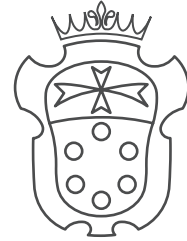




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DE GENÈVE**

FACULTÉ DES LETTRES



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NORMALE
SUPERIORE

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On the Nature of Impure Sets

A Defense of Abstractness

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Tu l'as voulu ton vélo?
Eh, bien, maintenant, pédale.

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Acknowledgments

On impure sets is a work announced in November 2018 at Scuola Normale Superiore in Pisa, where I started my Ph.D. in philosophy under the supervision of Professor Mario Piazza. My first idea was to further develop my Master thesis by working on the concept of set in Leśniewski's mereology. However, after a deep discussion with Mario Piazza, we agreed to take some more time to think about the project. The main challenge was to enhance my historical approach with some analytic tools to make my research less hyper-specific and elitist. For this reason, I spent the first year of my Ph.D. delving into contemporary analytic philosophy.

In 2019, under the advice of Professor Mario Piazza, Professor Marc Peeters, and Professor Achille Varzi, I decided to start a cotutelle (double agreement) with the University of Geneva under the supervision of Fabrice Correia. Together with Claudio Calosi, Fabrice Correia had me join *eidos*, the Center for Metaphysics at the University of Geneva.

In this context, the subject of my Ph.D. dissertation took its definitive form. First, I worked on Zalta's account and offered a defense and an extension of its Object Theory when applied to intentional contexts (published in *Metaphysica*, 2022). Then, I set out to investigate an application of that account to the notion of a set in general and to that of a concept set in particular. This latter concept became, as a matter of fact, the main topic of my dissertation.

What I find peculiar in this story is its, so to say, cyclic feature. Without realizing it, after abandoning the idea of working on Leśniewski's

concept of set, I did finally come up with a more comprehensive study of the concept of set through the lenses of analytic philosophy. I have therefore realized that the notion of set was my first urgence, which has come back to me after two years, more nourished and elaborated.

Now that this work has taken form, my new genuine urgence is to thank all the people who have helped me during these intense five years of work.

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Abstract

This dissertation aims to clarify the nature of impure sets. The classic definition takes impure sets as sets having concrete members or elements in their transitive closure. Concreteness is often understood in terms of spatiotemporal location: something is concrete *iff* it is spatiotemporally located. Also, concreteness is classically opposed to abstractness. They are mutually exclusive and exhaustive categories. My main purpose is to show that the definition of impure sets is misguided. Impure sets do not contain concrete objects but abstract ones. To corroborate this, I will first discuss and reject the idea that impure sets are abstract objects having concrete members or elements in their transitive closure. This idea is adopted by many authors, such as Cowling, Falguera, Fine, Lowe, and Parsons. I will focus on Parsons' and Fine's accounts which are the most detailed: Parsons takes impure sets as quasi-concrete objects, while Fine describes them as abstract objects having concrete elements as their parts. The arguments that I will raise against Parsons' idea address his criterion of identity and his notion of quasi-concrete objects. Concerning Fine's account, I will argue against the possibility of abstract objects composed of concrete ones. Once the idea of impure sets as abstract objects having concrete members or elements has been rejected, I will consider impure sets as fully concrete objects, *i. e.* concrete objects having concrete members or elements in their transitive closure. I will show that this thesis is misguided by focusing on Maddy's framework. Maddy's

framework is divided into two different but related claims: the ontological claim according to which impure sets are spatiotemporally located where their members or elements in their transitive closure are located, and the epistemic claim according to which we perceive impure sets. I will show that both claims are problematic. Given that impure sets are neither abstract objects having concrete members or elements, nor fully concrete objects, and that the idea that impure sets are concrete objects having abstract members or elements seems highly unintuitive, I will conclude that impure are fully abstract objects (*i. e.*, objects that are not spatiotemporally located having objects that are not spatiotemporally located as members or elements in their transitive closure). I will state that the most compelling strategy to argue that members or elements of impure sets are abstract objects can be found by exploring the relations between impure sets and the notions of concept, conceptualization, and conceptualized objects. I will conclude by arguing that Urelements of impure sets can be considered conceptualized objects modeled using Zalta's notion of individual concepts (or abstract copies) formulated in his Object Theory. Finally, I will discuss one application of this account of impure sets to the famous case of Fine's counterexample with Socrates and {Socrates}. By exploring this application, I will further justify the idea of impure sets as fully abstract objects.

Abstract

Cette thèse vise à clarifier la nature des ensembles impurs. La définition classique considère les ensembles impurs comme des ensembles ayant des membres ou des éléments concrets dans leur clôture transitive. L'être concret d'un objet est souvent compris en termes de localisation spatiotemporelle: quelque chose est concret s'il est localisé spatiotemporellement. Aussi, ceci s'oppose à l'abstrait. Il s'agit de catégories mutuellement exclusives et exhaustives. Dans cette thèse, mon objectif principal est de montrer que la définition des ensembles impurs est erronée. Les ensembles impurs ne contiennent pas d'objets concrets mais des objets abstraits. Pour corroborer cela, je discuterai et rejetterai d'abord l'idée selon laquelle les ensembles impurs sont des objets abstraits ayant des membres ou des éléments concrets dans leur clôture transitive. Cette idée est reprise par de nombreux auteurs, tels que Cowling, Falguera, Fine, Lowe et Parsons. Je me concentrerai sur les systèmes de Parsons et de Fine, qui sont les plus détaillés : Parsons considère les ensembles impurs comme des objets "quasi-concrets", tandis que Fine les décrit comme des objets abstraits ayant des éléments concrets comme parties. Les arguments que j'opposerai à l'idée de Parsons concernent son critère d'identité et sa notion d'objets quasi-concrets. Concernant Fine, j'argumenterai contre la possibilité d'objets abstraits composés d'objets concrets. Une fois rejetée l'idée d'ensembles impurs en tant qu'objets abstraits ayant des membres ou des éléments concrets, je considérerai les ensembles impurs

comme des objets entièrement concrets, c'est-à-dire des objets concrets ayant des membres ou des éléments concrets dans leur clôture transitive. Je montrerai que cette thèse est erronée en me concentrant sur l'idée d'ensemble impurs chez Maddy.

Celle-ci est divisée en deux thèses fondamentales : l'affirmation ontologique selon laquelle les ensembles impurs sont localisés spatiotemporellement là où se trouvent leurs membres ou éléments dans leur clôture transitive, et l'affirmation épistémique selon laquelle nous percevons les ensembles impurs. Je montrerai que les deux affirmations sont problématiques. Étant donné que les ensembles impurs ne sont ni des objets abstraits ayant des membres ou des éléments concrets, ni des objets entièrement concrets, et que l'idée selon laquelle les ensembles impurs sont des objets concrets ayant des membres ou des éléments abstraits semble très peu intuitive, je conclurai que les ensembles impurs sont des objets entièrement abstraits (c'est-à-dire des objets qui ne sont pas localisés spatiotemporellement ayant des objets qui ne sont pas localisés spatiotemporellement comme membres ou éléments dans leur clôture transitive). Je montrerai que la stratégie la plus convaincante pour affirmer que les membres ou éléments d'ensembles impurs sont des objets abstraits peut être trouvée en explorant les relations entre les ensembles impurs et les notions de concept, de conceptualisation et d'objets conceptualisés. Je conclurai en affirmant que les éléments d'ensembles impurs peuvent être considérés comme des objets conceptualisés modélisés à l'aide de la notion de concepts individuels (ou copies abstraites) de Zalta. Enfin, je montrerai une application de cette explication des ensembles impurs au cas

célèbre du contre-exemple de Fine avec Socrate et $\{\text{Socrate}\}$. En explorant cette application, je justifierai davantage l'idée d'ensembles impurs en tant qu'objets entièrement abstraits.

Introduction

This dissertation is about impure sets. Examples of impure sets are the singleton of Socrates, the set of human beings, the set of all physical points, and so on. Literature on the topic is rare. Impure sets have been neglected by scientists, probably because of their “hybrid” status. As it is stressed by Piazza, they are “too mathematical for the physicists but too physical for the mathematician” [Piazza, 2000], 74. More specifically, the mathematician is not interested in impure sets because they are not necessary to set theory, while the physicist does not directly deal with the set-theoretical principles appearing once the notion of impure sets is axiomatized.

Yet, this does not mean that scientists do not speak about impure sets. Physicists, for instance, often take spacetime as a set of physical non-material points. Contemporary philosophy is also full of examples involving impure sets. The most famous is [Fine, 1994]’s counterexample with {Socrates}. Also, both [Maddy, 1991] and [Parsons, 2008] conceive impure sets as starting points for all mathematical knowledge, and [Linnebo, 2018] takes impure sets to be emblematic cases of thin objects.

Despite this widespread use in scientific and philosophical speeches, a systematic study on the nature of impure sets is still missing. This dissertation aims to make up for this lack by clarifying the notion of impure sets. This study, which has, of course, an independent interest, will also be useful to further understand the scientific panorama involving impure sets.

I will approach the topic through the lens of analytic philosophy. Given the nature of the issue in question, the dissertation also contains elements of philosophy of mathematics. Here is a bunch of questions that I will discuss: Are impure sets outside of space and time? If not, do they share the location with their members? Or are they elsewhere? What kind of relation is membership relation? What is an Urelement? How are we supposed to get epistemic access to impure sets?

Answering these questions is not an easy task. As it is argued by Lewis, all issues about impure sets seem to result in “a choice of equal evil”. [Lewis, 1991], 32. The difficulty is further increased by the fact that the notion of impure set involves that of set, which is itself problematic. As it is well known, Cantor defines sets as “many (*jedes Viele*) which can be thought of as one, *i. e.* a totality of definite elements that can be combined into a whole by a law” [Cantor, 1883], 204. However, this definition does not accommodate singletons and the empty set. Another strategy is to just go by means of examples. But, of course, this approach is explanatory inert. Otherwise, it can be argued that, since the concept of set plays a fundamental role in our understanding of mathematics, it does not need to be defined. Nonetheless, the fact that sets are fundamental for mathematics is not a good reason for stopping trying to get a better explanation of what they are. Here is another sense in which my study is useful: clarifying what impure sets are will shed some light on the notion of set in set theory.

As mentioned, literature on impure sets is rare. This does mean that it is absent. In philosophy, the debate over impure sets is increasing.

For instance, [Cook, 2013] and [Effingham, 2013] discuss whether impure sets are or not located. Another place where the topic of impure sets is largely considered is [Maddy, 1991]. Other accounts that give some hints about the nature of impure sets are [Parsons, 2008] and [Fine, 2010]. Some secondary literature is also available, especially concerning Maddy's impure sets. Examples are [Chihara, 1991], [Balaguer, 1994], [Carons, 1996], and [Levine, 2005]. Among the authors who touch upon the topic of impure sets, we also have [Lowe, 1995], [Katz, 2000], and [Cowling, 2017].

In this dissertation, I will explore this literature focusing especially on Parsons', Fine's, and Maddy's accounts. Before going into the details of how this work is structured, let me face the big elephant in the room. I have said that impure sets are objects of the kind of singleton Socrates, the set of human beings, and the set of all physical points. So far, so good. But what is an impure set in the first place?

Some terminological clarifications are in order. Following [Cook, 2013], two definitions are available: a set is impure *iff* its members are concrete objects. Examples of impure sets are, therefore, {Socrates}, and {Plato, Socrates}. Then, this definition can be extended to cover cases of sets having concrete elements in their transitive closure such as {{Socrates}}, {{Plato}, {Socrates}}, and so on. The other less standard way to understand the notion of impure sets is to take them to be sets containing abstract objects that are not sets. Examples are {Sherlock Holmes} and the set of cardinal numbers.

As anticipated, this dissertation is about impure sets such as singleton

Socrates, the set of human beings, the set of all physical points, and so on. I will, therefore, ignore the second definition and focus on the first which I take as a starting point. To spoil a bit, after having largely explored the notion of impure sets, I will argue that the first definition is not adequate either. I will therefore offer a new definition of impure sets.

But let me deal with one thing at a time. Before getting to the heart of the topic, it is important to clarify the main terms appearing in the starting definition. The definition claims that impure sets are sets having concrete members or elements. What it is for something to be *concrete*? This is a main topic of contemporary philosophy. Classically, concreteness is opposed to abstractness. The distinction between abstract and concrete seems to be of fundamental significance for both metaphysics and epistemology. To use Falguera's words, "one doesn't go far in the study of what there is without encountering the view that every entity falls into one of two categories: concrete or abstract" [Falguera, 2001]. Yet, there is no uncontroversial account of how the distinction should be drawn.

In this dissertation, I do not aim to enter into this huge and complex debate which is, by the way, already well illustrated in [Falguera, 2001]. I will just limit myself to adopting some common tenets that are controversial but widely discussed and defended in the literature. One of these is the idea that something is concrete *iff* it is spatiotemporally located, while it is abstract if otherwise. For instance, my glasses are concrete in that they are spatiotemporally located, while numbers are

abstract because they are located nowhere. At an early stage, I take the notion of location to be primitive. Then, when it comes to considering some more specific arguments, I will adopt [Parsons, 2007] and [Gilmore, 2013]’s definitions.

I will assume that abstractness and concreteness are mutually exclusive and exhaustive categories. Roughly speaking, there is nothing “more than”, or “in between”, abstractness and concreteness. Once again, this idea is not uncontroversial. Exceptions are, for instance, [Zalta, 1983], [Katz, 2000], and [Williamson, 2007]. To be fair, in discussing my idea of impure sets, I will adopt Zalta’s account which displays an ontology composed of ordinary objects that can be both concrete and non-concrete, and abstract objects. As I will point out though, the fact that this framework does not look at concreteness and abstractness as mutually exclusive and exhaustive categories is not relevant to the solution that I will offer, and can therefore be ignored.

One last clarification is in order. Another criterion that is classically used to determine whether something is concrete or abstract is causality. Concrete objects are often taken to be causally efficacious, while abstract objects are not. I will nonetheless leave aside this criterion and focus only on spatiotemporality. This choice can be justified by some arguments from [Ismael, 2016] according to which causality is not a fundamental element of physical reality. Of course, this is a very controversial tenet. It has to be noticed, however, that most of the arguments that I will propose, and that use spatiotemporality as its main feature, can be reformulated by taking causality.

Now that all premisses have been stated, let me move on to presenting the content and the dialectic of this dissertation more in detail. One main issue concerns the relation between what is abstract and what is concrete. In Chapter 1, I will discuss the idea of impure sets as abstract objects having concrete members or elements in their transitive closure. Following the definition of concreteness above, this means that impure sets are not located, while their members or elements are located. I will call this idea “mainstream thesis” in that it is explicitly or implicitly adopted by many authors. Examples are [Fine, 1992], [Lowe, 1995], [Falguera, 2001], [Parsons, 2008], and [Cowling, 2017] .

The main aim of the chapter is to reject the mainstream thesis. To do this, I will focus on two different accounts which have the advantage of being explicit under different respects: [Parsons, 2008]’ and [Fine, 1992] [Fine, 2010]’s. Parsons wants impure sets to be quasi-concrete objects, *i. e.* abstract objects “embodied by concrete representations”, while Fine describes impure sets as abstract objects having concrete elements as their parts. The contents that I will discuss include the criteria of identity for impure sets, the notion of representation and intrinsicity, those of part and composition, and the concept of location.

The arguments that I will address against Fine’s and Parsons’ accounts generalize to all frameworks involving the mainstream thesis plus abstractness and concreteness as mutually exclusive and exhaustive categories defined using spatiotemporal location, and membership as composition or representation. Someone who desires to defend the mainstream

thesis has therefore the burden of proof of showing that other accounts are available.

Once the mainstream thesis has been rejected, other metaphysical hypotheses arise. In Chapter 2, I will discuss the idea that impure sets are fully concrete objects, *i. e.* concrete objects having concrete members or concrete elements in their transitive closure. To do this, I will focus on [Maddy, 1991]’s framework that divides into two different but related claims: the ontological claim according to which impure sets are spatiotemporally located where their members or elements in their transitive closure are located, and the epistemic claim according to which we perceive impure sets.

Maddy’s thesis has been sharply criticized by many commentators. Examples are [Chihara, 1991], [Balaguer, 1994], [Carons, 1996], [Piazza, 2000], and [Levine, 2005]. In discussing Maddy’s account, I will play the devil’s advocate showing that many of these objections can be resisted, but that, even when presented in the most compelling way, the idea of impure sets as fully concrete objects is still subjected to serious problems and has to be rejected. The contents that I will discuss involve the notion of identification and its application to impure sets, pluralism and monism, *qua*-object theory, and perception as an epistemic tool to grasp impure sets.

The outcomes of Chapters 1 and 2 lead me to formulate what I will call the “negative dialectic”. From Chapter 1, we know that the mainstream thesis is problematic; Chapter 2 shows that the idea of impure sets as fully concrete objects is subjected to counterexamples. Hence,

given that concreteness and abstractness are mutually exclusive and exhaustive, we have that impure sets are either fully abstract objects, (*i. e.* objects that are not spatiotemporally located having objects that are not spatiotemporally located as members or elements in their transitive closure) or concrete objects having abstract members or elements in their transitive closure (*i. e.* spatiotemporally located objects having objects that are not spatiotemporally located as members or elements).

To my knowledge, nobody adopts this last option, and I think that there are good reasons to not do so. Indeed, it would be highly counter-intuitive to claim that, for instance, {Socrates} is concrete while Socrates is abstract. In this dissertation, I will therefore not take this idea into account. Then, the only option left is to take impure sets to be fully abstract objects. In Chapter 3, I will discuss and defend this idea, which has to be considered the central issue of the dissertation.

This strategy has never been proposed and it is, of course, uncommon and controversial. The negative dialectic is not sufficient to justify the account which needs to be implemented with some positive reasons able to explain how impure sets can be understood as fully abstract objects. More specifically, the controversial part of the story is that members and elements of impure sets are abstract objects. This idea goes against the classic definition of impure sets as sets having concrete members or elements in their transitive closure. In other words, members or elements of impure sets, such as Socrates, Plato, and so on, are supposed to be abstract objects.

To defend this idea, I will, therefore, focus on the nature of the so-

called "Urelements", namely set-theoretical objects that are not sets, that do not contain sets, but that can belong to sets. I will argue that impure sets and their Urelements are strictly related to the notions of concept, conceptualization, and conceptualized objects. I will conclude by claiming that Urelements of impure sets can be considered cases of conceptualized objects, *i. e.* objects that result from a process of conceptualization which consists of considering objects, events, etc. as falling under some concepts.

The leitmotiv of the chapter, and, more generally, of all the dissertation, is what I will call the "Fregean argument" claiming that we cannot count objects, but that we rather count sets. This argument will take different forms (some of which may even be considered distant from Frege's doctrine). The form that interests me is meant to justify that Urelements are indeed conceptualized.

Then, I will defend the idea that these conceptualized objects are abstract. This means that the aforementioned members or elements of impure sets such as Socrates and Plato are conceptualized abstract objects. This is a delicate passage that I will treat with the help of some authority principles from [Balaguer, 1994] and [Parsons, 2008]. Then, I will try to further motivate it by spelling out my idea of how abstractness and concreteness are associated with objects.

I will argue though that conceptualized objects are related to the concrete. This means, for instance, that, given {Socrates}, Socrates is a conceptualized abstract object that has, nonetheless, a relation to something concrete. Here again, we deal with the main issue of how abstract

and concrete contents are related. Now the question is: What is this “concrete” to which Socrates is related? I will argue that Socrates conceptualized abstract object is related to Socrates concrete human being. I will also argue that this relation is a relation of representation which is, nonetheless, different from that formulated by Parsons and explored in Chapter 1.

More specifically, I will stress that this relation of representation is well modeled by using [Zalta, 1983]’s notion of individual concepts (or abstract copies) formulated in his Object Theory. Finally, I will briefly present a development in impure set theory concerning the notion of Urelement, and argue that conceptualism seems to be the most adequate epistemological framework to be associated with my account of impure sets as fully abstract objects.

The idea of modeling Urelements by Zalta’s Object Theory has never been proposed. This is a first attempt which raises, of course, a lot of doubts. Moreover, as mentioned, the account questions the classic definition of impure sets as sets having concrete members or elements, and needs, therefore, to be further motivated. At the end of the chapter, I will explore some objections and provide some replies. While exploring these replies, I will also point out some further advantages of my account over the mainstream thesis and the idea that impure sets are fully concrete objects.

Another way to valorize my account is to present one of its applications to contemporary metaphysics. Among the authors who quote cases of impure sets while discussing their philosophical claims, the most fa-

mous is [Fine, 1994]’s counterexample to the modal account of essence. The counterexample takes the case of Socrates and {Socrates} arguing that according to the modal account of essence, {Socrates} has essentially Socrates as its member and that Socrates essentially belongs to {Socrates}. Then, Fine argues that this last claim is unintuitive as long as there is nothing in the nature of Socrates that suggests that he belongs to {Socrates}. I will show that the validity of the counterexample rests on some philosophical assumptions about impure sets that my account may refuse. More specifically, I will point out that it can be resisted by postulating the existence of two different objects: ordinary objects and Urelements. Then, I will argue that my account with Urelements accommodates this idea.

The application of my account to Fine’s counterexample further justifies the idea of impure sets as fully abstract objects. I will argue that it also provides a better understanding of Fine’s essentialism, and suggests other interesting results in contemporary metaphysics. For instance, it has the effect of questioning the asymmetry of the ontological dependence relation which is classically taken to hold between sets and their members.

I will conclude by presenting [Wildman, 2013]’s and [Zalta, 2006]’s ways of dealing with Fine’s counterexample showing that they are subjected to some difficulties that my account can overcome.

At the end of the chapter, I will point out that my idea of Urelements as conceptualized abstract objects is also subjected to some problems that the discussion of Fine’s counterexample brings out. I will, nonethe-

less, argue for some ways out.

This work presents some original ideas about impure sets. I will show that some of these ideas are controversial. And that there still are open problems. The account that I will propose gives some hints on how these controversial points can be undermined. However, as it is only sketched, it needs to be further developed and justified. This dissertation is an attempt to clarify the notion of impure sets. In turn, this attempt has to be considered the first step of a more vast research on impure set theory that needs to be undertaken.

To take back Piazza's claim, impure sets are too mathematical for the physicists but too physical for the mathematician. I argue that philosophers are good candidates to discuss this issue and that, once this work is done, it appears plausible to claim that, contrary to what Lewis stresses, answering questions about impure sets is not a choice of equal evil. There certainly are evils that are worse than others, as well as ideas that seem to accommodate the main philosophical issues concerning impure sets.

Chapter 1

The *Mainstream Thesis*

Introduction

In this chapter, I will provide some reasons to reject the most accepted idea about the nature of impure sets which I will call the *mainstream thesis*. According to the supporters of the mainstream thesis, impure sets are abstract objects having concrete members or concrete elements in their transitive closure (see [Fine, 1992], [Lowe, 1995], [Falguera, 2001], [Parsons, 2008], and [Cowling, 2017] *inter alios*). Defining abstractness and concreteness by spatiotemporality, we can claim that impure sets are, therefore, objects that are not spatiotemporally located having members or elements in their transitive closure that are not spatiotemporally located. Classic examples: {Socrates} is abstract and not located, while Socrates is concrete and located. {{Plato}} is abstract and not located, while Plato is concrete and located.

Since the debate over impure sets has been neglected for a long time,

the mainstream thesis has very often taken an implicit character. Many of the philosophers just quoted have assumed that impure sets are abstract objects having concrete members or elements without providing a precise formulation of their metaphysical claims or philosophical implications.

To discuss the mainstream thesis, I will focus on two different accounts which have the advantage of being explicit in several respects. The first is [Parsons, 2008]' framework according to which impure sets are quasi-concrete objects; the second is presented in [Fine, 1992] and [Fine, 2010] and describes impure sets as abstract objects having concrete elements as their parts.

It is important to anticipate that the arguments that I will present against these accounts address the broadest formulation of the mainstream thesis when combined with certain particular philosophical claims, *i. e.* Parsons' and Fine's specific frameworks plus the classic idea of abstractness and concreteness as exclusive and exhaustive categories defined using spatiotemporal location. To be more precise, some of the arguments directly address the mainstream thesis plus the classic definition of abstractness and concreteness, while others apply mainly to the specific accounts at issue.

One last remark is in order. In this chapter, I will present no knock-down argument. I will rather limit myself to showing that we have good reasons to reject the mainstream thesis and that our opponent has the burden of proof to show how to escape such difficulties.

The chapter has four sections. Section 1 is devoted to Parsons' account. More specifically, I will first clarify his notion of quasi-concrete

objects. To do this, I will start by describing the context in which the notion appears, namely Parsons' *Mathematical Structuralism* (1.1.1). Then, I will provide a taxonomy of Parsons' ontology (1.1.2). After that, I will argue that Parsons' notion of quasi-concrete objects can be divided into two related claims: a general Definition and a criterion of Identity (1.1.3). After having clarified the meaning of the philosophical terms that the general definition and the criterion of identity contain, I will point out some problems related to the first (1.1.4) as well as some difficulties raised by the second (1.1.5).

Section 2 is devoted to Fine's idea of impure sets. In particular, I will present Fine's views on membership as parthood (1.2.1) and his idea of impure sets as non-homogeneously abstract objects (1.2.2). Then, I will propose a trilemma that friends of Fine's account have to face (1.2.3). After that, I will present two kinds of objections that can be addressed to the trilemma (1.2.4). The first is based on the concepts of location and concreteness, while the second addresses some principles presupposed by the Trilemma when combined with the notion of part. I will argue, nonetheless, that all these difficulties are surmountable.

Section 3 is devoted to some additional remarks. Some of these concern Parsons' account (1.3.1) and, more specifically, the notion of intrinsicity and that of mereological sum. One last remark is related instead to Fine's account and concerns the possibility of generalizing the trilemma (1.3.2).

Section 4 concludes.

1.1 Parsons' notion of quasi-concrete objects

In [Parsons, 1990], Parsons analyzes the notion of “being a member of” in terms of “being a representation of”. More specifically, he takes impure sets to be abstract objects “embodied by concrete representations”. To understand Parsons' view on impure sets, let me begin by presenting some essential features of his philosophy of mathematics. After that, I will clarify his ontology.

1.1.1 Parsons' *Mathematical Structuralism*

The context in which the mainstream thesis appears in Parsons' work is his *Mathematical Structuralism*.

In [Parsons, 2008], the author investigates what he calls the “mode of being” of mathematical objects. He defends Mathematical Structuralism. Roughly speaking, Mathematical Structuralism claims that what is essential to mathematical objects is the relations constituting the structures to which they belong. Classic example: the number 1 is exhaustively defined by the predicate “being the successor of 0” in the structure of the theory of natural numbers. To use [Burgess, 2008]'s suggestive expression, mathematical objects come “not in single spies, but in battalions”.

The debate on Mathematical Structuralism started with *inter alios* [Benacerraf, 1965], and [Putnam, 1967]. Parsons' version of Mathematical Structuralism is called *Non-Eliminative*. To use [Hellman, 1996]'s

terminology, Non-Eliminative Mathematical Structuralism is a “Structuralism with structures”, which means that what is not eliminated is the postulation of structures as abstract objects.¹

One of the main aims of [Parsons, 2008] is to grasp how we can acquire knowledge about mathematical structures to reply to the issue raised by [Benacerraf, 1973]’s dilemma. Given that our knowledge requires causal connections, how can we grasp mathematical entities having no spatiotemporal location or causal role? To use a Structuralist-flavored formulation, if structures are abstract objects – where the term “abstract” is understood in the classical way of lacking spatiotemporal location and causal power – and if what counts as knowledge is that some entities play an appropriate causal role in the generation of some beliefs, then the question arises of how mathematical structures can take part in any causal interaction whatsoever.

To answer that, Parsons introduces the notion of quasi-concrete objects which is meant to fill the gap between what is abstract and what is concrete. A quasi-concrete object is an abstract object which has an intrinsic relation to something concrete. Parsons’ examples are shapes, sense-qualities, and types. His cases for mathematical objects are impure sets.

Quasi-concrete objects are grasped by intuition. Intuition is, accord-

¹Other supporters of Non-Eliminative Mathematical Structuralism are [Shapiro, 1997], and [Resnik, 1997]. Eliminative Mathematical Structuralism is instead a “Structuralism without structure”: it claims that the grammar of places-are-objects statements does not reflect their underlying logical form. See, for instance, [Chihara, 2005], and [Feferman, 2014]. For an overview of Mathematical Structuralism, see [Reck, 2019].

ing to Parsons, a non-propositional faculty essentially tied with sense perception. More specifically, intuition is a very special kind of seeming that delivers immediate knowledge. Intuition links Mathematical Structuralism to the notion of quasi-concrete objects. Indeed, intuition allows us to "see" objects in structures, or, to use Parsons' terminology, to see something *as* something. For example, we grasp the object x by intuition when we perceive the concrete and perceivable token x as a token on a certain structure type. Similarly, we understand the nature of shapes and sense-qualities when we perceive them in a structure such that they are shapes and sense qualities *of* some objects. For the case that I am interested in, namely impure sets, intuition allows us to perceive these concrete objects as members or elements of some sets.

Parsons' quasi-concrete mathematical objects are "the most elementary mathematical objects". They constitute, therefore, the starting point for all mathematical knowledge. They have a privileged role in solving Benacerraf's dilemma because they make mathematical propositions true. More specifically, since we can perceive the structure of (at least) some perceivable objects thanks to their quasi-concrete nature, then Benacerraf's premise according to which our knowledge involves a causal component is satisfied. Examples of quasi-concrete mathematical objects that play this role are expression-types, sequences, impure sets, and geometrical figures.

However, a tension between Mathematical Structuralism and the notion of quasi-concrete may be pointed out. For the nature of quasi-concrete objects to be described, bare structures are not sufficient. More

specifically, Mathematical Structuralism does not seem appropriate for quasi-concrete objects because the relation between what is abstract and what is concrete is something additional to infrastructural relations. In other words, mathematical structures say nothing about abstractness and concreteness.

I do not aim to explore such difficulty. Since this dissertation aims to clarify the nature of impure sets, I am happy to discuss the plausibility of Parsons' specific formulation of the mainstream thesis. To do this, I will discuss Parsons' notion of quasi-concrete objects which I take to be an independent thesis concerning the metaphysics and the epistemology of impure sets. I will, therefore, also leave aside the more general and huge debate over Mathematical Structuralism.²

1.1.2 Parsons' ontology: Quasi-concrete, pure abstract, and mixed objects

As mentioned, Parsons analyzes the notion of “being a member of” in terms of “being a representation of”. More specifically, Parsons takes impure sets to be abstract objects embodied by concrete *representations*. To be precise, the author claims that

“some abstract objects are distinguished by the fact that they have an intrinsic relation to the concrete”. [Parsons, 2008],

33.

²For more on Parsons' particular kind of Mathematical Structuralism, see *inter alios* [Resnik, 1997], [Burgess, 2008], and [Smith, 2009]. For the more specific issue of Mathematical Structuralism and quasi-concrete objects, see [Smith, 2009].

These are quasi-concrete objects. Impure sets are examples of quasi-concrete objects. Other examples are types, sense-qualities, and shapes.

What distinguishes a quasi-concrete object from another is the concrete representation with which it is associated. To use Parsons' words,

“[a quasi-concrete object] goes with an intrinsic, concrete representation, such that different objects of the kind in question are distinguishable by having different representations”.

[Parsons, 2008], 34.

Hence, for instance, type a differs from type b because they have different tokens: a and b respectively; the shape of my laptop differs from that of my right hand because my laptop and my right hand are different concrete objects; impure sets such as $\{\text{Socrates}\}$ differs from $\{\text{Plato}\}$ because Socrates and Plato are different concrete objects.

Quasi-concrete objects are opposed to "pure abstract" objects that do not have the intrinsic property of being related to something concrete. Examples of pure abstract objects are numbers and sets containing no concrete elements.

However, quasi-concrete and pure abstract objects do not exhaust the domain of objects. There also are what Parsons calls "mixed objects". His example is the set containing von Neumann ordinal ω and his fountain pen, *i. e.* $\{\text{Parsons' fountain pen}, \omega\}$.

Hence, Parsons' ontology is divided into three categories of objects: quasi-concrete, fully abstract, and mixed objects.

Before focusing on quasi-concrete objects, it is worth noting that Parsons' ontology gives rise to some preliminary difficulties. The main one, I argue, concerns the fact that Parsons does not provide definition of mixed objects. It is, therefore, not clear whether for a set to be mixed, it has to necessarily contain at least a pure abstract object. Then, it is not clear whether, for instance, sets such as $\{\text{Socrates}, \{\text{Socrates}\}\}$ and $\{\{\text{Socrates}\}\}$ should count as quasi-concrete or as mixed objects.

More generally, Parsons' metaphysics are only sketched, and literature on this topic is rare. An exception is [Nefdt, 2019] who, nonetheless, does not clarify Parsons' ontology but employs the notion of quasi-concrete objects to solve some difficulties related to specific linguistic issues. Another exception is [Smith, 2009] who clarifies Parsons' notion of representation for quasi-concrete objects but says nothing about the viability of the mainstream thesis.

Concerning the lack of definition of mixed objects, I will show that taking impure sets such as $\{\text{Socrates}, \{\text{Socrates}\}\}$ and $\{\{\text{Socrates}\}\}$ to be mixed or quasi-concrete objects are both problematic.

Concerning the more general issue of Parsons' metaphysics and epistemology, I think that to discuss the viability of his notion of quasi-concrete objects, we need a more rigorous formulation. In what follows, I will start by providing one that I think to be close to Parsons' idea. After that, I will show that this formulation is, nonetheless, problematic.

To be honest, Parsons' idea could be presented in some more successful ways. After having pointed out the flaws of what I think to be the

most faithful formulation, I will present other more compelling ways of spelling out his principles. I will argue, however, that these new charitable formulations take a big departure from Parsons' original thoughts, and cannot, therefore, legitimately save his account of impure sets.

1.1.3 Parsons' General Definition and Criterion of Identity

The most plausible interpretation of Parsons' notion of quasi-concrete objects can be formulated using two different but strictly related claims:

General Definition: x is a quasi-concrete object *iff* 1) x is abstract and, 2) x has the intrinsic property of "being related to something concrete".

Criterion of Identity: x and y are the same quasi-concrete object *iff* for all z , z represents x *iff* z represents y .

One of the elusive features of Parsons' account is his starting point. Indeed, it is not straightforward to understand what it is to be an intrinsic property and what it is to be a concrete representation of something.

In what follows, I will, therefore, try to clarify the notions of intrinsic property and that of concrete representation.

Intrinsic properties

Up to now, we have taken for granted that, for instance, shapes have the intrinsic property of being related to the concrete objects having those shapes; sense-qualities to the concrete objects having those qualities; types to their tokens; and sets containing concrete elements to those elements. But what it is to have an intrinsic property in the first place?

Marshall offers a preliminary characterization:

“We have some of our properties purely in virtue of the way we are. (Our mass is an example.) We have other properties in virtue of the way we interact with the world. (Our weight is an example.) The former are intrinsic properties, the latter are extrinsic properties. This seems to be an intuitive enough distinction to grasp [. . .]. Unfortunately, when we look more closely at the intuitive distinction, we find reason to suspect that it conflates a few related distinctions, and that each of these distinctions is somewhat resistant to analysis” [Marshall, 2002].

In this dissertation, I do not aim to enter into this huge and complex debate. I will just limit myself to claiming that Parsons seems to embrace Marshall’s mentioned intuitive sense of intrinsicity for his ontology of quasi-concrete objects: Quasi-concrete objects have the property of being related to something concrete in virtue of the way they are.

Concrete representation

To complete the picture, it is important now to understand what it is to be a concrete representation of something. It goes without saying, the notion of representation has a remarkable philosophical pedigree. Just think about the Early Modern conceptions of representation³ or to its contemporary development in philosophy of mind.⁴ I argue that Parsons adopts a peculiar meaning that largely differs from the tradition. According to him, quasi-concrete objects are “determined by their concrete embodiments” [Parsons, 2008], 33; they are “directly ‘represented’ or ‘instantiated’ in the concrete” [Parsons, 1990], 304. Shortly, representations are embodiments or instantiations, and not pictures or propositions.⁵ Concrete representations determine the objects with which they are related. Shapes, sense-qualities, and types are “wholly present” (or “completely instantiated”) in the concrete object to which they are related, *e. g.* a particular triangle is wholly triangular; the inscription *a* entirely represents the type *a*.

To summarize, quasi-concrete objects are the kind of objects which are abstract and which have, by their nature, the property of “being determined by their concrete embodiments”. Then, it is in virtue of the way in which shapes, sense-qualities, types, and impure sets are that are determined by their concrete embodiments.

³In particular, I refer to Descartes’, Locke’s, Hume’s, and Kant’s ideas of representation.

⁴For an overview, see [Pitt, 2000].

⁵This is the sense in which, in [Smith, 2009], the author replaces the label “representation” with that of “representative”.

One last clarification is in order. It could be asked why I have put the story in terms of intrinsic properties, and why I have not used the notion of intrinsic relations instead, as Parsons does. I think that the reason becomes clear by considering intrinsicity as having the aforementioned intuitive flavor. Indeed, if quasi-concrete objects have intrinsic relations with the concrete, then both *relata* (the quasi-concrete and the concrete) have to be intrinsically related to one another. But what Parsons wants seems asymmetric: it is not in the nature of the concrete to be related to the quasi-concrete but the other way around. Intrinsic properties display this asymmetry.

Now that the terms contained in **General Definition** and **Criterion of Identity** are somehow clarified, let me move on discussing some problems of these two claims.

1.1.4 Some problems with Criterion of Identity

Categories of objects

One difficulty arises with **Criterion of Identity**. Indeed, **Criterion of Identity** is not sufficient to distinguish between different objects of different “kinds”.

As an example, {Socrates} and Socrates’ particular shape are plausibly both abstract and determined by all and only the same concrete object, namely Socrates. Yet, they are not the same object. Hence, **Criterion of Identity** does not apply to all quasi-concrete objects taken

together but has to be “localized” to each “kind” of quasi-concrete objects (*i. e.* types, shapes, sense-qualities, and impure sets).

Nonetheless, **Criterion of Identity** does not seem to be sufficient to distinguish between different impure sets either. $\{\text{Socrates}\}$ and $\{\text{Socrates}, \{\text{Socrates}\}\}$ are determined by the same concrete object, namely Socrates. Yet, they are different sets.

Here, it can be objected that these sets are not quasi-concrete but mixed objects. As mentioned, Parsons provides no definition of mixed objects but just mentions an example, *i. e.* $\{\text{Parsons' fountain pen}, \omega\}$. By the way, both alternatives are problematic. As mentioned, if sets such as $\{\text{Socrates}, \{\text{Socrates}\}\}$ and $\{\{\text{Socrates}\}\}$ are quasi-concrete objects, then **Criterion of Identity** fails; if they are not, Parsons' ontology can be accused of *ad hoc*-ness. What is the philosophical relevance of categorically distinguishing between sets such as $\{\text{Socrates}, \{\text{Socrates}\}\}$ and $\{\text{Socrates}\}$? The relation between concrete and abstract objects seems homogeneous: in both cases, Socrates is a concrete object belonging to an abstract set.

To sum up, even localizing **Criterion of Identity** to each kind of quasi-concrete object, Parsons' treatment of impure sets such as $\{\text{Socrates}, \{\text{Socrates}\}\}$ is not clear and all the options allowed by his account seem problematic. And, since Parsons' framework aims to provide a picture of how abstract and concrete entities are related, these issues are particularly heavy.

Representation and sets

Even conceding both that **Criterion of Identity** can be localized and that we have no problem with sets such as {Socrates, {Socrates}}, another issue arises.

Elements of impure sets do not represent the set in the same way in which tokens represent their types, or in which a concrete object represents a sense-quality or a shape. Example: type a is wholly instantiated in token a , while {Socrates, Plato} is not wholly instantiated in Socrates or Plato.

