

Research paper

Anomalous diffusion properties of stochastic transport by
heavy-tailed jump processes

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ABSTRACT

In this work, we investigate the large-scale transport properties of a passive scalar advected by a turbulent fluid, modelled as a superposition of divergence-free vector fields, each weighted by an independent symmetric α -stable-like process. Motivated by recent works [17,18] showing that complex small-scale spatial structures often lead to Brownian dispersion, we study if this principle persists when the driving noise exhibits heavy-tailed jump statistics. Our numerical results show a clear dichotomy linked with the tail behaviour of the noise. When considering standard α -stable processes, very large jumps survive the interaction with the spatial complexity and yield anomalous, super-diffusive transport. In contrast, when the α -stable noise is either truncated or exponentially tempered, suppressing extremely long jumps, the transport undergoes a transition to a classical diffusive regime.

1. Introduction

The problem of the diffusive properties of turbulent transport

$$\partial_t T + \mathbf{u} \cdot \nabla T = 0 \quad (1)$$

(possibly with a molecular diffusion term $\kappa \Delta T$) is a basic one in Physics (from the intuition of Boussinesq [1], used so often in fluid and plasma models), Engineering and Mathematical Physics [2,3]. Stochastic models of turbulent velocity fields, of the form

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} \boldsymbol{\sigma}_{\mathbf{k}}(\mathbf{x}) \xi_{\mathbf{k}}^k$$

where $\boldsymbol{\sigma}_{\mathbf{k}}(\mathbf{x})$ are vector fields, $\xi_{\mathbf{k}}^k$ are scalar (possibly generalized) stochastic processes and the sum is extended to suitable wave numbers $\mathbf{k}$, have been used for long time [4]. In recent years, these models have sparked renewed interest as a suitable description of small-scales of turbulent fluids [5–15], and many more authors and works. Most of the classical works cited above and others consider the case when $\xi_{\mathbf{k}}^k$ is a white noise (derivative of a Brownian motion) or an Ornstein-Uhlenbeck process, approximating the white noise in the small time correlation limit. The diffusive properties of such turbulent transport models have been thoroughly investigated in many of the above mentioned works.

Very recently, there has been an increasing interest in the case when $\xi_{\mathbf{k}}^k$ are either processes with memory or processes with large jumps. The reason is the kind of fluid regimes one aims to describe. In 3D turbulence, or quasi-two dimensional turbulence when the small eddies behave like in 3D, the turbulent vortex structures are small with short relaxation time, so that an approximation by white noise or Ornstein-Uhlenbeck process with small correlation time looks reasonable. But in the case of 2D turbulence, longer

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correlations appear and larger vortex structures start to dominate the energetics, due to the inverse cascade. Therefore, the description by white noise or close to white noise processes may miss important transport information. Up to now, it is not exactly clear which kind of processes is more suitable for describing such phenomena and thus it became important to investigate the diffusion properties of other processes, with memory or long jumps (mimicking long coherent excursions).

Concerning the case where ξ_t^k are related to Fractional Gaussian Noises (FGN) with Hurst parameter $H > 1/2$ (the persistence case), let us mention two works: in [16] a theoretical result is proved, which gives a strong indication of the fact that FGN with $H > 1/2$ produces stronger diffusion than classical white noise. Unfortunately, such result has been obtained only in the case $\mathbf{u}(\mathbf{x}, t)$ has a very simple space structure: a single field $\sigma_k(\mathbf{x})$ or when the $\sigma_k(\mathbf{x})$ are constant. Therefore it was not clear if the result was stable by space-complexity of $\mathbf{u}(\mathbf{x}, t)$, as it happens in turbulent fluids. This was investigated numerically in [17], where several small-scale $\sigma_k(\mathbf{x})$ fields (similar to those used in the white noise case) multiplied by independent FGN processes with $H > 1/2$ have been studied. The result has been quite unexpected: the behaviour is similar to the white noise case. A tracer in such a velocity field behaves like a Brownian motion, since it jumps from one component to the other of the family of σ_k and thus its displacements are governed by independent processes ξ_t^k .

The present paper aims to investigate the same regime as in [17] but in the case when ξ_t^k are α -stable processes with large jumps. Namely, we want to see whether the long-tails structure is preserved in the presence of a complex space structure. We shall investigate this problem numerically, after a proper theoretical formulation. Recently, a similar problem has been studied theoretically in [18], providing a rigorous result on the diffusion properties. However, the α -stable noise ξ_t^k in [18] satisfy a certain condition of small or moderate jumps, different from the one we have in mind. Moreover, such condition in [18] leads to a Brownian behaviour (namely a phenomenon similar to [17], but rigorously proved) in the limit.

We shall investigate three different cases, also motivated by the comparison with the theoretical result of [18]:

- (J_1) a cylindrical alpha stable noise, namely independent alpha-stable processes along each mode of the space-dependent noise;
- (J_2) truncation of the previous alpha-stable processes;
- (J_3) exponential tempering of them.

Our results about these three cases are summarized in Section 2.3, with the corresponding numerical simulations presented in Section 3. The processes and equations investigated here are introduced in Sections 2.1 and 2.2.

2. Model formulation and main results

Our aim is to investigate numerically a modification of the model which appeared in [17], in which the original fractional Brownian random weights $B_t^{H,k}$ are replaced by some heavy-tailed (non-Gaussian) noises. More precisely, we would like to understand under which assumptions on the time scales and the considered noise, one should expect an anomalous diffusion in the large wave-number limit, as opposed to the diffusive behaviour observed in [17] and [18].

We are interested in the following transport equation on $\mathbb{R}^2$:

$$\begin{cases} \partial_t T + \mathbf{u} \cdot \nabla_x T = 0; \\ T|_{t=0} = T_0, \end{cases} \quad (2)$$

where $\mathbf{u} = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t))$ represents a “turbulent” incompressible fluid. Here, we encode the turbulence as:

$$\mathbf{u}(\mathbf{x}, t) := uC(\eta, \tau, \alpha) \sum_{\mathbf{k} \in \mathbf{K}_\eta} \sigma_{\mathbf{k}}(\mathbf{x}) \dot{Z}_t^{\alpha, \mathbf{k}},$$

where u is the average velocity constant (with dimension $[L]/[T]$), $C(\eta, \tau, \alpha)$ is a normalizing constant whose precise expression will be given later, η indicates the space scale with dimension $[L]$ and the parameter τ appearing in $C(\eta, \tau, \alpha)$ represents the relaxation time of the fluid with dimension $[T]$. The index set $\mathbf{K}_\eta$ amounts for the wave numbers $\mathbf{k} \in \mathbb{Z}^2$ which are activated at a fixed space scale $\eta > 0$ and it is given by

$$\mathbf{K}_\eta := \left\{ \mathbf{k} \in \mathbb{Z}^2 : |\mathbf{k}| \in \left[\frac{1}{2\eta}, \frac{1}{\eta} \right] \right\}.$$

Note in particular that the limit for $\eta \rightarrow 0$ corresponds indeed to the large wave number limit. The divergence-free vector field σ_k are dimensionless and defined as

$$\sigma_{\mathbf{k}}(\mathbf{x}) := \begin{cases} \sqrt{2} \cos(\mathbf{k} \cdot \mathbf{x}) \frac{\mathbf{k}^\perp}{|\mathbf{k}|}, & \text{if } \mathbf{k} \in \mathbf{K}_\eta^+ \\ \sqrt{2} \sin(\mathbf{k} \cdot \mathbf{x}) \frac{\mathbf{k}^\perp}{|\mathbf{k}|}, & \text{if } \mathbf{k} \in \mathbf{K}_\eta^- \end{cases}$$

where $\mathbf{K}_\eta^+$ is the set of $\mathbf{k} = (k_1, k_2) \in \mathbf{K}_\eta$ such that either $k_1 > 0$ or $\{k_1 = 0, k_2 > 0\}$, and $\mathbf{K}_\eta^- = -\mathbf{K}_\eta^+$.

Above, each $\dot{Z}_t^{\alpha, \mathbf{k}}$ denotes the formal time derivative of $(Z_t^{\alpha, \mathbf{k}})_{t \geq 0}$, a symmetric “ α -stable-like” Lévy process on $\mathbb{R}$. Such processes are characterized by discontinuous trajectories. The parameter $\alpha \in (1, 2)$, known as the stability index, quantifies the frequency and relevance of large jumps: smaller values of α correspond to dynamics dominated by rare but very large jumps whereas in the formal limit as $\alpha \rightarrow 2$, one recovers the continuous paths of a Brownian motion. More precisely, we will test our model under three possible scenarios:

- (J_1) $(Z_t^{\alpha, \mathbf{k}})_{t \geq 0}$ are independent, standard α -stable processes;
 (J_2) $(Z_t^{\alpha, \mathbf{k}})_{t \geq 0}$ are independent, truncated α -stable processes with cut-off $\epsilon > 0$;
 (J_3) $(Z_t^{\alpha, \mathbf{k}})_{t \geq 0}$ are independent, exponentially tempered α -stable processes with tempering parameter $A > 0$.

