

# Adiabatic limits of closed orbits for some Newtonian systems in $\mathbb{R}^n$

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## Abstract

We deal with a Newtonian system like  $\ddot{x} + V'(x) = 0$ . We suppose that  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  possesses an  $(n - 1)$ -dimensional compact manifold  $M$  of critical points, and we prove the existence of arbitrarily slow periodic orbits. When the period tends to infinity these orbits, rescaled in time, converge to some closed geodesics on  $M$ .

*Key Words:* closed geodesics, slow motion, periodic solutions, limit trajectories

## 1 Introduction

Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth function and suppose that  $V$  possesses an  $(n - 1)$ -dimensional compact manifold  $M$  of critical points which is non degenerate, namely

$$(1) \quad \ker V''(x) = T_x M \quad \forall x \in M, \quad \text{or equivalently} \quad V''[n_x, n_x] \neq 0 \quad \forall x \in M,$$

where  $n_x$  is a normal vector to  $M$  at  $x$ .

We are interested in studying the existence of solutions to the Newtonian system

$$(T) \quad \begin{cases} \ddot{x} + V'(x) = 0; \\ x(\cdot) \text{ is } T\text{-periodic,} \end{cases}$$

when  $T$  is large and  $x(\cdot)$  is close to  $M$ . Equivalently, setting  $\varepsilon^2 = \frac{1}{T}$ , one looks for solutions to the problem

$$(P_\varepsilon) \quad \begin{cases} \ddot{x} + \frac{1}{\varepsilon} V'(x) = 0; \\ x(\cdot) \text{ is } 1\text{-periodic,} \end{cases}$$

for  $\varepsilon > 0$  sufficiently small.

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As  $T$  varies, problem (T) can possess some continuous families of solutions parametrized in  $T$ , and the fact that  $V$  is degenerate (in the sense of Morse) allows these solutions to have a non trivial limit behaviour. The case of solutions approaching a critical manifold of  $V$  has been considered for example in [6], [8] and [9]. It is known that if some smooth family  $x(t, \varepsilon)$  solves problem  $(P_\varepsilon)$  and if  $x(t, \varepsilon) \rightarrow M$  as  $\varepsilon \rightarrow 0$ , then the curve  $x(t, 0)$  is a geodesic on  $M$ . The curve  $x(\cdot, 0)$  is called adiabatic limit for the family  $x(\cdot, \varepsilon)$ .

The aim of this paper is to achieve some complementary result, namely to prove that for some closed geodesics on  $M$  there are indeed solutions of  $(P_\varepsilon)$  which approach these geodesics. Large period orbits with some limit behaviour have also been studied in [4] for planar systems.

Our main results are the following Theorems. The first one treats the case of a non-degenerate closed geodesic on  $M$ , see Definition 2.3.

**Theorem 1.1** *Suppose  $x_0 : S^1 \rightarrow M$  is a non-degenerate closed geodesic, and suppose  $V$  is repulsive w.r.t.  $M$ , namely that the following condition holds*

$$(2) \quad V''(x)[n_x, n_x] < 0 \quad \text{for all } x \in M.$$

*Then there exists  $T_0 > 0$  with the following property. For all  $T \geq T_0$  there exists a function  $u_T$  such that*

- (i)  $u_T$  is solution of problem (T);*
- (ii) as  $T \rightarrow +\infty$ ,  $u_T(T \cdot) \rightarrow x_0(\cdot)$  in  $C^1(S^1; \mathbb{R}^n)$ .*

The proof relies on the Local Inversion Theorem, which can be applied by the non-degeneracy of  $x_0$ . Since in  $(P_\varepsilon)$  appears the singular term  $\frac{1}{\varepsilon}$ , a quite accurate expansion of  $V'$  is needed.

If we want to prove the convergence of a sequence of trajectories, instead of the convergence of a one-parameter family, then we can remove any non-degeneracy assumption. With abuse of notation we will again call adiabatic limit the limit trajectory.

**Theorem 1.2** *If condition (2) holds, then for every sequence  $T_k \rightarrow +\infty$  there exists a sequence of solutions  $(u_k)_k$  to problem (T) corresponding to  $T = T_k$  such that up to subsequence  $u_k(T_k \cdot)$  converge in  $C^0(S^1, \mathbb{R}^n)$ . The adiabatic limit of  $u_k(T_k \cdot)$  is a non trivial closed geodesic  $x_0$  on  $M$ .*

**Remark 1.3** *(a) Since the adiabatic limit  $x_0$  in Theorem 1.2 can be degenerate, and so it is possible that it belongs to a family of degenerate geodesics, it is natural to expect convergence only along sequences of trajectories.*

*(b) The limit geodesic  $x_0$  can be characterized as follows. If  $\pi_1(M) \neq 0$ ,  $x_0$  realizes the infimum of the square length in some component of the closed loops in  $M$ . From the proof of Theorem 1.2 it follows that one can find adiabatic limits belonging to every element of  $\pi_1(M)$ , see also Remark 5.6. If  $\pi_1(M) = 0$  then the energy of  $x_0$  is the infimum in some suitable min-max scheme.*

The proof of Theorem 1.2 is based on a Lyapunov-Schmidt reduction on the Hilbert manifold  $H^1(S^1; M)$  of the closed loops in  $M$  of class  $H^1$ . Standard min-max arguments are applied to a suitable functional on  $H^1(S^1; M)$  which is a perturbation of the square length  $L_0$ , see formula (7). A similar approach has been used for example in [1] to perform reductions on finite-dimensional manifolds. The new feature of our method is that we perform a reduction of an infinite dimensional manifold.

If  $V$  is of attractive type, namely if  $V''(x)[n_x, n_x] > 0$  for all  $x \in M$ , then the situation is very different, since some phenomena of resonance may occur. As a consequence our hypotheses become stronger and we can prove convergence just for some suitable sequence  $T_k \rightarrow +\infty$ . Section 6 contains some results concerning this case. As an example we can state the following one.

**Theorem 1.4** *Suppose  $V$  satisfies the following conditions for some  $b_0 > 0$*

(i)  $V''(x)[n_x, n_x] = b_0$  for all  $x \in M$ ;

(ii)  $\frac{\partial^3 V}{\partial n_x^3}(x) = 0$  for all  $x \in M$ .

Then there exists a sequence  $T_k \rightarrow +\infty$  and there exists a sequence of solutions  $(u_k)_k$  to problem (T) corresponding to  $T = T_k$  such that up to subsequence  $u_k(T_k \cdot)$  converge in  $C^0(S^1, \mathbb{R}^n)$ . The adiabatic limit of  $u_k(T_k \cdot)$  is a non trivial closed geodesic  $x_0$  on  $M$ .

The paper is organized as follows: Section 2 is devoted to recalling some notations and preliminary facts. In Section 3 we prove Theorem 1.1. In Section 4 we study some linear ordinary differential equations, used to perform the reduction. In Section 5 we reduce the problem on  $H^1(S^1; M)$ , we study the reduced functional and we prove Theorem 1.2. Finally in Section 6 we treat the attractive case.

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## 2 Notations and Preliminaries

In this Section we recall some well known facts in Riemannian Geometry, we refer to [6] or [10] for the details. In particular we introduce the Levi-Civita connection, the Gauss' equations, the Hilbert manifold  $H^1(S^1; M)$ , and some properties of the square length functional  $L_0$  on  $H^1(S^1; M)$ .

It is given an orientable manifold  $M \subseteq \mathbb{R}^n$  of codimension 1, which inherits naturally a Riemannian structure from  $\mathbb{R}^n$ . On  $M$  it is defined the Gauss map  $n : M \rightarrow S^{n-1}$  which assigns to every point  $x \in M$  the unit versor  $n_x \in (T_x M)^\perp$ , where  $T_x M$  is the tangent space of  $M$  at  $x$ . The differential of  $n_x$  is given by

$$(3) \quad dn_x[v] = H(x)[v]; \quad \forall x \in M, \quad \forall v \in T_x M,$$

where  $H(x) : T_x M \rightarrow T_x M$  is a symmetric operator. Dealing with the operator  $H(x)$  we will also identify it with the corresponding symmetric bilinear form according to the relation

$$H(x)[v, w] = H(x)[v] \cdot w; \quad \forall v, w \in T_x M.$$

The bilinear form  $H(x)$  is called the *second fundamental form* of  $M$  at  $x$ .

If  $\mathcal{X}(M)$  denotes the class of the smooth vector fields on the manifold  $M$ , then for every  $x \in M$  the *Levi-Civita connection*  $\nabla : T_x M \times \mathcal{X}(M) \rightarrow T_x M$  is defined in the following way

$$(4) \quad \nabla_X Y = D_X \tilde{Y} - (n_x \cdot D_X \tilde{Y}) n_x.$$

Here  $\tilde{Y}$  is an extension of  $Y$  in a neighbourhood of  $x$  in  $\mathbb{R}^n$  and  $D_X$  denotes the standard differentiation in  $\mathbb{R}^n$  along the direction  $X$ .

**Remark 2.1** *The quantity  $\nabla_X Y$  defined in (4) is nothing but the projection of  $D_X \tilde{Y}$  onto the tangent space  $T_x M$ . Actually  $\nabla_X Y$  depends only on  $Y(x)$  and on the derivative of  $Y$  along the direction  $X$ . Hence formula (4) makes sense also when  $Y$  is defined just on a curve  $c$  on  $M$  for which  $c(t_0) = x$  and  $\dot{c}(t_0) = X$ . In the following this fact will be considered understood.*

The *Riemann tensor*  $R : T_x M \times T_x M \times T_x M \rightarrow T_x M$  is defined by

$$\nabla_{\tilde{X}} \nabla_{\tilde{Y}} \tilde{Z} - \nabla_{\tilde{Y}} \nabla_{\tilde{X}} \tilde{Z} - \nabla_{[\tilde{X}, \tilde{Y}]} \tilde{Z}.$$

Here  $\tilde{X}, \tilde{Y}$  and  $\tilde{Z}$  are smooth extensions of  $X, Y$  and  $Z$  respectively, and the symbol  $[\cdot, \cdot]$  denotes the usual Lie bracket. The above definition does not depend on the extensions of  $X, Y$  and  $Z$ .

Let  $i : M \rightarrow \mathbb{R}^n$  be the inclusion of  $M$  in  $\mathbb{R}^n$ : there exists a symmetric bilinear form  $s : T_x M \times T_x M \rightarrow \mathbb{R}$  which satisfies

$$s_x(X, Y) = \perp D_X \tilde{Y}, \quad \forall X, Y \in T_x M.$$

Here  $\tilde{Y}$  is any smooth extension of  $Y$  and  $\perp \nabla'_X \tilde{Y}$  is the component of  $D_X \tilde{Y}$  normal to  $T_x M$ . The *Gauss' equations* are the following

$$(5) \quad (R(X, Y)Z, W) = (s_x(Y, Z), s_x(X, W)) - (s_x(X, Z), s_x(Y, W)), \quad \forall X, Y, Z, W \in T_x M.$$

Given  $X, Y \in T_x M$ , consider a smooth extension  $\tilde{Y}$  of  $Y$  and an extension  $\Psi$  of  $n_x$  such that  $\Psi(p) \perp T_p M$  for all  $p \in M$  in some neighbourhood of  $x$ . Differentiating the relation  $(Y, \Psi) = 0$  along the direction  $X$ , there results

$$0 = D_X(Y, \Psi) = (D_X Y, \Psi) + (Y, D_X \Psi).$$

Hence it follows that  $s_x(X, Y)$  is given by

$$s_x(X, Y) = -H(x)[X, Y], \quad \forall X, Y \in T_x M.$$

In particular equation (5) becomes

$$(6) \quad (R(X, Y)Z, W) = (H(x)(Y, Z), H(x)(X, W)) - (H(x)(X, Z), H(x)(Y, W)), \quad \forall X, Y, Z, W \in T_x M.$$

In order to define the manifold  $H^1(S^1; M)$ , we recall first the differentiable structure of a smooth  $k$ -dimensional manifold. This is given by a family of local charts  $(U_\alpha, \varphi_\alpha)_\alpha$ , where  $\varphi_\alpha : \mathbb{R}^k \rightarrow \mathbb{R}^k$  are diffeomorphisms such that the compositions  $\varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(U_\alpha) \cap \varphi_\beta^{-1}(U_\beta) \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^k$  are smooth functions.

**Definition 2.2** A closed curve  $c : S^1 \rightarrow M$  is said to be of class  $H^1(S^1; M)$  if for some chart  $(U, \varphi)$  of  $M$  the map  $\varphi \circ c : S^1 \rightarrow \mathbb{R}^k$  is of class  $H^1$ .

This definition does not depend on the choice of the chart  $(U, \varphi)$ , since the composition of an  $H^1$  map in  $\mathbb{R}^k$  with a smooth diffeomorphism is still of class  $H^1$ . The class  $H^1(S^1; M)$  constitutes an infinite dimensional Hilbert manifold. We recall briefly its structure. Given a curve  $c \in C^\infty(S^1; M)$  we denote by  $c^*TM$  the pull-back of  $TM$  through  $c$ , namely the family of vector fields  $X : S^1 \rightarrow TM$  such that

$$X(t) \in T_{c(t)}M \quad \text{for all } t \in S^1.$$

We define also  $\mathcal{H}_c$  to be the sections  $\xi$  of  $c^*TM$  for which

$$\int_{S^1} |\xi(t)|^2 dt + \int_{S^1} |\nabla_{\dot{c}(t)} \xi(t)|^2 < +\infty.$$

There is a neighbourhood  $\mathcal{U}$  of the zero section of  $TM$  where the exponential map  $exp : TM \rightarrow M$  is well defined. For  $\xi \in \mathcal{U} \cap \mathcal{H}_c$ , the curve  $exp \xi$  belongs to  $H^1(S^1; M)$ , and viceversa every curve in  $H^1(S^1; M)$  can be obtained in this way for a suitable  $c \in C^\infty(S^1; M)$ . Hence the family  $(\mathcal{U} \cap \mathcal{H}_c, exp)$ ,  $c \in C^\infty(S^1; M)$  constitutes an atlas for  $H^1(S^1; M)$ .

The tangent space of  $H^1(S^1; M)$  can be described as follows: if  $h \in H^1(S^1; M)$ , we consider the class of curves  $\xi(\cdot)$  such that  $\xi(t) \in T_{h(t)}M$  for all  $t \in S^1$  and such that

$$\int_{S^1} |\xi(t)|^2 dt + \int_{S^1} |\nabla_{\dot{h}(t)} \xi(t)|^2 < +\infty.$$

By means of the Hölder inequality one can define the scalar product on  $T_h H^1(S^1; M)$  as

$$(\xi, \eta)_1 = \int_{S^1} (\xi(t), \eta(t)) dt + \int_{S^1} (\nabla_{\dot{h}(t)} \xi(t), \nabla_{\dot{h}(t)} \eta(t)) dt, \quad \forall \xi, \eta \in T_h H^1(S^1; M).$$

This scalar product determines a positive definite bilinear form on  $T_h H^1(S^1; M)$  and hence a Riemannian structure on  $H^1(S^1; M)$ .