It may be argued that this fact says nothing about the good standing of the notion of representation for impure sets. However, it is clear that, according to Parsons, the way in which types, shapes, and sense-qualities are represented by the concrete is more adequate than that of impure sets. This happens because the fact that impure sets are not wholly instantiated in each concrete object in their transitive closure asks for a process of individuation that is more complex than that of other quasi-concrete objects. To better see the problem, consider Parsons' example of the set consisting of the prime minister of all states of the European Union on January 1, 1999. He ironically asks why we should accept one, says Gerhard Schröder, as representing it (see [Parsons, 2008], 25).

His solution is to claim that impure sets are not represented by each element taken separately but by their mereological sums. Example: the set of all human beings is not represented by Parsons or by Socrates or by someone else but by the mereological sum of all human beings. However,

the same mereological sums can represent different sets. Parsons takes this fact as self-evident and provides no example. I think that this issue deserves to be clarified. In what follows, I will present and reject a case that could unduly be thought of as meeting such a requirement. Then, I will propose two good candidates.

Consider the Quinean case of the set containing all the things having a heart and the set containing all the things having a kidney. Since all the individuals having a heart also have a kidney and vice versa, then the same mereological sum of the members of those sets can be taken to represent both the set of things having a heart and the set of things having a kidney. But, by **Extensionality**, “these sets are” the very same set.⁶ The mereological sum of its members represents, therefore, this unique set. Hence, the Quinean set is not a case in which the same mereological sum represents different sets.

Consider now the set containing the member states of the European Union. This set has cardinality 27. As Parsons would suggest, such a set is a quasi-concrete object represented by the mereological sum of the 27 member states. So far, so good. What happens with the set containing the mereological sum of the 27 member states? Plausibly, this set is different from the set containing the member states of the European Union in that it has cardinality 1. However, both sets are represented by the same concrete multiplicity, *i. e.* the mereological sum of 27 member states of the European Union.

⁶By **Extensionality**, I mean the standard principle of set theory according to which X and Y are the same set *iff* they have all and only the same elements.

Perhaps, this case can be ruled out maintaining that mereological sums are abstract objects, and so that sets containing mereological sums are not quasi-concrete but abstract or mixed objects. But if mereological sums are abstract objects, and, as Parsons proposes, impure sets are represented by mereological sums, then impure sets are represented by abstract objects, and so they cannot be quasi-concrete objects.

Even leaving this issue aside, we have other cases of mereological sums representing different sets. Consider the set containing Macron and Draghi. This set has cardinality 2 and is represented by the mereological sum of Macron and Draghi. Now, consider the set containing the same Macron and Draghi, this time, though, divided into a certain number of parts. Such a set will contain all Macron's parts plus all Draghi's parts, and so will plausibly have more than 2 as cardinality. The two sets are different but represented by the same mereological sum, *i. e.* the mereological sum of Macron and Draghi. Given these cases, how can mereological sums adequately represent sets if they sometimes represent different sets?

To reply, Parsons adopts what I call the *bullet-biting strategy*. This strategy can be formulated as a *Tu Quoque*: it is true that the same mereological sum can represent different sets but it is also true that such a phenomenon arises for types as well. To be honest, I do not think that this is true. To decide about this issue we have, of course, to first clarify the meaning of token-type which is, nonetheless, not an easy work.⁷ I

⁷To get an idea of the complexity of the topic, see, for instance, [Wetzel, 2006] and [Wetzel, 2009].

will not get into this debate here. What it is sure is that even if types suffer from the same problem of sets, Parsons' **Criterion of Identity** would still not apply. Another device showing how both sets and types can be distinguished from one another is, therefore, required.

After proposing the bullet-biting strategy, Parsons adds that the case of sets is *sui generis* because of their nature necessarily involving many-one relations. Of course, the point is well taken – even though we could object that many-one relations do not hold for every set (the problematic cases would be singletons and the empty set). Nevertheless, this *sui generis* status does not make Parsons' explanation for quasi-concrete more adequate: we still cannot distinguish sets by their concrete representations.

To summarize, Parsons' idea of representation seems inadequate for the case of impure sets. In other words, it is not clear how concrete representations are supposed to provide a criterion of identity for sets such as the set containing Macron and Draghi and the set containing Macron's parts and Draghi's parts (or the set containing the member states of the European Union and the set containing the mereological sum of member state of the European Union). And the same problem seems to appear in cases such as that of {Socrates} and {{Socrates}}.

Alternative formulations of Criterion of Identity

It can be argued that the problems with Parsons' **Criterion of Identity** can be dismissed by modifying it. In what follows, I will present and

discuss some proposals by Correia (2023, personal communication). The first proposal goes as follows:

Criterion of Identity*: x is the same quasi-concrete object as y *iff*, for all relations R and all concrete objects C , x intrinsically bears R to C *iff* y intrinsically bears R to C .

Roughly speaking, **Criterion of Identity*** claims that quasi-concrete objects are identical if they have the very same relations with the same concrete things.

Criterion of Identity* allows us to distinguish between $\{\text{Socrates}\}$ and $\{\text{Socrates}, \{\text{Socrates}\}\}$ in that $\{\text{Socrates}\}$ is related to Socrates by having him as its sole member, while $\{\text{Socrates}, \{\text{Socrates}\}\}$ is related to Socrates by the relation “being a set with two members one of those is Socrates and the other is $\{\text{Socrates}\}$ ”. The general form of these relations can be spelled out as follows:

$$x R y \text{ iff } x = \{y, \{y\}\}$$

$$x R y \text{ iff } x = \{y\}$$

Moreover, **Criterion of Identity*** provides a simpler explanation of why $\{\text{Socrates}\}$ and $\{\text{Socrates}, \{\text{Socrates}\}\}$ are distinct. Let R be the relation defined as follows:

$x R y$ iff for some set S , x contains S as a member, and S contains y as a member.

Then, $\{\text{Socrates}, \{\text{Socrates}\}\}$ has intrinsically the property of R -ing Socrates, while $\{\text{Socrates}\}$ does not have this property.

Also, **Criterion of Identity*** allows us to distinguish between different kinds of quasi-concrete objects. Consider again the case of Socrates' particular shape and $\{\text{Socrates}\}$. Given **Criterion of Identity***, they are different objects: Socrates' shape is related to Socrates by the relation of "being Socrates' shape", while $\{\text{Socrates}\}$ is related to Socrates by the relation "having Socrates as a unique member".

Another solution that looks attractive but that Parsons does not embrace consists in taking the plurality of elements to be representative of the sets in question. Impure sets are represented by pluralities of things. This would solve the problems that Parsons raises: The plurality of the member states of the European Union is different from that of their mereological sum; the plurality of Macron and Draghi is different from that of Draghi's parts and Macron's parts.

Moreover, there is a way in which we can actually provide a good **Criterion of Identity** by using the plurality of concrete representations plus the relations between those representations and the quasi-concrete objects they represent. The strategy exploits the notion of tree of determination.

Say that $\{\text{Socrates}, \{\text{Socrates}\}\}$ is determined by the plurality "Socrates,

{Socrates}”. The first member of the plurality is concrete and therefore determined by nothing. The second member is determined by Socrates. Thus, {Socrates, {Socrates}} gives rise to a ”tree of determination”, with {Socrates, {Socrates}} at its root. Then, we have two children nodes: one occupied by Socrates, and the other by {Socrates}. This latter node also has a child, occupied by Socrates. Socrates occupies all the leaves of the tree, and so it can be said to ”mediately” determine {Socrates, {Socrates}}. On that view, {Socrates, {Socrates}} and {Socrates} are different in terms of determination: they have different determination trees.

All these suggestions show that Parsons’ **Criterion of Identity** for impure sets can be successfully modified or interpreted. I argue, nonetheless, that these modifications imply a big departure from Parsons’ idea so that if it is true that we can somehow “save” his account of impure sets, it is also true that these accommodations are incompatible with his original idea of abstract objects as determined by the concrete. For instance, the fact that **Criterion of Identity*** successfully replaces **Criterion of Identity** could be taken as suggesting that concrete objects are, so to say, ”not enough” to individuate sets. Relations are required. But relations are classically taken to be abstract objects. The same problem can be raised for trees of determination that are structures, and structures are arguably abstract. Same for pluralities. And otherwise, if we hold that pluralities are concrete, then it is hard to distinguish sets from universals or aggregates. And, since according to Parsons’ ontology, impure sets are

supposed to ground mathematics, they better have to maintain a status that differs from that of other entities that do not ground mathematics.⁸

To summarize, to individuate impure sets, we need "something more" than their concrete members or elements in their transitive closure. We need, for instance, structures, relations, collections of things, trees determinations, and so on. All objects that can arguably taken to be abstract.

At this point, the opponent can state a new objection. This objection is suggested by Correia (2023, personal communication), and goes as follows: Consider the set $\{\text{Socrates}\}$ and the set $\{\text{Socrates}, \{\text{Socrates}\}\}$. Now, consider the following linguistic concrete inscriptions written down on a blackboard:

$\{\text{Socrates}\}$

$\{\text{Socrates}, \{\text{Socrates}\}\}$

There is a clear sense in which the first one represents (only) $\{\text{Socrates}\}$, and the second one represents (only) $\{\text{Socrates}, \{\text{Socrates}\}\}$. Hence, objects such as structures or relations are not required to distinguish between quasi-concrete objects. Concrete inscriptions suffice.

This solution, however, puts the cart before the horses. It seems to me that it is not the case that, since $\{\text{Socrates}\}$ and $\{\text{Socrates}, \{\text{Socrates}\}\}$ have different inscriptions, they are different objects. It is the other

⁸For the importance of the difference between sets and universal, see [Lewis, 1989].

way around: since they are different objects, they have different inscriptions. Distinguishing between different impure sets by their different inscriptions seem to also imply a set of objections related to the fact that sets can be transfinite, while concrete inscriptions are countable. Moreover, it seems clear that, according to Parsons, what is representative are concrete objects of the kind of Socrates, European members, and not inscriptions.

Once having ruled out the inscriptionalist objection, we can go back to evaluating the meaning of the reformulations of **Criterion of Identity**. I think that the necessity of reformulating **Criterion of Identity** says something important concerning the nature of impure sets as well as the way in which they are grasped. To anticipate a bit, I think that the fact that **Criterion of Identity** for impure sets is not easily reducible to the concrete follows from the fact that impure sets are intimately related to concepts. This is a point that is important to bear in mind both in considering Maddy's account of impure sets as fully concrete objects in the next chapter, as well as for the discussion of my own position in Chapter 3.

Before proceeding in this direction, I will point out another difficulty of Parsons' account. This difficulty directly affects the mainstream thesis.

1.1.5 Some problems with General Definition: Slippery Slope

According to Parsons' idea, concrete representations are concrete objects. In this section, I will argue that this fact can be at least controversial.

To do this, I will start by presenting a problem that some opponents of Parsons' account have pointed out. According to Parsons' ontology, numbers are pure abstract objects. Then, he accepts that numbers are represented by sets or sequences of those numbers. But — the opponents claim — if these sequences or sets are concrete objects, numbers are not pure abstract objects but quasi-concrete objects.

Parsons replies that what is concrete are not the sequences or sets to which numbers are associated but the terms of such sequences or sets. Sets or sequences are, therefore, quasi-concrete objects in that they are determined by their concrete representations. Thus, numbers have no intrinsic relations with concrete objects. They can at most be classified as “quasi-quasi-concrete objects”. In other words, their “concreteness”, if any, is second order in that it is achieved in two steps.⁹

Parsons corroborates this reply by using what I will call **Fregean Argument**. **Fregean Argument** can be formulated as follows:

- 1) Numbers do not apply to objects;

⁹To be more precise, Parsons claims that the notion of quasi-quasi-concrete objects does not display the generality which characterizes numbers. Indeed, numbers number anything, including purely abstract objects. The mediate relation of numbers to concrete terms of sequences or concrete members of sets is, therefore, even not indispensable. Numbers are, therefore, better understood to be pure abstract objects. However, this is irrelevant to the scope of my argument.

2)Numbers apply to sets.

To motivate 1) and 2), Frege imagines a situation in which someone points at something in front of them and asks someone else to count what they see. Frege supposes that in front of the two subjects, there is a tree. How can be sure that what one has to count is the tree, and not, for instance, the foliage, the branches, or, more generally, all the parts of that tree? Basically, we cannot count physical objects. We can instead count sets because they are associated with some concepts providing a number property.¹⁰

Fregean Argument applies to numbers and sequences. To see why, consider Parsons' example of the number 3 and the concrete representation III. For III to be a representation of the number 3, it would already be seen as a set or a sequence of strokes, rather than, for instance, as a unique grid. III is, therefore, quasi-concrete because it involves conceptual choices about its nature.

Now, I argue that **Fregean Argument** associated with Parsons' ontology ironically suggests a revenge problem that questions the idea that concrete representations are fully concrete objects: How can I know that, for instance, Socrates is a unique object and not a certain number of molecules or atoms? To claim that Socrates is an individual, I plausibly need to consider him in a certain way. But if this is the case, then according to Parsons' intuition, Socrates would be a quasi-concrete object that is abstract in the sense of lacking spatio-temporal locations.

¹⁰See [Frege, 1884].

This argument shows that whoever wants to defend that impure sets are abstract objects having concrete elements has to block **Slippery Slope** that can be triggered using **Fregean Argument**. Of course, there can be options. The natural move is to throw out **Fregean Argument**. **Fregean Argument** seems, nonetheless, central to the notion of set. Another option is to reject the idea that if something is conceptualized, then it is abstract. Concepts are classically considered abstract entities.¹¹ Of course, this is not uncontroversial, but someone who desires to argue against this fact has the burden of proof. Moreover, Parsons' argument for numbers presupposes the implication above.

Another option consists of arguing that the fact that concepts are abstract is compatible with the view that what falls under them is, at least in some cases, not abstract. Following the example, we would have Socrates' abstract concept and Socrates concrete object falling under that concept. Then, what participates in the definition of {Socrates} as a quasi-concrete is Socrates concrete object, and not Socrates abstract concept. However, once adopted **Fregean Argument** plus Parsons' application to sequences as well as **Slippery Slope**, we are called to explain and justify the existence of these concrete *unconceptualized* objects. Once again, the burden of proof seems to be on friends of Parsons' quasi-concrete objects and, more generally, on friends of the mainstream thesis.

¹¹See, for instance, [Peacocke, 1992] and [Zalta, 2001].

Conclusions on Parsons' quasi-concrete objects

Given all these considerations, I conclude that Parsons' account for impure sets is problematic. To sum up, this happens for two different reasons. The first is that, as I have shown in Section 1.3, his **Criterion of Identity** has to be revised. Nonetheless, it is not sure that the revision goes without problems. The second is that, as I have shown in Section 1.4, the ontological dignity of quasi-concrete objects is not compatible with **Fregean Argument**, and so that **General Definition** is misguided.

Incidentally, it has to be noticed that Parsons' claim that sequences of concrete objects are quasi-concrete objects increases the viability of my objection concerning his **Criterion of Identity**. In Section 1.1.4, I have said that **Criterion of Identity** can be successfully replaced by other formulations involving pluralities of objects, relations, structures, and so on. However, following Parsons' adoption of **Fregean Argument** for sequences, it is plausible to claim that such objects also are quasi-concrete rather than concrete.

Up to here, I have analyzed Parsons' formulation of the mainstream thesis. Some of my arguments generalize, nonetheless, to any account of impure sets as abstract objects instantiated by concrete elements (see, for instance, [Lowe, 1995]).

Let me move on now to the second part of the chapter by considering Fine's suggestions for impure sets.

1.2 Impure sets and membership: Fine's account

1.2.1 Fine's operational account

Contrary to Parsons, Fine has no explicit ontological account for impure sets. Nonetheless, we can find some interesting claims in [Fine, 1992] and [Fine, 2010].

In [Fine, 1992], he suggests that impure sets are cases of heterogeneous Aristotelian composition:

“There is the question of whether non-physical objects can have physical objects as their matter. I am inclined to think that the answer is yes [...]: for the members of a set, which is non-physical, may be physical”. [Fine, 1992], 36, footnote 3.

I argue that the case in which the members of a set are physical is the case of impure sets. Hence, Fine seems to implicitly embrace the mainstream thesis: objects such as {Socrates} and {Plato} are abstract; objects such as Socrates and Plato are concrete.¹²

Also, he is committed to the idea that the relation that is in play between sets and members is a relation of composition. This thesis is

¹²Perhaps, it can be objected that while it is true that a physical object is concrete because for it to be subjected to physical laws, it has to spatiotemporally exist, it is not equally sure that something non-physical is something abstract. Indeed, we could have concrete non-physical things, such as Cartesian thinking substances, or miracles. However, these are really specific cases and do not concern the objects we are interested in, namely impure sets.

developed by [Fine, 2010] in which the author proposes what he calls an *operational account of parts-whole*. The idea is to outline a general framework dealing with questions of parts-whole in which the operation of composition is primitive. In this sense, the paper can be seen as an extension of [Fine, 1999] in which he argues that standard mereology is not adequate to account for composite objects. [Fine, 2010]’s account aims to apply to all cases of composition. For the scope of our discussion, I will focus on sets and, in particular, on impure sets.

For the account to be applied to sets, the uncontroversial presupposition is, of course, that membership is a case of parthood. Classically, this is not the case. The natural argument against this idea is that parthood is transitive while membership is not. But Fine’s interest is not to focus on the stipulated notion of part. He aims to display its intuitive appeal.¹³ There are, he argues, a lot of ways in which it is plausible to claim that something is a part of something else. Following his examples, a pint is a part of a gallon of milk, the head soldier is part of the soldier (p. 561). Hence, he concludes that “being a part” applies only to concrete objects is a “philosopher’s myth”. Cases of parthood not including concrete objects are sentences types being composed of words types and sets being composed of members [Fine, 2010]. This sense of being a part has to do with composition.

As Fine argues,

“There is a strong *prima facie* case in favor of taking the

¹³The main features of parthood are given by classic axiomatic mereology according to which part is reflexive, antisymmetric, and transitive. See [Varzi, 2003].

members of a set to be parts. For we do indeed talk of a set containing its members and of its being composed or being built up from its members”. [Fine, 2010], 563.

Given all these considerations, we can conclude that, in Fine’s account, impure sets are abstract objects having concrete members as their parts.

Fine’s view on parts and whole is discussed in recent literature. What is at issue is, above all, whether his alternative is superior to standard mereology (see, for instance, [Koslicki, 2007] and Fine’s reply in [Fine, 2007]). Another issue related to Fine’s mereological claims that has received a lot of attention is the problem of knowing whether wholes are something over and above their parts (see, for instance, [Cotnoir and Baxter, 2013]). In what follows, I will address Fine’s account of parts and whole focusing on the main problem of the chapter, namely the mainstream thesis. The results generalize to all accounts embracing the mainstream thesis plus membership as composition.

1.2.2 Impure sets are non-homogeneously abstract objects

Generally speaking, to embrace the classic conception of abstractness and concreteness plus the mainstream thesis, means to embrace that

- 1) Impure sets are not spatiotemporally located;

And that

2) Impure sets necessarily have as their members things that exist in space and time.¹⁴

Now, combining 1) and 2) plus the idea that membership is a relation of composition, we have that

3) Impure sets are non-homogeneously abstract objects, (where an object is homogeneous when its constituents each have the same ontological nature as the object itself, while it is non-homogenous if otherwise).

And so that

3*) Impure sets lack spatiotemporal location but necessarily have spatiotemporally located concrete constituents (Call this **NHA**).

Fine does not explicitly embrace 3*) but, generally speaking, friends of the mainstream thesis take 3*) to be unproblematic. In [Lowe, 1995], the author claims, for instance, that

”a set of concrete objects necessarily has as its members things that exist in space and time, and yet that seems to be no good reason for saying that such a set itself partici-

¹⁴That this is a necessity is given by the fact that sets are defined by their members.

pates in spatiotemporal relations.” [Lowe, 1995], 519.

In what follows, I will point out what I think to be a good reason to reject 3*).

1.2.3 Trilemma on location

Consider the following plausible glosses:

Weak Location: x is weakly located at r iff r is not completely free of x (see [Parsons, 2007], 203).

Exact Location: x is exactly located at r iff x has exactly the same shape and size as r and stands in all the same spatiotemporal relations to other entities as does r (see [Gilmore, 2013], §2.1).¹⁵

Constituency: If y is a constituent of x and y is exactly located at r , then x is weakly located at r – where “being a constituent of” is a generic relation that includes “being a part of”, “being a member of”, “being one of”, and so on).

If we accept **Weak Location**, **Exact Location**, and **Constituency**, then the following **Trilemma** holds:

¹⁵Given this definition of **Exact location**, notice that **Weak location** can be formulated using **Exact location** and overlap in the following way: x is weakly located at r iff x has an exact location that overlaps y (see [Gilmore, 2013]).

a) Sets are non-homogeneously abstract objects (from **NHA** account);

b) Concrete objects have spatiotemporal location (from **Definition of concrete objects**);

c) Every object is at least weakly located where its concrete constituents are exactly located (from **Constituency**).¹⁶

Now, given that a), b), and c) are jointly inconsistent claims, to maintain a), friends of **NHA** have to reject either b) or c). But b) seems uncontroversial, and c) seems a highly plausible principle.¹⁷

To sum up, if it is the case that b) and c) are difficult to reject, then **Trilemma** questions the idea of impure sets as abstract objects having spatiotemporally located components by claiming that since these objects have at least one located component, then they are “somehow” located. This means that they are either concrete or they belong to a third category of objects between abstract and concrete. In this chapter, I have discussed the mainstream thesis combined with the classical idea of abstractness and concreteness as mutually exclusive and exhaustive categories. As mentioned in the Introduction, other conceptions are possible. However, friends of the mainstream thesis owe us an explanation

¹⁶Notice also that c) can be reformulated using only **Weak location**, *i. e.* every object is at least weakly located where its concrete constituents are weakly located.

¹⁷To simplify the argument, we could also take the definition of **Weak location** by **Exact location** and overlap which already entails that if a thing is weakly located somewhere, then it is exactly located somewhere. This is what [Gilmore, 2013] calls **Exactness**.

of how they can be successfully adopted.¹⁸

1.2.4 Against Trilemma on location

The silent assumption on location and concreteness

It can be objected that **Trilemma** is based on the controversial assumption that concrete objects display **Exact Location**. A standard counterexample to this assumption is given by a gunky space occupied by things that are not gunky, *e. g.* point-like material objects (see, for instance, [Parsons, 2007] and [Gilmore, 2013]). This is nonetheless a very specific metaphysical assumption on space and objects that necessitates being argued, while **Exact Location** for concrete objects seems more straightforward.

Someone who is not convinced by the intuitiveness of **Exact Location** can still reformulate c) (and **Constituency**) in terms of **Weak location** as suggested in footnote 15. However, the opponent can still insist on arguing that **Weak Location** is not sufficient for concreteness and that for something to be concrete, it has to be exactly located. Examples of objects that are weakly located but intuitively not concrete are certain mereological sums. For instance, the mereological sum of me and the number 5 is not concrete and is weakly located where I am located.

However, these cases of mereological sums that the opponent uses as counterexamples to the definition of concreteness by **Weak Location** can also be used as counterexamples to the definition of concreteness

¹⁸I would like to thank Claudio Calosi for discussing this argument with me.

by **Exact Location**. Take again the case of mereological sums. It can be argued that such sums are exactly located where their components are located. Then, the sum of me and the number 5 is exactly located where I and the number 5 are exactly located. Hence, plausibly, since the number 5 is not located, then the sum is exactly located where I am located.

This shows that the problem that the opponent of the **Trilemma** points out concerns concreteness in general, and not concreteness when defined by **Weak Location**.¹⁹

Some further objections

Some further objections can be raised against **Trilemma**. For instance, it may be objected that it does not take into account that there are different ways of being a part. This fact — which has a clear Aristotelian pedigree — is pointed out by [Fine, 2010] who claims that

“Even though a soldier is part of a regiment and his head is part of the soldier, it is odd to say that his head is part of the regiment. Likewise, even though a gallon of milk may be part of the singleton of the gallon and a pint of milk may be a part of the gallon, it is odd to say that the pint is a part of the singleton”. [Fine, 2010], 579.

Such an oddity, Fine claims, follows from the fact that there are a lot of ways of being a part of something. He also suggests that this

¹⁹I would like to thank Fabrice Correia for raising these objections.

should not lead one to claim that the relevant objects are not parts. Perhaps, different parts suit different principles. Then, the objection goes: The way in which concrete members are part of sets does not satisfy **Constituency**. However, I think that someone who desires to argue that impure sets are *sui generis* cases has the burden of proof of showing why the principle does not hold and which relation applies instead.

Another similar but perhaps less *ad hoc* strategy is to generalize the argument claiming that **Constituency** holds only between concrete *relata*. Then, the case of non-homogeneously abstract object does not meet this requirement. Someone can claim, for instance, that a committee is an abstract object whose members are concrete human beings but that the committee is not located where these human beings are located. I am under the impression that this intuition can be resisted simply by arguing that there is evidence in favor of the idea that the committee is a concrete object located where its members are located: the committee often has a headquarter, it meets in a specific room at a specific time, it has a causal role, and so on. Of course, the opponent can come up with other counterexamples. Yet, once again, someone who desires to argue that non-homogeneously abstract objects do not suit **Constituency** has the burden of proof to show why and which relation applies instead.

To escape **Trilemma**, there also are bullet-biting strategies. For instance, it can be argued that impure sets occupy spacetime but in a derivative manner. This option is presented, for instance, in [Falguera, 2001]:

“{Peter, Paul} occupies a location in virtue of the fact that its concrete elements, Peter and Paul, together occupy that location. The set does not occupy the location in its own right.”

However, someone who regards ‘non-spatiality’ as a definitional feature of abstract objects but who allows that some abstract objects may have non-trivial spatial properties, owes us an account of the distinctive relation to spacetime that these objects are supposed to display.

Another option is to claim that, following [Black, 1971]’s view, since sets are not entities, it does not make any sense to ask where they are located in the first place. Once again, though, the mainstream thesis wants sets to be something, namely abstract objects having concrete elements.

At this point, someone who wants to argue against **Trilemma** may change the definition of abstractness to include causality. Then, she may concede that sets are located but still abstract in that they lack a causal role. As I have said, I take causality to be problematic and less fundamental than spatio-temporality. But even conceding to the opponent that abstractness is defined using causality, **Trilemma** can be iterated in the following way:

Constituency* If y is a constituent of x and y is causally efficacious, then x is weakly efficacious — where “being a constituent of” is a generic relation which includes “being a part of”, “being a member of”, “being

one of", and so on.²⁰

Then, we can construct the following three jointly inconsistent claims:

- a) Sets are non-homogeneously abstract objects (from **NHA** account);
- b) Concrete objects are causally efficacious (from **Definition of concrete objects**);
- c) Every object is weakly efficacious when at least one of its constituents is causally efficacious (from **Constituency***).

Another way to resist the argument is, of course, to reject, *contra* Fine, the intuitive claim according to which concrete members are constituents of abstract impure sets. Composition is not frequently used for sets. Fine is an exception. Nonetheless, it is plausible to think that someone who claims that, for instance, {Socrates} is an impure set having Socrates as a member is also inclined to maintain that {Socrates} has Socrates as a constituent. Then, **Trilemma** goes.

Conclusions on Fine's impure sets

To summarize, Fine's account of impure sets is subjected to **Trilemma**.

Also, **Trilemma** generalizes to all accounts embracing the mainstream

²⁰Yet, it is not entirely clear how to define "being weakly efficacious" in the first place. Perhaps, it can be argued that x is weakly efficacious in relation to the event y when at least a constituent of x appears in the causal chain related to y .

thesis, the classic conception of abstractness and concreteness, plus membership as parthood. As mentioned, there are ways to argue against **Trilemma**. Yet, there also are ways to resist these arguments. Hence, friends of impure sets as abstract objects having concrete parts are called to clarify their tenets.

1.3 Additional remarks

1.3.1 Some additional remarks about Parsons' ontology

Intrinsicity

In this chapter, I have conceded to Parsons that objects such as {Socrates, Plato} and {Socrates} are intrinsically related to something concrete.

However, it could be objected that this is not the case if we adopt the idea that for a property to be intrinsic, it has to only concern the subject that exemplifies that property (see [Marshall, 2002]). To see why, take the case of me and my hand. The relation between me and my hand is intrinsic in that the hand is a part of me. However, it is not equally sure that the same relation holds between {Socrates, Plato} and Socrates and Plato. It seems that, for this to be the case, we have to assume that {Socrates, Plato} is composed of Socrates and Plato. But then **Trilemma** that I have applied to Fine's account extends to Parsons' framework.

Mereological sums

In Section 1.2.2, I have proposed two cases of different sets represented by the same mereological sum. The first is the case of {Macron, Draghi} compared to the set containing all the parts of Macron plus all the parts of Draghi “divided” in a certain way; the second is the case of the set containing all the members of the European Union compared to the set containing the mereological sum of that set. Then, I have argued that, if we take mereological sums to be abstract objects, then the second case does not meet Parsons’ requirement for quasi-concrete objects. The set containing the mereological sum of the set containing all the members of the European Union is an abstract object, or at most, a mixed object.

Now, it may be argued that the first case is not a good example either. This fact does not happen for the same reasons as for the second case. We can indeed go hyperintentional and argue that the mereological sum of Macron and Draghi and the mereological sum of all the parts of Macron plus all the parts of Draghi “divided” in a certain way are not the same mereological sum. The first is the mereological sum of two individuals, while the second is the mereological sum of the parts of those individuals, and these are two different concrete representations.

This solution seems incompatible with extensional mereology according to which wholes are identical to the sum of their parts. Yet, non-extensional mereologies which are sensitive to structures and arrangement of parts are available. Examples are [Garbacz, 2017] and [Sattig, 2019]. Alternatively, it can be argued that the objection does not hold in

the first place. Marcon and Draghi are identical to Macron's and Draghi's parts but we have different ways of representing them, and these ways are good candidates to account for sets.

1.3.2 Some additional remarks about Fine's account

In this chapter, I have focused on impure sets but **Trilemma** generalizes to all cases of non-homogeneously abstract objects (*e. g.* objects such as committees, events, and so on). Moreover, **Trilemma** can also be reversed so as to be applied to non-homogeneously concrete objects, if any. Examples of non-homogeneously concrete objects could be human beings, tables, and, more generally, all ordinary objects conceived as spatiotemporal entities having abstract components such as, the way in which their parts are arranged. **Trilemma*** against these objects would start with **Constituency***.

Constituency* If y is a constituent of x and y is not located, then x is not weakly located.

Classic example: Suppose that a given statue has a particular form. If the form of the statue is a constituent of the statue, and the form is not located, then the statue is not located. But this would be quite unintuitive.

Of course, this cannot be considered a know-down argument against non-homogeneously concrete objects. Nonetheless, together with **Trilemma**

on location, it increases the sensation that, generally speaking, the link between abstract and concrete is anything but straightforward.

In this chapter, I have focused on impure sets. I conclude by saying that, as it is clear from this last observation, the interest in discussing the mainstream thesis goes well beyond the problem of impure sets in that it concerns the more general issue of mixed objects which has a lot of impact on contemporary analytic philosophy (see, for instance, [Pedersen, 2022]).

1.4 Conclusions

In this chapter, I have shown that the mainstream thesis combined with the classic idea of abstractness and concreteness can be resisted. In particular, I have focused on Parsons' and Fine's particular formulations.

More specifically, in Section 1, I have clarified Parsons' notion of quasi-concrete arguing that it is divided into two claims: **General Definition** and **Criterion of Identity**. Then, I have pointed out some problems with these two claims. In particular, I have argued that **Criterion of Identity** is not appropriate to account for impure sets and that **General Definition** suffers from some difficulties related to the ontology of quasi-concrete objects.

In Section 2, I have presented Fine's account of impure sets as non-homogeneously abstract objects having concrete parts. Then, I have proposed a **Trilemma** with the scope of questioning Fine's idea.

In Section 3, I have presented some further observations concerning

the notion of intrinsicity. Then, I have discussed a possible extension of **Trilemma** to all non-homogeneously objects.

Generally speaking, the arguments that I have proposed in this chapter address Parsons' and Fine's specific accounts but also apply to all other frameworks involving the so-called mainstream thesis plus membership as composition or representation.

Once the mainstream thesis has been rejected, other metaphysical hypotheses arise. In what follows, I will evaluate which framework is appropriate to replace the mainstream thesis. Embracing the classic conception of abstractness and concreteness, **Trilemma** suggests that if sets are composed of spatially located components, then they also are located. In the next chapter, I will discuss [Maddy, 1991]'s account according to which impure sets are fully concrete objects, *i. e.* spatiotemporally located objects having spatiotemporally located members or elements in their transitive closure. I will nonetheless provide some reasons to believe that her account fails.

Chapter 2

Impure Sets as fully Concrete Objects

Introduction

In this chapter, I will discuss [Maddy, 1991]’s idea of impure sets as fully concrete objects, *i. e.* concrete objects having concrete members or concrete elements in their transitive closure.

Maddy’s thesis is not mainstream. It has been sharply criticized by many commentators. Examples are [Chihara, 1991], [Balaguer, 1994], [Carons, 1996], [Piazza, 2000], and [Levine, 2005]. Maddy proposes some replies. However, she does not provide a systematic defense of her account.

I am not sympathetic with Maddy’s view on impure sets, but, I will argue that many criticisms are too hasty, and the idea of impure sets as fully concrete objects is more “resistant” than what we are inclined to

believe. To dismiss a philosophical thesis, we have first to lead it to its best dialectical formulation. Such an exercise will allow me to rephrase the debate over impure sets in a more fine-grained way, as well as to consider some unexplored objections and replies. To spoil the end, I will argue that, despite the attractiveness that Maddy's account can exert, it has a bunch of serious problems.

The chapter has nine sections. Section 1 presents Maddy's account of impure sets as fully concrete objects. More specifically, I will present the framework in which Maddy's idea of impure sets is presented, namely her so-called *Compromise Platonism* (2.1.1). Then, I will divide Maddy's account into two different but related principles: Ontological Claim according to which impure sets are spatiotemporally located, and Epistemic Claim stating that we perceive impure sets (2.1.2). After that, I will explore Maddy's idea of perception (2.1.3).

All the other sections are devoted to the discussion of some objections to Maddy's account. These objections can be divided into two kinds: Objections against Epistemic Claim and objections against Ontological Claim.

Section 2 presents an objection against Epistemic Claim which is based on the fact that perception is not able to individuate every object of impure set theory. In particular, I will present [Chihara, 1991]'s case of singletons that do not perceptually differ from their members (2.2.1). Then, I will present Maddy's reply to Chihara which is based on the identification of singletons and their members (2.2.2).

In Section 3, I will linger on the notion of identification whose discus-

sion will occupy a substantial part of the chapter. This choice is justified by the fact that such a notion plays a central role in Maddy's philosophy of impure sets. Also, to my knowledge, the features of identification have not been sufficiently explored. Such an analysis will, therefore, be useful to characterize this notion in all its generality. In particular, I will present the classic proof against the identification (2.3.1). Then, I will propose a characterization of the notion of identification that could *prima facie* invalidate this proof (2.3.2). After that, I will explore the specific case of identification of singletons and Urelements (2.3.3). I will conclude, nonetheless, that this identification is difficult to justify (2.3.4).

Section 4 presents some other objections against Epistemic Claim based on the fact that perception is not able to individuate every object of impure set theory. More specifically, I will discuss a generalization of Chihara's objection to cases of different impure sets having the same perceptual content (2.4.1). I will present Maddy's reply which is based on pluralism (2.4.2) and argue that this solution does not go.

Section 5 is also about Epistemic Claim but this time the objection goes the other way around. Perception accounts for more sets than required. The good candidates to show this fact are sets having different but co-extensive predicates (2.5.1). Then, I will present the so-called *qua*-objects framework that could solve this flaw of perception. I will argue, nonetheless, that this framework is subjected to other substantial problems (2.5.2).

In Section 6, I will explore two further objections against Maddy's Epistemic Claim. The first is [Balaguer, 1994]'s argument which points

out a tension between perception and the nature of impure sets. The second is based on Maddy's confusion between aggregates and sets and takes some arguments from [Levine, 2005].

Section 7 presents some objections against Maddy's Ontological Claim. More specifically, I will present two objections: [Cook, 2013]'s argument according to which location of impure sets entails general unintuitive consequences (2.7.1), and [Parsons, 2008]'s counterexamples of impure sets having non-coexistent located members (2.7.2).

Section 8 includes two additional remarks: the first explores two alternative replies to Chihara's objection (2.8.1); the second concerns some applications of the debate between monism and pluralism to Maddy's examples for Compromise Platonism (2.8.2).

Section 9 concludes.

2.1 Maddy's account of impure sets

2.1.1 Maddy's *Compromise Platonism*: a two-tiered theory

Compromise Platonism is stated in [Maddy, 1991]. The main goal of Maddy's Compromise Platonism is to provide a philosophically suitable version of realism for mathematics. Some canonical formulations of realism coincide with Platonism. Both require the existence of mathematical objects playing a role in determining the truth of mathematical statements. Peculiarly, Platonism claims that such objects are abstract.

Doing so, it lends itself to Benacerraf’s dilemma: if the term “abstract” means lacking spatiotemporal location and causal power, and if what counts as knowledge is that some entities play an appropriate causal role in the generation of some beliefs, then the question arises of how mathematical objects can take part in any causal interaction whatsoever. This fact suggests that Platonism has some trouble in justifying mathematical knowledge.¹

Maddy’s Compromise Platonism is an attempt to avoid Benacerraf-style problems. Compromise Platonism takes the form of a two-tiered theory. The first tier argues for a theory of perception able to homogeneously explain our knowledge of both ordinary and mathematical objects. This reply represents a solution to Benacerraf’s problems as long as it allows, using Maddy’s words, to “bring [mathematical objects] into the world we know and into contact with our familiar cognitive apparatus” [Maddy, 1991], 48.

Here is where impure sets come on stage. Indeed, according to Maddy, impure sets are the particular kinds of mathematical objects that display this cognitive familiarity. Maddy’s first tier can be formulated by using the following conditional: if we perceive medium-sized physical objects, then we perceive impure sets — and perception works for medium-sized physical objects and impure sets in a similar way.

The second tier of Maddy’s theory is based on Quine-Putnam’s argument for the indispensability of mathematics. In a nutshell, mathematics is indispensable for sciences, and this is enough to justify mathematical

¹As mentioned, the *locus classicus* for this problem is [Benacerraf, 1973].

practice. More specifically, since advanced scientific theories such as quantum chemistry and quantum field theory have relevant mathematical components, then if science holds, mathematics holds as well. Maddy follows Quine in claiming that the indispensability argument also justifies the ontological commitment to mathematical objects that the realist aims to support. That is, we are committed to the existence of mathematical objects because these objects are indispensable to formulating our best theory of the world.

What are the relations between the first and the second tier? A way to put the story affirms the priority of the first over the second: if the first tier is valid, then mathematical objects exist and Benacerraf's dilemma is solved. Perhaps though, the two tiers have to be used in a "cooperative way" such that when we are not able to grant the first, we use the second. Actually, the situation is more complex than this. Indeed, Maddy argues that the validity of the first tier is not enough to ground mathematics, and so we also need to discuss the second tier, *i. e.* our practical reasons for postulating the existence of mathematical objects.

Despite its central importance for the philosopher of mathematics, such a discussion goes beyond my scope. What interests me is the status of impure sets. I am happy with discussing the plausibility of Maddy's first tier which I take as an independent thesis concerning the metaphysics of impure sets. Generally speaking, I will follow [Balaguer, 1994] in claiming that if Maddy succeeds in defending the first tier, then she is also in a good position to adopt Compromise Platonism and reject Benacerraf's dilemma. Yet, I will not argue for such a position. The

readers who desire to complete the picture can find some interesting observations in [Decocki, 2002], [Marfori, 2012], and [Park, 2018].

Before beginning my investigation, it is worth noting that, over time, Maddy has abandoned Compromise Platonism. She is now embracing a form of Naturalism according to which the indispensable use of mathematics in natural sciences does not entail an ontological commitment to the mathematical objects being used (see [Maddy, 1997]). In the present work, I will not address this issue either. My omission is justified by the fact that the reasons why she has changed her mind are not directly linked to the plausibility (or implausibility) of the first tier, but mostly rely on the second one.