The above cases represent “prototypical” examples of jump-driven noises and illustrate well how variations in the intensity and/or frequency of the jumps influence the model limit behaviour. Roughly speaking, a standard α -stable process exhibits a power-law jump statistics: jumps of arbitrarily large size may occur with a frequency determined by a power law distribution. Its dimension is $[T]^{1/\alpha}$, which is suggested by its α -self-similarity. In case (J_2), one starts again from the standard α -stable process above but then removes all jumps larger than the cut-off $\epsilon > 0$. Finally, the exponentially tempered α -stable process can be viewed as an interpolation between the two previous models: it retains jump activity at all scales but large jumps become exponentially rare, with a rate controlled by the parameter A . In both cases (J_2) and (J_3), the dynamics preserves heavy-tail features at small scales, while exhibiting Gaussian-like behaviour at large ones. This suggests that the dimension of $\dot{Z}_t^{\alpha, \mathbf{k}}$ is given by $[T]^{(\alpha-1)/\alpha}$. Hence, a dimensional analysis implies that the normalizing constant $C(\eta, \tau, \alpha)$ must have dimension $[T]^{(\alpha-1)/\alpha}$. More precisely, it is given by

$$C(\eta, \tau, \alpha) = [\text{Card}(\mathbf{K}_\eta)]^{-1/\alpha} \tau^{(\alpha-1)/\alpha}.$$

Note that when η is small enough, $\text{Card}(\mathbf{K}_\eta) \sim \pi \frac{1}{\eta^2} - \pi \frac{1}{4\eta^2} = \pi \frac{3}{4\eta^2}$.

We are thus interested in the following SPDE on $\mathbb{R}^2$:

$$\begin{cases} dT + uC(\eta, \tau, \alpha) \sum_{\mathbf{k} \in \mathbf{K}_\eta} \sigma_{\mathbf{k}}(\mathbf{x}) \cdot \nabla_x T \diamond dZ_t^{\alpha, \mathbf{k}} = 0; \\ T|_{t=0} = T_0. \end{cases} \quad (3)$$

The symbol $\diamond$ above means the Marcus stochastic integral, which is a generalization of the more well-known Stratonovich integral for discontinuous paths processes. As the Stratonovich one, the Marcus integral incorporates in its definition some corrector terms in order to recover the standard differentiation properties.

Precise definitions of the above stochastic processes, together with the meaning of all parameters, and the corresponding stochastic integration theory, are provided below.

2.1. Some backgrounds on stable Lévy processes and their modifications

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $d \in \mathbb{N}$ a generic dimension. An $\mathbb{R}^d$ -valued stochastic process $(Z_t)_{t \geq 0}$ is called a *Lévy process* if it starts from zero almost surely, has independent and stationary increments, and is càdlàg (i.e. it has right continuous paths with left limits $\mathbb{P}$ -almost surely). A fundamental tool in the analysis of Lévy processes is given by the Lévy-Kintchine formula (see for instance [19]). It allows to represent the Lévy symbol Φ of $(Z_t)_{t \geq 0}$, given by

$$\mathbb{E}[e^{i\xi \cdot Z_t}] = e^{t\Phi(\xi)}, \quad \xi \in \mathbb{R}^d$$

in terms of the generating triplet (b, Σ, ν) as:

$$\Phi(\xi) = ib \cdot \xi - \frac{1}{2} \Sigma \xi \cdot \xi + \int_{\mathbb{R}_0^d} (e^{i\xi \cdot z} - 1 - i\xi \cdot z \mathbb{1}_{B(0,1)}(z)) \nu(dz), \quad \xi \in \mathbb{R}^d,$$

where b is a vector in $\mathbb{R}^d$, Σ is a symmetric, non-negative definite matrix in $\mathbb{R}^d \otimes \mathbb{R}^d$ and ν is a Lévy measure on $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$, i.e. a σ -finite measure on $B(\mathbb{R}_0^d)$, the Borel σ -algebra on $\mathbb{R}_0^d$, such that $\int (1 \wedge |z|^2) \nu(dz)$ is finite. In particular, the law of any Lévy process is completely determined by its generating triplet (b, Σ, ν) . In this work, we restrict our attention to *symmetric pure-jump Lévy processes*, namely processes for which $\Sigma = 0$, $b = 0$ and the Lévy measure ν is symmetric.

The first class (i.e. case (J_1)) of noises considered in this work consists of α -stable Lévy processes, with stability index $\alpha \in (1, 2)$. These are symmetric pure-jump Lévy processes whose increments follow an α -stable distribution. Their Lévy measure takes the form:

$$\nu_\alpha(dz) := \mu(d\theta) \frac{dr}{r^{d+\alpha}}, \quad z = r\theta \in \mathbb{R}^d, \theta \in \mathbb{S}^d, r > 0,$$

where μ is a finite Borel measure on $\mathbb{S}^d$, the unitary sphere on $\mathbb{R}^d$, known as the *spectral measure* of Z_t . In particular, we will talk about an *isotropic α -stable process* when its spectral measure is rotational invariant. In such a case, its characteristic function reduces to

$$\mathbb{E}[\exp(iZ_t \cdot \xi)] = \exp[-\sigma_\alpha |\xi|^\alpha t], \quad \xi \in \mathbb{R},$$

for some constant $\sigma_\alpha > 0$, called the *scale parameter*. Indeed, if Z_t is an isotropic α -stable process with scale parameter σ_α , then for any $\lambda > 0$, the process λZ_t remains isotropic α -stable, with scale parameter $\sigma_\alpha \lambda^\alpha$. In the particular case where $\sigma_\alpha = 1$, the corresponding process is usually called a *standard (isotropic) α -stable process*.

As already noted above, α -stable processes are self-similar with index $1/\alpha$, i.e. $(Z_{ct})_{t \geq 0} \stackrel{d}{=} (c^{1/\alpha} Z_t)_{t \geq 0}$ for any $c > 0$. Unlike the Gaussian case, α -stable processes do not possess finite moments of all orders. More precisely, they admit finite fractional moments only up to order $\beta < \alpha$. In the isotropic case with scale parameter σ_α , one can actually obtain an explicit formula (cf. [20, Eq. (25.6)]) for $\beta \in (0, \alpha)$:

$$\mathbb{E}[|Z_t|^\beta] = \frac{2^\beta \Gamma(\frac{d+\beta}{2}) \Gamma(1 - \frac{\beta}{\alpha})}{\Gamma(\frac{d}{2}) \Gamma(1 - \frac{\beta}{2})} (\sigma_\alpha t)^{\beta/\alpha}. \quad (4)$$

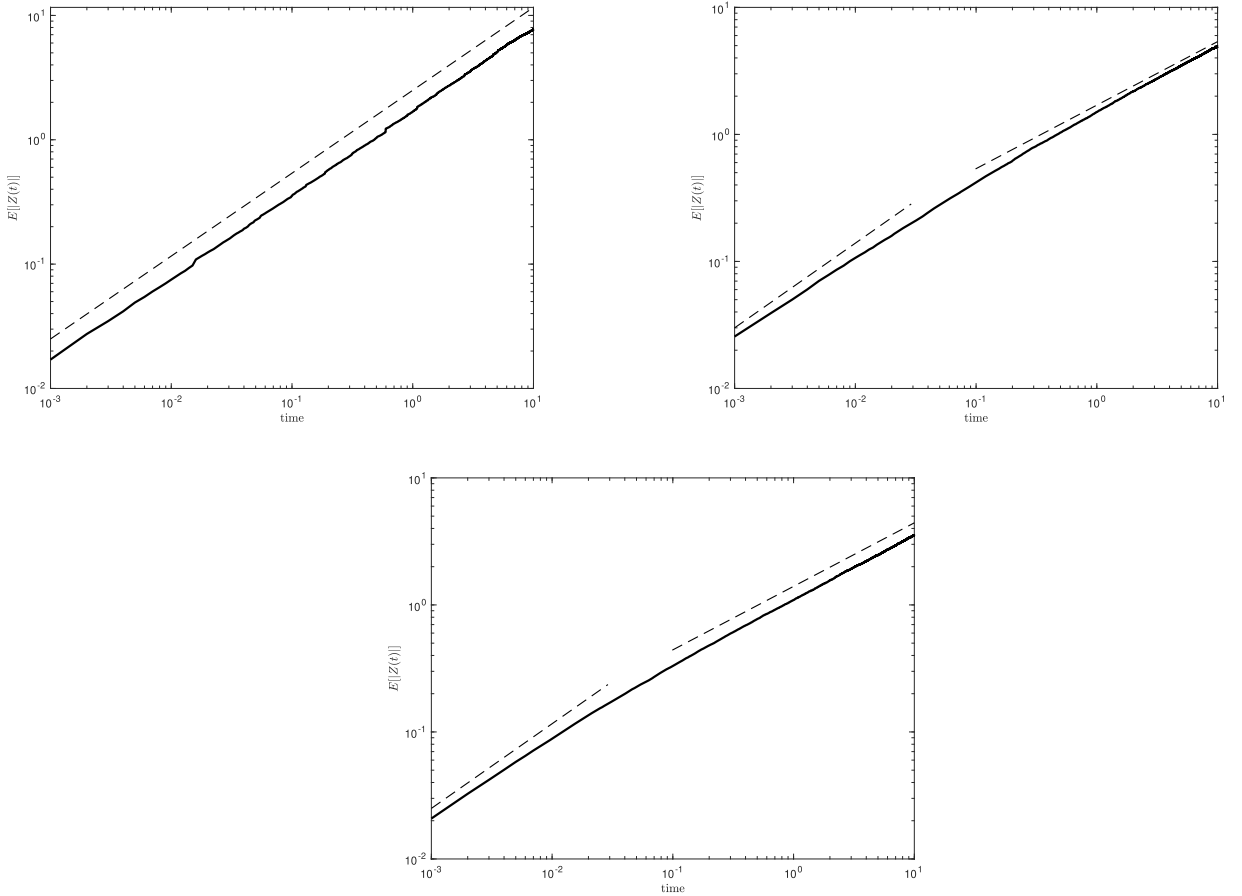


Fig. 1. $\mathbb{E}[|Z_t|]$ as a function of time for the α -stable process (top-left figure), the tempered process (top-right figure) and the truncated process (bottom figure). For the tempered process we set $A = 0.3$; for the truncated process we set $\epsilon = 10^{-3}$. Slopes $1/\alpha$ and $1/2$ are represented by the dashed lines.

