mass of the electron $A_r(e)$ has been determined by measuring cyclotron and spin-precession frequencies of hydrogen-like ions of carbon ($^{12}\text{C}^{5+}$) and silicon ($^{28}\text{Si}^{13+}$) using a Penning trap, in conjunction with the QED predictions for the g factor of a Coulomb-bound electron. The CODATA2022 recommended value of $A_r(e)$ is [734]

$$A_r(e) = 5.485\,799\,090\,441(97) \times 10^{-4} \quad [0.018 \text{ ppb}]. \quad (7.5)$$

The values of the fine-structure constant α from the atom-interferometer experiments are then updated by using Eqs. (7.1) and (7.3)–(7.5), and the defined value $c = 299\,792\,458 \text{ m/s}$ as

$$\alpha^{-1}(\text{Cs}) = 137.035\,999\,045(27) \quad [0.20 \text{ ppb}], \quad (7.6)$$

$$\alpha^{-1}(\text{Rb}) = 137.035\,999\,2052(97) \quad [0.071 \text{ ppb}]. \quad (7.7)$$

The last digit of $\alpha^{-1}(\text{Cs})$ in Ref. [68] is changed from 6 to 5, but the uncertainty is unchanged. The updated value of $\alpha^{-1}(\text{Rb})$ is smaller than that in Ref. [69] by 0.8×10^{-9} and the uncertainty is reduced from 11×10^{-9} to 9.7×10^{-9} . The difference between Eqs. (7.6) and (7.7) is

$$\alpha^{-1}(\text{Cs}) - \alpha^{-1}(\text{Rb}) = -0.160(29) \times 10^{-6}, \quad (7.8)$$

corresponding to a significance of 5.5σ .

7.2. Measurement of the electron anomalous magnetic moment a_e

The electron anomalous magnetic moment $a_e = (g_e - 2)/2$ has been measured using a single electron trapped in EM fields known as a Penning trap. The longstanding a_e measurement at Harvard University (HV) in 2008 [737] was superseded in 2022 by a new measurement from the same team, now based at Northwestern University (NW) [70]. The new measurement

$$a_e^{\text{exp}}(\text{NW22}) = 1\,159\,652\,180.59(13) \times 10^{-12} \quad [0.11 \text{ ppb}] \quad (7.9)$$

demonstrates a 2.2-fold improvement in precision, with a shift -0.14×10^{-12} from $a_e^{\text{exp}}(\text{HV08})$, which remains within the assigned uncertainties. As correlations between the two similar measurements are not obvious, the authors of Ref. [70] do not recommend averaging the two determinations. Stabilizing the magnetic field of a Penning trap typically takes weeks, but the NW team developed a faster method. This breakthrough enabled measurements at multiple magnetic field strengths for the first time. The consistency of results across different magnetic field strengths significantly enhanced the reliability of the measurement.

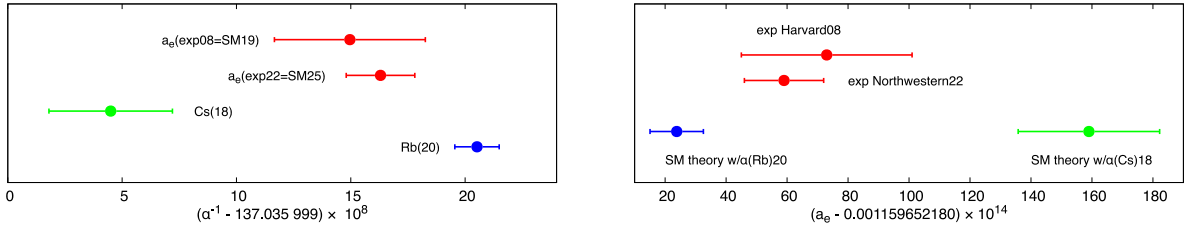


Fig. 78. The best three values of the fine-structure constant α (left) and a comparison of the measurement and SM prediction of a_e (right). The values of α are determined from the measurement and theory of a_e and the atom-mass measurements of Cs and Rb with atom interferometers. The two SM predictions for a_e are based on two different values of α . These two figures present the three experimental results and one theoretical prediction from different perspectives.

A value of the fine-structure constant α from a_e can be obtained by treating α as an unknown variable in the theoretical formula for a_e and equating it to the measured value in Eq. (7.9). The derived value is

$$\alpha^{-1}(a_e) = 137.035\,999\,163(15) \quad [0.11 \text{ ppb}], \quad (7.10)$$

which is smaller than the value reported in Ref. [70] by 3×10^{-9} . This shift, within the assigned uncertainties, is attributed to two factors. The first is a 13% reduction of the tenth-order QED term of a_e , as discussed in Section 7.3. The second is an updated evaluation of the HVP contribution, detailed in Section 7.4, which affects the result in the opposite direction. As a result, $\alpha^{-1}(a_e)$ remains nearly unchanged. The total uncertainty is entirely driven by the measurement in Eq. (7.9), while the uncertainties from the QED tenth-order term and the hadronic contributions are 0.44×10^{-9} and 0.36×10^{-8} , respectively.

The differences between the values of α determined by the atom-interferometer experiments and that obtained from a_e are

$$\begin{aligned} \alpha^{-1}(\text{Cs}) - \alpha^{-1}(a_e) &= -0.118(31) \times 10^{-6}, \\ \alpha^{-1}(\text{Rb}) - \alpha^{-1}(a_e) &= +0.042(18) \times 10^{-6}, \end{aligned} \quad (7.11)$$

corresponding to -3.8σ and $+2.3\sigma$, respectively. The relationships among the three values of α in Eqs. (7.6), (7.7) and (7.10) are visualized in Fig. 78.

7.3. The QED tenth-order mass-independent and universal term

The QED contribution involves only four particles: the photon, electron, muon, and τ lepton. Thus, the QED contribution to a_μ^{QED} can be divided according to the lepton-mass dependence as

$$a_\mu^{\text{QED}} = A_1 + A_2(m_\mu/m_e) + A_2(m_\mu/m_\tau) + A_3(m_\mu/m_e, m_\mu/m_\tau), \quad (7.12)$$

and each can be calculated by perturbation theory using Feynman diagrams as

$$A_i = \sum_{n=1,2,\dots} \left(\frac{\alpha}{\pi}\right)^n A_i^{(2n)}. \quad (7.13)$$

To match the current precision of a_μ , the QED contribution up to the tenth-order of perturbation theory are required. The second-, fourth-, sixth-, and eighth-order terms are well established and were reported in the QED section of WP20 [1].

All Feynman diagrams contributing to a_μ of the tenth-order terms $A_1^{(10)}$, the two $A_2^{(10)}$, and $A_3^{(10)}$, as well as the term $A_2^{(10)}(m_e/m_\mu)$ contributing to a_e were numerically calculated by a single team AHKN [64,738]. The τ -lepton contribution to a_e of the tenth order is negligible. The mass-independent and universal term $A_1^{(10)}$, which contributes equally to both a_e and a_μ , receives contributions from 12,672 vertex Feynman diagrams. Among these, 6,318 diagrams that include at least one fermion loop also contribute to $A_2^{(10)}$. Due to the complexity and enormity of the calculation, only relatively simple diagrams were independently cross-checked until recently [739,740].

In Ref. [65] and successively in Ref. [66], the contribution to $A_1^{(10)}$ from Set V, which consists of 6,354 vertex diagrams without a fermion loop, was obtained using numerical integration. The two results, both calculated by Volkov using different intermediate renormalization schemes, are consistent with each other. The best estimate is then obtained by statistically combining both results [66]:

$$A_1^{(10)}[\text{Set V : Volkov}] = 6.828(60). \quad (7.14)$$

This exhibited a 5σ discrepancy from the latest report 7.668(159) given in Ref. [741] by AHKN, which was used to determine the QED contribution to a_μ in WP20.

To resolve the discrepancy, AHKN performed a diagram-by-diagram numerical comparison with Volkov's results [67]. The 6,354 vertex diagrams were consolidated into 389 groups using the Ward–Takahashi identity. Because of different choices of intermediate renormalization schemes, the integrated magnetic moment amplitudes of the same group in Volkov's and AHKN's calculations show different values. However, the lower-order magnetic moment amplitudes and finite renormalization constants can account for this difference. No apparent inconsistencies were found between Volkov's and AHKN's results for any individual group. However, the sum of the results from 98 groups – each containing a single second-order self-energy subdiagram and no other self-energy subdiagrams – was found to be responsible for the 5σ discrepancy. The 98 integrals representing these 98 groups were then reevaluated by Monte-Carlo integration with substantially increased statistics. Although the old and new results of individual integrals are consistent with each other, the sum of the 98 integrals becomes smaller than before. As a consequence, AHKN's tenth-order contribution has been updated to

$$A_1^{(10)}[\text{Set V : AHKN}] = 6.800(128), \quad (7.15)$$

which is in good agreement with Eq. (7.14).

The contributions from the remaining 6,318 diagrams of the tenth order, which involve at least one fermion loop, were also calculated and reported in Ref. [66]:

$$A_1^{(10)}[\text{with fermion loops : Volkov}] = -0.9377(35). \quad (7.16)$$

The corresponding AHKN value was first reported in Ref. [738] and updated in Ref. [742]:

$$A_1^{(10)}[\text{with fermion loops : AHKN}] = -0.9304(36), \quad (7.17)$$

which is larger than Eq. (7.16) by 0.0073(50), indicating good agreement between the two. The mass-independent and universal $A_1^{(10)}$ terms of Volkov and AHKN then become

$$\begin{aligned} A_1^{(10)}[\text{all : Volkov}] &= 5.891(61), \\ A_1^{(10)}[\text{all : AHKN}] &= 5.870(128). \end{aligned} \quad (7.18)$$

Since they are in good agreement and independent, the weighted average of the two determinations gives the best estimate

$$A_1^{(10)} = 5.887(55), \quad (7.19)$$

which is used to derive the fine-structure constant $\alpha(a_e)$ in Eq. (7.10).

7.4. Theory of the electron anomalous magnetic moment a_e

To compare the measurement of a_e to the theoretical prediction, the SM contributions to a_e besides the QED contributions must be considered. The HVP contribution to the muon $g - 2$, $a_\mu^{\text{HVP, LO}}$, has been under discussion in light of recent lattice-QCD calculations and new measurements of the $\pi^+\pi^-$ cross section from CMD-3. The details are described in Sections 2 and 3. In connection with these new evaluations of HVP, the HVP contribution to a_e was recalculated in Ref. [32], replacing the $\pi^+\pi^-$ channel of KNT19 [31] by the corresponding CMD-3 data. Additionally, lattice-QCD predictions of $a_e^{\text{HVP, LO}}$ are available [448,452,743], though their precision is not yet comparable to data-driven calculations, whose values are, e.g.,

$$a_e^{\text{HVP, LO}} = \begin{cases} 1.8608(66) \times 10^{-12} & \text{KNT19,} \\ 1.920(9) \times 10^{-12} & \text{KNT19/CMD-3.} \end{cases} \quad (7.20)$$

The difference between KNT19 [31] and KNT19/CMD-3 [32] corresponds to a 6.2σ discrepancy.

The magnitude of this difference 0.059×10^{-12} is still much smaller than the difference of 1.35×10^{-12} arising from the discrepancy between the two theoretical predictions of a_e caused by the two values of α , see Eqs. (7.6), (7.7) and (7.23). To take a conservative approach, we adopt the simple mean of KNT19 and KNT19/CMD-3 and assign half of their difference as the uncertainty, in analogy to Section 2.9:

$$a_e^{\text{HVP, LO}} = 1.89(3) \times 10^{-12}, \quad (7.21)$$

which should cover all realistic HVP evaluations for a_e at the moment and is also consistent with the lattice-QCD predictions. Other hadronic and EW contributions are summarized as follows:

$$\begin{aligned} a_e^{\text{HVP, NLO}} &= -0.2263(35) \times 10^{-12}, \\ a_e^{\text{HVP, NNLO}} &= 0.027\,99(17) \times 10^{-12}, \\ a_e^{\text{HLbL}} &= 0.035\,1(23) \times 10^{-12}, \\ a_e^{\text{EW}} &= 0.030\,53(23) \times 10^{-12}, \end{aligned} \quad (7.22)$$

taken from Refs. [31,32], Ref. [744], Ref. [745], and Ref. [744], respectively. Eqs. (7.21) and (7.22) are used to derive the fine-structure constant $\alpha(a_e)$ in Eq. (7.10) as well as the SM prediction for a_e . With QED perturbation theory up to the tenth

order and the hadronic and EW contributions, the SM predictions a_e^{SM} are determined using the fine-structure constant α of Eq. (7.6) or Eq. (7.7) as

$$\begin{aligned} a_e^{\text{SM}}[\alpha(\text{Cs})] &= 1\,159\,652\,181.59(23)(0)(3) \times 10^{-12} & [0.20 \text{ ppb}] , \\ a_e^{\text{SM}}[\alpha(\text{Rb})] &= 1\,159\,652\,180.238(82)(4)(30) \times 10^{-12} & [0.075 \text{ ppb}] , \end{aligned} \quad (7.23)$$

where the uncertainties, listed from left to right, correspond to the fine-structure constant α , the tenth-order QED term $A_1^{(10)}$, and the hadronic contributions. While the uncertainties in α still dominate, a more reliable HVP contribution remains essential for probing BSM physics through a_e . The differences from the measurement Eq. (7.9) are

$$\begin{aligned} a_e^{\text{exp}}(\text{NW22}) - a_e^{\text{SM}}[\alpha(\text{Cs})] &= -1.00(26) \times 10^{-12} , \\ a_e^{\text{exp}}(\text{NW22}) - a_e^{\text{SM}}[\alpha(\text{Rb})] &= +0.35(16) \times 10^{-12} , \end{aligned} \quad (7.24)$$

corresponding to -3.8σ and $+2.2\sigma$, respectively. The comparison between the measurement and the SM predictions is shown in Fig. 78.

7.5. The QED contribution to the muon anomalous magnetic moment a_μ

The lepton-mass ratios necessary for a_μ^{SM} are unchanged since WP20 and they are [734]

$$\begin{aligned} m_e/m_\mu &= 4.836\,331\,70(11) \times 10^{-3} , \\ m_e/m_\tau &= 2.875\,85(19) \times 10^{-4} . \end{aligned} \quad (7.25)$$

As discussed earlier, the three values of α differ from each other by 2.4σ to 5.6σ , making it inappropriate to average them. Accordingly, we present the three values of the QED contribution to a_μ^{SM} using three values of α in Eqs. (7.6), (7.7) and (7.10):

$$\begin{aligned} a_\mu^{\text{QED}}[\alpha(\text{Cs})] &= 116\,584\,718.926(23)(7)(17)(6)(100)[104] \times 10^{-11} , \\ a_\mu^{\text{QED}}[\alpha(a_e)] &= 116\,584\,718.825(13)(7)(17)(6)(100)[103] \times 10^{-11} , \\ a_\mu^{\text{QED}}[\alpha(\text{Rb})] &= 116\,584\,718.789(8)(7)(17)(6)(100)[102] \times 10^{-11} , \end{aligned} \quad (7.26)$$

where the uncertainties from left to right arise from α , the τ -lepton mass, the QED eighth-order term, the QED tenth-order term, the estimated QED twelfth-order term, and the total combined uncertainties [1]. The difference between the largest and the smallest values of a_μ^{QED} is 0.137×10^{-11} , which is of the same order of magnitude as the estimated QED twelfth-order term. In view of Eq. (7.26), we use

$$a_\mu^{\text{QED}} = 116\,584\,718.8(2) \times 10^{-11} , \quad (7.27)$$

in the calculation of the complete SM prediction of a_μ .

8. The electroweak contributions to a_μ

D. Stöckinger, H. Stöckinger-Kim

By definition, the EW SM contributions to a_μ comprise all SM contributions that are not contained in the pure QED, HVP, or HLbL contributions. They are given by Feynman diagrams that contain at least one of the EW bosons W , Z , or the Higgs. Figs. 79–81 show sample one-loop and two-loop diagrams. Already in WP20 [1], the EW SM contributions a_μ^{EW} had a very small theoretical uncertainty of 10^{-11} , which had a negligible impact on the total SM theory uncertainty. Here we summarize the current status of the EW SM contributions and describe relevant recent updates. The updates are related to more accurate measurements of input parameters entering Feynman diagrams, leading to reduced parametric uncertainties, and to improved determinations of hadronic EW contributions. For further details, we refer to Ref. [1], the original papers described below, and to the review Ref. [573].

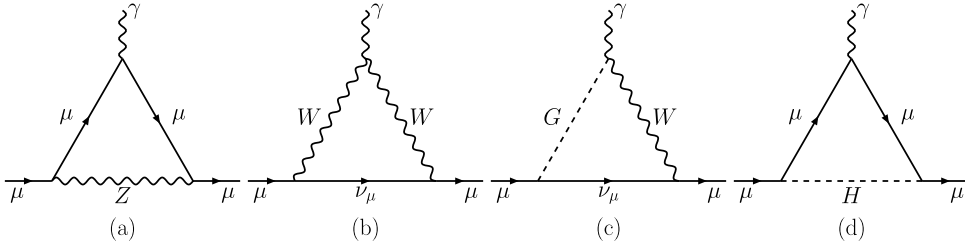
The overall magnitude of the EW contributions can be obtained from the essential factors entering the EW one-loop contribution from a W -boson loop,

$$a_\mu^{\text{EW}(1)} \propto \frac{g_2^2}{16\pi^2} \frac{m_\mu^2}{M_W^2} \sim 10^{-9} . \quad (8.1)$$

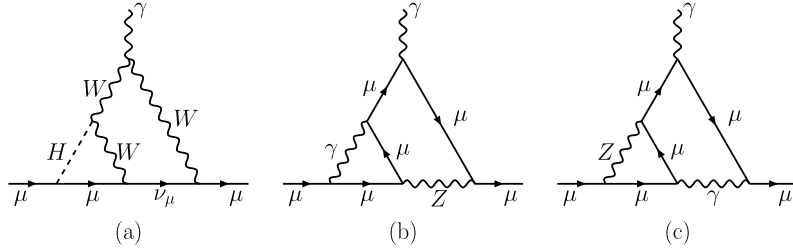
Here $g_2^2/16\pi^2$ corresponds to a loop factor involving the weak $\text{SU}(2)$ gauge coupling, and $m_\mu^2/M_W^2 \simeq 10^{-6}$ is a suppression factor arising from the heavy W -boson. The appearance of two powers of the ratio m_μ/M_W can be related to the need for a muon chirality flip and the need for spontaneous electroweak symmetry breaking to generate a nonvanishing muon mass and dipole moment, see e.g., Ref. [746].

For a precise and systematic evaluation of the EW contributions, the decomposition

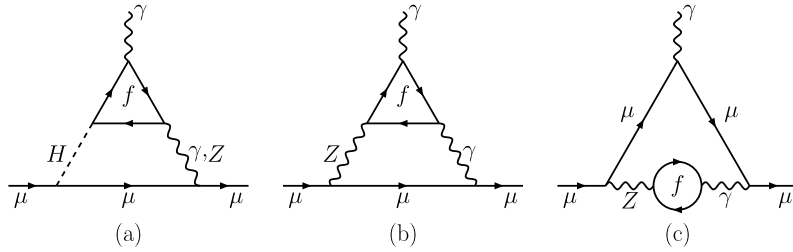
$$a_\mu^{\text{EW}} = a_\mu^{\text{EW}(1)} + a_{\mu;\text{bos}}^{\text{EW}(2)} + a_{\mu;\text{ferm}}^{\text{EW}(2)} + a_\mu^{\text{EW}(\geq 3)} \quad (8.2)$$

**Fig. 79.** One-loop Feynman diagrams contributing to a_μ^{EW} .

Source: Figures taken from Ref. [1].

**Fig. 80.** Sample bosonic two-loop Feynman diagrams contributing to a_μ^{EW} .

Source: Figures taken from Ref. [1].

**Fig. 81.** Sample fermionic two-loop Feynman diagrams contributing to a_μ^{EW} .

Source: Figures taken from Ref. [1].

is useful. Here $a_\mu^{\text{EW}(1)}$ are the one-loop contributions, $a_{\mu;\text{bos}}^{\text{EW}(2)}$ and $a_{\mu;\text{ferm}}^{\text{EW}(2)}$ are two-loop contributions, and $a_\mu^{\text{EW}(\geq 3)}$ are three-loop corrections and higher.

The EW one-loop contributions arise from diagrams with W -, Z -, or Higgs-boson exchange as shown in Fig. 79. Depending on the chosen gauge, diagrams with unphysical Goldstone bosons must also be included. The full one-loop result can be written as

$$a_\mu^{\text{EW}(1)} = \frac{G_F}{\sqrt{2}} \frac{m_\mu^2}{8\pi^2} \left[\frac{5}{3} + \frac{1}{3}(1 - 4s_W^2)^2 \right] = 194.79(1) \times 10^{-11}, \quad (8.3)$$

where the $\text{SU}(2)$ gauge coupling and the factor $1/M_W^2$ appearing in Eq. (8.1) have been replaced with the Fermi constant G_F , which is more precisely measured. The on-shell weak mixing angle $s_W^2 = 1 - M_W^2/M_Z^2$ is defined via the W - and Z -boson pole masses. In the evaluation of a_μ^{EW} , the W -boson mass is treated as an intermediate quantity that is itself predicted within the SM as a function of G_F , M_Z , and further inputs such as the Higgs-boson and top-quark masses. We use here the current best-fit value $M_W = 80.356(5) \text{ GeV}$ [225], which differs slightly from the value used in Ref. [1]. Though the W -boson mass is the result of a calculation and thus unaffected by discrepancies between different direct M_W measurements [225], it has an uncertainty due to missing higher-order corrections and due to the parametric dependence in particular on the top-quark mass. The numerical impact of the modification on the number quoted above is below 10^{-13} . The quoted uncertainty covers the parametric uncertainty from SM input parameters and neglected contributions that are suppressed by additional powers of $m_\mu^2 G_F$.

The bosonic two-loop contributions $a_{\mu;\text{bos}}^{\text{EW}(2)}$ are defined by two-loop and associated counterterm diagrams without closed fermion loop, as in Fig. 80. These contributions have been computed in Ref. [747] in the limit $M_H \gg M_W$, and with full Higgs-boson mass dependence in Ref. [748]. A seminumerical result has been obtained in Ref. [749] and a fully

numerical computation has been carried out in Ref. [750]. Here we use the analytical result of Ref. [748], updated in Ref. [72], with the PDG value $M_H = 125.20(11)$ GeV [225] and obtain

$$a_{\mu;\text{bos}}^{\text{EW}(2)} = -19.962(3) \times 10^{-11}. \quad (8.4)$$

The renormalization scheme used here reflects the chosen parameterization of the one-loop result in terms of the Fermi constant G_F . Thanks to the increased experimental accuracy of the SM input parameters it is possible to provide here an additional digit compared to Ref. [1]. The quoted uncertainties are the parametric uncertainties obtained from varying the Higgs-boson and W -boson masses by 1σ around their central values.

The fermionic two-loop contributions can be further subdivided into

$$a_{\mu;\text{ferm}}^{\text{EW}(2)} = a_{\mu;\text{f-rest,H}}^{\text{EW}(2)} + a_{\mu}^{\text{EW}(2)}(e, u, d; \mu, c, s; \tau, t, b) + a_{\mu;\text{f-rest,no H}}^{\text{EW}(2)}. \quad (8.5)$$

The first term on the RHS of Eq. (8.5) denotes the Higgs-dependent fermion-loop diagrams like in Fig. 81a. The second term denotes contributions from the diagrams in Fig. 81b with a $\gamma^*-\gamma-Z^*$ -subdiagram and fermions of the 1st, 2nd, and 3rd generation in the inner loop. For these diagrams, quarks and leptons must be combined because of gauge anomaly cancellation. The third term collects all remaining fermionic two-loop contributions, e.g., from the diagram in Fig. 81c.

First we focus on the Higgs-dependent fermion-loop corrections. They are given by the so-called Barr–Zee diagrams of Fig. 81a with Higgs– $\gamma-\gamma$ or Higgs– $\gamma-Z$ subdiagram. Various limits of the result have been computed in Ref. [751]; the exact result can be found, e.g., in Ref. [72]. These Higgs-dependent fermion-loop corrections are proportional to the Higgs–fermion Yukawa couplings and are therefore largest for the top quark. For the evaluation we use the quark masses of Ref. [225], in particular $m_t = 172.57(29)$ GeV. The full result for this class of contributions is then

$$a_{\mu;\text{f-rest,H}}^{\text{EW}(2)} = -1.500(2) \times 10^{-11}. \quad (8.6)$$

Compared to Ref. [1] the result has a slightly reduced magnitude due to the smaller top-quark mass. The indicated uncertainty is only the parametric uncertainty arising essentially from the uncertainty of the input parameters m_t and M_H . The theory uncertainty from missing higher-order corrections to these contributions could be estimated, e.g., by changing the renormalization scheme of the quark mass definitions and would be significantly larger. Since it will be covered by the general estimate of missing three-loop corrections below, it is not included in Eq. (8.6).

After the precise Higgs-boson mass measurement at the LHC, all the EW contributions discussed so far have a tiny uncertainty. The uncertainty of the remaining fermionic EW contributions appearing in Eq. (8.5) is larger because potentially large QCD corrections affect the two-loop diagrams with quark loops. It has become standard to generalize the perturbative loop counting and define the quantities $a_{\mu}^{\text{EW}(2)}(e, u, d; \mu, c, s; \tau, t, b) + a_{\mu;\text{f-rest,no H}}^{\text{EW}(2)}$ more generally as the EW contributions given by the respective two-loop diagrams plus appropriate higher-order, potentially nonperturbative, QCD corrections. Early results of such an improved treatment have been obtained in Refs. [71,752,753], and the results described and quoted in Ref. [1] were the ones of Ref. [71].

More recently, Ref. [73] further improved the evaluation of these hadronic EW contributions. The largest of these remaining contributions are the ones involving fermion loops inserted into a $\gamma-\gamma-Z$ subdiagram, shown in Fig. 81b. Because of the Furry theorem, here only the axial Z -boson coupling can contribute. Therefore, the subdiagram constitutes a VVA correlator and gauge anomaly cancellation is important. For 3rd-generation fermions, these diagrams can be evaluated perturbatively, and the two-loop calculation and its uncertainty due to missing higher orders have been discussed in Refs. [71,72]. Ref. [73] incorporated three-loop QCD corrections of Ref. [754] and obtained

$$a_{\mu}^{\text{EW}(2)}(\tau, t, b) = -8.12(1) \times 10^{-11}. \quad (8.7)$$

Compared to the pure two-loop result, the central value has shifted slightly within the previous uncertainty interval, but the uncertainty has been reduced by a factor 10.

For the first two generations of quarks it is appropriate to define the VVA Green functions $\langle 0 | T j^\mu(x) j^\nu(y) j_5^\rho(z) | 0 \rangle$, which can be expressed in terms of two scalar functions $w_{L,T}(Q^2)$ that only depend on the Z^* momentum scale Q^2 . The contribution to a_{μ} can then be obtained from integrals over $w_{L,T}$. Refs. [71,752,753] have investigated constraints on these functions from nonrenormalization theorems and from operator product expansions. For the actual calculation of a_{μ} , Ref. [71] used a model ansatz for the four light-quark functions $w_{L,T}^{[u,d],[s]}$ that is compatible with all constraints, while charm-quark and lepton loops are added perturbatively. Ref. [73] similarly treats the light quarks nonperturbatively but uses a dispersive calculation [51] for the VVA Green function with u, d -quark loops. The combined result for the 1st-generation quarks and leptons then reads

$$a_{\mu}^{\text{EW}(2)}(e, u, d) = -2.08(3) \times 10^{-11}. \quad (8.8)$$

Compared to Ref. [71] the uncertainty is reduced by almost an order of magnitude.⁴⁷

For the 2nd-generation contribution to the VVA Green function both Ref. [71] and Ref. [73] use an ansatz for the strange-quark contribution similar to the 1st-generation approach, combined with perturbation theory for the charm-quark and

⁴⁷ This result is consistent with the Regge model evaluation discussed in Section 5.7.2, which gives $a_{\mu}^{\text{EW}(2)}(e, u, d) = -1.98(9) \times 10^{-11}$.

the muon loop. Ref. [73] includes the perturbative three-loop correction of Ref. [754] to the charm-quark contribution. In total, Ref. [73] finds the 2nd-generation result

$$a_{\mu}^{\text{EW}(2)}(\mu, c, s) = -4.14(28) \times 10^{-11}. \quad (8.9)$$

Here the uncertainty is not significantly reduced compared to the result of Ref. [71] quoted in Ref. [1]. However, particularly the pQCD three-loop corrections have shifted the central value by more than the previously estimated uncertainty.

The non-Higgs-dependent contributions $a_{\mu; \text{f-rest, no H}}^{\text{EW}(2)}$ have first been computed in Ref. [751] neglecting the combination $1 - 4s_W^2 \rightarrow 0$ appearing in the vectorial muon–Z coupling. These leading contributions contain the enhancement factor m_t^2/M_W^2 . Subleading terms suppressed by $1 - 4s_W^2$ have been added in Ref. [71]. They arise in part via corrections to the ρ -parameter that are again enhanced by m_t^2 , and in part from the diagrams with γ –Z interaction shown in Fig. 81c, which are logarithmically enhanced. Based on Ref. [71], Ref. [72] provides the full analytic result. The required hadronic part of the γ –Z interaction, $8\pi^2 \tilde{\Pi}^{\gamma Z}(-M_Z^2)$, is again subject to nonperturbative QCD corrections. For the required result of this quantity, Ref. [1] used the value 6.88(50) based on Ref. [71] and references therein. More recent work shows that an improved data-based evaluation yields a significantly lower value, e.g., Ref. [660] quotes 5.87(4), and including lattice-QCD results [476] to quantify SU(3)-breaking corrections Ref. [73] obtains 6.0(1). Based on this last number, the full result can hence be written as

$$a_{\mu; \text{f-rest, no H}}^{\text{EW}(2)} = [-4.09 - 0.23 - 0.26(10)] \times 10^{-11} = -4.58(10) \times 10^{-11}. \quad (8.10)$$

Here the three individual results correspond to the terms without $1 - 4s_W^2$ suppression, the terms related to the ρ -parameter, and the terms involving the γ –Z mixing subdiagram, respectively. Compared to Ref. [1], the first result has changed because of the lower top-quark mass and the last term has changed as a result of the improvement in Ref. [73]. The change is smaller than the quoted uncertainty, which is unchanged and corresponds to an estimate of still neglected terms which are suppressed by a factor $1 - 4s_W^2$ or M_Z^2/m_t^2 .