One last observation is in order. Maddy seems to take concrete objects to be both spatiotemporally located and causally efficacious. Adding causality to the definition of concreteness has to do with Maddy dealing with Benacerraf's dilemma. As mentioned, if sets are causally efficacious, then we gain, at least in principle, epistemic access. In this work, I will address the question of causality only from the point of view of the supposedly perceptual interaction between epistemic subjects and impure sets. Incidentally, it can be said that, since Maddy's conditions for concreteness seem more exigent than Parsons', concreteness for sets is *prima facie* more difficult to defend.

2.1.2 Ontological and epistemic claims for impure sets

Let me start by presenting Maddy's definition of impure sets. Unlike Parsons' account, which is not clear about whether impure sets have to include sets with concrete elements in their transitive closure, Maddy's account is more explicit. According to her, impure sets are sets having concrete members or concrete elements in their transitive closure. Sets such as $\{\{\text{Socrates}\}\}$ and $\{\{\text{Socrates}\}, \text{Plato}\}$ count, therefore, as impure. And the same goes for sets that we have previously considered mixed such as $\{\text{Socrates}, \omega\}$ and $\{\text{Socrates}, \alpha\}$.

However, concreteness is not explicitly defined. As mentioned, what Maddy has in mind when she speaks about concreteness is the classic definition plus causality. Something is concrete *iff* it is spatiotemporally located and has causal power.

We are ready now to formulate Maddy's first tier which can be divided into two main theses that I will call the Ontological Claim and the Epistemic Claim:

Ontological Claim: sets having at least a concrete object in their transitive closure are spatiotemporally located precisely where this concrete object is located;

Epistemic Claim: we (do) perceive impure sets.

Ontological Claim and **Epistemic Claim** are strictly related. Maddy does not explicitly state how such a relation has to work. Since this point is important for my proposal, I will try to add some further elucidations.

As a first remark, we can say that perception plausibly involves a causal relation between the perceiver and the objects perceived such that it “locates” the objects in question. As a consequence, if Maddy succeeds in justifying **Epistemic Claim**, then she is also arguably right in supporting **Ontological Claim**. Taking our specific issue into account, if we perceive sets, then they are spatiotemporally located.

Of course, there can be cases of unperceived spatiotemporally located objects such as subatomic particles, galaxies, and, more basically, any other spatiotemporal object that is not in front of us. Hence, **Epistemic Claim** is a sufficient but not necessary condition for **Ontological Claim**. To claim this, it is to also state that, while there can be cases of unperceived spatiotemporally located objects, there cannot be cases of perceived non-spatiotemporally located objects. **Ontological Claim** is, therefore, taken to be a necessary but not sufficient condition for **Epistemic Claim**.

Of course, counterexamples arise if we take perception to be non-veridical. The most straightforward are hallucinations and, more generally, all phenomena where a subject erroneously perceives something. Then, it cannot be said that, necessarily, if they perceive something, then it is spatiotemporally located. **Epistemic Claim** does not necessarily entail **Ontological Claim**.

Such phenomena are, nonetheless, ruled out by Maddy’s idea of per-

ception. According to Maddy, perception is veridical. Hence, our perceptual activity is directed towards physical objects and produces true beliefs. Also, physical objects play an appropriate causal role in the generation of our perceptual beliefs. Thus, **Ontological Claim** and **Epistemic Claim** are such that if **Epistemic Claim** holds, then **Ontological Claim** holds as well (under Maddy's conditions for perception). Discussing Maddy's idea, I will charitably assume this high standard for perception that I will spell out in detail in the next section.

Maddy's **Ontological Claim** is not so popular. As largely mentioned in the previous chapter, the mainstream idea is that impure sets are not concrete but abstract objects having concrete members or concrete elements in their transitive closure. However, I have also largely shown that the mainstream thesis can be questioned. **Ontological Claim** deserves, therefore, to be seriously considered.

To my knowledge, there are two places in which we can find arguments *directly* addressed against Maddy's **Ontological Claim**: [Parsons, 2008], and [Cook, 2013]. I will focus on them at the end of the chapter.

Before doing this, I will explore some arguments against **Epistemic Claim**. More specifically, I will address [Chihara, 1991], [Balaguer, 1994], and [Levine, 2005]'s. As mentioned, if **Epistemic Claim** does not hold, it does not follow that **Ontological Claim** is false. It has, nonetheless, the effect of undermining it. Friends of Compromise Platonism are called to justify **Ontological Claim** otherwise.

Let me move on now presenting in detail Maddy's account for impure sets by focusing on the role of perception.

2.1.3 Perception and impure sets

Maddy's core idea goes as follows:

"Steve needs two eggs for a certain recipe. The egg carton he takes from the refrigerator feels ominously light. He opens the carton and sees, to his relief, three eggs there. My claim is that Steve has perceived a set of three eggs. [. . .] This requires that there be a set of three eggs in the carton, that Steve acquires perceptual beliefs about it, and that the set of eggs participates in the generation of these perceptual beliefs in the same way that my hand participates in the generation of my belief that there is a hand before me when I look at it in good light". [Maddy, 1991], 58.

Ontological Claim and **Epistemic Claim** are therefore stated. Concrete objects such as the three eggs in the carton are sets that can be perceived. Also, the realistic flavor of Compromise Platonism is explicitly stressed: sets are spatiotemporally located and they participate in the generation of perceptual beliefs in the way in which "standard" physical objects do. Shortly, they are causally efficacious.

At this point, we need a more precise idea of how perception is supposed to work, especially in the case of sets. Here, the analogy between perception of medium-sized physical objects and perception of sets plays a central role. Analogy is a symmetric relation but is used, in the case at issue, to explain how perception of sets works by assimilating it to that of medium-sized physical objects, and not the other way around (just

recall the direction of the implication "if we can perceive medium-sized physical objects, then we can perceive impure sets"). This happens because, even though perception of physical objects is a controversial issue, it enjoys a far clearer epistemological status than perception of sets.

Following [Hebb, 1949]'s neurophysiological theory, Maddy describes perception of physical objects as the result of a complex neurological process including innate and non-innate components. This process leads to the formation of the so-called "cell-assemblies", interconnected sets of neurons which enable us to "recognize" physical objects. As Maddy stresses,

"when I see a triangular figure, [. . .] I automatically see it as more like other triangles than like squares, I can recall it, recognize it, and call it and other similar figures 'triangles'. In the terminology we've adopted here, I acquire the perceptual belief that there is a triangle before me".[Maddy, 1991], 58.

It is clear that perception differs from simple sensation. In other words, perception makes its objects, so to say, *conceptually loaded*. This mechanism is in play also for perception of sets. As Maddy says,

"the younger child imagines that the number of elements in a set changes when it is rearranged, particularly when its elements are moved closer together or further apart. For older children, on the other hand, once a one-to-one correspondence between two sets has been established, the belief in their equinumerosity cannot be shaken; indeed the very question seems

silly to them. Once a perceptual belief about a set is gained, the thought that the set could change its number property when its elements are moved about (barring mishap) appears preposterous".[Maddy, 1991], 64.

That is, we perceive sets of eggs in the fridge in a similar way as we perceive triangular figures or other objects in front of us.

Maddy's first tier rests on three conditions under which her realistic account of perception works for both physical objects and sets:²

Veridicality condition: Our successful perceptual activity directed towards physical objects and sets produces true beliefs.

Directness condition: we do actually perceive physical objects, as opposed to sense-data, percepts, or representations of some kind. In the same way, we perceive sets as opposed to scattered members or representations or images of sets.

Causal condition: the physical objects and the sets in front of us play an appropriate causal role in the generation of our perceptual beliefs about them.

²These conditions are spelled out in [Levine, 2005]. They mainly derive from Maddy's adoption of [Pitcher, 1971]'s theory of perception.

The resulting picture of Compromise Platonism is a form of *monism* unifying mathematics and physics. As Maddy stresses,

”in place of the old picture — physical reality here and now, mathematical reality nowhere and nowhen — set theoretic monism offers a spatio-temporal reality inseparably physical and mathematical. Physics and mathematics, on this new picture, are two sciences, along with chemistry, biology, psychology, and the rest, that study aspects of this reality”.

[Maddy, 1991], 158.

After having investigated how perception of sets is supposed to work, Maddy focuses on how we acquire knowledge of the basic axioms of set theory (*e. g.* union, pairing, comprehension axiom). Since we arguably do not perceive those axioms in the same way in which we perceive sets of concrete things, another explanation is required. Maddy’s suggestion is to treat those axioms as results of higher-order levels of perceptions leading to general intuitive beliefs. The explanation is not so different from that which is used to account for “pure” perceptual cases:

“The development of higher-order cell-assemblies responsive to particular sets gives rise to an even higher-order assembly corresponding to the general concept of set. The structure of this general set assembly is then responsible for various intuitive beliefs about sets, for example, that they have number properties, that those number properties don’t change

when the elements are moved (barring mishap), that they have various subsets, that they can be combined, and so on. And these intuitions underlie the most basic axioms of our scientific theory of sets” [Maddy, 1991], 71.

In this chapter, I will not consider this issue. Two reasons why: the first is that intuition seems to be an easy way out: when perception fails, a more powerful (but arguably mysterious) faculty does the desired job; second, and more importantly, I am interested in discussing the viability of **Ontological Claim** and **Epistemic Claim** which directly regard the nature of impure sets, and deserve to be discussed independently from any other justification. I will nonetheless add, in 2.5.2, some remarks concerning the relation between perception and intuition.

2.2 Against Maddy’s Epistemic Claim: Perception is too weak

One of the most natural objections against Maddy’s proposal is that we do not perceive *every* impure sets. These are not only cases of sets of unperceivable objects such as galaxies or subatomic particles but cases of *prima facie* perceivable objects some of which are important to grant set theory. One of these cases is the singleton. For instance, the singleton of an egg. Other cases involve supposedly colocated sets such as the sets of four eggs that is now in my fridge and the set containing two pairs of eggs. Another interesting case is that of, for instance, the set

of three eggs, the set of that set, the set of the set of that set, and so on. Perception is indeed called to discriminate between these sets, and the fact that it does not suggests that it is not adequate to account for impure sets. This is the sense in which I claim that perception is "too weak".

In what follows, I will discuss these cases in detail.

2.2.1 Chihara's objection

The case of singletons is well argued in [Chihara, 1991]. Chihara's target is singletons having concrete things as members (*e. g.* the singleton of a duck, the singleton of an apple, etc.). As he argues, there is no perceptual difference between a given concrete object and the set containing exactly that concrete object. In treating Chihara's objection, I will use the terms "Urelements", "individuals", and "medium-sized physical objects" as all denoting these concrete objects.

Using Chihara formulation,

"what looks like a duck, waddles like a duck, quacks like a duck, smells like a duck, tastes like a duck, . . . but isn't a duck? It's the set whose only member is the duck". [Chihara, 1991], 201.

Hence, **Epistemic Claim** is undermined. Perception of singletons requires that of the difference between the singletons and the objects they contain.³ Nevertheless, it can still be the case that unperceived objects

³This claim can be questioned by arguing that it does not go without saying that

are spatiotemporally located. And Maddy's *desideratum* of **Ontological Claim** involves that concrete objects and their singletons are colocated. However, even though Chihara's objection does not automatically dismiss **Ontological Claim**, the burden of proof belongs to the set-theoretic realist.

Indeed, using, once again, Chihara's words, it is quite surprising that

"[the singleton set of an apple] looks, feels, smells and tastes exactly like the apple and is located in exactly the same spot and at exactly the same time – yet it is a distinct entity!"

[Chihara, 1991], 201.

Maddy sketches three replies, and finally validates only one of them. In what follows, I will focus on Maddy's favorite solution. I will, nonetheless, add some observations concerning the other two replies in Section 2.8.1.

2.2.2 Maddy's reply to Chihara's objection

According to Maddy, it can be argued that Chihara is right: we cannot distinguish between concrete objects and their singletons. Yet, this happens exactly because there is no such difference: Physical objects and their singletons have to be *identified*. To be clear, it is not the case that perception is too weak. The duck looks like, waddles like, and quacks

the perception of an object requires the perception of its constituents. However, Maddy's Compromise Platonism is based on the idea that perception grounds mathematics. How to ground the difference between $\{x\}$ and x if no perceptual difference is available? This is why, as we will see, Maddy takes seriously Chihara's objection. I would like to thank Giovanni Merlo for raising this point.

like the set whose only member is the duck precisely because *they are* the same perceivable and spatiotemporally located thing. **Ontological Claim** is safe and can be justified by **Epistemic Claim**. That is, Maddy bites Chihara's bullet.

However, there seems to be some trouble in identifying singletons with their Urelements. These troubles are expressed by a proof that has a longstanding philosophical pedigree. In the next section, I will make a detour and discuss the proof. If identification goes, then there is room to maintain that we actually perceive singletons. Or, at least, the burden of proof goes back to anyone asserting that we do not perceive singletons because we do not perceive any difference between singletons and their members (as Chihara does).

2.3 Identification of singletons and their members

2.3.1 Against the identification

Reductio ad absurdum

Pioneers of set theory have tended embracing the idea that singletons are different from the members they contain. However, we have exceptions. One of these can be found in [Peano, 1895]; another exception can be found in [Lesniewski, 1916]. This fact has pushed some philosophers to formulate a proof that aims to show that the identification between

singletons and their members leads to contradiction. This proof takes the form of a *reductio ad absurdum*. It points out that if we identify singletons with their Urelements, then we are also obliged to identify the elements of a pair. The first formulation of the proof can be found in [Frege, 1895], and goes as follows:

”If it were the case that a singleton coincides with the sole individual that can be conceived as belonging to it, then P would coincide with Q. Let us say that a and b are different objects that are conceived as the sole individuals belonging to P. Then, they would be conceived as belonging to Q. This would mean that a and b would coincide with P and so that a would coincide with b. However, this latter result goes against the legitimate assumption for which a and b are different”.

[Frege, 1895], 35.

The proof can be restated as follows:

- 1) $P = \{a, b\}$, where $a \neq b$;
- 2) Q is the singleton of P, *i. e.* $Q = \{P\}$;
- 3) $P = Q$.
- 4) Then, a and b belong to P and Q.
- 5) Hence, $a = b$. Contradiction.⁴

⁴To be fair, something similar to Frege’s *reductio* already appears in [Dedekind, 1888]. Dedekind argues as follows: ”Suppose that $\{a\}$ has distinct elements b and c, yet $\{a\} = a$; since the principle of extensionality had been assumed, it follows that $b = a = c$, which is a contradiction”. I argue, nonetheless, that this formulation is far from being clear. In particular, I am not sure to understand what Dedekind means by claiming that $\{a\}$ has distinct elements b and c.

Frege's proof has survived a century. A more recent formulation can be found in [Piazza, 2000]. Piazza's version deals directly with Maddy's account of sets and so introduces the perceptual component. Following [Piazza, 2000]'s formulation :

"Even though $\{x, y\}$ and $\{\{x, y\}\}$ produce the same perceptual phenomenon, we cannot say that they are the same object because this would lead to identifying x and y . Indeed, if $\{x, y\} = \{\{x, y\}\}$, then $x \in \{\{x, y\}\}$ and $y \in \{\{x, y\}\}$; so, $x = \{x, y\}$ and $y = \{x, y\}$. Then, $x = y$." [Piazza, 2000], 84.⁵

Other objections

Similar objections to the identification are recently stressed by Michels (2021 personal communication) and Correia (2021, personal communication).

Michels' objection Michels' objection points out that the identification leads to strange cases of self-referential membership. He simply argues that if $\{x, y\} = \{\{x, y\}\}$, then $\{x, y\}$ is a member of itself.

Correia's objection Correia suggests instead that the identification of x and $\{x\}$ leads to that of all the levels of the set-theoretical hierarchy. He argues as follows:

⁵Formulations including the perceptual component can be also found in [Balaguer, 1994].

- 1) $\{x\}$ has just one member, *i. e.* x .
- 2) $\{\{x\}\}$ has just one member, *i. e.* $\{x\}$
- 3) $x = \{x\}$
- 4) Hence, $\{x\} = \{\{x\}\}$.
- 5) 1-4) apply to all other levels of the set-theoretical hierarchy.

In what follows, I will discuss the validity of these objections. I will play the devil's advocate showing that Maddy has some persuasive arguments in favor of the identification. To be sure, I do not believe that such arguments go. Despite this, I will show that, even when presented in their most compelling form, they still are not conclusive. This will, therefore, corroborate the idea that the identification between singletons and their members is misguided.

2.3.2 The notion of identification

Maddy does not specify what she means by identification. A good way to discuss both the objections and their applications against Maddy's proposal, is to focus on this notion. The scope of this section goes, nonetheless, well beyond the significant question of knowing whether Maddy's account is compelling, and extends to the relevant issue of clarifying the notion of identification which is largely present in the contemporary debate but that has, I argue, not been clarified sufficiently.

A lot of unanswered questions can be spelled out. Here are some of them: Is identification a relation? Is it symmetric? Is identification the

same as identity?

Dictionary definition represents a good starting point: “Identification is the action or process of identifying someone or something (or someone/something with someone/something else)” where “to identify” can be understood as “to recognize”, “to pick out”, “to equate to”. Some examples are: The identification of a body; the mother-son identification; the identification of a person by her ID; the identification of a bird as a redbreast.⁶

Let me start by evaluating whether identification is the same as identity. Since the main formal features of identity are well established, this is, I argue, the easiest way to get into the problem. Classically speaking, identity is a relation. I do not want to get far into what it is to be a relation. For instance, I will stay neutral regarding the ontological dignity and the philosophical implications of this notion. I will simply stress that for something to be a relation, it has to hold between things. In this general sense, identification is also a relation.

Now, for a relation to be an identity relation, it has to be an equivalence relation (it has, therefore, to be reflexive, asymmetric, and transitive) and has to satisfy Leibniz’s Law (LL), that is their *relata* must exemplify all and only the same properties.⁷ If the dictionary definition is taken seriously, we already know that identification and identity are not the same. While identification is an action, identity is not. More specifically, identification seems to be an epistemic process and, as such,

⁶See “Identification” in *Oxford Languages*.

⁷LL is formally expressed as follows: $\forall x\forall y(x = y \rightarrow \forall F(Fx \leftrightarrow Fy))$.

it entails temporal and operational features. By contrast, identity states a metaphysical “atemporal” fact, *i. e.* that two *relata* have the same properties.

What is the relation between identification and identity (if any)? A good way to stress the issue is to ask whether “x can be identified with y *iff* x = y”. To answer that, let me consider the examples above. For the case of mother-son identification, it is plausible to claim that mothers and their children have different properties. By contrast, identification by ID identifies a person having a given set of properties (a name, some physical attributes, and so on) with the very person’s flesh and bones. LL is, therefore, satisfied. The case of body-identification is not so different. Suppose that I correctly identify a given body in front of me with the very body of, for instance, my friend Ugo: the body in front of me and Ugo’s body have the same properties. Things are different for bird-identification. This case is simply one of classification: to say that the bird has been identified as a redbreast is to say that it has been recognized as being of a certain species. Identity is clearly out of the picture. A given bird is just *one of* the redbreasts.⁸ Hence, from this survey, we have that not all cases of identification presuppose identity.

It is interesting to notice that even when LL is not satisfied, identification can still be understood as a process having identity *as a target*. In the case of mother-son, identification is a process in which two differ-

⁸I have assumed that the redbreast is an object that has ontological dignity. Not everyone takes general objects to be genuine objects. By the way, if the redbreast is not an object, then we can just say that a given specific bird has the property of being a redbreast, and no identity issues is raised.

ent objects share something that has to do with some main physiological and psychological mechanisms. Such mechanisms are, so to say, constitutive of the nature of the objects in question and so, once shared, make *as if* these objects *were* identical. Something similar happens in the bird-redbreast case. Identification is a process delivering some, so to say, essential properties that are necessary to the individuation of the bird. In this case, the sense of identification is expressed by the fact that the given bird cannot be what it is without being a redbreast.

To put the story using some classic categories, we can speak about *weaker* cases of identity. Things can be — to some degree — *qualitative* identical in that they share some properties (this is the case of the mother-son and the bird-redbreast). By contrast, LL displays *numerical* identity which requires absolute, or total qualitative identity, and can only hold between a thing and itself. Basically, when numerical identity holds, we have one unique object with a given set of properties. (This is the case of the ID and Ugo's body). Given this, it can be claimed that identification is an epistemic process that has qualitative or numerical identity as a target.

So far, so good. Is identification symmetric? In the case of mother-son identification, it can be assumed that symmetry holds: the mother identifies herself with her son and the son with her mother. Nevertheless, it is easy to find asymmetric cases of psychological identification: Consider, for instance, the American black comedy “Arsenic and Old Lace” in which Teddy identifies himself with Theodore Roosevelt. It is however not the case that Roosevelt (which is, by the way, already dead at

the time in which the comedy takes place) identifies himself with Teddy. The case of bird-identification is also asymmetric. It is not the redbreasts that are identified with a specific bird, but the other way around. What happens for body-identification? Once we recognize that a certain body is Ugo's body, we come to accept a true identity claim: this body (in front of us) is identical to Ugo's body. However, there still is a direction in which the identification goes: from the unknown body to Ugo's. Similarly, even though ID-identification provides identity, it shows an asymmetric direction: it is not me that I am going to be identified with the person who has a specific boarding card, but it is the person who has a specific boarding card that is going to be identified with me.

Identification is, therefore, non-symmetric in the sense that it allows for both symmetric and asymmetric cases. Furthermore, examples such as Ugo and ID-identification suggest that asymmetry of identification and numerical identity are compatible. This is the case because, as mentioned, the first is a metaphysical fact about the object of the identification, while the second is an epistemic feature of the process of identification.

Some new elements can, therefore, be added to the previous definition: Identification is a non-symmetric epistemic process having qualitative or numerical identity as a target.

One last crucial aspect needs to be clarified. It is plausible to claim that, before the identification, the identity between two *relata*, whether numerical or qualitative, is epistemically indeterminate. For instance, I just do not know whether the body in front of me is that of Ugo before

seeing some properties allowing me to assert this fact. Similarly, I do not know whether a bird is a redbreast before having individuated some relevant properties. The epistemic indeterminacy of the identity seems to be a constitutive starting point of the process of identification because such a process leads from a situation in which, at first glance, we do not know whether the *relata* are numerically or qualitatively identical (or we just think that they are not) to one in which we discover them to be (numerical or qualitative) identical. Nevertheless, from the metaphysical point of view, it is plausible to claim that the identity holds (or not) *before* the process of identification.⁹

Thus, I conclude by claiming that identification is a non-symmetric epistemic process having qualitative or numerical identity as a target. Such an identity was epistemically indeterminate (or not "believed") before the process of identification.

Given all these considerations, let me now focus on the specific case of Urelements-singletons identification.

2.3.3 The case of singletons and their members

Classically, singletons and Urelements are taken to be different objects. Given [Zermelo, 1908]'s definition, Urelements are objects that cannot be sets and that cannot contain sets or members; singletons are sets containing exactly one member. In other words, numerical identity does

⁹It can be pointed out that the case of mother-son identification does not meet this requirement because the identity is built up during the process of identification. However, this is a very special case of identification that has to do with some "metaphoric" identity. As mentioned, we act "as if" the objects were identical.

not hold. Of course, it can somehow be argued that qualitative identity is in play. Urelements and singletons share some properties such as “being set-theoretical objects” or “being logical objects”. However, for Maddy’s defense to work, the target of the identification has to be numerical identity. Qualitative identity would not suffice. To grant this, think about the root of the problem: Maddy replies to Chihara that we do not perceive any difference between $\{x\}$ and x because there is no such a difference. This claim is metaphysical and cannot, therefore, be associated with qualitative identity.¹⁰

Given this, it is clear that the identification between Urelements and their singletons is an epistemic process leading from a situation in which we thought that Urelements and singletons were different objects to one in which we state that they are the same unique object. Of course, this is possible only by embracing a revisionist approach. The first step to do is to change the grammar of set theory. Given that Urelements and their singletons are one sole thing, we will have just one notational sign. Plausibly, this means that we will have only one between x and $\{x\}$.

In what follows, I will discuss both options. To do this, I will use the long-standing proof against the identification presented in Section 2.3.1. More specifically, I will take Piazza’s formulation because it has the advantage of dealing with Maddy’s idea. After that, I will consider Michels’ and Correia’s objections.

¹⁰Incidentally, it can also be noticed that the fact that singletons and Urelements have incompatible properties prevents us from assimilating them to the bird-redbreast case: a singleton cannot be one of the Urelements, nor a Urelement one of the singletons.

**Disclaimer: Eliminative 1 – Eliminative 2 – Non-Eliminative 1
– Non-Eliminative 2**

Before starting, it is important to note that the change of the grammar is not the whole story. The other and perhaps more important issue is the genuinely metaphysical question of which properties the object is supposed to exemplify. I argue that we can take identification to be a process that allows for four different options. To illustrate the situation, take our case of $\{x\}$ and x . In identifying x and $\{x\}$, we can either keep all the x -properties, or all the $\{x\}$ -properties. Let me call these strategies **Eliminative 1** and **Eliminative 2** respectively. Otherwise, we can keep the sum of all the x -properties plus all the $\{x\}$ -properties (if compatible), or the sum of some of the x -properties plus some of the $\{x\}$ -properties (if compatible).¹¹ Let me call these two strategies **Non-Eliminative 1** and **Non-Eliminative 2** respectively.

Now, as it is clear from what has been said about the definition of Urelements and singletons, we cannot take all the $\{x\}$ -properties plus all the x -properties because we would end with something that is a set and not a set at the same time. And even though this object could have some dignity for the Neomeinongians, it is certainly not adequate to ground mathematics. Thus, **Non-Eliminative 1** is not an option.

It has also to be noticed that the grammar and the metaphysical

¹¹Of course, we can also take some of the x -properties plus all the $\{x\}$ -properties or vice versa.

issues are relatively independent from each other. Yet, they are not completely detachable. They can be discussed separately but the interaction is evident as long as some properties already appear in the notation in a way such that the manipulation in the grammar of the proof necessarily involves philosophical considerations. Also, as mentioned, for the case of singletons-Urelements, the change in the grammar has some philosophical reasons: it is justified by the fact that what a Maddian needs to point out is that Urelements and singletons are the same thing. Furthermore, the aforementioned metaphysical options can question certain strings of a given proof, simply by invoking philosophical considerations. This mechanism can in turn generate other bad consequences which oblige us to go back to the question of the validity of the metaphysical framework being used.

In what follows, I will consider the grammar and the metaphysical aspects insisting on their mutual influence as far as possible. More specifically, I will start by considering **Notational Survival of Urelements** and **Notational Survival of singletons**, pointing out some metaphysical considerations. Then, I will focus on **Eliminative 1**, **Eliminative 2**, and **Non-Eliminative 2**.

Grammar: Notational Survival of Urelement *vs* Notational Survival of singletons

As mentioned, some cases of identification show *prima facie* a straightforward direction. It is the case, for instance, of the unknown body. Looking at the unknown body we already know, before the identifica-

tion, that we do not have enough information about it, and that this lack would be filled by finding a specific object with which the body can be identified. Unfortunately, the case of singletons-Urelements is not so straightforward. This is why I will proceed to discuss all the alternatives.

●Notational Survival of the Urelements and Eliminative 1

Reply to Piazza's proof Piazza's proof claims that if $\{x,y\} = \{\{x,y\}\}$, then $x \in \{\{x,y\}\}$ and $y \in \{\{x,y\}\}$. So $x = \{x, y\}$ and $y = \{x, y\}$. Then, $x = y$. However, given that the grammar has to express that Urelements and singletons are now the same object and that what we are discussing is the elimination of singletons, then $\{x, y\} = \{\{x, y\}\}$ does not hold because $\{\{x, y\}\}$ is notationally meaningless. Indeed, $\{\{x, y\}\}$ becomes $\{x, y\}$, and so $\{x, y\} = \{\{x, y\}\}$ becomes $\{x, y\} = \{x, y\}$, which is trivially true. As a consequence, we cannot say that $x \in \{\{x, y\}\}$ nor that $y \in \{\{x, y\}\}$, which are, by the way, notationally meaningless. It follows that $x = \{x, y\}$ and $y = \{x, y\}$ are false as well. Yet, it is of course the case that $x \in \{x, y\}$ and that $y \in \{x, y\}$. In this way, we do not reach the problematic conclusion according to which $x = y$. Hence, contrary to what the proof aims to show, Maddy could identify singletons with their Urelements without falling into contradiction.

Reply to Michels' objection Michels stresses that if $\{x, y\} = \{\{x, y\}\}$, then $\{x, y\}$ is a member of itself. But since $\{\{x, y\}\}$ is meaningless, and has to be replaced by $\{x, y\}$, then we have, as before, that $\{x, y\} = \{x, y\}$, which is trivially true. However, it is not the case that

$\{x, y\}$ is a member of $\{x, y\}$. Members of $\{x, y\}$ just are x and y . Hence, Michels' objection does not hold.

Reply to Correia's objection Correia argues that if $\{x\}$ has just one member, *i. e.* x , and that $\{\{x\}\}$ has just one member, *i. e.* $\{x\}$, then if $x = \{x\}$, then $\{x\} = \{\{x\}\}$. But now $\{x\}$ has to be replaced by x which has no members at all. Then, $\{\{x\}\}$ has to be replaced by $\{x\}$ which is, in turn, replaced by x , which has no members at all. In this way, Correia's objection does not hold.

However, **Notational survival of Urelements** is highly problematic. In fact, if we do not have any $\{x\}$, we do not have any $\{\{x\}\}$ either, and the same goes for all the levels of the set hierarchy. Moreover, **Notational survival of Urelements** is incompatible with Pairing Axiom which implies that given x , we also have $\{x\}$.

These flaws have a metaphysical counterpart. If we embrace **Eliminative 1** according to which all the properties that we keep are those of the Urelements, we would end up having the counterintuitive conclusion that set theory contains objects that cannot be sets. Hence, **Eliminative 1** is not available.

To sum up, it is not true that if we identify singletons with their members, then we reach a contradiction. However, this identification leads eliminating sets from set theory, and it is, therefore, unsustainable.

Thus, from the metaphysical point of view, we have two options

left: either the object that is reached by the identification has all $\{x\}$ -properties (**Eliminative 2**), or it has some $\{x\}$ -properties and some x -properties (**Non-Eliminative 2**).

What are the relations between these two options and the grammar? To keep $\{x\}$ -notation instead of x -notation seems to presuppose **Eliminative 2**. However, we can also imagine using $\{x\}$ holding **Non-Eliminative 2**, and so taking $\{x\}$ to denote an object that has some properties of the singleton and some properties of the Urelement.¹² For the moment, I will discuss **Notational survival of singletons**. If the strategy is compelling, then I will move on and discuss which one between **Eliminative 2** and **Non-Eliminative 2** is the most plausible.

•Notational survival of the singletons

Reply to Piazza's proof Technically speaking, $\{x, y\} = \{\{x, y\}\}$ is, once again, not grammatically correct. Yet, this does not happen for the same reasons as before. This time, $\{x, y\}$ becomes $\{\{x\},\{y\}\}$, where $\{x\} \neq \{y\}$. What happens with $\{\{x, y\}\}$? Since the idea of $\{x, y\} = \{\{x, y\}\}$ was to stress the identification between $\{x, y\}$ and its singleton, and that we have replaced $\{x, y\}$ by $\{\{x\},\{y\}\}$, $\{\{x, y\}\}$ will become $\{\{\{x\},\{y\}\}\}$, *i. e.*, the singleton of $\{\{x\},\{y\}\}$. But $\{\{x\},\{y\}\}$ and $\{\{\{x\},\{y\}\}\}$ are different sets. As a consequence, it is the case that $\{x\} \in \{\{x\},\{y\}\}$ but not that $\{x\} \in \{\{\{x\},\{y\}\}\}$, and the same goes for

¹²Someone could ask how to justify the choice of **Notational survival of singletons** over **Notational survival of Urelements** if the object of identification takes also properties from Urelements. I will come back on this issue when discussing **Non-Eliminative 2**.

$\{y\}$. Also, it is not the case that $x = \{x, y\}$, nor that $y = \{x, y\}$ which are, by the way, notationally meaningless. Thus, we do not reach any problematic conclusion according to which $x = y$.

However, one may object that, in the expression “ $\{\{x\},\{y\}\}$ ”, the Urelement notation is still present: “ x ” and “ y ” are intended to denote Urelements. But if the fact of eliminating Urelements notation is taken seriously, then it will lead to the view that there are only expressions for pure sets, like “ $\{\}$ ”, “ $\{\{\},\{\}\}$ ”, etc.¹³ This objection invites me to be more precise. In **Notational Survival of singletons**, Urelements notation has to be intended to be, so to say, “subordinate” to that of singletons. In other words, x , y , and so on, can appear only when singleton notation appears (the first as “a compositant” of the second). Urelements as such are, therefore, incomplete symbols.

Has **Notational Survival of singletons** the same technical problem as **Notational Survival of Urelements**? The natural objection is that $\{\{x\},\{y\}\}$ and $\{\{\{x\},\{y\}\}\}$ are not different objects. According to the *desideratum* of the proof, they have to be identified. However, following the idea of identification stated above, $\{\{x\},\{y\}\}$ and $\{\{\{x\},\{y\}\}\}$ cannot both survive. In particular, this kind of identification tells us that $\{\{x\},\{y\}\}$ is replaced by $\{\{\{x\},\{y\}\}\}$. Then, we have no identity between $\{\{x\},\{y\}\}$ and $\{\{\{x\},\{y\}\}\}$; we just stay with $\{\{\{x\},\{y\}\}\}$.¹⁴

¹³I would like to thank Fabrice Correia for raising this objection.

¹⁴Notice that, even being charitable and conceding that $\{\{x\},\{y\}\}$ and $\{\{\{x\},\{y\}\}\}$ are identical, friends of the proof cannot state the revenge problem they would like to: Perhaps, it would be true that $\{x\} \in \{\{\{x\},\{y\}\}\}$ and that $\{y\} \in \{\{\{x\},\{y\}\}\}$. But it would certainly be false that $\{x\}$ and $\{y\}$ are identical to $\{\{x\},\{y\}\}$. Then, $\{x\}$ and $\{y\}$ will not be identical either.

But if this is true, then we do encounter the same uncomfortable result that we have found adopting **Notational survival Urelements**: If every time that we find a set, we identify it with its singleton, then we cannot have a set hierarchy. Indeed, in building the hierarchy, $\{x\}$ will be replaced by $\{\{x\}\}$, $\{\{x\}\}$ by $\{\{\{x\}\}\}$, $\{\{\{x\}\}\}$ by $\{\{\{\{x\}\}\}\}$, and so on. Roughly speaking, the process of building the hierarchy replacing objects by their singletons would entail the elimination of the set-theoretic objects that constitute this very hierarchy.

Maddy's restriction It has to be noticed though that Maddy says something important about the range of identification that could "save"

Notational Survival of singletons:

"We take it that the physical objects, x , the individuals from which the iterative hierarchy begins, are such that $x = \{x\}$. After that, the axiom of extensionality guarantees that sets formed at later stages will be distinct from their singletons."

[Maddy, 1991], 153.

This idea combined with **Eliminative 2** suggests that, while it is the case that x is identified with $\{x\}$, it is not the case that $\{x\}$ is identified with $\{\{x\}\}$, nor that $\{\{x\}\}$ is identified with $\{\{\{x\}\}\}$, and so on. The identification is, therefore, that of the Urelements with their singletons, and not that of all members with their singletons. Hence, **Extensionality** only applies to sets that are not singletons. $\{\{x\},\{y\}\}$ and $\{\{\{x\},\{y\}\}\}$ are different sets, and so, contrary to the objection

above, we can build a set hierarchy starting from $\{x\}$.

It has to be noticed that this way of thinking seems to suggest that, at least grammatically speaking, it is much better to identify Urelements with singletons than the other way around. And this is also clear from the fact that we cannot have the same restriction if identification goes the other way around.¹⁵

Reply to Michels' objection **Eliminative 2** plus Maddy's restriction allows us to block Michels' objection of self-referential membership. According to the identification, $\{x, y\}$ becomes $\{\{x\}, \{y\}\}$ and $\{\{x, y\}\}$ becomes $\{\{\{x\}, \{y\}\}\}$. Yet, following the restriction, $\{\{x\}, \{y\}\}$ and $\{\{\{x\}, \{y\}\}\}$ are different sets none of whose is a member of itself.

Reply to Correia's objection Given **Eliminative 2** plus Maddy's restriction, it happens that Correia's claim according to which $x = \{x\}$ becomes $\{x\} = \{x\}$, which is trivially true. In this way, we do not reach the counterintuitive conclusion according to which $\{x\} = \{\{x\}\}$. However, something strange happens with Correia's point according to which $\{x\}$ has x as its unique member: it becomes $\{x\}$ has $\{x\}$ as a unique member. We reach, therefore, a new version of Michels' objection.

Here, the supporter of Maddy's identification can bite the bullet conceding that $\{x\}$ belongs to both $\{x\}$ and $\{\{x\}\}$, and that it is also true

¹⁵Here, friends of **Notational survival of Urelements** can object that $\{x\}$ has to be identified with x but $\{\{x\}\}$ remains $\{\{x\}\}$. However, it would be very strange to take the set-hierarchy as passing from x to $\{\{x\}\}$. Indeed, it seems that if we do not have $\{x\}$, we do not have $\{\{x\}\}$ either, while if we do not have x , we can still have $\{x\}$, $\{\{x\}\}$, $\{\{\{x\}\}\}$, and so on. This strategy would also still be incompatible with Pairing Axiom.

that $\{\{x\}\}$ belongs to $\{\{\{x\}\}\}$ but that it is not the case that $\{x\}$ belongs to $\{\{\{x\}\}\}$. $\{x\}$ and $\{\{x\}\}$ have the same members but are not identical because they belong to different sets. Basically, the supporter of the identification is obliged to weaken **Extensionality** in the following way:

Extensionality* $X = Y \leftrightarrow \forall x((x \in X \leftrightarrow x \in Y) \wedge (X \in x \leftrightarrow Y \in x))$

But **Extensionality*** is cumbersome.

Up to here, we have seen that — *modulo* Correia’s objection — **Notational survival of singletons** works better than **Notational survival of Urelements**. As mentioned, the grammar and the metaphysical issue are related.

In what follows, I will discuss which one between **Eliminative 2** and **Non-Eliminative 2** is superior. To do this, I will first try to clarify more accurately what Maddy means by identification of Urelements and singletons. Incidentally, this topic also allows me to discuss a solution to a problem that can be addressed to Maddy’s restriction.

Metaphysical options

●**Philosophical meaning of identifying Urelements with singletons: Maddy’s Fregean argument** As I have said, Maddy is not explicit on what she means by identification of Urelements and single-

tons. It is, therefore, important to consider all the hypotheses. To some extents though, it can be argued that her preference goes for the identification of Urelements with singletons, and not the other way around.¹⁶ She is not explicit on which metaphysical options is superior either. Yet, she is pretty clear concerning the general meaning of the identification. This will be, therefore, my starting point.

It could be argued that the main information that the notational survival of $\{x\}$ over x delivers is the fact that objects are *individuated* things (see [Maddy, 1991], 53). Thus, the result of the identification is that every object is a singleton, namely a physical thing having 1 as a number property: *an* apple, *a* duck, *a* dissertation about singletons, and so on.

Maddy seems to support such a thesis by embracing what I have called **Fregean argument**. In Chapter 1, I have presented **Fregean argument** as follows:

- 1) Numbers do not apply to objects.
- 2) Numbers apply to sets.

Maddy is not clear about how to use **Fregean argument** to support identification. I argue that she may embrace the following version that I will call **Maddy's Fregean argument for identification**:

- 1) Objects are individuated things having some number properties.

¹⁶See [Maddy, 1991], 152-153.

2) Numbers apply to sets, and not to objects.

3) Hence, for objects to be individuated things, they have to be identified with their singletons.

However, as mentioned, Frege argues *against* the identification of Urelements and singletons. I think that the natural application of **Fregean argument** to the topic of the identification is **Fregean argument against identification**:

1) Objects are not individuated things having some number properties.

2) Numbers apply to sets, and not to objects.

3) Hence, for objects to not be considered individuated things, they do not have to be identified with their singletons.