A fundamental distinction from the Brownian case is that understanding the global behaviour of an α -stable process cannot be achieved by examining only its projections onto the coordinate axes. This limitation arises from the fact that, in general, the components of the process are not independent. Consequently, α -stable processes possess a substantially richer and more intricate geometric structure. In order to analyse the geometry of the support of the associated spectral measure, it is therefore necessary to consider projections along arbitrary directions (cf. [21, Thm. 2.1.5, Ex. 2.3.4] and [20, Prop. 11.10]). If $(Z_t)_{t \geq 0}$ is an α -stable process with spectral measure μ and $v \in \mathbb{S}^d$ is a direction, then $Z_t \cdot v$ is again an α -stable process (on $\mathbb{R}$) with scale parameter

$$\sigma_\alpha(v) = \int_{\mathbb{S}^d} |v \cdot \theta|^\alpha \mu(d\theta).$$

On the other hand, if for any $v \in \mathbb{S}^d$, $Z_t \cdot v$ is an α -stable process with scale parameter $\sigma_\alpha(v)$ and such a constant is actually independent from v , then one can conclude that Z_t is an isotropic α -stable process on $\mathbb{R}^d$ with scale parameter σ_α .

Starting from a given Lévy process, one can construct new Lévy processes by means of several standard modifications, such as linear transformations or subordination. In this work, we focus on two classical examples of such modifications, namely truncation and exponential tempering, which suppress large jumps while preserving α -stable behaviour at small scales.

A truncated α -stable process $(Z_t)_{t \geq 0}$ is obtained by removing large jumps from its Lévy measure. Given a cut-off parameter $\epsilon > 0$, it has the form:

$$\nu(dz) := \mathbb{1}_{\{|z| \leq \epsilon\}} \nu_\alpha(dz), \quad z \in \mathbb{R}^d,$$

where ν_α is an α -stable Lévy measure.

An exponentially tempered α -stable process modifies the Lévy measure by introducing an exponential damping of large jumps. For a tempering parameter $A > 0$, the Lévy measure is

$$\nu(dz) := e^{-A|z|} \nu_\alpha(dz), \quad z \in \mathbb{R}^d,$$

where ν_α is again some α -stable Lévy measure.

One of the key property of tempered and truncated α -stable processes is that, unlike α -stable process, they have finite moments.

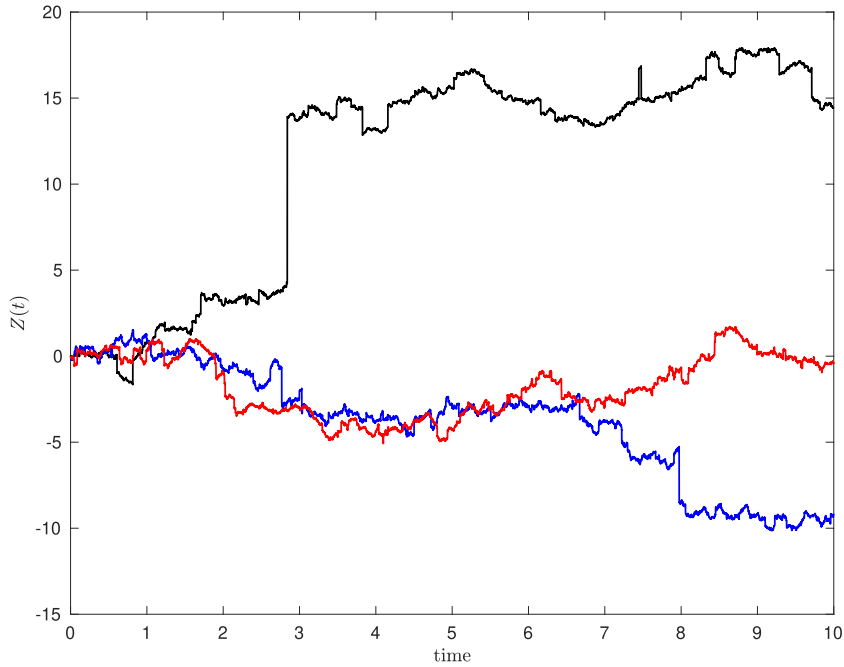


Fig. 2. Sample paths of the α -stable process (black line), the tempered process (blue line) and the truncated process (red line).

2.2. Solution concept for transport equation

In connection with the method of characteristics, it is natural in this context to interpret the stochastic transport Eq. (3) in the sense of Marcus stochastic integration (cf. [22]). Indeed, Itô interpretation is inappropriate in this context, as it does not preserve the classical chain rule and introduces artificial drift corrections unrelated to the transport mechanism. On the other hand, while Stratonovich integration provides a consistent framework for stochastic transport equation driven by continuous path noises (cf. [17]), it does not correctly describe the dynamics induced by discontinuous trajectories. Indeed, in the presence of jumps, Stratonovich integral models the effect of each jump as a purely instantaneous displacement, without taking into account the infinitesimal flow generated by the vector field during the jump. By contrast, the Marcus integral accounts for the internal dynamics of jumps by associating each jump of the driving process with the deterministic flow generated by the vector field. Such interpretation preserves the geometric structure of the transport equation and it is consistent with the characteristic method. Informally, a solution T to (3) is characterized as

$$T(\mathbf{X}_t^\eta(\mathbf{x}), t) = T_0(\mathbf{x}),$$

where $\mathbf{X}_t^\eta(\mathbf{x})$ is the flow map, solution to the characteristics equation:

$$\begin{cases} d\mathbf{X}_t^\eta(\mathbf{x}) = uC(\eta, \tau, \alpha) \sum_{\mathbf{k} \in \mathbb{K}_\eta} \sigma_{\mathbf{k}}(\mathbf{X}_t^\eta(\mathbf{x})) \diamond dZ_t^{\alpha, \mathbf{k}}; \\ \mathbf{X}_0^\eta(\mathbf{x}) = \mathbf{x}. \end{cases} \quad (5)$$

The symbol $\diamond$ above means the Marcus stochastic integral which is a generalization of the more well-known Stratonovich integral for discontinuous paths processes. For a precise formulation of this integral, see [23, Section 6.10]. Under our assumptions, i.e. smooth, divergence-free vector fields $\sigma_{\mathbf{k}}$ and symmetric pure-jump Lévy drivers with finite first moment (i.e. $\alpha > 1$ or truncated/tempered variants), it is well-known that the characteristic Eq. (5) admits a unique global càdlàg solution defining a stochastic flow of C^1 -diffeomorphisms almost surely (see, again [23, Section 6.10]). For the precise connection between the stochastic transport Eq. (3) and the associated characteristics Eq. (5) under our assumptions, as well as the well-posedness of such models, see [22].

2.3. Main conjectures

Here, we outline a brief description of the main results, which will be described in more details in Section 3. Roughly speaking, our numerical results show that the large-scale transport behaviour is strongly determined by how rare extremely long jumps are in the driving noise. When the velocity field is driven by a standard α -stable process (i.e. case (J_1)), large jumps survive the interaction with the complex spatial structure and lead to anomalous, super-diffusive transport: particle displacements scale like $t^{1/\alpha}$, and the limiting dynamics is consistent with an α -stable process. In contrast, when large jumps are either truncated or exponentially tempered (i.e. cases (J_2) and (J_3)), their effect is progressively suppressed, and the tracer dynamics shift to a classical diffusive regime. In these cases, after a short transient period dominated by jump-like behaviour, the particle motion becomes Gaussian with the standard $t^{1/2}$