Finally, $a_{\mu}^{\text{EW}(\geq 3)}$ collects corrections beyond the two-loop level which are not yet taken into account via the QCD corrections within $a_{\mu}^{\text{EW}(2)}(e, u, d; \mu, c, s; \tau, t, b)$ and $a_{\mu; \text{f-rest, no H}}^{\text{EW}(2)}$. These correspond to weak and QED corrections with and without fermion loops to $a_{\mu; \text{bos}}^{\text{EW}(2)}$ and weak, QED, and QCD corrections to $a_{\mu; \text{f-rest, H}}^{\text{EW}(2)}$. The leading-logarithmic corrections of the form $\simeq G_F \alpha^2 \log(M_Z/m_f) \log(M_Z/m_{f'})$, where $m_{f, f'}$ are light fermions, have been evaluated in Refs. [71, 755] using EFT and RG methods. A surprising numerical cancellation among the three-loop corrections was observed in Ref. [71] if the two-loop contributions are parameterized in terms of $G_F \alpha$ as done in all the above results. In this case the three-loop logarithms are numerically negligible. Hence,

$$a_{\mu}^{\text{EW}(\geq 3)} = 0.00(20) \times 10^{-11}. \quad (8.11)$$

The uncertainty estimate from remaining unknown higher-order contributions is from Ref. [71]. It is based on an analysis of subleading weak logarithmic three-loop corrections, but is expected to also cover nonlogarithmic three-loop QCD contributions to $a_{\mu; \text{f-rest, H}}^{\text{EW}(2)}$.

In total, the full EW SM contribution to a_{μ} is obtained by summing the one-loop contributions Eq. (8.3), the bosonic two-loop contributions Eq. (8.4), the fermionic two-loop contributions Eqs. (8.6)–(8.10), and the leading three-loop logarithms Eq. (8.11). The result is

$$a_{\mu}^{\text{EW}} = 154.4(4) \times 10^{-11}, \quad (8.12)$$

where we follow Ref. [73] and add the individual uncertainties in quadrature. After significant reduction of the hadronic EW uncertainties, the remaining uncertainty is dominated about equally by 2nd-generation hadronic contributions of the kind of Fig. 81b and by remaining unknown higher-order contributions beyond the two-loop level.

9. Conclusions and outlook

In this second edition of the White Paper on the muon $g - 2$, we have charted the progress that has been achieved since 2020 in evaluating the contributions from the electromagnetic (QED), electroweak (EW), and strong (QCD) interactions to a_{μ} .

Both the QED and EW contributions have changed very slightly since the first edition, as can be seen from Table 33. A discrepancy in the evaluation of a sub-class of the 10th-order QED contribution has been resolved [65–67], leading to a tiny shift in the central value of a_{μ}^{QED} . At the same time, the current tension in the experimental determination of the fine-structure constant α is reflected in an increase of the error and a change in the last decimal. The quoted uncertainty for the EW contribution has more than halved since WP20, thanks to a more precise determination of hadronic effects in the two-loop EW contributions [745], while also the precision in input parameters such as the top-quark and Higgs-boson masses has increased. Contributions to a_{μ} from QCD are still by far the dominant sources of uncertainty. Here, much of the current debate is centered on hadronic vacuum polarization (HVP). Significant developments in both data-driven, dispersive and lattice-QCD determinations of the HVP contribution to a_{μ} have fundamentally changed the picture since

Table 33

Comparison of the key results from this work (WP25), as given in Table 1, to the corresponding numbers from WP20 [1] (in units of 10^{-11}). Note that the “HLbL (lattice)” result from WP20 has been adapted to include the charm-loop contribution. The entry “HVP (LO + NLO + NNLO)” derives from HVP LO (lattice) [WP25] and HVP LO (e^+e^-) [WP20], respectively. The asterisk indicates that the LO HVP value from WP20 was based on e^+e^- data only, while in Table 5 we also include the current status for τ -based evaluations.

Contribution	WP25	WP20
HVP LO (lattice)	7132(61)	7116(184)
HVP LO (e^+e^- , τ)	Table 5	6931(40)*
HVP NLO (e^+e^-)	−99.6(1.3)	−98.3(7)
HVP NNLO (e^+e^-)	12.4(1)	12.4(1)
HLbL (phenomenology)	103.3(8.8)	92(19)
HLbL NLO (phenomenology)	2.6(6)	2(1)
HLbL (lattice)	122.5(9.0)	82(35)
HLbL (phenomenology + lattice)	112.6(9.6)	90(17)
QED	116 584 718.8(2)	116 584 718.931(104)
EW	154.4(4)	153.6(1.0)
HVP (LO + NLO + NNLO)	7045(61)	6845(40)
HLbL (phenomenology + lattice + NLO)	115.5(9.9)	92(18)
Total SM Value	116 592 033(62)	116 591 810(43)

WP20. This is reflected by the updated SM estimate, in which the result for the LO HVP contribution is now based on lattice-QCD calculations rather than the data-driven dispersive method.

For the data-driven approach, the CMD-3 measurements of the $e^+e^- \rightarrow \pi^+\pi^-$ cross section [95,96] are higher than those of all other data sets. Despite substantial efforts, see the detailed discussions in Section 2, the emerged discrepancies in this dominant channel are so far not understood and, unlike in WP20, cannot be accommodated any longer through any reasonable inflation of uncertainties in data combinations. The current situation is summarized in Fig. 26. Resolving the puzzles will require new measurements, together with an improved understanding of higher-order radiative corrections and their implementation in MC generators. For the former, new data analyses, in particular for the two-pion channel, are expected from several experiments. For the latter, efforts are underway by several groups, coordinated by the *RadioMonteCarLow 2* initiative. However, at this moment, the situation regarding the $\pi^+\pi^-$ data remains unresolved and prevents us from providing a common average for the data-driven dispersive estimate of $a_\mu^{\text{HVP, LO}}$. In addition, we re-examined the role of isospin-breaking corrections to hadronic τ spectral-function data, providing a detailed assessment of our current understanding thereof, of promising work in progress and avenues for future improvements. We are confident that the latter will allow the use of hadronic τ decays as additional input for the critical two-pion channel in the future. At the time of WP20 the precision of lattice-QCD calculations was not yet sufficient to impact the SM prediction, and the value quoted for HVP in the SM prediction of WP20 is entirely based on data-driven analyses of hadronic e^+e^- cross-section data. In the meantime lattice-QCD calculations have matured significantly, allowing for a precise and robust first-principles calculation of the HVP contribution. Two aspects are particularly important to achieve this. First, the introduction of window observables has proved instrumental for cross-checking and benchmarking lattice calculations of sub-contributions to HVP with a high level of precision. The individual windows isolate and separate the different technical challenges for lattice calculations and allow for tailored approaches for each window. A diverse set of methods with complementary systematic advantages and disadvantages employed by the different lattice-QCD collaborations has led to the consolidation of the individual window contributions one by one. While this process highlights the consistency of the lattice approaches, significant tensions are observed between lattice and data-driven estimates for the intermediate and long-distance window observables. These tensions appear to originate from the dominant $\pi^+\pi^-$ channel, and would disappear if only CMD-3 data were used. A second very important development in the lattice community is the broad adoption of blinding procedures to avoid confirmation bias. This is instrumental in establishing the reliability of the observed consolidation when comparing independent lattice-QCD results. The review of lattice-QCD results in WP25 is based on seventeen different papers from eight independent lattice-QCD collaborations [14–30], including three almost complete lattice calculations of the entire LO HVP contribution [16,26,27]. All available results are combined in various ways, yielding consistent averages for $a_\mu^{\text{HVP, LO}}$ —as our final SM prediction of the latter we take the average that includes the maximum number of independent lattice results from Refs. [14–30]. In summary, our consolidated average of lattice-QCD results provides a reliable determination of the LO HVP contribution to the SM prediction of a_μ .

The hadronic light-by-light (HLbL) contribution was already provided as an average of data-driven and lattice QCD results in WP20. Since then both data-driven and lattice evaluations have been developed further such that in this White Paper an update with reduced uncertainty can be provided. At the current level of precision the different lattice results as well as the lattice and data-driven average are consistent with each other (the latter two at the level of 1.5σ), see Fig. 82.

Adding the LO HVP average from lattice QCD, given in Eq. (3.37), to the NLO and NNLO iterations from e^+e^- data, given in Eqs. (2.47) and (2.48), we obtain for the total HVP contribution

$$a_\mu^{\text{HVP}} = 7045(61) \times 10^{-11}. \quad (9.1)$$

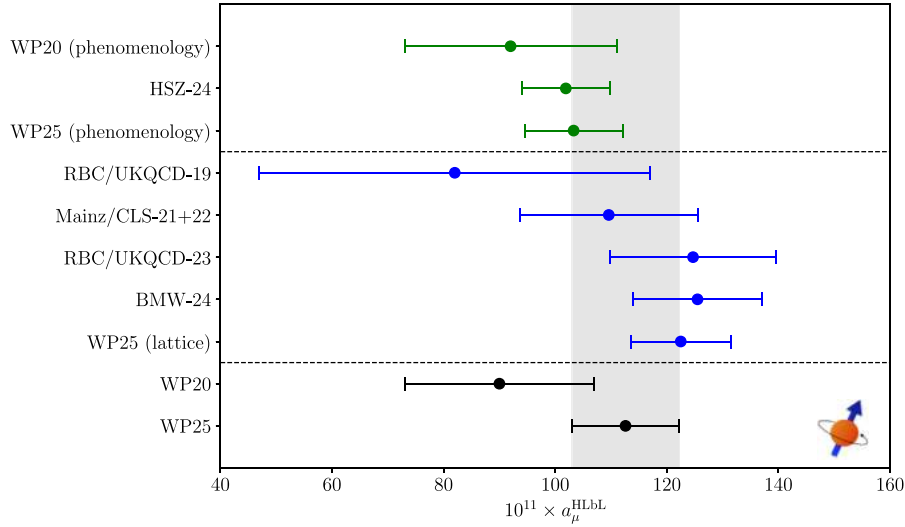


Fig. 82. Summary of HLbL evaluations, from data-driven methods (green), lattice QCD (blue), and combinations (black). The averages are from WP20 [1] and WP25, respectively, the other points refer to HSZ-24 [55,594], RBC/UKQCD-19 [59], Mainz/CLS-21+22 [60,61], RBC/UKQCD-23 [62], and BMW-24 [63].

Averaging the data-driven and lattice-QCD evaluations of the HLbL contribution, given in Eqs. (5.69) and (6.34), we obtain

$$a_{\mu}^{\text{HLbL}} = 112.6(9.6) \times 10^{-11} \quad (\text{phenomenology} + \text{lattice}), \quad (9.2)$$

where the uncertainty includes a scale factor $S = 1.5$. With this average, the NLO contribution in Eq. (5.70) slightly changes to $a_{\mu}^{\text{HLbL, NLO}}(\text{phenomenology} + \text{lattice}) = 2.8(6) \times 10^{-11}$, and the total HLbL contribution becomes

$$a_{\mu}^{\text{HLbL}} + a_{\mu}^{\text{HLbL, NLO}} = 115.5(9.9) \times 10^{-11} \quad (\text{phenomenology} + \text{lattice}). \quad (9.3)$$

Combining Eqs. (9.1) and (9.3) with the QED and EW contributions from Eqs. (7.27) and (8.12), we obtain for the final SM prediction

$$a_{\mu}^{\text{SM}} = 116\,592\,033(62) \times 10^{-11}, \quad (9.4)$$

which can be compared to the current experimental average [5–8,10–13]⁴⁸

$$a_{\mu}^{\text{exp}} = 116\,592\,071.5(14.5) \times 10^{-11}. \quad (9.5)$$

At the current level of precision there is no tension between the SM prediction and the experimental world average:

$$\Delta a_{\mu} \equiv a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}} = 38(63) \times 10^{-11}. \quad (9.6)$$

This marks a significant shift from the conclusions of WP20, which is driven by the developments relating to the HVP LO contribution, as can be seen in Table 33 and Fig. 83.

By comparing the uncertainties of Eqs. (9.4) and (9.5) it is apparent that the precision of the SM prediction must be improved by about a factor four to match the precision of the current experimental average after the final result from the E989 experiment at Fermilab. We expect progress on both data-driven and lattice methods applied to the hadronic contributions in the next few years. Resolving the tensions in the data-driven estimations of the HVP contribution is particularly important, and additional experimental results combined with further scrutiny of theory input such as from event generators should provide a path towards this goal. Further progress in the calculation of isospin-breaking corrections, from both data-driven and lattice-QCD methods, should enable a robust SM prediction from τ data as well. For lattice-QCD calculations of HVP continuing efforts by the world-wide lattice community are expected to yield further significant improvements in precision and, hopefully, even better consolidation thanks to a diversity of methods. The future focus will be, in particular, on more precise evaluations of isospin-breaking effects and the noisy contributions at long distances.

⁴⁸ This paper was posted on arXiv on May 28, 2025. Sections 2–8 and all numbers pertaining to the SM prediction have remained unchanged, but the experimental world average has been updated according to the E989 announcement on June 3, 2025 [8], and the description in Sections 0, 1 and 9 has been adapted accordingly. In particular, the experimental results in abstract, Table 1, Figs. 27, 40 and 83, and Section 9 have been updated.

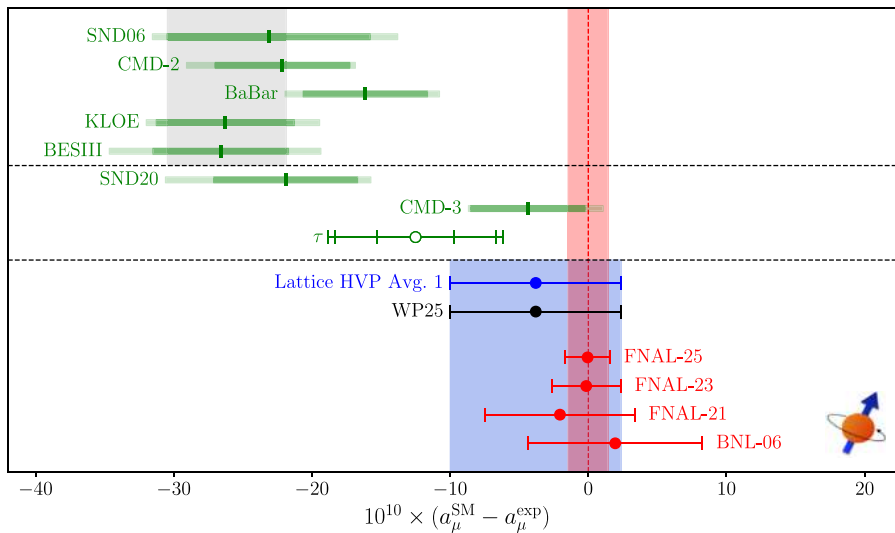


Fig. 83. Summary of the current SM prediction for a_μ in comparison to experiment (red band and data points). The final WP25 prediction is denoted in black and via the blue band, it derives from the LO HVP result defined by the lattice-QCD “Avg. 1” shown in blue, see Eq. (3.37). The gray band indicates the WP20 result, based on the e^+e^- experiments above the first dashed line. These experimental ranges, as well as the ones for SND20 and CMD-3 that appeared after WP20, are produced as in Fig. 27; they are meant to illustrate the current situation, but cannot be interpreted as uncertainties with a proper statistical meaning. The τ point refers to Eq. (2.23), the numerical results are collected in Table 5. In all cases except for the gray WP20 band the LO HVP results are combined with WP25 values for the remaining contributions, as summarized in Table 1. The figure has been updated after the announcement of the final results from the Fermilab experiment, including the corrections to the previous experimental points as detailed in Ref. [8].

The role of a_μ as a sensitive probe of the SM continues to evolve. We stress that, even though a consistent picture has emerged regarding lattice calculations of HVP, the case for a continued assessment of the situation remains very strong in view of the observed tensions among data-driven evaluations. New and existing data on e^+e^- hadronic cross sections from the main collaborations in the field, as well as new measurements of hadronic τ decays that will be performed at Belle II, will be crucial not only for resolving the situation but also for pushing the precision of the SM prediction for a_μ to that of the direct measurement. This must be complemented by new experimental efforts with completely different systematics, such as the MUonE experiment, aimed at measuring the LO HVP contribution, as well as an independent direct measurement of a_μ , which is the goal of the E34 experiment at J-PARC. The interplay of all these approaches, various experimental techniques and theoretical methods, may yield profound insights in the future, both regarding improved precision in the SM prediction and the potential role of physics beyond the SM. Finally, the subtleties in the evaluation of the SM prediction for a_μ will also become relevant for the anomalous magnetic moment of the electron, once the experimental tensions in the determination of the fine-structure constant are resolved.

In this second-edition White Paper, WP25, the Muon $g - 2$ Theory Initiative is presenting their new SM prediction for a_μ , which constitutes a major change from WP20. For the future, the Theory Initiative remains dedicated to continuing to support the wide-ranging efforts and to compile updated predictions.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We thank The University of Edinburgh for hosting the fifth plenary workshop [75]. This workshop was supported by the Higgs Centre for Theoretical Physics, the Edinburgh Parallel Computing Centre, the Institute for Particle Physics Phenomenology (UKRI grant ST/T001011/1), UKLFT (UKRI grant ST/T000813/1), and STRONG-2020. This workshop was also sponsored by Eviden (formerly ATOS), NVIDIA, and Storj. STRONG-2020 is a project part of the European Union’s Horizon 2020 research and innovation program with grant agreement no. 824093.

We thank the University of Bern for hosting the sixth plenary workshop [76]. This workshop was supported in part by the Albert Einstein Center for Fundamental Physics and the Swiss National Science Foundation (project nos. 200020_200553, 200021_200866, PCEFP2_181117, and PCEFP2_194272).

We thank KEK for hosting the seventh plenary workshop [77]. This workshop was supported in part by KEK Theory Center, KEK Institute of Particle and Nuclear Studies, the Kobayashi–Maskawa Institute and Flavor Physics International research center in Nagoya University, and the U.S.–Japan Science and Technology Cooperation Program in High Energy Physics. We also thank the Wilhelm and Else Heraeus Foundation, which generously supported the participation of early career scientists at the Simon Eidelman School on Muon Dipole Moments and Hadronic Effects. The school was organized at the Kobayashi–Maskawa Institute in Nagoya during the week preceding the seventh plenary workshop.

We thank M. Petschlies for discussions.

The work in this paper was supported in part by the Agence nationale pour la recherche (French National Research Agency, ANR) under contract ANR-22-CE31-0011 “HVP4NewPhys”, by the Austrian Science Fund (FWF) through grant DOI 10.55776/PAT2089624, 10.55776/PAT7221623, 10.55776/I3845, and the Doctoral Program Particles and Interactions, grant DOI 10.55776/W1252, by Beatriz-Galindo support, University of Huelva, Huelva, Spain, by CNPq (Brazilian Agency) grant no.308979/2021-4, by CNRS, by Conacyt, Mexico, by ICSC – Centro Nazionale di Ricerca in High Performance Computing, Big Data and Quantum Computing, funded by European Union – NextGenerationEU, by Coordinación de la Investigación Científica, Universidad Michoacana de San Nicolás de Hidalgo, Morelia, Mexico, grant 4.10, SECIHTI (Mexico), project CBF2023-2024-3544, by the Deutsche Forschungsgemeinschaft (German Research Foundation, DFG) through the Cluster of Excellence “Precision Physics, Fundamental Interactions and Structure of Matter” (PRISMA⁺ EXC 2118/1) funded within the German Excellence strategy (project no. 39083149), through the Collaborative Research Center 1660 “Hadrons and Nuclei as Discovery Tools”, through the Research Unit FOR5327 “Photon–photon interactions in the Standard Model and beyond: Exploiting the discovery potential from MESA to the LHC” (grant 458854507), through the Emmy Noether program (grant 449369623), through project HI 2048/1-3 (project no. 399400745), by the ERC grant MUON-101054515, by the European Union’s Horizon Europe Research and Innovation program under the Marie Skłodowska-Curie grant agreement no. 101106243, grant no. 824093 (H2020-INFRAIA-2018-1), and grant agreement no. 858199 (INTENSE), through the EuroHPC Joint Undertaking computing time allocated under grant EHPC-REG-2022R03-166, by the Excellence Initiative of Aix-Marseille University – A*MIDEX, a French “Investissements d’Avenir” program, through the Chaire d’Excellence program (AMX-22-RE-AB-052) and via grant AMX-22-RE-AB-052 “AMUalphaNP”, by FAPESP (São Paulo Research Foundation) grant nos. 2021/06756-6, 2020/15532-1, and 2022/02328-2, by FEDER UE through grants PID2023-147112NB-C21, through the “Unit of Excellence María de Maeztu 2020–2023” award to the Institute of Cosmos Sciences, grant CEX2019-000918-M, by Generalitat de Catalunya (AGAUR), Spain through grants 2021SGR01095 and 2021SGR00649, by the Serra Hùnter program, through the “PROMETEO” program grant CIPROM/2022/66, by Generalitat Valenciana (Spain) through the plan GenT program (CIDEIG/2023/12), through the “PROMETEO” program grant PROMETEO/2021/07, by the Istituto Nazionale di Fisica Nucleare (INFN), Italy, through research projects ENP (Exploring New Physics) and LQCD123, by the Japan Society for the Promotion of Science under grant nos. KAKENHI-16K05338, 20H05646, 20K03646, 20K03926, 20K03960, 20H05625, L23530, 22K21350, by Junta de Andalucía, Spain grant nos. POSTDOC_21_00136 and P18-FR-5057, by the Leverhulme Trust, United Kingdom, LIP-2021-014 (UK), by the Italian Ministry of University and Research (MUR) and the European Union (EU) – Next Generation EU, Mission 4, Component 1, PRIN 2022, CUP F53D23001480006, research project 2022TJFCYB – Nonperturbative aspects of fundamental interactions, in the Standard Model and beyond, by the Italian Ministero dell’Università e Ricerca (MUR) and European Union – Next Generation EU through the research grant 2022ENJMRS under the program PRIN 2022, by the Italian Ministero dell’Università e Ricerca (MUR) and European Union – Next Generation EU through the research grant no. 20225X52RA “MUS4GM2: Muon Scattering for $g-2$ ” under the program Prin 2022, by MICIU/AEI/10.13039/501100011033, by Spanish “Agencia Estatal de Investigación”, MCIN/AEI/10.13039/501100011033, through grant nos. PID2023-151418NB-I00, PID2020-114473GB-I00, PID2023-146220NB-I00, PID2020-114767GB-I00, PID2023-147072NB-I00, and CEX2023-001292-S, by Spanish Ministry of Science and Innovation (MICINN) grants PID2022-140440NB-C22, PID2023-146142NB-I00, by the Spanish MICIU (Ramon y Cajal program RYC2019-027605-I and project PID2022-136510NB-C31), by the National Natural Science Foundation of China (NSFC) under grant no.12125501, by the Natural Sciences and Engineering Research Council of Canada, Canada, by the “Rita Levi Montalcini” program for young researchers of the Italian Ministry of University and Research (MUR), The Royal Society (URF/R1/231503), by the Swiss National Science Foundation, Switzerland under grant nos. 200020_200553, 200020_208222, 200021_175761, PCEFP2_181117, PCEFP2_194272, TMCG-2_213690, 200020_207386, Ambizione program (grant PZ00P2_193383), by SECIHTI (Mexico), project CBF2023-2024-3226, by the UK Research and Innovation, Engineering and Physical Sciences Research Council, grant no.EP/X021971/1, by the UK Science and Technology Facilities Council (STFC) under grant nos. ST/S000925/, ST/T000600/1, ST/T000988/1, ST/X000494/1, ST/X000699/1, and ST/Y509759/1, by the U.S. Department of Energy, Office of Science, Office of High Energy Physics under award nos. DE-SC0010005, DE-SC0010120, DE-SC0010339, DE-SC0015655, DE-SC0013682, DE-SC0021147, and the “High Energy Physics Computing Traineeship for Lattice Gauge Theory” DE-SC0024053, by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics under award no. DE-FG02-00ER41132, and by the U.S. National Science Foundation under grant nos. PHY20-13064, PHY23-10571, PHY-2309135 (to the Kavli Institute for Theoretical Physics, KITP), OAC-2311430. This document was prepared in part using the resources of the Fermi National Accelerator Laboratory (Fermilab), a U.S. Department of Energy, Office of Science, Office of High Energy Physics HEP User Facility. Fermilab is managed by Fermi Forward Discovery Group, LLC, acting under United States Department of Energy contract no. 89243024CSC000002.