On the manifold  $H^1(S^1; M)$  is defined the square length functional  $L_0$  in the following way

$$(7) \quad L_0(u) = \frac{1}{2} \int_0^1 |\dot{u}(t)|^2 dt, \quad \text{for all } u \in H^1(S^1; M).$$

Given  $A > 0$  we define  $L_0^A \subseteq H^1(S^1; M)$  to be

$$L_0^A = \{u \in H^1(S^1; M) : L_0(u) \leq A\}.$$

It is a standard fact that the functional  $L_0$  is smooth on the manifold  $H^1(S^1; M)$  endowed with the above structure, and there results

$$(8) \quad DL_0(h)[k] = \int_{S^1} (\dot{h}, \nabla_{\dot{h}} k); \quad h \in H^1(S^1; M), k \in T_h H^1(S^1; M).$$

The critical points of  $L_0$  are precisely the closed geodesics on  $M$ . Furthermore, if  $h \in H^1(S^1; M)$  is a stationary point of  $L_0$  there results

$$(9) \quad D^2 L_0(h)[k, w] = \int_{S^1} (\nabla_{\dot{h}} k, \nabla_{\dot{h}} w) - \int_{S^1} R(k, \dot{x}_0, \dot{x}_0, w); \quad k, w \in T_h H^1(S^1; M),$$

where  $R$  is given by formula (6).

**Definition 2.3** A closed geodesic  $h \in H^1(S^1; M)$  is said to be non-degenerate if the kernel of  $D^2 L_0(h)$  is one dimensional, and hence coincides with the span of  $\dot{h} \in T_h H^1(S^1; M)$ .

One useful property of  $L_0$  is the following

$$(PS) \quad \text{the functional } L_0 \text{ satisfies the Palais Smale condition on } H^1(S^1; M),$$

namely every sequence  $(u_m)$  for which  $L_0(u_m) \rightarrow a \in \mathbb{R}$  and  $\|DL_0(u_m)\| \rightarrow 0$  admits a convergent subsequence.

Condition (PS) allows to apply the standard min-max arguments in order to prove existence of critical point of  $L_0$ . For example, if for some compact manifold  $M$  it is  $\pi_1(M) \neq 0$ , we can reason as follows. The negative gradient flow of  $L_0$  preserves the components of  $H^1(S^1; M)$  and the embedding  $H^1(S^1; M) \hookrightarrow C^0(S^1; M)$  is continuous. As an immediate application we have the following Theorem.

**Theorem 2.4** Let  $M$  be a compact manifold. Then for every  $\alpha \in \pi_1(M)$  there exists a non-constant closed geodesic  $u_\alpha : [0, 1] \rightarrow M$  such that  $[u_\alpha(\cdot)] = \alpha$ , and moreover

$$E(u_\alpha) = c_\alpha := \inf_{[u]=\alpha} E(u).$$

If  $M$  is simply connected, the proof of the existence of a closed geodesic is more involved and in its most general form it is due to Lusternik and Fet, see [7], by means of topological methods. The proof of our Theorem 1.2 follows that argument, and we will recall it later. A fundamental tool is the Hurewitz Theorem.

**Theorem 2.5** (Hurewitz) *Suppose  $M$  is a finite dimensional compact manifold such that  $\pi_1(M) = 0$ . Define  $q$  to be the smallest integer for which  $\pi_q(M) \neq 0$ , and define  $q'$  to be the smallest integer such that  $H_{q'}(M) \neq 0$ . Then  $q$  and  $q'$  are equal.*

**Remark 2.6** *In our case  $M$  is orientable, and there always results  $H_{n-1}(M) \simeq \mathbb{Z}$ , so it turns out that  $q \leq n - 1$ .*

We denote by  $E_0 : H^1(S^1; \mathbb{R}^n) \rightarrow \mathbb{R}$  the square length functional for the curves in  $\mathbb{R}^n$ , namely

$$E_0(u) = \frac{1}{2} \int_0^1 |\dot{u}(t)|^2 dt; \quad u \in H^1(S^1; \mathbb{R}^n),$$

and more in general, for every  $\varepsilon > 0$ , we define  $E_\varepsilon : H^1(S^1, \mathbb{R}^n)$  to be

$$E_\varepsilon(u) = \int_0^1 \left( \frac{1}{2} |\dot{u}(t)|^2 - \frac{1}{\varepsilon} V(u(t)) \right) dt, \quad \forall u \in H^1(S^1; \mathbb{R}^n).$$

The critical points of  $E_\varepsilon$  are precisely the solutions of problem  $(P_\varepsilon)$ .

Now we introduce some final notations. Given a covariant tensor  $T$  and a vector field  $X$ , we denote by  $\mathcal{L}_X T$  the Lie derivative. If  $h \in H^1(S^1; M)$ , we define the functions  $Q_h, P_h, B_h : S^1 \rightarrow \mathbb{R}$  in the following way

$$Q_h = |H(h(\cdot))[\dot{h}(\cdot)]|^2; \quad P_h = \dot{h}(\cdot) \cdot H(h(\cdot))[\dot{h}(\cdot)]; \quad B_h = b(h(\cdot)),$$

where we have set, for brevity

$$b(x) = V''(x)[n_x, n_x], \quad x \in M.$$

Since  $h \in H^1(S^1; M)$  it is easy to check that  $Q_h, P_h \in L^1(S^1)$  while, since  $h$  is continuous form  $S^1$  to  $M$ ,  $B_h \in C^0(S^1)$ . Finally we set

$$\bar{H} = \sup \{ \|H(x)\| : x \in M \}.$$

### 3 The case of a non degenerate geodesic

This Section is devoted to prove Theorem 1.1. The strategy is the following: since problems  $(T)$  and  $(P_\varepsilon)$  are equivalent, we are reduced to find critical points of  $E_\varepsilon$  for  $\varepsilon$  small. In order to do this, we first find some "pseudo" critical points for  $E_\varepsilon$ , in Lemma 3.1, and we prove the uniform invertibility of  $D^2 E_\varepsilon$  at these points in Lemma 3.2. Then, in Proposition 3.3 we use the Contraction Mapping Theorem to find "true" critical points of  $E_\varepsilon$ .

To carry out the first step of our procedure, let us consider the non-degenerate geodesic  $x_0$  in the statement of Theorem 1.1.  $x_0$  induces the smooth map  $n_{x_0(\cdot)} : S^1 \rightarrow S^{n-1}$ . Hence every curve  $y : S^1 \rightarrow \mathbb{R}^n$  can be decomposed into two parts, the first tangential to  $T_{x_0}M$ , and the second normal to  $T_{x_0}M$

$$(10) \quad y^T = y - (y \cdot n_{x_0}) n_{x_0}; \quad y^N = (y \cdot n_{x_0}) n_{x_0}.$$

When  $y$  is differentiable we can also decompose the derivatives of  $y^T$  and of  $y^N$  into a tangential part and a normal part. Setting  $y_n = y \cdot n_{x_0}$ , there results

$$(11) \quad \left( \frac{d}{dt} y^T \right)^T = \nabla_{\dot{x}_0} y^T; \quad \left( \frac{d}{dt} y^T \right)^N = -(y \cdot \dot{n}_{x_0}) n_{x_0} = -H(x_0)[y, \dot{x}_0] n_{x_0};$$

and

$$(12) \quad \left( \frac{d}{dt} y^N \right)^N = \dot{y}_n n_{x_0}; \quad \left( \frac{d}{dt} y^N \right)^T = (y \cdot n_{x_0}) \dot{n}_{x_0} = y_n H(x_0) \dot{x}_0.$$

Using equations (10), (11) and (12) one can easily deduce that for some constant  $C_0 > 0$  depending only on  $x_0$  there holds

$$(13) \quad \frac{1}{C_0} \cdot \left( \|z_n\|_{H^1(S^1)} + \|z^T\|_{T_{x_0} H^1(S^1; M)} \right) \leq \|z\|_{H^1(S^1; \mathbb{R}^n)} \leq C_0 \cdot \left( \|z_n\|_{H^1(S^1)} + \|z^T\|_{T_{x_0} H^1(S^1; M)} \right),$$

for all  $z \in H^1(S^1; M)$ . This means that  $H^1(S^1; \mathbb{R}^n) \simeq T_{x_0} H^1(S^1; M) \oplus H^1(S^1)$  and that the two norms  $\|\cdot\|_{H^1(S^1; \mathbb{R}^n)}$  and  $\|\cdot\|_{T_{x_0} H^1(S^1; M)} + \|\cdot\|_{H^1(S^1)}$  are equivalent.

Now, roughly, we want to solve the equation  $\varepsilon \cdot \ddot{x}_\varepsilon = -V'(x_\varepsilon)$  up to the first order in  $\varepsilon$ : expanding  $V'(x_\varepsilon) : S^1 \rightarrow \mathbb{R}^n$  as

$$V'(x_\varepsilon(t)) = \varepsilon \cdot \alpha(t) + \varepsilon^2 \cdot \beta(t) + O(\varepsilon^3),$$

we have to find  $f$  and  $g$  such that

$$(14) \quad \ddot{x}_0(t) = -\alpha(t); \quad \ddot{f} = -\beta(t).$$

In order to solve equation (14), we first find the explicit expressions of  $\alpha$  and  $\beta$  depending on  $f$  and  $g$ . Since  $M$  is assumed to be non-degenerate, the function  $V$  can be written as

$$V(x) = \frac{1}{2} b(x) d_x^2; \quad d_x = \text{dist}(x, M),$$

where  $b : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth negative function, see condition (2). Because of the factor  $\frac{1}{\varepsilon}$ , to expand  $\frac{1}{\varepsilon} V'(x_\varepsilon)$  up to the first order in  $\varepsilon$ , we need to take into account the derivatives of  $V$  up to the third order. We fix some point  $x_0(t)$ , and we consider an orthonormal frame  $(e_1, \dots, e_n)$  in  $x_0(t)$  such that  $e_1, \dots, e_{n-1}$  form an orthonormal basis for  $T_{x_0(t)} M$ , and  $e_n$  is orthogonal to  $M$ . With simple computations one can easily check that the only non-zero components of the second and the third differential of  $V$  at  $x_0(t)$  are

$$(15) \quad D_{inn}^3 V(x_0(t)) = D_i b(x_0(t)); \quad D_{ijn}^3 V(x_0(t)) = b(x_0(t)) \cdot D_{ij}^2 d(x_0(t)); \quad i, j = 1, \dots, n-1;$$

$$(16) \quad D_{nn}^2 V(x_0(t)) = b(x_0(t)), \quad D_{nnn}^3 V(x_0(t)) = 3D_n b(x_0(t)).$$

The second differential of  $d_x$  at  $x_0(t)$ , see [5] Appendix, is given by

$$D^2 d_x[v, w] = \sum_{i, j=1}^{n-1} H_{ij} v_i w_j; \quad v, w \in T_{x_0(t)} M.$$

Here the numbers  $H_{ij}$  denote the components of  $H(x_0(t))$  with respect to the basis  $(e_1, \dots, e_{n-1})$  of  $T_{x_0(t)} M$ . In particular from the last formula it follows that

$$D_{ijn}^3 V(x_0(t)) = b(x_0) \cdot D_{ij}^2 d(x_0) = b(x_0) \cdot H_{ij}(x_0), \quad i, j = 1, \dots, n-1.$$

Hence by expanding  $V'(x_0 + \varepsilon f + \varepsilon^2 g)$  in powers of  $\varepsilon$  we get

$$(17) \quad \alpha(t) \cdot z = b(x_0(t)) f_n z_n, \quad \forall z \in \mathbb{R}^n;$$

$$(18) \quad \beta(t) \cdot z = b(x_0) g_n \cdot z_n + \frac{1}{2} \cdot \left[ \sum_{i, j=1}^{n-1} b(x_0) \cdot H_{ij} (f_i f_j z_n + f_i f_n z_j + f_j f_n z_i) \right. \\ \left. + \sum_{i=1}^{n-1} D_i b(x_0) (2f_i f_n z_n + f_n^2 z_i) + 3D_n b(x_0) f_n^2 z_n \right], \quad \forall z \in \mathbb{R}^n.$$

Here  $f_i, f_n$ , etc. denote the components of the vectors with respect to the basis  $(e_1, \dots, e_n)$ . Taking into account (17) and (18), the equations in (14) become

$$(19) \quad \ddot{x}_0 = -b(x_0) f_n;$$

$$(20) \quad \begin{aligned} \dot{f} \cdot z &= -b(x_0) g_n z_n - \frac{1}{2} \left[ \left( b(x_0) \sum_{i,j=1}^{n-1} H_{ij} f_i f_j + 2 \sum_{i=1}^{n-1} D_i b(x_0) f_i f_n + 3D_n b(x_0) f_n^2 \right) z_n \right. \\ &+ \left. \sum_{i=1}^{n-1} \left( 2 \sum_{j=1}^{n-1} H_{ij} f_j f_n + D_i b(x_0) f_n^2 \right) z_i \right], \quad \forall z \in \mathbb{R}^n. \end{aligned}$$

Equation (19) can be solved in  $f_n$  with

$$(21) \quad f_n(t) = a(t) := \frac{1}{b(x_0(t))} \dot{x}_0(t) \cdot H(x_0(t)) \dot{x}_0(t) = \frac{1}{b(x_0(t))} H(x_0(t)) [\dot{x}_0(t), \dot{x}_0(t)].$$

In fact, since  $x_0$  is a geodesic, it turns out that  $(\ddot{x}_0)^T = \nabla_{\dot{x}_0} \dot{x}_0 = 0$ ; moreover by taking  $y^T = \dot{x}_0$  in formula (11), we can conclude that

$$(\ddot{x}_0)^T = 0; \quad (\ddot{x}_0)_n = -H(x_0) [\dot{x}_0, \dot{x}_0],$$

hence (21) follows. As far as (20) is concerned, we can write it in variational form, substituting the expression of  $f_n$  according to (21)

$$(22) \quad \begin{aligned} \int_{S^1} \dot{f} \dot{z} &= \int_{S^1} b(x_0) g_n z_n + \frac{1}{2} \int_{S^1} (b(x_0) H(x_0) [f^T, f^T] + 2a(t) \nabla^T b(x_0) f^T + 3D_n b(x_0) a^2(t)) z_n \\ &+ \int_{S^1} a(t) b(x_0(t)) H(x_0) [f^T, z^T] + \frac{1}{2} \int_{S^1} a^2(t) \nabla^T b(x_0) z^T; \quad \forall z \in H^1(S^1; \mathbb{R}^n). \end{aligned}$$

Here  $\nabla^T b$  is the tangential derivative of  $b$  on  $M$ .