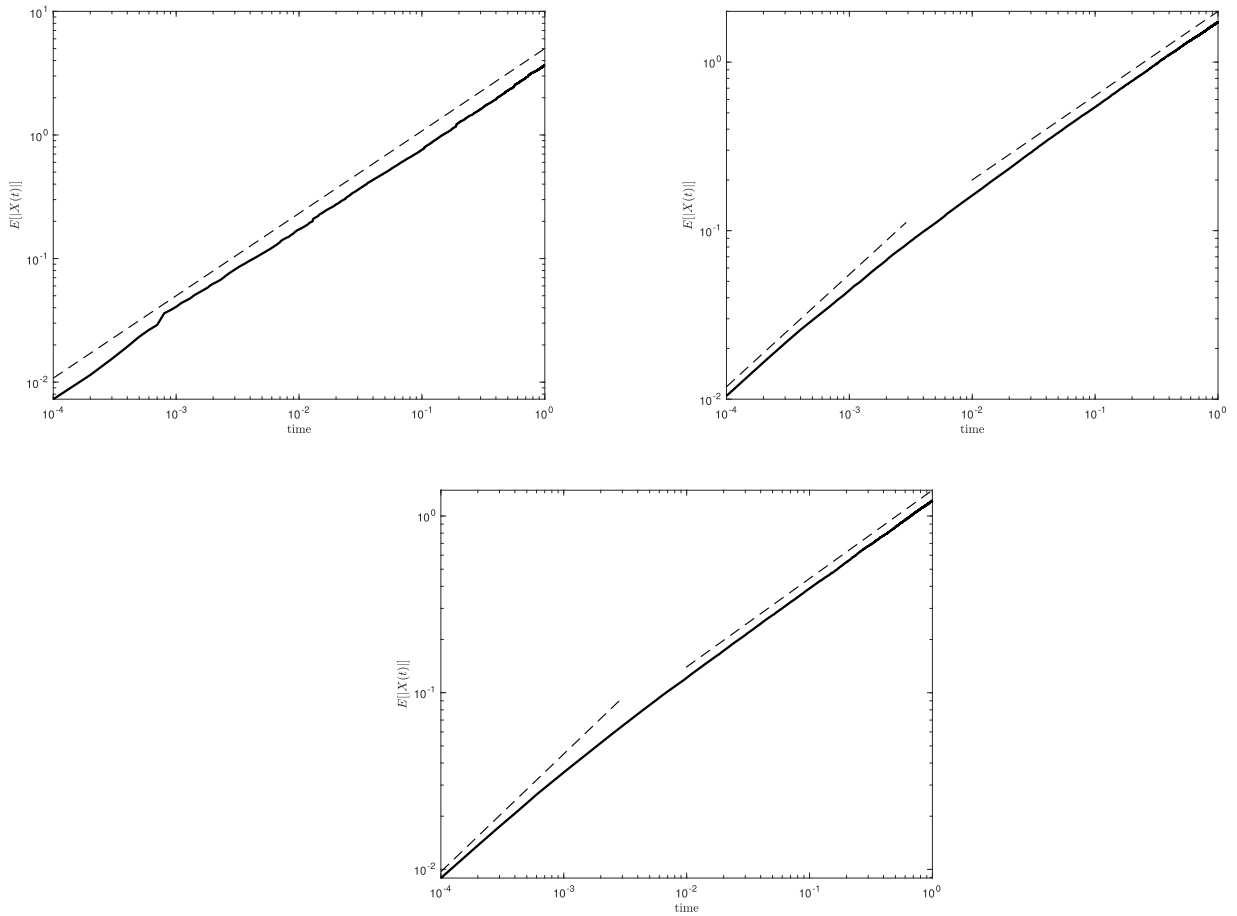


Fig. 3. $\mathbb{E}[|X_t|]$ as a function of time for the α -stable process (top-left figure), the tempered process (top-right figure) and the truncated process (bottom figure). The parameters of the driving stochastic processes are the same as in Fig. 1. Slopes $1/\alpha$ and $1/2$ are represented by the dashed lines.

mean displacement scaling, similarly to what is observed for white noise-driven models. Our results highlight how the tail behaviour of the driving noise, i.e. the presence or absence of a power-law jump distribution, plays a crucial role in selecting between anomalous and classical diffusive regimes.

Conjecture 1. Let $\mathbf{x} \in \mathbb{R}^2$. Then, as η goes to zero, the processes $(\mathbf{X}_t^\eta(\mathbf{x}))_{t \geq 0}$ converge in law to the 2-dimensional process $(\mathbf{x} + \bar{\mathbf{X}}_t)_{t \geq 0}$, where:

(J_1) : $(\bar{\mathbf{X}}_t)_{t \geq 0}$ is an isotropic α -stable process;

(J_2) : $(\bar{\mathbf{X}}_t)_{t \geq 0}$ is a centred Brownian motion;

(J_3) : $(\bar{\mathbf{X}}_t)_{t \geq 0}$ is a centred Brownian motion.

The validity of the above result remains an open question but we hope our formulation will motivate further theoretical and practical investigations in this direction. Only in case (J_2) , Luo and Teng in [18] obtained a theoretical result of the diffusive scaling limit. As a preliminary contribution, we provide the following numerical verification.

In the previous work [17], the main objective was to characterize the rate of information diffused by the velocity field, quantified via the mean squared displacement (MSD) of the particle position. While such a metric remains applicable for cases (J_2) and (J_3) , it is unsuitable for case (J_1) . Indeed, as the α -stable driving noise inherently lacks a finite second moment, the resulting particle variance is expected to diverge. In order to compare the three cases on the same ground, we propose instead to analyse the dynamics of the first absolute moment:

$$t \rightarrow \mathbb{E}[|\mathbf{X}_t^\eta(\mathbf{0})|].$$

As an intermediate step toward the validity of Conjecture 1, our numerical results show that under α -stable driving noise (i.e. case (J_1)), the particle trajectory exhibits super-diffusive behaviour characterized by the scaling $\mathbb{E}[|\mathbf{X}_t^\eta(\mathbf{0})|] \sim t^{1/\alpha}$. As explained in (4), this is the behaviour usually associated to α -stable processes. Conversely, for the truncated and tempered noise regimes, we observe that the process converges to a classical diffusive behaviour, namely $\mathbb{E}[|\mathbf{X}_t^\eta(\mathbf{0})|] \sim t^{1/2}$, after a short transient period.

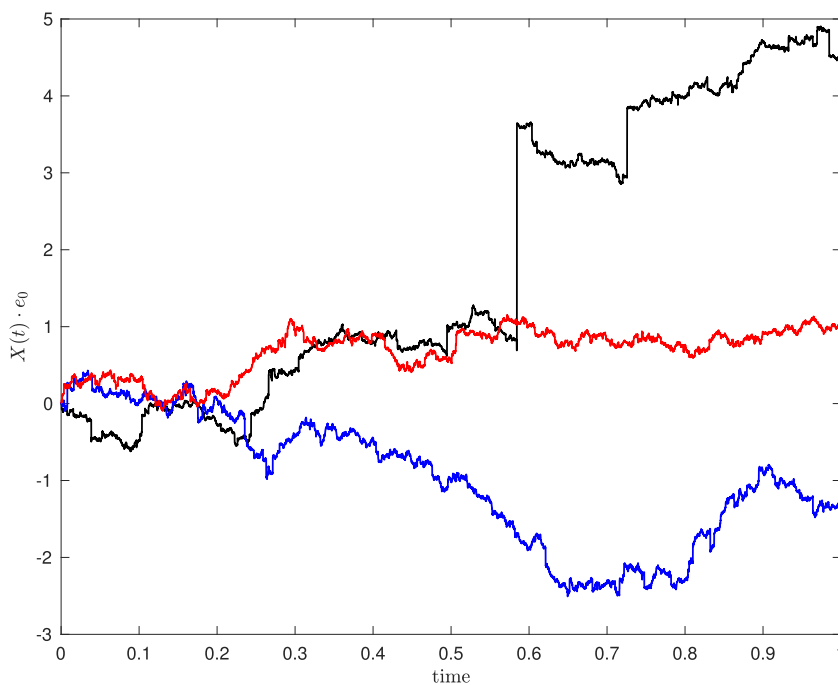


Fig. 4. Sample paths of X_t for driver Z_t being an α -stable process (black line), a tempered process (blue line) and a truncated process (red line).

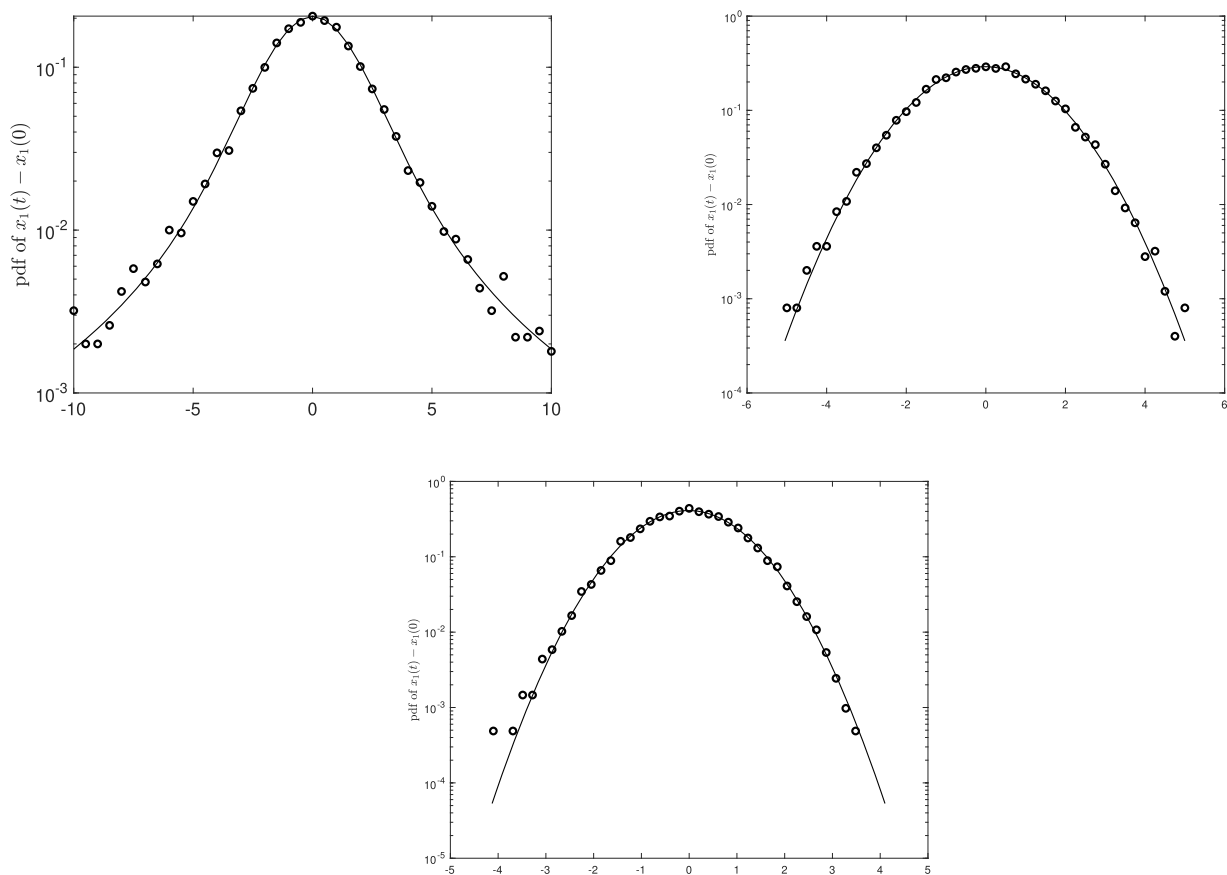


Fig. 5. Probability distribution function of X_t at the final time $t = 1$ for driver Z_t being an α -stable process (top-left figure), for driver Z_t being a tempered process (top-right figure) and for driver Z_t being a truncated process (bottom figure).