Thus, Maddy and Frege seem to use the same argument but take two opposite starting points. **Fregean argument against identification** points out that if objects acquire some features of sets, then we reach counterintuitive results, while **Maddy's Fregean argument for identification** claims that physical objects genuinely present some features of sets. For the moment, I will concede to Maddy that her version of the argument is compelling. I will come back on this by evaluating **Eliminative 2** more accurately.

In the meantime, it is important to note that **Maddy's Fregean argument for identification** can be said to have perceptual support in

that, once our cell-assemblies are developed, we perceive objects instead of stuffs. This is what we meant before when speaking about perception as making its objects, so to say, *conceptually loaded*. We can, therefore, ground **Notational survival of singletons** claiming that every time we see an object x , we already see something as having 1 as a number property, *i. e.* we see x as $\{x\}$. In this way, we will exploit the good direction of the conditional from **Epistemic Claim** to **Ontological Claim**: We perceive singletons, then singletons are spatiotemporally located.

Incidentally, this fact says something important concerning **Notational Survival of singletons**. Recall the problem of **Notational Survival of singletons**. It could be argued that if we take the elimination of Urelements notation seriously, then we only have expressions for pure sets, like “ $\{\}$ ”, “ $\{\{\},\{\}\}$ ”, etc. I have replied that Urelements notation has to be intended to be, so to say, “subordinate” to that of singletons. Hence, x , y , and so on, can appear only when singleton notation appears. Urelements as such are, therefore, incomplete symbols. It is clear now why **Notational Survival of singletons** does not have to rule out Urelements notation when appears “within” the singletons. Indeed, what Maddy deals with are concrete objects (which are represented by Urelement notation) but that have to be considered individuated objects having some number properties (which are, therefore, represented by singleton notation).

● **Ad hoc-ness of the restriction** Maddy’s idea of identification suggests to take a step back and discuss, once again, **Notational survival**

of singletons.

Up to here, it seems that, contrary to what the objections in Section 2.3.1 aim to demonstrate, Maddy's identification is plausible, modulo Correia's objection. As I have anticipated though, it may be objected that $\{\{x\},\{y\}\}$ and $\{\{\{x\},\{y\}\}\}$ are not different objects, and so that identification fails. But, as mentioned, Maddy has a way out that consists of restricting the identification to the Urelement-singletons level.

This strategy does not persuade everyone. The main problem is presented *inter alios* in [Balaguer, 1994], and concerns the *ad hoc*-ness of the restriction. The issue goes as follows: Why is identification supposed to hold only for singletons having concrete objects as members? Why does it fail for all other sets? To use Balaguer's formulation,

"the question is whether the double standard that Maddy employs here can be tolerated. Is it acceptable to claim that some singletons collapse into their transitive closures, while pairs and other singletons don't, and while certain other sets — *e. g.*, $\{\{\{Madonna\}\}, \{\{Madonna, Quine\}\}\}$ — suffer partial collapses? What non *ad hoc* reason could Maddy give for thinking that seemingly similar sets behave so differently?"

[Balaguer, 1994], 103, footnote 12.

To be fair, the way in which Balaguer formulates the problem is not correct. As mentioned, no singleton collapses in its transitive closure. It is the other way around. Also, pairs containing Urelements as members are not *sui generis* cases. They behave in the same way as other

sets containing Urelements as members (for instance, $\{x, y\}$ is replaced by $\{\{x\}, \{y\}\}$, and $\{x, y, z\}$ is replaced by $\{\{x\} \{y\}, \{z\}\}$). It is neither the case that some sets suffer partial collapse. By contrast, it is the case that, for instance, $\{\{\{Madonna\}\}, \{\{Madonna, Quine\}\}\}$ is replaced by $\{\{\{Madonna\}\}, \{\{Madonna\}, \{Quine\}\}\}$. This happens because $\{Madonna, Quine\}$ is meaningless, and has, therefore, to be replaced by $\{\{Madonna\}, \{Quine\}\}$.

Nevertheless, there still is a sense in which Balaguer's point seems well taken. Why $\{\{Madonna\}\}$ is not replaced by $\{\{\{Madonna\}\}\}$? We have seen that this cannot be accepted because otherwise, we would not be able to construct the set hierarchy. But, of course, this fact increases the *ad hoc* flavor of Maddy's restriction. So, Maddy's friends have to find some philosophical reasons why the identification has to concern all and only the Urelements-singletons cases.

To my knowledge, Maddy does not provide any reply. All that she says can be found in the passage that I have already quoted:

"We take it that the physical objects, x , the individuals from which the iterative hierarchy begins, are such that $x = \{x\}$. After that, the axiom of extensionality guarantees that sets formed at later stages will be distinct from their singletons".

[Maddy, 1991], 153.

Perhaps, this passage can be used to undermine Balaguer's objection. Indeed, it can be argued that **Extensionality** has a clear application for sets, but not for Urelements. That is, while all the levels of the set

hierarchy can be distinguished by **Extensionality**, this is not the case for x and $\{x\}$. **Extensionality** does not apply to x and $\{x\}$ because, contrary to $\{x\}$, x has no members. However, such a reply presupposes the difference between x and $\{x\}$ that Maddy is trying to avoid.

It may be asked what happens if we "force" somehow the application of **Extensionality** to Urelements. I think that the result does not help the Maddian either. Urelements having 0 members cannot be identical to their singletons which all have one member. Moreover, since all Urelements have 0 members, they all are identical among each other (and they also all are identical to the empty set).¹⁷ But this does not seem to be an acceptable result for an account that associates perception of sets with that of physical objects, and that focuses on the particular perceptual mechanisms through which we learn to distinguish and recognize things. To sum up, **Extensionality** does not solve Balaguer's difficulty.

Perhaps, the Maddian can state a more efficacious defense against the charge of *ad hoc*-ness. Here is where Maddy's idea of identification, and, more specifically, **Fregean argument** and **Maddy's Fregean argument for identification**, can be used. They may start by claiming that both arguments necessarily take objects as a target. Then, they can concede that, as mentioned, the meaning of **Maddy's Fregean argument for identification** is the very opposite of **Fregean argument**. They can still claim, however, that, despite this, it seems clear that the specific application of both arguments to objects is not *ad hoc* but has a clear philosophical rationale: **Fregean argument** wants to avoid that

¹⁷See, for instance, [Lewis, 1991].

objects are countable, while **Maddy's argument for identification** wants objects to be countable. Finally, they can claim that this fact shows that the restriction of the identification to Urelements-singletons is not *ad hoc* either.

To be clear, Maddy can stress what I will call **Maddy's Fregean argument for the restriction** that can be formulated as follows:

- 1) Objects are individuated things having a number properties.
- 2) To say that objects are singletons is to say that they are individuated things having some number properties.
- 3) By contrast, sets do not need to be identified with other sets because they already have number properties.
- 4) Hence, the identification has to be restricted to the Urelements-singletons cases.

To further clarify, consider again Frege's example of a tree in front of us that needs to be counted. Maddy can argue that it is no coincidence that the example works with concrete objects and that it does not work with sets. Sets are already conceptualized, they do not need a further conceptualization.

To sum up, Maddy's restriction of the identification of Urelements-singletons cases can be considered as a consequence of the specific application of **Fregean argument**, and so of **Maddy's Fregean argument for identification** to objects.

It is time now to ask whether Maddy embraces **Eliminative 2** or

Non-Eliminative 2.

•Eliminative 2

Eliminative 2 and Epistemic Claim Let me start by evaluating **Eliminative 2**. **Eliminative 2** prescribes that the object reached by the identification has all and only the properties of singletons. The first thing to notice is that there is a sense in which **Eliminative 2** is incompatible with Maddy's **Epistemic Claim**. More specifically, **Eliminative 2** takes the opposite direction than that prescribed by Maddy's analogy which grounds her **Epistemic Claim**. The analogy prescribes that we perceive sets in a similar way as we perceive medium-sized physical objects. As mentioned, Maddy's analogy is asymmetric in that it explains how perception of sets works by assimilating it with that of medium-sized physical objects, and not the other way around. By contrast, **Eliminative 2** displays no distinction between perceiving physical objects and perceiving singletons such that each time that we think to perceive the first, we actually perceive the second. How to explain perception of singletons by that of medium-sized physical objects if what we perceive is not medium-sized physical objects but singletons?

This objection can be generalized. It can be argued that the problem does not only concern **Eliminative 2** but all strategies of identification that lead to numerical identity. Indeed, if identification is a process that has as a result one unique object, how to make sense of the analogy essentially involving two different objects? I think, nonetheless, that there

is a sense in which this objection misses the point. Once again, metaphysical and epistemic features have to be distinguished. The analogy describes an epistemic process including two objects, but metaphysically speaking, we just have one object. For the case at issue, we think having medium-sized physical objects and sets, and this is good to explain perception of sets by that of objects, but then, throughout the process of identification, we realize that what we have is just perception of sets.

Eliminative 2 and Ontological Claim This problem applies also to Maddy's **Ontological Claim**. How to claim that sets are located where their physical objects are located if physical objects are sets? As it was the case for the analogy, this objection generalizes. It can be argued that Maddy's **Ontological Claim** is incompatible with no matter which identification that leads to numerical identity. **Ontological Claim** needs two objects, identification prescribes that we have one unique object. Can we reply in the same way as we did for the analogy? Not exactly. **Ontological Claim** is not an epistemic process; it is a metaphysical point. To be compatible with the identification, **Ontological Claim** has to be reformulated as follows: we have one object that is spatiotemporally located, and following **Eliminative 2**, this object has to be a set.

Some properties of objects? I argue nonetheless that, even letting aside the issue related to **Epistemic Claim** and **Ontological Claim**, if Maddy wants to maintain both the analogy and the identification, she should reject **Eliminative 2**. This is the case because, even consider-

ing that the analogy is just an epistemic process that is metaphysically misguided regarding how many objects we have in our ontology, the fact that we epistemically use some properties of medium-sized physical objects to understand sets strongly suggests that what we reach by the identification is not an object that has all and only the properties of sets.

This is also supported by the fact that Maddy's medium-sized physical objects do not seem to acquire all the properties we generally attribute to singletons. Her account does not prescribe that, for instance, physical objects are discussed by Peano, nor that they reply to Pairing Axiom, nor to other set-theoretical principles.

Moreover, consider the following argument from Correia (2022, personal communication): Suppose that I perceive Socrates' hand as a part of Socrates; If **Eliminative 2** holds, then it is plausible to claim that what I am perceiving is {Socrates' hand} as a part of {Socrates}. However, this is not a plausible description of what I am perceiving. The problem is, of course, that the perception of Socrates's hand as a part of Socrates could be adequate (*i. e.* it could satisfy Maddian high standard for perception) but misrepresented by {Socrates' hand} as a part of {Socrates}. Hence, Maddy owes us an explanation of which other description of Socrates's hand as a part of Socrates we can adopt embracing **Eliminative 2**.

To tell the truth, singletons seem to be what we normally think medium-sized physical objects are.¹⁸ Why not "eliminate" singletons

¹⁸To put it in more general terms, what is at odd is not the nature of medium-sized physical objects, but that of singletons. And this oddity has clear philosophical reasons. As mentioned in the Introduction, singletons have a, so to say, tortuous path in

instead of individuals? If it this were the case, then identification would fit **Eliminative 1** and not **Eliminative 2**. Nevertheless, this cannot be the case. First, because **Eliminative 1** is unavailable; second, because, following Maddy’s idea, it is not true that the object that we reach by the identification exemplifies all and only the properties of medium-sized physical objects. It rather takes at least the $\{x\}$ -property of “having 1 as a number property”.

It is clear now that **Eliminative 2** is not an option. Thus, identification does not have a unique direction. This leads me to discuss the only metaphysical option left *i. e.* **Non-Eliminative 2**.

●**Non-Eliminative 2** Let me finally discuss **Non-Eliminative 2**. **Non-Eliminative 2** prescribes that the singletons-Urelement case displays symmetrical identification such that it leads to an object having some features of sets and some features of medium-sized physical objects. This option seems suggested by the monistic flavor of Maddy’s Compromise Platonism. To use her words,

”every physical thing is already mathematical, and every mathematical thing is based on the physical. To appreciate just how closely the two are intertwined in this view, try to separate them. A purely mathematical world would be empty.

What would a purely physical world be like? As soon as

the history of set theory especially because they do not fit Cantor’s canonical characterization of sets as “many (*jedes Viele*) which can be thought of as one” [Cantor, 1883], 204. Since singletons present no “many”, their nature remains definitionally mysterious.

there are number properties, there are sets that bear them, so a world without mathematical things would have to be a world without any things, a completely amorphous mass: the Blob”. [Maddy, 1991], 158.

Applied to the case of singletons, this idea produces the following conclusion:

“To add even the structuring into individual physical objects is to admit singletons, to broach the mathematical. The only way to confine ourselves to the purely physical is to refrain from any differentiation whatsoever.” [Maddy, 1991], 158.¹⁹

Non-Eliminative 2 and Correia’s objection First of all, it has to be noticed that **Non-Eliminative 2** could be used to argue against Correia’s concern. As mentioned, if we apply **Notational survival of singletons**, then we reach the counterintuitive result according to which $\{x\}$ has $\{x\}$ as its unique member. Now, given **Non-Eliminative 2**, it can be argued that, since $\{x\}$ keeps some properties from medium-sized physical objects, then it may be considered as having parts, instead of members. If this is true, then since $\{x\}$ has no members, it is, of course, not the case that $\{x\}$ has $\{x\}$ as its unique member either. In this sense, **Non-Eliminative 2** is superior to **Eliminative 2**.²⁰

¹⁹It could be interesting to compare this idea with the so-called Pythagoreanism according to which, broadly speaking, “all is number”. See, for instance, [Segal, 2019].

²⁰Notice also that if this argument goes, then Maddy’s restriction is safe. Indeed, Correia’s objection aims *prima facie* to reject Maddy’s restriction.

Singletons which are not singletons? This situation displays the case we have anticipated before in which the metaphysical framework we embrace invalidates a string of the proof. As mentioned though, this mechanism could involve other bad consequences which in turn discredit the metaphysical framework. Is this the case? Perhaps it is. It could be argued, for instance, that if we take $\{x\}$ as having no members, then we reach the counterintuitive result according to which $\{x\}$ and $\{\{x\}\}$ are completely different objects, and that such a difference is not compatible with set theory. $\{x\}$ and $\{\{x\}\}$ would not be comparable using **Extensionality**, and, worse than this, since $\{x\}$ does not exemplify the plausibly definitional property of “being a set having exactly one member”, it may be claimed that $\{x\}$ is not a singleton any more. Then, what makes some sets a singleton?

Probably, the best reply is to insist that, given **Non-Eliminative 2**, a singleton does not have to exemplify all the properties that we attribute to it in set theory. As Maddy’s prescribes, a singleton just is an object that has 1 as a number property. No **Extensionality** is involved. Moreover, Maddy can bite the bullet claiming that, as for the philosophical relevance of the restriction, $\{x\}$ and $\{\{x\}\}$ are supposed to be different. $\{x\}$ is the result of a process of identification that leads $\{x\}$ to acquire some number properties which $\{\{x\}\}$ already exemplifies.

But, of course, the opponent of Maddy’s identification can still insist on complaining that, since the only set of properties that $\{x\}$ exemplifies are such as “having 1 as a number property” or “being the result of a process of acquiring a given number property”, it is difficult to make

sense of the notion of singleton in set theory. Here is where the debate could become a dialogue of the deaf. It seems clear to me that Maddy has the burden of proof.

***Ad hoc*-ness of Non-Eliminative 2?** To be fair, the problem with the identity of singletons seems to be just a symptom of a more general issue of **Non-Eliminative 2** which has to do with *ad hoc*-ness. Someone could wonder how do we choose which x-properties and which {x}-properties the object of the identification is supposed to exemplify. But a rigorous answer is difficult to find. Difficult does not mean impossible. However, *ad hoc*-ness seems to be a real danger.

This is also related to the specific problem that I have somehow anticipated: How to justify the choice of **Notational survival of singletons** over **Notational survival of Urelements** if the properties that the object of identification takes from singletons are just properties such as “having 1 as a number property” and “being the result of a process of acquiring a given number property”? Perhaps, Maddy can claim that this set of properties is essential to the revised notion of singleton. But then, the anything but uncontroversial question of essentialism risks further aggravating the confusion of Maddy’s picture. What is sure is that the supporter of **Non-Eliminative 2** still owes us a more accurate explanation concerning the properties that the object of identification has to exemplify and the *criteria* through which this choice is made.

Piazza's, Michels', Correia's objections and Non-Eliminative

2 Another interesting thing about **Non-Eliminative 2** can be spelled out by considering again Piazza's proof. Maddy could argue that Piazza's proof is based on a confusion between the identification of x with $\{x\}$ and that of $\{x\}$ with x . In particular, on the one hand, the entailment from $\{x, y\} = \{\{x, y\}\}$ to $x \in \{\{x, y\}\}$ and $y \in \{\{x, y\}\}$ is based on the identification of $\{\{x, y\}\}$ with $\{x, y\}$. In fact, $\{\{x, y\}\}$ takes the properties of "having x as a member" and "having y as a member" from $\{x, y\}$. This is also corroborated by the fact that membership is not a transitive relation, so x is not a member of $\{\{x, y\}\}$ (and the same goes for y). On the other hand, $x = \{x, y\}$ and $y = \{x, y\}$ follows from the fact that x and y are members of $\{\{x, y\}\}$. Indeed, since the sole member of $\{\{x, y\}\}$ is $\{x, y\}$, and that $x \in \{\{x, y\}\}$, then $x = \{x, y\}$ (and the same goes for y). But, here, the relevant property, *i. e.* " $\{x, y\}$ being the sole member of $\{\{x, y\}\}$ " is clearly issued from the membership relative to $\{\{x, y\}\}$, and not to $\{x, y\}$. Generally speaking, when we stress that $\{x, y\} = \{\{x, y\}\}$, we expect that $\{x, y\}$ and $\{\{x, y\}\}$ are both singletons with cardinality 1 and pairs with cardinality 2 at the same time. But, of course, this cannot be the case. The proof is, therefore, misguided.

This remark also applies to Michels' and Correia's objections. Indeed, for Michels' case, the counterintuitive conclusion according to which $\{x, y\} \in \{x, y\}$ results from the combination between $\{x, y\}$ as having the $\{\{x, y\}\}$ -property "having $\{x, y\}$ as a member" plus the $\{x, y\}$ -property "being the set containing x and y ". Concerning Correia's proof, it can be said that his conclusion according to which $\{x\}$ and $\{\{x\}\}$ are identical

follows from the fact that they both have the $\{x\}$ -property "having x as its unique member" and the $\{\{x\}\}$ -property "having $\{x\}$ as its unique member".

That seems to be another case in which philosophical considerations invalidate certain technical features of some proofs. Nonetheless, the confusion between x and $\{x\}$ can be invoked only if Maddy's friends embrace **Eliminative 1** or **Eliminative 2**, namely if the proof is supposed to take either all and only x -properties or all and only $\{x\}$ -properties. Since what they embrace is **Non-Eliminative 2**, which prescribes that the object of the identification takes some x -properties and some $\{x\}$ -properties, then, in the absence of a clarification about which properties the object is supposed to exemplify, the proof goes. Indeed, it precisely does what **Non-Eliminative 2** prescribes to do: it takes some x -properties and some $\{x\}$ -properties.²¹

This means that **Non-Eliminative 2** entails the risk of falling back into the proofs against the identification and that the properties that the object of the identification exemplifies have, therefore, to be spelled out. Hence, once again, Maddy owes us a more accurate explanation of how identification is supposed to work.

Other arguments addressed to the more general idea of identifying singletons and Urelements, and not to how identification is made, can

²¹Notice also that another way to invalidate the proof is to claim that it takes incompatible properties. As we have seen, **Non-Eliminative 1** is not an option. This answer applies to Correia's and Michels' cases but not to Piazza's one. Piazza's proof takes the properties "having x as a member", "having y as a member", and " $\{x, y\}$ being the unique member of $\{\{x, y\}\}$ " which are not incompatible.

be stated. For instance, it can be argued that the distinction between Urelements and singletons is indispensable because it displays some relevant facts: *e. g.* that, unlike its singleton, a given tree is green; that such a singleton is a set-theoretical object, while the tree is not, and so on. I did not focus on these objections. I have instead been charitable conceding to Maddy that her reply to Chihara has some philosophical and technical dignity. I have shown, however, that it is still subjected to a lot of problems.

2.3.4 Conclusive remarks about identification

To understand whether Maddy's reply to Chihara is plausible, I have investigated the notion of identification. It has turned out that identification is a non-symmetric epistemic process having qualitative or numerical identity as a target. I have argued that, in the case of Urelement-singleton, identification has numerical identity as a target.

Then, I have discussed the validity of the identification of x and $\{x\}$ by replacing, in the *reductio ad absurdum*, each occurrence of $\{x\}$ by x . I have shown that the results that we reach are implausible. Then, I have tried the other way around replacing each occurrence of x by $\{x\}$. I have argued that this strategy fits better with Maddy's philosophical claims and that, when adequately restricted, it produces consistent results that meet the relevant set-theoretical *desiderata*. However, I have shown that it entails the counterintuitive result according to which $\{x\}$ is a member of $\{x\}$. I have argued that this conclusion can be questioned after having

investigated the metaphysical issue of which properties the object of the identification is supposed to exemplify.

I have discussed **Eliminative 1**, **Eliminative 2**, and **Non-Eliminative 2**. I have argued that **Non-Eliminative 2** is the most compelling strategy. This means that, in the case of Urelements-singletons, identification has as a target an object that has some properties that we originally attribute to singletons and some properties that we originally attribute to Urelements. However, I have shown that **Non-Eliminative 2** also comes with problems. In absence of a clarification about the properties that the object of the identification is supposed to exemplify (and the rationale of such a choice), **Non-Eliminative 2** is *ad hoc*, and can even be inconsistent.

Also, and more generally, there seems to be a mismatch between the rigorous meaning of **Epistemic Claim** and **Ontological Claim** according to which we perceive impure sets and medium-sized physical objects (which are both taken to be spatiotemporally located), the rigorous sense of the identification according to which we just have one object, and the anything but clear sense of Maddy's Compromise Platonism according to which perception is supposed to be the starting point for acquiring mathematical knowledge.

Enough for singletons and Urelements. Let me move on now considering whether Chihara's objection applies to other cases of sets.

2.4 Against Maddy's Epistemic Claim: Perception is too weak (part 2)

2.4.1 Generalizing Chihara's objection

I have spent a lot of time analyzing the case of singletons-Urelements. As mentioned though, Chihara's point according to which we do not perceive singletons can be extended to other cases of colocated sets. I have quoted two other cases. The first case concerns sets such as the set of four eggs that is now in my fridge. This set occupies the same place at the same time as, for instance, the set containing two pairs of eggs. However, the perceptual difference between them is not straightforward. The second concerns certain higher-order sets such as the set of three eggs, the set of that set, the set of the set of that set, and so on. That Maddy admits these cases is clear from what follows:

“A set of higher order [...] would again be located where its members are [...]. In this way, even an extremely complicated set would have spatio-temporal location, as long as it has physical things in its transitive closure. And any number of different sets would be located in the same place; for example, the set of the set of three eggs and the set of two hands is located in the same place as the set of the set of two eggs and the set of the other egg and the two hands” [Maddy, 1991], 59.

We are, therefore, entitled to ask the question of how does perception

discriminate between this kind of sets.

2.4.2 Maddy's solution

Maddy's reply goes as follows:

"we've all been amused by the psychologist's examples in which we see a single picture first as an undifferentiated mass, then as representing a definite number of distinct objects, or the child's puzzle in which the homogeneous jungle foliage resolves itself into a pack of ferocious beasts. Where a book-binder sees a large set of individual books (so many perhaps that she has no perceptual access to the exact number), the encyclopedia salesman sees three of his rival's products; where I see a set of four shoes, you might see a set of two pairs. A microscopic image looks to me like an unorganized mess of Jackson Pollock drips, while the biologist sees three paramecia and an amoeba". [Maddy, 1991], 66.

The fact that we perceive one set instead of another depends on the context in which perception holds, on our background experiences, on our intentions, and so on. This reply links back **Ontological Claim** and **Epistemic Claim**: since certain phenomena related to some natural human mechanisms account for the perception of different colocated sets (and can eventually also explain the reasons why we contingently see a set instead of another), and since perception is always perception of spatiotemporal located objects, then sets can be colocated. Coming back

to our previous example, I could explain that I am seeing a set of four eggs in my fridge because I aim to make a omelet, while my roommate Anna just sees, looking at the same perceptual content, a set of two pairs of eggs because she aims to use a pair to make fried eggs and the other pair to make a cake. These two sets exist and coexist.

2.4.3 Objection to Maddy's solution

Unfourtunately though, it is not always easy to find adequate explanations related to intentions, background experiences, and other context-sensitive factors. For instance, there seems to be no plausible replies for the case of sets such as $\{\text{egg1}\} \neq \{\{\text{egg1}\}\}$ and $\{\{\{\text{egg1}\}\}\}$. Perhaps, Maddy can claim to perceive them as they were a “system of boxes”. For instance, the singleton of an egg in my fridge is the carton containing just one egg, the singleton of that set is another carton containing the previous one, and so on. But this way of putting the story is just metaphoric; we do not, strictly speaking, perceive any carton or box simply because there is none; or, to better say, for the case of eggs, there just is one carton and it is, of course, the one that we (can) perceive: the carton containing eggs.

It seems that the only plausible justification for the existence of these sets is mathematical experience: Steve learns that sets are built in a cumulative way starting from singletons (or Urelements). But then, the explanatory role of perception fails. It seems, therefore, true that perception is too weak for accounting for all the sets that we want to account

for in set theory.

To be fair, there is also room for an objection to Maddy's perceptual solution for sets such as {egg1, egg2, egg3, egg4} and {{egg1, egg2}, {egg3, egg4}}. Generally speaking, for Maddy's argument to work, her cases for perception of colocated objects have to all share the fact that from a unique perceptual content, we perceive different objects. This follows from the fact that the sets we are supposed to account for are different from each other (*e. g.*, the set containing four eggs has cardinality 4, while that containing two pairs of eggs has cardinality 2). But, even granting that we can see different patterns from the same perceptual content, it is plausible to maintain that what we are seeing are not different colocated objects, but *different aspects of a unique object*.

This debate partially mirrors that between monism and pluralism. Pluralists argue that we have counterexamples to Locke's Law according to which materially coinciding objects made up entirely of exactly similar particles, related in precisely the same way, in identical surroundings, are identical. Monists maintain that we have no such counterexamples. In what follows, I will argue that Maddy's examples are not cases of different colocated objects. If I am right, then her analogy called to justify colocated impure sets fails.

Single picture/undifferentiated mass For the case of a single picture that is first seen as an undifferentiated mass, then as representing a definite number of distinct objects, the passage from a perceptual pattern to another seems just to follow from an improvement in the perception

of the subject such that what he is seeing is the same thing, first in a quite indeterminate and partial way, then in a much more refined and adequate one. Maddy's *desideratum* is, therefore, not met.

Duck-rabbit illusion Another case is that of the child's puzzle in which the homogeneous jungle foliage resolves itself into a pack of ferocious beasts. I have never seen this puzzle in my life, but I think that it can be considered a case of a perceptually ambiguous image in which we see different objects at different times. The most famous case of perceptually ambiguous images is the duck-rabbit illusion. I think that the discussion of Maddy's case can, therefore, be reformulated in terms of this more famous example. This is the line also taken by Chihara who speaks about a piece of wood that can alternatively be seen as rabbit-shaped or duck-shaped. According to Chihara, it is intuitive to claim that no matter whether duck- or rabbit-shaped, what the subject sees is the same piece of wood. I think that he is right, but that this point is not as straightforward as he puts it. Indeed, a Maddian can insist by claiming that these are two colocated objects. The opponents have to better find some arguments.

A fact that makes monism more appealing for this case is that the perceptual game including seeing a rabbit or a duck works precisely because we know that we are dealing with one unique object. In other words, the paradoxical effect that we experiment with when we see a duck or a rabbit is reached properly *because* these different perceptions are perceptions of the same object (and that we know that one object

cannot be two different objects at the same time). Here, Maddy can react claiming that the paradoxical effect is rather given by the fact that they are two colocated objects. In other words, the strangeness would follow from the violation of Locke's law. But this reply is not very compelling. Indeed, other colocated objects, whose existence could be accepted by a charitable interlocutor, seem not to generate the same effect. For instance, there is nothing perceptually paradoxical in a statue occupying the same region as the material composing it, nor in a person occupying the same region as her body.

Perhaps, the pluralist can argue that the paradoxical effect arises from the mutually exclusive perceptual feature of the image. As a matter of principle though, I do not see why colocated perceptually mutually exclusive objects would necessarily produce such a paradoxical effect. On the contrary, it is clear why it would be paradoxical if a unique object were two different objects. Also, the monist could find other cases of objects producing paradoxical effects without being perceptually mutually exclusive. Consider, for instance, the case of consubstantiation which is a Christian theological doctrine holding that during the sacrament, the substance of the body and blood of Christ are present alongside the substance of the bread and wine. The paradoxical effect seems here not to follow from some perceptual items, rather from a conceptual contradiction: one sole object is two distinct objects at the same time.²²

²²Some further remarks concerning the rabbit-duck case can be found in Section 2.8.2.

Microscopic image/unorganized mess of Pollock drips Another case is that of the microscopic image that I see as an unorganized mess of Pollock drips and that the biologist sees as three paramecia, and an amoeba. Plausibly, what the image really is about is not Pollock drips, but three paramecia and an amoeba. Then, it seems to me that what I am looking at is, contrary to Maddy's claim, three paramecia and an amoeba. And that this happens even if, due to a lack of competence, I do not know that I am seeing them. To see why, consider the following fact. At a given moment, scientific developments has shown that shooting stars are not stars but glowing meteoroids, micrometeoroids, comets or asteroids. Now, to argue that what I see looking at three paramecia and an amoeba is not three paramecia and an amoeba is to say that what men have seen looking at glowing meteoroids, micrometeoroids, comets or asteroids are not meteoroids, micrometeoroids, comets or asteroids. And this seems at least at odd with Maddy's realism.

Roughly speaking, the microscope image does not change its perceptual properties when looked at by me or the biologist. And this fact suggests that what we are looking at is a unique object. Of course, this object can be confused with other objects. But this does not mean that it is both the object that it is and the object that a given subject erroneously sees²³

²³Moreover, the fact that I erroneously perceive Pollock drips instead of three paramecia and an amoeba does not fit with Maddy's conditions for perception according to which our successful perceptual activity directed towards physical objects produces true beliefs and that physical objects in front of us play an appropriate causal role in the generation of our perceptual beliefs about them. See Section 2.1.3.

Set of individual books/three books I think that Maddy's most compelling case is that of the bookbinder that sees a large set of individual books, while the encyclopedia salesman sees three of his rival's product. However, different perceptual possibilities mean infinite ones. In the same place at the same time we have a book, the quantity of paper composing it, but also the object composing its half right side plus its half left side, or the object which is composed of its upper plus its lower part, and so on.²⁴ And even though such proliferation could be useful to ground colocated sets (which are infinite), it seems ontologically implausible.

To avoid this objection, a Maddian is obliged to take the essential rather than the perceptual properties as those which are relevant to distinguish between objects. That is, to distinguish between the materially coincident objects a and b, we have to find some essential properties of a and b which differ. However, there seems to be no way to grasp these properties without circularly assuming them. That is, we would need to assume that books and encyclopedias are discernible²⁵

To sum up, Maddy's cases of supposedly colocated objects are better understood as cases of "single" objects. Of course, these objects can be conceptualized in many different ways, but this is not enough to ground

²⁴These are general objections to pluralism that apply to all counterexamples to Locke's Law. These objections are formulated *inter alios* in [Varzi, 2007] and [Olson, 2001].

²⁵More generally, it can be argued that the fact that a certain object can be a set of books or three encyclopedias mainly relies on some semantic and practical considerations which are question-begging in the face of Maddy will of having two objects instead of one.

the existence of colocated sets.

2.5 Against Maddy's Epistemic Claim: Perception is too strong

2.5.1 Co-extensional sets having different predicates

Up to here, I have pointed out that perception is too weak to discriminate between some different impure sets. I argue that the opposite phenomenon holds as well. Perception is, so to say, "too strong" in that it generates more impure sets than required. This is the case of co-extensional sets having different predicates. Different predicates plausibly correspond to different cell-assemblies. Yet, **Extensionality** seems indispensable for sets. Let me consider three cases of co-extensional sets.

Superman/Clark Kent The first is the set that Lois Lane built from her cell-assemblies "being Superman" and "being Clark Kent". The set containing Clark Kent and Superman is the very same set but Maddy's account displays two different singletons.

The President(s)? It may be objected that the case of Superman/Clark Kent is trivial because is based on an error of perception which causes an error in the individuation of the objects which leads, in turn, to an error in counting sets. Or perhaps, that it cannot be taken seriously because it involves fictional characters. The second case is not subjected to such difficulties. Consider the property "being the president of the Italian

republic in 2020” and the property “being the president of the Italian Superior Council of Magistracy in 2020”. Same as before, these properties correspond to two different cell-assemblies, but generate coextensive sets.

Having a heart/having a kidney It can, nonetheless, be objected that “being the president of the Italian republic in 2020” and “being the president of the Italian Superior Council of Magistracy in 2020” are not perceptual properties. They are not grasped by perception, but by conceptual analysis, and so they cannot be used against Maddy’s account of perception. It has to be noticed that if this argument goes, then Maddy owes us an explanation of how perception is supposed to grant these sets which seem to have concrete members. But even leaving this issue aside, we can still find other examples that seem to uncontroversially invoke perceptual properties. This is the case of the very known and already quoted Quinean example of the predicates “having a heart” and “having a kidney”. Indeed, it is plausible to claim that we have two different cell-assemblies for things having a kidney and having a heart. However, the set generated is the same. Same goes for the set containing all the objects having a tone and that of all objects having a timber or that of all objects having a color and all objects having an extension.²⁶

²⁶Of course, it could be objected that I always perceive all these properties at the same time when perceiving the objects in question, so that it makes no sense to speak about different perceptions. I do not think that perception works in this way, and, as it is clear from the cases of duck-rabbit, microscope image, and so on, Maddy’s account rather goes on the other direction: we can have different perceptions of the same object.

To summarize, there seems to be something paradoxical concerning perception: it is too weak (we do not perceive every sets) but also too strong (we perceive much more sets than required). In other words, there are cases in which we do not perceive sets that are supposed to be in front of us, but there also are cases in which we perceive more sets than there actually are. Now, as mentioned, these cases of sets are essential to impure set theory. It follows that perception is not adequate to grant impure set theory.

2.5.2 *Qua*-objects framework

Correia (2023, personal communication) points out that Maddy can embrace an epistemological and metaphysical package that defends pluralism and, more importantly, this framework could also be used to justify cases of colocated impure sets for which perception has been said to be too weak to account for. On the epistemological side, she can argue for the phenomenon called "seeing as" which can be taken to be all over the place. Then, she can combine this epistemological framework with Fine's broadly Aristotelian view about *qua*-objects: whenever there is an object x with the property F , there is the further object x *qua* F . Fine calls x the basis and F the gloss. For Fine, the basis is sometimes plural (for instance, the piece of ham and the two slices of bread *qua* arranged sandwichwise).

Then, all the aforementioned Maddian cases of supposedly colocated perceivable medium-sized physical objects can be treated as cases of *qua*-

objects. For instance, *qua*-objects framework allows for the object *x qua* rabbit and the object *x qua* duck. And these are two different objects.

For impure sets, the idea is to use, as a gloss, the property of “being considered as one”. This option somehow corresponds to how Cantor describes sets. Maddy can therefore argue that if she perceives, for instance, egg1 and egg2, she can perceive them as forming a unity, and when she does so, there is such a thing as the two eggs *qua* considered as forming a unity. Applied to the case of {egg1, egg2, egg 3, egg 4} and {{egg1, egg2}, {egg 3, egg 4}}, we could say that, in the first case, we perceive egg1, egg2, egg 3, egg 4 as forming a unity, while in the second case, these objects form another different and more complex unity, *i. e.* another different and more complex object. Then, they are different *qua*-objects.

However, I am not sure that this framework goes without problems. One can indeed insist that the *qua*-objects framework is not able to justify the aforementioned perceptual cases where the fact of having one object instead of many has a main explanatory role. As mentioned, to explain the paradoxical effect of the duck-rabbit, we need to consider the object as one sole object. Of course, *qua*-objects framework allows for all the object rabbit, the object duck, as well as the object *qua* considered as one object. However, to explain the paradox, the object considered as one has to be, so to say, “ontological superior” to both the object *qua*-rabbit and the object *qua*-duck. And, more importantly, for this framework to save Maddy’s **Epistemic Claim**, it has to also grant that the “ontological superiority” of the object *qua* considered as one has to

be *perceptual*, which is not straightforward as long as what we actually perceive is a rabbit or a duck.

Friends of *qua*-objects can argue that even if the framework is not able to justify certain very specific perceptual phenomena such as the duck-rabbit paradox, it still justifies perception of colocated impure sets, and this is the important thing. However, even conceding that *qua*-objects framework can solve all the cases in which we perceive less objects than required, I argue that it has troubles explaining the reversed phenomenon of perceiving more sets than required. To see why, take again the case of sets having coextensional predicates such as the set of all the things having a heart and all the things having a kidney. Friends of *qua*-objects would say that we have different objects: objects *qua* “having a heart”, objects *qua* “having a kidney”, as well as objects “having a heart” and “having a kidney” *qua* considered as one. To account for this set-theoretical specific phenomenon though, they also have to argue that only one between these *qua*-objects is a good candidate for granting set theory, i. e. “having a heart” and “having a kidney” *qua* considered as one. But, as for the duck-rabbit paradox, friends of *qua*-objects owe us an explanation of why to chose that object instead of another.

For the case of sets having co-extensional predicates, the natural explanation is to claim that we do not want to reject **Extensionality**. But **Extensionality** is not a perceptual principle. More generally, it seems that, for the epistemic phenomenon of ”seeing as” to grant impure sets, it has to involve some crucial conceptual choices on its corresponding *qua*-objects that take a big departure from perception. And, even if

Maddy takes perception to be conceptually loaded, the conceptual work of **Extensionality** is not easy to be reduced to perceptual experiences.

To be fair, Maddy introduces intuition to solve all the problems related to perception of impure sets. Intuition grants **Extensionality** as well as other essential principles of set theory. Perception is, therefore, only the first step. Then, we need to appeal to other higher-order mechanisms. It is interesting to notice that, even when we do not have more objects than required, the conceptual “proliferation” that *qua*-objects framework involves seems incompatible with Maddy’s epistemic realism. I think that it is not a case that Maddy speaks about Steve opening his fridge and seeing a carton of three eggs, and not about Steve opening his fridge and seeing an infinity of *qua*-objects. In this sense, Maddy and *qua*-objects friends adopt two different strategies to reply to the same phenomenon: *Qua*-objects frameworks tries to “perceptualize” objects, while Maddy opts for intuition.

Generally speaking, as I have argued, the validity of intuition over perception seems a too easy shortcut: when perception fails, a more powerful (but arguably mysterious) faculty does the desired job. More importantly, my scope is to discuss the validity of Maddy’s **Epistemic Claim** as an independent principle. I have left therefore aside the issue of intuition discussing **Epistemic Claim** using instead the most successful epistemic and metaphysical framework of *qua*-objects, and showing that it still leads to many difficulties.

To sum up, a nice way to grant impure set theory by **Epistemic**

Claim is to embrace *qua*-objects framework. However, when applied to impure set theory (as well as to cases of other perceptual phenomena), it suggests that we need some additional conceptual choices that are not reducible to perception. In Chapter 1, I have raised a similar issue concerning impure sets. In particular, I have suggested that concrete objects are, so to say, "not enough" to individuate sets, and so that something more is required to grasp their nature. In the next section, I will present two further objections against Maddy's **Epistemic Claim**. The first is [Balaguer, 1994]'s objection which is to some extent similar to both *qua*-objects objection and to the argument explored in Chapter 1 in that it points out a tension between perception and the nature of impure sets. The second is based on Maddy's confusion between aggregates and sets, and takes some arguments from [Levine, 2005].