References

- [1] T. Aoyama, et al., Phys. Rep. 887 (2020) 1, <http://dx.doi.org/10.1016/j.physrep.2020.07.006>, arXiv:2006.04822 [hep-ph].
- [2] G. Charpak, F.J.M. Farley, R.L. Garwin, Phys. Lett. 1 (1962) 16, [http://dx.doi.org/10.1016/0031-9163\(62\)90263-9](http://dx.doi.org/10.1016/0031-9163(62)90263-9).
- [3] J. Bailey, W. Bartl, G. Von Bochmann, R.C.A. Brown, F.J.M. Farley, H. Joestlein, E. Picasso, R.W. Williams, Phys. Lett. B 28 (1968) 287, [http://dx.doi.org/10.1016/0031-9163\(62\)90263-9](http://dx.doi.org/10.1016/0031-9163(62)90263-9).
- [4] J. Bailey, et al., (CERN-Mainz-Daresbury), Nuclear Phys. B 150 (1979) 1, [http://dx.doi.org/10.1016/0370-2693\(68\)90261-X](http://dx.doi.org/10.1016/0370-2693(68)90261-X).
- [5] G.W. Bennett, et al., (Muon $g - 2$), Phys. Rev. D 73 (2006) 072003, <http://dx.doi.org/10.1103/PhysRevD.73.072003>, arXiv:hep-ex/0602035.
- [6] B. Abi, et al., (Muon $g - 2$), Phys. Rev. Lett. 126 (2021) 141801, <http://dx.doi.org/10.1103/PhysRevLett.126.141801>, arXiv:2104.03281 [hep-ex].
- [7] D.P. Aguillard, et al., (Muon $g - 2$), Phys. Rev. Lett. 131 (2023) 161802, <http://dx.doi.org/10.1103/PhysRevLett.131.161802>, arXiv:2308.06230 [hep-ex].
- [8] D.P. Aguillard, et al., (Muon $g - 2$), Phys. Rev. Lett. 135 (2025) 101802, <http://dx.doi.org/10.1103/7clf-sm2v>, arXiv:2506.03069 [hep-ex].
- [9] M. Abe, et al., PTEP 2019 (2019) 053C02, <http://dx.doi.org/10.1093/ptep/ptz030>, arXiv:1901.03047 [physics.ins-det].
- [10] D.P. Aguillard, et al., (Muon $g - 2$), Phys. Rev. D 110 (2024) 032009, <http://dx.doi.org/10.1103/PhysRevD.110.032009>, arXiv:2402.15410 [hep-ex].
- [11] T. Albahri, et al., (Muon $g - 2$), Phys. Rev. D 103 (2021) 072002, <http://dx.doi.org/10.1103/PhysRevD.103.072002>, arXiv:2104.03247 [hep-ex].
- [12] T. Albahri, et al., (Muon $g - 2$), Phys. Rev. A 103 (2021) 042208, <http://dx.doi.org/10.1103/PhysRevA.103.042208>, arXiv:2104.03201 [hep-ex].
- [13] T. Albahri, et al., (Muon $g - 2$), Phys. Rev. Accel. Beams 24 (2021) 044002, <http://dx.doi.org/10.1103/PhysRevAccelBeams.24.044002>, arXiv:2104.03240 [physics.acc-ph].
- [14] T. Blum, P.A. Boyle, V. Gülpers, T. Izubuchi, L. Jin, C. Jung, A. Jüttner, C. Lehner, A. Portelli, J.T. Tsang, (RBC and UKQCD), Phys. Rev. Lett. 121 (2018) 022003, <http://dx.doi.org/10.1103/PhysRevLett.121.022003>, arXiv:1801.07224 [hep-lat].
- [15] D. Giusti, V. Lubicz, G. Martinelli, F. Sanfilippo, S. Simula, (ETM), Phys. Rev. D 99 (2019) 114502, <http://dx.doi.org/10.1103/PhysRevD.99.114502>, arXiv:1901.10462 [hep-lat].
- [16] S. Borsányi, et al., Nature 593 (2021) 51, <http://dx.doi.org/10.1038/s41586-021-03418-1>, arXiv:2002.12347 [hep-lat].
- [17] C. Lehner, A.S. Meyer, Phys. Rev. D 101 (2020) 074515, <http://dx.doi.org/10.1103/PhysRevD.101.074515>, arXiv:2003.04177 [hep-lat].
- [18] G. Wang, T. Draper, K.-F. Liu, Y.-B. Yang, (χ QCD), Phys. Rev. D 107 (2023) 034513, <http://dx.doi.org/10.1103/PhysRevD.107.034513>, arXiv:2204.01280 [hep-lat].
- [19] C. Aubin, T. Blum, M. Golterman, S. Peris, Phys. Rev. D 106 (2022) 054503, <http://dx.doi.org/10.1103/PhysRevD.106.054503>, arXiv:2204.12256 [hep-lat].
- [20] M. Cè, et al., Phys. Rev. D 106 (2022) 114502, <http://dx.doi.org/10.1103/PhysRevD.106.114502>, arXiv:2206.06582 [hep-lat].
- [21] C. Alexandrou, et al., (ETM), Phys. Rev. D 107 (2023) 074506, <http://dx.doi.org/10.1103/PhysRevD.107.074506>, arXiv:2206.15084 [hep-lat].
- [22] T. Blum, et al., (RBC, UKQCD), Phys. Rev. D 108 (2023) 054507, <http://dx.doi.org/10.1103/PhysRevD.108.054507>, arXiv:2301.08696 [hep-lat].
- [23] S. Kuberski, M. Cè, G. von Hippel, H.B. Meyer, K. Ottnad, A. Risch, H. Wittig, JHEP 03 (2024) 172, [http://dx.doi.org/10.1007/JHEP03\(2024\)172](http://dx.doi.org/10.1007/JHEP03(2024)172), arXiv:2401.11895 [hep-lat].
- [24] A. Boccaletti, et al., 2024, arXiv:2407.10913 [hep-lat].
- [25] S. Spiegel, C. Lehner, Phys. Rev. D 111 (2025) 114517, <http://dx.doi.org/10.1103/mj3d-yq87>, arXiv:2410.17053 [hep-lat].
- [26] T. Blum, et al., (RBC and UKQCD), Phys. Rev. Lett. 134 (2025) 201901, <http://dx.doi.org/10.1103/PhysRevLett.134.201901>, arXiv:2410.20590 [hep-lat].
- [27] D. Djukanovic, G. von Hippel, S. Kuberski, H.B. Meyer, N. Miller, K. Ottnad, J. Parrino, A. Risch, H. Wittig, JHEP 04 (2025) 098, [http://dx.doi.org/10.1007/JHEP04\(2025\)098](http://dx.doi.org/10.1007/JHEP04(2025)098), arXiv:2411.07969 [hep-lat].
- [28] C. Alexandrou, et al., (ETM), Phys. Rev. D 111 (2025) 054502, <http://dx.doi.org/10.1103/PhysRevD.111.054502>, arXiv:2411.08852 [hep-lat].
- [29] A. Bazavov, et al., (Fermilab Lattice, HPQCD, MILC), Phys. Rev. D 111 (2025) 094508, <http://dx.doi.org/10.1103/PhysRevD.111.094508>, arXiv:2411.09656 [hep-lat].
- [30] A. Bazavov, et al., (Fermilab Lattice, HPQCD, MILC), Phys. Rev. Lett. 135 (2025) 011901, <http://dx.doi.org/10.1103/d583-yhfs>, arXiv:2412.18491 [hep-lat].
- [31] A. Keshavarzi, D. Nomura, T. Teubner, Phys. Rev. D 101 (2020) 014029, <http://dx.doi.org/10.1103/PhysRevD.101.014029>, arXiv:1911.00367 [hep-ph].
- [32] L. Di Luzio, A. Keshavarzi, A. Masiero, P. Paradisi, Phys. Rev. Lett. 134 (2025) 011902, <http://dx.doi.org/10.1103/PhysRevLett.134.011902>, arXiv:2408.01123 [hep-ph].
- [33] A. Kurz, T. Liu, P. Marquard, M. Steinhauser, Phys. Lett. B 734 (2014) 144, <http://dx.doi.org/10.1016/j.physletb.2014.05.043>, arXiv:1403.6400 [hep-ph].
- [34] G. Colangelo, M. Hoferichter, M. Procura, P. Stoffer, JHEP 09 (2015) 074, [http://dx.doi.org/10.1007/JHEP09\(2015\)074](http://dx.doi.org/10.1007/JHEP09(2015)074), arXiv:1506.01386 [hep-ph].
- [35] P. Masjuan, P. Sánchez-Puertas, Phys. Rev. D 95 (2017) 054026, <http://dx.doi.org/10.1103/PhysRevD.95.054026>, arXiv:1701.05829 [hep-ph].
- [36] G. Colangelo, M. Hoferichter, M. Procura, P. Stoffer, JHEP 04 (2017) 161, [http://dx.doi.org/10.1007/JHEP04\(2017\)161](http://dx.doi.org/10.1007/JHEP04(2017)161), arXiv:1702.07347 [hep-ph].
- [37] M. Hoferichter, B.-L. Hoid, B. Kubis, S. Leupold, S.P. Schneider, JHEP 10 (2018) 141, [http://dx.doi.org/10.1007/JHEP10\(2018\)141](http://dx.doi.org/10.1007/JHEP10(2018)141), arXiv:1808.04823 [hep-ph].
- [38] G. Eichmann, C.S. Fischer, E. Weil, R. Williams, Phys. Lett. B 797 (2019) 134855, <http://dx.doi.org/10.1016/j.physletb.2019.134855>, arXiv:1903.10844 [hep-ph]; Phys. Lett. B 799 (2019) 135029, Erratum.
- [39] J. Bijnens, N. Hermansson-Truedsson, A. Rodríguez-Sánchez, Phys. Lett. B 798 (2019) 134994, <http://dx.doi.org/10.1016/j.physletb.2019.134994>, arXiv:1908.03331 [hep-ph].
- [40] J. Leutgeb, A. Rebhan, Phys. Rev. D 101 (2020) 114015, <http://dx.doi.org/10.1103/PhysRevD.101.114015>, arXiv:1912.01596 [hep-ph].
- [41] L. Cappiello, O. Catà, G. D'Ambrosio, D. Greynat, A. Iyer, Phys. Rev. D 102 (2020) 016009, <http://dx.doi.org/10.1103/PhysRevD.102.016009>, arXiv:1912.02779 [hep-ph].
- [42] P. Masjuan, P. Roig, P. Sánchez-Puertas, J. Phys. G 49 (2022) 015002, <http://dx.doi.org/10.1088/1361-6471/ac3892>, arXiv:2005.11761 [hep-ph].
- [43] J. Bijnens, N. Hermansson-Truedsson, L. Laub, A. Rodríguez-Sánchez, JHEP 10 (2020) 203, [http://dx.doi.org/10.1007/JHEP10\(2020\)203](http://dx.doi.org/10.1007/JHEP10(2020)203), arXiv:2008.13487 [hep-ph].
- [44] J. Bijnens, N. Hermansson-Truedsson, L. Laub, A. Rodríguez-Sánchez, JHEP 04 (2021) 240, [http://dx.doi.org/10.1007/JHEP04\(2021\)240](http://dx.doi.org/10.1007/JHEP04(2021)240), arXiv:2101.09169 [hep-ph].
- [45] I. Danilkin, M. Hoferichter, P. Stoffer, Phys. Lett. B 820 (2021) 136502, <http://dx.doi.org/10.1016/j.physletb.2021.136502>, arXiv:2105.01666 [hep-ph].
- [46] D. Stamen, D. Hariharan, M. Hoferichter, B. Kubis, P. Stoffer, Eur. Phys. J. C 82 (2022) 432, <http://dx.doi.org/10.1140/epjc/s10052-022-10348-3>, arXiv:2202.11106 [hep-ph].
- [47] J. Leutgeb, J. Mager, A. Rebhan, Phys. Rev. D 107 (2023) 054021, <http://dx.doi.org/10.1103/PhysRevD.107.054021>, arXiv:2211.16562 [hep-ph].
- [48] M. Hoferichter, B. Kubis, M. Zanke, JHEP 08 (2023) 209, [http://dx.doi.org/10.1007/JHEP08\(2023\)209](http://dx.doi.org/10.1007/JHEP08(2023)209), arXiv:2307.14413 [hep-ph].
- [49] M. Hoferichter, P. Stoffer, M. Zillinger, JHEP 04 (2024) 092, [http://dx.doi.org/10.1007/JHEP04\(2024\)092](http://dx.doi.org/10.1007/JHEP04(2024)092), arXiv:2402.14060 [hep-ph].
- [50] E.J. Estrada, S. González-Solís, A. Guevara, P. Roig, JHEP 12 (2024) 203, [http://dx.doi.org/10.1007/JHEP12\(2024\)203](http://dx.doi.org/10.1007/JHEP12(2024)203), arXiv:2409.10503 [hep-ph].

- [51] J. Lütke, M. Procura, P. Stoffer, JHEP 04 (2025) 130, [http://dx.doi.org/10.1007/JHEP04\(2025\)130](http://dx.doi.org/10.1007/JHEP04(2025)130), arXiv:2410.11946 [hep-ph].
- [52] O. Deineka, I. Daniilkin, M. Vanderhaeghen, Phys. Rev. D 111 (2025) 034009, <http://dx.doi.org/10.1103/PhysRevD.111.034009>, arXiv:2410.12894 [hep-ph].
- [53] G. Eichmann, C.S. Fischer, T. Haeuser, O. Regenfelder, Eur. Phys. J. C 85 (2025) 445, <http://dx.doi.org/10.1140/epjc/s10052-025-14055-7>, arXiv:2411.05652 [hep-ph].
- [54] J. Bijnens, N. Hermansson-Truedsson, A. Rodríguez-Sánchez, JHEP 03 (2025) 094, [http://dx.doi.org/10.1007/JHEP03\(2025\)094](http://dx.doi.org/10.1007/JHEP03(2025)094), arXiv:2411.09578 [hep-ph].
- [55] M. Hoferichter, P. Stoffer, M. Zillinger, JHEP 02 (2025) 121, [http://dx.doi.org/10.1007/JHEP02\(2025\)121](http://dx.doi.org/10.1007/JHEP02(2025)121), arXiv:2412.00178 [hep-ph].
- [56] S. Holz, M. Hoferichter, B.-L. Hoid, B. Kubis, JHEP 04 (2025) 147, [http://dx.doi.org/10.1007/JHEP04\(2025\)147](http://dx.doi.org/10.1007/JHEP04(2025)147), arXiv:2412.16281 [hep-ph].
- [57] L. Cappiello, J. Leutgeb, J. Mager, A. Rebhan, JHEP 07 (2025) 033, [http://dx.doi.org/10.1007/JHEP07\(2025\)033](http://dx.doi.org/10.1007/JHEP07(2025)033), arXiv:2501.09699 [hep-ph].
- [58] G. Colangelo, M. Hoferichter, A. Nyffeler, M. Passera, P. Stoffer, Phys. Lett. B 735 (2014) 90, <http://dx.doi.org/10.1016/j.physletb.2014.06.012>, arXiv:1403.7512 [hep-ph].
- [59] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner, Phys. Rev. Lett. 124 (2020) 132002, <http://dx.doi.org/10.1103/PhysRevLett.124.132002>, arXiv:1911.08123 [hep-lat].
- [60] E.-H. Chao, R.J. Hudspeth, A. Gérardin, J.R. Green, H.B. Meyer, K. Ottnad, Eur. Phys. J. C 81 (2021) 651, <http://dx.doi.org/10.1140/epjc/s10052-021-09455-4>, arXiv:2104.02632 [hep-lat].
- [61] E.-H. Chao, R.J. Hudspeth, A. Gérardin, J.R. Green, H.B. Meyer, Eur. Phys. J. C 82 (2022) 664, <http://dx.doi.org/10.1140/epjc/s10052-022-10589-2>, arXiv:2204.08844 [hep-lat].
- [62] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner, C. Tu, (RBC, UKQCD), Phys. Rev. D 111 (2025) 014501, <http://dx.doi.org/10.1103/PhysRevD.111.014501>, arXiv:2304.04423 [hep-lat].
- [63] Z. Fodor, A. Gérardin, L. Lellouch, K.K. Szabó, B.C. Toth, C. Zimmermann, Phys. Rev. D 111 (2025) 114509, <http://dx.doi.org/10.1103/wdrk-7nrt>, arXiv:2411.11719 [hep-lat].
- [64] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Phys. Rev. Lett. 109 (2012) 111808, <http://dx.doi.org/10.1103/PhysRevLett.109.111808>, arXiv:1205.5370 [hep-ph].
- [65] S. Volkov, Phys. Rev. D 100 (2019) 096004, <http://dx.doi.org/10.1103/PhysRevD.100.096004>, arXiv:1909.08015 [hep-ph].
- [66] S. Volkov, Phys. Rev. D 110 (2024) 036001, <http://dx.doi.org/10.1103/PhysRevD.110.036001>, arXiv:2404.00649 [hep-ph].
- [67] T. Aoyama, M. Hayakawa, A. Hirayama, M. Nio, Phys. Rev. D 111 (2025) L031902, <http://dx.doi.org/10.1103/PhysRevD.111.L031902>, arXiv:2412.06473 [hep-ph].
- [68] R.H. Parker, C. Yu, W. Zhong, B. Estey, H. Müller, Science 360 (2018) 191, <http://dx.doi.org/10.1126/science.aap7706>, arXiv:1812.04130 [physics.atom-ph].
- [69] L. Morel, Z. Yao, P. Cladé, S. Guellati-Khélifa, Nature 588 (2020) 61, <http://dx.doi.org/10.1038/s41586-020-2964-7>.
- [70] X. Fan, T.G. Myers, B.A.D. Sukra, G. Gabrielse, Phys. Rev. Lett. 130 (2023) 071801, <http://dx.doi.org/10.1103/PhysRevLett.130.071801>, arXiv:2209.13084 [physics.atom-ph].
- [71] A. Czarnecki, W.J. Marciano, A. Vainshtein, Phys. Rev. D 67 (2003) 073006, <http://dx.doi.org/10.1103/PhysRevD.67.073006>, arXiv:hep-ph/0212229 [hep-ph]; Phys. Rev. D 73 (2006) 119901, Erratum.
- [72] C. Gnendiger, D. Stöckinger, H. Stöckinger-Kim, Phys. Rev. D 88 (2013) 053005, <http://dx.doi.org/10.1103/PhysRevD.88.053005>, arXiv:1306.5546 [hep-ph].
- [73] M. Hoferichter, J. Lütke, L. Naterop, M. Procura, P. Stoffer, Phys. Rev. Lett. 134 (2025) 201801, <http://dx.doi.org/10.1103/PhysRevLett.134.201801>, arXiv:2503.04883 [hep-ph].
- [74] Fourth Plenary Workshop of the Muon $g - 2$ Theory Initiative, 2021, online hosted by KEK, June 28–July 2, <https://agenda.hepl.phys.nagoya-u.ac.jp/indico/conferenceTimeTable.py?confId=1691#all>.
- [75] Fifth Plenary Workshop of the Muon $g - 2$ Theory Initiative, 2022, held at the University of Edinburgh, Edinburgh, United Kingdom, September 5–9, <https://indico.ph.ed.ac.uk/event/112/>.
- [76] Sixth Plenary Workshop of the Muon $g - 2$ Theory Initiative, 2023, held at the University of Bern, Bern, Switzerland, September 4–8, <https://indico.cern.ch/event/1258310/>.
- [77] Seventh Plenary Workshop of the Muon $g - 2$ Theory Initiative, 2024, held at KEK, Tsukuba, Japan, September 9–13, <https://conference-indico.kek.jp/event/257/>.
- [78] The hadronic vacuum polarization from lattice QCD at high precision, 2020, hosted online, November 16–20, <https://indico.global/event/6556/>.
- [79] Scientific seminar and discussion on new CMD-3 result, 2023, hosted online, March 27, <https://indico.fnal.gov/event/59052/>.
- [80] Second CMD-3 discussion meeting, 2023, hosted online, July 20, <https://indico.jlab.in2p3.fr/event/9697/>.
- [81] Muon $g - 2$ Theory Initiative Spring 2024 meeting, 2024, hosted online, April 15–23, <https://indico.global/event/1373/>.
- [82] Virtual mini workshop on τ decays, 2024, 2025, hosted online, November 8, December 9, January 30, <https://indico.global/event/1378/>.
- [83] Eighth Plenary Workshop of the Muon $g - 2$ Theory Initiative, 2025, to be held at IJCLab, Paris, France, September 8–12, <https://indico.jlab.in2p3.fr/event/11652/>.
- [84] G. Colangelo, et al., 2022, arXiv:2203.15810 [hep-ph].
- [85] C. Bouchiat, L. Michel, J. Phys. Radium 22 (1961) 121, <http://dx.doi.org/10.1051/jphysrad:01961002202012101>.
- [86] S.J. Brodsky, E. de Rafael, Phys. Rev. 168 (1968) 1620, <http://dx.doi.org/10.1103/PhysRev.168.1620>.
- [87] B.E. Lautrup, E. de Rafael, Phys. Rev. 174 (1968) 1835, <http://dx.doi.org/10.1103/PhysRev.174.1835>.
- [88] M. Gourdin, E. de Rafael, Nuclear Phys. B 10 (1969) 667, [http://dx.doi.org/10.1016/0550-3213\(69\)90333-2](http://dx.doi.org/10.1016/0550-3213(69)90333-2).
- [89] V.M. Aulchenko, et al., CMD-2M Detector Project, Tech. Rep., BINP, 2001, <https://inspirehep.net/literature/579332>.
- [90] B. Khazin, Nucl. Phys. B Proc. Suppl. 181–182 (2008) 376, <http://dx.doi.org/10.1016/j.nuclphysbps.2008.09.068>.
- [91] M.N. Achasov, et al., Nucl. Instrum. Meth. A 598 (2009) 31, <http://dx.doi.org/10.1016/j.nima.2008.08.012>.
- [92] P.Y. Shatunov, et al., Phys. Part. Nucl. Lett. 13 (2016) 995, <http://dx.doi.org/10.1134/S154747711607044X>.
- [93] D. Schwartz, et al., PoS ICHEP2016 (2016) 054, <http://dx.doi.org/10.22323/1.282.0054>.
- [94] F.V. Ignatov, et al., (CMD-3), EPJ Web Conf. 212 (2019) 04001, <http://dx.doi.org/10.1051/epjconf/201921204001>.
- [95] F.V. Ignatov, et al., (CMD-3), Phys. Rev. Lett. 132 (2024) 231903, <http://dx.doi.org/10.1103/PhysRevLett.132.231903>, arXiv:2309.12910 [hep-ex].
- [96] F.V. Ignatov, et al., (CMD-3), Phys. Rev. D 109 (2024) 112002, <http://dx.doi.org/10.1103/PhysRevD.109.112002>, arXiv:2302.08834 [hep-ex].
- [97] R.R. Akhmetshin, et al., (CMD-3), Phys. Lett. B 768 (2017) 345, <http://dx.doi.org/10.1016/j.physletb.2017.03.022>, arXiv:1612.04483 [hep-ex].
- [98] R.R. Akhmetshin, et al., (CMD-3), Phys. Lett. B 723 (2013) 82, <http://dx.doi.org/10.1016/j.physletb.2013.04.065>, arXiv:1302.0053 [hep-ex].
- [99] R.R. Akhmetshin, et al., (CMD-3), Phys. Lett. B 792 (2019) 419, <http://dx.doi.org/10.1016/j.physletb.2019.04.007>, arXiv:1902.06449 [hep-ex].
- [100] S.S. Gribov, et al., (CMD-3), JHEP 01 (2020) 112, [http://dx.doi.org/10.1007/JHEP01\(2020\)112](http://dx.doi.org/10.1007/JHEP01(2020)112), arXiv:1907.08002 [hep-ex].
- [101] R.R. Akhmetshin, et al., (CMD-3), Phys. Lett. B 773 (2017) 150, <http://dx.doi.org/10.1016/j.physletb.2017.08.019>, arXiv:1706.06267 [hep-ex].
- [102] E.A. Kozyrev, et al., (CMD-3), Phys. Lett. B 779 (2018) 64, <http://dx.doi.org/10.1016/j.physletb.2018.01.079>, arXiv:1710.02989 [hep-ex].
- [103] D.N. Shemyakin, et al., (CMD-3), Phys. Lett. B 756 (2016) 153, <http://dx.doi.org/10.1016/j.physletb.2016.02.072>, arXiv:1510.00654 [hep-ex].
- [104] E.A. Kozyrev, et al., (CMD-3), Phys. Lett. B 760 (2016) 314, <http://dx.doi.org/10.1016/j.physletb.2016.07.003>, arXiv:1604.02981 [hep-ex].

- [105] R.R. Akhmetshin, et al., (CMD-3), Phys. Lett. B 836 (2023) 137606, <http://dx.doi.org/10.1016/j.physletb.2022.137606>, arXiv:2207.04615 [hep-ex].
- [106] R.R. Akhmetshin, et al., (CMD-3), Phys. Lett. B 804 (2020) 135380, <http://dx.doi.org/10.1016/j.physletb.2020.135380>, arXiv:1912.05751 [hep-ex].
- [107] V.L. Ivanov, et al., (CMD-3), Phys. Lett. B 798 (2019) 134946, <http://dx.doi.org/10.1016/j.physletb.2019.134946>, arXiv:1906.08006 [hep-ex].
- [108] R.R. Akhmetshin, et al., (CMD-3), Phys. Lett. B 759 (2016) 634, <http://dx.doi.org/10.1016/j.physletb.2016.04.048>, arXiv:1507.08013 [hep-ex].
- [109] M.N. Achasov, et al., (SND), JHEP 01 (2021) 113, [http://dx.doi.org/10.1007/JHEP01\(2021\)113](http://dx.doi.org/10.1007/JHEP01(2021)113), arXiv:2004.00263 [hep-ex].
- [110] V.L. Auslander, G.I. Budker, J.N. Pestov, V.A. Sidorov, A.N. Skrinsky, A.G. Khabakhpashev, Phys. Lett. B 25 (1967) 433, [http://dx.doi.org/10.1016/0370-2693\(67\)90169-4](http://dx.doi.org/10.1016/0370-2693(67)90169-4).
- [111] V.L. Auslander, G.I. Budker, E.V. Pakhtusova, Y.N. Pestov, V.A. Sidorov, A.N. Skrinsky, A.G. Khabakhpashev, Yad. Fiz. 9 (1969) 114, <https://inspirehep.net/literature/57008>.
- [112] V.E. Balakin, G.I. Budker, E.V. Pakhtusova, V.A. Sidorov, A.N. Skrinsky, G.M. Tumaiyin, A.G. Khabakhpashev, Phys. Lett. B 34 (1971) 328, [http://dx.doi.org/10.1016/0370-2693\(71\)90617-4](http://dx.doi.org/10.1016/0370-2693(71)90617-4).
- [113] V.E. Balakin, G.I. Budker, L.M. Kurdadze, A.P. Onuchin, E.V. Pakhtusova, S.I. Serednyakov, V.A. Sidorov, A.N. Skrinsky, Phys. Lett. B 41 (1972) 205, [http://dx.doi.org/10.1016/0370-2693\(72\)90462-5](http://dx.doi.org/10.1016/0370-2693(72)90462-5).
- [114] D. Benakass, G. Cosme, B. Jean-Marie, S. Jullian, F. Laplanche, J. LeFrancois, A.D. Liberman, G. Parrou, J.P. Repellin, G. Sauvage, Phys. Lett. B 39 (1972) 289, [http://dx.doi.org/10.1016/0370-2693\(72\)90799-X](http://dx.doi.org/10.1016/0370-2693(72)90799-X).
- [115] G. Cosme, et al., Measurement of the electron-positron annihilation cross-section into $\pi^+\pi^-$ at the energies 915 MeV, 990 MeV, and 1076 MeV, 1976, <https://inspirehep.net/literature/109771>.
- [116] D. Bollini, P. Giusti, T. Massam, L. Monari, F. Palmonari, G. Valenti, A. Zichichi, Lett. Nuovo Cimento 14 (1975) 418, <http://dx.doi.org/10.1007/BF02746868>.
- [117] A. Quenzer, et al., Phys. Lett. B 76 (1978) 512, [http://dx.doi.org/10.1016/0370-2693\(78\)90918-8](http://dx.doi.org/10.1016/0370-2693(78)90918-8).
- [118] B. Esposito, et al., Phys. Lett. B 67 (1977) 239, [http://dx.doi.org/10.1016/0370-2693\(77\)90113-7](http://dx.doi.org/10.1016/0370-2693(77)90113-7).
- [119] B. Esposito, et al., Lett. Nuovo Cimento 28 (1980) 337, <http://dx.doi.org/10.1007/BF02804624>.
- [120] I.B. Vasserman, P.M. Ivanov, G.Y. Kezerashvili, I.A. Koop, A.P. Lysenko, Y.N. Pestov, A.N. Skrinsky, G.V. Fedotov, Y.M. Shatunov, Sov. J. Nucl. Phys. 33 (1981) 368, <https://inspirehep.net/literature/167191>; Yad. Fiz. 33 (1981) 709.
- [121] S.R. Amendolia, et al., Phys. Lett. B 138 (1984) 454, [http://dx.doi.org/10.1016/0370-2693\(84\)91938-5](http://dx.doi.org/10.1016/0370-2693(84)91938-5).
- [122] D. Bisello, et al., (DM2), Phys. Lett. B 220 (1989) 321, [http://dx.doi.org/10.1016/0370-2693\(89\)90060-9](http://dx.doi.org/10.1016/0370-2693(89)90060-9).
- [123] L.M. Barkov, et al., Nuclear Phys. B 256 (1985) 365, [http://dx.doi.org/10.1016/0550-3213\(85\)90399-2](http://dx.doi.org/10.1016/0550-3213(85)90399-2).
- [124] R.R. Akhmetshin, et al., (CMD-2), Phys. Lett. B 527 (2002) 161, [http://dx.doi.org/10.1016/S0370-2693\(02\)01168-1](http://dx.doi.org/10.1016/S0370-2693(02)01168-1), arXiv:hep-ex/0112031.
- [125] R.R. Akhmetshin, et al., (CMD-2), Phys. Lett. B 578 (2004) 285, <http://dx.doi.org/10.1016/j.physletb.2003.10.108>, arXiv:hep-ex/0308008.
- [126] V.M. Aul'chenko, et al., (CMD-2), JETP Lett. 82 (2005) 743, <http://dx.doi.org/10.1134/1.2175241>, arXiv:hep-ex/0603021.
- [127] R.R. Akhmetshin, et al., (CMD-2), Phys. Lett. B 648 (2007) 28, <http://dx.doi.org/10.1016/j.physletb.2007.01.073>, arXiv:hep-ex/0610021.
- [128] V.M. Aul'chenko, et al., (CMD-2), JETP Lett. 84 (2006) 413, <http://dx.doi.org/10.1134/S0021364006200021>, arXiv:hep-ex/0610016.
- [129] M.N. Achasov, et al., (SND), J. Exp. Theor. Phys. 101 (2005) 1053, <http://dx.doi.org/10.1134/1.2163921>, arXiv:hep-ex/0506076 [hep-ex]; Zh. Eksp. Teor. Fiz. 128 (2005) 1201.
- [130] M.N. Achasov, et al., (SND), J. Exp. Theor. Phys. 103 (2006) 380, <http://dx.doi.org/10.1134/S106377610609007X>, arXiv:hep-ex/0605013 [hep-ex]; Zh. Eksp. Teor. Fiz. 130 (2006) 437.
- [131] A. Aloisio, et al., (KLOE), Phys. Lett. B 606 (2005) 12, <http://dx.doi.org/10.1016/j.physletb.2004.11.068>, arXiv:hep-ex/0407048.
- [132] F. Ambrosino, et al., (KLOE), Phys. Lett. B 670 (2009) 285, <http://dx.doi.org/10.1016/j.physletb.2008.10.060>, arXiv:0809.3950 [hep-ex].
- [133] F. Ambrosino, et al., (KLOE), Phys. Lett. B 700 (2011) 102, <http://dx.doi.org/10.1016/j.physletb.2011.04.055>, arXiv:1006.5313 [hep-ex].
- [134] D. Babusci, et al., (KLOE), Phys. Lett. B 720 (2013) 336, <http://dx.doi.org/10.1016/j.physletb.2013.02.029>, arXiv:1212.4524 [hep-ex].
- [135] A. Anastasi, et al., (KLOE-2), JHEP 03 (2018) 173, [http://dx.doi.org/10.1007/JHEP03\(2018\)173](http://dx.doi.org/10.1007/JHEP03(2018)173), arXiv:1711.03085 [hep-ex].
- [136] T.K. Pedlar, et al., (CLEO), Phys. Rev. Lett. 95 (2005) 261803, <http://dx.doi.org/10.1103/PhysRevLett.95.261803>, arXiv:hep-ex/0510005.
- [137] K.K. Seth, S. Dobb, Z. Metreveli, A. Tomaradze, T. Xiao, G. Bonvicini, Phys. Rev. Lett. 110 (2013) 022002, <http://dx.doi.org/10.1103/PhysRevLett.110.022002>, arXiv:1210.1596 [hep-ex].
- [138] T. Xiao, S. Dobb, A. Tomaradze, K.K. Seth, G. Bonvicini, Phys. Rev. D 97 (2018) 032012, <http://dx.doi.org/10.1103/PhysRevD.97.032012>, arXiv:1712.04530 [hep-ex].
- [139] J.P. Lees, et al., (BABAR), Phys. Rev. D 86 (2012) 032013, <http://dx.doi.org/10.1103/PhysRevD.86.032013>, arXiv:1205.2228 [hep-ex].
- [140] M. Ablikim, et al., (BESIII), Phys. Lett. B 753 (2016) 629, <http://dx.doi.org/10.1016/j.physletb.2015.11.043>, arXiv:1507.08188 [hep-ex]; Phys. Lett. B 812 (2021) 135982, Erratum.
- [141] F. Ignatov, R.N. Lee, Phys. Lett. B 833 (2022) 137283, <http://dx.doi.org/10.1016/j.physletb.2022.137283>, arXiv:2204.12235 [hep-ph].
- [142] G. Colangelo, M. Hoferichter, J. Monnard, J. Ruiz de Elvira, JHEP 08 (2022) 295, [http://dx.doi.org/10.1007/JHEP08\(2022\)295](http://dx.doi.org/10.1007/JHEP08(2022)295), arXiv:2207.03495 [hep-ph]; JHEP 03 (2025) 217, Erratum.
- [143] E. Budassi, C.M. Carloni Calame, M. Ghilardi, A. Gurgone, G. Montagna, M. Moretti, O. Nicosini, F. Piccinini, F.P. Ucci, JHEP 05 (2025) 196, [http://dx.doi.org/10.1007/JHEP05\(2025\)196](http://dx.doi.org/10.1007/JHEP05(2025)196), arXiv:2409.03469 [hep-ph].
- [144] G. Abbiendi, et al., STRONG2020 virtual workshop space-like and time-like determination of the hadronic leading order contribution to the Muon $g - 2$, 2022, arXiv:2201.12102 [hep-ph].
- [145] R. Aliberti, et al., SciPost Phys. Comm. Rep. 9 (2025) 1, <http://dx.doi.org/10.21468/SciPostPhysCommRep.9>, arXiv:2410.22882 [hep-ph].
- [146] T.P. Leplumey, P. Stoffer, 2025, arXiv:2501.09643v1 [hep-ph].
- [147] M.N. Achasov, et al., (SND), Phys. Rev. D 110 (2024) 072001, <http://dx.doi.org/10.1103/PhysRevD.110.072001>, arXiv:2407.15140 [hep-ex].
- [148] M.N. Achasov, et al., (SND), Phys. At. Nucl. 87 (2024) 747, <http://dx.doi.org/10.1134/S106377882400728>.
- [149] J.P. Lees, et al., (BABAR), Phys. Rev. D 104 (2021) 112003, <http://dx.doi.org/10.1103/PhysRevD.104.112003>, arXiv:2110.00520 [hep-ex].
- [150] M.N. Achasov, et al., (SND), Eur. Phys. J. C 80 (2020) 1139, <http://dx.doi.org/10.1140/epjc/s10052-020-08719-9>, arXiv:2007.04527 [hep-ex].
- [151] B. Aubert, et al., (BABAR), Phys. Rev. D 77 (2008) 092002, <http://dx.doi.org/10.1103/PhysRevD.77.092002>, arXiv:0710.4451 [hep-ex].
- [152] M.N. Achasov, et al., (SND), Eur. Phys. J. C 82 (2022) 761, <http://dx.doi.org/10.1140/epjc/s10052-022-10696-0>, arXiv:2206.13047 [hep-ex].
- [153] M.N. Achasov, et al., (SND), Phys. At. Nucl. 86 (2023) 1165, <http://dx.doi.org/10.1134/S1063778823060054>, arXiv:2309.05241 [hep-ex].
- [154] M.N. Achasov, et al., (SND), Phys. At. Nucl. 87 (2024) 604, <http://dx.doi.org/10.1134/S106377882400637>, arXiv:2407.15308 [hep-ex].
- [155] M.N. Achasov, et al., (SND), Eur. Phys. J. C 80 (2020) 1008, <http://dx.doi.org/10.1140/epjc/s10052-020-08556-w>, arXiv:2006.05465 [hep-ex].
- [156] M.N. Achasov, et al., (SND), Eur. Phys. J. C 82 (2022) 168, <http://dx.doi.org/10.1140/epjc/s10052-022-10102-9>, arXiv:2110.05845 [hep-ex].
- [157] M.N. Achasov, et al., (SND), Phys. At. Nucl. 86 (2023) 1157, <http://dx.doi.org/10.1134/S1063778823060042>, arXiv:2306.12734 [hep-ex].
- [158] M.N. Achasov, et al., (SND), Phys. Rev. D 110 (2024) 072004, <http://dx.doi.org/10.1103/PhysRevD.110.072004>, arXiv:2406.17357 [hep-ex].
- [159] G. Venanzoni, (KLOE-2), EPJ Web Conf. 166 (2018) 00021, <http://dx.doi.org/10.1051/epjconf/201816600021>, arXiv:1705.10365 [hep-ex].
- [160] H. Czyz, J.H. Kühn, G. Rodrigo, PHOKHARA Monte Carlo event generator, 2020, <https://looptreedom.csc.es/phokhara/>.
- [161] S.E. Müller, The PHOKHARA Monte Carlo event generator and its use in the KLOE analyses, 2023, Sixth Plenary Workshop of the Muon $g - 2$ Theory Initiative, Bern, https://indico.cern.ch/event/1258310/contributions/5559977/attachments/2708102/4701921/smueller_Bern_TL_workshop.pdf.