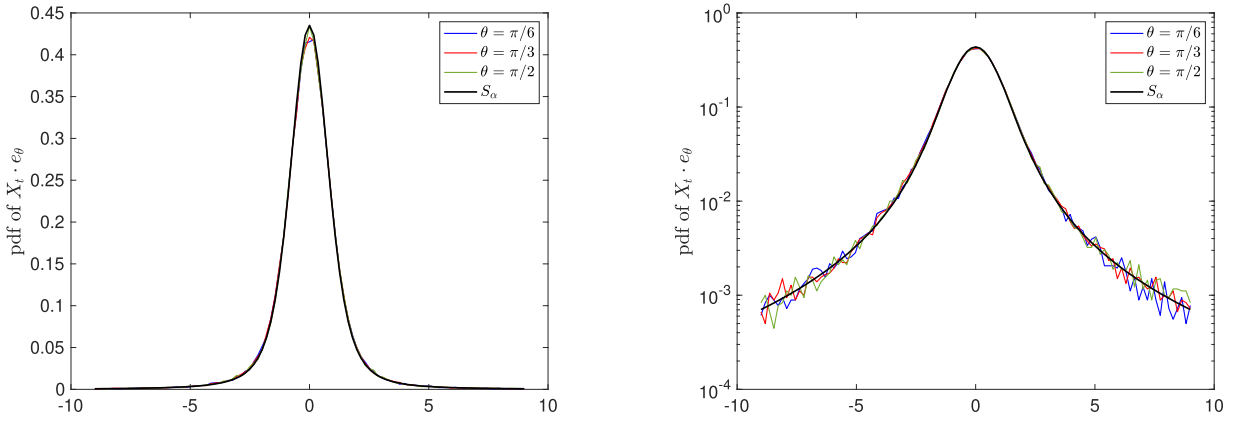


Fig. 6. Probability distribution function of $X_t \cdot e_\theta$, at the final time $t = 1$, using an α -stable driver Z_t for $\theta = \pi/6, \pi/3, \pi/2$. The black line is a reference $S_{1.5}$ distribution.

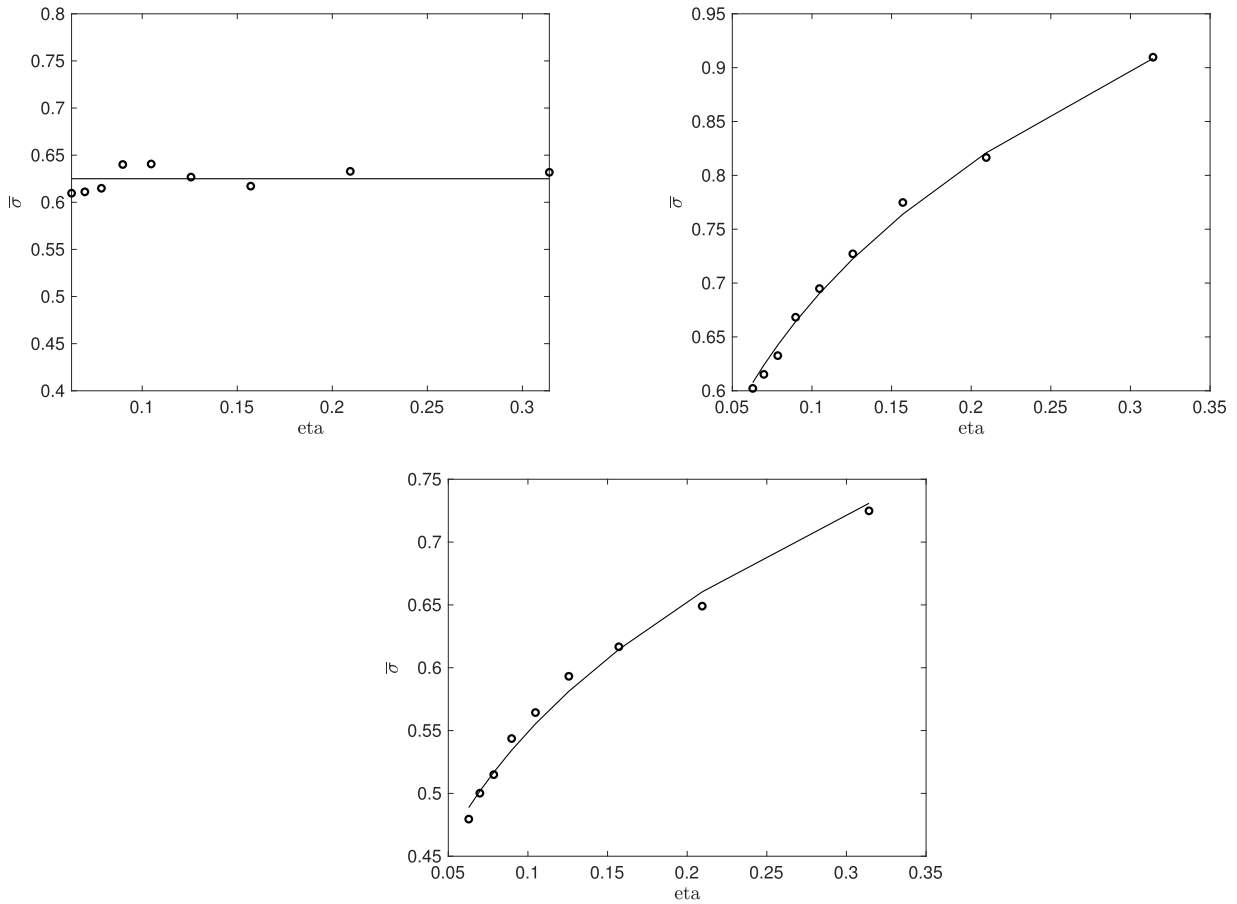


Fig. 7. Parameter $\bar{\sigma}$ as a function of η for driver Z_t being an α -stable process (top-left), a tempered one (top-right) or a truncated one (bottom), computed numerically (dots) and analytically (solid line) from (6). The coefficients of the solid line are fitted from the numerical data.

While such information already points towards [Conjecture 1](#), we provide additional confirmation by computing the probability density function (PDF) of the particle displacement along the x -axis at time $t = 1$. By visually comparing their graph with the expected ones, we show that

- (J_1): the distribution $\mathcal{L}$ is α -stable;
- (J_2): the distribution $\mathcal{L}$ is centred Gaussian;

(J_3): the distribution $\mathcal{L}$ is centred Gaussian.

where $\mathcal{L}$ is the probability distribution function of $\mathbf{X}_1^\eta(\mathbf{0}) \cdot \mathbf{e}_1$.

The above claim already yields strong evidence in support of the diffusive limits stated in [Conjecture 1](#) for cases (J_2) and (J_3). Consequently, we restrict our subsequent analysis to case (J_1). To better understand the properties of the limiting process, we investigate the geometric structure of the associated spectral measure.

More precisely, we fix various directions $\mathbf{v} \in \mathbb{S}^2$ and compare the distribution of the tracer projected along these directions: $\mathbf{X}_1^\eta(\mathbf{0}) \cdot \mathbf{v}$. The numerical results presented below indicate that the limiting process is isotropic.

Furthermore, as part of a more detailed analysis, we also aim to estimate the coefficients $\bar{\sigma}$ appearing in front of the limiting processes.

Conjecture 2. Let $\mathbf{x} \in \mathbb{R}^2$. Then, as η goes to zero, the processes $(\mathbf{X}_t^\eta(\mathbf{x}))_{t \geq 0}$ converge in law to the 2-dimensional process $(\mathbf{x} + \bar{\sigma} \bar{\mathbf{X}}_t)_{t \geq 0}$, where:

- $(\bar{\mathbf{X}}_t)_{t \geq 0}$ is a standard α -stable process in case (J_1) and a centred Brownian motion in the other ones;
- the scaling parameter is given by

$$\bar{\sigma} \sim \lambda^{1-\alpha/H} u^{\alpha/H} \eta^{1-\alpha/H} \tau^{(\alpha-1)/H}, \quad (6)$$

where, as before, $H = \alpha$ in case (J_1) and $H = 2$ in the other cases and the parameter $\lambda > 0$ depends on the features of the model (i.e. A , ϵ , σ_k).

The heuristic derivation of the scaling parameter $\bar{\sigma}$ is detailed in [Appendix A](#) and relies on a suitable time discretization of the tracer dynamics. As expressed above, the “approximate” formula for $\bar{\sigma}$ depends however on an additional parameter $\lambda > 0$. Below in [Section 3](#), such a parameter is computed from numerical simulations and then compared against the theoretical prediction. We have not found an argument to predict the coefficient λ , but we can show numerically that there exists a value providing a good fit between this formula and numerical experiments.