The quantity  $\int_{S^1} \dot{f} \dot{z}$  can be expressed in a suitable way by decomposing  $f$  and  $z$  into their tangent and normal parts. Using equations (11) and (12) corresponding to  $f$  and  $z$ , and taking into account (21) we have

$$\begin{aligned} \int_{S^1} \dot{f} \dot{z} &= \int_{S^1} \nabla f^T \nabla z^T + \int_{S^1} H[f^T, \dot{x}_0] H[z^T, \dot{x}_0] + \int_{S^1} \dot{a} \dot{z}_n + \int_{S^1} a z_n H[\dot{x}_0] \cdot H[\dot{x}_0] \\ &+ \int_{S^1} z_n H[\dot{x}_0, \nabla f^T] + \int_{S^1} a H[\dot{x}_0, \nabla z^T] - \int_{S^1} H[\dot{x}_0, f^T] \dot{z}_n - \int_{S^1} H[\dot{x}_0, z^T] \dot{a}. \end{aligned}$$

From equations (6) and (9) it follows that

$$D^2 L_0(x_0) [z^T, f^T] = \int_{S^1} \nabla f^T \nabla z^T + \int_{S^1} H[f^T, \dot{x}_0] H[z^T, \dot{x}_0] - \int_{S^1} H[\dot{x}_0, \dot{x}_0] H[z^T, f^T],$$

hence there holds

$$\begin{aligned} \int_{S^1} \dot{f} \dot{z} &= D^2 L_0(x_0) [z^T, f^T] + \int_{S^1} H[\dot{x}_0, \dot{x}_0] H[z^T, f^T] + \int_{S^1} \dot{a} \dot{z}_n + \int_{S^1} a z_n H^2[\dot{x}_0, \dot{x}_0] \\ &+ \int_{S^1} z_n H[\dot{x}_0, \nabla f^T] + \int_{S^1} a H[\dot{x}_0, \nabla z^T] - \int_{S^1} H[\dot{x}_0, f^T] \dot{z}_n - \int_{S^1} H[\dot{x}_0, z^T] \dot{a}. \end{aligned}$$



Taking into account that by the definition of  $a(t)$ , it is

$$\int_{S^1} a(t) b(x_0(t)) H(x_0)[f^T, f^T] = \int_{S^1} H(x_0)[\dot{x}_0, \dot{x}_0] H(x_0)[f^T, f^T],$$

equation (22) assumes the form

$$\begin{aligned} D^2 L_0(x_0)[z^T, f^T] &+ \int_{S^1} \dot{a} \dot{z}_n + \int_{S^1} a z_n H^2[\dot{x}_0, \dot{x}_0] + \int_{S^1} a H[\dot{x}_0, \nabla z^T] - \int_{S^1} H[\dot{x}_0, f^T] \dot{z}_n \\ (23) \quad &= \frac{1}{2} \int_{S^1} (b(x_0) H(x_0)[f^T, f^T] + 2a(t) \nabla^T b(x_0) f^T + 3D_n b(x_0) a^2(t) - 2H[\dot{x}_0, \nabla f^T]) z_n \\ &+ \frac{1}{2} \int_{S^1} a^2(t) \nabla^T b(x_0) z^T + \int_{S^1} b(x_0) g_n z_n + \int_{S^1} H[\dot{x}_0, z^T] \dot{a}; \quad \forall z \in H^1(S^1; \mathbb{R}^n). \end{aligned}$$

Now we claim that we can find  $f^T$  satisfying the following conditions

$$(24) \quad (f^T, \dot{x}_0)_{T_{x_0} H^1(S^1; M)} = 0;$$

$$\begin{aligned} D^2 L_0(x_0)[z^T, f^T] &= - \int_{S^1} a H[\dot{x}_0, \nabla z^T] + \frac{1}{2} \int_{S^1} a^2(t) \nabla^T b(x_0) z^T + \int_{S^1} H[\dot{x}_0, z^T] \dot{a}; \\ (25) \quad &\text{for all } z^T \in T_{x_0} H^1(S^1; M). \end{aligned}$$

In fact, since  $D^2 L_0(x_0)$  is non degenerate on  $(\dot{x}_0)^\perp$ , there exists  $f^T$  satisfying (24) and satisfying (25) for all  $z^T \in (\dot{x}_0)^\perp$ . But since  $D^2 L_0(x_0)[\dot{x}_0, f^T] = 0$  and also

$$- \int_{S^1} a H[\dot{x}_0, \nabla \dot{x}_0] + \frac{1}{2} \int_{S^1} a^2(t) \nabla^T b(x_0) \dot{x}_0 + \int_{S^1} H[\dot{x}_0, \dot{x}_0] \dot{a} = 0,$$

as one can check with simple computations, indeed  $f^T$  satisfies equation (25) for all  $z^T \in T_{x_0} H^1(S^1; M)$ . We note that from standard regularity theory for ordinary differential equations, it turns out that  $f^T$  is smooth. Then, choosing  $g_n$  such that

$$\begin{aligned} b(x_0) g_n &= -\frac{1}{2} (b(x_0) H(x_0)[f^T, f^T] + 2a(t) \nabla^T b(x_0) f^T + 3D_n b(x_0) a^2(t) - 2H[\dot{x}_0, \nabla f^T]), \\ (26) \quad &- \ddot{a} + a H^2[\dot{x}_0, \dot{x}_0] + \frac{d}{dt} (H[\dot{x}_0, f^T]), \end{aligned}$$

with the above choice of  $f^T$  equation (23) holds true. In conclusion, we have solved (19) and (20), so also (14) is satisfied.

We can summarize the above discussion in the following Lemma.

**Lemma 3.1** *Let  $f_n$  and  $f^T$  be given by equations (21) and (24) and (25) respectively. Let also  $g_n$  satisfy (26) and let  $g^T \equiv 0$ . Then, setting*

$$(27) \quad x_\varepsilon = x_0 + \varepsilon f + \varepsilon^2 g,$$

there exists  $C_1 > 0$  such that for  $\varepsilon$  sufficiently small

$$\|DE_\varepsilon(x_\varepsilon)\| \leq C_1 \cdot \varepsilon^2.$$

*Proof.* For all  $z \in H^1(S^1; M)$  there results

$$\int_{S^1} \dot{x}_\varepsilon \cdot \dot{z} - \frac{1}{\varepsilon} \int_{S^1} V'(x_\varepsilon)[z] = - \int_{S^1} \left( \ddot{x}_\varepsilon + \frac{1}{\varepsilon} V'(x_\varepsilon) \right) \cdot z.$$

Furthermore by (14) one has

$$\ddot{x}_\varepsilon + \frac{1}{\varepsilon} V'(x_\varepsilon) = \ddot{x}_0 + \varepsilon \ddot{f} + \varepsilon^2 \ddot{g} + \alpha(t) + \beta(t) + O(\varepsilon^2),$$

hence it follows that

$$\|DE_\varepsilon(x_\varepsilon)\| \leq C_0 \cdot \varepsilon^2 \cdot \|z\|; \quad \text{for all } z \in H^1(S^1; M).$$

This concludes the proof of the Lemma.  $\square$

The group  $S^1$  induces naturally an action on the closed curves given by  $\theta : x(\cdot) \rightarrow x(\cdot + \theta)$ ,  $\theta \in S^1$ . If  $x \in H^1(S^1; M)$ , or  $x \in H^1(S^1; \mathbb{R}^n)$  is a non-constant map, then in a small neighbourhood of  $x$ , the quotients  $H^1(S^1; M)/S^1$ ,  $H^1(S^1; \mathbb{R}^n)/S^1$  are smooth manifolds. The functionals  $L_0$  and  $E_\varepsilon$  are invariant under the action of  $S^1$ , so one can expect to have invertibility of  $D^2L_0$  and  $D^2E_\varepsilon$  at some point only passing to the quotient spaces. In particular, taking into account Definition 2.3 one has that if  $h$  is a non trivial geodesic, then  $D^2L_0(h)$  has zero kernel on  $T_h(H^1(S^1; M)/S^1)$ .

We want to prove the uniform invertibility of  $D^2E_\varepsilon(x_\varepsilon)$ . Hence, in Lemma 3.2 and Proposition 3.3 below, it will be understood that we are considering the quotients of  $H^1(S^1; M)$  and of  $H^1(S^1; \mathbb{R}^n)$ , without writing it explicitly.

**Lemma 3.2** *Let  $x_\varepsilon$  be given by formula (27), where  $f$  and  $g$  are the functions in the statement of Lemma 3.1. Then there exists  $\varepsilon_0 > 0$  and  $C_2 > 0$  with the following properties. For  $\varepsilon \in (0, \varepsilon_0)$  the operator  $D^2E_\varepsilon(x_\varepsilon)$  is invertible and*

$$(28) \quad \|(D^2E_\varepsilon(x_\varepsilon))^{-1}\| \leq C_2; \quad \varepsilon \in (0, \varepsilon_0).$$

*Proof.* Given  $z, w \in H^1(S^1; \mathbb{R}^n)$  there holds

$$(29) \quad D^2E_\varepsilon(x_\varepsilon)[z, w] = \int_{S^1} \dot{z} \dot{w} - \frac{1}{\varepsilon} V''(x_\varepsilon)[z, w].$$

Reasoning as in the proof of Lemma 3.1 we can write the relation

$$\begin{aligned} \int_{S^1} \dot{z} \dot{w} &= \int_{S^1} \nabla z^T \nabla w^T + \int_{S^1} H[z^T, \dot{x}_0] H[w^T, \dot{x}_0] + \int_{S^1} \dot{z}_n \dot{w}_n + \int_{S^1} z_n w_n H^2[\dot{x}_0, \dot{x}_0] \\ &+ \int_{S^1} w_n H[\dot{x}_0, \nabla z^T] + \int_{S^1} z_n H[\dot{x}_0, \nabla w^T] - \int_{S^1} H[\dot{x}_0, z^T] \dot{w}_n - \int_{S^1} H[\dot{x}_0, w^T] \dot{z}_n. \end{aligned}$$

Moreover, expanding  $V''(x_\varepsilon)$  one has

$$V''(x_\varepsilon) = V''(x_0) + \varepsilon V'''(x_0)[f] + O(\varepsilon^2),$$

so, taking into account formulas (15) and (16) one can check that there exist  $C_3 > 0$  and some smooth functions  $\overline{C}_i : S^1 \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n$ , such that

$$(30) \quad \left| \frac{1}{\varepsilon} V''(x_\varepsilon)[z, w] - \frac{1}{\varepsilon} b(x_0) z_n w_n - f_n b(x_0) \sum_{i,j=1}^{n-1} H_{ij}(x_0) z_i w_j - \sum_{i=1}^{n-1} \overline{C}_i(z_i w_n + w_i z_n) \right| \leq C_3 \cdot \varepsilon \cdot |z| \cdot |w|.$$

Hence, since  $f_n$  satisfies equation (21), there results

$$\begin{aligned}
(31) \quad D^2 E_\varepsilon(x_\varepsilon)[z, w] &= D^2 L_0(x_0)[z, w] + \int_{S^1} \dot{z}_n \dot{w}_n + \int_{S^1} z_n w_n H^2[\dot{x}_0, \dot{x}_0] - \sum_{i=1}^{n-1} \int_{S^1} \bar{C}_i (z_i w_n + w_i z_n) \\
&+ \int_{S^1} w_n H[\dot{x}_0, \nabla z^T] + \int_{S^1} z_n H[\dot{x}_0, \nabla w^T] - \int_{S^1} H[\dot{x}_0, z^T] \dot{w}_n - \int_{S^1} H[\dot{x}_0, w^T] \dot{z}_n \\
&- \frac{1}{\varepsilon} \int_{S^1} b(x_0) z_n w_n + O(\varepsilon) \cdot \|z\| \cdot \|w\|.
\end{aligned}$$

Since  $x_0$  is a non-degenerate critical point for  $L_0$ , there exist subspaces  $W^+, W^- \subseteq T_{x_0} H^1(S^1; M)$  with the following properties

$$(i) \quad W^+ \oplus W^- = T_{x_0} H^1(S^1; M), \quad W^+ \cap W^- = 0;$$

(ii)  $D^2 L_0(x_0)$  is positive definite (resp. negative definite) on  $W^+$  (resp.  $W^-$ ).

Now, taking into account that  $H^1(S^1; \mathbb{R}^n)$  can be decomposed as  $H^1(S^1; \mathbb{R}^n) \simeq T_{x_0} H^1(S^1; M) \oplus H^1(S^1)$ , we equip it with the equivalent scalar product  $(\cdot, \cdot)_\#$

$$(v, w)_\# = (v^T, w^T)_{T_{x_0} H^1(S^1; M)} + (v_n, w_n)_{H^1(S^1)}.$$

We set also

$$X^+ = W^+ \oplus H^1(S^1); \quad X^- = W^- \oplus \{0\},$$

and we define

$$P^+ : H^1(S^1; \mathbb{R}^n) \rightarrow X^+; \quad P^- : H^1(S^1; \mathbb{R}^n) \rightarrow X^-$$

to be the orthogonal projections, with respect to  $(\cdot, \cdot)_\#$  onto the subspaces  $X^+$  and  $X^-$ .

From equation (31) there results

$$(32) \quad D^2 E_\varepsilon(x_\varepsilon)|_{X^-} = D^2 L_0(x_0)|_{W^-} + o(1).$$

So, since  $D^2 E_\varepsilon(x_\varepsilon)$  is self-adjoint it follows that also

$$(D^2 E_\varepsilon(x_\varepsilon)v, w)_\# = (v, D^2 E_\varepsilon(x_\varepsilon)w)_\# = o(1) \|v\| \|w\|; \quad v \in X^+, w \in X^-.$$

This implies that

$$(33) \quad D^2 E_\varepsilon(x_\varepsilon)|_{X^+} = P^+ \circ D^2 E_\varepsilon(x_\varepsilon)|_{X^+} + o(1).$$

Our aim is to prove that there exists  $C_4 > 0$  and  $\varepsilon_1 < \varepsilon_0$  such that the following properties hold

$$(j) \quad (D^2 E_\varepsilon(x_\varepsilon)y, y)_\# \leq -C_4 \|y\|^2; \quad y \in X^-, \varepsilon \in (0, \varepsilon_1);$$

$$(jj) \quad (D^2 E_\varepsilon(x_\varepsilon)y, y)_\# \geq C_4 \|y\|^2; \quad y \in X^+, \varepsilon \in (0, \varepsilon_1).$$

There results

$$(D^2 E_\varepsilon(x_\varepsilon)y, y)_\# = D^2 E_\varepsilon(x_\varepsilon)[y, y]; \quad y \in X^-,$$

so condition (j) follows immediately from (32) and (ii). On the other hand, for  $z \in W^+ \oplus H^1(S^1)$ , one can write

$$\int_{S^1} H[\dot{x}_0, z^T] \dot{z}_n = - \int_{S^1} z_n \frac{d}{dt} (H[\dot{x}_0, z^T]).$$

So, inserting this relation into (31) one can easily see that there exists  $\hat{C} > 0$  such that

$$(34) \quad D^2 E_\varepsilon(x_\varepsilon)[z, z] \geq D^2 L_0(x_0)[z, z] + \int_{S^1} z_n^2 - \frac{1}{\varepsilon} \int_{S^1} b_* z_n^2 - \hat{C} \left( \int_{S^1} z_n^2 \right)^{\frac{1}{2}} \cdot \|z\|.$$