- [162] J. Gasser, H. Leutwyler, Ann. Phys. 158 (1984) 142, [http://dx.doi.org/10.1016/0003-4916\(84\)90242-2](http://dx.doi.org/10.1016/0003-4916(84)90242-2).
- [163] S. Actis, et al., (Working Group on Radiative Corrections, Monte Carlo Generators for Low Energies), Eur. Phys. J. C 66 (2010) 585, <http://dx.doi.org/10.1140/epjc/s10052-010-1251-4>, arXiv:0912.0749 [hep-ph].
- [164] F. Campanario, H. Czyż, J. Gluza, T. Jeliński, G. Rodrigo, S. Tracz, D. Zhuridov, Phys. Rev. D 100 (2019) 076004, <http://dx.doi.org/10.1103/PhysRevD.100.076004>, arXiv:1903.10197 [hep-ph].
- [165] G. Venanzoni, Investigation of radiative corrections with KLOE 2π analysis, 2024, Seventh Plenary Workshop of the Muon $g - 2$ Theory Initiative, KEK, <https://conference-indico.kek.jp/event/257/contributions/5777/>.
- [166] M. Davier, A. Hoecker, A.-M. Lutz, B. Malaescu, Z. Zhang, Eur. Phys. J. C 84 (2024) 721, <http://dx.doi.org/10.1140/epjc/s10052-024-12964-7>, arXiv:2312.02053 [hep-ph].
- [167] S. Jadach, Acta Phys. Pol. B 36 (2005) 2387, arXiv:hep-ph/0506180 [hep-ph], <https://www.actaphys.uj.edu.pl/fulltext?series=Reg&vol=36&page=2387>.
- [168] E. Zaid, Status report on the KLOE $\pi\pi\gamma/\mu\mu\gamma$ analysis, 2024, Seventh Plenary Workshop of the Muon $g - 2$ Theory Initiative, KEK, <https://conference-indico.kek.jp/event/257/contributions/5763/>.
- [169] L. Punzi, Towards a high precision measurement of $\sigma(e^+e^- \rightarrow \pi^+\pi^-\gamma)$ and the two-pion contribution to the muon $g - 2$ with the KLOE Detector (Master's thesis), Università di Pisa, 2023, <https://etd.adm.unipi.it/t/etd-10052023-111842>.
- [170] B. Cao, X.L. Kang, (KLOE-2), 18th International Conference on Hadron Spectroscopy and Structure, 2020, pp. 701–704, http://dx.doi.org/10.1142/9789811219313_0120.
- [171] J.P. Lees, et al., (BABAR), Phys. Rev. D 103 (2021) 092001, <http://dx.doi.org/10.1103/PhysRevD.103.092001>, arXiv:2102.01314 [hep-ex].
- [172] J.P. Lees, et al., (BABAR), Phys. Rev. D 104 (2021) 112004, <http://dx.doi.org/10.1103/PhysRevD.104.112004>, arXiv:2110.00823 [hep-ex].
- [173] J.P. Lees, et al., (BABAR), Phys. Rev. D 107 (2023) 072001, <http://dx.doi.org/10.1103/PhysRevD.107.072001>, arXiv:2207.10340 [hep-ex].
- [174] B. Aubert, et al., (BABAR), Phys. Rev. Lett. 103 (2009) 231801, <http://dx.doi.org/10.1103/PhysRevLett.103.231801>, arXiv:0908.3589 [hep-ex].
- [175] M. Davier, e^+e^- results from BABAR and implications for the muon $g - 2$, 2017, First Workshop of the Muon $g - 2$ Theory Initiative, Fermilab, <https://indico.fnal.gov/event/13795/session/10/contribution/47/material/slides/0.pdf>.
- [176] J.P. Lees, et al., (BABAR), Phys. Rev. D 108 (2023) L111103, <http://dx.doi.org/10.1103/PhysRevD.108.L111103>, arXiv:2308.05233 [hep-ex].
- [177] C.M. Carloni Calame, G. Montagna, O. Nicosini, F. Piccinini, Nucl. Phys. B Proc. Suppl. 131 (2004) 48, <http://dx.doi.org/10.1016/j.nuclphysbps.2004.02.008>, arXiv:hep-ph/0312014.
- [178] S. Jadach, B.F.L. Ward, Z. Wąs, Phys. Rev. D 63 (2001) 113009, <http://dx.doi.org/10.1103/PhysRevD.63.113009>, arXiv:hep-ph/0006359.
- [179] M. Ablikim, et al., (BESIII), Phys. Rev. Lett. 128 (2022) 062004, <http://dx.doi.org/10.1103/PhysRevLett.128.062004>, arXiv:2112.11728 [hep-ex].
- [180] B. Andersson, The Lund Model, vol. 7, Cambridge University Press, 1998, <http://dx.doi.org/10.1017/9781009401296>.
- [181] B. Andersson, H. m. Hu, 1999, arXiv:hep-ph/9910285.
- [182] R.-G. Ping, et al., Chin. Phys. C 40 (2016) 113002, <http://dx.doi.org/10.1088/1674-1137/40/11/113002>, arXiv:1605.09208 [hep-ex].
- [183] R.-G. Ping, Chin. Phys. C 38 (2014) 083001, <http://dx.doi.org/10.1088/1674-1137/38/8/083001>, arXiv:1309.3932 [hep-ph].
- [184] I. Adachi, et al., (Belle-II), Chin. Phys. C 49 (2025) 013001, <http://dx.doi.org/10.1088/1674-1137/ad806c>, arXiv:2407.00965 [hep-ex].
- [185] I. Adachi, et al., (Belle-II), Phys. Rev. D 110 (2024) 112005, <http://dx.doi.org/10.1103/PhysRevD.110.112005>, arXiv:2404.04915 [hep-ex].
- [186] M. Hoferichter, B.-L. Hoid, B. Kubis, D. Schuh, JHEP 08 (2023) 208, [http://dx.doi.org/10.1007/JHEP08\(2023\)208](http://dx.doi.org/10.1007/JHEP08(2023)208), arXiv:2307.02546 [hep-ph].
- [187] R. Alemany, M. Davier, A. Hoecker, Eur. Phys. J. C 2 (1998) 123, <http://dx.doi.org/10.1007/s100520050127>, arXiv:hep-ph/9703220.
- [188] S. Schael, et al., (ALEPH), Phys. Rep. 421 (2005) 191, <http://dx.doi.org/10.1016/j.physrep.2005.06.007>, arXiv:hep-ex/0506072.
- [189] M. Davier, A. Hoecker, B. Malaescu, C.-Z. Yuan, Z. Zhang, Eur. Phys. J. C 74 (2014) 2803, <http://dx.doi.org/10.1140/epjc/s10052-014-2803-9>, arXiv:1312.1501 [hep-ex].
- [190] K. Ackerstaff, et al., (OPAL), Eur. Phys. J. C 7 (1999) 571, <http://dx.doi.org/10.1007/s100529901061>, arXiv:hep-ex/9808019.
- [191] S. Anderson, et al., (CLEO), Phys. Rev. D 61 (2000) 112002, <http://dx.doi.org/10.1103/PhysRevD.61.112002>, arXiv:hep-ex/9910046.
- [192] M. Fujikawa, et al., (Belle), Phys. Rev. D 78 (2008) 072006, <http://dx.doi.org/10.1103/PhysRevD.78.072006>, arXiv:0805.3773 [hep-ex].
- [193] M. Davier, A. Hoecker, Z. Zhang, Rev. Modern Phys. 78 (2006) 1043, <http://dx.doi.org/10.1103/RevModPhys.78.1043>, arXiv:hep-ph/0507078.
- [194] M. Davier, A. Hoecker, G. López Castro, B. Malaescu, X.H. Mo, G. Toledo, P. Wang, C.Z. Yuan, Z. Zhang, Eur. Phys. J. C 66 (2010) 127, <http://dx.doi.org/10.1140/epjc/s10052-009-1219-4>, arXiv:0906.5443 [hep-ph].
- [195] V. Cirigliano, G. Ecker, H. Neufeld, Phys. Lett. B 513 (2001) 361, [http://dx.doi.org/10.1016/S0370-2693\(01\)00764-X](http://dx.doi.org/10.1016/S0370-2693(01)00764-X), arXiv:hep-ph/0104267.
- [196] V. Cirigliano, G. Ecker, H. Neufeld, JHEP 08 (2002) 002, <http://dx.doi.org/10.1088/1126-6708/2002/08/002>, arXiv:hep-ph/0207310 [hep-ph].
- [197] M. Davier, S. Eidelman, A. Hoecker, Z. Zhang, Eur. Phys. J. C 27 (2003) 497, <http://dx.doi.org/10.1140/epjc/s2003-01136-2>, arXiv:hep-ph/0208177 [hep-ph].
- [198] A. Sirlin, Nuclear Phys. B 196 (1982) 83, [http://dx.doi.org/10.1016/0550-3213\(82\)90303-0](http://dx.doi.org/10.1016/0550-3213(82)90303-0).
- [199] W.J. Marciano, A. Sirlin, Phys. Rev. Lett. 56 (1986) 22, <http://dx.doi.org/10.1103/PhysRevLett.56.22>.
- [200] W.J. Marciano, A. Sirlin, Phys. Rev. Lett. 61 (1988) 1815, <http://dx.doi.org/10.1103/PhysRevLett.61.1815>.
- [201] W.J. Marciano, A. Sirlin, Phys. Rev. Lett. 71 (1993) 3629, <http://dx.doi.org/10.1103/PhysRevLett.71.3629>.
- [202] E. Braaten, C.-S. Li, Phys. Rev. D 42 (1990) 3888, <http://dx.doi.org/10.1103/PhysRevD.42.3888>.
- [203] J. Erler, Rev. Mex. Fis. 50 (2004) 200, arXiv:hep-ph/0211345.
- [204] V. Cirigliano, W. Dekens, E. Mereghetti, O. Tomalak, Phys. Rev. D 108 (2023) 053003, <http://dx.doi.org/10.1103/PhysRevD.108.053003>, arXiv:2306.03138 [hep-ph].
- [205] F. Flores-Báez, A. Flores-Tlalpa, G. López Castro, G. Toledo, Phys. Rev. D 74 (2006) 071301, <http://dx.doi.org/10.1103/PhysRevD.74.071301>, arXiv:hep-ph/0608084.
- [206] J.A. Miranda, P. Roig, Phys. Rev. D 102 (2020) 114017, <http://dx.doi.org/10.1103/PhysRevD.102.114017>, arXiv:2007.11019 [hep-ph].
- [207] J. Schwinger, Particles, Sources, and Fields, vol. 3, Taylor and Francis, 1998, <http://dx.doi.org/10.1201/9780429498466>.
- [208] M. Drees, K.-I. Hikasa, Phys. Lett. B 252 (1990) 127, [http://dx.doi.org/10.1016/0370-2693\(90\)91094-R](http://dx.doi.org/10.1016/0370-2693(90)91094-R).
- [209] G. López Castro, A. Miranda, P. Roig, Phys. Rev. D 111 (2025) 073004, <http://dx.doi.org/10.1103/PhysRevD.111.073004>, arXiv:2411.07696 [hep-ph].
- [210] A. Flores-Tlalpa, F. Flores-Báez, G. López Castro, G. Toledo, Nucl. Phys. B Proc. Suppl. 169 (2007) 250, <http://dx.doi.org/10.1016/j.nuclphysbps.2007.03.011>, arXiv:hep-ph/0611226.
- [211] F.V. Flores-Báez, G. López Castro, G. Toledo, Phys. Rev. D 76 (2007) 096010, <http://dx.doi.org/10.1103/PhysRevD.76.096010>, arXiv:0708.3256 [hep-ph].
- [212] P. Masjuan, A. Miranda, P. Roig, Phys. Lett. B 850 (2024) 138492, <http://dx.doi.org/10.1016/j.physletb.2024.138492>, arXiv:2305.20005 [hep-ph].
- [213] G. Ecker, J. Gasser, A. Pich, E. de Rafael, Nuclear Phys. B 321 (1989) 311, [http://dx.doi.org/10.1016/0550-3213\(89\)90346-5](http://dx.doi.org/10.1016/0550-3213(89)90346-5).
- [214] G. Ecker, J. Gasser, H. Leutwyler, A. Pich, E. de Rafael, Phys. Lett. B 223 (1989) 425, [http://dx.doi.org/10.1016/0370-2693\(89\)91627-4](http://dx.doi.org/10.1016/0370-2693(89)91627-4).
- [215] V. Cirigliano, G. Ecker, M. Eidemüller, R. Kaiser, A. Pich, J. Portolés, Nuclear Phys. B 753 (2006) 139, <http://dx.doi.org/10.1016/j.nuclphysb.2006.07.010>, arXiv:hep-ph/0603205.
- [216] K. Kampf, J. Novotný, Phys. Rev. D 84 (2011) 014036, <http://dx.doi.org/10.1103/PhysRevD.84.014036>, arXiv:1104.3137 [hep-ph].
- [217] R. Escribano, J.A. Miranda, P. Roig, Phys. Rev. D 109 (2024) 053003, <http://dx.doi.org/10.1103/PhysRevD.109.053003>, arXiv:2303.01362 [hep-ph].

- [218] M.A. Arroyo-Ureña, G. Hernández-Tomé, G. López Castro, P. Roig, I. Rosell, Phys. Rev. D 104 (2021) L091502, <http://dx.doi.org/10.1103/PhysRevD.104.L091502>, arXiv:2107.04603 [hep-ph].
- [219] M.A. Arroyo-Ureña, G. Hernández-Tomé, G. López Castro, P. Roig, I. Rosell, JHEP 02 (2022) 173, [http://dx.doi.org/10.1007/JHEP02\(2022\)173](http://dx.doi.org/10.1007/JHEP02(2022)173), arXiv:2112.01859 [hep-ph].
- [220] F. Guerrero, A. Pich, Phys. Lett. B 412 (1997) 382, [http://dx.doi.org/10.1016/S0370-2693\(97\)01070-8](http://dx.doi.org/10.1016/S0370-2693(97)01070-8), arXiv:hep-ph/9707347.
- [221] D. Gómez Dumm, P. Roig, Eur. Phys. J. C 73 (2013) 2528, <http://dx.doi.org/10.1140/epjc/s10052-013-2528-1>, arXiv:1301.6973 [hep-ph].
- [222] S. González-Solís, P. Roig, Eur. Phys. J. C 79 (2019) 436, <http://dx.doi.org/10.1140/epjc/s10052-019-6943-9>, arXiv:1902.02273 [hep-ph].
- [223] G.J. Gounaris, J.J. Sakurai, Phys. Rev. Lett. 21 (1968) 244, <http://dx.doi.org/10.1103/PhysRevLett.21.244>.
- [224] J.H. Kühn, A. Santamaria, Z. Phys. C 48 (1990) 445, <http://dx.doi.org/10.1007/BF01572024>.
- [225] S. Navas, et al., (Particle Data Group), Phys. Rev. D 110 (2024) 030001, <http://dx.doi.org/10.1103/PhysRevD.110.030001>.
- [226] G. Colangelo, M. Hoferichter, P. Stoffer, JHEP 02 (2019) 006, [http://dx.doi.org/10.1007/JHEP02\(2019\)006](http://dx.doi.org/10.1007/JHEP02(2019)006), arXiv:1810.00007 [hep-ph].
- [227] J. Jegerlehner, R. Szafron, Eur. Phys. J. C 71 (2011) 1632, <http://dx.doi.org/10.1140/epjc/s10052-011-1632-3>, arXiv:1101.2872 [hep-ph].
- [228] G. Colangelo, M. Cottini, M. Hoferichter, S. Holz, in preparation.
- [229] J. Bijnens, G. Ecker, J. Gasser, Nuclear Phys. B 396 (1993) 81, [http://dx.doi.org/10.1016/0550-3213\(93\)90259-R](http://dx.doi.org/10.1016/0550-3213(93)90259-R), arXiv:hep-ph/9209261.
- [230] S. Weinberg, Phys. Rev. 166 (1968) 1568, <http://dx.doi.org/10.1103/PhysRev.166.1568>.
- [231] J. Gasser, H. Leutwyler, Nuclear Phys. B 250 (1985) 465, [http://dx.doi.org/10.1016/0550-3213\(85\)90492-4](http://dx.doi.org/10.1016/0550-3213(85)90492-4).
- [232] R. Urech, Nuclear Phys. B 433 (1995) 234, [http://dx.doi.org/10.1016/0550-3213\(95\)90707-N](http://dx.doi.org/10.1016/0550-3213(95)90707-N), arXiv:hep-ph/9405341.
- [233] M. Knecht, H. Neufeld, H. Rupertsberger, P. Talavera, Eur. Phys. J. C 12 (2000) 469, <http://dx.doi.org/10.1007/s100529900265>, arXiv:hep-ph/9909284.
- [234] X. Feng, M. Gorchtein, L.-C. Jin, P.-X. Ma, C.-Y. Seng, Phys. Rev. Lett. 124 (2020) 192002, <http://dx.doi.org/10.1103/PhysRevLett.124.192002>, arXiv:2003.09798 [hep-lat].
- [235] P.-X. Ma, X. Feng, M. Gorchtein, L.-C. Jin, C.-Y. Seng, Phys. Rev. D 103 (2021) 114503, <http://dx.doi.org/10.1103/PhysRevD.103.114503>, arXiv:2102.12048 [hep-lat].
- [236] J.-S. Yoo, T. Bhattacharya, R. Gupta, S. Mondal, B. Yoon, Phys. Rev. D 108 (2023) 034508, <http://dx.doi.org/10.1103/PhysRevD.108.034508>, arXiv:2305.03198 [hep-lat].
- [237] G. Colangelo, M. Hoferichter, B. Kubis, P. Stoffer, JHEP 10 (2022) 032, [http://dx.doi.org/10.1007/JHEP10\(2022\)032](http://dx.doi.org/10.1007/JHEP10(2022)032), arXiv:2208.08993 [hep-ph].
- [238] J. Monnard, Radiative corrections for the two-pion contribution to the hadronic vacuum polarization contribution to the muon $g - 2$ (Ph.D. thesis), Universität Bern, 2021, <https://boristheses.unibe.ch/2825/>.
- [239] M.T. Hansen, H.B. Meyer, D. Robaina, Phys. Rev. D 96 (2017) 094513, <http://dx.doi.org/10.1103/PhysRevD.96.094513>, arXiv:1704.08993 [hep-lat].
- [240] M. Hansen, A. Lupo, N. Tantalo, Phys. Rev. D 99 (2019) 094508, <http://dx.doi.org/10.1103/PhysRevD.99.094508>, arXiv:1903.06476 [hep-lat].
- [241] G. Bails, S. Hashimoto, T. Ishikawa, PTEP 2020 (2020) 043B07, <http://dx.doi.org/10.1093/ptep/ptaa044>, arXiv:2001.11779 [hep-lat].
- [242] A. Lupo, L. Del Debbio, M. Panero, N. Tantalo, PoS LATTICE2023 (2024) 004, <http://dx.doi.org/10.22323/1.453.0004>, arXiv:2311.18125 [hep-lat].
- [243] T. Bergamaschi, W.I. Jay, P.R. Oare, Phys. Rev. D 108 (2023) 074516, <http://dx.doi.org/10.1103/PhysRevD.108.074516>, arXiv:2305.16190 [hep-lat].
- [244] M. Bruno, L. Giusti, M. Saccardi, Phys. Rev. D 111 (2025) 094515, <http://dx.doi.org/10.1103/PhysRevD.111.094515>, arXiv:2407.04141 [hep-lat].
- [245] M. Bruno, T. Izubuchi, C. Lehner, A. Meyer, PoS LATTICE2018 (2018) 135, <http://dx.doi.org/10.22323/1.334.0135>, arXiv:1811.00508 [hep-lat].
- [246] M. Bruno, Report on lattice QCD program on τ data for $(g - 2)_\mu$, 2024, Virtual mini-workshop on tau decays, https://indico.global/event/1378/contributions/27704/attachments/14031/21460/tau_bruno.pdf.
- [247] M. Hoferichter, G. Colangelo, B.-L. Hoid, B. Kubis, J. Ruiz de Elvira, D. Schuh, D. Stamen, P. Stoffer, Phys. Rev. Lett. 131 (2023) 161905, <http://dx.doi.org/10.1103/PhysRevLett.131.161905>, arXiv:2307.02532 [hep-ph].
- [248] A.J. Buras, P.H. Weisz, Nuclear Phys. B 333 (1990) 66, [http://dx.doi.org/10.1016/0550-3213\(90\)90223-Z](http://dx.doi.org/10.1016/0550-3213(90)90223-Z).
- [249] B. Moussallam, Eur. Phys. J. C 73 (2013) 2539, <http://dx.doi.org/10.1140/epjc/s10052-013-2539-y>, arXiv:1305.3143 [hep-ph].
- [250] J. Bijnens, P. Gosdzinsky, Phys. Lett. B 388 (1996) 203, [http://dx.doi.org/10.1016/0370-2693\(96\)01147-1](http://dx.doi.org/10.1016/0370-2693(96)01147-1), arXiv:hep-ph/9607462.
- [251] M. Davier, B. Malaescu, Z. Zhang, 2025, arXiv:2504.13789 [hep-ph].
- [252] Y. Ulrich, et al., N^3 LO kick-off workshop / thinkstart, 2022, IPPP, Durham University, <https://conference.ippp.dur.ac.uk/event/1104/>.
- [253] A. Kupś, A. Signer, Y. Ulrich, G. Venanzoni, et al., Radiative corrections and Monte Carlo tools for STRONG2020, 2023, 5th Workshop / Thinkstart, University of Zurich, <https://indico.psi.ch/event/13707/>.
- [254] C. Carloni Calame, M. Passera, L. Trentadue, G. Venanzoni, et al., Consolidation of the MUonE experiment and recent developments in low energy e^+e^- data, 2024, MITP Topical Workshop, <https://indico.mtp.uni-mainz.de/event/352/>.
- [255] D.R. Yennie, S.C. Frautschi, H. Suura, Ann. Phys. 13 (1961) 379, [http://dx.doi.org/10.1016/0003-4916\(61\)90151-8](http://dx.doi.org/10.1016/0003-4916(61)90151-8).
- [256] S. Jadach, B.F.L. Ward, Comput. Phys. Comm. 56 (1990) 351, [http://dx.doi.org/10.1016/0010-4655\(90\)90020-2](http://dx.doi.org/10.1016/0010-4655(90)90020-2).
- [257] F. Krauss, A. Price, M. Schönherr, SciPost Phys. 13 (2022) 026, <http://dx.doi.org/10.21468/SciPostPhys.13.2.026>, arXiv:2203.10948 [hep-ph].
- [258] F.V. Ignatov, Measurement of the Pion Form-Factor At 1.04-1.38 GeV Energy Range with the CMD-2 Detector (Ph.D. thesis), BINP, Novosibirsk, 2008, <https://inspirehep.net/files/9bf185bdab9fbeca7ff2e05770736e15>.
- [259] F. Ignatov, Vacuum polarization, 2022, <http://cmd.inp.nsk.su/~ignatov/vpl/>.
- [260] M. Davier, A. Hoecker, B. Malaescu, C.Z. Yuan, Z. Zhang, Eur. Phys. J. C 66 (2010) 1, <http://dx.doi.org/10.1140/epjc/s10052-010-1246-1>, arXiv:0908.4300 [hep-ph].
- [261] M. Davier, A. Hoecker, B. Malaescu, Z. Zhang, Eur. Phys. J. C 71 (2011) 1515, <http://dx.doi.org/10.1140/epjc/s10052-010-1515-z>, arXiv:1010.4180 [hep-ph]; Eur. Phys. J. C 72 (2012) 1874, Erratum.
- [262] M. Davier, A. Hoecker, B. Malaescu, Z. Zhang, Eur. Phys. J. C 77 (2017) 827, <http://dx.doi.org/10.1140/epjc/s10052-017-5161-6>, arXiv:1706.09436 [hep-ph].
- [263] M. Davier, A. Hoecker, B. Malaescu, Z. Zhang, Eur. J., 80 (2020) 241, <http://dx.doi.org/10.1140/epjc/s10052-020-7792-2>, arXiv:1908.00921 [hep-ph]; Eur. Phys. J. C 80 (2020) 410, Erratum.
- [264] B. Malaescu, Dispersive HVP updates from DHMZ – sources of the existing tensions among the $e^+e^- \rightarrow \pi^+\pi^-$ measurements and implications for the dispersive combinations, 2024, Seventh Plenary Workshop of the Muon $g - 2$ Theory Initiative, KEK, https://conference-indico.kek.jp/event/257/contributions/5770/attachments/3705/5083/DispersiveHVP_amu_TL_KEK24_Malaescu.pdf.
- [265] A. Keshavarzi, D. Nomura, T. Teubner, Phys. Rev. D 97 (2018) 114025, <http://dx.doi.org/10.1103/PhysRevD.97.114025>, arXiv:1802.02995 [hep-ph].
- [266] A. Keshavarzi, D. Nomura, T. Teubner, A. Wright, Phys. Rev. D 111 (2025) L011901, <http://dx.doi.org/10.1103/PhysRevD.111.L011901>, arXiv:2409.02827 [hep-ph].
- [267] B. Ananthanarayan, I. Caprini, D. Das, Phys. Rev. D 98 (2018) 114015, <http://dx.doi.org/10.1103/PhysRevD.98.114015>, arXiv:1810.09265 [hep-ph].
- [268] M. Hoferichter, B.-L. Hoid, B. Kubis, JHEP 08 (2019) 137, [http://dx.doi.org/10.1007/JHEP08\(2019\)137](http://dx.doi.org/10.1007/JHEP08(2019)137), arXiv:1907.01556 [hep-ph].
- [269] R. Omnès, Nuovo Cim. 8 (1958) 316, <http://dx.doi.org/10.1007/BF02747746>.
- [270] P. Stoffer, G. Colangelo, M. Hoferichter, JINST 18 (2023) C10021, <http://dx.doi.org/10.1088/1748-0221/18/10/C10021>, arXiv:2308.04217 [hep-ph].
- [271] S.M. Roy, Phys. Lett. B 36 (1971) 353, [http://dx.doi.org/10.1016/0370-2693\(71\)90724-6](http://dx.doi.org/10.1016/0370-2693(71)90724-6).