Remark. It appears plausible that, in order to obtain a mathematically rigorous scaling limit, similar to the ones obtained in [\[6,18\]](#), an additional assumption on the interplay between the parameters of the model is required. More precisely, one may assume that the mean velocity $u := u(\tau, \eta)$ is chosen in such a way that

$$u^{\alpha/H} \eta^{1-\alpha/H} \tau^{(\alpha-1)/H} \sim 1, \quad \text{as } \eta \rightarrow 0, \quad (7)$$

where $H = \alpha$ in case (J_1) and $H = 2$ in cases (J_2) – (J_3). This scaling condition is designed to ensure that the scaling parameter $\bar{\sigma}$ of the limit process is independent from η and thus non-trivial in the limit, as η goes to 0. Without such a balance condition, one would not expect the emergence of a meaningful macroscopic transport behaviour, since the limiting dynamics would either collapse to a degenerate regime or diverge.

3. Numerical simulations

In this section we perform numerical simulations of transport [Eq. \(1\)](#) with advection velocity $\mathbf{u}(\mathbf{x}, t)$ given by the stochastic model [\(3\)](#). Analogously to our related work [\[17\]](#), a Monte Carlo method is employed where particle trajectories are simulated by numerical integration of the equations of characteristics [\(5\)](#). In all the results presented below, stochastic differential equations are interpreted in the sense of Marcus integrals and solved numerically via integration of the induced flow ODE for jumps. Expected values and probabilities are approximated by appropriate ensemble averages over 10^4 realisations.

We consider the following three cases for the stochastic process Z_t :

- (J_1): $\Delta Z_t \sim \Delta t^{1/\alpha} S_\alpha(1, 0, 0)$ sampled from the Chambers-Mallows-Stuck algorithm;
- (J_2): ΔZ_t sampled from the truncated distribution by a Gaussian approximation of the small jumps;
- (J_3): ΔZ_t sampled from the tempered distribution by the Bauemer-Merchaer algorithm.

Further details regarding the computational algorithms utilized in simulating these families of stochastic processes can be found in [Appendix B](#) (cf. [Algorithms 2, 4](#) and [6](#)).

As a preliminary step, we show the scaling of $\mathbb{E}[|Z_t|]$ as a function of time for the aforementioned cases. Exponential tempering is adopted in order to construct an effective Lévy measure whose tails are tamed by a factor e^{-A} . Strong jumps become more and more rare for large values of A . In the truncated measure a parameter ϵ is introduced to suppress all jumps of size larger than ϵ . This behaviour is summarised in [Fig. 1](#).

Clearly, the scaling $t^{1/\alpha}$ is shown by the α -stable process at all times, as expected from theory. For the tempered and the truncated process, after an initial power law with exponent $1/\alpha$, a transition to Gaussianity takes place with its characteristic scaling $t^{1/2}$ at large enough t . The time at which the latter scaling takes over depends on the parameters A and ϵ of the tempered and truncated measure, respectively. The main observation here is that for small t , the process is dominated by jumps while for large t rare events are either tamed or cut out such that diffusion is recovered. An illustration of the sample paths of Z_t is shown in [Fig. 2](#).

Next, we simulate particle trajectories by numerical integration of [\(5\)](#). The reference length scale η is set to $2\pi/20$ and the total simulation time to $T = 1$. We consider the same three cases detailed above, namely the driving stochastic processes are the α -stable process and its tempered and truncated counterpart. [Fig. 3](#) shows $\mathbb{E}[|X_t|]$ as a function of time.

Overall the particle position X_t behaves as the driver stochastic process Z_t . When the tempered or the truncated processes are used in the vector field, the same transition from the scaling $t^{1/\alpha}$ at small times to the scaling $t^{1/2}$ at large times is observed. We remark here that the typical time scale at which this transition takes place is entirely related to the choice of the tempering parameter A and of the truncation threshold ϵ , but not to the physical time t^* , defined in [17] and linked to the length η . What is remarkable here is, rather the fact that X_t maintains the same scaling law of Z_t when the latter is an α -stable process, despite the complex interactions among the vector field components σ_k . An illustration of the sample paths of X_t is shown in Fig. 4.

The scaling of the first absolute moment of $t^{1/\alpha}$ and $t^{1/2}$ suggest α -stable and Gaussian displacement of the particle X_t , respectively. As a further confirmation of this hypothesis, we present in Fig. 5 the probability distribution function of X_t at the final time $t = 1$.

When driving X_t by either the tempered or the truncated process, the computed pdf values (dots) closely adhere to the Gaussian (solid line) having the same mean and standard deviation. Differently, when Z_t is α -stable the numerical pdf approximates an α -stable distribution with parameter σ numerically computed from $E[|X_t|]$.

In order to support Conjecture 1, we compute the pdf of X_t , driven by the α -stable process, projected along the direction e_θ for $\theta = \pi/6, \pi/3, \pi/2$.

As shown in Fig. 6, at all values of θ the numerical pdfs closely match the α -stable distribution $S_{1.5}$. To further strengthen our claim, we performed a Kolmogorov–Smirnov test and found a maximum cumulative distribution function (CDF) deviation of approximately 0.5% between the numerical and reference distributions. Furthermore, to better emphasise the distribution tails, we carried out the Anderson–Darling test, which produced a statistic of 10.39 compared to a critical value of 2.49, thus rejecting the null hypothesis. This indicates that while good accuracy is found in the bulk of the distribution, the tails present a deviation from the reference density. However, the numerical tests that we have conducted show that the accuracy of the tails increases with the number of sample paths simulated. Therefore, we believe that the small discrepancy observed in the tails is due to the finite sample size rather than a systematic deviation from the postulated probability distribution.

As stated in Conjecture 2, the truncated and tempered noise regimes converge to classical (Brownian) diffusion behaviour for sufficiently large times, while the pure α -stable case remains stable. An expression for the scaling parameter $\bar{\sigma}$ is provided in (6). A numerical verification of the validity of the latter would require a series of simulations in the parameter space u, η, τ . Furthermore, the parameter λ depends as well on parameters of the driving noise, such as the tempering and truncation factors A and ϵ . While an analysis of this kind is in principle possible, it becomes quickly computationally unfeasible. This is the reason why in this work we limit ourselves to the verification of the dependence of $\bar{\sigma}$ on the length scale η . Fig. 7 depicts $\bar{\sigma}$ as a function of η in the three noise regimes. A close agreement is observed between the numerical values and the postulated theoretical function, further strengthening our claim.

In conclusion, we would like also to emphasise that the numerical methods employed here come with shortcomings. For example, as with any Monte Carlo method, the accuracy of the results is limited by the size of the sample. The bounds of such errors are often too complex to be derived. Consequently, the validity of our conjectures can only be assessed numerically. Secondly, samples from the truncated and tempered distributions are not exact; rather, they are drawn from approximate distributions. While this simplification makes the simulations in this paper computationally feasible, it also introduces an additional source of error. Bearing these limitations in mind, we provided compelling analytical and numerical evidence in support of our claim.

4. Conclusions

In this work we have investigated the diffusion effects of certain stochastic models of turbulent fluids, a problem of major importance for applications like the anomalous diffusion in confined fusion plasma.

These stochastic models, also in the works quoted below, are made of the superposition of divergence-free highly oscillating vector fields, each weighted by an independent stochastic process. The difference between the various papers is mostly in the choice of the stochastic processes.

A main technical motivation for this work comes from two previous papers, Cifani and Flandoli [17], Luo and Teng [18]. In both of them it is shown that a classical Brownian behaviour of the diffusion arises in spite of the fact that the time-structure of the stochastic turbulence model was not Brownian, but fractional Brownian with $H > 1/2$ in one case and α -stable (with a bound on the tails) in the second one. In the present work we have shown that this is not always the case.

Our main result is that, if the driving processes of the turbulence model exhibit heavy-tailed jump statistics, then this property persists for the passive scalar advection. The “Brownianization” observed and proved in [17,18] does not take place.

For comparison, we also investigated the cases when the driving processes, still α -stable-like, are either truncated or exponentially tempered. In this case Brownianization takes place.

We may summarize the intuitions behind the quoted results and the new ones presented here in the following way. Two elements of a turbulence model (of the form said above, namely superposition of divergence-free highly oscillating vector fields weighted by independent stochastic processes) which concur to produce a Brownian behaviour of tracers are:

- the high number of highly oscillating space components, combined with the independence of the stochastic multipliers. In such a case, the tracer is subject to the effect of several independent inputs, some stronger and some weaker, depending on its relative position with respect to the oscillation of the components and changing very rapidly the relative strength of these components, hence restoring some form of chaoticity of the background;
- some limitation in the strength or frequency of very large inputs by the stochastic multipliers.

In the case of classical α -stable cylindrical turbulent fields, the second property above does not hold while the first one is not sufficient to restore a Brownian behaviour. We observe an α -stable behaviour of the tracer, namely an anomalous, super-diffusive transport.