Here we have set  $b_* = \sup_{x \in M} b(x) < 0$ . Given an arbitrary  $\delta > 0$ , by the Newton inequality there results

$$\left( \int_{S^1} z_n^2 \right)^{\frac{1}{2}} \cdot \|z\| \leq \left( \frac{1}{2} \cdot \delta \cdot \|z\|^2 + \frac{1}{2} \cdot \frac{1}{\delta} \cdot \int_{S^1} z_n^2 \right),$$

hence, by equation (13) it follows that

$$(35) \quad \left( \int_{S^1} z_n^2 \right)^{\frac{1}{2}} \cdot \|z\| \leq \left( \frac{C_0}{2} \cdot \delta \cdot \|z^T\|_{T_{x_0} H^1(S^1, M)}^2 + \frac{C_0}{2} \cdot \delta \cdot \|z_n\|_{H^1(S^1)}^2 + \frac{1}{2} \cdot \frac{1}{\delta} \cdot \int_{S^1} z_n^2 \right).$$

Hence, since  $D^2 L_0(x_0)$  is positive definite on  $W^+$ , we can choose  $\delta$  to be so small that

$$\delta \cdot \hat{C} \cdot C_0 \leq \min \{1, \inf \{D^2 L_0(x_0)[w, w] : w \in W^+, \|w\| = 1\}\}.$$

With this choice of  $\delta$ , equations (34) and (35) imply the existence of  $\varepsilon_2 < \varepsilon_1$  for which

$$(36) \quad D^2 E_\varepsilon(x_\varepsilon)[z, z] \geq \frac{1}{2} \cdot \left( L_0(x_0)[z^T, z^T] + \int_{S^1} z_n^2 - \frac{1}{\varepsilon} \cdot b_* \cdot \int_{S^1} z_n^2 \right), \quad \varepsilon \in (0, \varepsilon_2), z \in X^+.$$

Equation (36) together with (32) implies (jj) and concludes the Proof of the Lemma.  $\square$

**Proposition 3.3** *For  $\varepsilon$  small, problem  $(P_\varepsilon)$  admits an unique solution  $\bar{x}_\varepsilon$  which satisfies*

$$(37) \quad \|x_\varepsilon - \bar{x}_\varepsilon\|_{C^0(S^1)} \leq C_5 \cdot \varepsilon^2; \quad \text{for some } C_5 > 0.$$

*Proof.* We prove the Proposition by using the Contraction Mapping Theorem. Actually we want to find  $y \in H^1(S^1; \mathbb{R}^n)$  which satisfies

$$DE_\varepsilon(x_\varepsilon + y) = 0; \quad \|y\|_{H^1(S^1; \mathbb{R}^n)} \leq C_5 \cdot \varepsilon^2.$$

We can write

$$DE_\varepsilon(x_\varepsilon + y) = DE_\varepsilon(x_\varepsilon + y) - DE_\varepsilon(x_\varepsilon) - D^2 E_\varepsilon(x_\varepsilon)[y] + DE_\varepsilon(x_\varepsilon) + D^2 E_\varepsilon(x_\varepsilon)[y].$$

Hence it turns out that

$$DE_\varepsilon(x_\varepsilon + y) = 0 \quad \Leftrightarrow \quad y = F_\varepsilon(y),$$

where  $F_\varepsilon : H^1(S^1; \mathbb{R}^n) \rightarrow H^1(S^1; \mathbb{R}^n)$  is defined by

$$F_\varepsilon(z) := - \left( D^2 E_\varepsilon(x_\varepsilon) \right)^{-1} \left[ DE_\varepsilon(x_\varepsilon) - \left( DE_\varepsilon(x_\varepsilon + z) - DE_\varepsilon(x_\varepsilon) - D^2 E_\varepsilon(x_\varepsilon)[z] \right) \right].$$

We show that the map  $F_\varepsilon$  is a contraction in some ball  $\bar{B}_\rho = \{z \in H^1(S^1; \mathbb{R}^n) : \|z\| \leq \rho\}$ . In fact, if  $z \in \bar{B}_\rho$ , by equation (28) there results

$$(38) \quad \|F_\varepsilon(z)\| \leq C_2 \cdot \left( \|DE_\varepsilon(x_\varepsilon)\| + \|DE_\varepsilon(x_\varepsilon + z) - DE_\varepsilon(x_\varepsilon) - D^2 E_\varepsilon(x_\varepsilon)[z]\| \right).$$

With a straightforward calculation one obtains that for all  $w \in H^1(S^1; \mathbb{R}^n)$  there holds

$$DE_\varepsilon(x_\varepsilon + z)[w] - DE_\varepsilon(x_\varepsilon)[w] - D^2 E_\varepsilon(x_\varepsilon)[z, w] = \frac{1}{\varepsilon} \int_{S^1} (V''(x_\varepsilon)[z, w] - V'(x_\varepsilon + z)[w] + V'(x_\varepsilon)[w]).$$

Since  $V$  is a smooth function, there exists  $C_6 > 0$  such that

$$\|V''(x_\varepsilon(t))[z, w] + V'(x_\varepsilon(t))[w] - V'(x_\varepsilon(t) + z)[w]\| \leq C_6 \cdot \|z\|^2 \cdot \|w\|; \quad w, z \in \mathbb{R}^n, \|z\| \leq 1, t \in S^1,$$

so it follows that for  $\rho$  sufficiently small

$$(39) \quad \|DE_\varepsilon(x_\varepsilon + z) - DE_\varepsilon(x_\varepsilon) - D^2E_\varepsilon(x_\varepsilon)[z]\| \leq \frac{1}{\varepsilon} C_6 \cdot \|z\|_{H^1(S^1; \mathbb{R}^n)}^2; \quad \|z\|_{H^1(S^1; \mathbb{R}^n)} \leq \rho.$$

Hence, by using equations (38), (28) and (39), for  $\rho$  sufficiently small there holds

$$(40) \quad \|F_\varepsilon(z)\| \leq C_2 \cdot \left( C_1 \varepsilon^2 + \frac{1}{\varepsilon} C_6 \cdot \rho^2 \right); \quad \|z\| \leq \rho.$$

Now consider two functions  $z, z' \in H^1(S^1; \mathbb{R}^n)$ : for all  $y \in H^1(S^1; \mathbb{R}^n)$  there results

$$\begin{aligned} & DE(x_\varepsilon + z)[y] - D^2(x_\varepsilon)[z, y] - DE(x_\varepsilon + z')[y] + D^2(x_\varepsilon)[z', y] \\ &= \frac{1}{\varepsilon} \int_{S^1} (V''(x_\varepsilon(t))[z, y] - V'(x_\varepsilon(t) + z)[y] - V''(x_\varepsilon(t))[z', y] + V'(x_\varepsilon(t) + z')[y]) \, ds. \end{aligned}$$

So, taking into account that

$$\begin{aligned} & V''(x_\varepsilon(t))[z] - V'(x_\varepsilon(t) + z) - V''(x_\varepsilon(t))[z'] + V'(x_\varepsilon(t) + z') \\ &= \int_0^1 (V''(x_\varepsilon(t) + z + s(z' - z)) - V''(x_\varepsilon(t))) [z - z'] \, ds, \end{aligned}$$

there results

$$(41) \quad |(F_\varepsilon(z) - F_\varepsilon(z'), w)| \leq \frac{1}{\varepsilon} C_2 \sup_{t \in S^1, s \in [0,1]} \|V''(x_\varepsilon(t) + z + s(z' - z)) - V''(x_\varepsilon(t))\| \cdot \|z' - z\|_\infty \cdot \|w\|_\infty.$$

Choosing

$$(42) \quad \rho = C_7 \cdot \varepsilon^2,$$

with  $C_7$  sufficiently large, by equations (40) and (41) the map  $F_\varepsilon$  turns out to be a contraction in  $\overline{B}_\rho$ . This concludes the proof.  $\square$

**Remark 3.4** *By equation (42), it follows that  $\|y\| = O(\varepsilon^2)$ . On the other hand, the proof of Lemma 3.1 determines uniquely the normal component  $g_n$  of  $g$ . In other words, this means that the following condition must be satisfied*

$$(43) \quad \|y_n\|_{C^0(S^1)} = o(\varepsilon^2).$$

Actually we can prove that (43) holds true. In fact, by the proof of Proposition 3.3, the fixed point  $y$  solves

$$(44) \quad y = (D^2E_\varepsilon(x_\varepsilon))^{-1} z,$$

with

$$z = - [DE_\varepsilon(x_\varepsilon) - (DE_\varepsilon(x_\varepsilon + y) - DE_\varepsilon(x_\varepsilon) - D^2E_\varepsilon(x_\varepsilon)[y])].$$

By using equations (32), (33) and (36) one can show that  $y_n$  satisfies the inequality

$$\|y_n\|_{L^2(S^1)} \leq \overline{C} \cdot \varepsilon \cdot \|P^+ z\|_{H^1(S^1; \mathbb{R}^n)} \leq \overline{C} \cdot \varepsilon \cdot \|z\|_{H^1(S^1; \mathbb{R}^n)},$$

for some fixed  $\bar{C} > 0$ . Since  $\|z\|_{H^1(S^1; \mathbb{R}^n)} = O(\varepsilon^2)$ , see Proposition 3.3, from the Interpolation Inequality (see for example [2]) it follows that

$$\|y_n\|_{C^0(S^1)} \leq \bar{C} \|y_n\|_{L^2(S^1)}^{\frac{1}{2}} \cdot \|y_n\|_{H^1(S^1)}^{\frac{1}{2}} = o(\varepsilon^2).$$

Hence (43) is proved.

*Proof of Theorem 1.1* We define  $u_T$  as  $u_T(T \cdot) = \bar{x}_\varepsilon(\cdot)$ ,  $T, \varepsilon^2 = 1$ , see Proposition 3.3. Property (i) follows immediately from Lemma 3.1 and Proposition 3.3. As far as property (ii) is concerned, we note that formulas (27) and (37) imply that  $\|\bar{x}_\varepsilon - x_0\|_\infty = O(\varepsilon)$ , hence  $\frac{1}{\varepsilon} \|V'(\bar{x}_\varepsilon)\|_\infty = O(1)$ , uniformly in  $\varepsilon$ . This means that  $\|\ddot{\bar{x}}_\varepsilon\|_\infty = O(1)$  uniformly in  $\varepsilon$ . The conclusion follows from the Ascoli Theorem.  $\square$

## 4 About some linear ODE's

The purpose of this section is to perform a preliminary study in order to reduce the problem, in Section 5, on the manifold  $H^1(S^1; M)$ . The arguments are elementary, and perhaps our estimates are well known, but for the reader's convenience we collect here the proofs.

We start by studying the equation

$$(45) \quad \begin{cases} \ddot{v}(t) + \lambda_0 \cdot v(t) = \sigma(t), & \text{in } [0, 2\pi], \\ v(0) = v(2\pi), \quad \dot{v}(0) = \dot{v}(2\pi), \end{cases}$$

where  $\lambda_0 \in \mathbb{R}$  is a fixed constant and  $\sigma(t) \in L^1([0, 2\pi])$ . By the Fredholm alternative Theorem, problem (45) admits a unique solution if  $\lambda_0$  is not an eigenvalue of the associated homogeneous problem. The eigenvalues are precisely the numbers  $\{k^2\}, k \in \mathbb{N}$ . Since the behaviour of the solutions of (45) changes qualitatively when  $\lambda_0$  is positive or negative, we distinguish the two cases separately. The former ( $\lambda_0 < 0$ ) is related to condition (2), namely to the repulsive case. The latter ( $\lambda_0 > 0$ ) is instead related to the attractive case.

**Case  $\lambda_0 < 0$**

Let  $G(t)$  be the Green function for problem (45), namely the solution  $v(t)$  corresponding to  $\sigma(t) = \delta_0(t)$ . One can verify with straightforward computations that  $G(t)$  is given by

$$(46) \quad G(t) = \frac{1}{2\sqrt{|\lambda_0|} \sinh(\pi\sqrt{|\lambda_0|})} \cosh\left(\sqrt{|\lambda_0|}(t - \pi)\right), \quad t \in [0, 2\pi].$$

The solution for a general function  $\sigma$  is obtained by convolution, namely one has

$$(47) \quad v(t) = \int_{S^1} G(t - s) \sigma(s) ds, \quad t \in [0, 2\pi].$$

In particular the following estimate holds

$$(48) \quad \|v\|_{L^\infty} \leq \|G\|_{L^\infty} \cdot \|\sigma\|_{L^1} = \frac{1}{2\sqrt{|\lambda_0|}} \cdot \|\sigma\|_{L^1}.$$

Furthermore, if  $\sigma \in L^\infty$  one can deduce

$$(49) \quad \|v\|_{L^\infty} \leq \|G\|_{L^1} \cdot \|\sigma\|_{L^\infty} = \frac{1}{|\lambda_0|} \cdot \|\sigma\|_{L^\infty}.$$

The last two estimates hold true if, more in general, the constant  $\lambda_0$  is substituted by a function bounded above by  $\lambda_0$ . This is the content of the following Lemma.

**Lemma 4.1** *Let  $\lambda(t)$  be a negative continuous and periodic function on  $[0, 2\pi]$  such that  $\lambda(t) \leq \lambda_0 < 0$ , and let  $\sigma(t) \in L^1([0, 2\pi])$ . Then the solution  $v(\cdot)$  of problem*

$$(50) \quad \begin{cases} \ddot{v}(t) + \lambda(t)v(t) = \sigma(t), & \text{in } [0, 2\pi], \\ v(0) = v(2\pi), \quad \dot{v}(0) = \dot{v}(2\pi). \end{cases}$$

*satisfies the estimates (48) and (49).*

*Proof.* The existence (and the uniqueness) of a solution is an easy consequence of the Lax-Milgram Theorem. Let  $\bar{v}(\cdot)$  denote the unique solution of (50) corresponding to  $\lambda \equiv \lambda_0$ . We start by supposing that  $f \geq 0$ . In this way, by the maximum principle, it must be  $v(t), \bar{v}(t) \leq 0$  for all  $t$ . Define  $y(t) = v(t) - \bar{v}(t)$ : it follows immediately by subtraction that  $y$  is a  $2\pi$ -periodic solution of the equation

$$(51) \quad \ddot{y}(t) = \lambda_0 \bar{v}(t) - \lambda(t)v(t) \leq |\lambda_0| \cdot y(t).$$

We claim that it must be  $y(t) \geq 0$  for all  $t$ . Otherwise, there is some  $t_0$  for which  $y(t_0) < 0$ , and  $y'(t_0) = 0$ , since the function  $y$  is periodic. It then follows from (51) that  $y(t)$  should be strictly decreasing in  $t$ , contradicting its periodicity. Hence we deduce that

$$(52) \quad \bar{v}(t) \leq v(t) \leq 0, \quad \text{for all } t \in [0, 2\pi].$$

For a general  $\sigma$ , we write  $\sigma = \sigma^+ - \sigma^-$ , where  $\sigma^+$  and  $\sigma^-$  are respectively the positive and the negative part of  $\sigma$ . Let also  $v^\pm, \bar{v}^\pm$  denote the solutions corresponding to  $\sigma^\pm$ . By linearity it is  $v(t) = v^+(t) - v^-(t)$  and  $\bar{v}(t) = \bar{v}^+(t) - \bar{v}^-(t)$ , so, since  $v^\pm$  and  $\bar{v}^\pm$  have definite sign, it turns out that

$$\|v\|_{L^\infty} \leq \max\{\|v^+\|_{L^\infty}, \|v^-\|_{L^\infty}\} \leq \max\{\|\bar{v}^+\|_{L^\infty}, \|\bar{v}^-\|_{L^\infty}\} \leq \|\bar{v}\|_{L^\infty}.$$

This implies immediately the estimates (54) and (55) for  $v(t)$ . □

We want to prove that the estimates in (48) and in (49) are stable under bounded  $L^1$  perturbations of the function  $\lambda$ . Precisely we consider the following problem, where  $\gamma \in L^1(S^1)$ .

$$(53) \quad \begin{cases} \ddot{v}(t) + (\lambda + \gamma(t)) \cdot v(t) = \sigma(t), & \text{in } [0, 2\pi], \\ v(0) = v(2\pi), \quad \dot{v}(0) = \dot{v}(2\pi), \end{cases}$$

for which it is well known the existence and the uniqueness of a solution  $v(t)$ .