- [272] B. Ananthanarayan, G. Colangelo, J. Gasser, H. Leutwyler, Phys. Rep. 353 (2001) 207, [http://dx.doi.org/10.1016/S0370-1573\(01\)00009-6](http://dx.doi.org/10.1016/S0370-1573(01)00009-6), [arXiv:hep-ph/0005297](#) [hep-ph].
- [273] R. García-Martín, R. Kamirski, J.R. Peláez, J. Ruiz de Elvira, F.J. Ynduráin, Phys. Rev. D 83 (2011) 074004, <http://dx.doi.org/10.1103/PhysRevD.83.074004>, [arXiv:1102.2183](#) [hep-ph].
- [274] I. Caprini, G. Colangelo, H. Leutwyler, Eur. Phys. J. C 72 (2012) 1860, <http://dx.doi.org/10.1140/epjc/s10052-012-1860-1>, [arXiv:1111.7160](#) [hep-ph].
- [275] H. Leutwyler, Contin. Adv. QCD 2002 (2002) 23, http://dx.doi.org/10.1142/9789812776310_0002, [arXiv:hep-ph/0212324](#) [hep-ph].
- [276] B. Ananthanarayan, I. Caprini, I.S. Imsong, Phys. Rev. D 83 (2011) 096002, <http://dx.doi.org/10.1103/PhysRevD.83.096002>, [arXiv:1102.3299](#) [hep-ph].
- [277] E. Ruiz Arriola, P. Sánchez-Puertas, Phys. Rev. D 110 (2024) 054003, <http://dx.doi.org/10.1103/PhysRevD.110.054003>, [arXiv:2403.07121](#) [hep-ph].
- [278] T.N. Truong, R.V. Mau, P.X. Yem, Phys. Rev. 172 (1968) 1645, <http://dx.doi.org/10.1103/PhysRev.172.1645>.
- [279] B.V. Geshkenbein, Phys. Rev. D 61 (2000) 033009, <http://dx.doi.org/10.1103/PhysRevD.61.033009>, [arXiv:hep-ph/9806418](#).
- [280] B. Ananthanarayan, I. Caprini, D. Das, I. Sentitemsu Imsong, Phys. Rev. D 89 (2014) 036007, <http://dx.doi.org/10.1103/PhysRevD.89.036007>, [arXiv:1312.5849](#) [hep-ph].
- [281] F. Niecknig, B. Kubis, S.P. Schneider, Eur. Phys. J. C 72 (2012) 2014, <http://dx.doi.org/10.1140/epjc/s10052-012-2014-1>, [arXiv:1203.2501](#) [hep-ph].
- [282] S.P. Schneider, B. Kubis, F. Niecknig, Phys. Rev. D 86 (2012) 054013, <http://dx.doi.org/10.1103/PhysRevD.86.054013>, [arXiv:1206.3098](#) [hep-ph].
- [283] M. Hoferichter, B. Kubis, D. Sakkas, Phys. Rev. D 86 (2012) 116009, <http://dx.doi.org/10.1103/PhysRevD.86.116009>, [arXiv:1210.6793](#) [hep-ph].
- [284] M. Hoferichter, B. Kubis, S. Leupold, F. Niecknig, S.P. Schneider, Eur. Phys. J. C 74 (2014) 3180, <http://dx.doi.org/10.1140/epjc/s10052-014-3180-0>, [arXiv:1410.4691](#) [hep-ph].
- [285] M. Niehus, M. Hoferichter, B. Kubis, JHEP 12 (2021) 038, [http://dx.doi.org/10.1007/JHEP12\(2021\)038](http://dx.doi.org/10.1007/JHEP12(2021)038), [arXiv:2110.11372](#) [hep-ph].
- [286] N.N. Khuri, S.B. Treiman, Phys. Rev. 119 (1960) 1115, <http://dx.doi.org/10.1103/PhysRev.119.1115>.
- [287] S. Holz, C. Hanhart, M. Hoferichter, B. Kubis, Eur. Phys. J. C 82 (2022) 434, <http://dx.doi.org/10.1140/epjc/s10052-022-10247-7>, [arXiv:2202.05846](#) [hep-ph]; Addendum: Eur. Phys. J. C 82 (2022) 1159.
- [288] M.N. Achasov, et al., (SND), Phys. Rev. D 63 (2001) 072002, <http://dx.doi.org/10.1103/PhysRevD.63.072002>, [arXiv:hep-ex/0009036](#) [hep-ex].
- [289] M.N. Achasov, et al., (SND), Phys. Rev. D 66 (2002) 032001, <http://dx.doi.org/10.1103/PhysRevD.66.032001>, [arXiv:hep-ex/0201040](#) [hep-ex].
- [290] M.N. Achasov, et al., (SND), Phys. Rev. D 68 (2003) 052006, <http://dx.doi.org/10.1103/PhysRevD.68.052006>, [arXiv:hep-ex/0305049](#).
- [291] M.N. Achasov, et al., (SND), Eur. Phys. J. C 80 (2020) 993, <http://dx.doi.org/10.1140/epjc/s10052-020-08524-4>, [arXiv:2007.14595](#) [hep-ex].
- [292] R.R. Akhmetshin, et al., (CMD-2), Phys. Lett. B 364 (1995) 199, [http://dx.doi.org/10.1016/0370-2693\(95\)01394-6](http://dx.doi.org/10.1016/0370-2693(95)01394-6).
- [293] R.R. Akhmetshin, et al., (CMD-2), Phys. Lett. B 434 (1998) 426, [http://dx.doi.org/10.1016/S0370-2693\(98\)00826-0](http://dx.doi.org/10.1016/S0370-2693(98)00826-0).
- [294] R.R. Akhmetshin, et al., (CMD-2), Phys. Lett. B 642 (2006) 203, <http://dx.doi.org/10.1016/j.physletb.2006.09.041>.
- [295] J. Wess, B. Zumino, Phys. Lett. 37B (1971) 95, [http://dx.doi.org/10.1016/0370-2693\(71\)90582-X](http://dx.doi.org/10.1016/0370-2693(71)90582-X).
- [296] E. Witten, Nuclear Phys. B 223 (1983) 422, [http://dx.doi.org/10.1016/0550-3213\(83\)90063-9](http://dx.doi.org/10.1016/0550-3213(83)90063-9).
- [297] J. Bijnens, A. Bramon, F. Cornet, Phys. Lett. B 237 (1990) 488, [http://dx.doi.org/10.1016/0370-2693\(90\)91212-T](http://dx.doi.org/10.1016/0370-2693(90)91212-T).
- [298] M. Hoferichter, B. Kubis, M. Zanke, Phys. Rev. D 96 (2017) 114016, <http://dx.doi.org/10.1103/PhysRevD.96.114016>, [arXiv:1710.00824](#) [hep-ph].
- [299] M. Hoferichter, B.-L. Hoid, B. Kubis, JHEP 07 (2025) 095, [http://dx.doi.org/10.1007/JHEP07\(2025\)095](http://dx.doi.org/10.1007/JHEP07(2025)095), [arXiv:2504.13827](#) [hep-ph].
- [300] B.-L. Hoid, M. Hoferichter, B. Kubis, Eur. Phys. J. C 80 (2020) 988, <http://dx.doi.org/10.1140/epjc/s10052-020-08550-2>, [arXiv:2007.12696](#) [hep-ph].
- [301] M.N. Achasov, et al., (SND), Eur. Phys. J. C 12 (2000) 25, <http://dx.doi.org/10.1007/s100529900222>.
- [302] M.N. Achasov, et al., (SND), Phys. Lett. B 559 (2003) 171, [http://dx.doi.org/10.1016/S0370-2693\(03\)00336-8](http://dx.doi.org/10.1016/S0370-2693(03)00336-8), [arXiv:hep-ex/0302004](#).
- [303] R.R. Akhmetshin, et al., (CMD-2), Phys. Lett. B 605 (2005) 26, <http://dx.doi.org/10.1016/j.physletb.2004.11.020>, [arXiv:hep-ex/0409030](#).
- [304] M.N. Achasov, et al., (SND), Phys. Rev. D 93 (2016) 092001, <http://dx.doi.org/10.1103/PhysRevD.93.092001>, [arXiv:1601.08061](#) [hep-ex].
- [305] M.N. Achasov, et al., (SND), Phys. Rev. D 98 (2018) 112001, <http://dx.doi.org/10.1103/PhysRevD.98.112001>, [arXiv:1809.07631](#) [hep-ex].
- [306] R.L. Workman, et al., (Particle Data Group), PTEP 2022 (2022) 083C01, <http://dx.doi.org/10.1093/ptep/ptac097>.
- [307] W. Qin, L.-Y. Dai, J. Portolés, JHEP 03 (2021) 092, [http://dx.doi.org/10.1007/JHEP03\(2021\)092](http://dx.doi.org/10.1007/JHEP03(2021)092), [arXiv:2011.09618](#) [hep-ph].
- [308] M. Benayoun, L. DelBuono, F. Jegerlehner, Eur. Phys. J. C 82 (2022) 184, <http://dx.doi.org/10.1140/epjc/s10052-022-10096-4>, [arXiv:2105.13018](#) [hep-ph].
- [309] S.-J. Wang, Z. Fang, L.-Y. Dai, JHEP 07 (2023) 037, [http://dx.doi.org/10.1007/JHEP07\(2023\)037](http://dx.doi.org/10.1007/JHEP07(2023)037), [arXiv:2302.08859](#) [hep-ph].
- [310] B.-H. Qin, W. Qin, L.-Y. Dai, Phys. Rev. D 111 (2025) 034025, <http://dx.doi.org/10.1103/PhysRevD.111.034025>, [arXiv:2403.14294](#) [hep-ph].
- [311] S. Blatnik, J. Stahov, C.B. Lang, Lett. Nuovo Cimento 24 (1979) 39, <http://dx.doi.org/10.1007/BF02725742>.
- [312] P. Büttiker, S. Descotes-Genon, B. Moussallam, Eur. Phys. J. C 33 (2004) 409, <http://dx.doi.org/10.1140/epjc/s2004-01591-1>, [arXiv:hep-ph/0310283](#).
- [313] J.R. Peláez, A. Rodas, Eur. Phys. J. C 78 (2018) 897, <http://dx.doi.org/10.1140/epjc/s10052-018-6296-9>, [arXiv:1807.04543](#) [hep-ph].
- [314] J.R. Peláez, A. Rodas, Phys. Rep. 969 (2022) 1, <http://dx.doi.org/10.1016/j.physrep.2022.03.004>, [arXiv:2010.11222](#) [hep-ph].
- [315] E.B. Dally, et al., Phys. Rev. Lett. 45 (1980) 232, <http://dx.doi.org/10.1103/PhysRevLett.45.232>.
- [316] S.R. Amendolia, et al., (NA7), Phys. Lett. B 178 (1986) 435, [http://dx.doi.org/10.1016/0370-2693\(86\)91407-3](http://dx.doi.org/10.1016/0370-2693(86)91407-3).
- [317] R.R. Akhmetshin, et al., (CMD-2), Phys. Lett. B 669 (2008) 217, <http://dx.doi.org/10.1016/j.physletb.2008.09.053>, [arXiv:0804.0178](#) [hep-ex].
- [318] J.P. Lees, et al., (BABAR), Phys. Rev. D 88 (2013) 032013, <http://dx.doi.org/10.1103/PhysRevD.88.032013>, [arXiv:1306.3600](#) [hep-ex].
- [319] W.N. Cottingham, Ann. Phys. 25 (1963) 424, [http://dx.doi.org/10.1016/0003-4916\(63\)90023-X](http://dx.doi.org/10.1016/0003-4916(63)90023-X).
- [320] G. Colangelo, M. Hoferichter, P. Stoffer, Phys. Lett. B 814 (2021) 136073, <http://dx.doi.org/10.1016/j.physletb.2021.136073>, [arXiv:2010.07943](#) [hep-ph].
- [321] B. Hyams, et al., Nuclear Phys. B 64 (1973) 134, [http://dx.doi.org/10.1016/0550-3213\(73\)90618-4](http://dx.doi.org/10.1016/0550-3213(73)90618-4).
- [322] S.D. Protopopescu, M. Alston-Garnjost, A. Barbaro-Galtieri, S.M. Flatté, J.H. Friedman, T.A. Lasinski, G.R. Lynch, M.S. Rabin, F.T. Solmitz, Phys. Rev. D 7 (1973) 1279, <http://dx.doi.org/10.1103/PhysRevD.7.1279>.
- [323] J.R. Peláez, P. Rabán, J. Ruiz de Elvira, Phys. Rev. D 111 (2025) 074003, <http://dx.doi.org/10.1103/PhysRevD.111.074003>, [arXiv:2412.15327](#) [hep-ph].
- [324] G. Chanturia, PoS Regio2021 (2022) 030, <http://dx.doi.org/10.22323/1.412.0030>.
- [325] L.A. Heuser, G. Chanturia, F.-K. Guo, C. Hanhart, M. Hoferichter, B. Kubis, Eur. Phys. J. C 84 (2024) 599, <http://dx.doi.org/10.1140/epjc/s10052-024-12884-6>, [arXiv:2403.15539](#) [hep-ph].
- [326] J. Koponen, F. Bursa, C.T.H. Davies, R.J. Dowdall, G.P. Lepage, Phys. Rev. D 93 (2016) 054503, <http://dx.doi.org/10.1103/PhysRevD.93.054503>, [arXiv:1511.07382](#) [hep-lat].
- [327] X. Feng, Y. Fu, L.-C. Jin, Phys. Rev. D 101 (2020) 051502, <http://dx.doi.org/10.1103/PhysRevD.101.051502>, [arXiv:1911.04064](#) [hep-lat].
- [328] G. Wang, J. Liang, T. Draper, K.-F. Liu, Y.-B. Yang, (χ QCD), Phys. Rev. D 104 (2021) 074502, <http://dx.doi.org/10.1103/PhysRevD.104.074502>, [arXiv:2006.05431](#) [hep-ph].

- [329] X. Gao, N. Karthik, S. Mukherjee, P. Petreczky, S. Syritsyn, Y. Zhao, Phys. Rev. D 104 (2021) 114515, <http://dx.doi.org/10.1103/PhysRevD.104.114515>, arXiv:2102.06047 [hep-lat].
- [330] M. Passera, W.J. Marciano, A. Sirlin, Phys. Rev. D 78 (2008) 013009, <http://dx.doi.org/10.1103/PhysRevD.78.013009>, arXiv:0804.1142 [hep-ph].
- [331] A. Crivellin, M. Hoferichter, C.A. Manzari, M. Montull, Phys. Rev. Lett. 125 (2020) 091801, <http://dx.doi.org/10.1103/PhysRevLett.125.091801>, arXiv:2003.04886 [hep-ph].
- [332] A. Keshavarzi, W.J. Marciano, M. Passera, A. Sirlin, Phys. Rev. D 102 (2020) 033002, <http://dx.doi.org/10.1103/PhysRevD.102.033002>, arXiv:2006.12666 [hep-ph].
- [333] B. Malaescu, M. Schott, Eur. Phys. J. C 81 (2021) 46, <http://dx.doi.org/10.1140/epjc/s10052-021-08848-9>, arXiv:2008.08107 [hep-ph].
- [334] F.-K. Guo, C. Hanhart, F.J. Llanes-Estrada, U.-G. Meißner, Phys. Lett. B 678 (2009) 90, <http://dx.doi.org/10.1016/j.physletb.2009.05.052>, arXiv:0812.3270 [hep-ph].
- [335] G. Colangelo, M. Hoferichter, B. Kubis, M. Niehus, J. Ruiz de Elvira, Phys. Lett. B 825 (2022) 136852, <http://dx.doi.org/10.1016/j.physletb.2021.136852>, arXiv:2110.05493 [hep-ph].
- [336] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, M.E. Sainio, Phys. Lett. B 374 (1996) 210, [http://dx.doi.org/10.1016/0370-2693\(96\)00165-7](http://dx.doi.org/10.1016/0370-2693(96)00165-7), arXiv:hep-ph/9511397.
- [337] M. Niehus, M. Hoferichter, B. Kubis, J. Ruiz de Elvira, Phys. Rev. Lett. 126 (2021) 102002, <http://dx.doi.org/10.1103/PhysRevLett.126.102002>, arXiv:2009.04479 [hep-ph].
- [338] J. Bijnens, G. Colangelo, P. Talavera, JHEP 05 (1998) 014, <http://dx.doi.org/10.1088/1126-6708/1998/05/014>, arXiv:hep-ph/9805389.
- [339] G. Colangelo, J. Gasser, H. Leutwyler, Nuclear Phys. B 603 (2001) 125, [http://dx.doi.org/10.1016/S0550-3213\(01\)00147-X](http://dx.doi.org/10.1016/S0550-3213(01)00147-X), arXiv:hep-ph/0103088.
- [340] M. Hoferichter, G. Colangelo, B.-L. Hoid, B. Kubis, J. Ruiz de Elvira, D. Stamen, P. Stoffer, Pos, LATTICE 2022 (2022) 316, <http://dx.doi.org/10.22323/1.430.0316>, arXiv:2210.11904 [hep-ph].
- [341] R. Urech, Phys. Lett. B 355 (1995) 308, [http://dx.doi.org/10.1016/0370-2693\(95\)00749-B](http://dx.doi.org/10.1016/0370-2693(95)00749-B), arXiv:hep-ph/9504238.
- [342] C.L. James, R. Lewis, K. Maltman, Phys. Rev. D 105 (2022) 053010, <http://dx.doi.org/10.1103/PhysRevD.105.053010>, arXiv:2109.13729 [hep-ph].
- [343] A. Hofer, J. Gluza, F. Jegerlehner, Eur. Phys. J. C 24 (2002) 51, <http://dx.doi.org/10.1007/s100520200916>, arXiv:hep-ph/0107154 [hep-ph].
- [344] H. Czyż, A. Grzelińska, J.H. Kühn, G. Rodrigo, Eur. Phys. J. C 39 (2005) 411, <http://dx.doi.org/10.1140/epjc/s2004-02103-1>, arXiv:hep-ph/0404078 [hep-ph].
- [345] J. Gluza, A. Hofer, S. Jadach, F. Jegerlehner, Eur. Phys. J. C 28 (2003) 261, <http://dx.doi.org/10.1140/epjc/s2003-01146-0>, arXiv:hep-ph/0212386 [hep-ph].
- [346] Y.M. Bystritskiy, E.A. Kuraev, G.V. Fedotov, F.V. Ignatov, Phys. Rev. D 72 (2005) 114019, <http://dx.doi.org/10.1103/PhysRevD.72.114019>, arXiv:hep-ph/0505236 [hep-ph].
- [347] G. Colangelo, M. Cottini, J. Ruiz de Elvira, Isospin-breaking in the $\pi\pi$ scattering amplitude I: Effects due to the pion mass difference, 2025, in preparation.
- [348] G. Colangelo, et al., Isospin-breaking in the $\pi\pi$ scattering amplitude II: Effects due to photons, 2025, in preparation.
- [349] B. Kubis, U.-G. Meißner, Nucl. Phys. A 671 (2000) 332, [http://dx.doi.org/10.1016/S0375-9474\(99\)00823-4](http://dx.doi.org/10.1016/S0375-9474(99)00823-4), arXiv:hep-ph/9908261; Nucl. Phys. A 692 (2001) 647, Erratum.
- [350] S. Descotes-Genon, M. Knecht, Eur. Phys. J. C 72 (2012) 1962, <http://dx.doi.org/10.1140/epjc/s10052-012-1962-9>, arXiv:1202.5886 [hep-ph].
- [351] R. García-Martín, B. Moussallam, Eur. Phys. J. C 70 (2010) 155, <http://dx.doi.org/10.1140/epjc/s10052-010-1471-7>, arXiv:1006.5373 [hep-ph].
- [352] M. Hoferichter, D.R. Phillips, C. Schat, Eur. Phys. J. C 71 (2011) 1743, <http://dx.doi.org/10.1140/epjc/s10052-011-1743-x>, arXiv:1106.4147 [hep-ph].
- [353] I. Danilkin, M. Vanderhaeghen, Phys. Lett. B 789 (2019) 366, <http://dx.doi.org/10.1016/j.physletb.2018.12.047>, arXiv:1810.03669 [hep-ph].
- [354] M. Hoferichter, P. Stoffer, JHEP 07 (2019) 073, [http://dx.doi.org/10.1007/JHEP07\(2019\)073](http://dx.doi.org/10.1007/JHEP07(2019)073), arXiv:1905.13198 [hep-ph].
- [355] I. Danilkin, O. Deineka, M. Vanderhaeghen, Phys. Rev. D 101 (2020) 054008, <http://dx.doi.org/10.1103/PhysRevD.101.054008>, arXiv:1909.04158 [hep-ph].
- [356] D. Boito, M. Golterman, K. Maltman, S. Peris, Phys. Rev. D 105 (2022) 093003, <http://dx.doi.org/10.1103/PhysRevD.105.093003>, arXiv:2203.05070 [hep-ph].
- [357] D. Boito, M. Golterman, K. Maltman, S. Peris, Phys. Rev. D 107 (2023) 074001, <http://dx.doi.org/10.1103/PhysRevD.107.074001>, arXiv:2211.11055 [hep-ph].
- [358] G. Benton, D. Boito, M. Golterman, A. Keshavarzi, K. Maltman, S. Peris, Phys. Rev. Lett. 131 (2023) 251803, <http://dx.doi.org/10.1103/PhysRevLett.131.251803>, arXiv:2306.16808 [hep-ph].
- [359] G. Benton, D. Boito, M. Golterman, A. Keshavarzi, K. Maltman, S. Peris, Phys. Rev. D 109 (2024) 036010, <http://dx.doi.org/10.1103/PhysRevD.109.036010>, arXiv:2311.09523 [hep-ph].
- [360] G. Benton, D. Boito, M. Golterman, A. Keshavarzi, K. Maltman, S. Peris, Phys. Rev. D 111 (2025) 034018, <http://dx.doi.org/10.1103/PhysRevD.111.034018>, arXiv:2411.06637 [hep-ph].
- [361] M. Della Morte, A. Jüttner, JHEP 11 (2010) 154, [http://dx.doi.org/10.1007/JHEP11\(2010\)154](http://dx.doi.org/10.1007/JHEP11(2010)154), arXiv:1009.3783 [hep-lat].
- [362] T. Blum, P.A. Boyle, T. Izubuchi, L. Jin, A. Jüttner, C. Lehner, K. Maltman, M. Marinković, A. Portelli, M. Spraggs, Phys. Rev. Lett. 116 (2016) 232002, <http://dx.doi.org/10.1103/PhysRevLett.116.232002>, arXiv:1512.09054 [hep-lat].
- [363] G. Colangelo, A.X. El-Khadra, M. Hoferichter, A. Keshavarzi, C. Lehner, P. Stoffer, T. Teubner, Phys. Lett. B 833 (2022) 137313, <http://dx.doi.org/10.1016/j.physletb.2022.137313>, arXiv:2205.12963 [hep-ph].
- [364] J.P. Lees, et al., (BABAR), Phys. Rev. D 98 (2018) 032010, <http://dx.doi.org/10.1103/PhysRevD.98.032010>, arXiv:1806.10280 [hep-ex].
- [365] P.A. Baikov, K.G. Chetyrkin, J.H. Kühn, Phys. Rev. Lett. 101 (2008) 012002, <http://dx.doi.org/10.1103/PhysRevLett.101.012002>, arXiv:0801.1821 [hep-ph].
- [366] D. Boito, M. Golterman, A. Keshavarzi, K. Maltman, D. Nomura, S. Peris, T. Teubner, Phys. Rev. D 98 (2018) 074030, <http://dx.doi.org/10.1103/PhysRevD.98.074030>, arXiv:1805.08176 [hep-ph].
- [367] D. Boito, M. Golterman, K. Maltman, S. Peris, M.V. Rodrigues, W. Schaaf, Phys. Rev. D 103 (2021) 034028, <http://dx.doi.org/10.1103/PhysRevD.103.034028>, arXiv:2012.10440 [hep-ph].
- [368] D. Boito, M. Golterman, K. Maltman, S. Peris, Phys. Rev. D 107 (2023) 034512, <http://dx.doi.org/10.1103/PhysRevD.107.034512>, arXiv:2210.13677 [hep-lat].
- [369] A. Bazavov, et al., (Fermilab Lattice, HPQCD, MILC), Phys. Rev. D 107 (2023) 114514, <http://dx.doi.org/10.1103/PhysRevD.107.114514>, arXiv:2301.08274 [hep-lat].
- [370] J. Calmet, S. Narison, M. Perrottet, E. de Rafael, Phys. Lett. 61B (1976) 283, [http://dx.doi.org/10.1016/0370-2693\(76\)90150-7](http://dx.doi.org/10.1016/0370-2693(76)90150-7).
- [371] A.V. Nesterenko, J. Phys. G 49 (2022) 055001, <http://dx.doi.org/10.1088/1361-6471/ac5d0a>, arXiv:2112.05009 [hep-ph]; J. Phys. G 50 (2022) 029401, Addendum.
- [372] E. Balzani, S. Laporta, M. Passera, Phys. Lett. B 834 (2022) 137462, <http://dx.doi.org/10.1016/j.physletb.2022.137462>, arXiv:2112.05704 [hep-ph].
- [373] E. Balzani, S. Laporta, M. Passera, Phys. Lett. B 858 (2024) 139040, <http://dx.doi.org/10.1016/j.physletb.2024.139040>, arXiv:2406.17940 [hep-ph].