2.6 Other arguments against Maddy's Epistemic Claim

2.6.1 Balaguer's objection

In [Balaguer, 1994], the author argues that Maddy's Compromise Platonism is inconsistent in that it wants mathematical objects to be both abstract and perceivable. Then, Compromise Platonism is, so to say, physical and non-physical at the same time. To use Balaguer words,

"To naturalize platonism by bringing abstract objects into space-time is like naturalizing theism by taking God to be

the Lincoln Tunnel”. [Balaguer, 1994].²⁷

Balaguer is not completely clear concerning the mechanism that makes Maddy’s Compromise Platonism inconsistent. I argue that the story can be put in the following way: As mentioned, Maddy’s impure sets are perceivable individuated things. For something to be individuated, it has to have a number property. Perception of impure sets is, therefore, perception of things having some number properties. But to say that impure sets are “individuated things” is to claim that they are abstract to some extent.

This happens because for impure sets to be individuated things having a number properties, they need to be conceptualized. And concepts are classically taken to be abstract in that they have no causal role, and do not occupy space. In other words, even conceding to Maddy that impure sets are not abstract in the classic sense of not being spatiotemporally located, she is still obliged to embrace another non-traditional sense of abstractness as “being individuated”.

Balaguer argues therefore for a tension between Maddy’s **Epistemic Claim** and her **Ontological Claim**. This tension is triggered by her use of **Fregean Argument** for perception. To sum up, Balaguer’s objection can be formulated as a dilemma. Given that impure sets, as described by Maddy, seem to be individuated abstract things, then either perception of sets is perception of abstract objects (which is implausible and raises against Benacerraf’s problem), or Maddy is not right in arguing that we

²⁷This inconsistency is also pointed out in [Katz, 2000] in which the author claims that Maddy’s metaphysics is Aristotelian rather than Platonic.

perceive impure sets.

It has to be noticed that Balaguer's argument is somehow similar to **Slippery Slope** that I have proposed against Parsons' ontology (see Section 1.1.5). Both Maddy and Parsons seem obliged to accept some unwanted abstract objects in their ontologies. However, contrary to Parsons', Maddy's account does not prescribe that "being individuated" requires abstractness. Then, the Maddian can still argue that individuated things are fully concrete objects.

Nevertheless, a lot has been said about the difficulties that arise in taking concreteness to be representative of impure sets. From the epistemological point of view, what is at issue is, once again, perception. In this sense, Balaguer's argument relates somehow to that that I have addressed against friends of *qua*-objects. Maddy claims that perception is conceptually loaded. However, impure set theory presupposes conceptual choices that are not reducible to perception. And since perception is perception of spatiotemporally located objects, and impure sets are not reducible to perception, then they plausibly are not concrete things. As largely mentioned though, this argument is not conclusive. Impure sets can still count as un-perceivable but concrete objects. However, all that has been said in the previous sections against perception of impure sets suggests that this is not the case. In the next Chapter, I will further argue that individuated things are better understood to be abstract.

In what follows, I will address one last objection against Maddy's **Epistemic Claim**. As anticipated, this objection is based on Maddy's confusion between aggregates and sets.

2.6.2 Levine's argument

In [Levine, 2005], the author argues that the objections addressed to **Epistemic Claim** miss the point. If he is right, then Maddy's account is compelling. Despite this, Levine's paper does not have to be considered a defense of Maddy's account. After having argued against the objections often addressed to Maddy's framework, he highlights a tension between the realistic and the empiricist features of her account. I think that his objection is well taken but I do not focus on this point here. Since my aim is to dismiss Maddy's account, I will present Levine's argument in defense of **Epistemic Claim** and show that it meets some difficulties.

Levine refers to the objections stressed by [Chihara, 1991], [Balaguer, 1994], and [Carons, 1996] *inter alios*. He describes the general schema of the objections as follows: Since we do not perceive the essential properties of sets (such as **Extensionality** and **Existence of singletons**), then what we perceive are not sets. Hence, **Epistemic Claim** fails. To put the dialectic in more general terms, if a subject S that perceives an object x does not perceive some essential properties P that make $x = a$, then what she is perceiving is not a.

Levine argues against the conclusion of the argument taking back the analogy with physical objects. The fact that I do not perceive the essential properties of, for instance, a given table, does not entail that I do not perceive that table. In other words, the perception of things does not require the perception of their essential properties. Thus, it is possible for S to perceive a, even though she does not perceive its

essential properties P. **Epistemic Claim** is, therefore, still in place.

To tell the truth, [Chihara, 1991], [Balaguer, 1994], and [Carons, 1996] do not put the objection in this specific terms. Despite this, it seems to me a good way to formulate their ideas. A crucial thing is missing though. They do not just argue that we do not perceive these properties of sets; they also point out that what S is perceiving are aggregates instead of sets, namely collections of spatiotemporally located objects that often change in time. To take back Maddy's case, what Steve perceives in his fridge would be an aggregate of eggs, and not a set.

This fact is important because it suggests that Levine's position cannot be used to argue against [Chihara, 1991]'s, and [Balaguer, 1994]'s, and [Carons, 1996]'s objections. Indeed, it is plausible to argue that, in the situation at issue, what is relevant for understanding which object the subject S is perceiving is not that S is not perceiving P but that S perceives another set of properties F that suggests that $x = b$. In particular, the idea would be that S perceives a collections of spatio-temporally located objects that often change in time, and, since these properties belong to the definition of the term "aggregate", they seem to be essential to aggregates. Then, the schema of the argument can be reformulated as follows: since we do not perceive the essential properties P that make $x = a$, and that we perceive the essential properties F that make $x = b$, then what we are perceiving is b and not a.

Nonetheless, Levine could still object that this new formulation is misleading. Indeed, consider two objects a and b which are different from each other. Suppose also that a has P as essential properties and

that *b* has P as non-essential properties. Suppose that S perceives P. There seems to be no way of being sure that S is perceiving *a* and not *b*. Examples of these cases are not very easy to find. One of these goes as follows: Suppose that you have a bunch of painted fruits. Now, suppose that you look at this object from a point of view such that it looks as it were a painting. What you are perceiving is an essential property of paintings (*i. e.* "being painted") but also a non-essential property of fruits. In short, essentiality is not sufficient condition for concluding that we are perceiving a given object.²⁸

Then, the question is: what do we need to conclude that a given set of properties individuate aggregates, and not sets? Or, to put it in epistemic terms, which features a given set of properties have to exemplify to allow us to conclude that what a given subject is perceiving are aggregates and not sets?

The obvious reply is that the given set of properties have to include all the properties that allow us to individuate aggregates in a, so to say, univocal way. Another possibility is that, if, as in this case, the choice is between two specific objects, it is sufficient for the subject to perceive a set of properties that is not compatible with some of the properties that one of the two objects in question necessarily exemplify.

Does Maddy's case meet these requirements? To understand this point, let me first stress what Maddy says about the choice between aggregates and sets, which goes as follows:

²⁸Perhaps, it could also be argued that essentiality is not a necessary condition for concluding that we are perceiving a given object. Indeed, there can be other conclusive reasons related to the context in which perception holds.

”To decide the case between sets, aggregates, [...] and whatever else, we need to look, not to our perceptual experiences, but to our overall theory of the world, and we must ask which of these is best suited to playing the role of the most fundamental mathematical entity. [...] On this score, sets win going away; they are extremely simple and manageable entities that form the basis for a surprisingly effective and efficient mathematical theory”. [Maddy, 1991], 61.

Maddy appeals to the best theory’s explanation. Arguably, this strategy is compelling only if the alternatives among which we are called to choose have the same plausibility. And this is, I think, what she argues for. To use her words,

“Asking Steve to look and see whether he’s perceiving a set or an aggregate is like asking him to look and see whether the egg is solid or mainly an empty space littered with atoms”
[Maddy, 1991], 61.

Now, this idea of taking sets and aggregates to be equally plausible is, of course, heterodox. All these examples that are called to justify **Ontological Claim** and **Epistemic Claim** are issued from medium-sized physical objects occupying spatiotemporal regions by having parts, which is a typical behavior of aggregates. To take back again the problem of justifying colocated sets, Maddy argues as follows:

“none of this [case of colocated sets] is any more surprising

than that fifty-two cards can be located in the same place as a deck” [Maddy, 1991], 59.

Yet, the standard view on sets would take these analogies to be unhelpful, if not misleading. For the last one, it could be said that, even conceding to the pluralist that the deck and the cards composing it are different objects, it is not surprising that fifty-two cards can be located in the same place as a deck because a deck is a concrete object having concrete cards as proper parts. By contrast, sets classically have no parts, but members. And the notion of membership is not transitive. We cannot consider, for instance, the set of the set of three eggs as a concrete object containing the eggs and the set of eggs as its proper parts. Indeed, to claim that sets have parts dangerously put together the notion of sets and that of aggregates.

However, Maddy could still reply that her account aims to precisely dismiss these differences between aggregates and sets. Indeed, as we have largely shown, sets have certain characteristics that we normally attribute to aggregates. Then, the opponent cannot argue against **Epistemic Claim** by saying that what the given subject S is perceiving are properties that cannot be attributed to sets. Also, in this perspective, the fact that sets have parts as members could be embraced.²⁹

Nonetheless, the whole dialectic suffers from the same problem related to the idea of identification. Maddy wants to argue that perception of sets is analogous to that of medium-sized physical objects, and that sets have

²⁹Maddy is not clear on this point, but, as mentioned in Chapter 1, Section 1.2, literature on this point is available (See [Fine, 2010]).

certain characteristics of medium-sized physical objects. But then, to reply to the objection according to which what we perceive does not have the characteristics that we normally attribute to sets, she seems to hold that sets indeed have these properties. However, since I do not perceive any difference between sets and aggregates, and that the properties that I perceive are those which are classically taken to belong to aggregates, Maddy has the burden of proof of justifying that the two hypotheses have the same plausibility.³⁰

To sum up, Levine is right in undermining [Chihara, 1991], [Balaguer, 1994]’s and [Carons, 1996]’s objections, but the opponent of Maddy’s account can argue for a more compelling version of them.

Until now, I have discussed some objections addressed to Maddy’s **Epistemic Claim**. All these objections suggest that **Epistemic Claim** is not adequate. However, as largely mentioned, the fact that **Epistemic Claim** is not adequate does not mean that Maddy’s view of impure sets as fully concrete objects has to be rejected. Objections against **Epistemic Claim** just undermines Maddy’s ontology of impure sets. Given this, let me move on discussing some objections directly addressed to Maddy’s **Ontological Claim**. The first one is [Cook, 2013]’s objection; the second one is [Parsons, 1990]’s.

³⁰Notice that this conclusion is similar to that that I have proposed speaking about Chihara’ objection: since perception of an object does not necessarily require that of its constituents, perception of $\{x\}$ does not require any perceptual difference between $\{x\}$ and x . However, Maddy’s Compromise Platonism is based on the idea that perception grounds mathematics. How to ground the difference between $\{x\}$ and x if no perceptual difference is available?

2.7 Against Maddy's Ontological Claim

2.7.1 Cook's argument on location

In [Cook, 2013], the author proposes an argument against Maddy's **Ontological Claim**. He argues that some objects are abstract objects of concepts and have the following form: the @ of C, where C is a concept and @ a function that goes from the concepts to the corresponding adequate abstract objects. Examples of these objects are numbers and sets. Then, the author takes a Uniformity Thesis (UT) as a premise. UT claims that the relations between the spatial location of the set of the concept C and the spatial locations of objects falling under C should mirror the spatial location of the number of the concept C and the spatial location of objects falling under C. Then, he shows that UT plus Maddy's **Ontological Claim** leads to contradiction. To see why, he proposes to consider the following two concepts:

- 1) Being identical to A or B

- 2) Being identical to C or D

Then, from Maddy's **Ontological Claim**, it follows that

- 3) The set of the concept "being identical to A or B" is spatially located where A and B are located. And the same goes for the set of the

concept “being identical to C or D”.

From UT, it follows that

4) The number of the concept “being identical to A or B” is spatially colocated where A and B are colocated, i. e. The number 2 is spatially colocated where A and B are colocated. And the same goes for C and D.

Conclusion:

A and B are colocated where C and D are located. Contradiction.

However, there is room to state that Cook’s argument does not go. In [Effingham, 2013], the author discusses Cook’s objection by disambiguating the notion of “being located (and colocated)”. In particular, he argues that there is at least a sense of location that avoids Cook’s contradiction. This is the case of **Exact Location** when applies to many-one relation.³¹ And this happens because this notion of location does not display the transitivity that Cook needs for his argument to be conclusive.

The classic example of **Exact Location** applied to many-one relation is that of immanent universals and their instances: the universal is located where its instances are located. However, it not the case that all

³¹Effingham takes **Exact Location** to be primitive. As I have mentioned in Chapter 1, it can be defined by using [Gilmore, 2013]’s strategy.

instances are located at the same place. For instance, the Redness is located where all the red things are located. However, it is not the case that all red things are located at the same place. Then, Effingham argues that numbers behave similarly in that they are multiply instantiated. Yet, their instances are not colocated. To take back Cook's example, the number 2 can be colocated with A, B, C, and D. But, since transitivity is blocked, we do not reach the unwanted result according to which A, B, C, and D are colocated.

Nevertheless, I argue that Effingham's argument leads to other implausible consequences. I have conceded that it can be the case that the number 2 is located where B and C are located, while B and C are not colocated. As mentioned, transitivity does not hold for certain many-one relations. Now, the question is: Does symmetry hold? Classically, it is the case: the water in the sponge is located where the sponge is located, and viceversa. And, by the way, it can even be argued that the very meaning of colocation seems intimately related to symmetry (two objects are located at the same place when the first is located where the second is located, and viceversa). But symmetry of location applied to Effingham's argument leads to bad consequences. Consider again the case of Redness and their instances. For instance, Redness is located in a given red strawberry, but it is not the case that strawberry is located where Redness is located.

Effingham has therefore to block not only transitivity but also symmetry. He can, of course, insist that it is in the very nature of location for many-one relations such as universals and their instances, or num-

bers and their concrete instantiations to “take only one direction”: from the universals/numbers to the instances. However, it can be argued that asymmetry goes against Maddy’s naturalistic monism according to which “every physical thing is already mathematical, and every mathematical thing is based in the physical”, which suggests a symmetric relation between numbers and physical things having those numbers.

Another problem deserves to be pointed out, and concerns sets having both concrete and abstract members such as {Socrates, the golden mountain}. Following Effingham’s idea of numbers and their instances, the number 2 is located where Socrates and the golden mountain are located. But the golden mountain is abstract and so it is not located. Location for many-one relations necessitates, therefore, to be further clarified.

To sum up, Cook’s argument against Maddy’s **Ontological Claim** can be questioned using Effingham’s idea of location and many-one relations. However, this is still room for pointing out some flaws in Effingham’s defense. Given this, I conclude that both Cook’s and Effingham’s arguments are not conclusive. In what follows, I will present Parsons’ argument against Maddy’s **Ontological Claim**.

2.7.2 Parsons’ objection

Sets with spatiotemporally non-coexistent members

As Parsons remarks, Maddy’s **Ontological Claim** seems inconsistent with our talk of certain particular sets, namely sets with spatiotemporally

non-coexistent members. Parsons addresses such a worry in the following way:

”Consider a set of objects no two of whose elements exist at the same time, say the set consisting in the Roman Empire, the Holy Roman Empire, and the German Empire proclaimed in 1871. If this set existed in say, 1900, our principle would imply that the Roman Empire and the Holy Roman Empire existed at that time. We might avoid this result by saying that a set could change its elements over time; the same reasons supporting the rigidity of membership in modal contexts count against that. For example, our set, which intuitively has three members, would at any time being identical either with a one-element set or (as now) with the empty set”. [Parsons, 2008], 17.

Then, Parsons concedes that Maddy could avoid such unintuitive results embracing a tenseless fourdimensional metaphysics (Fourdimensionalism). He does not furnish the details, but we can imagine how the situation would work: the set containing the Roman Empire, the Holy Roman Empire, and the German Empire proclaimed in 1871 just is the set containing the temporal sections at which its members disjointly exist. Then, during its existence, this set does not strictly speaking lose nor gain any member.

Extensionality and Change

However, Fourdimensionalism does not solve all the problems that arise with Maddy's idea of sets. Furthermore, the application of Fourdimensionalism to her account seems problematic. To see this, let me point out a difficulty affecting Maddy's account when associated with Three-dimensionalism. After that, I will show that Fourdimensionalism allows us to overcome such a difficulty. However, the solution seems to take a big departure from Maddy's **Epistemic Claim**, and more generally, from Maddy's analogy between medium-sized physical objects and impure sets.

Plausibly, **Epistemic Claim** and Maddy's analogy suggest that concrete spatiotemporally located sets can change their members in time. To see this, imagine, for instance, a situation in which Steve opens his fridge and sees a carton of three eggs. Then, his roommate picks one egg. After that, Steve opens his fridge again and sees a carton of two eggs. Plausibly, he recognizes the given carton of eggs as being the same object losing an egg across time. Then, given the analogy with medium-sized physical objects, we would say that the set of three eggs that Steve has seen in his fridge is the very same object of the set of two eggs that he has seen in his fridge after his roommate has picked one. In other words, we keep the intuition according to which we have the very same object that loses a member: $\{\text{egg1}, \text{egg2}, \text{egg3}\}$ becomes $\{\text{egg1}, \text{egg2}\}$.

Hence, generally speaking, perception of sets is perception of some objects preserving their identity in time even when affected by some

changes. Of course, the debate over what kind of change an ordinary object can undergo without losing its identity is a very complex one, and some people even deny that objects can change. Despite this, taking for granted the commonsensical view according to which physical objects can undergo at least to some changes (as Maddy seems to do), we can imagine that a similar situation arises for sets.

Moreover, conceding to Maddy that we can perceive different collocated sets, in seeing the same carton of eggs, Steve could also perceive two different sets: $\{\text{egg1}\}$ and $\{\text{egg2}, \text{egg3}\}$. Then, imagine that his roommate picks egg1. Steve opens the fridge and sees a set containing two eggs, *i. e.* $\{\text{egg2}, \text{egg3}\}$. Compare this case with the previous one of Steve first perceiving $\{\text{egg1}, \text{egg2}, \text{egg3}\}$, then $\{\text{egg1}, \text{egg2}\}$. We have claimed that the set loses a member. Now, if we want to keep a similar intuition as that of $\{\text{egg1}\}$ and $\{\text{egg2}, \text{egg3}\}$, then the two sets, namely $\{\text{egg1}\}$ and $\{\text{egg2}, \text{egg3}\}$, becomes one set, *i. e.* $\{\text{egg2}, \text{egg3}\}$. Perception of sets allows, therefore, for both cases of sets changing their members, and cases of one set becomes many different sets (or viceversa).

This is, of course, incompatible with **Extensionality** which states constancy of sets. According to Maddy, constancy of sets is not perceptual but results from higher-order processes related to intuition. It is a purely conceptual insight about sets which is gained from understanding an aspect of how sets behave. As mentioned, Maddy claims that perception is just a first step, and that it has to be implemented with intuition. But here perception and intuition, which are both supposed to build our knowledge of sets, lead to incompatible results. Then, assuming intuition

as a second steps means to question the validity of perception. Hence, **Epistemic Claim** is strongly undermined.³² If we rather want to maintain that perception of sets is analogous to perception of physical objects, then what we have to rule out is **Extensionality**. But **Extensionality** is an identity criterion for sets, and seems, therefore, indispensable.

Now, Fourdimensionalism could accommodate the tension between **Extensionality** and the intuition according to which sets can change their members. According to Fourdimensionalism, the set of eggs that Steve sees in his fridge before and after his roommate steals one is just a unique object having different temporal parts. Then, **Extensionality** is not localized to each temporal part of the object in question, but applies to all its fourdimensional extension. The “change” in the object is explained by the change in its temporal parts. {egg1, egg2, egg3} always has cardinality 3. It is, nonetheless, a set that has the temporal part of not having egg1. So far, so good. What happens for the previous case of {egg1, egg2} and {egg1} that becomes {egg1, egg2}? These are two different fourdimensional objects one of which, {egg1}, has a shorter life than {egg1, egg2}. Fourdimensionalism has, therefore, the advantage of reconciling certain perceptual phenomena with **Extensionality**.

Many problems can be raised against Fourdimensionalism.³³ I argue

³²Also, as mentioned, the validity of intuition over perception seems somehow a too easy shortcut: when perception fails, a more powerful (but arguably mysterious) faculty does the desired job. Other arguments against Maddy’s use of perception associated with intuition can be offered. For instance, in [Balaguer, 1994], the author claims that the epistemic passage from perceivable impure sets to pure sets grasped by intuition is suspicious, and, in [Parsons, 1990], the author suggests that the gap is due to the own nature of the objects we are dealing with. That is, higher order set theory is not a physically possible structure.

³³For an overview of these problems, see [Sider, 2001].

that, even leaving aside this huge metaphysical debate, certain major difficulties of combining Fourdimensionalism with Maddy's framework can be pointed out. One of these is that given that sets are not determined by the objects that they present in each temporal section, but by all their fourdimensional extension, then what we need is a criterion of identity that determine that given object. Generally speaking, if Maddy wants to maintain the analogy between sets and physical objects, she owes us a criterion of identity for the temporal extension of sets. Why two temporal sequences s_1 at t_1 and s_2 at t_2 are both supposed to be temporal sequences of the same object? Of course, it can be said that, generally speaking, criteria for diachronic identity are not easy to individuate. I argue, nonetheless, that there is something in Maddy's framework that makes Fourdimensionalism less adequate than Three-dimensionalism. More specifically, Fourdimensionalism does not seem easy to combine with the perception-claims allowing us to justify **Ontological Claim** by **Epistemic Claim**: we do not perceive sets containing temporal disjointed sections of concrete objects. Generally speaking, as it is described, Maddy's theory of perception seems to take objects to be three-dimensional.

To sum up, Maddy can reply to Parsons' objection to **Ontological Claim** by adopting Fourdimensionalism. However, the application of Fourdimensionalism to Maddy's account does not go without problems. More specifically, it seems difficult to find good criteria of identity for fourdimensional impure sets, and Maddy's theory of perception seems to rather speak about three-dimensional objects. This means that there is

still room for Parsons' objection.

I conclude by claiming that both Cook's and Parsons' arguments against Maddy's **Ontological Claim** are not conclusive. They can, nonetheless, be appropriately defended. Together with all the arguments against Maddy's **Epistemic Claim** that I have proposed in this chapter, they seem to suggest that her account for impure sets as fully concrete objects is inadequate. In what follows, I will conclude by pointing out some additional remarks concerning Maddy's alternative reply to Chihara, as well as some further clarification of monism and pluralism.

2.8 Additional remarks about Maddy's account

2.8.1 Alternative replies to Chihara's objection

In Section 2.2.2, I have presented Maddy's favorite reply to Chihara's objection. This reply consists in a biting-bullet strategy claiming that Urelements have to be identified with their singletons. I have also shown that this solution comes with problems. For the sake of completeness, it is worth noting that Maddy considers two other replies.

Perceptual difference between singletons and Urelements

The first consists in arguing that, contrary to the expectations, a perceptual difference is there. The singleton corresponds to an individuated object which has 1 as a number property. By contrast, the object itself

has no determinate number property: it can be considered either a physical mass, or many cells, many molecules, many atoms, and so on. This reply corresponds to **Fregean Argument**. Nevertheless, Maddy quickly dismisses this alternative arguing that once our cell-assemblies are sufficiently developed for seeing objects as individuated things, the difference between the objects and their singletons “evaporates”. In what follows, I will try to take this reply more seriously. However, such an attempt will lead us to add some further and, I argue, stronger reasons to not account for singletons by perceptual differences.

The opponents of the distinction could argue that a physical mass, many cells, or molecules are just different ways of referring precisely to a given object having 1 as a number property. They can argue that we can see a given object as many cells, many atoms, and so on, *only if* we have previously individuated that object as having 1 as a number property. How can we even speak about many cells, molecules, atoms, and so on, without presupposing something having such cells, molecules, atoms, and so on?

At this point, the friends of the difference will perhaps reply that the perception of objects as having no determinate number properties is not the perception of molecules, atoms, etc. It is more like the perception of a child who has not yet achieved the competence of seeing the objects around him, and that just sees undifferentiated stuff which are not well-distinguished from the surrounding.³⁴ The perceptual difference between

³⁴Such a perception captures bundles of properties having no substrate, where the expression substrate has to be intended as something which counts as an object having 1 as a number property.

the objects and their singletons is, therefore, explained by comparing the child's perception with that of an adult who is able to immediately perceive individuated objects. As a consequence, we can no more claim that the perception of singletons is made impossible by the fact that we are unable to perceptually distinguish between singletons and objects.

However, this solution violates Maddy's **Directness condition** according to which we do actually perceive physical objects, as opposed to sense data, percepts, or representations of some kind. Then, we come back to what Maddy has quickly sketched: once our cell-assemblies are sufficiently developed for seeing objects as individuated things, the difference between the objects and their singletons "evaporates".

However, it could be argued that to dismiss the perceptual distinction, these arguments are not enough. We do not have to just assume **Directness condition**, we have to also show that we do not possibly perceive undifferentiated stuffs. In other words, we have to argue that, once our cell-assembly is sufficiently developed, we do not just perceive objects as individuated things as opposed to sense data, but we also cannot avoid doing so.³⁵ This can be considered a modally loaded version of **Directness condition** suggesting that the child's perception is out of reach.³⁶

³⁵Speaking about modality, I refer here to physical necessities and possibilities. It is impossible for an adult S to perceive some Fs means that in a world in which physical laws obtain, there is no S perceiving some Fs.

³⁶Moreover, Maddy's rejection of the distinction works if we presuppose that what is worth is just the adult's perception. I agree with such a requirement. It would indeed be quite strange that a difference that concerns mathematical objects depends upon the development of our cell-assemblies such that once we grow up, we are not anymore able to justify it.

If I am right in stressing Maddy's dialectic in this way, then we can still imagine a really obstinate defender of the distinction simply arguing against the modally-loaded version of **Directness condition**. She can claim that even though we do not have a common experience of perceived undifferentiated stuffs, we still could make an effort in order to reach again such a perception. To be charitable, we can indeed concede that I see singletons, that the child sees some stuffs, and that I am also in the privileged position of possibly perceiving both.

The question is now: are we able to perceive singletons and objects as colocated things? Perhaps we are. Perhaps, I can "locate" the rediscovered child's perception exactly where our spontaneous adult perception takes place. If this is the case, then we have two perceptually distinct and plausibly colocated items.

Nonetheless, a problem arises. Even conceding that we perceive both individuated things and undifferentiated stuffs, it is highly plausible to claim that these are two different perceptions of the same object: we perceive the very same object as an individuated thing or as an undifferentiated physical mass. But physical things and their singletons are supposed to be two different objects. Moreover, singletons having physical things as members plausibly are singletons of individuated objects, and not singletons of undifferentiated stuffs. That is, when we speak about for instance {Socrates}, we are speaking about the set containing the individual Socrates as a member, and not about the singleton of some undifferentiated Socrates-stuff that we cannot even distinguish from the

surrounding.³⁷

To sum up, Maddy is right in claiming that her first reply to Chihara's objection according to which we do perceive singletons because we perceive a difference between undifferentiated stuff and individuated objects fails. However, this does not happen for the reasons that she mentions. The problem is not that we cannot reach the perception of undifferentiated things, or that we cannot locate this perception in the same "place" as that of individuated objects. The problem is rather that the perceptual difference in question does not seem to correspond to that between physical objects and their singletons.

Unperceivable difference between singletons and Urelements

After having dismissed this first reply, Maddy reformulates Chihara's objection in the following way:

"While the set theoretic realist has a ready answer to one question — what distinguishes a physical mass from a set? — Chihara is asking another—what distinguishes an individuated physical object from its unit set? The answer to this new question cannot be that one has an unambiguous number property and the other doesn't, because both the single object and the singleton have the same number property: one".

[Maddy, 1991], 151.

At this point, the other way to go consists in postulating a difference

³⁷This problem is the same that we have found for the identification. See Section 2.3.3.

between the objects and their singletons, conceding to Chihara that such a difference is not perceivable. Following Chihara's example, this move consists in claiming that even though the duck looks like, waddles like, quacks like the set whose only member is the duck, they still are different colocated objects. In other words, **Ontological Claim** holds even when it cannot be justified by **Epistemic Claim**.

The "unperceivable"-strategy can be justified by extending the previous analogy between medium-sized physical things and sets: the fact that there is an unperceivable difference between objects and their singletons is not that strange. Indeed, other differences between physical things are not perceivable either. As Maddy remarks, we cannot for instance immediately distinguish between gold and fool's gold, or between water and heavy water.

However, there seems to be some relevant inconsistencies between unperceivable differences for the case of singletons-objects and for those of physical things. While, in science, it could be the case that unperceivable differences are detected by more sensitive instruments, it is of course implausible to claim that the unperceivable difference between the objects and their singletons can be reached in a similar way. We can for instance perceive, with the help of some specific instruments, that the composition of water is H₂O, while that of heavy water is D₂O; or that gold has 79 as atomic number, while fool's gold has 26. By contrast, we can hardly imagine an instrument able to detect the difference between physical things and their singletons.

But, in science, there is another strategy to justify unperceivable

differences. They can be implied by well-supported theories. This means that we postulate a difference because either it serves as an explanation for other perceivable phenomena, or it is implied by other perceivable phenomena. Can this be the case for objects and singletons? It can be argued that such differences can be accounted for by our philosophy of mathematics or by other theoretical constructions.

Nevertheless, the difference between the objects and their singletons seems not to explain nor to be explained by any other perceivable phenomena. More generally speaking, for unperceivable differences between physical things, we keep alive the hope of, one day or another, perceiving them. Yet, this seems not to be the case for the difference between objects and their singletons. To put it in another way, it seems not only that the difference between singletons and physical objects is not perceived, but that it is, by its nature, not perceivable either. Reasonably, we have no hope to perceive it one day or another.

The difficulty in maintaining the analogy between singletons-objects and cases of unperceivable differences between physical things seems to follow from what Maddy calls the “inviolable differences between concrete and abstract, or between mathematical and physical”. (see [Maddy, 1991], 152). However, such a claim jettisons the entire effort of formulating a monistic theory which replies to Benacerraf’s dilemma by defending the idea of perceivable physical sets. Thus, this strategy ends up having similar consequences than those raised by Chihara’s objection: **Epistemic Claim** fails for singletons cases and so **Ontological Claim** remains unjustified. As largely mentioned, even though this does

not mean that **Ontological Claim** is false, the burden of the proof is turned back to the set-theoretic realist.

2.8.2 More on monism and pluralism

In Section 2.4.3, I have argued that in the case of the duck-rabbit representation, monism is superior to pluralism because the paradoxical effect comes from the fact that the same object is perceived as two different objects. I think that the monist has a further argument.

The classical way to argue in favor of the possibility of different collocated objects is to show that they have different temporal and modal properties. But this seems not to be the case for the duck/rabbit-shaped piece of wood. There seems to be no possible alteration that the duck would undergo without this alteration affecting the rabbit as well, and vice-versa. For instance, if I pull out a certain quantity of wood from the piece of wood, both the rabbit and the duck will undergo some mutilations.

The pluralist could still maintain that certain mutilations lead to the annihilation of only one between the rabbit and the duck. We can indeed think that losing, for instance, a "perceptually essential part" of the duck that eliminates its perceptibility does not lead to the annihilation of the rabbit. However, such a position is not uncontroversial. To claim that losing a perceptually essential part of the given duck leads to its annihilation requires some precise criteria of identity for the representation of the duck. Similarly, to claim that after such a change, the rabbit

survives requires some precise criteria of identity granting that, after the change, the representation of the rabbit continues to be the very same representation.

This fact has to do with the thorny question concerning the identity criteria for representations. In general, we can assume that seeing a rabbit, but not a duck after the change is a necessary condition for granting the identity in question. In other words, x and y are the same representation if they represent the same thing. However, this principle may have some counterexamples. For instance, if I steal the Mona Lisa and I draw some big donkey ears on top of it, what the picture represents is not the Mona Lisa anymore. However, we may still have the intuition that the picture is the very same because, for instance, it maintains all its proper parts during that time or because the police will identify the picture in question as the stoled Mona Lisa. To be charitable, I will concede that for a picture to remain the same, it is necessary that what it represents does not change.

Nonetheless, this cannot be taken to be a sufficient condition. To see why, imagine to draw a design representing a rabbit. Then, you are not satisfied, so you delete it and draw a new one. Plausibly, even if both designs represent a rabbit (so both representations are representations of a rabbit), they are two different objects.³⁸

³⁸Someone could argue that the supposedly different designs are actually one sole design considered at two different moments, and so that the counterexample does not work. However, other counterexamples can be found. Imagine, for instance, two different statues both representing Garibaldi, one in Pisa, and the other in Rome. These statues represent the same object but are, of course, not the same object. What makes this case more compelling is the fact that the statues, contrary to the designs, are located in different places at the same time.

With this in mind, it is easy to see that the pluralist is missing a piece. Holding that, for an object x to represent y , it has to necessarily (but not sufficiently) be what it is, the pluralist can grant that, after the change, the duck is no more there. Yet, he cannot explain the survival of the rabbit. Then, the monist could argue that the sufficient condition for the identity of representations has to hold for all representations, and so that it is independent of the content which is represented.

What a criterion of identity for representations looks like? Let me assume that a representation remains the same *iff* none of its boundary lines are modified. This is a completely arbitrary way to stress the criterion but is useful to show that the monist can easily resist the fact that, in the previous example, the duck and the rabbit have different temporal and modal parts. Indeed, given that a representation remains the same *iff* none of its boundary lines are modified, we have that the monist *desideratum* is achieved: after modifying the boundaries of the representation, even if we could claim to see a rabbit and not a duck, what is sure is that the resulting representation is different from the previous one. We cannot, therefore, modify the duck without modifying the rabbit, or vice-versa. No difference in temporal or modal properties between the duck and the rabbit is, therefore, involved.

Perhaps, the pluralist can still insist that the criterion of identity for representations *does* involve the objects which are represented. He can even claim that having a criterion for representations that is independent of the content which is represented is already to stress that the duck/rabbit actually is a unique object. If this is true, then the monistic

defense is circular. Then, the pluralist can find some other compelling sufficient criteria for representations involving a difference in the temporal and modal properties of the objects in question. Or perhaps, he can simply refuse to speak about representations. This kind of debate is far to be closed and deserves deeper attention. Yet, this is not the right place to start such an inquiry. I will limit myself claiming again that, in the case of the duck/rabbit illusion, the pluralist owes us an explanation.

2.9 Conclusions

In this chapter, I have presented Maddy's account of impure sets.

In Section 1, I have argued that Maddy's idea can be divided into two principles: **Ontological Claim**, according to which impure sets are spatiotemporally located objects, and **Epistemic Claim** stating that we perceive impure sets. Then, I have presented Maddy's idea of perception.

In discussing Maddy's account, I have played the devil's advocate showing that, even when it is presented in the most compelling way, it is still subjected to serious problems.

In Section 2, I have discussed Chihara's objection according to which it is impossible to perceptually distinguish between singletons and their members. I have considered Maddy's bullet-biting strategy which consists in identifying singletons and their members.

In Section 3, I have tried to defend this strategy by deeply analyzing the notion of identification. I have, nonetheless, shown that, even embracing the most adequate notion of identification for singletons and

members, Maddy's bullet-biting strategy is subjected to a lot of difficulties.

Section 4 is dedicated to some generalizations of Chihara's objection that concern other cases of impure sets. I have presented Maddy's defense, and shown that it does not account for those cases. I have concluded that, even embracing Maddy's high standard, perception is too weak for account for all the impure sets that we need to account for.

Section 5 points out that Maddy's account is also subjected to the opposite problem. Perception is too strong to justify and explain impure set theory because it generates more sets than required. I have argued that the good candidates to show this fact are sets having different but co-extensive predicates. After that, I have presented the so-called *qua*-objects framework that could solve this problem. I have argued, nonetheless, that this framework is subjected to other substantial difficulties.

In Section 6, I have presented two further arguments against Maddy's **Epistemic Claim**. The first is Balaguer's objection which suggests the existence of abstract objects in Maddy's ontology of sets. In particular, I have suggested that Maddy's idea of perception as conceptually loaded can raise doubts about the concreteness of impure sets. The second argument against **Epistemic Claim** is based on Maddy's confusion between aggregates and sets. In particular, I have presented Levine's point according to which Maddy can safely claim that what we perceive are sets and not aggregates, and shown that such an argument can be undermined.

In Section 7, I have presented some objections against Maddy's **On-**

tological Claim. More specifically, I have presented Cook’s argument according to which location of impure sets entails general unintuitive consequences. After that, I have presented Effingham’s objection to Cook’s argument which is based on the non-transitivity of location for many-one relations. I have nonetheless shown that Effingham’s argument has some flaws. Then, I have considered Parsons’ counterexample with sets having non-coexistent concrete members. I have shown that Maddy can defend her account by adopting Fourdimensionalism. However, this picture has the effect of increasing the tension between Maddy’s **Epistemic Claim** and her idea of perception.

In Section 8, I have added two remarks: the first explores two alternative replies to Chihara’s objection (8.1); the second concerns some applications of the debate between monism and pluralism to Maddy’s examples for Compromise Platonism (8.2).

To sum up, we can say that finding a knockdown argument against Maddy’s proposal is not an easy task. Despite this, Compromise Platonism has a bunch of serious problems. The story can be summarized using [Miller, 2002] dialectic. According to Miller, we have two ways in which a realist metaphysics can be questioned: by pointing out some inadequacy within the metaphysics itself, or by showing that it cannot be integrated into a plausible realistic worldview. I think that Maddy’s account suffers from both problems: the first because **Ontological Claim** can be seriously questioned; the second because **Epistemic Claim** does not seem plausible.

At this point, here is my dialectic: 1) In Chapter 1, I have shown that the mainstream thesis is problematic; 2) In Chapter 2, I have pointed out that the idea of impure sets as fully concrete objects is subjected to counterexamples. 3) Hence, since the idea that impure sets are concrete objects having abstract members or elements is highly unintuitive, it would be better to evaluate the claim that impure sets are fully abstract object.

This is, of course, a negative procedure, and has to be implemented with some positive reasons. Indeed, the idea of impure sets as fully abstract objects is uncommon and controversial. It goes against the classic meaning of the definition of impure sets as sets containing concrete elements. It necessitates, therefore, to be adequately supported.

I think that some relevant philosophical reasons already appear by comparing Chapters 1 and 2, some of whose conclusions mirror. Chapter 1 shows that members or elements conceived as concrete objects do not offer an adequate **Criterion of identity** for sets. More specifically, I have concluded that if concrete representations are just concrete, then they are, so to say, “not enough” to represent sets. Chapter 2 points out that Maddy’s perception has to be considered conceptually loaded. And that this is necessary (but, I have argued, not sufficient) for accounting for impure sets. The natural move is, therefore, to investigate the notion of impure sets by clarifying the role of the conceptual detail. Chapter 3 is devoted to this issue, and more specifically, to the relation between abstractness and conceptualization that I have just taken for granted until now.

Chapter 3

Impure Sets as fully Abstract Objects

Introduction

In this chapter, I will discuss and defend the idea that impure sets are fully abstract objects, namely objects that are not spatiotemporally located having objects that are not spatiotemporally located as members or elements in their transitive closure. For instance, according to this account, both $\{\text{Socrates}\}$ and Socrates are abstract objects.

As mentioned, this idea is uncommon and controversial. More specifically, the controversial part is that members and elements are abstract objects in that it goes against the classic definition of impure sets as sets having concrete members or elements.

Nonetheless, this claim can be justified by employing the negative dialectic that I have pointed out at the end of Chapter 3: Impure sets

have to be fully abstract objects because they are neither abstract objects having concrete elements, nor fully concrete objects. This means that the classic definition of impure sets has to be revised.