**Lemma 4.2** *Let  $A > 0$  be a fixed constant,  $\lambda_0 < 0$  and  $\lambda(t) \in C^0(S^1)$  satisfy  $\lambda(t) \leq \lambda_0$  for all  $t$ . Let also  $\gamma \in L^1(S^1)$ : then, given any number  $\delta > 0$ , if  $|\lambda_0|$  is sufficiently large the solution  $v(\cdot)$  of (53) satisfies the inequality*

$$(54) \quad \|v\|_\infty \leq (1 + \delta) \cdot \frac{1}{2\sqrt{|\lambda_0|}} \cdot \|\sigma\|_{L^1}.$$

*Proof.* The solution  $v$  satisfies the equation

$$\ddot{v}(t) + \lambda v(t) = \sigma(t) - \gamma(t)v(t), \quad \text{in } [0, 2\pi],$$

with periodic boundary conditions, then by Lemma 4.1 there holds

$$\|v\|_{L^\infty} \leq \frac{1}{2\sqrt{|\lambda_0|}} \cdot (\|\sigma\|_{L^1} + M \cdot \|v\|_{L^\infty}) \leq \frac{1}{2\sqrt{|\lambda_0|}} \cdot (\|\sigma\|_{L^1} + A \cdot \|v\|_{L^\infty}).$$

This implies immediately (54) and concludes the proof. □

**Lemma 4.3** *Let  $A > 0$  be a fixed constant, let  $\lambda_0 < 0$ , and let  $\lambda(t) \in C^0(S^1)$  satisfy  $\lambda(t) \leq \lambda_0$  for all  $t$ . Suppose also that  $\sigma \in L^\infty(S^1)$ . Then given any number  $\delta > 0$ , if  $|\lambda_0|$  is sufficiently large the solution  $v(\cdot)$  of problem (53) satisfies the inequality*

$$(55) \quad \|v\|_{L^\infty} \leq (1 + \delta) \cdot \frac{1}{|\lambda_0|} \cdot \|\sigma\|_{L^\infty}.$$

*Proof.* Let  $v_0(t)$  be the solution of problem (53) corresponding to  $\gamma \equiv 0$ , so in particular, by Lemma 4.1, there holds

$$(56) \quad \|v_0\|_{L^\infty} \leq \frac{1}{|\lambda_0|} \cdot \|\sigma\|_{L^\infty}.$$

Let  $y : [0, 2\pi] \rightarrow \mathbb{R}$  be defined by  $y(t) = v_0(t) - v(t)$ . By subtraction one infers that  $y(t)$  is a solution of the problem

$$\begin{cases} \ddot{y}(t) + by(t) = c(t)v(t), & \text{in } [0, 2\pi], \\ y(0) = y(2\pi), \quad \dot{y}(0) = \dot{y}(2\pi). \end{cases}$$

Hence, by applying inequality (48) one deduce sthat

$$(57) \quad \|y\|_{L^\infty} \leq \frac{1}{2\sqrt{|\lambda_0|}} \cdot \|\gamma(\cdot) \cdot v(\cdot)\|_{L^1} \leq \frac{1}{2\sqrt{|\lambda_0|}} \cdot \|v\|_{L^\infty} \cdot \|\gamma\|_{L^1}.$$

So, since by Lemma 4.2 the function  $v(t)$  satisfies inequality (54), it follows that

$$\|y\|_{L^\infty} \leq (1 + \delta) \cdot \frac{1}{2|\lambda_0|} \cdot A \cdot \|\sigma\|_{L^\infty}.$$

Now, taking into account formulas (54) and (56) it follows that

$$(58) \quad \|v\|_{L^\infty} \leq \|v_0\|_{L^\infty} + \|y\|_{L^\infty} \leq \frac{1}{|\lambda_0|} \cdot \|\sigma\|_{L^\infty} + (1 + \delta) \cdot \frac{1}{2|\lambda_0|} \cdot A \cdot \|\sigma\|_{L^\infty} \leq \frac{(1 + \delta)}{|\lambda_0|} \cdot \left(1 + \frac{A}{2}\right).$$

For  $|\lambda_0|$  large this is a better estimate than (55), and inserting it in formula (57) we obtain

$$\|y\|_{L^\infty} \leq \frac{1}{2\sqrt{|\lambda_0|}} \cdot \|v\|_{L^\infty} \cdot \|\gamma\|_{L^1} \leq \frac{(1 + \delta)}{2} \cdot A \cdot \frac{1}{|\lambda_0|^{\frac{3}{2}}} \cdot \left(1 + \frac{A}{2}\right) \cdot \|\sigma\|_{L^\infty}.$$

Using this estimate in (58) we finally deduce, if  $|\lambda_0|$  is sufficiently large

$$\|v\|_{L^\infty} \leq \frac{1}{|\lambda_0|} \cdot \|\sigma\|_{L^\infty} + \frac{(1 + \delta)}{2} \cdot A \cdot \frac{1}{|\lambda_0|^{\frac{3}{2}}} \cdot \left(1 + \frac{A}{2}\right) \cdot \|\sigma\|_{L^\infty} \leq (1 + 2\delta) \cdot \frac{1}{|\lambda_0|} \cdot \|\sigma\|_{L^\infty}.$$

This concludes the proof. □

### Case $\lambda_0 > 0$

We recall that the estimates of this case will be applied to the study of the attractive case. We will always take for simplicity  $\lambda_0$  of the form  $\lambda_0 = \left(k + \frac{1}{2}\right)^2$ . This is in order to assure that  $\lambda_0$  is not an eigenvalue of the problem and that the distance of  $\lambda_0$  from the spectrum is always of order  $\sqrt{\lambda_0}$ . Let



$\bar{G}(t)$  be the Green function for problem (45), namely the solution  $v(t)$  corresponding to  $\sigma(t) = \delta_0(t)$ . One easily verifies that  $\bar{G}(t)$  is given by

$$(59) \quad \bar{G}(t) = \frac{1}{2\sqrt{\lambda_0}} \sin(\sqrt{\lambda_0}t), \quad t \in [0, 2\pi].$$

The solution for a general function  $\sigma$  is obtained again by convolution, namely one has

$$(60) \quad v(t) = \int_{S^1} \bar{G}(t-s)\sigma(s)ds, \quad t \in [0, 2\pi].$$

In particular the following estimate can be immediately deduced

$$(61) \quad \|v\|_{L^\infty} \leq \|\bar{G}\|_{L^\infty} \cdot \|\sigma\|_{L^1} = \frac{1}{2\sqrt{\lambda_0}} \cdot \|\sigma\|_{L^1}.$$

If moreover  $\sigma \in L^\infty(S^1)$ , one can further deduce

$$(62) \quad \|v\|_{L^\infty} \leq \|\bar{G}\|_{L^1} \cdot \|\sigma\|_{L^\infty} = \frac{2}{\sqrt{\lambda_0}} \cdot \|\sigma\|_{L^\infty}.$$

**Remark 4.4** *We note that, differently from the case of  $\lambda_0 < 0$ , the constant is changed only by a factor 4 from (61) to (62), and is not by a power of  $\lambda_0$ , as in the preceding case. However, if  $\sigma \in L^\infty$ , (62) could be a better estimate than (61), since  $\|\sigma\|_{L^1} \leq 2\pi \|\sigma\|_{L^\infty}$ .*

The following analogue of Lemmas 4.2 and 4.3 holds, the proof follows the same arguments.

**Lemma 4.5** *Let  $A > 0$  be a fixed constant, and suppose  $\lambda_0$  is a constant of the form  $\lambda_0 = (m + 1/2)^2$ . Suppose that  $\sigma \in L^1(S^1)$ . Then if  $\lambda_0$  is sufficiently large, problem*

$$(63) \quad \begin{cases} \ddot{v}(t) + (\lambda_0 + \gamma(t)) \cdot v(t) = \sigma(t), & \text{in } [0, 2\pi], \\ v(0) = v(2\pi), \quad \dot{v}(0) = \dot{v}(2\pi). \end{cases}$$

*possesses an unique solution for all  $\gamma(t) \in L^1(S^1)$  with  $\|\gamma\|_{L^1} \leq A$ . Moreover, given any number  $\delta > 0$ , if  $\lambda_0$  is sufficiently large then the solution  $v(\cdot)$  satisfies the inequality*

$$(64) \quad \|v\|_{L^\infty} \leq (1 + \delta) \cdot \frac{1}{2\sqrt{\lambda_0}} \cdot \|\sigma\|_{L^1}.$$

*If moreover  $\sigma \in L^\infty(S^1)$ , then for  $\lambda_0$  is sufficiently large there holds*

$$(65) \quad \|v\|_{L^\infty} \leq (1 + \delta) \cdot \frac{2}{\sqrt{\lambda_0}} \cdot \|\sigma\|_{L^\infty}.$$

## 5 Proof of Theorem 1.2

The goal of this section is to prove Theorem 1.2. Two are the main ingredients: the first is the reduction on the manifold  $H^1(S^1; M)$ , treated in Subsection 5.1. The second is the study of the reduced functional, carried out Subsection 5.2.

## 5.1 The reduction on $H^1(S^1, M)$

In this subsection we perform a Lyapunov-Schmidt reduction of problem (T) on the manifold  $H^1(S^1, M)$  of the closed  $H^1$  loops on  $M$ . A fundamental tool are the estimates of Section 4.

Solutions of problem  $(P_\varepsilon)$ , and hence of problem (T), can be found as critical points of the functional  $E_\varepsilon : H^1(S^1; \mathbb{R}^n) \rightarrow \mathbb{R}$ .

If  $u \in \mathbb{R}^n$  is in a sufficiently small neighbourhood of  $M$ , say if  $\text{dist}(u, M) \leq \rho_0$  for some  $\rho_0 > 0$ , then are uniquely defined  $h(u) \in M$  and  $v(u) \in \mathbb{R}$  such that

$$(66) \quad u = h(u) + v(u) \cdot n_{h(u)}; \quad \text{dist}(u, M) \leq \rho_0.$$

It is clear that  $h$  and  $v$  depend smoothly on  $u$ . In particular, if  $u(\cdot) \in H^1(S^1; \mathbb{R}^n)$ , and if  $\text{dist}(u(t); M) \leq \rho_0$  for all  $t$ , then  $h(u(\cdot))$  and  $v(u(\cdot))$  are of class  $H^1(S^1; M)$  and  $H^1(S^1)$  respectively. In the sequel we will often omit the dependence on  $u$  of  $h(u)$  and  $v(u)$ . Viceversa, given  $h \in M$ , and  $v \in \mathbb{R}$ ,  $|v| \leq \rho_0$ , then the point  $u \in \mathbb{R}^n$ ,  $u = h + v \cdot n_h$  depends smoothly on  $h$  and  $v$ .

If  $u(\cdot) \in H^1(S^1; \mathbb{R}^n)$ , then there holds

$$(67) \quad \dot{u} = \dot{h} + \dot{v} \cdot n + v \cdot \dot{n} = \dot{h} + \dot{v} \cdot n + v \cdot H(h)[\dot{h}] = (Id + v H(h))[\dot{h}] + \dot{v} \cdot n; \quad \text{a. e. in } S^1.$$

From the last expression it follows in particular that

$$(68) \quad |\dot{u}|^2 = |\dot{h}|^2 + |\dot{v}|^2 + v^2 |H(h)[\dot{h}]|^2 + 2v\dot{h} \cdot H(h)[\dot{h}].$$

We define  $\bar{E}_0, \bar{E}_\varepsilon : H^1(S^1; M) \times H^1(S^1) \rightarrow \mathbb{R}$  and  $\bar{V} : M \times (-\rho_0, \rho_0) \rightarrow \mathbb{R}$  to be

$$\begin{aligned} \bar{E}_0(h, v) &= E_0(u(h, v)), & \bar{E}_\varepsilon(h, v) &= E_\varepsilon(u(h, v)); & h &\in H^1(S^1; M), v \in H^1(S^1), \|v\|_{L^\infty} \leq \rho_0, \\ \bar{V}(h, v) &= V(u(h, v)); & h &\in M, v \in (-\rho_0, \rho_0). \end{aligned}$$

Hence, by means of formula (68), the functional  $\bar{E}_0$  assumes the expression

$$\begin{aligned} \bar{E}_0(h, v) &= \frac{1}{2} \int_{S^1} \left( |\dot{h}|^2 + |\dot{v}|^2 + v^2 |H(h)[\dot{h}]|^2 + 2v\dot{h} \cdot H(h)[\dot{h}] \right) \\ &= L_0(h) + \frac{1}{2} \int_{S^1} \left( |\dot{v}|^2 + v^2 |H(h)[\dot{h}]|^2 + 2v\dot{h} \cdot H(h)[\dot{h}] \right). \end{aligned}$$

If we differentiate  $\bar{E}_0$  with respect to a variation  $k \in T_h H^1(S^1; M)$  of  $h$ , we obtain

$$(69) \quad \begin{aligned} D_h \bar{E}_0(h, v)[k] &= DL_0(h)[k] + \int_{S^1} v^2 \left( H(h)[\dot{h}] \cdot H(k)[\dot{k}] + H(h)[\dot{h}] \cdot \mathcal{L}_k H(h)[\dot{h}] \right) \\ &\quad + \int_{S^1} v \left( 2H(h)[\dot{h}, \dot{k}] + \dot{h} \cdot \mathcal{L}_k H(h)[\dot{h}] \right). \end{aligned}$$

Here  $\mathcal{L}_k H$ , see Notations, denotes the Lie derivative of  $H$  in the direction  $k$ . Similarly, if we differentiate  $\bar{E}_0$  with respect to a variation  $w \in H^1(S^1)$  of  $v$ , we have

$$(70) \quad D_v \bar{E}_0(h, v)[w] = \int_{S^1} \dot{v} \dot{w} dt + \int_{S^1} v w |H(h)[\dot{h}]|^2 dt + \int_{S^1} w \dot{h} \cdot H(h)[\dot{h}] dt.$$

From equations (66) and (67) it follows that if  $u(\cdot) \in H^1(S^1; \mathbb{R}^n)$ , and if  $\text{dist}(u(t), M) \leq \rho_0$  for all  $t \in S^1$ , then the condition  $D\bar{E}_\varepsilon(h, v) = 0$  is equivalent to the system

$$(71) \quad \begin{cases} D_h \bar{E}_\varepsilon(h, v)[k] = 0 & \forall k \in T_h H^1(S^1; M), \\ D_v \bar{E}_\varepsilon(h, v)[w] = 0 & \forall w \in H^1(S^1). \end{cases}$$

Hence, in order to find critical points of  $\bar{E}_\varepsilon$  (and hence of  $E_\varepsilon$ ), we first solve the second equation in (71). Then, denoting by  $v(h)$  this solution, we solve in  $h$  the equation

$$D_h \bar{E}_\varepsilon(u(h, v(h)))[k] = 0, \quad \forall k \in T_h H^1(S^1; M).$$

By (70), the equation in  $v$   $D_w \bar{E}_\varepsilon(u(h, v)) = 0$  means that  $v : S^1 \rightarrow \mathbb{R}$  is solution of the following problem

$$(72) \quad \begin{cases} \ddot{v} - |H(h)[\dot{h}]|^2 v = \dot{h} \cdot H(h)[\dot{h}] - \frac{1}{\varepsilon} \frac{\partial \bar{V}}{\partial v}(h, v). & \text{in } [0, 2\pi], \\ v(0) = v(2\pi), \quad \dot{v}(0) = \dot{v}(2\pi). \end{cases}$$

**Proposition 5.1** *Suppose that the potential  $V$  is of repulsive type, namely that (2) holds, and let  $A$  be a fixed positive constant. Then there exists  $\varepsilon_A > 0$  and  $C_A > 0$  such that for all  $\varepsilon \in (0, \varepsilon_A)$  and for all  $h \in L_0^A$  equation (72) admits an unique solution  $v(h)$  which satisfies*

$$(73) \quad \|v(h)\|_{C^0(S^1)} \leq C_A \cdot \sqrt{\varepsilon}.$$

Moreover the application  $h \rightarrow v(h)$  from  $L_0^A$  to  $C^0(S^1)$  is of class  $C^1$  and compact, namely if  $(h_m)_m \subseteq L_0^A$ , then up to a subsequence  $v(h_m)$  converge in  $C^0(S^1)$ .