- [374] B. Chakraborty, C.T.H. Davies, J. Koponen, G.P. Lepage, R.S. Van de Water, *Phys. Rev. D* 98 (2018) 094503, <http://dx.doi.org/10.1103/PhysRevD.98.094503>, [arXiv:1806.08190](https://arxiv.org/abs/1806.08190) [hep-lat].
- [375] M. Hoferichter, T. Teubner, *Phys. Rev. Lett.* 128 (2022) 112002, <http://dx.doi.org/10.1103/PhysRevLett.128.112002>, [arXiv:2112.06929](https://arxiv.org/abs/2112.06929) [hep-ph].
- [376] C.M. Carloni Calame, M. Passera, L. Trentadue, G. Venanzoni, *Phys. Lett. B* 746 (2015) 325, <http://dx.doi.org/10.1016/j.physletb.2015.05.020>, [arXiv:1504.02228](https://arxiv.org/abs/1504.02228) [hep-ph].
- [377] G. Abbiendi, et al., (MUonE), *Eur. Phys. J. C* 77 (2017) 139, <http://dx.doi.org/10.1140/epjc/s10052-017-4633-z>, [arXiv:1609.08987](https://arxiv.org/abs/1609.08987) [hep-ex].
- [378] G. Abbiendi, et al., (MUonE), Letter of Intent: The MUonE Project, Tech. Rep., CERN, Geneva, 2019, <https://cds.cern.ch/record/2677471>.
- [379] G. Abbiendi, *Phys. Scr.* 97 (2022) 054007, <http://dx.doi.org/10.1088/1402-4896/ac6297>, [arXiv:2201.13177](https://arxiv.org/abs/2201.13177) [physics.ins-det].
- [380] M. Alacevich, C.M. Carloni Calame, M. Chiesa, G. Montagna, O. Nicosini, F. Piccinini, *JHEP* 02 (2019) 155, [http://dx.doi.org/10.1007/JHEP02\(2019\)155](http://dx.doi.org/10.1007/JHEP02(2019)155), [arXiv:1811.06743](https://arxiv.org/abs/1811.06743) [hep-ph]; *JHEP* 02 (2022) 201, Erratum.
- [381] P. Mastrolia, M. Passera, A. Primo, U. Schubert, *JHEP* 11 (2017) 198, [http://dx.doi.org/10.1007/JHEP11\(2017\)198](http://dx.doi.org/10.1007/JHEP11(2017)198), [arXiv:1709.07435](https://arxiv.org/abs/1709.07435) [hep-ph].
- [382] S. Di Vita, S. Laporta, P. Mastrolia, A. Primo, U. Schubert, *JHEP* 09 (2018) 016, [http://dx.doi.org/10.1007/JHEP09\(2018\)016](http://dx.doi.org/10.1007/JHEP09(2018)016), [arXiv:1806.08241](https://arxiv.org/abs/1806.08241) [hep-ph].
- [383] M. Fael, M. Passera, *Phys. Rev. Lett.* 122 (2019) 192001, <http://dx.doi.org/10.1103/PhysRevLett.122.192001>, [arXiv:1901.03106](https://arxiv.org/abs/1901.03106) [hep-ph].
- [384] M. Fael, *JHEP* 02 (2019) 027, [http://dx.doi.org/10.1007/JHEP02\(2019\)027](http://dx.doi.org/10.1007/JHEP02(2019)027), [arXiv:1808.08233](https://arxiv.org/abs/1808.08233) [hep-ph].
- [385] C.M. Carloni Calame, M. Chiesa, S.M. Hasan, G. Montagna, O. Nicosini, F. Piccinini, *JHEP* 11 (2020) 028, [http://dx.doi.org/10.1007/JHEP11\(2020\)028](http://dx.doi.org/10.1007/JHEP11(2020)028), [arXiv:2007.01586](https://arxiv.org/abs/2007.01586) [hep-ph].
- [386] P. Banerjee, et al., *Eur. Phys. J. C* 80 (2020) 591, <http://dx.doi.org/10.1140/epjc/s10052-020-8138-9>, [arXiv:2004.13663](https://arxiv.org/abs/2004.13663) [hep-ph].
- [387] P. Banerjee, T. Engel, A. Signer, Y. Ulrich, *SciPost Phys.* 9 (2020) 027, <http://dx.doi.org/10.21468/SciPostPhys.9.2.027>, [arXiv:2007.01654](https://arxiv.org/abs/2007.01654) [hep-ph].
- [388] R. Bonciani, et al., *Phys. Rev. Lett.* 128 (2022) 022002, <http://dx.doi.org/10.1103/PhysRevLett.128.022002>, [arXiv:2106.13179](https://arxiv.org/abs/2106.13179) [hep-ph].
- [389] E. Budassi, C.M. Carloni Calame, M. Chiesa, C.L. Del Pio, S.M. Hasan, G. Montagna, O. Nicosini, F. Piccinini, *JHEP* 11 (2021) 098, [http://dx.doi.org/10.1007/JHEP11\(2021\)098](http://dx.doi.org/10.1007/JHEP11(2021)098), [arXiv:2109.14606](https://arxiv.org/abs/2109.14606) [hep-ph].
- [390] E. Budassi, C.M. Carloni Calame, C.L. Del Pio, F. Piccinini, *Phys. Lett. B* 829 (2022) 137138, <http://dx.doi.org/10.1016/j.physletb.2022.137138>, [arXiv:2203.01639](https://arxiv.org/abs/2203.01639) [hep-ph].
- [391] A. Broggio, et al., *JHEP* 01 (2023) 112, [http://dx.doi.org/10.1007/JHEP01\(2023\)112](http://dx.doi.org/10.1007/JHEP01(2023)112), [arXiv:2212.06481](https://arxiv.org/abs/2212.06481) [hep-ph].
- [392] M. Fael, F. Lange, K. Schönwald, M. Steinhauser, *Phys. Rev. Lett.* 128 (2022) 172003, <http://dx.doi.org/10.1103/PhysRevLett.128.172003>, [arXiv:2202.05276](https://arxiv.org/abs/2202.05276) [hep-ph].
- [393] M. Fael, F. Lange, K. Schönwald, M. Steinhauser, *Phys. Rev. D* 106 (2022) 034029, <http://dx.doi.org/10.1103/PhysRevD.106.034029>, [arXiv:2207.00027](https://arxiv.org/abs/2207.00027) [hep-ph].
- [394] M. Fael, F. Lange, K. Schönwald, M. Steinhauser, *Phys. Rev. D* 107 (2023) 094017, <http://dx.doi.org/10.1103/PhysRevD.107.094017>, [arXiv:2302.00693](https://arxiv.org/abs/2302.00693) [hep-ph].
- [395] S. Badger, J. Kryś, R. Moodie, S. Zoia, *JHEP* 11 (2023) 041, [http://dx.doi.org/10.1007/JHEP11\(2023\)041](http://dx.doi.org/10.1007/JHEP11(2023)041), [arXiv:2307.03098](https://arxiv.org/abs/2307.03098) [hep-ph].
- [396] T. Dave, W.J. Torres Bobadilla, *Phys. Rev. D* 111 (2025) 036024, <http://dx.doi.org/10.1103/PhysRevD.111.036024>, [arXiv:2411.07063](https://arxiv.org/abs/2411.07063) [hep-ph].
- [397] T. Engel, *JHEP* 07 (2023) 177, [http://dx.doi.org/10.1007/JHEP07\(2023\)177](http://dx.doi.org/10.1007/JHEP07(2023)177), [arXiv:2304.11689](https://arxiv.org/abs/2304.11689) [hep-ph].
- [398] T. Engel, *JHEP* 03 (2024) 004, [http://dx.doi.org/10.1007/JHEP03\(2024\)004](http://dx.doi.org/10.1007/JHEP03(2024)004), [arXiv:2311.17612](https://arxiv.org/abs/2311.17612) [hep-ph].
- [399] R. Plestid, M.B. Wise, *Phys. Rev. D* 110 (2024) 056032, <http://dx.doi.org/10.1103/PhysRevD.110.056032>, [arXiv:2403.12184](https://arxiv.org/abs/2403.12184) [hep-ph].
- [400] R. Plestid, M.B. Wise, *Phys. Rev. D* 110 (2024) 113007, <http://dx.doi.org/10.1103/PhysRevD.110.113007>, [arXiv:2407.21752](https://arxiv.org/abs/2407.21752) [hep-ph].
- [401] G. Abbiendi, E. Budassi, C.M. Carloni Calame, A. Gurgone, F. Piccinini, *Phys. Lett. B* 854 (2024) 138720, <http://dx.doi.org/10.1016/j.physletb.2024.138720>, [arXiv:2401.06077](https://arxiv.org/abs/2401.06077) [hep-ph].
- [402] F. Ignatov, R.N. Pilato, T. Teubner, G. Venanzoni, *Phys. Lett. B* 848 (2024) 138344, <http://dx.doi.org/10.1016/j.physletb.2023.138344>, [arXiv:2309.14205](https://arxiv.org/abs/2309.14205) [hep-ph].
- [403] D. Greynat, E. de Rafael, *JHEP* 05 (2022) 084, [http://dx.doi.org/10.1007/JHEP05\(2022\)084](http://dx.doi.org/10.1007/JHEP05(2022)084), [arXiv:2202.10810](https://arxiv.org/abs/2202.10810) [hep-ph].
- [404] D. Boito, C.Y. London, P. Masjuan, C. Rojas, *Phys. Rev. D* 110 (2024) 074012, <http://dx.doi.org/10.1103/PhysRevD.110.074012>, [arXiv:2405.13638](https://arxiv.org/abs/2405.13638) [hep-ph].
- [405] A. Tumasyan, et al., (CMS), <http://dx.doi.org/10.17181/CERN.QZ28.FLHW>.
- [406] G. Hall, U. Marconi, C. Matteuzzi, D. Počanić, (MUonE), Proposal for Phase 1 of the MUonE Experiment, Tech. Rep., CERN, Geneva, 2024, <https://cds.cern.ch/record/2896293>.
- [407] E. Spedicato, *PoS ICHEP2024* (2025) 345, <http://dx.doi.org/10.22323/1.476.0345>.
- [408] R.N. Pilato, *PoS ICHEP2024* (2025) 387, <http://dx.doi.org/10.22323/1.476.0387>.
- [409] G. D'Agostini, *Nucl. Instrum. Meth. A* 346 (1994) 306, [http://dx.doi.org/10.1016/0168-9002\(94\)90719-6](http://dx.doi.org/10.1016/0168-9002(94)90719-6).
- [410] V. Blobel, *EConf C030908* (2003) MOET002, <https://www.slac.stanford.edu/econf/C030908/papers/MOET002.pdf>.
- [411] A. Denig, BESIII investigations of radiative corrections, 2024, Seventh Plenary Workshop of the Muon $g - 2$ Theory Initiative, KEK, <https://conference-indico.kek.jp/event/257/contributions/5778/>.
- [412] J. Bijmns, J. Refelors, *JHEP* 12 (2017) 114, [http://dx.doi.org/10.1007/JHEP12\(2017\)114](http://dx.doi.org/10.1007/JHEP12(2017)114), [arXiv:1710.04479](https://arxiv.org/abs/1710.04479) [hep-lat].
- [413] C. Aubin, T. Blum, P. Chau, M. Golterman, S. Peris, C. Tu, *Phys. Rev. D* 93 (2016) 054508, <http://dx.doi.org/10.1103/PhysRevD.93.054508>, [arXiv:1512.07555](https://arxiv.org/abs/1512.07555) [hep-lat].
- [414] C. Aubin, T. Blum, C. Tu, M. Golterman, C. Jung, S. Peris, *Phys. Rev. D* 101 (2020) 014503, <http://dx.doi.org/10.1103/PhysRevD.101.014503>, [arXiv:1905.09307](https://arxiv.org/abs/1905.09307) [hep-lat].
- [415] C. Aubin, T. Blum, M. Golterman, S. Peris, *Phys. Rev. D* 102 (2020) 094511, <http://dx.doi.org/10.1103/PhysRevD.102.094511>, [arXiv:2008.03809](https://arxiv.org/abs/2008.03809) [hep-lat].
- [416] B. Chakraborty, C.T.H. Davies, P.G. de Oliveira, J. Koponen, G.P. Lepage, R.S. Van de Water, (HPQCD), *Phys. Rev. D* 96 (2017) 034516, <http://dx.doi.org/10.1103/PhysRevD.96.034516>, [arXiv:1601.03071](https://arxiv.org/abs/1601.03071) [hep-lat].
- [417] H.B. Meyer, *Phys. Rev. Lett.* 107 (2011) 072002, <http://dx.doi.org/10.1103/PhysRevLett.107.072002>, [arXiv:1105.1892](https://arxiv.org/abs/1105.1892) [hep-lat].
- [418] A. Francis, B. Jäger, H.B. Meyer, H. Wittig, *Phys. Rev. D* 88 (2013) 054502, <http://dx.doi.org/10.1103/PhysRevD.88.054502>, [arXiv:1306.2532](https://arxiv.org/abs/1306.2532) [hep-lat].
- [419] M. Della Morte, A. Francis, V. Gülpers, G. Herdoíza, G. von Hippel, H. Horch, B. Jäger, H.B. Meyer, A. Nyffeler, H. Wittig, *JHEP* 10 (2017) 020, [http://dx.doi.org/10.1007/JHEP10\(2017\)020](http://dx.doi.org/10.1007/JHEP10(2017)020), [arXiv:1705.01775](https://arxiv.org/abs/1705.01775) [hep-lat].
- [420] M.T. Hansen, A. Patella, *Phys. Rev. Lett.* 123 (2019) 172001, <http://dx.doi.org/10.1103/PhysRevLett.123.172001>, [arXiv:1904.10010](https://arxiv.org/abs/1904.10010) [hep-lat].
- [421] M.T. Hansen, A. Patella, *JHEP* 10 (2020) 029, [http://dx.doi.org/10.1007/JHEP10\(2020\)029](http://dx.doi.org/10.1007/JHEP10(2020)029), [arXiv:2004.03935](https://arxiv.org/abs/2004.03935) [hep-lat].
- [422] D. Giusti, V. Lubicz, G. Martinelli, F. Sanfilippo, S. Simula, *PoS CD2018* (2019) 063, <http://dx.doi.org/10.22323/1.317.0063>, [arXiv:1909.01962](https://arxiv.org/abs/1909.01962) [hep-lat].
- [423] Y. Aoki, et al., (FLAG), [arXiv:2411.04268](https://arxiv.org/abs/2411.04268) [hep-lat].
- [424] S. Borsányi, S. Dürer, Z. Fodor, C. Hoelbling, S.D. Katz, S. Krieg, T. Kurth, L. Lellouch, T. Lippert, C. McNeile, (BMW), *JHEP* 09 (2012) 010, [http://dx.doi.org/10.1007/JHEP09\(2012\)010](http://dx.doi.org/10.1007/JHEP09(2012)010), [arXiv:1203.4469](https://arxiv.org/abs/1203.4469) [hep-lat].
- [425] D. Bernecker, H.B. Meyer, *Eur. Phys. J. A* 47 (2011) 148, <http://dx.doi.org/10.1140/epja/i2011-11148-6>, [arXiv:1107.4388](https://arxiv.org/abs/1107.4388) [hep-lat].

- [426] C. Aubin, T. Blum, M. Golterman, S. Peris, Phys. Rev. D 86 (2012) 054509, <http://dx.doi.org/10.1103/PhysRevD.86.054509>, arXiv:1205.3695 [hep-lat].
- [427] B. Chakraborty, C.T.H. Davies, G.C. Donald, R.J. Dowdall, J. Koponen, G.P. Lepage, T. Teubner, (HPQCD), Phys. Rev. D 89 (2014) 114501, <http://dx.doi.org/10.1103/PhysRevD.89.114501>, arXiv:1403.1778 [hep-lat].
- [428] D. Giusti, S. Simula, PoS LATTICE2021 (2022) 189, <http://dx.doi.org/10.22323/1.396.0189>, arXiv:2111.15329 [hep-lat].
- [429] A.H. Hoang, M. Jezabek, J.H. Kühn, T. Teubner, Phys. Lett. B 338 (1994) 330, [http://dx.doi.org/10.1016/0370-2693\(94\)91387-0](http://dx.doi.org/10.1016/0370-2693(94)91387-0), arXiv:hep-ph/9407338.
- [430] M. Della Morte, R. Sommer, S. Takeda, Phys. Lett. B 672 (2009) 407, <http://dx.doi.org/10.1016/j.physletb.2009.01.059>, arXiv:0807.1120 [hep-lat].
- [431] M. Cè, T. Harris, H.B. Meyer, A. Toniato, C. Török, JHEP 12 (2021) 215, [http://dx.doi.org/10.1007/JHEP12\(2021\)215](http://dx.doi.org/10.1007/JHEP12(2021)215), arXiv:2106.15293 [hep-lat].
- [432] R. Sommer, L. Chimirri, N. Husung, PoS LATTICE2022 (2023) 358, <http://dx.doi.org/10.22323/1.430.0358>, arXiv:2211.15750 [hep-lat].
- [433] A.O.G. Källén, A. Sabry, Kong. Dan. Vid. Sel. Mat. Fys. Med. 29N17 (1955) 1, http://dx.doi.org/10.1007/978-3-319-00627-7_93.
- [434] K.G. Chetyrkin, J.H. Kühn, M. Steinhauser, Phys. Lett. B 371 (1996) 93, [http://dx.doi.org/10.1016/0370-2693\(95\)01593-0](http://dx.doi.org/10.1016/0370-2693(95)01593-0), arXiv:hep-ph/9511430.
- [435] K.G. Chetyrkin, J.H. Kühn, A. Kwiatkowski, Phys. Rep. 277 (1996) 189, [http://dx.doi.org/10.1016/S0370-1573\(96\)00012-9](http://dx.doi.org/10.1016/S0370-1573(96)00012-9), arXiv:hep-ph/9503396.
- [436] J. Erler, P. Masjuan, H. Spiesberger, PoS CHARM2020 (2021) 012, <http://dx.doi.org/10.22323/1.385.0012>.
- [437] B. Colquhoun, R.J. Dowdall, C.T.H. Davies, K. Hornbostel, G.P. Lepage, Phys. Rev. D 91 (2015) 074514, <http://dx.doi.org/10.1103/PhysRevD.91.074514>, arXiv:1408.5768 [hep-lat].
- [438] D. Hatton, C.T.H. Davies, J. Koponen, G.P. Lepage, A.T. Lytle, Phys. Rev. D 103 (2021) 054512, <http://dx.doi.org/10.1103/PhysRevD.103.054512>, arXiv:2101.08103 [hep-lat].
- [439] G.M. de Divitiis, et al., (RM123), JHEP 04 (2012) 124, [http://dx.doi.org/10.1007/JHEP04\(2012\)124](http://dx.doi.org/10.1007/JHEP04(2012)124), arXiv:1110.6294 [hep-lat].
- [440] G.M. de Divitiis, R. Frezzotti, V. Lubicz, G. Martinelli, R. Petronzio, G.C. Rossi, F. Sanfilippo, S. Simula, N. Tantalo, (RM123), Phys. Rev. D 87 (2013) 114505, <http://dx.doi.org/10.1103/PhysRevD.87.114505>, arXiv:1303.4896 [hep-lat].
- [441] C. Alexandrou, et al., (ETM), Phys. Rev. D 104 (2021) 074520, <http://dx.doi.org/10.1103/PhysRevD.104.074520>, arXiv:2104.06747 [hep-lat].
- [442] A. Bazavov, et al., (MILC), Phys. Rev. D 93 (2016) 094510, <http://dx.doi.org/10.1103/PhysRevD.93.094510>, arXiv:1503.02769 [hep-lat].
- [443] R.J. Dowdall, C.T.H. Davies, G.P. Lepage, C. McNeile, Phys. Rev. D 88 (2013) 074504, <http://dx.doi.org/10.1103/PhysRevD.88.074504>, arXiv:1303.1670 [hep-lat].
- [444] B. Strassberger, et al., PoS LATTICE2021 (2022) 135, <http://dx.doi.org/10.22323/1.396.0135>, arXiv:2112.06696 [hep-lat].
- [445] S. Itatani, H. Fukaya, S. Hashimoto, Phys. Rev. D 111 (2025) 114507, <http://dx.doi.org/10.1103/PhysRevD.111.114507>, arXiv:2411.05413 [hep-lat].
- [446] D. Hatton, C.T.H. Davies, B. Galloway, J. Koponen, G.P. Lepage, A.T. Lytle, (HPQCD), Phys. Rev. D 102 (2020) 054511, <http://dx.doi.org/10.1103/PhysRevD.102.054511>, arXiv:2005.01845 [hep-lat].
- [447] D. Giusti, F. Sanfilippo, S. Simula, Phys. Rev. D 98 (2018) 114504, <http://dx.doi.org/10.1103/PhysRevD.98.114504>, arXiv:1808.00887 [hep-lat].
- [448] D. Giusti, S. Simula, PoS LATTICE2019 (2019) 104, <http://dx.doi.org/10.22323/1.363.0104>, arXiv:1910.03874 [hep-lat].
- [449] A. Gérardin, M. Cè, G. von Hippel, B. Hörz, H.B. Meyer, D. Mohler, K. Otnad, J. Wilhelm, H. Wittig, Phys. Rev. D 100 (2019) 014510, <http://dx.doi.org/10.1103/PhysRevD.100.014510>, arXiv:1904.03120 [hep-lat].
- [450] C. DeTar, et al., Fermilab Lattice, HPQCD, MILC, PoS LATTICE2019 (2019) 070, <http://dx.doi.org/10.22323/1.363.0070>, arXiv:1912.04382 [hep-lat].
- [451] E. Shintani, Y. Kuramashi, Phys. Rev. D 100 (2019) 034517, <http://dx.doi.org/10.1103/PhysRevD.100.034517>, arXiv:1902.00885 [hep-lat].
- [452] S. Borsányi, et al., (BMW), Phys. Rev. Lett. 121 (2018) 022002, <http://dx.doi.org/10.1103/PhysRevLett.121.022002>, arXiv:1711.04980 [hep-lat].
- [453] T. Blum, Phys. Rev. Lett. 91 (2003) 052001, <http://dx.doi.org/10.1103/PhysRevLett.91.052001>, arXiv:hep-lat/0212018.
- [454] C. Aubin, T. Blum, Phys. Rev. D 75 (2007) 114502, <http://dx.doi.org/10.1103/PhysRevD.75.114502>, arXiv:hep-lat/0608011.
- [455] X. Feng, K. Jansen, M. Petschlies, D.B. Renner, (ETM), Phys. Rev. Lett. 107 (2011) 081802, <http://dx.doi.org/10.1103/PhysRevLett.107.081802>, arXiv:1103.4818 [hep-lat].
- [456] F. Burger, X. Feng, G. Hotzel, K. Jansen, M. Petschlies, D.B. Renner, (ETM), JHEP 02 (2014) 099, [http://dx.doi.org/10.1007/JHEP02\(2014\)099](http://dx.doi.org/10.1007/JHEP02(2014)099), arXiv:1308.4327 [hep-lat].
- [457] C.T.H. Davies, et al., (Fermilab Lattice, HPQCD, MILC), Phys. Rev. D 101 (2020) 034512, <http://dx.doi.org/10.1103/PhysRevD.101.034512>, arXiv:1902.04223 [hep-lat].
- [458] J. Parrino, V. Bilošytyskyi, E.-H. Chao, H.B. Meyer, V. Pascalutsa, JHEP 07 (2025) 201, [http://dx.doi.org/10.1007/JHEP07\(2025\)201](http://dx.doi.org/10.1007/JHEP07(2025)201), arXiv:2501.03192 [hep-lat].
- [459] M. Di Carlo, D. Giusti, V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo, S. Simula, N. Tantalo, Phys. Rev. D 100 (2019) 034514, <http://dx.doi.org/10.1103/PhysRevD.100.034514>, arXiv:1904.08731 [hep-lat].
- [460] J. Gasser, A. Rusetsky, I. Scimemi, Eur. Phys. J. C 32 (2003) 97, <http://dx.doi.org/10.1140/epjc/s2003-01383-1>, arXiv:hep-ph/0305260.
- [461] G. Ray, et al., PoS LATTICE2022 (2023) 329, <http://dx.doi.org/10.22323/1.430.0329>, arXiv:2212.12031 [hep-lat].
- [462] M. Hayakawa, S. Uno, Progr. Theoret. Phys. 120 (2008) 413, <http://dx.doi.org/10.1143/PTP.120.413>, arXiv:0804.2044 [hep-ph].
- [463] J. Green, N. Asmussen, O. Gryniuk, G. von Hippel, H.B. Meyer, A. Nyffeler, V. Pascalutsa, PoS LATTICE2015 (2016) 109, <http://dx.doi.org/10.22323/1.251.0109>, arXiv:1510.08384 [hep-lat].
- [464] X. Feng, L. Jin, Phys. Rev. D 100 (2019) 094509, <http://dx.doi.org/10.1103/PhysRevD.100.094509>, arXiv:1812.09817 [hep-lat].
- [465] Z. Davoudi, J. Harrison, A. Jüttner, A. Portelli, M.J. Savage, Phys. Rev. D 99 (2019) 034510, <http://dx.doi.org/10.1103/PhysRevD.99.034510>, arXiv:1810.05923 [hep-lat].
- [466] M. Di Carlo, M.T. Hansen, N. Hermansson-Truedsson, A. Portelli, 2025, arXiv:2501.07936 [hep-lat].
- [467] M.G. Endres, A. Shindler, B.C. Tiburzi, A. Walker-Loud, Phys. Rev. Lett. 117 (2016) 072002, <http://dx.doi.org/10.1103/PhysRevLett.117.072002>, arXiv:1507.08916 [hep-lat].
- [468] A. Bussone, M. Della Morte, T. Janowski, EPJ Web Conf. 175 (2018) 06005, <http://dx.doi.org/10.1051/epjconf/201817506005>, arXiv:1710.06024 [hep-lat].
- [469] M.A. Clark, M. Della Morte, Z. Hall, B. Hörz, A. Nicholson, A. Shindler, J.T. Tsang, A. Walker-Loud, H. Yan, PoS LATTICE2021 (2022) 281, <http://dx.doi.org/10.22323/1.396.0281>, arXiv:2201.03251 [hep-lat].
- [470] B. Lucini, A. Patella, A. Ramos, N. Tantalo, JHEP 02 (2016) 076, [http://dx.doi.org/10.1007/JHEP02\(2016\)076](http://dx.doi.org/10.1007/JHEP02(2016)076), arXiv:1509.01636 [hep-th].
- [471] A. Altherr, et al., PoS LATTICE2025 (2025) 119, <http://dx.doi.org/10.22323/1.466.0119>, arXiv:2502.14845 [hep-lat].
- [472] M. Bruno, Isospin breaking in τ data for $(g-2)_\mu$, 2023, Sixth Plenary Workshop of the Muon $g-2$ Theory Initiative, Bern, https://indico.cern.ch/event/1258310/contributions/5516452/attachments/2710996/4707742/bern_gM2_bruno.pdf.
- [473] M. Bruno, Hadronic τ data and lattice QCD+QED simulations for the muon $(g-2)$, 2024, Seventh Plenary Workshop of the Muon $g-2$ Theory Initiative, KEK, https://conference-indico.kek.jp/event/257/contributions/5782/attachments/3713/5182/kek24_bruno.pdf.
- [474] F. Burger, K. Jansen, M. Petschlies, G. Pientka, JHEP 11 (2015) 215, [http://dx.doi.org/10.1007/JHEP11\(2015\)215](http://dx.doi.org/10.1007/JHEP11(2015)215), arXiv:1505.03283 [hep-lat].
- [475] A. Francis, V. Gülpers, G. Herdoíza, H. Horsch, B. Jäger, H.B. Meyer, H. Wittig, PoS LATTICE2015 (2015) 110, <http://dx.doi.org/10.22323/1.251.0110>, arXiv:1511.04751 [hep-lat].
- [476] M. Cè, A. Gérardin, G. von Hippel, H.B. Meyer, K. Miura, K. Otnad, A. Risch, T. San José, J. Wilhelm, H. Wittig, JHEP 08 (2022) 220, [http://dx.doi.org/10.1007/JHEP08\(2022\)220](http://dx.doi.org/10.1007/JHEP08(2022)220), arXiv:2203.08676 [hep-lat].
- [477] F. Jegerlehner, CERN Yellow Reports: Monographs, vol. 3, 2020, p. 9, <http://dx.doi.org/10.23731/CYRM-2020-003.9>.