CRedit authorship contribution statement

Paolo Cifani: Writing – review & editing, Writing – original draft; **Franco Flandoli:** Writing – review & editing, Writing – original draft; **Lorenzo Marino:** Writing – review & editing, Writing – original draft.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Scaling parameter for limiting process

The main aim of this section is to derive an heuristic formula for the scaling coefficient $\bar{\sigma}$ appearing in [Conjecture 2](#). We recall that the particle dynamics, when starting from zero, is defined by (5):

$$\mathbf{X}_t^\eta = uC(\eta, \tau, \alpha) \sum_{\mathbf{k} \in \mathbf{K}_\eta} \int_0^t \sigma_{\mathbf{k}}(\mathbf{X}_s^\eta) \diamond dZ_s^{\alpha, \mathbf{k}}$$

where, for simplicity, we omitted the dependency from the initial condition $\mathbf{0}$. We denote by t_η the average time the tracer is influenced mostly by a certain Fourier component before changing to another one. It is clear that since for $\eta \ll 1$, more and more Fourier components are added to the dynamics, the average time $t_\eta \ll 1$ is small. Then, it is well-known that one can approximate:

$$\mathbf{X}_{t+t_\eta}^\eta - \mathbf{X}_t^\eta \sim uC(\eta, \tau, \alpha) \sum_{\mathbf{k} \in \mathbf{K}_\eta} \sigma_{\mathbf{k}}(\mathbf{X}_t^\eta) (Z_{t+t_\eta}^{\alpha, \mathbf{k}} - Z_t^{\alpha, \mathbf{k}}).$$

The validity of such a Wong-Zakai approximation of the Marcus stochastic integral has been corroborated by various studies (cf. [\[24,25\]](#)). Since $\mathbf{X}_t^\eta$ wanders through space and the Fourier modes $\sigma_{\mathbf{k}}$ are highly oscillatory, the value of $|\sigma_{\mathbf{k}}(\mathbf{X}_t^\eta)|^\alpha$ is “typically” close to its spatial average:

$$\langle \sigma_{\mathbf{k}} \rangle^\alpha := \lim_{R \rightarrow +\infty} \int_{[-\frac{R}{2}, \frac{R}{2}]^2} |\sigma_{\mathbf{k}}(x)|^\alpha d\mathbf{x} = \frac{1}{2\pi} \int_0^{2\pi} |\sin(t)|^\alpha dt \sim \frac{\alpha-1}{\sqrt{\pi\alpha}} \frac{\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{\alpha}{2})}.$$

Hence, the tracer mean displacement can be roughly approximated by:

$$\mathbb{E} \left[\left| \mathbf{X}_{t+t_\eta}^\eta - \mathbf{X}_t^\eta \right| \right] \sim u\tau^{(\alpha-1)/\alpha} \mathbb{E} \left[\left| Z_{t+t_\eta}^{\alpha, \mathbf{k}} - Z_t^{\alpha, \mathbf{k}} \right| \right] \sim u\tau^{(\alpha-1)/\alpha} t_\eta^{\frac{1}{\alpha}}.$$

Notice now that the increments

$$\mathbf{X}_{t_\eta}^\eta, \mathbf{X}_{2t_\eta}^\eta - \mathbf{X}_{t_\eta}^\eta, \mathbf{X}_{3t_\eta}^\eta - \mathbf{X}_{2t_\eta}^\eta, \dots$$

are approximately independent, since each Fourier mode is affected by an independent process. Moreover, each increment has a length $|\mathbf{X}_{(i+1)t_\eta}^\eta - \mathbf{X}_{it_\eta}^\eta|$ of order $\lambda\eta$, for a certain parameter $\lambda > 0$. Indeed, these increments measure the distance the tracer travels when affected by a certain Fourier component, before “jumping” on another one. Such a typical “distance” to travel in order to jump from one to the other is of the order of the wave-length of the sinusoidal components of the noise, possibly modified by a factor λ . The intuition behind is that the tracer is on the “top” of a cosine function, where the function takes approximately the value ± 1 . By moving a little bit, just a portion of the wave-length η , it will be no more on the top of that cosine component, but closer to the top of another component. Summarizing, at each time step it_η , we can approximate $\mathbf{X}_{it_\eta}^\eta$ by a random walk with mean displacement of size $i\lambda\eta$.

To conclude our heuristic derivation, we now split the argument between the different cases. When we claim a diffusive limit (i.e. cases (J_2) and (J_3)) so that approximately

$$\mathbf{X}_t^\eta \sim \bar{\sigma} \mathbf{W}_t,$$

for a Brownian motion $\{\mathbf{W}_t\}_{t \geq 0}$ on $\mathbb{R}^2$, the random walk approximation of $\mathbf{X}_{it_\eta}^\eta$ has finite square-average distance from the origin given by $i(\lambda\eta)^2$. Since it should be also equal to $\bar{\sigma}^2 it_\eta$, it follows that

$$\bar{\sigma} \sim \frac{\lambda\eta}{\sqrt{it_\eta}}.$$

By the arguments above, we also have that

$$\lambda\eta \sim u\tau^{(\alpha-1)/\alpha} t_\eta^{\frac{1}{\alpha}}$$

so that

$$t_\eta \sim \left(\frac{\lambda\eta}{u} \right)^\alpha \tau^{1-\alpha}.$$

Finally, we obtain that

$$\bar{\sigma} \sim \lambda^{1-\alpha/2} u^{\alpha/2} \eta^{1-\alpha/2} \tau^{(\alpha-1)/2}.$$

Note that by our assumption (7), the parameter $\bar{\sigma}$ is actually independent from η .

The heuristic derivation for the case (J_1) follows a similar line of reasoning, relying directly on Formula 4 for the mean displacement.

Appendix B. Numerical simulation algorithms

Here we resume the algorithms we used to implement our numerical simulations.

We begin by presenting the classical Chambers-Mallows-Stuck (CMS) algorithm [26], which simulates an α -stable distribution $S_\alpha(\sigma, \beta, 0)$. While the stability index $\alpha \in (1, 2)$ and the scale parameter $\sigma > 0$ were already introduced above (cf. Section 2.1), we recall that the skewness parameter $\beta \in [-1, 1]$ represents a measure of asymmetry. When $\beta = 0$, the distribution is symmetric, while $\beta > 0$ and $\beta < 0$ indicate right-skewed and left-skewed distributions, respectively. See [21, Sect. 1.7] for additional discussions on the topic.

Algorithm 1: Chambers–Mallows–Stuck (CMS) for $S_\alpha(\sigma, \beta, 0)$.

Require: $\alpha \in (1, 2)$, $\sigma > 0$, $\beta \in [-1, 1]$

Step 1. Sample independently $U \sim \text{Unif}(-\frac{\pi}{2}, \frac{\pi}{2})$ and $W \sim \text{Exp}(1)$.

Step 2: Set

$$B_{\alpha,\beta} = \frac{1}{\alpha} \arctan(\beta \tan(\frac{\pi\alpha}{2})), \quad S_{\alpha,\beta} = \left(1 + \beta^2 \tan^2(\frac{\pi\alpha}{2})\right)^{\frac{1}{2\alpha}}$$

Step 3: Set

$$Z = S_{\alpha,\beta} \frac{\sin(\alpha(U + B_{\alpha,\beta}))}{(\cos U)^{1/\alpha}} \left(\frac{\cos(U - \alpha(U + B_{\alpha,\beta}))}{W} \right)^{\frac{1-\alpha}{\alpha}}$$

return σZ

Now, the simulation of the sample paths for a one-dimensional α -stable Lévy process (i.e. case (J_1)) is straightforward since we can take advantage of the fact that its increments are independent and identically distributed. By discretising the time horizon and sequentially accumulating these discrete steps, the process's trajectories can be accurately reconstructed using the algorithm detailed below.

Algorithm 2: Sampling paths of α -stable Lévy process $(Z_t)_{t \geq 0}$ on time grid $\{0, \Delta t, \dots, n\Delta t\}$.

Require: $\alpha \in (1, 2)$, $\sigma > 0$, $\Delta t > 0$

Step 1. Set $Y_0 = 0$

Step 2. Sample n independent copies $(Y_i)_{i=1}^n$ of $S_\alpha(\sigma, 0, 0)$ using Algorithm 1

Step 3. Set $\Delta Z_i = (\Delta t)^{1/\alpha} Y_i$ for any $i = 0, \dots, n$

return $\left(\sum_{i=1}^{\ell} \Delta Z_i \right)_{\ell=0}^n$

To simulate the increments of a truncated α -stable process (with truncation parameter c) as in case (J_2) , one must handle again the infinite activity of its small jumps. In the pure α -stable case, such an issue is circumvented by relying on the self-similarity property

of the process. Here, one way to achieved it is by splitting the process at a small approximation threshold $\epsilon \in (0, c)$. The simulation method then relies on a Gaussian approximation of the “smaller” jumps to represent the truncated distribution. More precisely, the infinite sum of small jumps (those with an absolute value less than ϵ) is approximated by a normal random variable with a matched variance, while the large jumps (those bounded between ϵ and c) are generated exactly using a compound Poisson process. [Algorithm 3](#) outlines how to generate a single combined increment over a time step Δt . We refer to [\[27\]](#) for a detailed explanation on the validity of such diffusion approximation.

Algorithm 3: Diffusive approximation of small jumps increments of truncated α -stable process.

Require: $\alpha \in (1, 2)$, $\sigma > 0$, $c > 0$, $\epsilon \in (0, c)$ and $\Delta t > 0$

Step 1. Set

$$B = \frac{2\sigma}{2-\alpha} \epsilon^{2-\alpha}, \quad M = \frac{2\sigma}{\alpha} (\epsilon^{-\alpha} - c^{-\alpha})$$

Step 2. Sample independently $S \sim \mathcal{N}(0, B)$ and $N \sim \text{Poi}(M \Delta t)$

Step 3. Sample $2N$ independent copies $(U_j, V_j)_{j=1}^N$ of $\text{Unif}(0, 1)$

Step 4. For $j = 1, \dots, N$, set

$$Y_j = \text{sgn}(U_j - \frac{1}{2}) (\epsilon^{-\alpha} - V_j (\epsilon^{-\alpha} - c^{-\alpha}))^{-1/\alpha}$$

return $((\Delta t)^{1/2} S, \sum_{j=1}^N Y_j)$

Now that we have obtained a robust method of generating the individual increments, the full trajectory of a truncated α -stable process can be constructed in the same way as before.