*Proof.* Equation (72) can be written it in the form ( $Q_h$ ,  $P_h$  and  $B_h$  are defined in Section 2)

$$(74) \quad \ddot{v} + \left( \frac{1}{\varepsilon} B_h - Q_h \right) v = P_h + \frac{1}{\varepsilon} \left( B_h v - \frac{\partial \bar{V}}{\partial v}(h, v) \right),$$

Since  $V$  is of repulsive type and since every curve  $h \in H^1(S^1, M)$  is continuous, the function  $B_h$  is a negative continuous function of  $t$  and is bounded above by the negative constant  $b_* = \sup_{x \in M} b(x)$ . From Lemma 4.2 it follows that the resolutive operator  $\Sigma_{h, \varepsilon} : L^1(S^1) \rightarrow C^0(S^1)$  for the  $2\pi$ -periodic problem

$$\ddot{v} + \left( \frac{1}{\varepsilon} B_h - Q_h \right) v = \sigma(t),$$

is well defined, whenever  $h \in H^1(S^1; M)$ ,  $\sigma \in L^1(S^1)$  and  $\varepsilon > 0$  is sufficiently small. Moreover, by Lemma 4.2 (resp. Lemma 4.3) it follows that, given  $\delta > 0$ , the solution of problem (72) satisfies the following estimate, provided  $\varepsilon$  is small enough

$$(75) \quad \|\Sigma_{h, \varepsilon} \sigma\|_{L^\infty} \leq (1 + \delta) \cdot \frac{1}{2} \frac{\sqrt{\varepsilon}}{\sqrt{|b_*|}} \cdot \|\sigma\|_{L^1(S^1)} \quad \left( \text{resp. } \|\Sigma_{h, \varepsilon} \sigma\|_{L^\infty} \leq (1 + \delta) \cdot \frac{\varepsilon}{|b_*|} \cdot \|\sigma\|_{L^\infty(S^1)} \right).$$

For  $v \in C^0(S^1)$ , let

$$\sigma_v = P_h + \frac{1}{\varepsilon} \left( B_h v - \frac{\partial \bar{V}}{\partial v}(h, v) \right),$$

and let  $\Theta_{h, \varepsilon} : v \rightarrow \Sigma_{h, \varepsilon} \sigma_v$ . Then, as observed in the Notations it is

$$(76) \quad P_h \in L^1(S^1); \quad \frac{1}{\varepsilon} \left( B_h v - \frac{\partial \bar{V}}{\partial v}(h, v) \right) \in L^\infty(S^1).$$

We show that  $\Theta_{h, \varepsilon}$  is a contraction in some suitable ball  $\mathcal{B}_\rho(0) = \{v \in C^0(S^1) : \|v\|_{L^\infty} \leq \rho\}$ , where  $\rho$  will be chosen appropriately later.

If  $\|v\|_{L^\infty} \leq \rho$ , from (76), (75) and from the linearity of equation (74) it follows that

$$\|\Sigma_{h, \varepsilon} \sigma_v\|_{L^\infty} \leq (1 + \delta) \cdot \frac{1}{2} \frac{\sqrt{\varepsilon}}{\sqrt{|b_*|}} \cdot \int_{S^1} P_h + (1 + \delta) \cdot \frac{\varepsilon}{|b_*|} \cdot \frac{1}{\varepsilon} \cdot \left\| B_h v - \frac{\partial \bar{V}}{\partial v}(h, v) \right\|_{L^\infty}.$$

By the definition of  $\overline{H}$  (see Section 2) it follows that we can estimate  $\|P_h\|_{L^1(S^1)}$  in this way

$$(77) \quad \|P_h\|_{L^1(S^1)} = \int_{S^1} \dot{h} \cdot H(h)[\dot{h}] \leq 2\overline{H} \cdot L_0(h) \leq 2\overline{H} \cdot A.$$

Moreover, since  $B_h = \frac{\partial^2 \overline{V}}{\partial v^2}(h, 0)$ , it turns out that

$$(78) \quad B_h v - \frac{\partial \overline{V}}{\partial v}(h, v) = \psi(h, v) \cdot v^2; \quad |v| \leq \rho_0,$$

where  $\psi(h, v)$  is a smooth (and hence bounded) function. So by equations (77) and (78) there holds

$$(79) \quad \|\Sigma_{h,\varepsilon} \sigma_v\|_{L^\infty} \leq (1 + \delta) \frac{\sqrt{\varepsilon}}{\sqrt{|b_*|}} \cdot \overline{H} \cdot A + (1 + \delta) \frac{1}{|b_*|} \cdot \|\psi\|_{L^\infty} \cdot \rho^2.$$

Furthermore, if we consider two functions  $v, v' \in C^0(S^1)$ , it turns out that

$$\|\sigma_v - \Sigma_{h,\varepsilon} \sigma_{v'}\|_{L^\infty} = \frac{1}{\varepsilon} \|\psi(h, v) v^2 - \psi(h, v') (v')^2\|_{L^\infty}.$$

Writing

$$\psi(h, v) v^2 - \psi(h, v') (v')^2 = \psi(h, v) (v^2 - (v')^2) + (v')^2 (\psi(h, v) - \psi(h, v')),$$

one can deduce that

$$\|\psi(h, v) v^2 - \psi(h, v') (v')^2\|_{L^\infty} \leq \|\psi\|_{L^\infty} \cdot \|v + v'\|_{L^\infty} \cdot \|v - v'\|_{L^\infty} + \|v'\|_{L^\infty}^2 \cdot \|D\psi\|_{L^\infty} \cdot \|v - v'\|_{L^\infty}.$$

Hence, if  $\|v\|_{L^\infty}, \|v'\|_{L^\infty} \leq \rho$ , then by formula (75) it follows that

$$(80) \quad \|\Sigma_{h,\varepsilon} \sigma_v - \Sigma_{h,\varepsilon} \sigma_{v'}\|_{L^\infty} \leq (1 + \delta) \cdot \frac{1}{b_*} \cdot (2\rho \cdot \|\psi\|_{L^\infty} + \rho^2 \cdot \|D\psi\|_{L^\infty}) \cdot \|v - v'\|_{L^\infty}.$$

In conclusion, if we choose  $\rho = C \cdot \sqrt{\varepsilon}$ , with  $C > 0$  sufficiently large, it follows from equations (79) and (80) that  $\Theta_{h,\varepsilon}$  maps  $\mathcal{B}_\rho(0)$  in itself and is a contraction. Hence we obtain the existence and the local uniqueness of the solution. It is a standard fact that  $h \rightarrow v(h)$  is of class  $C^1$ .

It remains to prove the compactness assertion. But this is an immediate consequence of equation (72) which implies that  $\ddot{v}_n$  is bounded in  $W^{2,1}(S^1)$ , so the compactness follows. This concludes the proof of the Proposition.  $\square$

## 5.2 Study of the reduced functional

In this Subsection we study the functional  $E_\varepsilon$ , after the reduction of Subsection 5.1. Roughly, for  $\varepsilon$  small, the reduced functional turns out to be a perturbation of  $L_0$  and the standard min-max arguments can be used to find critical points.

Let  $A > 0$ , and let  $h \in L_0^A$ . By Proposition 5.1 there exists  $v(h)$  such that  $D_v \overline{E}_\varepsilon(h, v(h)) = 0$ . Define  $L_\varepsilon, G_\varepsilon : L_0^A \rightarrow \mathbb{R}$  in the following way

$$L_\varepsilon(h) = \overline{E}_\varepsilon(h, v(h)), \quad G_\varepsilon(h) = L_\varepsilon(h) - L_0(h); \quad h \in L_0^A, \varepsilon \in (0, \varepsilon_A).$$

**Proposition 5.2** *Let  $A$  be a fixed positive constant, and let  $\varepsilon_A$  be given by Proposition 5.1. Then there exists  $\overline{\varepsilon}_A, 0 < \overline{\varepsilon}_A \leq \varepsilon_A$ , such that  $L_\varepsilon : L_0^A \rightarrow \mathbb{R}$  is of class  $C^1$ . Moreover, there exists  $\overline{C}_A > 0$  such that*

$$(81) \quad |G_\varepsilon(h)| \leq \overline{C}_A \cdot \sqrt{\varepsilon}, \quad |DG_\varepsilon(h)| \leq \overline{C}_A \cdot \sqrt{\varepsilon}; \quad \forall h \in L_0^A, \forall \varepsilon \in (0, \overline{\varepsilon}_A).$$

*Proof.* Setting for brevity  $v = v(h)$ , equation (72) assumes the form

$$\int_{S^1} \dot{v} \dot{w} + \int_{S^1} v w Q_h + \int_{S^1} 2w P_h - \frac{1}{\varepsilon} \int_{S^1} \frac{\partial \bar{V}}{\partial v}(h, v) w = 0, \quad \forall w \in H^1(S^1).$$

In particular, by taking  $w = v$  in the last formula it turns out that  $v$  satisfies the relation

$$(82) \quad \int_{S^1} \dot{v}^2 + \int_{S^1} v^2 Q_h + \int_{S^1} 2v P_h = \frac{1}{\varepsilon} \int_{S^1} \frac{\partial \bar{V}}{\partial v}(h, v) v.$$

Since  $\bar{E}_\varepsilon(h, v)$  has the expression

$$E_\varepsilon(u(h, v)) = \frac{1}{2} \left( \int_{S^1} |\dot{h}|^2 + |\dot{v}|^2 + v^2 Q_h + 2v P_h \right) - \frac{1}{\varepsilon} \int_{S^1} \bar{V}(h, v) dt,$$

we can take into account formula (82) to obtain

$$(83) \quad G_\varepsilon(h) = \frac{1}{\varepsilon} \int_{S^1} \left( \frac{1}{2} v \cdot \frac{\partial \bar{V}}{\partial v}(h, v) - \bar{V}(h, v) \right).$$

Since  $\bar{V}$  is smooth, it turns out that

$$\frac{1}{2} v \cdot \frac{\partial \bar{V}}{\partial v}(h, v) - \bar{V}(h, v) = \bar{\psi}(h, v) \cdot v^3, \quad |v| \leq \rho_0,$$

for some regular function  $\bar{\psi}$ . Hence, from equations (73) and (83) one infers that

$$|G_\varepsilon(h)| \leq \frac{1}{\varepsilon} \cdot \|\bar{\psi}\|_{L^\infty} \cdot \|v(h)\|_{L^\infty}^3 \leq C_A^3 \cdot \|\bar{\psi}\|_{L^\infty} \cdot \sqrt{\varepsilon}.$$

This proves the first inequality in (81).

To show that  $G_\varepsilon$  is of class  $C^1$ , we start proving that for some  $\hat{C}_A > 0$  and for  $\varepsilon$  sufficiently small,

$$(84) \quad \bar{\Sigma}_\varepsilon : L_0^A \rightarrow C^0(S^1), \quad h \rightarrow v_h \quad \text{is of class } C^1, \quad \text{and } \|D_h \bar{\Sigma}_\varepsilon(h)\| \leq \hat{C}_A \cdot \sqrt{\varepsilon}.$$

Consider two elements  $h$  and  $l$  of  $H^1(S^1; M)$  and the corresponding solutions  $v(h)$  and  $v(l)$  (which for brevity we denote with  $v_h$  and  $v_l$ ) of problem (72). By subtraction of the equations there results

$$(85) \quad \begin{aligned} \ddot{y} &+ \left( \frac{1}{\varepsilon} B_h - Q_h \right) y + \left( \frac{1}{\varepsilon} \Delta B + \Delta Q \right) v_l = \Delta P \\ &+ \frac{1}{\varepsilon} \cdot (v_h + v_l) \cdot \psi(h, v_h) y + \frac{1}{\varepsilon} \cdot [\Delta_1 \psi + \Delta_2 \psi] v_l^2, \end{aligned}$$

where we have set

$$\begin{aligned} y &= v_h - v_l; & \Delta B &= B_h - B_l; & \Delta Q &= Q_h - Q_l; \\ \Delta P &= P_h - P_l; & \Delta_1 \psi &= \psi(h, v_h) - \psi(l, v_h); & \Delta_2 \psi &= \psi(l, v_h) - \psi(l, v_l). \end{aligned}$$

Since  $\psi$  is smooth, there holds

$$\Delta_2 \psi = \psi(l, v_h) - \psi(l, v_l) = \tilde{\psi}(l, v_h, v_l) \cdot y,$$

for another smooth function  $\tilde{\psi}$ . Hence we can write equation (85) in the form

$$(86) \quad \begin{aligned} \ddot{y} &+ \left( \frac{1}{\varepsilon} B_h - Q_h - \frac{1}{\varepsilon} \cdot (v_h + v_l) \cdot \psi(h, v_h) - \tilde{\psi}(l, v_h, v_l) \right) y \\ &= - \left( \frac{1}{\varepsilon} \Delta B + \Delta Q \right) v_l + \Delta P + \frac{1}{\varepsilon} \cdot \Delta_1 \psi v_l^2. \end{aligned}$$