- [478] S. Eidelman, F. Jegerlehner, A.L. Kataev, O. Veretin, Phys. Lett. B 454 (1999) 369, [http://dx.doi.org/10.1016/S0370-2693\(99\)00389-5](http://dx.doi.org/10.1016/S0370-2693(99)00389-5), arXiv:hep-ph/9812521 [hep-ph].
- [479] F. Jegerlehner, Nucl. Phys. Proc. Suppl. 181–182 (2008) 135, <http://dx.doi.org/10.1016/j.nuclphysbps.2008.09.010>, arXiv:0807.4206 [hep-ph].
- [480] K.G. Chetyrkin, J.H. Kühn, M. Steinhauser, Nuclear Phys. B 482 (1996) 213, [http://dx.doi.org/10.1016/S0550-3213\(96\)00534-2](http://dx.doi.org/10.1016/S0550-3213(96)00534-2), arXiv:hep-ph/9606230.
- [481] J. Erler, R. Ferro-Hernández, JHEP 12 (2023) 131, [http://dx.doi.org/10.1007/JHEP12\(2023\)131](http://dx.doi.org/10.1007/JHEP12(2023)131), arXiv:2308.05740 [hep-ph].
- [482] M. Davier, D. Díaz-Calderón, B. Malaescu, A. Pich, A. Rodríguez-Sánchez, Z. Zhang, JHEP 04 (2023) 067, [http://dx.doi.org/10.1007/JHEP04\(2023\)067](http://dx.doi.org/10.1007/JHEP04(2023)067), arXiv:2302.01359 [hep-ph].
- [483] J. de Blas, M. Ciuchini, E. Franco, A. Gonçalves, S. Mishima, M. Pierini, L. Reina, L. Silvestrini, Phys. Rev. D 106 (2022) 033003, <http://dx.doi.org/10.1103/PhysRevD.106.033003>, arXiv:2112.07274 [hep-ph].
- [484] J. Haller, A. Hoecker, R. Kogler, K. Mönig, T. Peiffer, J. Stelzer, Eur. Phys. J. C 78 (2018) 675, <http://dx.doi.org/10.1140/epjc/s10052-018-6131-3>, arXiv:1803.01853 [hep-ph].
- [485] J. De Blas, et al., Eur. Phys. J. C 80 (2020) 456, <http://dx.doi.org/10.1140/epjc/s10052-020-7904-z>, arXiv:1910.14012 [hep-ph].
- [486] W.J. Marciano, A. Sirlin, Phys. Rev. D 29 (1984) 75, <http://dx.doi.org/10.1103/PhysRevD.29.75>; Phys. Rev. D 31 (1985) 213, Erratum.
- [487] F. Jegerlehner, Z. Phys. C 32 (1986) 195, <http://dx.doi.org/10.1007/BF01552495>.
- [488] F. Jegerlehner, Conf. Proc. C 900603 (1990) 476, <https://inspirehep.net/literature/316501>.
- [489] W.J. Marciano, 21st annual SLAC summer institute on particle physics: Spin structure in high-energy processes, 1993, pp. 35–56, (School: 26 July–3 August, Topical Conference: 4–6 August) (SSI 93), <https://inspirehep.net/literature/363961>.
- [490] J. Erler, R. Ferro-Hernández, S. Kuberski, Phys. Rev. Lett. 133 (2024) 171801, <http://dx.doi.org/10.1103/PhysRevLett.133.171801>, arXiv:2406.16691 [hep-ph].
- [491] H.B. Meyer, Eur. Phys. J. C 77 (2017) 616, <http://dx.doi.org/10.1140/epjc/s10052-017-5200-3>, arXiv:1706.01139 [hep-lat].
- [492] E.-H. Chao, H.B. Meyer, J. Parrino, Phys. Rev. D 107 (2023) 054505, <http://dx.doi.org/10.1103/PhysRevD.107.054505>, arXiv:2211.15581 [hep-lat].
- [493] V. Biloshytskiy, E.-H. Chao, A. Gérardin, J.R. Green, F. Hagelstein, H.B. Meyer, J. Parrino, V. Pascalutsa, JHEP 03 (2023) 194, [http://dx.doi.org/10.1007/JHEP03\(2023\)194](http://dx.doi.org/10.1007/JHEP03(2023)194), arXiv:2209.02149 [hep-lat].
- [494] J. Gasser, H. Leutwyler, A. Rusetsky, Eur. Phys. J. C 80 (2020) 1121, <http://dx.doi.org/10.1140/epjc/s10052-020-08615-2>, arXiv:2008.05806 [hep-ph].
- [495] B.E. Lautrup, A. Peterman, E. de Rafael, Phys. Rep. 3 (1972) 193, [http://dx.doi.org/10.1016/0370-1573\(72\)90011-7](http://dx.doi.org/10.1016/0370-1573(72)90011-7).
- [496] A.V. Nesterenko, J. Phys. G 51 (2024) 015005, <http://dx.doi.org/10.1088/1361-6471/ad0b33>, arXiv:2306.16392 [hep-ph].
- [497] T. DeGrand, Phys. Rev. D 106 (2022) 014504, <http://dx.doi.org/10.1103/PhysRevD.106.014504>, arXiv:2203.04393 [hep-lat].
- [498] M. Davier, Z. Fodor, A. Gérardin, L. Lellouch, B. Malaescu, F.M. Stokes, K.K. Szabó, B.C. Toth, L. Varnhorst, Z. Zhang, Phys. Rev. D 109 (2024) 076019, <http://dx.doi.org/10.1103/PhysRevD.109.076019>, arXiv:2308.04221 [hep-ph].
- [499] C.T.H. Davies, A.S. Kronfeld, G.P. Lepage, C. McNeile, R.S. Van de Water, Phys. Rev. D 111 (2025) 014513, <http://dx.doi.org/10.1103/PhysRevD.111.014513>, arXiv:2410.23832 [hep-lat].
- [500] C. Alexandrou, et al., (ETM), Phys. Rev. Lett. 130 (2023) 241901, <http://dx.doi.org/10.1103/PhysRevLett.130.241901>, arXiv:2212.08467 [hep-lat].
- [501] J. Bijnens, E. Pallante, J. Prades, Nuclear Phys. B 626 (2002) 410, [http://dx.doi.org/10.1016/S0550-3213\(02\)00074-3](http://dx.doi.org/10.1016/S0550-3213(02)00074-3), arXiv:hep-ph/0112255 [hep-ph].
- [502] M. Hayakawa, T. Kinoshita, 2001, arXiv:hep-ph/0112102 [hep-ph].
- [503] K. Melnikov, A. Vainshtein, Phys. Rev. D 70 (2004) 113006, <http://dx.doi.org/10.1103/PhysRevD.70.113006>, arXiv:hep-ph/0312226 [hep-ph].
- [504] W.A. Bardeen, W.K. Tung, Phys. Rev. 173 (1968) 1423, <http://dx.doi.org/10.1103/PhysRev.173.1423>; Phys. Rev. D 4 (1971) 3229, Erratum.
- [505] R. Tarrach, Nuovo Cim. A 28 (1975) 409, <http://dx.doi.org/10.1007/BF02894857>.
- [506] J. Aldins, T. Kinoshita, S.J. Brodsky, A.J. Dufner, Phys. Rev. D 1 (1970) 2378, <http://dx.doi.org/10.1103/PhysRevD.1.2378>.
- [507] J. Lüdtkke, M. Procura, P. Stoffer, JHEP 04 (2023) 125, [http://dx.doi.org/10.1007/JHEP04\(2023\)125](http://dx.doi.org/10.1007/JHEP04(2023)125), arXiv:2302.12264 [hep-ph].
- [508] M. Ablikim, et al., (BESIII), Phys. Rev. D 109 (2024) 072001, <http://dx.doi.org/10.1103/PhysRevD.109.072001>, arXiv:2401.09136 [hep-ex].
- [509] M. Ablikim, et al., (BESIII), Phys. Rev. D 92 (2015) 012001, <http://dx.doi.org/10.1103/PhysRevD.92.012001>, arXiv:1504.06016 [hep-ex].
- [510] P. Adlarson, et al., (A2), Phys. Rev. C 95 (2017) 035208, <http://dx.doi.org/10.1103/PhysRevC.95.035208>, arXiv:1609.04503 [hep-ex].
- [511] R. Arnaldi, et al., (NA60), Phys. Lett. B 757 (2016) 437, <http://dx.doi.org/10.1016/j.physletb.2016.04.013>, arXiv:1608.07898 [hep-ex].
- [512] M. Ablikim, et al., (BESIII), Phys. Rev. D 105 (2022) 112010, <http://dx.doi.org/10.1103/PhysRevD.105.112010>, arXiv:2203.12229 [hep-ex].
- [513] A. Hayrapetyan, et al., (CMS), Phys. Rev. Lett. 131 (2023) 091903, <http://dx.doi.org/10.1103/PhysRevLett.131.091903>, arXiv:2305.04904 [hep-ex].
- [514] T. Petri, Anomalous Decays of Pseudoscalar Mesons (Master's thesis), Universität Bonn, 2010, arXiv:1010.2378 [nucl-th].
- [515] M. Ablikim, et al., (BESIII), Phys. Rev. Lett. 129 (2022) 122001, <http://dx.doi.org/10.1103/PhysRevLett.129.122001>, arXiv:2203.13782 [hep-ex].
- [516] H. Czyż, J.H. Kühn, S. Tracz, Phys. Rev. D 94 (2016) 034033, <http://dx.doi.org/10.1103/PhysRevD.94.034033>, arXiv:1605.06803 [hep-ph].
- [517] M. Ablikim, et al., (BESIII), Phys. Rev. D 109 (2024) 032006, <http://dx.doi.org/10.1103/PhysRevD.109.032006>, arXiv:2311.12895 [hep-ex].
- [518] F.-K. Guo, B. Kubis, A. Wirzba, Phys. Rev. D 85 (2012) 014014, <http://dx.doi.org/10.1103/PhysRevD.85.014014>, arXiv:1111.5949 [hep-ph].
- [519] M. Ablikim, et al., (BESIII), Phys. Rev. D 103 (2021) 092005, <http://dx.doi.org/10.1103/PhysRevD.103.092005>, arXiv:2011.07902 [hep-ex].
- [520] M. Ablikim, et al., (BESIII), Phys. Rev. D 103 (2021) 072006, <http://dx.doi.org/10.1103/PhysRevD.103.072006>, arXiv:2012.04257 [hep-ex].
- [521] M. Ablikim, et al., (BESIII), JHEP 07 (2024) 135, [http://dx.doi.org/10.1007/JHEP07\(2024\)135](http://dx.doi.org/10.1007/JHEP07(2024)135), arXiv:2402.01993 [hep-ex].
- [522] M. Ablikim other, (BESIII), Phys. Rev. D 110 (2024) 032005, <http://dx.doi.org/10.1103/PhysRevD.110.032005>, arXiv:2401.14711 [hep-ex].
- [523] M. Ablikim, et al., (BESIII), Phys. Rev. D 108 (2023) L111101, <http://dx.doi.org/10.1103/PhysRevD.108.L111101>, arXiv:2310.10452 [hep-ex].
- [524] M. Ablikim, et al., (BESIII), Phys. Lett. B 813 (2021) 136059, <http://dx.doi.org/10.1016/j.physletb.2020.136059>, arXiv:2009.08099 [hep-ex].
- [525] M. Ablikim, et al., (BESIII), JHEP 07 (2024) 093, [http://dx.doi.org/10.1007/JHEP07\(2024\)093](http://dx.doi.org/10.1007/JHEP07(2024)093), arXiv:2404.07436 [hep-ex].
- [526] M. Ablikim, et al., (BESIII), Phys. Rev. D 103 (2021) 072007, <http://dx.doi.org/10.1103/PhysRevD.103.072007>, arXiv:2012.07360 [hep-ex].
- [527] J. Bijnens, N. Hermansson-Truedsson, A. Rodríguez-Sánchez, JHEP 02 (2023) 167, [http://dx.doi.org/10.1007/JHEP02\(2023\)167](http://dx.doi.org/10.1007/JHEP02(2023)167), arXiv:2211.17183 [hep-ph].
- [528] G.P. Lepage, S.J. Brodsky, Phys. Lett. B 87 (1979) 359, [http://dx.doi.org/10.1016/0370-2693\(79\)90554-9](http://dx.doi.org/10.1016/0370-2693(79)90554-9).
- [529] S.J. Brodsky, G.P. Lepage, Phys. Rev. D 24 (1981) 1808, <http://dx.doi.org/10.1103/PhysRevD.24.1808>.
- [530] B. Ioffe, A. Smilga, Nuclear Phys. B 232 (1984) 109, [http://dx.doi.org/10.1016/0550-3213\(84\)90364-X](http://dx.doi.org/10.1016/0550-3213(84)90364-X).
- [531] I.I. Balitsky, A.V. Yung, Phys. Lett. B 129 (1983) 328, [http://dx.doi.org/10.1016/0370-2693\(83\)90676-7](http://dx.doi.org/10.1016/0370-2693(83)90676-7).
- [532] N.I. Usyukina, A.I. Davydychev, Phys. Lett. B 332 (1994) 159, [http://dx.doi.org/10.1016/0370-2693\(94\)90874-5](http://dx.doi.org/10.1016/0370-2693(94)90874-5), arXiv:hep-ph/9402223.
- [533] F. Chavez, C. Duhr, JHEP 11 (2012) 114, [http://dx.doi.org/10.1007/JHEP11\(2012\)114](http://dx.doi.org/10.1007/JHEP11(2012)114), arXiv:1209.2722 [hep-ph].
- [534] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub, P. Stoffer, Phys. Rev. D 101 (2020) 051501, <http://dx.doi.org/10.1103/PhysRevD.101.051501>, arXiv:1910.11881 [hep-ph].
- [535] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub, P. Stoffer, JHEP 03 (2020) 101, [http://dx.doi.org/10.1007/JHEP03\(2020\)101](http://dx.doi.org/10.1007/JHEP03(2020)101), arXiv:1910.13432 [hep-ph].
- [536] K. Melnikov, A. Vainshtein, 2019, arXiv:1911.05874 [hep-ph].
- [537] J. Lüdtkke, M. Procura, Eur. Phys. J. C 80 (2020) 1108, <http://dx.doi.org/10.1140/epjc/s10052-020-08611-6>, arXiv:2006.00007 [hep-ph].

- [538] M. Knecht, JHEP 08 (2020) 056, [http://dx.doi.org/10.1007/JHEP08\(2020\)056](http://dx.doi.org/10.1007/JHEP08(2020)056), arXiv:2005.09929 [hep-ph].
- [539] L. Cappiello, O. Catà, G. D'Ambrosio, Phys. Rev. D 105 (2022) 056020, <http://dx.doi.org/10.1103/PhysRevD.105.056020>, arXiv:2110.05962 [hep-ph].
- [540] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub, P. Stoffer, Eur. Phys. J. C 81 (2021) 702, <http://dx.doi.org/10.1140/epjc/s10052-021-09513-x>, arXiv:2106.13222 [hep-ph].
- [541] J. Leutgeb, J. Mager, A. Rebhan, Eur. Phys. J. C 81 (2021) 1008, <http://dx.doi.org/10.1140/epjc/s10052-021-09780-8>, arXiv:2110.07458 [hep-ph].
- [542] J. Leutgeb, A. Rebhan, Phys. Rev. D 104 (2021) 094017, <http://dx.doi.org/10.1103/PhysRevD.104.094017>, arXiv:2108.12345 [hep-ph].
- [543] M. Zanke, M. Hoferichter, B. Kubis, JHEP 07 (2021) 106, [http://dx.doi.org/10.1007/JHEP07\(2021\)106](http://dx.doi.org/10.1007/JHEP07(2021)106), arXiv:2103.09829 [hep-ph].
- [544] G. Colangelo, M. Hoferichter, M. Procura, P. Stoffer, JHEP 09 (2014) 091, [http://dx.doi.org/10.1007/JHEP09\(2014\)091](http://dx.doi.org/10.1007/JHEP09(2014)091), arXiv:1402.7081 [hep-ph].
- [545] G. Colangelo, M. Hoferichter, B. Kubis, M. Procura, P. Stoffer, Phys. Lett. B 738 (2014) 6, <http://dx.doi.org/10.1016/j.physletb.2014.09.021>, arXiv:1408.2517 [hep-ph].
- [546] M. Hoferichter, G. Colangelo, M. Procura, P. Stoffer, Int. J. Mod. Phys. Conf. Ser. 35 (2014) 1460400, <http://dx.doi.org/10.1142/S2010194514604001>, arXiv:1309.6877 [hep-ph].
- [547] V. Pauk, M. Vanderhaeghen, Phys. Rev. D 90 (2014) 113012, <http://dx.doi.org/10.1103/PhysRevD.90.113012>, arXiv:1409.0819 [hep-ph].
- [548] M. Hoferichter, B.-L. Hoid, B. Kubis, S. Leupold, S.P. Schneider, Phys. Rev. Lett. 121 (2018) 112002, <http://dx.doi.org/10.1103/PhysRevLett.121.112002>, arXiv:1805.01471 [hep-ph].
- [549] A. Gérardin, H.B. Meyer, A. Nyffeler, Phys. Rev. D 100 (2019) 034520, <http://dx.doi.org/10.1103/PhysRevD.100.034520>, arXiv:1903.09471 [hep-lat].
- [550] S. Holz, M. Hoferichter, B.-L. Hoid, B. Kubis, Phys. Rev. Lett. 134 (2025) 171902, <http://dx.doi.org/10.1103/PhysRevLett.134.171902>, arXiv:2411.08098 [hep-ph].
- [551] F. Stollenwerk, C. Hanhart, A. Kupść, U.-G. Meißner, A. Wirzba, Phys. Lett. B 707 (2012) 184, <http://dx.doi.org/10.1016/j.physletb.2011.12.008>, arXiv:1108.2419 [nucl-th].
- [552] C. Hanhart, A. Kupść, U.-G. Meißner, F. Stollenwerk, A. Wirzba, Eur. Phys. J. C 73 (2013) 2668, <http://dx.doi.org/10.1140/epjc/s10052-013-2668-3>, arXiv:1307.5654 [hep-ph]; Eur. Phys. J. C 75 (2015) 242, Erratum.
- [553] B. Kubis, J. Plenter, Eur. Phys. J. C 75 (2015) 283, <http://dx.doi.org/10.1140/epjc/s10052-015-3495-5>, arXiv:1504.02588 [hep-ph].
- [554] S. Holz, J. Plenter, C.-W. Xiao, T. Dato, C. Hanhart, B. Kubis, U.-G. Meißner, A. Wirzba, Eur. Phys. J. C 81 (2021) 1002, <http://dx.doi.org/10.1140/epjc/s10052-021-09661-0>, arXiv:1509.02194 [hep-ph].
- [555] S. Holz, The Quest for the η and η' Transition Form Factors: A Stroll on the Precision Frontier (Ph.D. thesis), Universität Bonn, 2022, <https://hdl.handle.net/20.500.11811/10336>.
- [556] M. Hoferichter, B.-L. Hoid, B. Kubis, J. Lüdtkke, Phys. Rev. Lett. 128 (2022) 172004, <http://dx.doi.org/10.1103/PhysRevLett.128.172004>, arXiv:2105.04563 [hep-ph].
- [557] D. Babusci, et al., (KLOE), Phys. Lett. B 718 (2013) 910, <http://dx.doi.org/10.1016/j.physletb.2012.11.032>, arXiv:1209.4611 [hep-ex].
- [558] M. Ablikim, et al., (BESIII), Phys. Rev. Lett. 120 (2018) 242003, <http://dx.doi.org/10.1103/PhysRevLett.120.242003>, arXiv:1712.01525 [hep-ex].
- [559] J. Gasser, A. Rusetsky, Eur. Phys. J. C 78 (2018) 906, <http://dx.doi.org/10.1140/epjc/s10052-018-6378-8>, arXiv:1809.06399 [hep-ph].
- [560] A. Dobado, M.J. Herrero, T.N. Truong, Phys. Lett. B 235 (1990) 134, [http://dx.doi.org/10.1016/0370-2693\(90\)90109-J](http://dx.doi.org/10.1016/0370-2693(90)90109-J).
- [561] T.N. Truong, Phys. Rev. Lett. 67 (1991) 2260, <http://dx.doi.org/10.1103/PhysRevLett.67.2260>.
- [562] A. Dobado, J.R. Peláez, Phys. Rev. D 47 (1993) 4883, <http://dx.doi.org/10.1103/PhysRevD.47.4883>, arXiv:hep-ph/9301276.
- [563] L. Gan, B. Kubis, E. Passemar, S. Tulin, Phys. Rep. 945 (2022) 1, <http://dx.doi.org/10.1016/j.physrep.2021.11.001>, arXiv:2007.00664 [hep-ph].
- [564] H.J. Behrend, et al., (CELLO), Z. Phys. C 49 (1991) 401, <http://dx.doi.org/10.1007/BF01549692>.
- [565] J. Gronberg, et al., (CLEO), Phys. Rev. D 57 (1998) 33, <http://dx.doi.org/10.1103/PhysRevD.57.33>, arXiv:hep-ex/9707031.
- [566] M. Acciarri, et al., (L3), Phys. Lett. B 418 (1998) 399, [http://dx.doi.org/10.1016/S0370-2693\(97\)01219-7](http://dx.doi.org/10.1016/S0370-2693(97)01219-7).
- [567] P. del Amo Sanchez, et al., (BABAR), Phys. Rev. D 84 (2011) 052001, <http://dx.doi.org/10.1103/PhysRevD.84.052001>, arXiv:1101.1142 [hep-ex].
- [568] G.P. Lepage, S.J. Brodsky, Phys. Rev. D 22 (1980) 2157, <http://dx.doi.org/10.1103/PhysRevD.22.2157>.
- [569] A. Khodjamirian, Eur. Phys. J. C 6 (1999) 477, <http://dx.doi.org/10.1007/s100520005357>, arXiv:hep-ph/9712451.
- [570] V.A. Novikov, M.A. Shifman, A.I. Vainshtein, M.B. Voloshin, V.I. Zakharov, Nuclear Phys. B 237 (1984) 525, [http://dx.doi.org/10.1016/0550-3213\(84\)90006-3](http://dx.doi.org/10.1016/0550-3213(84)90006-3).
- [571] V.A. Nesterenko, A.V. Radyushkin, Sov. J. Nucl. Phys. 38 (1983) 284, <https://inspirehep.net/literature/182367>; Yad. Fiz. 38 (1983) 476.
- [572] A.S. Gorsky, Sov. J. Nucl. Phys. 46 (1987) 537, <https://inspirehep.net/literature/256640>; Yad. Fiz. 46 (1987) 938.
- [573] F. Jegerlehner, A. Nyffeler, Phys. Rep. 477 (2009) 1, <http://dx.doi.org/10.1016/j.physrep.2009.04.003>, arXiv:0902.3360 [hep-ph].
- [574] A. Nyffeler, Phys. Rev. D 94 (2016) 053006, <http://dx.doi.org/10.1103/PhysRevD.94.053006>, arXiv:1602.03398 [hep-ph].
- [575] C. Alexandrou, et al., (ETM), Phys. Rev. D 108 (2023) 054509, <http://dx.doi.org/10.1103/PhysRevD.108.054509>, arXiv:2212.06704 [hep-lat].
- [576] A. Gérardin, W.E.A. Verplanke, G. Wang, Z. Fodor, J.N. Guenther, L. Lellouch, K.K. Szabó, L. Varnhorst, Phys. Rev. D 111 (2025) 054511, <http://dx.doi.org/10.1103/PhysRevD.111.054511>, arXiv:2305.04570 [hep-lat].
- [577] K. Ottnad, C. Urbach, (ETM), Phys. Rev. D 97 (2018) 054508, <http://dx.doi.org/10.1103/PhysRevD.97.054508>, arXiv:1710.07986 [hep-lat].
- [578] G.S. Bali, V. Braun, S. Collins, A. Schäfer, J. Simeth, RQCD, JHEP 08 (2021) 137, [http://dx.doi.org/10.1007/JHEP08\(2021\)137](http://dx.doi.org/10.1007/JHEP08(2021)137), arXiv:2106.05398 [hep-lat].
- [579] B. Aubert, et al., (BABAR), Phys. Rev. D 76 (2007) 092005, <http://dx.doi.org/10.1103/PhysRevD.76.092005>, arXiv:0708.2461 [hep-ex]; Phys. Rev. D 77 (2008) 119902, Erratum.
- [580] J.P. Lees, et al., (BABAR), Phys. Rev. D 97 (2018) 052007, <http://dx.doi.org/10.1103/PhysRevD.97.052007>, arXiv:1801.02960 [hep-ex].
- [581] L.-Y. Dai, X.-W. Kang, U.-G. Meißner, X.-Y. Song, D.-L. Yao, Phys. Rev. D 97 (2018) 036012, <http://dx.doi.org/10.1103/PhysRevD.97.036012>, arXiv:1712.02119 [hep-ph].
- [582] M. Benayoun, L. DelBuono, F. Jegerlehner, Eur. Phys. J. C 84 (2024) 295, <http://dx.doi.org/10.1140/epjc/s10052-024-12608-w>, arXiv:2309.15050 [hep-ph].
- [583] C. Hanhart, S. Holz, B. Kubis, A. Kupść, A. Wirzba, C.-W. Xiao, Eur. Phys. J. C 77 (2017) 98, <http://dx.doi.org/10.1140/epjc/s10052-017-4651-x>, arXiv:1611.09359 [hep-ph]; Eur. Phys. J. C 78 (2018) 450, Erratum.
- [584] G. Colangelo, M. Hoferichter, M. Procura, P. Stoffer, Phys. Rev. Lett. 118 (2017) 232001, <http://dx.doi.org/10.1103/PhysRevLett.118.232001>, arXiv:1701.06554 [hep-ph].
- [585] E.J. Estrada, J.M. Márquez, D. Portillo-Sánchez, P. Roig, Phys. Rev. D 111 (2025) 093008, <http://dx.doi.org/10.1103/PhysRevD.111.093008>, arXiv:2411.07115 [hep-ph].
- [586] G. Eichmann, C.S. Fischer, R. Williams, Phys. Rev. D 101 (2020) 054015, <http://dx.doi.org/10.1103/PhysRevD.101.054015>, arXiv:1910.06795 [hep-ph].
- [587] I. Danilkin, O. Deineka, M. Vanderhaeghen, Phys. Rev. D 103 (2021) 114023, <http://dx.doi.org/10.1103/PhysRevD.103.114023>, arXiv:2012.11636 [hep-ph].
- [588] M. Knecht, S. Narison, A. Rabemananjara, D. Rabetiarivony, Phys. Lett. B 787 (2018) 111, <http://dx.doi.org/10.1016/j.physletb.2018.10.048>, arXiv:1808.03848 [hep-ph].

- [589] G.A. Schuler, F.A. Berends, R. van Gulik, Nuclear Phys. B 523 (1998) 423, [http://dx.doi.org/10.1016/S0550-3213\(98\)00128-X](http://dx.doi.org/10.1016/S0550-3213(98)00128-X), arXiv:hep-ph/9710462.
- [590] M. Hoferichter, P. Stoffer, JHEP 05 (2020) 159, [http://dx.doi.org/10.1007/JHEP05\(2020\)159](http://dx.doi.org/10.1007/JHEP05(2020)159), arXiv:2004.06127 [hep-ph].
- [591] V. Pauk, M. Vanderhaeghen, Eur. Phys. J. C 74 (2014) 3008, <http://dx.doi.org/10.1140/epjc/s10052-014-3008-y>, arXiv:1401.0832 [hep-ph].
- [592] S. Uehara, et al., (Belle), Phys. Rev. D 80 (2009) 032001, <http://dx.doi.org/10.1103/PhysRevD.80.032001>, arXiv:0906.1464 [hep-ex].
- [593] S. Uehara, et al., (Belle), PTEP 2013 (2013) 123C01, <http://dx.doi.org/10.1093/ptep/ptt097>, arXiv:1307.7457 [hep-ex].
- [594] M. Hoferichter, P. Stoffer, M. Zillinger, Phys. Rev. Lett. 134 (2025) 061902, <http://dx.doi.org/10.1103/PhysRevLett.134.061902>, arXiv:2412.00190 [hep-ph].
- [595] S. Bellucci, J. Gasser, M.E. Sainio, Nuclear Phys. B 423 (1994) 80, [http://dx.doi.org/10.1016/0550-3213\(94\)90566-5](http://dx.doi.org/10.1016/0550-3213(94)90566-5), arXiv:hep-ph/9401206; Nuclear Phys. B 431 (1994) 413, Erratum.
- [596] V. Mathieu, A. Pilloni, M. Albaladejo, L. Bibrzycki, A. Celentano, C. Fernández-Ramírez, A.P. Szczepaniak, (JPAC), Phys. Rev. D 102 (2020) 014003, <http://dx.doi.org/10.1103/PhysRevD.102.014003>, arXiv:2005.01617 [hep-ph].
- [597] I. Danilkin, M. Vanderhaeghen, Phys. Rev. D 95 (2017) 014019, <http://dx.doi.org/10.1103/PhysRevD.95.014019>, arXiv:1611.04646 [hep-ph].
- [598] P. Achard, et al., (L3), Phys. Lett. B 526 (2002) 269, [http://dx.doi.org/10.1016/S0370-2693\(01\)01477-0](http://dx.doi.org/10.1016/S0370-2693(01)01477-0), arXiv:hep-ex/0110073.
- [599] M.N. Achasov, et al., (SND), Phys. Lett. B 800 (2020) 135074, <http://dx.doi.org/10.1016/j.physletb.2019.135074>, arXiv:1906.03838 [hep-ex].
- [600] P. Achard, et al., (L3), JHEP 03 (2007) 018, <http://dx.doi.org/10.1088/1126-6708/2007/03/018>.
- [601] S.S. Agaev, V.M. Braun, N. Offen, F.A. Porkert, A. Schäfer, Phys. Rev. D 90 (2014) 074019, <http://dx.doi.org/10.1103/PhysRevD.90.074019>, arXiv:1409.4311 [hep-ph].
- [602] J.M. Maldacena, Internat. J. Theoret. Phys. 38 (1999) 1113, <http://dx.doi.org/10.1023/A:1026654312961>, arXiv:hep-th/9711200 [hep-th].
- [603] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, Y. Oz, Phys. Rep. 323 (2000) 183, [http://dx.doi.org/10.1016/S0370-1573\(99\)00083-6](http://dx.doi.org/10.1016/S0370-1573(99)00083-6), arXiv:hep-th/9905111 [hep-th].
- [604] T. Sakai, S. Sugimoto, Progr. Theoret. Phys. 113 (2005) 843, <http://dx.doi.org/10.1143/PTP.113.843>, arXiv:hep-th/0412141 [hep-th].
- [605] T. Sakai, S. Sugimoto, Progr. Theoret. Phys. 114 (2005) 1083, <http://dx.doi.org/10.1143/PTP.114.1083>, arXiv:hep-th/0507073 [hep-th].
- [606] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 505, <http://dx.doi.org/10.4310/ATMP.1998.v2.n3.a3>, arXiv:hep-th/9803131 [hep-th].
- [607] J. Erlich, E. Katz, D.T. Son, M.A. Stephanov, Phys. Rev. Lett. 95 (2005) 261602, <http://dx.doi.org/10.1103/PhysRevLett.95.261602>, arXiv:hep-ph/0501128 [hep-ph].
- [608] L. Da Rold, A. Pomarol, Nuclear Phys. B 721 (2005) 79, <http://dx.doi.org/10.1016/j.nuclphysb.2005.05.009>, arXiv:hep-ph/0501218 [hep-ph].
- [609] J. Hirn, V. Sanz, JHEP 12 (2005) 030, <http://dx.doi.org/10.1088/1126-6708/2005/12/030>, arXiv:hep-ph/0507049 [hep-ph].
- [610] K. Ghoroku, N. Maru, M. Tachibana, M. Yahiro, Phys. Lett. B 633 (2006) 602, <http://dx.doi.org/10.1016/j.physletb.2005.12.004>, arXiv:hep-ph/0510334.
- [611] A. Karch, E. Katz, D.T. Son, M.A. Stephanov, Phys. Rev. D 74 (2006) 015005, <http://dx.doi.org/10.1103/PhysRevD.74.015005>, arXiv:hep-ph/0602229 [hep-ph].
- [612] H.J. Kwee, R.F. Lebed, JHEP 01 (2008) 027, <http://dx.doi.org/10.1088/1126-6708/2008/01/027>, arXiv:0708.4054 [hep-ph].
- [613] U. Gürsoy, E. Kiritsis, JHEP 02 (2008) 032, <http://dx.doi.org/10.1088/1126-6708/2008/02/032>, arXiv:0707.1324 [hep-th].
- [614] U. Gürsoy, E. Kiritsis, F. Nitti, JHEP 02 (2008) 019, <http://dx.doi.org/10.1088/1126-6708/2008/02/019>, arXiv:0707.1349 [hep-th].
- [615] P. Colangelo, F. De Fazio, F. Giannuzzi, F. Jugeau, S. Nicotri, Phys. Rev. D 78 (2008) 055009, <http://dx.doi.org/10.1103/PhysRevD.78.055009>, arXiv:0807.1054 [hep-ph].
- [616] T. Gherghetta, J.I. Kapusta, T.M. Kelley, Phys. Rev. D 79 (2009) 076003, <http://dx.doi.org/10.1103/PhysRevD.79.076003>, arXiv:0902.1998 [hep-ph].
- [617] T. Branz, T. Gutsche, V.E. Lyubovitskij, I. Schmidt, A. Vega, Phys. Rev. D 82 (2010) 074022, <http://dx.doi.org/10.1103/PhysRevD.82.074022>, arXiv:1008.0268 [hep-ph].
- [618] P. Colangelo, F. De Fazio, J.J. Sanz-Cillero, F. Giannuzzi, S. Nicotri, Phys. Rev. D 85 (2012) 035013, <http://dx.doi.org/10.1103/PhysRevD.85.035013>, arXiv:1108.5945 [hep-ph].
- [619] S.J. Brodsky, G.F. de Teramond, H.G. Dosch, J. Erlich, Phys. Rep. 584 (2015) 1, <http://dx.doi.org/10.1016/j.physrep.2015.05.001>, arXiv:1407.8131 [hep-ph].
- [620] O. Domènech, G. Panico, A. Wulzer, Nucl. Phys. A 853 (2011) 97, <http://dx.doi.org/10.1016/j.nuclphysa.2011.02.002>, arXiv:1009.0711 [hep-ph].
- [621] F. Brünner, A. Rebhan, Phys. Rev. Lett. 115 (2015) 131601, <http://dx.doi.org/10.1103/PhysRevLett.115.131601>, arXiv:1504.05815 [hep-ph].
- [622] E. Katz, M.D. Schwartz, JHEP 08 (2007) 077, <http://dx.doi.org/10.1088/1126-6708/2007/08/077>, arXiv:0705.0534 [hep-ph].
- [623] R. Casero, E. Kiritsis, A. Paredes, Nuclear Phys. B 787 (2007) 98, <http://dx.doi.org/10.1016/j.nuclphysb.2007.07.009>, arXiv:hep-th/0702155.
- [624] P. Colangelo, F. Giannuzzi, S. Nicotri, Phys. Lett. B 840 (2023) 137878, <http://dx.doi.org/10.1016/j.physletb.2023.137878>, arXiv:2301.06456 [hep-ph].
- [625] H.R. Grigoryan, A.V. Radyushkin, Phys. Rev. D 76 (2007) 115007, <http://dx.doi.org/10.1103/PhysRevD.76.115007>, arXiv:0709.0500 [hep-ph].
- [626] H.R. Grigoryan, A.V. Radyushkin, Phys. Rev. D 77 (2008) 115024, <http://dx.doi.org/10.1103/PhysRevD.77.115024>, arXiv:0803.1143 [hep-ph].
- [627] H.R. Grigoryan, A.V. Radyushkin, Phys. Rev. D 78 (2008) 115008, <http://dx.doi.org/10.1103/PhysRevD.78.115008>, arXiv:0808.1243 [hep-ph].
- [628] D.K. Hong, D. Kim, Phys. Lett. B 680 (2009) 480, <http://dx.doi.org/10.1016/j.physletb.2009.09.026>, arXiv:0904.4042 [hep-ph].
- [629] L. Cappiello, O. Catà, G. D'Ambrosio, Phys. Rev. D 83 (2011) 093006, <http://dx.doi.org/10.1103/PhysRevD.83.093006>, arXiv:1009.1161 [hep-ph].
- [630] J. Leutgeb, J. Mager, A. Rebhan, Phys. Rev. D 100 (2019) 094038, <http://dx.doi.org/10.1103/PhysRevD.100.094038>, arXiv:1906.11795 [hep-ph]; Phys. Rev. D 104 (2021) 059903, Erratum.
- [631] B. Melić, D. Müller, K. Passek-Kumerički, Phys. Rev. D 68 (2003) 014013, <http://dx.doi.org/10.1103/PhysRevD.68.014013>, arXiv:hep-ph/0212346 [hep-ph].
- [632] J. Leutgeb, A. Rebhan, M. Stadlbauer, Phys. Rev. D 105 (2022) 094032, <http://dx.doi.org/10.1103/PhysRevD.105.094032>, arXiv:2203.16508 [hep-ph].
- [633] M. Järvinen, E. Kiritsis, JHEP 03 (2012) 002, [http://dx.doi.org/10.1007/JHEP03\(2012\)002](http://dx.doi.org/10.1007/JHEP03(2012)002), arXiv:1112.1261 [hep-ph].
- [634] J. Mager, J. Leutgeb, A. Rebhan, in preparation.
- [635] J. Leutgeb, J. Mager, A. Rebhan, Phys. Rev. D 111 (2025) 114001, <http://dx.doi.org/10.1103/PhysRevD.111.114001>, arXiv:2411.10432 [hep-ph].
- [636] L.D. Landau, Dokl. Akad. Nauk. Ser. Fiz. 60 (1948) 207, <http://dx.doi.org/10.1016/B978-0-08-010586-4.50070-5>.
- [637] C.-N. Yang, Phys. Rev. 77 (1950) 242, <http://dx.doi.org/10.1103/PhysRev.77.242>.
- [638] V. Pascalutsa, V. Pauk, M. Vanderhaeghen, Phys. Rev. D 85 (2012) 116001, <http://dx.doi.org/10.1103/PhysRevD.85.116001>, arXiv:1204.0740 [hep-ph].
- [639] P. Roig, P. Sánchez-Puertas, Phys. Rev. D 101 (2020) 074019, <http://dx.doi.org/10.1103/PhysRevD.101.074019>, arXiv:1910.02881 [hep-ph].
- [640] A.E. Dorokhov, A.E. Radzhabov, A.S. Zhevlakov, Eur. Phys. J. C 71 (2011) 1702, <http://dx.doi.org/10.1140/epjc/s10052-011-1702-6>, arXiv:1103.2042 [hep-ph].
- [641] A.E. Radzhabov, A.S. Zhevlakov, A.P. Martynenko, F.A. Martynenko, Phys. Rev. D 108 (2023) 014033, <http://dx.doi.org/10.1103/PhysRevD.108.014033>, arXiv:2301.12641 [hep-ph].
- [642] A.E. Radzhabov, A.S. Zhevlakov, arXiv:2504.13588 [hep-ph].