In this chapter, I will first present the negative dialectic in more detail. Then, I will argue that to consider Urelements as abstract objects, we have to explore the notions of concept, conceptualization, and conceptualized objects, which are intimately related to impure sets. More specifically, I will argue that Urelements of impure sets are conceptualized abstract objects. I will also argue that these objects are, nonetheless, related to the concrete. After having motivated this metaphysics of Urelements, I will show that [Zalta, 1983]’s Object Theory (OT) is particularly prominent in modeling this idea.

This Chapter has 7 sections. In Section 1, I will present the negative dialectic. In Section 2, I will investigate how the notions of concept, conceptualization, and conceptualized objects are related to impure set theory. To do this, I will explore the aforementioned Fregean Argument (3.2.1). After that, I will present [Jackson, 1998]’s metaphor (3.2.2). I will conclude that Urelements of impure sets have to be understood as conceptualized objects.

In Section 3, I will argue that conceptualized objects, and so Urelements of impure sets, are abstract. To do this, I will exploit Balaguer’s and Parsons’ inference from conceptualized objects to abstractness (3.3.1). After that, I will offer some alternative interpretations of conceptualized objects conceived as abstract (3.3.2). In Section 4, I will raise some objections against this idea, and provide some replies. More

specifically, I will argue that if conceptualized objects are abstract, then nothing is concrete (3.4.1). I will show, however, that there is a way in which we can safely maintain both conceptualized abstract objects and concrete ones (3.4.2). After that, I will argue these two kinds of objects are strictly related to one another (3.4.3).

In Section 5, I will present Zalta's Object Theory as a model for impure set theory. To do this, I will start by presenting its main features (3.5.1). Then, I will show that it accounts for a theory of intentionality that involves conceptualized objects (3.5.2). After that, I will argue that it has also the resources to model Urelements of impure sets. More specifically, I will argue that Urelements of impure sets are well-modeled by Zalta's *individual concepts* (or abstract copies) (3.5.3) plus the logical notion of Urelement (3.5.4). Then, I will briefly present a development in impure set theory concerning the notion of Urelement (3.5.5), lingering on the case of singletons (3.5.6) and the empty set (3.5.7). I will conclude this section by claiming that *conceptualism* seems to be the most adequate epistemological framework for impure sets conceived as conceptualized objects (3.5.8).

My strategy has never been proposed. To further defend it, I will explore, in Section 6, some objections that can be raised against it, and I will try to provide some significant replies. More specifically, I will address some problems of taking Urelements as conceptualized objects (3.6.1). After that, I will discuss some technical problems of Zalta's OT (3.6.2) and some perplexities concerning the relation between what is abstract and what is concrete (3.6.3). While exploring the replies, I will also

point out some further advantages of my account over the mainstream thesis and the idea that impure sets are fully concrete objects. Section 7 concludes.

3.1 The negative dialectic

This is the negative strategy that pertains to the general dialectic of this dissertation:

1) Impure sets cannot be abstract objects having concrete members or elements in their transitive closure;

2) Impure sets cannot be fully concrete objects;

3) Therefore, impure sets have abstract members or abstract elements in their transitive closure.

To grant 1), I have explored [Parsons, 1990] and [Fine, 2010]’s accounts of impure sets arguing that they are subjected to some difficulties that can be generalized to all accounts taking impure sets to be abstract objects having concrete components or representations. 2) follows instead from the arguments against Maddy’s account of impure sets as fully concrete objects that can be perceived. Hence, since impure sets are neither fully concrete nor abstract objects having concrete elements, we can conclude that these members or elements are abstract objects.

As mentioned, I will not discuss the idea of impure sets as concrete objects having abstract members or elements because it seems unintuitive. This means that I take the negative strategy as suggesting not only that Uelements of impure sets are abstract, but also that impure sets are fully abstract objects. The idea that Uelements of impure sets are abstract is nonetheless uncommon in that it goes against the classic definition of impure sets as sets containing concrete members or elements. In what follows, I will further explain and motivate this position by providing a new definition of impure sets.

3.2 Concept, conceptualization, conceptualized objects

I think that the most compelling strategy to justify that impure sets are fully abstract objects is to investigate the notion of concept. Indeed, as it is clear from Chapters 1 and 2, impure sets are intimately related to concepts.

Of course, the debate about the nature of concepts is vast and intricate. I will not provide an exhaustive explanation of this notion. I will rather try to offer an intuitive understanding of its main features by considering its role in certain activities related to impure set theory such as counting objects and replying to some questions about their nature. In doing so, I will also explore other notions in the vicinity of that of

concept, such as that of conceptualization, and that of conceptualized object that are central to my proposal.

3.2.1 Fregean argument

Let me start by recalling **Fregean Argument** that I have spelled out in Chapters 1 and 2. **Fregean Argument** goes as follows:

- 1) Numbers do not apply to objects;
- 2) Numbers apply to sets.

I have claimed that the notion of concept plays a crucial role in **Fregean Argument**: sets are countable because they are associated with some concepts providing a number property. For instance, I can claim that a given set of trees has 3 as cardinality because its three members fall under the concept of “being a tree”. In the absence of this concept, I cannot say whether that set is rather a set of foliages, of branches, and so on. Thus, roughly speaking, ”putting something in a set” requires what we can call a “process of conceptualization” which consists of considering objects, events, etc. as falling under some concepts. In other words, to say that x belongs to the set of all F s is to conceptualize x as falling under F . And this is the necessary condition to count objects.

3.2.2 Jackson's analogy

It can be argued that concepts have a similar role when sets are not involved. To see how, let me stress [Jackson, 1998]'s analogy between metaphysicians and bounty hunters:

“When bounty hunters go searching, they are searching for a person and not a handbill. But they will not get very far if they fail to attend to the representational properties of the handbill on the wanted person. These properties give them their target, or, if you like, define the subject of their search. Likewise, metaphysicians will not get very far with questions like: Are there Ks? Are Ks nothing over and above Js? [...] in the absence of some conception of what counts as a K, and what counts as a J.” Jackson 1998, 30-31.

In other words, concepts seem to be essential to individuate objects, and to reply to questions related to them. Jackson speaks about kinds of objects (*e. g.* the Ks, the Js). However, his claim can also be applied to individuals such as Socrates, Plato, and so on. Slightly modifying Jackson's metaphor, when the bounty hunters are searching for Socrates, they have to attend to at least some of Socrates' representational properties. In other words, it may be claimed that considering Socrates, is already to take him as having some properties such as, for instance, "being a human being", "being a philosopher", "being Greek", and so on.

Sets are not involved. However, this is exactly the process of conceptualization at work in impure set theory. As mentioned, to say that

x belongs to the set of all F s is to conceptualize x as falling under F . Hence, to claim that, for instance, Socrates is a philosopher means to conceptualize Socrates as a philosopher. Then, Socrates belongs to the set of all the philosophers.

This is the sense in which the notions of concept and conceptualization play an important role in impure set theory. I have provide no definition of these notions. Intuitively, I take that conceptualization can be defined by the notion of concept: it is the process of considering some objects as falling under some concepts. The output will be what I have called a *conceptualized object*. Example: from the concept “being a philosopher” we reach Socrates as a philosopher, which is a conceptualized object; the process that leads to consider Socrates as a philosopher is a process of conceptualization. Of course, this is just an intuitive formulation. Things will be clearer after having presented Zalta’s OT in Section 5.¹

For the time being, I will conclude that Urelements of impure sets are conceptualized objects. In what follows, I will suggest that these entities are abstract.

¹One preliminary observation is in order. In [Zalta, 2001], Zalta argues that Fregean modes of presentation and concepts can be identified using his Object Theory. As anticipated, I will also use Zalta’s Object Theory to model Urelements of impure sets. In discussing my account, I will nonetheless leave aside Zalta’s results concerning concepts, and focus on my idea of Urelements. These topics are strictly related and a more detailed investigation on their relations needs to be undertaken. I will leave this work for another time.

3.3 Conceptualized objects are abstract

Given the negative dialectic, the *desideratum* is that Urelements have to be abstract objects. For this to be the case, conceptualized objects have to be abstract. There are some plausible ways to justify this tenet. One of these is to just claim that many authors accept the inference from what is conceptualized to what is abstract. In what follows, I will quote two authors whose arguments are already presented in Chapters 1 and 2.

3.3.1 Authority principle

Balaguer's argument Recall Balaguer's argument against Maddy's ontology for impure sets discussed in Chapter 2, Section 2.6.1. Maddy claims that impure sets are individuated perceivable objects having some number properties. Perception of impure sets is, therefore, perception of things that have some number properties. This is also the sense in which perception is supposed to be conceptually loaded. Balaguer objects that perception of sets is inconsistent with the idea that they are individuated things. This happens because to say that impure sets are conceptually loaded is to claim that they are abstract, to some extent. His dialectic can be, therefore, summarized as follows: For impure sets to be individuated things having some number properties, they need to be conceptualized. Then, impure sets are abstract. Balaguer speaks about impure sets, but his argument can be applied to the case of Urelements: Urelements have to be abstract because they are conceptualized.

Parsons' argument Another author who adopts the inference from what is conceptualized to what is abstract is Parsons. As mentioned in 1.1.5, Parsons uses **Fregean Argument** to claim that numbers are not intrinsically related to any concrete objects whatsoever because of the nature of the sequences to which they are related. Indeed, for something to be a sequence of a given number, it has to be considered a sequence of that number, and not, for instance, a unique grid. Roughly, since the very nature of sequences involves a conceptual choice, they cannot be concrete.

Nevertheless, it can be argued for **Slippery Slope**: if we accept Parsons' argument, then Urelements of impure sets also are abstract objects. Recall the case of Socrates: what allows us to claim that Socrates is an individual, and not a certain number of molecules or atoms is his conceptualization. Then, Socrates is abstract.² More generally, the inference from what is conceptualized to abstractness is not restricted to sequences but plausibly applies to all conceptualized objects.

Of course, the fact that some authors accept the inference from what is conceptualized to abstractness is not, strictly speaking, a conclusive argument in favor of such an inference. Nonetheless, it has to be considered a clue in favor of this idea.

²Notice also that this is precisely what impure set theory presupposes: to claim that an object x belongs to a given set of F s requires a conceptualization of x (*e. g.* the fact that Socrates is considered a human being allows us to claim that he belongs to the set of all the human beings).

3.3.2 Alternative models for conceptualized objects

Another way to stress that conceptualized objects are abstract is to just claim that they *are* concepts. Concepts are indeed often taken to be abstract objects. I did not say what a concept is but we could just state that, for instance, “Socrates as a philosopher” is an object of the same kind as “being a philosopher”. Perhaps, conceptualized objects can be considered *qua*-objects à la Fine, namely objects having the form “*x qua F*” where *x* is the underlying object, and *F* is the gloss (see Chapter 2, Section 2.5.2).³ Then, **Trilemma*** applies: since *qua*-objects have abstract components, they cannot be concrete.

All these are available options. In what follows, I will explore another strategy that takes some tools from [Zalta, 1983]’s Object Theory (OT). This way of conceiving conceptualized objects relates somehow to the debate about what concepts are. Literature on the topic shows two main factions: those who claim that concepts are mental representations, and those who claim that they are abstract objects. Concepts are often said to be abstract in that they are the meanings (or “contents”) of words and phrases (*e. g.*, [Peacocke, 1992] and [Zalta, 2001]). Concepts as mental representations are instead understood in the context of a representational theory of mind where for each episode of believing that *P*, there is a corresponding mental representation which means that *P* (*e. g.*, [Sutton, 2004]).

³I would like to thank Fabrice Correia for suggesting me this idea.

These two tenets are, nonetheless, not mutually exclusive, and can be combined in a successful way. According to [Margolis and Laurence, 1999], for instance, concepts are mental representations typed in terms of the senses they express. In what follows, I will show that Zalta's OT allows conceiving conceptualized objects in the vicinity of a non-mutually exclusive theory: conceptualized objects are abstract mental representations.

Before doing so, I will consider some difficulties of taking conceptualized objects to be abstract. My replies to those difficulties lay the groundwork for further clarifying the nature of conceptualized objects, as well as to show that Zalta's OT describes them adequately.

3.4 Objections and replies

3.4.1 A bad consequence of Slippery Slope

Paradoxically, a specific interpretation of **Slippery Slope** could suggest that Urelements are concrete. Consider again Socrates. As mentioned, Socrates is conceptualized, and conceptualized objects are abstract. Socrates is therefore abstract. But then, the same would apply to all entities that are classically considered concrete (mulberry trees, my laptop, my glass of wine, etc.). In this way, we reach the outrageous conclusion that the world in which we live is abstract in all its parts. And, of course, the distinction between abstractness and concreteness disappears.

The problem is therefore that **Slippery Slope** could be taken to suggest that we have no concrete entities at all. And, since this consequence is highly unintuitive, it can be argued not only that the argument does not go, but also that it points out a serious problem in taking objects such as conceptualized objects to be abstract.

3.4.2 Abstract and concrete objects

There is a reply to this objection which is crucial for my proposal because it says something important about the nature of conceptualized objects and, more specifically, about that of the Urelements of impure sets. My reply goes as follows: The use of conceptualized objects does not entail the non-existence of concrete objects flesh and blood. The situation goes as follows: both abstract and concrete objects play a role. In speaking about concrete objects "flesh and blood", we use abstract objects. And these two objects are, of course, different. For instance, in speaking about Socrates concrete object, we use a conceptualization of Socrates, which is not flesh and blood.

Of course, it can be argued that objects such as Socrates or Plato flesh and blood fall under some concepts, such as the concept of being concrete, and being flesh and blood. Then, they have to be conceptualized objects. The idea is, nonetheless, that whenever we "interact" with these entities flesh and blood, we use conceptualizations, but this fact does not entail the non-existence of concrete *unconceptualized* objects. Roughly speaking, the concrete does not *eo ipso* fall under concepts but

our way of referring to it is always conceptualized.

This idea is also motivated by a general fact about scientific theories. It is common for any scientific theory to use a specific and independent language but to aim to speak about real objects in the world. Thus, scientific theories make extensive use of mixed *formulae* in which mathematical and empirical terms interact. An example is measurement in which something concrete is represented by some abstract numbers. Another good case is impure set theory in which pure and impure sets of axioms coexist and interact. It seems therefore a mistake to take a theory postulating abstract objects as *prima facie* incompatible with the theories speaking about concrete objects.

Furthermore, in the case of impure set theory, the presence of both abstract and concrete objects is suggested by some ontological basic facts: Urelements and concrete objects exemplify different properties. Also, certain set-theoretical properties seem extraneous to concrete objects. For instance, the property of “being a member of a set”, “being a logical object”, “being subjected to set-theoretical axioms”, etc., seem to have nothing to do with concrete objects such as Socrates, Plato, tables, etc.

It can be objected that we do not need to have two different objects. We could just embrace the previously mentioned *qua*-objects framework and claim that the properties related to set theory are not exemplified by concrete objects strictly speaking but by those objects *qua* set-theoretical objects: for instance, it is not Socrates concrete object that exemplifies the property “being a logical object” but Socrates concrete object *qua* object of set theory.

However, the *qua*-framework seems misguided because Urelements and concrete objects do not only exemplify different properties but also exemplify different and *incompatible* properties. For instance, Socrates Urelement is a logical unanalyzable (atomic) object of set theory, while Socrates concrete object is arguably made out of parts. Similarly, in contemporary metaphysics of grounding, it can be argued that the empty set is 0-grounded, while Urelements are un-grounded. However, Socrates concrete object is plausibly grounded in something, *e. g.* in its parts.⁴

Of course, it can be objected that generic objects can be taken to exemplify incompatible properties. For instance, a generic triangle can disjunctively exemplify “being isosceles” and “being scalene”. However, Socrates, Plato, etc. are not generic objects, but individuals.

To summarize, it may be argued that **Slippery Slope** plus the idea that all that is conceptualized is also abstract suggests that we have no concrete objects at all. However, the fact that all can be conceptualized (and that our relation to objects involves such a conceptualization) does not entail that concrete objects flesh and blood do not exist. Furthermore, the existence of both conceptualized objects and concrete objects flesh and blood is suggested by the fact that Urelements and concrete objects flesh and blood are usually taken to exemplify incompatible properties. The nature of these properties and, more specifically, the fact that

⁴Fine stresses a similar argument in [Fine, 2012], but he takes grounding to relate facts and not objects. His argument can, nonetheless, be modified to apply to objects, and, more specifically, to set-theoretical and non-set-theoretical objects such as Socrates Urelement and Socrates human being.

concrete objects are extraneous to them corroborates the idea that our ontology of Urelements should involve abstract objects.

3.4.3 Abstract conceptualized objects *represent* concrete objects

It is also natural to argue that abstract objects having set-theoretical properties and concrete objects cannot be completely unrelated. If this were the case, then impure set theory and concrete objects would be completely unrelated too, which sounds like a very bad consequence. Similarly, other abstract concepts such as my conceptualization of Socrates or that of Plato, should not be completely unrelated with respect to Socrates and Plato. I will therefore claim that abstract conceptualized objects have *some relations* to the concrete. It is important now to understand the nature of these relations.

Generally speaking, there are many ways to relate concrete objects of physical worlds to abstract conceptualizations. It is also plausible to say that the relations that appear in impure set theory just represent a way in which an abstract object can refer to abstract and concrete contents: modal thoughts, states of affairs, situations and worlds, objective and cognitive content of natural language, fictional objects are other examples.

It can also be said that we speak about concepts or conceptualized objects and concrete objects *to which they refer*. To analyze the way in which conceptualized objects refer to concrete ones, I propose to use

Jackson's aforementioned metaphor as a starting point. As the author suggests, the person that the bounty hunter is searching for is not identical to the handbill that gives the target of the identity of the person in question. What is the relation between the person and the handbill? It is plausible to claim that it is a relation of *representation*: the handbill represents the person. Of course, this is just a metaphor, but it successfully describes the way in which conceptualized objects are related to the concrete objects to which they refer.

In what follows, I will argue that this relation of representation can be modeled using [Zalta, 1983]'s Object Theory. More specifically, I will address two different cases displaying this relation: that of the so-called *Noemata* (*i. e.* Husserlian intentional objects), and that of Zalta's "Individual concepts" (or "abstract copies"). After that, I will show that Zalta's individual concepts are good candidates to account for Urelements of impure sets.

Zalta's theory of intentionality is only sketched. It therefore needs to be further extended and explored. In [Luporini, 2022], I have proposed a defense and extension. For the aim of our scope though, a basic version largely suffices. Also, Zalta's individual concepts have never been used to model Urelements of impure sets. Zalta's use rather refers to Leibniz's framework (see [Zalta, 2000b]). My proposal has therefore to be considered a first attempt. As such, it still presents some difficulties. After having explained how it works, I will explore these difficulties and propose some solutions.

3.5 Zalta’s Object Theory (OT) for intentionality and impure set theory

3.5.1 The main features of OT

Let me begin by presenting Zalta’s Object Theory (OT). The main aim of OT is to solve some problems affecting Meinong’s theory of objects.⁵ However, OT also applies to many other philosophical items such as Fregean senses, Platonic forms, and Leibnizian concepts.⁶

OT quantifies over two domains: that of objects and that of n -place relations. The domain of objects is exhaustively divided into two mutually exclusive subdomains: that of abstract objects (such as numbers, sets, ideas, Sherlock Holmes, etc.), and that of ordinary ones (such as Socrates, my mother, my laptop, etc.). Ordinary objects can be concrete or non-concrete. The meaning of concreteness is given by [Linsky and Zalta, 1994]. Their actualist interpretation of Quantified Modal Logic sets up a fixed domain of objects: ordinary and abstract objects exist at every world. Existence is nonetheless distinct from concreteness: the latter requires spatiotemporal locations, while the former does not. That is, the theorem that states that “everything necessarily exists”, *i. e.* $\forall x \Box \exists y (y = x)$, is taken to involve an existentially loaded quantifier with no spatiotemporal connotations. Hence, ordinary objects exist at every possible world, but cannot be concrete at all of them.

Given such a framework, Zalta defines the property of “being ordi-

⁵For an overview of these problems, see [Reicher, 2006].

⁶See [Zalta, 2001], [Pelletier and Zalta, 2000], and [Zalta, 2000a].

nary” ($O!$) and “being abstract” ($A!$) by using a second-order modal language:

$$O! =_{def} \lambda x \diamond E!x$$

$$A! =_{def} \lambda x \neg \diamond E!x$$

where $E!x$ means “ x is concrete”.

Ordinary objects exemplify properties (for example, my laptop exemplifies the property of “being grey”, and I exemplify the property of “being hungry”). Unlike ordinary objects, abstract ones do not only exemplify but also encode some properties. An abstract object x encodes F – (formally, xF) – *iff* F is predicable of x and determines/characterizes it; by contrast, an abstract object exemplifies x – (formally, Fx) – *iff* F is predicable of x but does not determine/characterize it.⁷ Following Zalta’s examples, the empty set encodes all the properties assigned to it in ZF (such as “being a set with no members”, “being a subset of all other sets”, etc.) and exemplifies the properties that we do not find in ZF (such as “being abstract”, “not having a mass”, etc.); Holmes encodes all the properties assigned to him in Doyle’s novel (such as “being a detective”, “living in London”, etc.), and exemplifies properties that do not appear in the novel (such as “being fictional”, “being admired by modern criminologists”, etc). In general, abstract objects necessarily exemplify the negation of all concreteness-entailing properties (such

⁷This idea comes from [Mally, 1912], and can also be found in [Castañeda, 1973] and [Rapaport, 1978].

as “being red”, “being human”, and so on)⁸ and contingently exemplify some converse intentional properties (such as “being imagined by George Clooney”, “being loved by me”, etc.).⁹

The principal axiom for abstract objects is **Comprehension Principle** that asserts the conditions under which abstract objects exist and encode some properties: for any expressible condition ϕ that is satisfiable (in Tarski’s sense) by properties F , there exists an abstract object that encodes exactly the properties F satisfying ϕ . Formally speaking,

$$\exists x(A!x \wedge \forall F(xF \equiv \phi))$$

Then, the particular mathematical theories/fictional stories are analyzed as abstract entities that encode some propositional content. The relation between OT and a particular theory T /a particular story S is given by **Theoretical Identification Principle** that asserts that the object K of a theory T /a story S is the abstract object that only encodes the properties F s exemplified by Kt / Ks according to the theory T /story S . Formally speaking,

$$K_T/K_S = (\iota x A!x \wedge \forall F(xF \equiv T/S \models FK_{T/S})).$$

⁸Concreteness-entailing properties are defined as follows: $CE(F) = \Box \forall x(Fx \rightarrow E!x)$ *i. e.* F is a concreteness-entailing property *iff* it is necessary that for all x , if x exemplifies F , then x is concrete. Notice however that these properties can also be encoded. Example: Holmes encodes “being a man”.

⁹As before, these properties could also be encoded. If, for instance, according to the story S , the character c is imagined by Clooney, then c would encode the property “being imagined by Clooney”. Also, encoding and exemplifying are not mutually exclusive modes. This fact will be particularly relevant when discussing Fine’s counterexample in Chapter 4.

From this principle, Zalta derives **Equivalence Theorem** which says that mathematical/fictional objects encode all and only the properties according to their governing mathematical theory/story. Formally speaking,

$$K_T/K_S F \equiv (T/S \models F K_T/S).$$

In addition, the encoded properties of abstract objects can be relevantly entailed by those explicitly presented. For example, even though it is not explicitly stated in Doyle’s novel that Holmes is, for instance, a human being, that property is relevantly entailed by, for instance, the encoded property of “being a detective”.

Two more things about the encoding mode. First, an abstract object is complete regarding its exemplified properties but notoriously incomplete regarding its encoded ones. For instance, Holmes does not encode the property of “having a mole on one’s left foot” nor its negation since such a property does not appear in the novel at all. By contrast, for ordinary objects (which encode no properties) the exemplification is always complete. Second, abstract objects do not “lose” their encoded properties from one world to another; so, the encoded properties are rigidly encoded. This is called **Modal Axiom of Encoding**. Formally speaking,

$$\diamond xF \rightarrow \square xF.$$

Then, we have **Principle of Identity for Abstract Objects**: Two abstract objects are identical *iff* they encode the same properties. Formally,

$$x =_a y =_{df} A!x \wedge A!y \wedge \forall F(xF \leftrightarrow yF)$$

Encoding is, therefore, the relevant mode for abstract objects. As Zalta claims,

”it is important to know that abstract objects that necessarily (always) exemplify the same properties need not to be identical. There are so many abstract objects generated by the theory that the traditional notion of exemplification is incapable of distinguishing them all. Encoding is the only real guide to the identity of abstract objects.” [Zalta, 1988], 31.

It can be pointed out that, given Leibniz’s indiscernibility of identicals, if x and y are abstract, and they encode but do not exemplify the same properties, then Zalta’s principle fails. However, these kinds of counterexamples are very difficult to find. The easiest way is to use converse intentional properties plus other special cases of epistemic contexts in which the subject lacks a piece of information. For instance, it could be said that Superman but not Clark Kent is thought by me. Nevertheless, both converse intentional properties and these epistemic contexts

are problematic independently from Zalta's encoding-exemplifying distinction and his criterion of identity for abstract objects.¹⁰

By contrast, for ordinary objects, what counts is the exemplification. **Principle of Identity for Ordinary Objects** claims that two ordinary objects are identical *iff* they exemplify the same properties. Formally,

$$x =_o y =_{df} O!x \wedge O!y \wedge \forall F(Fx \leftrightarrow Fy).$$

Before looking at the application of OT to impure set theory, an observation is in order. OT describes a metaphysics that is different from the one adopted until now: abstractness and concreteness are not exhaustive categories anymore. This choice expresses the necessity of accounting for objects that might have been something (which had no ontological dignity in Kripkean models with rigid designation). Following [Linsky and Zalta, 1994]'s examples, the sister of *b* that *b* does not have, or Quine's possible fat man in the doorway. Moreover, by implementing the classic notion of existence along with that of concreteness, the account avoids the unintuitive conclusion according to which all objects necessarily exist. Of course, this idea is not uncontroversial. In what follows, I will adopt all the items associated with OT. However, it is important to note that my solution does not necessarily integrate all these features.

Now that I have spelled out Zalta's OT main features, let me move on to explaining how it accounts for conceptualized objects. As mentioned,

¹⁰I would like to thank Fabrice Correia for raising this point.

one way to do this is to model them by using Zalta’s theory of intentionality. I have anticipated nonetheless that the best candidate to account for conceptualized objects as Urelements are individual concepts. Zalta’s theory of intentionality may be considered a detour. I argue however that OT for intentionality and OT for impure set theory have some remarkable common features. In this sense, Zalta’s theory of intentionality is a propaedeutic to discussing the viability of individual concepts as models for Urelements of impure sets.

3.5.2 OT for intentionality

Zalta’s account of intentionality in [Zalta, 1988] stems from the attempt at reconciling Mally’s theory of abstract objects and Husserl’s notion of *Noemata*. The common background is Brentano’s starting point according to which each intentional state is directed towards something (I think about something, she dreams something, and so on). However, the “content” of intentionality is different from the object we refer to. Husserl’s canonical example is the perceived tree that is not the tree of the “external world”, but rather the sense of the perception of that tree.¹¹

In particular, Husserl posits the so-called *Noemata* which organize every intentional state “as if” they were of some objects. The “as if” is crucial because it points out a continuity with Brentano’s tradition for which “every mental phenomenon is characterized by [...] the inexistence of an object, and what we might call [...] direction toward an object” [Brentano, 1973], 88. The *Noemata* are therefore intermediate

¹¹See [Husserl, 1901], 11-12.

representational items that objectify some, so to speak, “non-concrete” contents.

Now, OT prescribes that the intentional act directed towards an ordinary object has abstract objects as content. The ordinary object and the abstract content are nonetheless related. The relation is expressed by the properties that these objects exemplify and encode.

To see how the relation goes, consider the previous case of the external tree. This tree is an ordinary object that exemplifies properties (for instance, “being partially green” and “being 3 meters tall”). Then, a subject’s intentional state directed towards that tree is associated with an abstract object which encodes properties that the tree exemplifies. For instance, if I have perceived the tree as being partially green and 3 meters tall, my *Noema* directed towards that tree would be an abstract object encoding “being partially green” and “being 3 meters tall”.

This is the general structure of Zalta’s intentionality theory. As it is clear, intentionality, as described by Zalta, suggests the existence of some abstract objects that are related to concrete ones. The theory has therefore the advantage of describing the fact that abstract and concrete objects can be related by a representational relation: *Noemata* ”represent” concrete objects.

To better grasp the nature of these objects, let me now explore the difference between *Noemata* and ordinary objects they refer to.

***Noemata* and ordinary objects they refer to**

The first difference between *Noemata* and ordinary objects they refer to is that the former are abstract objects, while the latter can be concrete. Also, they differ both in properties and in the way in which they have them. As mentioned, while, for instance, the tree in front of me exemplifies the property of being partially green, my *Noema* will *encode* that property. Also, my *Noema* of the tree encodes the property of being abstract, while the tree exemplifies the negation of that property.

Also, as mentioned, for ordinary objects, exemplification is complete (which is, as Santambrogio suggestively stresses, the seal of their reality¹²); by contrast, the *Noemata* are notoriously incomplete in terms of the encoding mode (for instance, my *Noema* of the tree that I have seen yesterday encodes neither “being old” nor its negation).

That there is no one-to-one correspondence of properties between *Noemata* and the ordinary objects to which they refer is easy to state. It is interesting to notice that the lack of a one-to-one correspondence also concerns the objects themselves: As it is often the case, we have a lot of different *Noemata* that refer to the same ordinary objects. The following passage further clarifies that this is the case in common epistemic situations:

”A situation in which a person has had a single perceptual encounter with an object. Suppose a particular A-object is the

¹²See [Santambrogio, 1992], 140-141: “An object i is real if, for every predicate $F(x)$, either $F(i)$ or not- $F(i)$ holds. *Tertium non datur* then is the seal of reality [...] since it expresses the property of being determined under all the respects”. (My translation).

content of the person's mental state during that encounter (it may, for example, encode just the perceptual properties available to the observer from a certain visual perspective). Suppose further that the person acquires no new information about the object. It now seems plausible to suggest that future mental states directed towards this object, whether they be rememberings, imaginings, fears, etc., will be mediated by the A-object in question. [...] Of course, if new information about the object is acquired, some other A-object may come to serve as the content of states directed towards this same object".¹³[Zalta, 1988], 111.

I take this possibility as modeling the intuitive fact that we have a lot of different mental representations of the same thing.

Possibility of errors

It is now important to note that this framework accounts for errors and situations in which we just are unsure about something. Errors and doubts are indeed plausibly essential to intentionality. As mentioned, in thinking about an ordinary object, we form an abstract object directed towards it. This abstract object encodes some properties that could (or not) be (some of) the properties that the ordinary object in question exemplifies. Then, if I erroneously attribute some properties to a given object, I will just have a partially wrong *Noema* of that object. If otherwise I do not know whether a given object has certain properties, my

¹³For "A-object", Zalta means an abstract object.

Noema directed towards that object will be incomplete. Consider, for instance, the case of someone who thinks that Socrates is not snub-nosed. This person will simply have a *Noema* directed towards Socrates which encodes the negation of the property “being snub-nosed”. If otherwise this person does not know whether Socrates is or not snub-nosed, then their *Noema* would be incomplete.

Incidentally, it has to be noticed that this framework allows us to account for counterfactual situations involving ordinary objects, those in which, for instance, Socrates is not snub-nosed. Furthermore, it can even be argued that this allows us to account for counteressential situations, those in which we are wrong about some supposedly essential properties of an ordinary object. To see how, suppose that water plausibly is essentially H₂O and imagine someone who mistakenly attributes a chemical composition different from H₂O to water. Simply, the person would have an abstract object directed towards water that encodes properties that water necessarily does not exemplify.¹⁴

The problem of reference

It becomes now quite natural to ask for a criterion that states when a given *Noema* can be considered the *Noema* directed towards a given object. As Zalta claims, intentionality makes as if “the mind were construed as a mental bow whose arrows could be properly aimed at different targets” [Zalta, 1988], 10. But what makes an arrow to be aimed at a target (and not at another)?

¹⁴For more on this, see [Luporini, 2022].

As already pointed out, the link between *Noemata* and objects is not individuated by a one-to-one correspondence of properties. For such cases of *Noemata* directed towards ordinary objects that we perceive (or we have perceived), the relation is not so difficult to individuate. As Zalta proposes,

”The content of the state itself is not the place to look to determine philosophically what the state is about. Rather, the place to look is the contextual and historical facts that give rise to the state. The contextual/historical fact that a particular object stands at the source of this perceptual state is what makes it true to say that that state is about, or directed towards, that object”. [Zalta, 1988], 110–111.

Unfortunately, the relation is not always so easy to individuate. And it is also hard to deny that the property-correspondence has no impact at all. What happens when I have no straightforward historical perceptual element and, say, my *Noema* of something involves all and only properties that the object to which it is supposed to refer does not encode or exemplify? Rephrasing [Burge, 1979]’s example, imagine that I have never had arthritis in my life and that I do not know anyone with arthritis. How can my *Noema* directed towards arthritis and encoding, for instance, “being a disease of the thighs”, still refer to arthritis (and not to other inflammatory disorders affecting thighs)?¹⁵ Even if it is true, the fact that my *Noema* exemplifies the property of “being a *Noema* directed

¹⁵See [Burge, 1979], 77).

towards arthritis” is not very informative, especially because, given OT, the salient properties of abstract objects are the encoded ones.

Of course, these are limiting cases. But they either suggest that the properties-correspondence is not completely irrelevant (and we have to draw a line somewhere), or that we have to search for a criterion that involves no appeal to properties at all. To avoid suspiciously *ad hoc* lines, it is maybe better to opt for the second strategy: what counts as the main clue has to be searched for amongst the agent’s intentions as well as amongst contextual facts. All this is supposed to warrant that someone’s *Noema* is directed towards a given object, no matter what they know about it.¹⁶

As it is clear, Zalta’s theory of intentionality combines the idea of conceptualized objects as mental representations and that of conceptualized objects as abstract objects. As anticipated, *Noemata* are, nonetheless, not good candidates for accounting for Urelements of impure sets. In what follows, I will explain why and try to model Urelements of impure sets by using Zalta’s individual concepts (or abstract copies).

¹⁶I do not think that such criteria are indisputably efficacious. A deeper debate on the problem of reference needs to be undertaken. However, this is not the right place.

3.5.3 Urelements and Zalta's individual concepts (or abstract copies)

Urelements are not *Noemata*

Up to now, I have presented Zalta's theory of intentionality showing how it relates abstract with concrete objects. Previously, I have suggested that Urelements of impure sets are conceptualized abstract entities that refer to concrete objects. As mentioned, it would be strange if Urelements were completely extraneous to ordinary objects they refer to. It could therefore be thought that Urelements can also be modeled by using Zalta's theory of intentionality.

I argue nonetheless that, even before clarifying how OT is supposed to apply to the case of Urelements, it does not seem a good idea to just take *Noemata* as models for Urelements. As I have shown, this theory includes the possibility of errors, which is good news when we aim to account for our way of intentionally referring to objects, but could be problematic when the focus is set theory. The problem is, of course, that if supposedly concrete members of impure sets are objects of intentionality formalized with Zalta's Object Theory, then set theory has a subjective flavor. To see why, suppose that I have a hallucination, and I see the previously mentioned tree as being partially red. Then, my *Noema* would wrongly encode "being partially red". Then, plausibly, impure set theory would somehow "inherit" this error.

Zalta's individual concepts or abstract copies

To model Urelements of impure set theory, I propose to use another device from Zalta's OT, namely his previously mentioned individual concepts or abstract copies. Individual concepts or abstract copies are introduced by [Zalta, 1983] and used by [Pelletier and Zalta, 2000] to deal with the Leibnizian notion of individual concepts. Let me briefly explain what individual concepts or abstract copies are. To do this, recall **Comprehension Schema** which asserts the existence of an abstract object corresponding to any condition on properties expressible in the language of OT:

$$\exists x(A!x \wedge \forall F(xF \leftrightarrow \phi))$$

Where $A!$ stands for "being abstract", xF for " x encodes F ". Then, if we take s as a constant denoting an individual, let me say, Socrates, then, using Russell's definite descriptions, we obtain a copy in the following way:

$$Cs = \iota x A!x \wedge \forall F(xF \leftrightarrow F(s))$$

This means that, for instance, Socrates' copy (Cs) is a unique object x which is abstract ($A!x$), and for all properties F , x encodes F (xF) iff Socrates exemplifies F (Fs).¹⁷

¹⁷Notice that Individual concepts and abstract copies are always individual concepts or abstract copies *of something*.

In this way, we obtain a unique object that *represents* Socrates flesh and blood. Which properties are predicable of this object? For example, *Cs* encodes “being a philosopher”, “being Greek” but also exemplifies properties such as “being abstract”, and “being Socrates’ copy”.

The philosophical meaning of taking Urelements as abstract copies of ordinary objects is simple and intuitive. Consider what has been said about scientific theories having mixed *formulae*: when a philosopher or a metaphysician develops impure set theory with Urelements, he is thinking about a formal theory containing some abstract objects. But he aims to refer to some concrete ones. In other words, the mathematical language is made out for abstract conceptualized objects some of which represent ordinary objects. To put it in other terms, we never find Socrates flesh and blood inside of a handbook of set theory but we can find an abstract conceptualized object strictly resembling him. Abstract copies accommodate this idea.

Recall also that one reason to think that objects of impure set theory should be different from objects of flesh and blood was that they have different and incompatible properties. Thanks to Zalta’s double predication, these incompatible properties can co-exist. For instance, *Cs* is not, itself spatiotemporally located (it indeed exemplifies the property of not being spatiotemporally located). By contrast, *Cs*, as representing Socrates flesh and blood, it is spatiotemporally located (it indeed encodes the property of being spatiotemporally located).¹⁸

¹⁸Zalta takes the distinction between encoding and exemplifying mode as primitive.

To my knowledge, the idea of abstract copies representing ordinary objects has never been developed. An interesting case that can somehow be assimilated to the need of having some abstract objects representing concrete ones can be found in [Weir, 2007] in which the author discusses how Carnap’s postulationism could apply to the empirical world:

“There could be a theory T^* structurally isomorphic to the empirical theory T which occurs in the Carnap conditional but such that no term in T^* has any empirical meaning. Postulation with respect to T^* might then be legitimate, but the objects of the postulationist theory are not the empirical entities which T characterizes but abstract surrogates. The domains of these surrogates are structurally isomorphic with physical or psychological structures but distinct because, for one thing, they are causally inert. Given a plenitudinous conception of the abstract realm, why should we not believe in the existence of abstract structures corresponding to any empirical structures we might discover?” [Weir, 2007], 96.

His suggestion is that this happens because we need a notion of empirical concept, which is, nonetheless, problematic. In this sense, I think that Zalta’s abstract copies can be considered good starting points in that they describe how the relation between theories and ordinary objects can be understood.

Many authors argue against this idea (see, for instance, [Berto, 2013], [Jacquette, 2015], and [Priest, 2016]) claiming that it is *ad hoc*. Zalta defends the distinction by showing some of their philosophical applications ([Bueno and Zalta, 2017]). I will discuss this defense in Section 3.7.2.

Analogies and differences between Zalta’s theory of intentionality and Zalta’s abstract copies applied to set theory are now clear: In both cases, we have abstract objects representing ordinary ones. I have argued that *Noemata* are, nonetheless, not the best candidate to account for Urelements of impure sets. Abstract copies are preferable because we want those Urelements to be objects that *fully* represent ordinary ones. However, *Noemata* can also be employed in impure set theory when we refer to intentional objects.