Now fix  $h \in H^1(S^1; M)$ , and let  $l \rightarrow h$ ; we can suppose that both  $h, l \in L_0^A$ . If  $\varepsilon \in (0, \varepsilon_A)$  (see Proposition 5.1), then  $v_h$  and  $v_l$  satisfy inequality (73). Hence by choosing  $\varepsilon$  sufficiently small there results

$$\frac{1}{\varepsilon} B_h - \frac{1}{\varepsilon} (v_h + v_l) \cdot \psi(h, v_h) - \tilde{\psi}(l, v_h, v_l) \leq \frac{1}{2} \frac{b_*}{\varepsilon}.$$

Moreover, with computations similar to (77) one can easily prove that  $\|Q_h\|_{L^1(S^1)} \leq 2\overline{H}^2 \cdot A$ , so Lemmas 4.2 and 4.3 yield, for  $\delta$  fixed and  $\varepsilon$  sufficiently small

$$\begin{aligned} \|y\|_{L^\infty} &\leq (1 + \delta) \cdot \frac{1}{2} \frac{\sqrt{2\varepsilon}}{\sqrt{|b_*|}} (C_A \cdot \sqrt{\varepsilon} \cdot \|\Delta Q\|_{L^1(S^1)}) \\ &+ (1 + \delta) \cdot \frac{2\varepsilon}{|b_*|} \left( C_A \cdot \sqrt{\varepsilon} \cdot \frac{1}{\varepsilon} \cdot \|\Delta B\|_{C^0(S^1)} + C_A^2 \cdot \frac{1}{\varepsilon} \cdot \varepsilon \cdot \|\Delta_1 \psi\|_{C^0(S^1)} \right). \end{aligned}$$

The last quantity tends to 0 as  $l \rightarrow h$ . This proves the continuity of  $\overline{\Sigma}_\varepsilon : h \rightarrow v(h)$  from  $L_0^A$  to  $C^0(S^1)$ . The differentiability follows from the same reasoning, taking into account that the maps

$$h \rightarrow Q_h \quad \text{and} \quad h \rightarrow P_h \quad \text{from } H^1(S^1; M) \text{ to } L^1(S^1; M)$$

$$h \rightarrow B_h \quad \text{from } H^1(S^1; M) \text{ to } C^0(S^1; M) \quad \text{and} \quad \psi(h, v) \text{ from } M \times (-\rho_0, \rho_0) \text{ to } \mathbb{R}$$

are differentiable. We remark that the differentials of these functions are uniformly bounded for  $h \in L_0^A$ . Passing to the incremental ratio in (86), one obtains that the directional derivative  $D\overline{\Sigma}_\varepsilon(h)[k]$  along the direction  $k \in T_{v_h} H^1(S^1; M)$  is the unique  $2\pi$ -periodic solution of the equation (linear in  $y$ )

$$\begin{aligned} (87) \quad \ddot{y} &+ \left( \frac{1}{\varepsilon} B_h - Q_h - \frac{2}{\varepsilon} \psi(h, v) v - \frac{1}{\varepsilon} v^2 D_v \psi(h, v) \right) y \\ &= DP_h[k] + \frac{1}{\varepsilon} D_h \psi(h, v)[k] v^2 - \frac{1}{\varepsilon} DB_h[k] v - DQ_h[k] v. \end{aligned}$$

From this formula (using for example a local chart on  $H^1(S^1; M)$ ) one deduces immediately that the differential  $D\overline{\Sigma}_\varepsilon$  is continuous on  $L_0^A$ . Now we prove the estimate in (84). From equation (73) and from the uniform boundedness on the differentials of  $P_h$  and  $Q_h$  one can deduce that, for some constant  $C$  independent on  $h \in L_0^A$  there holds

$$\begin{aligned} \|DP_h[k]\|_{L^1(S^1)} &\leq C \cdot \|k\|; \quad \left\| \frac{1}{\varepsilon} \cdot v^2 \cdot D_h \psi(h, v)[k] \right\|_{C^0(S^1)} \leq C \cdot \|k\|; \\ \left\| \frac{1}{\varepsilon} \cdot v \cdot DB_h[k] \right\|_{C^0(S^1)} &\leq C \cdot \frac{1}{\sqrt{\varepsilon}} \cdot \|k\|; \quad \|v DQ_h[k]\|_{L^1(S^1)} \leq C \cdot \sqrt{\varepsilon} \cdot \|k\|. \end{aligned}$$

Moreover the coefficient of  $x$  in formula (87) for  $\varepsilon$  sufficiently small can be estimated as

$$\frac{1}{\varepsilon} B_h - Q_h - \frac{2}{\varepsilon} \psi(h, v) v - \frac{1}{\varepsilon} v^2 D_v \psi(h, v) \leq \frac{1}{2} \cdot \frac{b_*}{\varepsilon},$$

Hence one can apply Lemmas 4.2 and 4.3 to obtain

$$D\overline{\Sigma}_\varepsilon(h)[k] \leq \hat{C}_A \cdot \sqrt{\varepsilon} \cdot \|k\|, \quad \forall h \in L_0^A, \forall k \in T_h H^1(S^1; M),$$

which is the estimate in (84).

To prove the second inequality in (81) one can write  $G_\varepsilon$  in the form  $G_\varepsilon(h) = \frac{1}{\varepsilon} \int_{S^1} \overline{\psi}(h, v_h) \cdot v_h^3$ , and differentiate with respect to  $h$

$$(88) \quad DG_\varepsilon(h)[k] = \frac{1}{\varepsilon} \int_{S^1} \left( 3\overline{\psi}(h, v) v^2 Dv(h)[k] + v^3 \frac{\partial \overline{\psi}}{\partial h}[k] + v^3 \frac{\partial \overline{\psi}}{\partial v} Dv(h)[k] \right).$$

Now it is sufficient to apply the second inequality in (84). This concludes the proof.  $\square$

**Lemma 5.3** *Let  $(u_m)_m \subseteq H^1(S^1; \mathbb{R}^n)$  be a Palais Smale sequence for  $E_\varepsilon$  which for some  $R > 0$  satisfies the condition*

$$(89) \quad \|u_m(t)\| \leq R, \quad \forall m \in \mathbb{N}, \forall t \in S^1.$$

*Then, passing to a subsequence,  $u_m$  converges strongly in  $H^1(S^1; \mathbb{R}^n)$ .*

*Proof.* To prove this claim we first note that by condition (89), the sequence of numbers  $\int_{S^1} V(u_m)$  is bounded. Moreover there holds

$$\frac{1}{2} \int_{S^1} \dot{u}_m^2 = E_\varepsilon(u_m) + \frac{1}{\varepsilon} \int_{S^1} V(u_m),$$

so from the convergence of  $E_\varepsilon(u_m)$  and condition (89) one deduces that  $(u_m)_m$  is bounded in  $H^1(S^1; \mathbb{R}^n)$ . Hence, passing to a subsequence,  $u_m \rightharpoonup u_0$  weakly in  $H^1(S^1; \mathbb{R}^n)$ , and strongly in  $C^0(S^1; \mathbb{R}^n)$ .

As a consequence one has

$$(90) \quad V(u_m(\cdot)) \rightarrow V(u_0(\cdot)) \quad \text{and} \quad V'(u_m(\cdot)) \rightarrow V'(u_0(\cdot)) \quad \text{uniformly on } S^1.$$

From the fact  $DE_\varepsilon(u_m) \rightarrow 0$ , it follows that

$$\int_{S^1} \dot{u}_m \dot{z} - \frac{1}{\varepsilon} \int_{S^1} V'(u_m)[z] = o(1) \cdot \|z\|, \quad \forall z \in H^1(S^1; \mathbb{R}^n),$$

where  $o(1) \rightarrow 0$  as  $m \rightarrow +\infty$ . In particular, taking as test function  $z = u_m$ , one has

$$(91) \quad \int_{S^1} |\dot{u}_m|^2 - \frac{1}{\varepsilon} \int_{S^1} V'(u_m)[u_m] = o(1) \cdot \|u_m\| = o(1), \quad m \rightarrow +\infty.$$

On the other hand, taking  $z = u_0$

$$(92) \quad \int_{S^1} |\dot{u}_0|^2 - \frac{1}{\varepsilon} \int_{S^1} V'(u_0)[u_0] = \lim_m \int_{S^1} \dot{u}_m \dot{u}_0 - \frac{1}{\varepsilon} \int_{S^1} V'(u_m)[u_m] = 0, \quad m \rightarrow +\infty.$$

From equations (91) and (92) it follows that

$$\lim_m \int_{S^1} |\dot{u}_m|^2 = \int_{S^1} |\dot{u}_0|^2,$$

and hence  $u_m \rightarrow u_0$  strongly in  $H^1(S^1; \mathbb{R}^n)$ . □

**Corollary 5.4** *The functional  $L_\varepsilon : L_0^A \rightarrow \mathbb{R}$ ,  $\varepsilon \in (0, \varepsilon_A)$ , satisfies the Palais Smale condition.*

*Proof.* Let  $(h_m)_m \subseteq L_0^A$  be a Palais Smale sequence for  $L_\varepsilon$ , namely a sequence which satisfies

- (i)  $L_\varepsilon(h_m) \rightarrow c \in \mathbb{R}$ , as  $m \rightarrow +\infty$ ;
- (ii)  $\|DL_\varepsilon(h_m)\| \rightarrow 0$ , as  $m \rightarrow +\infty$ .

We want to prove that  $h_m$  converges up to a subsequence to some function  $h_0 \in H^1(S^1; M)$ . By Lemma 5.3 it is sufficient to show that the sequence  $u_m = u(h_m, v_{h_m})$  is a Palais Smale sequence for  $E_\varepsilon$ . In fact, by the continuity of  $h(u)$  the convergence of  $u_m$  implies that of  $h_m$ .

To prove that  $u_m$  is a Palais Smale sequence for  $E_\varepsilon$ , it is sufficient to take into account that, by the choice of  $v_{h_n}$ , it is  $D_v \bar{E}_\varepsilon(h_n, v_{h_n}) = 0$ , so one has

$$D_h L_\varepsilon(h)[k] = D_h \bar{E}_\varepsilon(h, v_h)[k] + D_v \bar{E}_\varepsilon(h, v_h)[Dv(h)[k]] = D_h \bar{E}_\varepsilon(h, v_h)[k].$$

Hence, it turns out that

$$DE_\varepsilon(u_n)[w] = D_h \bar{E}_\varepsilon(h_n, v_{h_n})[(Dh(u_n)[w]) + D_v \bar{E}_\varepsilon(h_n, v_{h_n})[Dv(u_n)[w]]] = DL_\varepsilon(h_n)[Dh(u_n)[w]].$$

It is immediate to check that the differential  $Dh(u)$  is uniformly bounded for  $|v| \leq \rho_0$ , so  $\|DL_\varepsilon(h_n)\| \rightarrow 0$  implies that also  $\|DE_\varepsilon(u_n)\| \rightarrow 0$ . This concludes the proof. □

Since  $L_\varepsilon$  is of class  $C^1$ , then it is possible to prove, see for example [3], that there exists a pseudo gradient  $\Omega_\varepsilon$  for  $L_\varepsilon$ , namely a  $C^{1,1}$  vector field which satisfies the conditions

$$(\Omega_\varepsilon(h), -DL_\varepsilon(h)) \geq \|DL_\varepsilon(h)\|^2, \quad \|\Omega_\varepsilon\| \leq 2 \|DL_\varepsilon\|; \quad \forall h \in L_0^A.$$

$\Omega_\varepsilon$  induces locally a flow  $\phi_\varepsilon^t$ ,  $t \geq 0$ , for which  $L_\varepsilon$  is non decreasing, and which is strictly decreasing whenever  $DL_\varepsilon \neq 0$ .

**Lemma 5.5** *Let  $A > 0$  be a fixed constant. Then there exists  $\tilde{\varepsilon}_A$  such that for all  $\varepsilon \in (0, \tilde{\varepsilon}_A)$  and for all  $h \in L_0^A$  the flow  $\phi_\varepsilon^t h$  is defined for all  $t \geq 0$ .*

*Proof.* By Proposition 5.1 there exists  $\varepsilon_{2A}$  such that for  $\varepsilon \in (0, \varepsilon_{2A})$  the functional  $L_\varepsilon$  is well defined on  $L_0^{2A}$ . Moreover, by Proposition 5.2 we can choose  $\bar{\varepsilon}_{2A} \leq \varepsilon_{2A}$  such that

$$(93) \quad |G_\varepsilon(h)| \leq \bar{C}_{2A} \cdot \sqrt{\varepsilon}, \quad \|DG_\varepsilon\| \leq \bar{C}_{2A} \cdot \sqrt{\varepsilon}; \quad \varepsilon \in (0, \bar{\varepsilon}_{2A}), h \in L_0^{2A}.$$

We choose  $\tilde{\varepsilon}_A$  satisfying

$$\tilde{\varepsilon}_A \leq \bar{\varepsilon}_{2A}; \quad \bar{C}_{2A} \cdot \sqrt{\tilde{\varepsilon}_A} \leq \frac{1}{4} A.$$

So, fixed  $h \in L_0^{2A}$ , by our choice of  $\tilde{\varepsilon}_A$  and by equation (93), there holds

$$L_0(h) \leq A \quad \Rightarrow \quad L_\varepsilon(h) \leq A + \frac{1}{4} A \quad \Rightarrow \quad L_\varepsilon(\phi_\varepsilon^t h) \leq A + \frac{1}{4} A \quad \Rightarrow \quad L_0(\phi_\varepsilon^t h) \leq A + \frac{1}{2} A < 2A,$$

for  $\varepsilon \in (0, \tilde{\varepsilon}_A)$  and whenever  $\phi_\varepsilon^t h$  is well defined. Hence  $\phi_\varepsilon^t h$  belongs to the domain of  $L_\varepsilon$  whenever  $\phi_\varepsilon^t h$  is well defined.

By the Hölder inequality and by equation (93) there results

$$\|DL_\varepsilon(h)\| \leq \|DG_\varepsilon(h)\| + \|DL_0(h)\| \leq \bar{C}_{2A} \cdot \sqrt{\tilde{\varepsilon}_A} + \int_{S^1} |\dot{h}|^2 \leq \bar{C}_{2A} \cdot \sqrt{\tilde{\varepsilon}_A} + A.$$

Hence the curve  $t \rightarrow \phi_\varepsilon^t h$  is globally Lipschitz, so by the local existence and uniqueness of the solutions it can be extended for all  $t \geq 0$ .  $\square$

Now we are in position to prove Theorem 1.2; the arguments rely on classical topological methods, see [6], which we recall for the reader's convenience.