- [643] F. Hechenberger, J. Leutgeb, A. Rebhan, Phys. Rev. D 107 (2023) 114020, <http://dx.doi.org/10.1103/PhysRevD.107.114020>, arXiv:2302.13379 [hep-ph].
- [644] J. Mager, L. Cappiello, J. Leutgeb, A. Rebhan, Phys. Rev. Lett. 135 (2025) 091901, <http://dx.doi.org/10.1103/dxwr-gpsl>, arXiv:2501.19293 [hep-ph].
- [645] E. Katz, A. Lewandowski, M.D. Schwartz, Phys. Rev. D 74 (2006) 086004, <http://dx.doi.org/10.1103/PhysRevD.74.086004>, arXiv:hep-ph/0510388.
- [646] M. Masuda, et al., (Belle), Phys. Rev. D 93 (2016) 032003, <http://dx.doi.org/10.1103/PhysRevD.93.032003>, arXiv:1508.06757 [hep-ex].
- [647] B. Aubert, et al., (BABAR), Phys. Rev. D 80 (2009) 052002, <http://dx.doi.org/10.1103/PhysRevD.80.052002>, arXiv:0905.4778 [hep-ex].
- [648] S. Uehara, et al., (Belle), Phys. Rev. D 86 (2012) 092007, <http://dx.doi.org/10.1103/PhysRevD.86.092007>, arXiv:1205.3249 [hep-ex].
- [649] J.P. Lees, et al., (BABAR), Phys. Rev. D 98 (2018) 112002, <http://dx.doi.org/10.1103/PhysRevD.98.112002>, arXiv:1808.08038 [hep-ex].
- [650] C. Alexandrou, et al., (ETM), Phys. Rev. D 108 (2023) 094514, <http://dx.doi.org/10.1103/PhysRevD.108.094514>, arXiv:2308.12458 [hep-lat].
- [651] G. 't Hooft, Nuclear Phys. B 72 (1974) 461, [http://dx.doi.org/10.1016/0550-3213\(74\)90154-0](http://dx.doi.org/10.1016/0550-3213(74)90154-0).
- [652] P. Roig, A. Guevara, G. López Castro, Phys. Rev. D 89 (2014) 073016, <http://dx.doi.org/10.1103/PhysRevD.89.073016>, arXiv:1401.4099 [hep-ph].
- [653] A. Guevara, P. Roig, J.J. Sanz-Cillero, JHEP 06 (2018) 160, [http://dx.doi.org/10.1007/JHEP06\(2018\)160](http://dx.doi.org/10.1007/JHEP06(2018)160), arXiv:1803.08099 [hep-ph].
- [654] T. Kadavý, K. Kampf, J. Novotný, JHEP 10 (2020) 142, [http://dx.doi.org/10.1007/JHEP10\(2020\)142](http://dx.doi.org/10.1007/JHEP10(2020)142), arXiv:2006.13006 [hep-ph].
- [655] V.A. Nesterenko, A.V. Radyushkin, Phys. Lett. B 128 (1983) 439, [http://dx.doi.org/10.1016/0370-2693\(83\)90935-8](http://dx.doi.org/10.1016/0370-2693(83)90935-8).
- [656] S.J. Brodsky, G.R. Farrar, Phys. Rev. Lett. 31 (1973) 1153, <http://dx.doi.org/10.1103/PhysRevLett.31.1153>.
- [657] P. Roig, J.J. Sanz Cillero, Phys. Lett. B 733 (2014) 158, <http://dx.doi.org/10.1016/j.physletb.2014.04.034>, arXiv:1312.6206 [hep-ph].
- [658] T. Feldmann, P. Kroll, B. Stech, Phys. Rev. D 58 (1998) 114006, <http://dx.doi.org/10.1103/PhysRevD.58.114006>, arXiv:hep-ph/9802409.
- [659] M. Anderson, EPJ Web Conf. 303 (2024) 01001, <http://dx.doi.org/10.1051/epjconf/202430301001>.
- [660] F. Jegerlehner, Springer Tracts Mod. Phys. 274 (2017) 1, <http://dx.doi.org/10.1007/978-3-319-63577-4>.
- [661] P. Masjuan, E. Ruiz Arriola, W. Broniowski, Phys. Rev. D 85 (2012) 094006, <http://dx.doi.org/10.1103/PhysRevD.85.094006>, arXiv:1203.4782 [hep-ph].
- [662] I.M. Nugent, T. Przedziński, P. Roig, O. Shekhovtsova, Z. Waś, Phys. Rev. D 88 (2013) 093012, <http://dx.doi.org/10.1103/PhysRevD.88.093012>, arXiv:1310.1053 [hep-ph].
- [663] T. Hahn, Comput. Phys. Comm. 168 (2005) 78, <http://dx.doi.org/10.1016/j.cpc.2005.01.010>, arXiv:hep-ph/0404043.
- [664] E.J. Estrada, P. Roig, arXiv:2504.00448 [hep-ph].
- [665] M.Q. Huber, Phys. Rev. D 101 (2020) 114009, <http://dx.doi.org/10.1103/PhysRevD.101.114009>, arXiv:2003.13703 [hep-ph].
- [666] G. Eichmann, J.M. Pawłowski, J.M. Silva, Phys. Rev. D 104 (2021) 114016, <http://dx.doi.org/10.1103/PhysRevD.104.114016>, arXiv:2107.05352 [hep-ph].
- [667] A.C. Aguilar, F. De Soto, M.N. Ferreira, J. Papavassiliou, F. Pinto-Gómez, J. Rodríguez-Quintero, L.R. Santos, Phys. Lett. B 858 (2024) 139065, <http://dx.doi.org/10.1016/j.physletb.2024.139065>, arXiv:2408.06135 [hep-ph].
- [668] G. Eichmann, H. Sanchis-Alepuz, R. Williams, R. Alkofer, C.S. Fischer, Prog. Part. Nucl. Phys. 91 (2016) 1, <http://dx.doi.org/10.1016/j.ppnp.2016.07.001>, arXiv:1606.09602 [hep-ph].
- [669] M. Ding, C.D. Roberts, S.M. Schmidt, Particles 6 (2023) 57, <http://dx.doi.org/10.3390/particles6010004>, arXiv:2211.07763 [hep-ph].
- [670] K. Raya, A. Bashir, D. Binosi, C.D. Roberts, J. Rodríguez-Quintero, Few Body Syst. 65 (2024) 60, <http://dx.doi.org/10.1007/s00601-024-01924-2>, arXiv:2403.00629 [hep-ph].
- [671] G. Eichmann, C.S. Fischer, Phys. Rev. D 87 (2013) 036006, <http://dx.doi.org/10.1103/PhysRevD.87.036006>, arXiv:1212.1761 [hep-ph].
- [672] G. Eichmann, C.S. Fischer, W. Heupel, R. Williams, AIP Conf. Proc. 1701 (2016) 040004, <http://dx.doi.org/10.1063/1.4938621>, arXiv:1411.7876 [hep-ph].
- [673] G. Eichmann, C.S. Fischer, W. Heupel, Phys. Rev. D 92 (2015) 056006, <http://dx.doi.org/10.1103/PhysRevD.92.056006>, arXiv:1505.06336 [hep-ph].
- [674] K. Raya, A. Bashir, P. Roig, Phys. Rev. D 101 (2020) 074021, <http://dx.doi.org/10.1103/PhysRevD.101.074021>, arXiv:1910.05960 [hep-ph].
- [675] T. Goecke, C.S. Fischer, R. Williams, Phys. Lett. B 704 (2011) 211, <http://dx.doi.org/10.1016/j.physletb.2011.09.019>, arXiv:1107.2588 [hep-ph].
- [676] G. Eichmann, C.S. Fischer, E. Weil, R. Williams, Phys. Lett. B 774 (2017) 425, <http://dx.doi.org/10.1016/j.physletb.2017.10.002>, arXiv:1704.05774 [hep-ph].
- [677] E. Weil, G. Eichmann, C.S. Fischer, R. Williams, Phys. Rev. D 96 (2017) 014021, <http://dx.doi.org/10.1103/PhysRevD.96.014021>, arXiv:1704.06046 [hep-ph].
- [678] A. Miramontes, A. Bashir, K. Raya, P. Roig, Phys. Rev. D 105 (2022) 074013, <http://dx.doi.org/10.1103/PhysRevD.105.074013>, arXiv:2112.13916 [hep-ph].
- [679] K. Raya, A. Bashir, A.S. Miramontes, P. Roig, Rev. Mex. Fis. Suppl. 3 (2022) 020709, <http://dx.doi.org/10.31349/SuplRevMexFis.3.020709>, arXiv:2204.01652 [hep-ph].
- [680] A.S. Miramontes, H. Sanchis-Alepuz, R. Alkofer, Phys. Rev. D 103 (2021) 116006, <http://dx.doi.org/10.1103/PhysRevD.103.116006>, arXiv:2102.12541 [hep-ph].
- [681] A.S. Miramontes, A. Bashir, Phys. Rev. D 107 (2023) 014016, <http://dx.doi.org/10.1103/PhysRevD.107.014016>, arXiv:2212.10800 [hep-ph].
- [682] A. Miramontes, A. Bashir, Rev. Mex. Fis. Suppl. 4 (2023) 021114, <http://dx.doi.org/10.31349/SuplRevMexFis.4.021114>.
- [683] A.S. Miramontes, K. Raya, A. Bashir, P. Roig, G. Paredes-Torres, 2024, arXiv:2411.02218 [hep-ph], <http://dx.doi.org/10.1088/1674-1137/add259>.
- [684] J.S. Schwinger, Proc. Nat. Acad. Sci. 72 (1975) 1, http://dx.doi.org/10.1007/978-3-7091-8424-0_9, <http://dx.doi.org/10.1073/pnas.72.1.1>; Acta Phys. Austriaca Suppl. 14 (1975) 471.
- [685] F. Hagelstein, V. Pascalutsa, Phys. Rev. Lett. 120 (2018) 072002, <http://dx.doi.org/10.1103/PhysRevLett.120.072002>, arXiv:1710.04571 [hep-ph].
- [686] F. Hagelstein, V. Pascalutsa, PoS CD2018 (2019) 066, <http://dx.doi.org/10.22323/1.317.0066>, arXiv:1907.06927 [hep-ph].
- [687] V. Biloshytskiy, Hadronic Contributions To Light-By-Light Scattering and Muon ($G - 2$) (Ph.D. thesis), Universität Mainz, 2024, <http://dx.doi.org/10.25358/openscience-11464>.
- [688] V. Biloshytskiy, F. Hagelstein, V. Pascalutsa, 2025, in preparation.
- [689] A. Gasparian, et al., A Precision Measurement of the η Radiative Decay Width Via the Primakoff Effect, Tech. Rep., JLab, 2009, https://www.jlab.org/exp_prog/proposals/10/PR12-10-011.pdf.
- [690] D. Smith, A Measurement of the Eta Meson Radiative Decay Width Via the Primakoff Effect (Ph.D. thesis), Duke University (main), 2024, <https://hdl.handle.net/10161/30944>.
- [691] A. Accardi, et al., Eur. Phys. J. A 60 (2024) 173, <http://dx.doi.org/10.1140/epja/s10050-024-01282-x>, arXiv:2306.09360 [nucl-ex].
- [692] C.F. Redmer, (BESIII), EPJ Web Conf., 212 (2019) 04004, <http://dx.doi.org/10.1051/epjconf/201921204004>.
- [693] H. Czyz, P. Kiszka, Comput. Phys. Comm. 234 (2019) 245, <http://dx.doi.org/10.1016/j.cpc.2018.07.021>, arXiv:1805.07756 [hep-ph].
- [694] V.P. Druzhinin, L.V. Kardapoltsev, V.A. Tayursky, Comput. Phys. Comm. 185 (2014) 236, <http://dx.doi.org/10.1016/j.cpc.2013.07.017>.
- [695] S. Ong, P. Kessler, Phys. Rev. D 38 (1988) 2280, <http://dx.doi.org/10.1103/PhysRevD.38.2280>.
- [696] M. Lellmann, (BESIII), EPJ Web Conf. 303 (2024) 01017, <http://dx.doi.org/10.1051/epjconf/202430301017>.
- [697] W. Altmannshofer, et al., (Belle-II), PTEP 2019 (2019) 123C01, <http://dx.doi.org/10.1093/ptep/ptz106>, arXiv:1808.10567 [hep-ex]; PTEP 2020 (2020) 029201, Erratum.

- [698] I. Danilkin, C.F. Redmer, M. Vanderhaeghen, Prog. Part. Nucl. Phys. 107 (2019) 20, <http://dx.doi.org/10.1016/j.pnpnp.2019.05.002>, arXiv:1901.10346 [hep-ph].
- [699] M. Lellmann, M. Vanderhaeghen, I. Danilkin, A. Denig, J. Muskalla, C.F. Redmer, X.L. Ren, HadroTOPS: A Monte Carlo event generator for hadronic two-photon scattering in electron positron collisions, 2025, in preparation.
- [700] V.M. Budnev, I.F. Ginzburg, G.V. Meledin, V.G. Serbo, Phys. Rep. 15 (1975) 181, [http://dx.doi.org/10.1016/0370-1573\(75\)90009-5](http://dx.doi.org/10.1016/0370-1573(75)90009-5).
- [701] X.-L. Ren, I. Danilkin, M. Vanderhaeghen, Phys. Rev. D 110 (2024) 094043, <http://dx.doi.org/10.1103/PhysRevD.110.094043>, arXiv:2409.07235 [hep-ph].
- [702] M. Hayakawa, T. Blum, PoS LAT2005 (2006) 353, <http://dx.doi.org/10.22323/1.020.0353>, arXiv:hep-lat/0509016.
- [703] T. Blum, S. Chowdhury, M. Hayakawa, T. Izubuchi, Phys. Rev. Lett. 114 (2015) 012001, <http://dx.doi.org/10.1103/PhysRevLett.114.012001>, arXiv:1407.2923 [hep-lat].
- [704] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Lehner, Phys. Rev. D 93 (2016) 014503, <http://dx.doi.org/10.1103/PhysRevD.93.014503>, arXiv:1510.07100 [hep-lat].
- [705] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner, Phys. Rev. Lett. 118 (2017) 022005, <http://dx.doi.org/10.1103/PhysRevLett.118.022005>, arXiv:1610.04603 [hep-lat].
- [706] N. Asmussen, J. Green, H.B. Meyer, A. Nyffeler, PoS LATTICE2016 (2016) 164, <http://dx.doi.org/10.22323/1.256.0164>, arXiv:1609.08454 [hep-lat].
- [707] N. Asmussen, A. Gérardin, H.B. Meyer, A. Nyffeler, EPJ Web Conf. 175 (2018) 06023, <http://dx.doi.org/10.1051/epjconf/201817506023>, arXiv:1711.02466 [hep-lat].
- [708] N. Asmussen, A. Gérardin, J. Green, O. Gryniuk, G. von Hippel, H.B. Meyer, A. Nyffeler, V. Pascalutsa, H. Wittig, EPJ Web Conf. 179 (2018) 01017, <http://dx.doi.org/10.1051/epjconf/201817901017>, arXiv:1801.04238 [hep-lat].
- [709] N. Asmussen, E.-H. Chao, A. Gérardin, J.R. Green, R.J. Hudspith, H.B. Meyer, A. Nyffeler, PoS LATTICE2019 (2019) 195, <http://dx.doi.org/10.22323/1.363.0195>, arXiv:1911.05573 [hep-lat].
- [710] N. Asmussen, E.-H. Chao, A. Gérardin, J.R. Green, R.J. Hudspith, H.B. Meyer, A. Nyffeler, JHEP 04 (2023) 040, [http://dx.doi.org/10.1007/JHEP04\(2023\)040](http://dx.doi.org/10.1007/JHEP04(2023)040), arXiv:2210.12263 [hep-lat].
- [711] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner, Phys. Rev. D 96 (2017) 034515, <http://dx.doi.org/10.1103/PhysRevD.96.034515>, arXiv:1705.01067 [hep-lat].
- [712] A. Gérardin, H.B. Meyer, A. Nyffeler, Phys. Rev. D 94 (2016) 074507, <http://dx.doi.org/10.1103/PhysRevD.94.074507>, arXiv:1607.08174 [hep-lat].
- [713] T. Lin, M. Bruno, X. Feng, L.-C. Jin, C. Lehner, C. Liu, Q.-Y. Luo, Rep. Progr. Phys. 88 (2025) 080501, <http://dx.doi.org/10.1088/1361-6633/adb147>, arXiv:2411.06349 [hep-lat].
- [714] N. Kalntis, G. Kanwar, M. Petschlies, S. Romiti, U. Wenger, PoS LATTICE2024 (2025) 228, <http://dx.doi.org/10.22323/1.466.0228>, arXiv:2412.12320 [hep-lat].
- [715] N. Asmussen, A. Gérardin, A. Nyffeler, H.B. Meyer, SciPost Phys. Proc. 1 (2019) 031, <http://dx.doi.org/10.21468/SciPostPhysProc.1.031>, arXiv:1811.08320 [hep-lat].
- [716] E.-H. Chao, A. Gérardin, J.R. Green, R.J. Hudspith, H.B. Meyer, Eur. Phys. J. C 80 (2020) 869, <http://dx.doi.org/10.1140/epjc/s10052-020-08444-3>, arXiv:2006.16224 [hep-lat].
- [717] M. Bruno, et al., JHEP 02 (2015) 043, [http://dx.doi.org/10.1007/JHEP02\(2015\)043](http://dx.doi.org/10.1007/JHEP02(2015)043), arXiv:1411.3982 [hep-lat].
- [718] E.-H. Chao, R.J. Hudspith, A. Gérardin, J.R. Green, H.B. Meyer, K. Ottnad, PoS LATTICE2021 (2022) 209, <http://dx.doi.org/10.22323/1.396.0209>.
- [719] J. Bijmns, J. Releforts, JHEP 09 (2016) 113, [http://dx.doi.org/10.1007/JHEP09\(2016\)113](http://dx.doi.org/10.1007/JHEP09(2016)113), arXiv:1608.01454 [hep-ph].
- [720] V.B. Berestetskii, O.N. Krokhnin, A.K. Khlebnikov, JETP 3 (1956) 761, <http://jetp.ras.ru/cgi-bin/e/index/r/30/5/p788?a=list>; Zh. Eksp. Teor. Fiz. 30 (1956) 788.
- [721] W.S. Cowland, Nucl. Phys. 8 (1958) 397, [http://dx.doi.org/10.1016/0029-5582\(58\)90171-8](http://dx.doi.org/10.1016/0029-5582(58)90171-8).
- [722] X. d. Ji, C. w. Jung, Phys. Rev. Lett. 86 (2001) 208, <http://dx.doi.org/10.1103/PhysRevLett.86.208>, arXiv:hep-lat/0101014 [hep-lat].
- [723] X. Feng, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko, J. i. Noaki, E. Shintani, Phys. Rev. Lett. 109 (2012) 182001, <http://dx.doi.org/10.1103/PhysRevLett.109.182001>, arXiv:1206.1375 [hep-lat].
- [724] J. Koponen, A. Gérardin, H.B. Meyer, K. Ottnad, G. von Hippel, PoS LATTICE2023 (2024) 254, <http://dx.doi.org/10.22323/1.453.0254>, arXiv:2311.07330 [hep-lat].
- [725] N. Christ, X. Feng, L. Jin, C. Tu, Y. Zhao, Phys. Rev. Lett. 130 (2023) 191901, <http://dx.doi.org/10.1103/PhysRevLett.130.191901>, arXiv:2208.03834 [hep-lat].
- [726] Y. Meng, X. Feng, C. Liu, T. Wang, Z. Zou, Sci. Bull. 68 (2023) 1880, <http://dx.doi.org/10.1016/j.scib.2023.07.041>, arXiv:2109.09381 [hep-lat].
- [727] I. Larin, et al., (PrimEx), Phys. Rev. Lett. 106 (2011) 162303, <http://dx.doi.org/10.1103/PhysRevLett.106.162303>, arXiv:1009.1681 [nucl-ex].
- [728] I. Larin, et al., (PrimEx-II), Science 368 (2020) 506, <http://dx.doi.org/10.1126/science.aay6641>.
- [729] R. Frezzotti, G.C. Rossi, JHEP 08 (2004) 007, <http://dx.doi.org/10.1088/1126-6708/2004/08/007>, arXiv:hep-lat/0306014.
- [730] R. Frezzotti, G.C. Rossi, JHEP 10 (2004) 070, <http://dx.doi.org/10.1088/1126-6708/2004/10/070>, arXiv:hep-lat/0407002.
- [731] S.A. Burri, et al., PoS LATTICE2022 (2023) 306, <http://dx.doi.org/10.22323/1.430.0306>, arXiv:2212.10300 [hep-lat].
- [732] C.F. Redmer, (BESIII), 2018, arXiv:1810.00654 [hep-ex].
- [733] J.S. Schelfhout, T.M. Hird, K.M. Hughes, C.J. Foot, Phys. Rev. A 110 (2024) 053309, <http://dx.doi.org/10.1103/PhysRevA.110.053309>, arXiv:2403.10225 [physics.atom-ph].
- [734] P.J. Mohr, D.B. Newell, B.N. Taylor, E. Tiesinga, Rev. Modern Phys. 97 (2025) 025002, <http://dx.doi.org/10.1103/RevModPhys.97.025002>, arXiv:2409.03787 [hep-ph].
- [735] W.J. Huang, M. Wang, F.G. Kondev, G. Audi, S. Naimi, Chin. Phys. C 45 (2021) 030002, <http://dx.doi.org/10.1088/1674-1137/abdbb0>.
- [736] M. Wang, W.J. Huang, F.G. Kondev, G. Audi, S. Naimi, Chin. Phys. C 45 (2021) 030003, <http://dx.doi.org/10.1088/1674-1137/abddaf>.
- [737] D. Hanneke, S. Fogwell, G. Gabrielse, Phys. Rev. Lett. 100 (2008) 120801, <http://dx.doi.org/10.1103/PhysRevLett.100.120801>, arXiv:0801.1134 [physics.atom-ph].
- [738] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Phys. Rev. Lett. 109 (2012) 111807, <http://dx.doi.org/10.1103/PhysRevLett.109.111807>, arXiv:1205.5368 [hep-ph].
- [739] S. Laporta, Phys. Lett. B 328 (1994) 522, [http://dx.doi.org/10.1016/0370-2693\(94\)91513-X](http://dx.doi.org/10.1016/0370-2693(94)91513-X), arXiv:hep-ph/9404204 [hep-ph].
- [740] P.A. Baikov, A. Maier, P. Marquard, Nuclear Phys. B 877 (2013) 647, <http://dx.doi.org/10.1016/j.nuclphysb.2013.10.020>, arXiv:1307.6105 [hep-ph].
- [741] T. Aoyama, T. Kinoshita, M. Nio, Atoms 7 (2019) 28, <http://dx.doi.org/10.3390/atoms7010028>.
- [742] T. Aoyama, T. Kinoshita, M. Nio, Phys. Rev. D 97 (2018) 036001, <http://dx.doi.org/10.1103/PhysRevD.97.036001>, arXiv:1712.06060 [hep-ph].
- [743] D. Giusti, S. Simula, Phys. Rev. D 102 (2020) 054503, <http://dx.doi.org/10.1103/PhysRevD.102.054503>, arXiv:2003.12086 [hep-lat].
- [744] F. Jegerlehner, EPJ Web Conf. 218 (2019) 01003, <http://dx.doi.org/10.1051/epjconf/201921801003>, arXiv:1711.06089 [hep-ph].
- [745] M. Hoferichter, P. Stoffer, M. Zillinger, Phys. Lett. B 866 (2025) 139565, <http://dx.doi.org/10.1016/j.physletb.2025.139565>, arXiv:2504.10582 [hep-ph].
- [746] D. Stöckinger, H. Stöckinger-Kim, Front. Phys. 10 (2022) 944614, <http://dx.doi.org/10.3389/fphy.2022.944614>.

- [747] A. Czarnecki, B. Krause, W.J. Marciano, Phys. Rev. Lett. 76 (1996) 3267, <http://dx.doi.org/10.1103/PhysRevLett.76.3267>, arXiv:hep-ph/9512369 [hep-ph].
- [748] S. Heinemeyer, D. Stöckinger, G. Weiglein, Nuclear Phys. B 699 (2004) 103, <http://dx.doi.org/10.1016/j.nuclphysb.2004.08.014>, arXiv:hep-ph/0405255.
- [749] T. Gribouk, A. Czarnecki, Phys. Rev. D 72 (2005) 053016, <http://dx.doi.org/10.1103/PhysRevD.72.053016>, arXiv:hep-ph/0509205.
- [750] T. Ishikawa, N. Nakazawa, Y. Yasui, Phys. Rev. D 99 (2019) 073004, <http://dx.doi.org/10.1103/PhysRevD.99.073004>, arXiv:1810.13445 [hep-ph].
- [751] A. Czarnecki, B. Krause, W.J. Marciano, Phys. Rev. D 52 (1995) 2619, <http://dx.doi.org/10.1103/PhysRevD.52.R2619>, arXiv:hep-ph/9506256.
- [752] S. Peris, M. Perrottet, E. de Rafael, Phys. Lett. B 355 (1995) 523, [http://dx.doi.org/10.1016/0370-2693\(95\)00768-G](http://dx.doi.org/10.1016/0370-2693(95)00768-G), arXiv:hep-ph/9505405.
- [753] M. Knecht, S. Peris, M. Perrottet, E. de Rafael, JHEP 11 (2002) 003, <http://dx.doi.org/10.1088/1126-6708/2002/11/003>, arXiv:hep-ph/0205102 [hep-ph].
- [754] K. Melnikov, Phys. Lett. B 639 (2006) 294, <http://dx.doi.org/10.1016/j.physletb.2006.06.031>, arXiv:hep-ph/0604205.
- [755] G. Degrandi, G. Giudice, Phys. Rev. D 58 (1998) 053007, <http://dx.doi.org/10.1103/PhysRevD.58.053007>, arXiv:hep-ph/9803384.