Incidentally, it has also to be noticed that abstract copies are not the best candidate for accounting for intentionality either. This happens because it is implausible that a given intentional act concerns all the properties a given ordinary object exemplifies. In the best-case scenario, we have a *Noema* which is a partial copy of an ordinary object, *i. e.* an abstract object that encodes a subset of properties that the corresponding ordinary object exemplifies.¹⁹

3.6 Urelements as functions saturated by abstract copies

To be fair, modeling Urelements of impure sets as abstract copies of ordinary objects does not go without problems. The main one, I argue, concerns the properties that these objects are supposed to express. In

¹⁹This is what is called an “incomplete blueprint” or a “weak correlate” in [Rapaport, 1978].

particular, while, as I have pointed out, abstract copies exemplify some properties that are extraneous to ordinary objects (such as being spatiotemporally located, or being causally efficacious), it is not clear how they express the more specific set-theoretical properties (*e. g.* “being a member of its singleton”, “being a Urelement”, “being a logical atomic object”, and so on). Do they exemplify or encode these properties? If they encode these properties, and the encoding mode is somehow coherent with the role that has in relating abstract copies and ordinary objects, then abstract copies cannot encode this set of properties because they are not properties that the corresponding ordinary objects exemplify. As mentioned, they rather are extraneous to ordinary objects. For instance, Socrates flesh and blood plausibly does not exemplify the property of being a logical object of set theory). Perhaps, it can be argued that abstract copies *exemplify* some set-theoretical properties. However, it could be objected that there is no reason why the conceptualization of someone (or something) should involve properties such as “being a logical object” or “being a member of some sets”.

A better solution can be found by considering the nature of the objects of set theory. In our specific case, it is useful to investigate the nature of the notion of Urelement more in detail. Following [Zermelo, 1908]’s first formulation of set theory, an Urelement is an object that cannot be a set, that cannot contain sets or members, but that can belong to some sets. As mentioned, it is also a logical atomic object of set theory. Given OT as a model, Urelements are abstract objects and, as such, they encode a set of properties (*e. g.*, ”not containing sets or members”, ”possibly

belonging to sets”), and exemplify other properties (*e. g.* ”being abstract objects of set theory, ”being thought by Zermelo”).²⁰

So far, so good. Now, how Urelements relate to abstract copies? I argue that Urelements have a particular relation with some other objects. ”Urelement” is indeed just a generic term. If we consider each Urelement, we have that they are specific objects (*e. g.* number 1, Socrates, Plato, etc.). However, Zermelo’s definition says nothing about this fact, which, I argue, can rather be modeled by considering Urelements as functions that have to be saturated by some objects. More specifically, Urelements can be saturated by other abstract objects (*e. g.* number 1) or by abstract copies (*e. g.* Socrates Urelement).

Thus, generally speaking, Urelements are functions that take abstract objects and “give them back” with additional conceptual content, namely the set-theoretical properties that we need. The generic form of Urelement is therefore that of a function (let me call this UR) that can be saturated by some abstract objects (let me call this *a*). Hence, Urelements have the following generic form: UR(*a*). As an example, Socrates Urelement will take the form UR(*Cs*), where UR stands for the function, *Cs* for the abstract copy of Socrates, and *s* for Socrates ordinary object. UR(*Cs*) is therefore a logical object encoding some set-theoretical properties derived from Zermelo’s definition of Urelements (*e. g.* ”not being

²⁰It can be wondered why properties such as “not containing sets or members” and ”possibly belonging to sets” are encoded and not exemplified. Recall **Theoretical Identification Principle** that asserts that the object *K* of a theory *T* /a story *S* is the abstract object which only encodes the properties *Fs* exemplified by *Kt* / *Ks* according to the theory *T*/story *S*. Then, since these properties appear in Zermelo’s theory, they are encoded by Urelements.

a set”, ”not containing sets or members”, and ”(possibly) belonging to some sets”), exemplifying other properties (*e. g.* ”being abstract”), encoding some properties exemplified by Socrates ordinary object such as “being a philosopher”, “being Greek”, and so on.²¹

It could sound strange that neither $UR(Cs)$ nor Socrates ordinary object exemplifies properties such as ”being a member of {Socrates, Plato}”. Indeed, this means that nothing is, in the usual sense, a member of that set. But, of course, this account involves a revision of the language of impure set theory according to which membership is rather delivered by the encoding mode relative to Urelements. This is motivated by the fact that both ordinary objects and their abstract copies can be said to be extraneous to set-theoretical properties. In chapter 4, this point will be further motivated by considering the application of this account to Fine’s counterexample of Socrates essentially belonging to {Socrates}.

Similarly, it can be asked why the function Urelement cannot directly apply to ordinary objects rather than to abstract copies. The reason why can be found by recalling the rationale of using abstract copies in impure set theory instead of ordinary objects: When a philosopher or a metaphysician develops an impure set theory with Urelements, he is thinking about a formal theory containing some abstract objects. The mathe-

²¹I am not sure whether $UR(a)$ is supposed to also exemplify the properties that a exemplifies. I have the tendency to reply that this is not the case. For instance, it could indeed be that Socrates’ abstract copy exemplifies the property of being thought by me, while its Urelement does not. Generally speaking, the relevant properties are those that are encoded by the abstract copies because they are those that are taken from the ordinary objects to which they refer.

mathematical language that he employs is made out of abstract conceptualized objects. Even when he aims to speak about, for instance, Socrates flesh and blood, he is speaking about an abstract conceptualized object strictly resembling Socrates. Hence, UR function applies to these objects.

It could be also asked what happens when UR applies to something that is already of that form. In other words, it could be asked whether there are things such as $UR(UR(a))$ $UR(UR(UR(a)))$, and so on. I think that these cases have to be rejected by claiming that if $a = UR(b)$ for some b , then $UR(a) = a$. Hence, $UR(UR(UR(a))) = UR(a)$.

Of course, this “metaphysical” characterization of Urelements does not explicitly appear in Zermelo’s definition. Zermelo’s interest is primarily formal. From a set-theoretical perspective, any “non-set-theoretical” question about Urelements is uninteresting; it is simply beyond the grasp of the theory. Since our interest is here mainly metaphysical, I will consider this feature relevant.

One last remark is in order. What is argued for Urelements of impure sets also holds for Urelements of pure sets. For instance, the natural numbers do not belong to sets. What belongs to sets is the result of applying the function UR to the natural numbers: $UR(0)$, $UR(1)$, etc. In this sense, my view differs from the “naïve” mathematician’s view even when concrete objects are completely left aside. The philosophical meaning is, once again, that UR is needed whenever we want to display some set-theoretical properties.²²

²²I would like to thank Fabrice Correia for raising these last two points.

3.6.1 A development in impure set theory

I have shown that Urelements are objects that possibly belong to some sets. We need now to state a criterion to establish to which sets they belong. To do this, let me sketch a theory in which each particular membership (or its negation) is decided *via* the properties that the abstract objects or the abstract copies encode or exemplify. The distinction between exemplifying and encoding mode plausibly gives rise to two different sets to which Urelements belong.

Given that a denotes abstract objects or abstract copies,

$[P]$ is the set of objects a which encode P

$]P[$ is the set of objects a which exemplify P

Then, we have two principles. The first concerns the membership condition for the [encoding sets]:

i) $\text{UR}(a) \in [P] \text{ iff } aP$

i. e., $\text{UR}(a)$ belongs to $[P]$ iff a is an abstract object and encodes P (where P is the predicate forming the set).²³

The second concerns the membership condition for the]exemplifying

²³Notice that for the abstract copies (where $a = Cx$), the clause holds iff x exemplifies P .

sets[:

ii) $UR(a) \in]P[$ iff Pa

i. e., $UR(a)$ belongs to $]P[$ iff a is an abstract object and a exemplifies P (where P is the predicate forming the set).

Consider two examples. The first takes an abstract object such as number 1; the second takes Socrates' abstract copy (Cs). Let me take A as standing for "being abstract", O for "being odd", and P for "being a philosopher". Then,

a) $A1 \rightarrow UR(1) \in]A[$ – for ii) *i. e.*, since number 1 exemplifies the property of being an abstract object, then $UR(1)$ belongs to the]set[of abstract objects.

b) $1D \rightarrow UR(1) \in [D]$ – for i) *i. e.*, since number 1 encodes the property of being odd, then $UR(1)$ belongs to the [set] of odd numbers.

c) $ACs \rightarrow UR(Cs) \in]A[$ – for ii) *i. e.*, Cs exemplifies the property of being abstract, then $UR(Cs)$ belongs to the]set[of abstract objects.

d) $CsP \rightarrow UR(Cs) \in [P]$ – for i) *i. e.*, since Cs encodes the property of being a philosopher, then $UR(Cs)$ belongs to the [set] of philosophers.

3.6.2 Problematic cases

Singletons

Among the different sets that we can find in set theory, the case of singletons seems to occupy a special place. The notion of singleton is problematic because it does not satisfy Cantor's canonical definition of sets as "many (*jedes Viele*) which can be thought of as one, *i. e.* a totality of definite elements that can be combined into a whole by a law". [Cantor, 1883], 204 & 443. Basically, since we have no "many", we do not know how we must build a set by a combination of elements. In terms of properties, singletons are defined as "sets having exactly one member". Then, a singleton has its member in virtue of the property of "being a singleton of that member". But, of course, this position is exposed to circularity.

To put the story differently, while we can say, for instance, that Socrates Urelement belongs to the [set] of philosophers because Socrates was a philosopher, the predicate that allows an analogous reconstruction for Socrates Urelement belonging to its singleton by some Socrates' properties is not so easy to find. To formulate the question in a general way: how can we say that $UR(a)$ belongs to its singleton?

My answer is coherent with the standard metaphysics of sets. Looking again at what has been established for the other sets, we can derive the relevant property "being a member of its singleton" from the membership condition for the [encoding sets], plus the self-identity of every abstract object a . The fact that $UR(a)$ belongs to the [set] containing a and only

a (namely, $[a]$) follows from the fact that $\text{UR}(a)$ belongs to the $[\text{set}]$ of all x such that x is identical to a , *i. e.* $[x:x=a]$. This is the case because a encodes the property of being self-identical *i. e.*, $(= a)$. In this way, the account would also preserve homogeneity. To formalize,

$$a (= a) \rightarrow \text{UR}(a) \in [x: x=a]$$

Hence,

$$\text{UR}(a) \in [a]$$

Similarly, we can derive the property “being a member of its singleton” from the membership condition for the $[\text{exemplifying sets}]$, plus the self-identity of every abstract object a . The fact that $\text{UR}(a)$ belongs to the $[\text{set}]$ containing a and only a (namely, $]a[$) follows from the fact that $\text{UR}(a)$ belongs to the $[\text{set}]$ of all x such that x is identical to a , *i. e.* $]x:x=a[$. This is the case because a exemplifies the property of being self-identical *i. e.*, $(= a)$.

$$(= a) a \rightarrow \text{UR}(a) \in]x: x=a[$$

Hence,

$$\text{UR}(a) \in]a[$$

The same holds for singletons belonging to other singletons. For example, since $\{a\}$ is an abstract object exemplifying and encoding the property of “being self-identical”, then it belongs to the set $[x: x=\{a}]$ and to the set $]x:x=\{a}[$, *i. e.*, to $[\{\{a\}\}]$, and to the $] \{\{a\} [$.

This way of understanding singletons is more explanatory than just claiming that singletons are sets having exactly one member, and is also coherent with the standard metaphysics of sets according to which the singleton of a is the set that is the extension of the property of being identical to a .

Empty set

Another problematic case is that of the empty set. As it is clear, the empty set has no members, and so does not satisfy Cantor’s definition either. Now, from what has been said, we can claim that the empty set just is the function Urelement when is not saturated. In other words, there is no abstract object that encodes properties that some ordinary objects exemplify, and there is no abstract object encoding some properties that saturate the function. Despite this, the empty set still is an object encoding and exemplifying some properties (as Zalta suggests, it encodes properties such as “being a subset of all the other sets” and exemplifies “being abstract”). More specifically, the empty set can be built up by Urelements and exemplification in the following way: there is no $UR(x)$ such as it belongs to the $]empty\ set[$ (namely, $] \emptyset [$) *iff* x is not identical to x , *i. e.* $]x: x \neq x[$.

Formally,

$$x (\neq x) \rightarrow \text{UR}x \in]x: x \neq x[$$

Hence,

$$\neg \exists x \text{ t.c. } \text{UR}x \in]\emptyset[$$

In other words, the empty set cannot be defined by its elements, but it is of course defined by the fact that it has no elements. Once again, this is coherent with the standard metaphysics of sets according to which the empty set is the set that is the extension of the property of something being different from itself.

3.6.3 Epistemology for impure sets

An important point concerns the epistemic access to the abstract objects I have introduced. How do we know *Noemata* and abstract copies? Or, to put the story in Benacerraf terms again: How do we get epistemic access to all these abstract objects that are plausibly causally inert?

3.6.4 Epistemic access to *Noemata*

Noemata look epistemically familiar as long as they are central to intentionality. Hence, our access to these *Noemata* follows from our way of referring to objects flesh and blood. Basically, we do have mental

representations, and we also have a cognitive relation with these contents. This relation could have empirical components (recall the case of someone who perceives a tree), or not (Zalta's theory of intentionality also applies to purely abstract objects modeling situations such as me thinking about Pegasus).²⁴

3.6.5 Epistemic access to abstract copies

For abstract copies, the situation is more complicated. As mentioned, abstract copies inherit all their encoded properties from ordinary objects. It could then be thought that they have empirical origins: the subject perceives ordinary objects, and they build up abstract copies from their experience. However, contrary to *Noemata*, this would be problematic. Abstract copies do not have subjective flavor, and by the way, empirical sources are not sufficient for accounting for all the abstract copies that we need in impure set theory (for instance, how we have epistemic access to sets containing abstract copies of non-perceivable ordinary objects such as the set containing all the stars of the universe?). Moreover, it is not even plausible to claim that we have empirical access to abstract copies because these copies have all the properties of ordinary objects, and this kind of epistemic access is certainly not empirical.

Zalta does not explicitly associate OT with a particular epistemic framework. However, when he speaks about abstract copies, it is clear that he takes them to be mind-independent objects. Since they are built up starting from the properties of their elements, impure sets are also

²⁴This is shown and defended in [Luporini, 2022].

mind-independent. Now, the debate on epistemic access to abstract objects, and more specifically, to objects related to mathematics, is one of the main vast and intricate. Benacerraf's dilemma seems even more dangerous when we deal with mind-independent objects. In what follows, I will just limit myself to presenting a framework that, I argue, can be meaningfully associated with my account. This framework consists in a form of conceptualism which has nonetheless no psychologist flavor.

A particular kind of conceptualism

The idea comes from [Bellotti, 2010] which proposes to reframe a Kantian-flavored account. Mathematics is the world of concepts and of the possibility of concepts. It is therefore not about mental activities, but about what is thought independently from any subject. However, mathematics is not Platonic either. It is about the conditions of any possible reality.

In this sense, it is true that concepts are humanly graspable but the term "humanly" does not have a subjective connotation. Mathematical structures are essentially understandable; they are nothing else than the forms assumed by the act of understanding. Hence, as conceptual structures, mathematical theories are both autonomous and accessible to human beings. Shortly, the problem of subjectivism for this kind of conceptualism is reformulated in this way: how do human beings manage to successfully develop and use conceptual heritage?

To use [Bellotti, 2010]'s words,

“Just in this properly human activity, the concepts which

arise are radically transcendent over the activity itself: once they are reached, they have no connection whatsoever with human minds, not in the sense that they cannot be grasped (...) but in the sense that nothing in their content presupposes anything having to do with cognition". [Bellotti, 2010], 3.

Even though this framework is not yet completely developed, I argue that it fits with Zalta's abstract copies. As argued, abstract copies are conceptualized objects, and they are both understandable and mind-independent. Abstract copies are reached throughout human activity (by conceptualization of abstract and concrete items). Nonetheless, their content does not presuppose such a cognition. This form of conceptualism seems therefore to be compatible with the realistic framework we are working with.

To sum up, Urelements are conceptualized objects that are better understood as abstract rather than concrete. This means that, contrary to the classic definition, impure sets are fully abstract objects (*i.e.* non-spatiotemporal objects having non-spatiotemporal elements). Impure sets are nonetheless related to the concrete. This relation is warranted by a revised notion of Urelements plus some devices from Zalta's OT. More specifically, Urelements of impure sets are taken to be functions that can be saturated by some abstract copies, *i. e.* abstract objects that encode all and only the properties that some ordinary objects exemplify.

Here is my new definition of impure sets. As anticipated in the In-

trodition, this outcome says something about the general notion of set. Sets are indeed fully abstract objects. Also, they are intimately related to concepts, conceptualization, and conceptualized objects. Their identity is given by Extensionality which is a principle reached by conceptual tools. More specifically, sets are fully abstract objects containing other sets and/or Urelement-functions some of which are saturated by abstract objects or abstract copies of ordinary objects. Singletons are fully abstract objects containing a set or a Urelement-function saturated by objects encoding the property “being such that $x = a$ ”. Empty set is a fully abstract object containing all the Urelements saturated by some objects encoding the properties of not being self-identical, namely no object.

The idea of modeling Urelements by OT has never been proposed. In the next section, I will consider some objections that can be raised against this account, and provide some replies.

3.7 Objections and replies

It is worth anticipating that not all objections are new. Some of them already apply to the mainstream thesis and to Maddy’s account for impure sets. In reply to such objections, I will also show that my framework is superior.

3.7.1 Objects instead of conceptualized objects

One of the perplexities that can be raised against my account concerns the fact that Urelements are conceptualized entities. It is plausible to

claim that someone who desires to go against this tenet, would insist that Urelements are objects.

There are many ways in which this objection can be stated. In what follows, I will present three of them: the first argues that if Urelements are conceptualized objects, then we reach either too many or not enough sets; the second is a more general objection and points out that our language does not refer to conceptualized objects but to objects; the third points out that conceptualized objects give rise to an account with too many items.

I honestly think that these objections have to be taken seriously. I will show, nonetheless, that friends of OT applied to impure sets can successfully reply.

Counting problem

OT and realism To stress the so-called "counting problem", I need to first discuss a further point. As mentioned, OT can apply to different philosophical items such as Platonic Ideas, Fregean senses, and individual concepts. However, such applications may give some philosophical features back to OT. For instance, when applied to Zalta's theory of intentionality, OT seems to involve realism: *Noemata* and objects they refer to exist and have some properties. Such realism is mirrored from a more basic epistemic point of view: it is natural to claim that our knowledge of these ordinary objects is more or less adequate depending on how many "right" properties our *Noemata* encode.

My treatment of impure set theory by OT also involves realism. To see

why, recall that *Noemata* can be different from ordinary objects they refer to (e. g. my *Noema* of a tree can wrongly encode “being red”). Thus, I have argued that they are not sufficient to account for Urelements. The implicit clause is of course that impure set theory is supposed to describe reality. Abstract copies, which replace *Noemata*, allow us to incorporate this realistic flavor: they exist, and have all the properties of existing ordinary objects. This leads us to think that impure set theory also has a realist flavor: impure sets exist and result from the properties of their Urelements and their abstract copies (which, in turn, result from the objects of which they are copies). Hence, roughly speaking, a realistic account for impure sets aims to describe objects *as they are*.

Nonetheless, some incompatibilities between realism and the development of impure set theory by OT can be pointed out. The main one is that we end up having too many sets than those required in set theory. In what follows, I will consider this objection starting by individuating its source.

Overpopulation by double predication One of the reasons why my account allows for too many sets comes from Zalta’s double predication: For instance, $UR(Cs)$ both encodes and exemplifies “being self-identical”, and so belongs to both the $[set]$ and the $]set[$ of self-identical objects. Hence, Zalta’s double predication is a conceptual tool that leads to cases in which the same object or collection of objects give rise to two different kinds of sets with exactly the same members. But this goes against **Extensionality**.

However, there are good reasons why double predication is in play: it describes the double nature of abstract objects. More specifically, for the case of Urelements, it allows them to express different and even incompatible properties that the relation of representation between copies and ordinary objects involves. Examples were: $\{\text{Socrates}\}$ contains an element which is a human being and which is not a human being at the same time (*i. e.* an abstract object that encodes the property of being a human being and exemplifies the negation of that property). Then, if we accept Zalta's double predication, its treatment in terms of sets just seems a natural extension. Zalta's double predication has its opponents. [Bueno and Zalta, 2017] provides a defense. In Section 3.7.2, I will discuss in detail some of the items of this debate.

Overpopulation by co-extensional predicates Even conceding that double predication is good news, one can still insist claiming that the problem of the overpopulation of sets follows from the more general fact that what we are counting are concepts or conceptualized objects instead of objects. As explored in Chapter 2, Maddy's account has the same problem. Maddy could end up with too many sets (recall the case of co-extensional predicates such as "having a heart" and "having a kidney" generating two sets instead of one). In Maddy's case, I have argued that this was a bad consequence of taking perception to be a good tool to account for impure set theory. But the same phenomenon can of course happen with concepts and conceptualizations: to take back the example, "having a heart" and "having a kidney" are two different predicates

corresponding to two different concepts. Yet, they build up a unique set. Roughly speaking, both perception and conceptualization seem to produce bad results for set theory.

However, conceptualization is superior to perception because it brings up a device that rules out the problem of co-extentional concepts: **Extensionality**. To claim that **Extensionality** is conceptual is to claim that constancy of sets is not perceptual but that it comes from a purely conceptual insight about sets which is gained from understanding an aspect of how sets behave.²⁵

Overpopulation by conceptualization of objects considered under different points of view Another problem that arises from taking conceptualization as the main source of impure set theory concerns objects considered under different points of view. Once again, we could end up having different sets for the same object.

However, it is easy to remark that the treatment with concepts is superior to that with perception. To see why, recall some cases proposed in Chapter 2 in which a unique object produces perceptually different objects, *e. g.* the duck-rabbit piece of wood. We obtained two sets from one object. And even conceding to the pluralist that we really had two different objects, we can always find some more problematic cases.

²⁵It can be objected that this is circular: It is true that **Extensionality** is justified by conceptualization, and not by perception. But it can also be argued that what suggests that the target of impure sets theory are concepts and conceptualized objects instead of objects is nothing but **Extensionality**. Moreover, and more importantly, the fact that conceptual tools justify **Extensionality** for set theory follows from the fact that the concept of set involves **Extensionality**. However, I think that, even if justification by conceptualization is circular, it is still better than justification by perception which is impossible.

Consider a door in front of us. Potentially, there can be infinite different perceptions of this object. However, it would be wrong to assume that these perceptions correspond to different objects (and so to different sets).

This conclusion is avoided by Maddy's conditions on perception and, more specifically, by both **Directness Condition** claiming that we perceive physical objects as opposed to sense-data, and **Causal Condition** claiming that physical objects and sets in front of us play an appropriate causal role in the generation of our perceptual beliefs about them. However, it is clear that perception does not always work in this way. Sometimes, even often, we fail in perception. Maddy's procedure is therefore circular: realism justifies conditions on perception which aims to justify realism in their turn. Thus, perception is not enough to account for set theory.

Conceptualizations allow a better treatment of these cases of overpopulation of objects. We can indeed have potentially infinite conceptualizations of the same object. These conceptualizations will correspond to sets. But this is not problematic because it does not directly involve realism as perception did. The fact that we have a lot (and even incompatible) ways of conceptualizing objects can be modeled using Zalta's theory of intentionality: These sets derive from the function Urelement saturated by some *Noemata*. To take back the example, the door is an ordinary object to which different *Noemata* can refer. Starting from this, we can have different impure sets: for instance, the set of Valentina's perception of the door, the set of someone else perception of the door, and so on. Hence, even without bothering **Extensionality**, conceptualization

seems superior to perception.

It could be objected that we have chosen abstract copies instead of *Noemata* to prevent impure set theory to have subjective flavor. Now, it is clear that there are exceptions, and these exceptions come into play when what we want to account for is precisely the subjective flavor of conceptualization. This means that the use of *Noemata* does not rule out realism for impure set theory. Then, impure set theory both has *Noemata* and abstract copies. However, these two have different roles: *Noemata* are related to ordinary objects, and come into play only when we want to account for intentional acts, subjective perceptions (which can refer to ordinary objects flesh and blood), while abstract copies directly account for ordinary objects flesh and blood.

Underpopulation by weakness of conceptualization Maddy's account was also affected by the opposite problem: the underpopulation of objects. Basically, perception is too weak to account for all the sets that we want to account for: Singletons are perceptually identical to their members, $\{\{\text{Socrates}\}, \text{Socrates}\}$ is identical to $\{\text{Socrates}\}$, and so on. It can be argued that my account suffers from the same problem: Conceptualization is too weak to account for all the sets that we want to account for. Basically, we do not have enough concepts for the sets we need in set theory. For instance, it can be argued that we do not have any concepts for sets having uncountable many objects.

However, it is plausible to claim that this problem follows from a theory of concepts that has subjective flavor: since we are finite subjects,

we cannot figure out uncountable many objects. But, as mentioned, concepts are supposed to be mind-independent. Sets are therefore not necessarily grasped by some particular subjects. The relevant thing is to have a *graspable* concept. And for the case of uncountable sets, it would be the following: sets that cannot be put in a one-to-one correspondence with natural numbers.

Of course, the opponent can still argue that some sets do not correspond to any graspable concepts. Examples could be sets such as uncountable sets of countable sequences. Then, among all the sequences of natural numbers, we necessarily have sequences that are not individuated by any law (*i. e.*, they are not identified by something which can be formulated in a finite way). This is therefore a case of set without concept. However, the counterexample only applies when considering the subject doing mathematics as a Turing machine delivering finite instructions, and can be ruled out just by considering its infinitary extension.

Generally speaking, even conceding to the opponent that conceptualization alone is not enough to account for all the sets in set theory, it is of course superior to perception which fails to account for all the aforementioned kinds of sets.

Verbal or genuine disagreement? It could be argued that the disagreement between Maddy and me is only verbal. As mentioned, while speaking about perception, Maddy refers to conceptually loaded perpetual contents. Then, one can conclude that perception and conceptualization are the same thing. To claim that concepts are needed for impure

set theory is analogous to claim that for perceiving something, we need at least some hints about what we are perceiving.²⁶

I actually do not think that the disagreement is merely verbal though. It is indeed clear from Maddy's examples that she gives such importance to perception as an empirical faculty (recall the case of Steve perceiving the set of eggs by opening the fridge). But conceptualization does not incorporate this purely empirical feature, at least *prima facie*. Hence, I think that claiming that perception is identical to conceptualization involves category mistakes.

Also, there are some contexts in which conceptualization is superior to perception. To use Hale's powerful image, "we may see the hungry with our eyes, but we see that we should feed them with the mind's eye". [Hale, 2013], 73. And this is of course the case for impure set theory: one has not only to "see" sets. They also have to understand their origins, their functions, their roles in some given theories, and so on. Now, I argue that one of the main differences between our eyes and our mind's eye specifically concerns the aforementioned empirical component which is essential to the first but not to the second.

Here the opponent could insist that perception is not conceptualization but still requires it. However, as largely shown, conceptualization plays the main role in impure set theory. Thus, I do not see any particular reason to privilege the analysis of perception instead of that of conceptualization.

Finally, it may be objected that there is a sense in which Maddy would

²⁶For this kind of arguments, see, for instance, [Siegel, 2019].

agree on this. According to her, set theory requires intuition which is employed to account for higher-order operations concerning conceptualizations. First of all, this idea would already involve a difference between concepts and perception so that the previous disagreement would not be only verbal; second, as already argued, to postulate intuition means to postulate an arguably mysterious faculty that providentially intervenes when perception fails. Thus, if this is the case, the role of intuition has to be justified as I did for conceptualization.

Names of concepts or names of things?

Another way to argue that there is something misguided in taking concepts instead of objects for impure set theory concerns some more general issues about language and reference. Let me consider a huge topic of the 17th history of philosophy: Locke's idea of words as standing for ideas instead of objects.

In particular, he claims that

"A name is a word taken at pleasure to serve as a mark that may raise in our mind a thought like some thought we had before, and which being pronounced to others gives them a sign of what thought the speaker has before in his mind."

[Locke, 1689], 3, § 2.

Hobbes also thinks that language refers to concepts. Here is the passage:

“Since, as has been defined, names ordered in speech are signs of conceptions, it is obvious that they are not signs of things themselves; for in what sense can the sound of the vocal sound “stone” be understood to be a sign of a stone, other than that whoever might have heard this vocal sound will gather that the speaker has thought of a stone?” [Hobbes, 1655], 1, §5.

However, this idea is not uncontroversial. The infamous criticism comes from [Mill, 1843]:

“Names are intended not only to make the hearer conceive what we conceive but also to inform him of what we believe. Now, when I use a name to express a belief, it’s a belief about the thing itself, not about my idea of it. When I say ‘The sun causes daylight’ I don’t mean that my idea of the sun causes in me the idea of daylight; I mean that a certain physical fact (the sun’s presence) causes another physical fact, namely daylight.” [Mill, 1843], 20-21.

More recently, this position is adopted by Frege who claims that when we speak about the moon, we actually refer to the planetary mass object, and not to our mental representation of it.

Mill’s objection to Locke’s and Hobbes’s idea can apply to my account in that it claims that, for instance, the fact of speaking about Socrates brings into play a concept instead of a human being flesh and blood. And the same goes for Socrates when he appears in {Socrates}.

Now, Mill's argument is not uncontroversial and literature on the topic is vast.²⁷ My defense will go as follows: the fact that we use concepts while speaking does not entail that we do not aim to refer to objects flesh and blood. Even Mill is clear about this phenomenon:

“This simple definition of a name, as a word or phrase that serves as a mark to recall to ourselves the likeness of a former thought, and a sign to make it known to others, seems to be just right”. [Mill, 1843], 23.

Then, taking the example of a stone and people referring to, he claims that

“if the point is merely that what is brought back into the speaker's mind or conveyed to the hearer is not a stone, there's no denying it.” [Mill, 1843], 23.

Basically, what we have in mind are not, strictly speaking, concrete objects to which we aim to refer. Following my account, words refer to conceptualized objects which sometimes refer to these concrete objects. And, as deeply argued, conceptualized objects and concrete objects are distinct things.

Occam's Razor?

Another objection that can be addressed against the idea of taking concepts instead of objects for impure set theory concerns the fact that my

²⁷Santambrogio claims that Locke's idea is completely out of date. Not everyone agrees on this. See for instance, [Fodor, 1975].

account mobilates too many objects: ordinary objects, their copies, and Urelements. Then, even if we reach the right number of sets, Occam's Razor is not satisfied.

I have largely shown the rationale of my account. In this sense, I think that it does not multiply the entities without or beyond necessity because it delivers a relevant philosophical meaning that other more economic theories do not incorporate. I therefore privilege a luxurious account which is able to describe what I take to be the nature of impure set theory, than one which has the advantage of being thin, but not sufficiently explanatory.

The account can also be taken to be too cumbersome, especially if one considers how Urelements and abstract objects are supposed to interact each other. I honestly think that, despite all the technicalities that I have tried to sketch, the general idea remains quite simple and intuitive. Once again, when someone develops impure set theory, he is thinking about a formal theory with some abstract objects having logical properties. Yet, he also aims to speak about ordinary objects. Here, the importance of my account that relates all these aspects: abstract objects having logical features (*i. e.* Urelements), and abstract objects having relations with ordinary ones (*i. e.* abstract copies).

3.7.2 Technical problems with OT

Another set of objections concerns some unintuitive outcomes that OT involves. Some of these outcomes appear when OT applies to impure set

theory, others directly follows from OT taken alone, and also apply to OT when used to model impure set theory. For the aim of my scope, I will treat these outcomes by considering OT when applied to impure set theory.

More specifically, the first problem is that abstract copies resembles to fictional objects; the second concerns the rationale of the double predication, and the third a restriction on the encoding mode that Zalta proposes.

Fictional objects and abstract copies

One of the difficulties is that abstract copies of ordinary objects and abstract objects “tout-court” are dangerously similar. Consider Sherlock Holmes and the abstract copy of a flesh and blood French detective: Marcel Guillaume. They both encode “being a detective” and exemplify “being abstract”. When it comes for set theory, we will have Marcel Guillaume’s abstract copy which saturates the function Urelement. Now, from what has been said, Sherlock Holmes and Marcel Guillaume Urelement both belong to the very same [set] of detectives. Yet, we would like to distinguish between the two: the first is a detective of a novel; the second is an abstract copy of a physical one. This distinction is quite difficult to draw in terms of encoded and exemplified properties. But the distinction between abstract objects and abstract copies can be drawn in another way: Sherlock Holmes and Marcel Guillaume do not encode the property of being a detective for the same reasons: While Marcel Guillaume’s copy encodes it because of its corresponding ordinary ob-

ject exemplifying it, Sherlock Holmes encodes it in virtue of its narrative characterization.

Notice also that the difference between an abstract object and an abstract copy is warranted if we do not only take encoded properties as a criterion of identity for abstract objects as Zalta did. Consider the borderline case of the (strange but possible) existence of a real detective named Sherlock Holmes sharing all the properties of Doyle's Sherlock Holmes (living in London, having 60 years old in 1914, being addicted to morphine, etc). Basically, a physical doubleganger of Sherlock Holmes abstract object. In impure set theory, this object will become an abstract copy that saturates an Urelement: $UR(Csh)$. How can we distinguish between $UR(Csh)$ and Sherlock Holmes Urelement, *i. e.* $UR(Sh)$? The answer is straightforward: $UR(Csh)$ does not exemplify the property of "being a character of Doyle's novel", while $UR(Sh)$ certainly does.

Here, it can be argued that if we imagine a case in which Doyle has effectively written about Sherlock Holmes taking the inspiration from Sherlock Holmes human being, or, even better, imagine that the novel is a biography of Sherlock Holmes human being that Doyle has followed step by step, then we would perhaps claim that both $UR(Csh)$ and $UR(Sh)$ exemplify the property of "being a character of Doyle's novel". Yet, we can still distinguish between the two in various ways. Just to mention two of these. First, for $UR(Sh)$, the encoding mode can be incomplete, while for $UR(Csh)$, is always complete. For instance, $UR(Csh)$ must encode the property of "having a mole on one's left foot" or its negation, while it is possible that $UR(Sh)$ does not encode this property nor its

negation. Second, $UR(Csh)$ and $UR(Sh)$ arguably differ from the point of view of their essential properties, if any. It can be said for instance that $UR(Sh)$ is essentially a detective, but that $UR(Csh)$ is not. Or that $UR(Csh)$ encodes the essential property of being human and being born from its parents, while $UR(Sh)$ does not.

To sum up, OT can be problematic when it comes to treat abstract objects *tout-court* that are similar to abstract copies. Nonetheless, such a difficulty is not knockdown. Conceptualized objects related to ordinary objects and abstract objects can indeed have properties in common and the same mode of predication, but can still be distinguished by either considering some other properties, or the reasons why they have these properties.²⁸

Non-homogeneity of the encoding mode

From what has been said in the section above, it may be argued that the encoding mode is not homogeneous: Encoding some properties that ordinary objects exemplify is different from just encoding some properties: Sherlock Holmes and Guillaume's abstract copy both encode "being a detective". Sherlock Holmes encodes it because of Doyle's story, while Guillaume really *is* a detective. I think that this difficulty is just a symptom of a more general problem with Zalta's double predication.

²⁸It could sound strange to consider Sherlock Holmes in set theory in the first place. Set theory aims to speak about existing objects, while Sherlock Holmes is a fictional object. Following Zalta, I take Sherlock Holmes to be an abstract object, and I put no restriction of this kind in set theory.

This problem is treated in contemporary literature on Neo-meinonghianism and, more specifically, is incorporated by the discussion between supporters of Modal Meinonghianism and OT (see, for instance, [Bueno and Zalta, 2017] and [Berto et al., 2020]). I do not aim to consider all the aspects of the debate which is huge and complex. In what follows, I will focus on encoding and exemplifying mode by showing some crucial and still unexplored philosophical items. In particular, I will argue that the main arguments pro and cons the distinction are not conclusive. Then, I will propose a sense in which the distinction can be understood.

***Ad hoc*-ness of double predication** Zalta's double predication is taken to be *ad hoc* by many philosophers (see, for instance, [Jacquette, 2015], and [Priest, 2016]). In [Bueno and Zalta, 2017], the authors argue that the distinction is not *ad hoc* because it provides a unified approach to a vast range of phenomena, such as Platonic Forms, Leibnizian concepts, Fregean extensions, and so on.

To better grasp the debate, it is important to agree on what *ad hoc*-ness means. Following some dictionary definitions, a solution is *ad hoc* when is "designed for a specific problem, non-generalizable, and not intended to be adapted to other purposes".²⁹ Then, [Bueno and Zalta, 2017]'s reply makes perfectly sense. However, and quite ironically, it seems that what the opponents of Zalta's double predication have in mind goes the other way around. It is precisely because double predication applies to a vast range of very different phenomena that we stick

²⁹See *Cambridge Dictionary*.

with the feeling that their homogeneous treatment is arbitrary. In this second and, I argue, more compelling interpretation of the charge, something is *ad hoc* when it is, so to say, not philosophically relevant. Then, [Bueno and Zalta, 2017]’s reply increases the charge. Of course, it can still be argued that the notion of philosophical relevance is obscure, but the supporter of the encoding-exemplification seems to have the burden of proof.

Other similar distinctions [Bueno and Zalta, 2017] argues that the distinction is philosophically relevant in that it already appears in the history of philosophy, though under different names. Examples are [Boo-los, 1984]’ interpretation of Frege’s distinction between ”falling under” (relating some objects to some concepts) and ”being in” (relating some concepts to some objects); Other examples are [Inwagen, 1977]’s difference between having a property and holding a property, [Kripke, 1973]’s distinction for fictional objects, and [Meinwald, 2016]’s distinction between *pros ta alla* and *pros heauto*.

It has to be noticed that all these examples have to be considered analogies in that their relations with Zalta’s double predication involve some differences. Thus, [Bueno and Zalta, 2017]’s reply requires a clarification about the role of the analogy in the defense of double predication. Is the relation between encoding and exemplifying and these other philosophical items able to explain and justify double predication? In [Berto et al., 2020], the authors argue that it is not the case and that is rather leads to unpalatable consequences. To be honest, I do not

think that this is the case. Nevertheless, even conceding to [Berto et al., 2020] that their arguments go, the author do not consider Meinwald’s distinction, which deserves to be briefly analyzed. To anticipate a bit, I do not think that Meinwald’s distinction fully accommodates Zalta’s encoding-exemplifying modes. Nonetheless, this debate has an independent interest, and also leads me to introduce my alternative explanation.

Meinwald’s distinction [Meinwald, 2016] attributes a distinction between two kinds of predication to the second part of the Plato’s *Parmenides*: x is *F pros ta alla* (*i.e.*, in relation to the others) and x is *F pros heauto* (*i.e.*, in relation to itself). This distinction, similar to that between intrinsic/extrinsic properties, is used in connection with the well-known One-Many problem. Roughly speaking, the existing things would have some properties *pros ta alla*. Then, for any of those properties, we have a corresponding Platonic Form which has the same property *pros heauto*. An example: The Form of the Triangle is triangular *pros heauto*; existing triangular objects are triangular *pros ta alla*. Translating this distinction into Zalta’s OT, we would say that the Form of the Triangle is an abstract object encoding the property of “being a triangle”, while existing triangular objects are concrete and would exemplify it. Hence, predication *pros heauto* is supposed to explain the encoding mode, while predication *pros ta alla* is supposed to explain the exemplifying mode: properties that x encodes are encoded in relation to x itself, while properties that x exemplifies are exemplified in relation to other objects.

It seems though that not all the exemplified properties presented in

Zalta's framework are *pros ta alla*. Consider, for instance, the case of the Form of Triangle that supposedly exemplifies "being abstract". Arguably, since abstractness is part of the nature of the Form of Triangle, it has this property in relation to itself. This property is therefore exemplified and *pros heauto*. It may be objected that this property is not *pros heauto* but *pros ta alla* because it is predicated of the Form of Triangle in relation to the Form of Abstractness. Similar cases can be found in Meinwald. For instance, the Form of Triangle is intelligible *pros ta alla* because it participates in the Form of Intelligibility (*i. e.* it has this property in relation to the Form of Intelligibility). Similarly, some properties that ordinary objects seem to exemplify in relation to themselves such as "being self-identical", are exemplified in virtue of their Forms.

Nonetheless, Zalta's opponents could still propose some cases of abstract objects that encode some properties *pros ta alla*. One interesting case is that of abstract copies. For instance, Socrates' abstract copy encodes all the properties of Socrates ordinary object. But the encoding mode here is plausibly *pros ta alla* (the copy encodes all these properties in relation to Socrates). Another case concerns mathematical objects. Mathematical objects count as abstract in Zalta's ontology. As mentioned, they encode some properties in relation to the system with which they are associated. For instance, the empty set encodes the property of "being a set with no members" in relation to ZF. Then, such properties are encoded but *pros ta alla*.