*Proof of Theorem 1.2.* Suppose first that  $\pi_1(M) \neq 0$ , and let  $[\alpha]$  be a non trivial element of  $\pi_1(M)$ . Define

$$\alpha_0 = \inf_{h \in [\alpha] \cap H^1(S^1; M)} L_0(h);$$

since the curves belonging to  $[\alpha]$  are non contractible, then it is  $\alpha_0 > 0$ . By Proposition 5.1 there exists  $\varepsilon_{2\alpha_0} > 0$  such that for  $\varepsilon \in (0, \varepsilon_{2\alpha_0})$  the functional  $L_\varepsilon$  is defined on  $L_0^{2\alpha_0}$ . Hence for  $\varepsilon$  sufficiently small it makes sense to define also

$$\alpha_\varepsilon = \inf_{h \in [\alpha] \cap H^1(S^1; M)} L_\varepsilon(h).$$

By using standard arguments, based on Corollary 5.4 and Lemma 5.5, one can prove that the infimum  $\alpha_\varepsilon$  is achieved by some critical point  $h_\varepsilon$  of  $L_\varepsilon$ , and that  $\alpha_\varepsilon \rightarrow \alpha_0$  as  $\varepsilon \rightarrow 0$ . Now let  $\varepsilon_k \rightarrow 0$ : since by Proposition 5.2 it is  $\|DG_{\varepsilon_k}(h_{\varepsilon_k})\| \leq O(\sqrt{\varepsilon_k})$ , then  $h_{\varepsilon_k}$  is a Palais Smale sequence for  $L_0$ . Hence, passing to a subsequence,  $h_{\varepsilon_k}$  must converge to a critical  $x_0$  point of  $L_0$  at level  $\alpha_0$ . This concludes the proof in the case  $\pi_1(M) \neq 0$ .



Now consider the case of  $\pi_1(M) = 0$ . By Theorem 2.5 and by Remark 2.6, there exists  $\omega : S^{l+1} \rightarrow M$  which is not contractible. To the function  $\omega$  one can associate the map

$$F_\omega : (B^l, \partial B^l) \rightarrow (H^1(S^1; M), C_0(S^1; M)),$$

where  $C_0(S^1; M)$  denotes the class of constant maps from  $S^1$  to  $M$ .  $F_\omega$  is defined in the following way. First, identify the closed  $l$ -ball  $B^l$  with the half equator on  $S^{l+1} \subseteq \mathbb{R}^{l+2}$  given by

$$\{y = (y_0, \dots, y_{l+1}) \in S^{l+1} : y_0 \geq 0, y_{l+1} = 0\}.$$

Denote by  $c_p(t)$ ,  $0 \leq t \leq 1$ , the circle which starts out from  $p \in B^l$  orthogonally to the hyper plane  $\{y_1 = 0\}$  and enters the half sphere  $\{y_1 \geq 0\}$ . With this we put

$$F_\omega = \{f \circ c_p(t) | 0 \leq t \leq 1\}.$$

The correspondence  $\omega \rightarrow F_\omega$  is clearly bijective, and hence one can define the value

$$\beta_0 = \inf_{F_l, t \in [\omega]} \sup_{p \in S^l} L_0(F_l(p)).$$

Since  $\omega$  is non contractible, it is possible to prove that  $\beta_0 > 0$  so, as above, for  $\varepsilon$  sufficiently small one can define the quantity

$$\beta_\varepsilon = \inf_{F_l, t \in [\omega]} \sup_{p \in S^l} L_\varepsilon(F_l(p)).$$

The number  $\beta_\varepsilon$  turns out to be a critical value for  $L_\varepsilon$  and, again, if  $\varepsilon_k \rightarrow 0$  we can find critical points  $h_{\varepsilon_k}$  of  $L_{\varepsilon_k}$  at level  $\beta_{\varepsilon_k}$  converging to some geodesic  $x_0$  with  $L_0(x_0) = \beta_0$ . This concludes the Proof of Theorem 1.2.  $\square$

**Remark 5.6** In [4] is studied the system (T) on  $\mathbb{R}^2$ . It is assumed, roughly, that  $V$  possesses a non-contractible set of maxima, and are proved some existence and multiplicity results for large-period orbits. Dealing with two-dimensional systems, we suppose that  $V$  is non-degenerate in the sense of (1). In particular, this implies that  $M$  is a simple closed curve. When  $M$  is a manifold of maxima for  $V$ , this corresponds to the case treated in Theorem 1.1. The non-degeneracy of  $V$  allows us to describe in a quite precise way the asymptotic behaviour of the trajectories. Furthermore, according to Remark 1.3, we could also obtain existence of an arbitrarily large number of adiabatic limits, since we can apply the above argument to each different element of  $\pi_1(M)$ . Note also that we can address the case in which  $M$  is a manifold of minima, see the next Section.

## 6 Attractive potentials

In this section we describe how the arguments of Section 5 can be modified to handle the attractive case, namely that in which  $V''(x)[n_x, n_x]$  is positive for all  $x \in M$ . Since attractive potentials may cause some resonance phenomena, the reduction procedure could fail. To overcome this difficulty we need to make stronger assumptions on  $V$ . In particular we assume that the following condition holds

$$(94) \quad \frac{\partial^2 V}{\partial n_x^2}(x) = b_0 > 0; \quad \text{for all } x \in M.$$

We also set

$$(95) \quad \Lambda = \sup_{x \in M} \left| \frac{\partial^3 V}{\partial n_x^3}(x) \right|.$$

**Proposition 6.1** *Suppose that the potential  $V$  satisfies condition (94), and let  $A > 0$  be a fixed constant. Then if  $\Lambda$  satisfies*

$$(96) \quad 4\Lambda \cdot \bar{H} \cdot A < b_0,$$

*then there exists  $\varepsilon_k \rightarrow 0$  such that equation (72) admits a solution  $v(h)$  for every  $m$  sufficiently large and for every  $h \in L_0^A$ . Moreover, there exists  $C_A > 0$  such that  $v(h)$  satisfies*

$$(97) \quad \|v(h)\|_{C^0(S^1)} \leq C_A \cdot \sqrt{\varepsilon_k}; \quad \text{for } m \text{ large and for } h \in L_0^A.$$

*Furthermore the application  $h \rightarrow v(h)$  from  $H^1(S^1; M)$  to  $C^0(S^1)$  is compact.*

**Remark 6.2** *One can easily verify, using rescaling arguments, that condition (96) is invariant under translation, dilation and rotation in  $\mathbb{R}^n$  of problem (T).*

*Proof.* The proof is very similar to that of Proposition 3.3 and is again based on the Contraction Mapping Theorem. We choose  $\varepsilon_k$  such that

$$\frac{b_0}{\varepsilon_k} = \left(k + \frac{1}{2}\right)^2.$$

In this way, we can apply the estimates of Section 4 with  $\lambda_0 = \frac{b_0}{\varepsilon_k}$ ; we will use the same notations of Section 5.

Since now  $B_h \equiv b_0$ , equation (72) can be written as

$$(98) \quad \ddot{v} + \left(\frac{b_0}{\varepsilon_k} - Q_h\right) v = \sigma_v := P_h + \frac{1}{\varepsilon_k} \psi(h, v) \cdot v^2.$$

The definition of  $\psi$  and (95) imply

$$(99) \quad |\psi(x, 0)| = \frac{1}{2} \left| \frac{\partial^3 \bar{V}}{\partial n_x^3}(x, 0) \right| \leq \frac{1}{2} \Lambda, \quad \forall x \in M.$$

So, given an arbitrary number  $\delta > 0$ , if one takes  $\|v\| \leq \rho$  with  $\rho$  sufficiently small and if  $h \in L_0^A$ , then by equations (64) and (65) one has

$$\begin{aligned} \|\Sigma_{h, \varepsilon} \sigma_v\|_{C^0(S^1)} &\leq (1 + \delta) \cdot \frac{1}{2} \frac{\sqrt{\varepsilon_k}}{\sqrt{b_0}} \cdot \left( \|P_h\|_{L^1(S^1)} + 4 \frac{1}{\varepsilon_k} \|\psi(h, v) v^2\|_{L^\infty} \right) \\ &\leq (1 + \delta) \cdot \frac{\sqrt{\varepsilon_k}}{\sqrt{b_0}} \cdot \bar{H} \cdot A + (1 + \delta) \frac{1}{\sqrt{b_0} \sqrt{\varepsilon_k}} \cdot \Lambda \cdot \rho^2. \end{aligned}$$

So, choosing  $\rho = C \cdot \sqrt{\varepsilon_k}$ , if the following equation is satisfied

$$(100) \quad (1 + \delta) \frac{1}{\sqrt{b_0}} \cdot \Lambda \cdot C^2 + (1 + \delta) \frac{1}{\sqrt{b_0}} \cdot \bar{H} \cdot A \leq C,$$

then  $\|\Sigma_{h, \varepsilon} \sigma_v\|_{C^0(S^1)} \leq \rho$ , so  $\mathcal{B}_\rho$  is mapped into itself by  $\Theta_{h, \varepsilon}$ . If  $\delta$  is chosen to be small enough, then (100) is solvable with

$$(101) \quad C = \left( 1 - \sqrt{1 - 4 \frac{\Lambda \bar{H} A}{b_0}} \right) \frac{\sqrt{b_0}}{4\Lambda} + o_\delta(1).$$

Here  $o_\delta(1)$  denotes a quantity which tends to 0 as  $\delta$  tends to 0.

Now we show that  $\Theta_{h,\varepsilon}$  turns out to be a contraction. In fact, given two functions  $v, v' \in C^0(S^1)$ , there holds

$$\sigma_v - \sigma_{v'} = \frac{1}{\varepsilon} \left( (\psi(h, v) - \psi(h, v')) v^2 + \psi(h, v') (v + v') (v - v') \right).$$

Hence from formula (65) it follows that

$$\begin{aligned} \|\Sigma_{h,\varepsilon}\sigma_v - \Sigma_{h,\varepsilon}\sigma_{v'}\|_{C^0(S^1)} &\leq \left( \frac{1}{\varepsilon_k} \|D\psi\|_{L^\infty} C^2 \rho^2 2(1+\delta) \frac{\sqrt{\varepsilon_k}}{\sqrt{b_0}} \right. \\ &\quad \left. + \frac{2}{\sqrt{b_0}} \sqrt{\varepsilon_k} \frac{\Lambda}{\varepsilon_k} \left( 1 - \sqrt{1 - 8 \frac{\Lambda \bar{H} A}{b_0}} \right) \frac{\sqrt{b_0}}{2A} \sqrt{\varepsilon_k} + o_\delta(1) \right) \cdot \|v - v'\|_{C^0(S^1)}. \end{aligned}$$

If  $\delta$  and  $\varepsilon_k$  are sufficiently small, then the coefficient of  $\|v - v'\|_{C^0(S^1)}$  in the last formula is strictly less than 1, hence  $F$  is a contraction in  $B_\rho$ . This concludes the proof of the existence. The compactness can be proved in the same way as before.  $\square$

About the functional  $L_\varepsilon(h) = \bar{E}_\varepsilon(h, v(h))$  we have the following analogous of Proposition 5.2.

**Proposition 6.3** *Suppose condition (94) holds true, and suppose that  $\Lambda$  satisfies inequality (96). Then for  $\varepsilon_k$  sufficiently small the functional  $G_\varepsilon$  is of class  $C^1$  on  $L_0^A$  and there exists  $\bar{C}_A > 0$  such that*

$$(102) \quad |G_\varepsilon(h)| \leq \bar{C}_A \cdot \sqrt{\varepsilon}, \quad |DG_\varepsilon(h)| \leq \bar{C}_A \cdot \sqrt{\varepsilon_k}; \quad \forall h \in L_0^A, \forall \varepsilon \in (0, \bar{\varepsilon}_A).$$

*Proof.* The proof is analogous to that of Proposition 5.2. The only difference is the estimate of  $\|D\bar{\Sigma}_{\varepsilon_k}(h)[k]\|_{C^0(S^1)}$ , namely the norm of the  $2\pi$  periodic solution of equation

$$(103) \quad \begin{aligned} \ddot{y} + \left( \frac{1}{\varepsilon_k} B_h - Q_h - \frac{2}{\varepsilon_k} \psi(h, v) v - \frac{1}{\varepsilon_k} v^2 D_v \psi(h, v) \right) y \\ = DP_h[k] + \frac{1}{\varepsilon_k} D_h \psi(h, v)[k] v^2 - \frac{1}{\varepsilon_k} DB_h[k] v - DQ_h[k] v, \end{aligned}$$

which under assumption (94) takes the form

$$(104) \quad \begin{aligned} \ddot{y} + \left( \frac{1}{\varepsilon_k} b_0 - Q_h - \frac{2}{\varepsilon_k} \psi(h, v) v - \frac{1}{\varepsilon_k} v^2 D_v \psi(h, v) \right) y \\ = \theta(t) := DP_h[k] + \frac{1}{\varepsilon_k} D_h \psi(h, v)[k] v^2 - DQ_h[k] v \in L^1(S^1). \end{aligned}$$

The study of this equation requires some modifications of the arguments in Section 4. The reason is that the coefficient of  $y$  is not uniformly close (in  $L^1(S^1)$ ) to the constant function  $\frac{1}{\varepsilon_k} b_0$ .

Equation (104) can be written in the form

$$\ddot{y} + \frac{1}{\varepsilon_k} b_0 y = \theta + \left( Q_h + \frac{2}{\varepsilon_k} \psi(h, v) v + \frac{1}{\varepsilon_k} v^2 D_v \psi(h, v) \right) y.$$

So, applying equations (64) and (65) and taking into account of (99) one can deduce

$$\begin{aligned} \|y\|_{C^0(S^1)} &\leq (1+\delta) \frac{\sqrt{\varepsilon_k}}{2\sqrt{b_0}} \cdot \|y\|_{C^0(S^1)} \cdot \left( \|Q_h\|_{L^1(S^1)} + 4\Lambda C \sqrt{\varepsilon_k} \frac{2}{\sqrt{\varepsilon_k}} + 4 \frac{1}{\varepsilon_k} C^2 \varepsilon_k \cdot \|D\psi\|_{L^\infty} \right) \\ &\quad + (1+\delta) \frac{\sqrt{\varepsilon_k}}{2\sqrt{b_0}} \cdot \|g\|_{L^1(S^1)}. \end{aligned}$$

If the constant  $C$  is given by formula (101), then one can show that for  $\delta$  and  $\varepsilon_k$  sufficiently small the coefficient of  $\|y\|_{C^0(S^1)}$  on the right hand side is strictly less than 1, so estimate (102) holds true. This concludes the proof.  $\square$

Corollary 5.4 holds without changes also for this case, and having Proposition 6.3 one can easily prove the analogous of Lemma 5.5. These facts allow to apply the reduction on  $H^1(S^1; M)$  below the level  $\frac{b_0}{4\bar{H}\Lambda}$ . As a consequence we have the following result.

**Theorem 6.4** *Suppose  $\pi_1(M) \neq 0$  (resp.  $\pi_1(M) = 0$ ), and let  $\alpha_0$  (resp.  $\beta_0$ ) be the value which appears in the proof of Theorem 1.2. Suppose the following condition is satisfied*

$$4\alpha_0 \cdot \bar{H} \cdot \Lambda < b_0 \quad (\text{ resp. } 4\beta_0 \cdot \bar{H} \cdot \Lambda < b_0).$$

*Then there exists a sequence  $T_k \rightarrow +\infty$  and there exists a sequence of solutions  $(u_k)_k$  to problem (T) corresponding to  $T = T_k$  such that up to subsequence  $u_k(T_k \cdot)$  converge in  $C^0(S^1, \mathbb{R}^n)$ . The adiabatic limit of  $u_k(T_k \cdot)$  is a non trivial closed geodesic  $x_0$  on  $M$  at level  $\alpha_0$  (resp.  $\beta_0$ ).*

*Proof of Theorem 1.4.* It is an immediate consequence of Theorem 6.4, since condition (ii) implies that  $\Lambda = 0$ . □

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