Algorithm 4: Sampling paths of a truncated α -stable Lévy process $(Z_t)_{t \geq 0}$ on time grid $\{0, \Delta t, \dots, n\Delta t\}$.

Require: $\alpha \in (1, 2)$, $\sigma > 0$, $c > 0$, $\epsilon \in (0, c)$ and $\Delta t > 0$

Step 1. Set $\Delta Y_0^s = \Delta Y_0^\ell = 0$

Step 2. Sample n independent copies $(\Delta Y_i^s, \Delta Y_i^\ell)_{i=1}^n$ using [Algorithm 3](#).

Step 3. Set $\Delta Z_i = \Delta Y_i^s + \Delta Y_i^\ell$ for any $i = 0, \dots, n$

return $(\sum_{i=0}^\ell \Delta Z_i)_{\ell=0}^n$

We can finally focus on the numerical simulation of sample paths for a tempered α -stable process, as it was introduced in case (J_3) . Although a variety of algorithms have been suggested in the literature for sampling tempered stable processes on the real line, the Baeumer-Meerschaert acceptance-rejection method [\[28\]](#) seems to be the more reliable option. This algorithm essentially involves simulating first the increments of a totally skewed (i.e. $\beta = 1$) pure α -stable process. These increments are then subjected to a acceptance-rejection criterion to accurately impose the desired exponential tempering effect on the generated trajectories as detailed in the next algorithm.

Algorithm 5: Baeumer-Meerschaert algorithm for Tempered α -stable increments.

Require: $\alpha \in (1, 2)$, $\sigma > 0$, $A \geq 0$, $\Delta t > 0$, $c > 0$

Step 1. Generate independently $U \sim \text{Unif}(0, 1)$ and $S \sim S_\alpha((\Delta t)^{1/\alpha} \sigma, 1, 0)$

Step 2. if $U \leq e^{-A(S+c)}$ **then**

return $S - \Gamma(1 - \alpha) \sigma A^{\alpha-1} \Delta t$

else

 Return to Step 1.

Analogous to the pure α -stable case, constructing the full sample paths for a one-dimensional tempered α -stable Lévy process is simply obtained by discretizing the time horizon and sequentially accumulating the discrete increments generated by the Baeumer-Meerschaert algorithm. For a more comprehensive exposition and further discussions about this topic, the interested reader is referred to [\[29\]](#) and [\[30\]](#).

Algorithm 6: Sampling paths of a tempered α -stable Lévy process $(Z_t)_{t \geq 0}$ on time grid $\{0, \Delta t, \dots, n\Delta t\}$.

Require: $\alpha \in (1, 2)$, $\sigma > 0$, $A \geq 0$, $\Delta t > 0$, $c > 0$

Step 1. Set $\Delta Y_0^+ = \Delta Y_0^- = 0$

Step 2. Sample n independent copies $(\Delta Y_i^+, \Delta Y_i^-)_{i=1}^n$ using Algorithm 5.

Step 3. Set $\Delta Z_i = \Delta Y_i^+ - \Delta Y_i^-$ for any $i = 0, \dots, n$

return $\left(\sum_{i=0}^{\ell} \Delta Z_i\right)_{\ell=0}^n$

References

- [1] Boussinesq JV. Essai sur la théorie des eaux courantes. Mémoires présentés par divers savants à l'académie des sciences, Tome XXIII, No. 1 1877;1–680.
- [2] Berselli LC, Iliescu T, Layton WJ. *Mathematics of large eddy simulation of turbulent flows*. Scientific computation; Springer-Verlag, Berlin; 2006.
- [3] Majda AJ, Kramer PR. Simplified models for turbulent diffusion: theory, numerical modelling, and physical phenomena. *Phys Rep* 1999;314(4-5):237–574. [https://doi.org/10.1016/S0370-1573\(98\)00083-0](https://doi.org/10.1016/S0370-1573(98)00083-0)
- [4] Kraichnan RH. Anomalous scaling of a randomly advected passive scalar. *Phys Rev Lett* 1994;72:1016–19. <https://doi.org/10.1103/PhysRevLett.72.1016>
- [5] Majda AJ, Timofeyev I, Vanden Eijnden E. A mathematical framework for stochastic climate models. *Comm Pure Appl Math* 2001;54(8):891–974. <https://doi.org/10.1002/cpa.1014>
- [6] Galeati L. On the convergence of stochastic transport equations to a deterministic parabolic one. *Stoch Partial Differ Equ Anal Comput* 2020;8(4):833–68. <https://doi.org/10.1007/s40072-019-00162-6>
- [7] Flandoli F, Galeati L, Luo D. Eddy heat exchange at the boundary under white noise turbulence. *Philos Trans R Soc A* 2022;380(2219):20210096 13.
- [8] Flandoli F, Luongo E. Stochastic partial differential equations in fluid mechanics; vol. 2330 of Lecture notes in mathematics. Springer, Singapore; 2023. <https://doi.org/10.1007/978-981-99-0385-6>
- [9] Coghi M, Maurelli M. Existence and uniqueness by Kraichnan noise for 2D Euler equations with unbounded vorticity. 2023, [arXiv:2308.03216](https://arxiv.org/abs/2308.03216).
- [10] Holm DD, Luesink E, Pan W. Stochastic mesoscale circulation dynamics in the thermal ocean. *Phys Fluids* 2021;33(4):46603. <https://doi.org/10.1063/5.0040026>
- [11] Resseguier V, Mémín E, Chapron B. Geophysical flows under location uncertainty, part I random transport and general models. *Geophys Astrophys Fluid Dyn* 2017;111 (3):149–76. <https://doi.org/10.1080/03091929.2017.1310210>
- [12] Luo D. Convergence of stochastic 2D inviscid Boussinesq equations with transport noise to a deterministic viscous system. *Nonlinearity* 2021;34(12):8311–30. <https://doi.org/10.1088/1361-6544/ac3145>
- [13] Butori F, Mayorcas A, Morlacchi S. Homogenisation of a passive scalar transported by locally supported white noise. 2025, [arXiv:2509.06878](https://arxiv.org/abs/2509.06878).
- [14] Ephrati SR, Cifani P, Luesink E, Geurts BJ. Data-driven stochastic lie transport modeling of the 2D Euler equations. *J Adv Model Earth Syst* 2023;15(1):e2022MS003268. <https://doi.org/10.1029/2022MS003268>
- [15] Ephrati SR, Jansson E, Papini A. Diffusive behavior of transport noise on $\mathbb{S}^2$. *J Comput Dyn* 2026;14:1–16. <https://doi.org/10.3934/jcd.2025008>
- [16] Flandoli F, Russo F. Stochastic transport by Gaussian noise. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.* 2026. <https://doi.org/10.4171/RLM/1090>
- [17] Cifani P, Flandoli F. Diffusion properties of small-scale fractional transport models. *J Stat Phys* 2025;192(11):152 20. <https://doi.org/10.1007/s10955-025-03534-6>
- [18] Luo D, Teng F. Eddy viscosity by Lévy transport noises. 2025, [arXiv:2510.13463v1](https://arxiv.org/abs/2510.13463v1).
- [19] Jacob N. Pseudo differential operators and Markov processes. Vol. I. Imperial College Press, London; 2001. <https://doi.org/10.1142/9781860949746>
- [20] Sato K-I. Lévy processes and infinitely divisible distributions; vol. 68 of Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge; 2013.
- [21] Samorodnitsky G, Taqqu MS. Stable non-Gaussian random processes. Stochastic modeling; Chapman & Hall, New York; 1994. Stochastic models with infinite variance.
- [22] Hartmann L-S, Pavlyukevich I. First-order linear Marcus SPDEs. *Stoch. Dyn.* 2023;23(6):2350054 32. <https://doi.org/10.1142/S0219493723500545>
- [23] Applebaum D. Lévy processes and stochastic calculus; vol. 116 of Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge; second ed.; 2009. <https://doi.org/10.1017/CBO9780511809781>
- [24] Marcus SI. Modeling and analysis of stochastic differential equations driven by point processes. *IEEE Trans Inf Theory* 1978;24(2):164–72. <https://doi.org/10.1109/TIT.1978.1055857>
- [25] Kosenkova T, Kulik A, Pavlyukevich I. First order convergence of weak Wong-Zakai approximations of Lévy-driven Marcus SDEs. *Theory Stoch Process* 2019;24 (2):32–60.
- [26] Chambers JM, Mallows CL, Stuck BW. A method for simulating stable random variables. *J Am Stat Assoc* 1976;71 (354):340–4.
- [27] Asmussen S, Rosiński J. Approximations of small jumps of Lévy processes with a view towards simulation. *J Appl Probab* 2001;38(2):482–93. <https://doi.org/10.1017/s0021900200019987>
- [28] Baeumer B, Meerschaert MM. Tempered stable Lévy motion and transient super-diffusion. *J Comput Appl Math* 2010;233(10):2438–48. <https://doi.org/10.1016/j.cam.2009.10.027>
- [29] Kawai R, Masuda H. On simulation of tempered stable random variates. *J Comput Appl Math* 2011;235(8):2873–87. <https://doi.org/10.1016/j.cam.2010.12.014>
- [30] Jalal T. Stable and tempered stable distributions and processes: an overview toward trajectory simulation. 2024; [arXiv:2412.06374](https://arxiv.org/abs/2412.06374).