To sum up, we do not have a one-to-one correspondence between *pro heauto-pros ta alla* and encoding-exemplifying mode. Also, *pros heauto-*

pros ta alla seems mutually exclusive, while encoding-exemplifying is not. Hence, I conclude by claiming that the analogy with Meinwald's modes of predication works only for certain kinds of objects, but not for all of them. It seems therefore not able to cover the whole extension and meaning of Zalta's distinction. I think therefore that the opponent of Zalta's double predication could argue that the analogy may have the effect of increasing the confusion about the philosophical meaning of Zalta's double predication.³⁰

Encoding as representation What is the philosophical relevance of Zalta's double predication? A more accurate work on its rationale deserves to be done. For what is worth, I have tried to show that Zalta's distinction explains the double nature of abstract objects. As mentioned, thanks to Zalta's double predication, some incompatible properties of Urelements can co-exist. To be fair, there are other ways of reaching the same result. Impossible worlds and nuclear and extra-nuclear properties are good examples (see, for instance, [Berto, 2013] and [Jacquette, 2015]). None of these come without problems. I do not aim to argue for the superiority of Zalta's distinction here, but I will propose what I think to be a plausible philosophical interpretation of the encoding-exemplifying mode which comes from considering the behavior of abstract objects.

Let me take the specific problem of the non-homogeneity of the en-

³⁰After that, [Bueno and Zalta, 2017] argue that the distinction is primitive and has therefore not to be fully explained. The analogy this time is with the set-membership relation. However, as [Berto et al., 2020] pointed out, to introduce \in as the primitive of a theory of abstract objects like set theory seems a natural move, while to propose that \in can be used to analyze predication seems more controversial.

coding mode. To my knowledge, no literature on the specific issue of the difference between encoding mode for abstract copies and for abstract objects which are not copies is available. It seems to me that in both cases, the encoding mode highlights the *representational* feature of abstract objects: abstract copies represent ordinary objects by encoding properties, abstract objects which are not abstract copies represent some of their characteristics that cannot be exemplified. Examples are Socrates abstract copies that represents Socrates' humanity by encoding the property of being a human being; if any, Platonic Idea of redness that represents its own characteristic of being red by encoding that property without existing in space and time; Sherlock Holmes that represents someone who is a detective, and Guillaume abstract copy that represents Guillaume ordinary object being a detective. By contrast, exemplification is for both ordinary and abstract objects the kind of predication which refers to some, so to say, *natural way* of having properties (*i. e.* the way in which we usually use the copula "is". Examples are: Socrates' abstract copy is abstract, Socrates is a human being). I conclude, therefore, by claiming that [Bueno and Zalta, 2017] are right in defending Zalta's double predication and that their defense could be accompanied by this new notion of encoding as representation.

Encoding as representation has to be considered just as a first step towards the understanding of the philosophical relevance of Zalta's double predication. This idea has also to be integrated to all other meanings (*i. e.* the fact that encoding characterizes abstract objects, that encoding and exemplification can be related to theories, and so on). Since the

aim of my dissertation is to understand impure sets, and encoding as representation allows us to describe Urelements of impure sets, I will let this work for another time.

Identity encoded

The last objection is technical. It has to do with the encoding mode and the property of "being self-identical". As anticipated, encoding-exemplifying mode are not exclusive. In Pirandello's "Six Characters in Search of an Author", for instance, the characters play other fictional characters. They therefore encode and exemplify properties such as "being fictional", "being characters", etc. Another example was that of Urelements of singletons that encode and exemplify the property of being self-identical.

However, this property seems problematic for the encoding mode. This happens because the consistency of OT, established by two different kinds of models, *i. e.* Scott-models (see [Zalta, 1983], Appendix) and Aczel-models (see [Zalta, 1999]), is secured by the one restriction on the formation of λ -expressions: that encoding subformulas may not appear in the matrix of such expressions. As a result of this constraint on λ -expressions, the general notion of identity ($x = y$) cannot appear in λ -expressions when contains encoding subformulas.

This is problematic for the case of singletons. As mentioned, the fact that $UR(a)$ belongs to the [set] containing a and only a (namely, $[a]$) follows from the fact that $UR(a)$ belongs to the [set] of all x such that x is identical to a , *i. e.* $[x: x=a]$, because a encodes the property of being

self-identical *i. e.*, ($= a$). But since it is not allowed to have encoding subformulas, how to account for singletons?

A solution is to just stay with exemplification. For exemplification, $[\lambda xy x=y]$ is a well-formed expression and denotes a relation. However, philosophically speaking, this would mean that, in the case of singletons containing abstract copies, the relevant property (*i.e.* self-identity) only comes from abstract copies, and not from the ordinary objects to which they refer. In other words, Socrates belongs to $\{\text{Socrates}\}$ because Socrates' copy is self-identical, and not because Socrates ordinary object is self-identical.

Another solution can be proposed just by looking at the rationale of Zalta's restriction. Zalta's solution avoids paradoxes in that, by rejecting all identity statements in which the encoding mode appears, it also rejects those which are problematic. However, this does not mean that it is impossible to save at least some of the identity statements with the encoding mode. Roughly, his restriction seems to be *ad hoc* and too strong. To better grasp this idea just think to Tarski's treatment of the liar paradox: his restriction throw away problematic sentences such as "this sentence is false", but it also throw away innocuous sentences such as "this sentence is true". In this work, I do not aim to show how identity plus encoding can be saved. I just limit myself claiming that these subformulas do not seem to lead to paradoxes *prima facie*.

3.7.3 Problematic relation between abstract and concrete objects

The last set of objections concerns the relation between abstract and concrete items. As largely shown in Chapter 1, these relations may be problematic. Recall the mainstream thesis. According to the mainstream thesis, impure sets are abstract objects having concrete elements. But this was problematic. Now, it can be argued that my account is not so different from the mainstream thesis: impure sets have, strictly speaking, no concrete elements, but they still have a relation with the concrete, or, adopting Zalta's idea of concreteness, with something that can be concrete. Then, someone who wants to dismiss my account would claim that it suffers of the same problems raised by mixing concreteness with abstractness.

Inheriting the problems of the mainstream thesis

Formulated in this way, the objection is vague. It does not take into account how the relation between abstract and concrete objects is supposed to work. For instance, my account does not suffer of the problem that I have pointed out for Fine's framework simply because it does not involve the notion of parthood or, more generally, that of component: abstract copies are not composed of ordinary objects to which they are copies. They rather represent them. Hence, **Trilemma** does not hold.

Perhaps, the account suffers of the difficulties raised against Parsons' framework. More specifically, it can be argued for **Slippery Slope**. But,

as mentioned, my account still allows for a distinction between what is abstract and what is concrete.

It could still be objected that, as for Parsons' account, the notion of representation is vague and fails in individuating impure sets. However, this is not the case. Representation is not vague because it is expressed by Zalta's logic of abstract copies. Concerning individuation, abstract objects have criteria of identity, and impure sets are just subjected to **Extensionality**.

Homogeneity

Finally, it could be noticed that my account presents another advantage over Parsons'. Parsons' treatment of sets having supposedly concrete elements as members and sets having supposedly concrete elements in their transitive closure is unclear. Let me recall some examples. Parsons claims that sets containing both concrete and non-concrete elements are mixed. His example is: $\{\omega, \text{his fountain pen}\}$. However, it is not clear whether for a set to be mixed it has to necessarily contain at least a fully abstract object. Then, it is not clear whether sets such as $\{\text{Socrates}, \{\text{Socrates}\}\}$ and $\{\{\text{Socrates}\}\}$ should count as quasi-concrete or as mixed objects. By the way, both alternatives are problematic. As mentioned, if sets such as $\{\text{Socrates}, \{\text{Socrates}\}\}$ and $\{\{\text{Socrates}\}\}$ are quasi-concrete objects, then **Criterion of Identity** fails; if they are not, Parsons' ontology can be accused of *ad hoc*-ness (see Chapter 1, Section 1.1.4).

In this dissertation, I did not focus on higher-order sets such as $\{\text{Socrates}, \{\text{Socrates}\}\}$, and $\{\{\text{Socrates}\}\}$. I have rather explored the

case of Urelements of impure sets. Nonetheless, the account allows us to claim that $\{\text{Socrates}, \{\text{Socrates}\}\}$, $\{\{\text{Socrates}\}\}$, and so on, are all impure sets. Following the new definition, impure sets are abstract entities containing Urelements saturated by abstract copies or other abstract objects. $\{\text{Socrates}, \{\text{Socrates}\}\}$ is abstract and contains functions saturated by Socrates' abstract copy as well as other abstract objects, *i. e.* $\{\text{Socrates}\}$, while $\{\text{Socrates}\}$ is an abstract object containing a Urelement saturated by Socrates' abstract copy. Homogeneity is therefore preserved.

3.8 Conclusions

In this chapter, I have discussed and defended the idea that impure sets are fully abstract objects, *i. e.* objects that are not spatiotemporally located having members or elements in their transitive closure which are not spatiotemporally located.

In Section 1, I have presented the negative dialectic suggesting that, given what has been said in Chapters 1 and 2, impure sets have to be fully abstract objects because they are neither abstract objects having concrete elements, nor fully concrete objects. In Section 2, I have shown that the notions of concept and conceptualization are intimately related to impure set theory. More specifically, I have argued that Urelements are conceptualized objects, *i. e.* objects that result from a process of conceptualization which consists in considering objects, events, etc. as falling under some concepts.

According to the *desideratum* of the negative dialectic, Urelements have to be abstract. In Section 3, I have argued in favor of this claim by using some arguments from Parsons and Balaguer. After that, I have briefly presented some alternatives such as considering conceptualized objects as concepts or Finean *qua*-objects.

In Section 4, I have shown that the risk of taking conceptualized objects to be abstract is that all objects become abstract. In replying to this objection, I have further clarified the nature of conceptualized objects arguing that they are strictly related to the concrete by a relation of representation.

Here is where Zalta's Object Theory comes into play. In Section 5, I have argued that OT is a good candidate to model this framework. More specifically, I have shown that Zalta's version of Husserl's *Noemata* well describes intentionality as made of conceptualized objects, while Urelements are better explained with the help of his individual concepts or abstract copies. After that, I have shown that individual concepts are nonetheless not enough to account for Urelements which necessitate a further clarification concerning their logical nature. I have concluded by claiming that Urelements of impure sets are functions saturated by some abstract copies.

The idea of modeling Urelements by OT has never been proposed. This is therefore a first attempt which raises, of course, a lot of doubts. In Section 6, I have explored some objections and provided some replies showing also some advantages of my account over the mainstream thesis and the idea of impure sets as fully concrete objects.

In the next and last chapter, I will explore one significant application of this account of impure sets to contemporary metaphysics. In particular, I will explore how my account applies to the very famous case of [Fine, 1994]’s counterexample with Socrates and {Socrates}. This application aims to both better understand the issue of Fine’s essentialism and to further justify the idea of impure sets as fully abstract objects.

Chapter 4

Applying my Account to Fine's Counterexample

Introduction

As mentioned in the Introduction, analytic contemporary philosophy is full of examples involving impure sets. In this chapter, I will explore one application of my account to one of the most famous use of impure sets in the contemporary debate: [Fine, 1994]'s counterexample with {Socrates}.

The chapter has six sections. In Section 1, I will present the counterexample and Fine's way of dealing with it. To do this, I will briefly describe the framework in which the counterexample is formulated. This means to speak about the more general tenets of Modalism and Essentialism (4.1.1). After that, I will explain the reasons why the counterexample is taken to be compelling (4.1.2) and formulate it in more detail (4.1.3).

Finally, I will present Fine's way to get out of the counterexample (4.1.4).

In Section 2, I will argue that the validity of the counterexample strictly depends on the philosophical assumptions that we adopt. To do this, I will recall [Fine, 1992]'s idea of impure sets as abstract objects having concrete components (4.2.1). Then, I will argue that Fine's counterexample can be resisted by postulating two different objects: ordinary objects and Urelements. After that, I will argue that my account with Urelements as functions saturated by abstract copies accommodates this idea (4.2.2).

In Section 3, I will show that this move has the effect of questioning the asymmetry of the ontological dependence relation which is classically taken to hold between sets and their members. More specifically, I will show that either this relation is symmetric (4.3.1), or that it is a case of joint ontological dependence (4.3.2).

In Section 4, I will present [Wildman, 2013]'s way of dealing with Fine's counterexample. More specifically, I will discuss his metaphysics of impure sets (4.4.1) showing how it applies to Fine's case (4.4.2). After that, I will point out some difficulties that this framework involves (4.4.3) arguing for the superiority of my account (4.4.4).

Section 5 is dedicated to [Zalta, 2006]'s reply to Fine's counterexample. I will present Zalta's account of essence (4.5.1). Then, I will argue for some difficulties that it has to overcome, and that my account can at least partially rule out (4.5.2). Finally, I will consider a revenge problem that is more problematic to solve, and try to limit its damages (4.5.3).

Section 6 concludes.

4.1 Fine's counterexample with {Socrates}

4.1.1 Modalism and Essentialism

To start, let me briefly present the framework in which Fine's counterexample with {Socrates} is formulated. [Fine, 1994] presents several counterexamples to the modal account of essence. Generally speaking, the modal account of essence (M) claims that essentiality is derived from necessity.¹ M has two variants which can be informally presented as follows:

Modal account of essence (M*): an object x has F as an essential property *iff* x has F as a necessary property, *i. e.* x has F in every possible world.

or

Modal account of essence with variable domains (M):** an object x has F as an essential property *iff* x has F in all and only the possible worlds in which it exists.

Examples are: Socrates is essentially self-identical *iff* he is self-identical in all the possible worlds* (in which he exists**); Socrates is essentially

¹It is clear that the term "Modalism" is not to be intended as a theory according to which alethic metaphysical modality is primitive and so not reducible to quantification over possible worlds. For a general presentation of this kind of Modalism, I suggest to see [Bueno and Shalkowski, 2009].

human *iff* he is human in all the possible worlds* (in which he exists**).²

As pointed out in [Fine, 1994], M has counterexamples. It leads to some counterintuitive results. For instance, M states that it is essential for Socrates to be different from the Eiffel Tower and to belong to {Socrates}. Other counterexamples concern necessary facts. Since a conditional having a necessary truth as a consequent ends up being necessarily true, then, following M, it is also essentially true (for instance, it is essential that if Socrates exists, then 2 is a natural number). Moreover, since for any necessary fact N, necessarily, if, for instance, Socrates exists, he is such that N obtains, then this holds essentially.

According to Fine, M is explanatory inert and produces counterintuitive outcomes because if we look at the necessity of a proposition, we find that all the objects are equally considered as possible grounds for its truth. To use Fine's words, "they are all grist to the necessitarian mill" ([Fine, 1994], 7).

Fine proposes to replace M with its Essentialism (E). The idea is that necessity follows from essentiality, and not the other way around. In particular, essentiality enables us to localize the specific source of necessity. E allows for two versions that can be informally stressed as follows:

Essentialist reduction (E*): an object x has F as a necessary

²See [Mill, 1843] and [Stalnaker, 1979]. Examples taken from Quantified Modal Logic are [Marcus, 1967], [Plantinga, 1974], [Stalnaker, 1979], and [Kripke, 1980].

property *iff* x has F in virtue of the essence (or nature) of x , *i. e.* F appears in the definition of x ;

and

Essentialist collective reduction (E):** an object x has F as a necessary property *iff* x has F in virtue of the essence (or nature) of several objects $y, z, w...$ taken together, *i. e.* F appears in the definition of several objects $y, z, w...$ taken together.

Since E allows for the localization of the source of the necessity, it is not subjected to the counterexamples that can be addressed to M. Socrates and the Eiffel Tower are necessarily distinct objects because it is in virtue of the nature of Socrates and of the Eiffel Tower taken together that they are distinct objects. Similarly, it is true in virtue of the essence of {Socrates}, but not of that of Socrates, that he necessarily belongs to {Socrates}; it is true in virtue of the nature of the conditional that it is necessarily true if the consequent is true; it is true in virtue of the nature of N if propositions containing necessary facts obtain.

Thus, E is the best candidate and M has to be rejected.

4.1.2 Fine's counterexample is compelling

[Fine, 1994] has raised a huge debate. Many authors have proposed arguments pro and against both views. Also, some new solutions are now

about to be suggested. What is interesting for my proposal is that Fine's counterexample with {Socrates} is often considered the most compelling one (see, for instance, [Correia, 2008]).

One of the reasons is that the validity of the other counterexamples strictly depends on some specific metaphysical assumptions and can therefore be rejected by ruling them out independently from M or E. For instance, the counterexample with Socrates and the Eiffel Tower is based on essential distinctness, and can be ruled out, for instance, by taking self-identity as a trivial and non-essential property (see [della Rocca, 1996]); that with the conditional having necessary consequent by using Fine's "generalizing away" strategy (see [Fine, 1995]); that involving essential existence can be rejected by assuming M^{**} instead of M^* . By contrast, the counterexample with {Socrates} seems to lie on a more solid and shared framework: Modal set theory with Urelements.

However, I have largely argued that the nature of impure sets is highly controversial. Fine's counterexample with {Socrates} is therefore not exempted from philosophical interpretations. In what follows, I will argue that, given my account, Fine's counterexample with {Socrates} can be neutralized.

Another reason suggesting that the counterexample with {Socrates} is the most compelling one lies in the fact that it exhibits a relation of asymmetric ontological dependence that other cases do not involve and that M fails to represent. I am referring to {Socrates} depending on Socrates which is expressed by E and, more specifically, by the fact that it is in virtue of the essence of {Socrates} that, necessarily, it has Socrates

as a member.

Nevertheless, I do not think that this asymmetric relation of ontological dependence genuinely represents the case of $\{\text{Socrates}\}$ and Socrates, nor that of impure sets in general. Of course, this position is not mainstream. In what follows, I will show that my interpretation of impure sets leads to a plausible reading of set-theoretical objects which does not involve the asymmetry of this relation.

Let me now move on to presenting Fine's counterexample with $\{\text{Socrates}\}$ in detail.

4.1.3 The counterexample in detail

In [Fine, 1994], we read that

“it is [...] necessary, according to standard views within modal set theory, that Socrates belongs to singleton Socrates if he exists; for, necessarily, the singleton exists if Socrates exists and, necessarily, Socrates belongs to singleton Socrates if both Socrates and the singleton exist. It therefore follows according to the modal criterion that Socrates essentially belongs to singleton Socrates. ” [Fine, 1994], 3.

Fine's informal argument allows different formulations. Let me present it in the way I think could be the most faithful to the text:

- 1) Necessarily, $\{\text{Socrates}\}$ exists if Socrates does;
- 2) Necessarily, Socrates belongs to $\{\text{Socrates}\}$ if both exist;

Hence,

3) Socrates essentially belongs to $\{\text{Socrates}\}$.

Points 1) and 2) follow from modal set theory. In particular, the main role is accorded to the definition of the singleton as a set containing exactly one member. Thus, it becomes impossible to conceive any singleton without its own member (both existing and belonging to it). Finally, the conclusion in point 3) derives from points 1) and 2) plus M. Yet, Fine stresses that the conclusion in point 3) is unintuitive:

“It is no part of the essence of Socrates to belong to the singleton. Strange as the literature on personal identity may be, it has never been suggested that in order to understand the nature of a person one must know to which sets he belongs. There is nothing in the nature of a person, if I may put it this way, which demands that he belongs to this or that set or which even demands that there be any sets”. [Fine, 1994],
7.

This problem seems to arise not only with singletons but also with other relevant sets. Consider for instance Socrates and his brain B, and suppose that necessarily, if Socrates exists, then B also exists. Now, given modal set theory, we have that necessarily, if Socrates exists, then Socrates belongs to the set $\{\text{Socrates}, B\}$. Then, by M, we finally should

say that it is essential to Socrates to belong to $\{\text{Socrates}, B\}$. But, again, there seems to be nothing in the nature of Socrates that demands that he belongs to that set.³

In what follows, I will consider $\{\text{Socrates}\}$ as a paradigmatic case. Nevertheless, the solution that I will propose for Fine's counterexample with $\{\text{Socrates}\}$ must be taken as ranging over all other impure sets.

4.1.4 Fine's solution

As anticipated, according to Fine, the unwanted guest is M which allows the unpleasant passage from the necessity of a fact (Socrates belonging to $\{\text{Socrates}\}$), to the unintuitive essentiality of a property (Socrates containing in his essence the fact that he belongs to $\{\text{Socrates}\}$). So, M is not able to account for the asymmetry between $\{\text{Socrates}\}$ essentially containing Socrates and Socrates necessary – but not essentially! – belonging to $\{\text{Socrates}\}$.

As anticipated, given E, Fine's reply to his own counterexample goes as follows: Socrates necessarily (but not essentially) belongs to $\{\text{Socrates}\}$ in virtue of the essence (or nature) of $\{\text{Socrates}\}$. Thus, point 3) does not hold anymore; point 1) follows from the definition of $\{\text{Socrates}\}$, and point 2) is true thanks to the "localization" of the source of the necessity. M is therefore completely replaced.

³This example can be found in [Correia, 202x].

4.2 Two different Socrates

4.2.1 Fine's *implicit* philosophical assumption

It is now important to take a step back wondering about which metaphysics of impure sets Fine adopts. In other words, we need to wonder about the nature of Fine's Socrates, and, following Lewis' expression, we have to finally choose an "evil".

In [Fine, 1994], we find no explicit suggestions. Nonetheless, we deal first with {Socrates} and its member Socrates necessarily belonging to it (for laws of set theory or Finean reduction), and then with Socrates human being who does not essentially belong to any singleton (for "laws of essence"). Now the question is: are Socrates member and Socrates human being the very *same* Socrates? Contrary to my account, I think that Fine would say that they are.

As usual, the devil hides in the details. As mentioned in Chapter 1, Fine says something important about his metaphysics of impure sets in [Fine, 1992]. [Fine, 1992] is an attempt to build a unified picture of the "versatile" behavior of the Aristotelian *hylé* which applies both to physical and non-physical objects. The philosophical interesting point appears in the aforementioned footnote where Fine pushes far the "comprehensive" feature of the *hylé*:

"There is the question of whether non-physical objects can have physical objects as their matter and of whether physical objects can have non-physical objects as their matter. I am

inclined to think that the answer is yes in both cases: for the members of a set, which is non-physical, may be physical; and the meaning of a word is, arguably, as much an "ingredient" of the word as the underlying token. However, there is no indication that Aristotle would have allowed either of these two possibilities". [Fine, 1992], 36, footnote 3.

Thus, according to Fine's intuition, there are cases in which the matter of a non-physical object is physical and, I argue, {Socrates} seems a good candidate: for {Socrates}, which is non-physical, the member Socrates is physical. It seems therefore that Fine distinguish the two Socrates because he thinks that there is not: Socrates member is Socrates human being, *i. e.* a very physical entity.⁴

From what has been said, it seems that Fine's counterexample with {Socrates} attributes some properties to the very same Socrates. However, as mentioned, in points 1) and 2), Fine exploits some set-theoretical properties belonging to Socrates as a member, while in point 3), he underlines a fact related to Socrates as a human being.

In Chapter 3, I have pointed out some reasons to distinguish between these two objects and given an account that describes their relations. It is important now to understand what happens to Fine's counterexample if we choose another "evil" explicitly distinguishing between the two Socrates.

⁴To say that the non-physical members of a set are its matter could sound strange. But this because Aristotelian hylé applies both to physical and non-physical objects. To use Fine's example, "not only will the token letters constitute the matter of a token expression, the letter types will also constitute the matter of the expression type". [Fine, 1992], 37.

4.2.2 A solution to Fine’s counterexample by my account

As mentioned, Urelements of impure sets are conceptualized abstract objects referring to concrete ones. Applying this idea to Fine’s counterexample, we obtain the following reformulation:

“Strange as the literature on [impure set theory] may be, it has never been suggested that in order to understand the nature of an [Urelement] one must know to which sets it belongs. There is nothing in the nature of an [Urelement], if I may put it this way, which demands that it belongs to this or that set or which even demands that there be any sets”.

I argue that, given this reformulation, all of this does not sound so “strange” anymore. Let me consider Fine’s claim dividing it into three questions and exploring them one by one.

First: Is there something in the nature of an Urelement which demands that it belongs to this or that set? Arguably, it does. For instance, the very nature of $UR(Cs)$ suggests that it belongs to the set of all philosophers (because Cs encodes that property). More generally, $UR(Cs)$ contains all information derived from Socrates’ abstract copy (and so from Socrates human being) allowing some memberships and excluding some others. It can also be argued that there is something in the generic nature of Urelements that suggests that they belong to some sets. Indeed, Urelements are objects that are not a set, that do not contain any set, or members, but that can be members of some sets.

This fact leads to the second point: Is there something in the nature of an Urelement that demands that there be any set? I think it does. This point is perhaps more complicated to maintain. The idea is that the aforementioned definition of an Urelement seems to involve that of sets.

Last point: Must we know to which sets an Urelement belongs to understand its nature? Of course, not everything of its nature is related to its set-theoretical properties, but the fact that $UR(Cs)$ belongs or not belongs to certain sets is certainly not epistemically irrelevant. To take back the example above, knowing that $UR(Cs)$ belongs to the set of all philosophers says something about $UR(Cs)$.

We are ready now to formulate the solution to Fine's counterexample: $UR(Cs)$ both necessary and essentially belongs to its singleton because it belongs to the]set[and to the [set] of all x such that x is identical to Cs , *i. e.*]x:x=Cs[and [x:x=Cs] , because Cs exemplifies and encodes the property of being self-identical *i. e.*, $(= Cs)$.

In other words, Fine's point 3) — Socrates essentially belongs to $\{\text{Socrates}\}$ — holds if we consider Socrates as a conceptualized abstract object of the kind that I have described in Chapter 3.⁵

As mentioned, Fine's counterexample can be generalized. Remember the case of Socrates and his brain B. M leads to the counterintuitive

⁵The solution that I have proposed holds only if we take self-identity as an essential property of objects, which is controversial (see [della Rocca, 1996]). Anyone who is not happy with this can rather consider the nature of UR saying that UR encodes or exemplifies the essential property of "belonging to the sets to which it belongs when it is saturated".

conclusion that it is essential to Socrates to belong to $\{\text{Socrates}, B\}$. My solution applies. What essentially belongs to $\{\text{Socrates}, B\}$ is $UR(Cs)$ and UR (Socrates' brain copy). Then, this can be generalized to all Urelements of impure sets.

In this work, I aim to stay neutral regarding the debate between E and M. Previously, I have conceded to Fine his essentialist framework and shown that even holding E, his counterexample with $\{\text{Socrates}\}$ can be resisted. This shows that M is, at least in this case, not necessarily the problematic point.

4.3 Impure sets and members: symmetric or asymmetric dependence?

4.3.1 Symmetric ontological dependence

Incidentally, my solution suggests something important concerning a topic that is central in contemporary metaphysics: the relation of ontological dependence of sets upon their members.

As mentioned, one of the reasons of taking Fine's counterexample with $\{\text{Socrates}\}$ as the most compelling one is that, contrary to M, E accounts for the asymmetric relation of ontological dependence that is classically taken to describe the case of sets and their members: sets depend upon their members but it is not the case that members depend upon the sets to which they belong. Transposing this into essentialist language, $\{\text{Socrates}\}$ essentially has Socrates as a member, but Socrates

does not essentially belong to {Socrates}. This idea is widely accepted.⁶

Now, my solution suggests that such a relation of asymmetric ontological dependence does not hold, at least for cases of impure sets and Urelements such as {Socrates} and Socrates. As mentioned, {Socrates} essentially has Socrates as a member and Socrates essentially belongs to {Socrates}. Also, as suggested, this symmetry is supposed to hold for every impure set.

To better see why, let me take a step back considering again the notions of concept, conceptualization, and conceptualized objects as presented in Chapter 3. As largely argued, impure sets are intimately related to all these notions. More specifically, the operation of individuating and counting sets and their members rests on concepts and conceptualizations, and Urelements can be considered a special kind of conceptualized object. More generally, I have pointed out that the process of conceptualization is all over the place.

Concerning the relation of ontological dependence, it can be argued that the conceptualization of a given object deeply depends on other more general conceptualizations in a way that the object very persistence, identity, and distinctness derive from these very conceptualizations. To corroborate this point, consider [Blok et al., 2005] who focus on concrete objects. Their example is similar to Frege's:

“The concept of a tree, for example, may dictate when your neighbor's mulberry tree begins its life and finishes it, where

⁶For a defense and an overview of this idea, see, for instance, [Wigglesworth, 2015].

the mulberry tree’s spatial extent ends and the ground begins, and whether the mulberry-as-a-sapling is the numerically same item as the mulberry-as-a-mature-tree”. [Blok et al., 2005], 131.

It follows that conceptualization is essential to determine the identity of concrete objects.

Now, transposing this idea to impure set theory, we have that the very nature of Urelements requires conceptualization, which is the main process generating sets. Furthermore, as mentioned, the notion of Urelement is defined in a way that involves that of the set: Urelements are objects that are not sets, that do not contain any set or members, but that can be members of some sets. I argue, therefore, that there is room to claim that Urelements also depend upon the sets to which they belong. In this way, for instance, $UR(Cs)$ depends upon the set of human beings because its nature depends on this fact. If this is the case, then ontological dependence between impure sets and their members is not asymmetric: sets depend upon their Urelements and their Urelements depend upon the sets to which they belong.

4.3.2 Joint ontological dependence

It can be objected that the case of impure sets and their Urelements is not a genuine case of symmetric ontological dependence. It is at most a case of what is sometimes called “joint dependence” (see [Barnes, 2018]). Joint dependence is in play when two things look as if they depend on each

other because they each depend on the same further thing. Following the example above, sets and their Urelements both depend on concepts: both set operations and Urelements depend on concepts and conceptualization.

Now, it seems clear to me that, even assuming joint ontological dependence, symmetric dependence is not automatically dismissed. To see why, take the case of cold and warm. They both plausibly depend on subatomic physics. Then, it can be argued that they also both depend on each other (as Fine did in [Fine, 1995]). For impure sets and their Urelements, I think that joint ontological dependence on concepts suggests symmetry. This happens because the notion of Urelement conceptually requires that of set (and that of impure set involves that of Urelements).

The debate over ontological dependence is vast and intricate. More details need to be spelled out. However, this is not the right place to pursue such a debate. I will just conclude by claiming that symmetric ontological dependence seems also more coherent with Fine's conception of set theory presented in [Fine, 2005a] according to which, given an understanding of the ontology of classes, we successively carve out extensions of the membership predicate by using conditions on the domain of classes to specify which further membership relations should obtain. To use Fine's metaphor, sets are boxes. Then, we have God that has a complete knowledge of all the boxes (including their content), while Gabriel does not. Nevertheless, Gabriel can reach step-by-step God's knowledge by choosing a concept and asking God to "open its relative box". Urelements are therefore already "inside of the box"; Gabriel can see the set (the box), but can grasp its content (the members) only by

choosing its relative concept. Paraphrasing this, the access to Urelements depends on that of sets and concepts.

4.4 Comparing Wildman’s strategy with my account

4.4.1 Wildman’s bullet-biting strategy

In [Wildman, 2013], the author presents a way to resist Fine’s counterexample. His outcome is similar to mine: Socrates essentially and necessarily belongs to $\{\text{Socrates}\}$. According to Wildman, Fine’s rejection of point 3), *i. e.*, “Socrates essentially belongs to $\{\text{Socrates}\}$ ”, derives from what he calls the “priority thesis”. The priority thesis holds in the context of a constructivist approach to mathematics. The idea is that the development of set theory follows a temporal item from which we derive a particular hierarchy of objects, entertaining some relation of ontological dependence with each other. Taking into consideration Fine’s relevant case, Socrates would come first and, only later, we would construct its singleton. Thus, if this is true, it is arguably plausible to say that Socrates does not essentially belong to $\{\text{Socrates}\}$ because of the “priority” of the former over the latter.

Wildman suggests that things radically change choosing another metaphysics for mathematics. He proposes to adopt the Platonic Iterative Conception (PIC) according to which all mathematical objects are already in some particular relation with each other. The temporal con-

struction is just “metaphorical” and has, at most, the epistemic task of retracing the mental activity that we employ doing axiomatic set theory.

To use Wildman’s words,

“it isn’t as if the ur-elements exist prior in time, sitting around like so much unused lumber” [Wildman, 2013], 76.

In this sense, Socrates “already” belongs to {Socrates}.

Wildman adopts sparse modalism as a strategy to reconstruct essences. A property is essential if it is sparse and necessary.⁷ Then, given that the membership relation is a sparse property because it characterizes the objects in set theory, and that Socrates necessarily belongs to {Socrates}, we can safely say that Socrates essentially belongs to {Socrates}.

Wildman’s PIC provides no specific information concerning the nature of Urelements. I argue that if we take Urelements as concrete objects, then Wildman’s arguments are weakened, if not rejected. In what follows, I will examine this issue.

4.4.2 Wildman’s arguments against Fine’s asymmetry

The dialectic used by Wildman to show that constructivism is the only way to draw the asymmetry between Socrates and {Socrates} consists in arguing for the impossibility of other sources. In what follows, I do not aim to explore all the possibilities that he presents. I will just point out

⁷The notion of sparseness originally comes from [Lewis, 1989]. The debate on Sparse modalism is one of the most influential events in contemporary analytic philosophy. See, for instance, [Melo, 2019] and [Lee, 2020].

those that can be questioned if Socrates is a human being rather than an abstract object. This will show that the asymmetry is more resistant than Wildman claims, and that, to be rejected, it requires other metaphysical presupposes such as Socrates as an abstract object of set theory.

One of Wildman's arguments is that, given PIC, the asymmetry cannot derive from a difference in the individuation conditions because, even if the standard individuation condition for sets invokes their members⁸, while that for objects does not involve sets, it is always possible to provide an individuation condition for objects in terms of sets as follows: objects x and y are identical *iff* necessarily, all and only the same sets have them as members.

Then, given the reformulation of the individuation condition for objects in terms of sets, it can neither be argued that the asymmetry comes from the fact that this reformulation is not metaphysically relevant as that for sets, because it would mean to circularly justify the asymmetry of essence in terms of an asymmetry of essence.

Finally, he claims that the asymmetry cannot be derived from what Fine calls the "sense of nature" according to which it does not lie in the nature of Socrates and {Socrates} because it is not clear, excluding the priority thesis, where this sense of nature comes from.

⁸Wildman refers here to the old-fashioned principle of extensionality in modal set theory saying that sets x and y are identical *iff* necessarily, all and only the same entities are their members.

4.4.3 Against Wildman's arguments

I argue that Wildman's reformulation of the individuation conditions for objects in terms of sets does not seem very persuasive if we take Socrates as a human being. First of all, Wildman does not specify what an "individuation condition" is. Perhaps, it is an epistemic principle saying how we can identify objects. However, it seems hard to maintain that Socrates' individuation epistemically requires that of all sets to which he necessarily belongs. First, since these sets are plausibly infinite, the criterion would be highly costly. Second, the predicates expressed by these sets are often intuitively "unrelated" to the object for which they supply the individuation. Does the individuation for Socrates require to know for instance that he belongs to the sets of objects which are different from the cup of coffee that is now on my table? Plausibly not.

Perhaps, Wildman's criterion is an identity condition claiming how things are. Then, it would be a second-level criterion. The second-level criteria of identity quantify over some functional terms which are items of a different kind from that of the objects for which they supply a criterion of identity. The typical example is Frege's identity of the directions of lines a and b given by the property of "being parallel with one other". Nevertheless, the reliability of this kind of criteria is very controversial especially if the objects that must be identified are human beings. This point is well argued by Lowe:

"Consider, for instance, the problem of personal identity: the trouble is that there is no standard functional mode of refer-

ence to persons as there is to directions and numbers and sets. Directions are directions of lines, and numbers are numbers of objects satisfying some condition, and so are sets. But persons aren't at all obviously person "of" anything at all in this sense; in short, it isn't obvious what domain of entities ought to be invoked in order that an equivalence relation on them may be cited as a criterion of identity of persons". [Lowe, 1998], 44.

If this is the case, then it is not true that the asymmetry cannot come from the fact that the reformulation of the criterion is not metaphysically relevant as that for sets. The reformulation does not fit with Socrates precisely because of the asymmetry in nature between Socrates and {Socrates}: Socrates is a human being, and so is not *prima facie* individuated via the sets to which he belongs (especially if the sets in question are not essentially related to his nature); by contrast, {Socrates} is a logical abstract object whose individuation plausibly requires that of its unique member. In other words, the criterion is metaphysically more relevant for the set than for the Urelement.

The circularity evoked by Wildman is also innocuous: the asymmetry does not explain itself; a metaphysical inquiry distinguishing between relevant properties of ordinary and abstract objects is at work and Fine's Essentialism aims to do this job accounting for what the objects are, and *a fortiori*, for the natural differences between them. For similar reasons, Wildman last argument seems false as well. We need no priority thesis

to know that it is not in the nature of Socrates to belong to {Socrates}.
Definitional knowledge of Socrates human being widely suffices.

This fact seems confirmed by [Fine, 1994]:

“We want to say that it is essential to the singleton to have Socrates as a member, but that it is not essential to Socrates to be a member of the singleton. But there is nothing in the ”logic” of the situation to justify an asymmetric judgment of relevance; the difference lies entirely in the nature of the objects in question.” [Fine, 1994], 33.

4.4.4 Against Fine’s asymmetry by my account

The arguments above show that Wildman’s claim according to which the asymmetry is ruled out just by taking PIC instead of constructivism is controversial.⁹ I have argued that the problem lies in taking Socrates as a concrete individual. I indeed think that PIC plus Urelements as conceptualized abstract objects of set theory is sufficient. As mentioned, $UR(Cs)$ is defined as an object of set theory that belongs to some sets. For this reason, to consider the sets to which it belongs seems a good way to individuate it.

It may be objected that it is not strictly speaking in the definition of Urelements to belong to the sets to which they belong; they just *possibly*

⁹Notice incidentally that [Fine, 2005b]’s mathematical framework, which is called ”procedural logicism”, does not involve constructivism in a strong sense. Using Fine’s words, ”Postulation works by literally creating new objects which then enter into the domain of quantification. The objects that are introduced through postulation existed prior to all acts of postulation [...] and would have existed even if there had been no postulation or people to postulate.” [Fine, 2005a], 91

belong to them. But this is precisely where PIC is involved: Urelements already belong to the sets to which they belong. In this sense, the modal feature is just metaphoric. Hence, Wildman's other argument for invalidating the asymmetry is now in play: for Urelements to belong to sets is no less metaphysically relevant than for sets to have the members they have.

I therefore conclude by saying that adopting a bullet-biting strategy works only if the "body" hit by the bullet has deeply changed: Socrates is not a human being, but a conceptualized abstract object of set theory.

One last point needs to be stressed. It can be argued that the argument that I have proposed goes against my own idea of Urelements as conceptualized objects. It is precisely because objects are conceptualized that this conceptualization says something important about them (recall the case of the tree which can be individuated only by conceptualizing it as a tree). Then, it would be plausible to argue that Wildman's reformulation of individuation conditions for objects is meaningful.

However, I think that this conceptualization has levels. We first have concrete unconceptualized objects, and then we have abstract copies or *Noemata* of those objects, then we have more complex objects such as Urelements. My point here is that what we have is a stratification: individuation conditions in terms of sets are good criteria for Urelements; concrete objects are rather individuated by their conceptualization. In this sense, we have a difference between, for instance, Socrates individuated by some concepts, and {Socrates} individuated by some concepts plus the set to which it belongs.

4.5 Comparing Zalta's strategy with my account

4.5.1 Zalta's solution to Fine's counterexample

In Chapter 3, I have used Zalta's abstract copies to model Urelements of impure sets. Zalta does not directly address this issue. Nevertheless, he proposes a reply to Fine's counterexamples from which it is possible to extrapolate some relevant information concerning impure sets. It has to be noticed that, even though I use Zalta's OT, my solution differs from Zalta's. In what follows, I will compare them.

First of all, recall Zalta's main features of OT. OT prescribes a fixed domain of objects that can be abstract or ordinary. Ordinary objects can be concrete or nonconcrete. Abstract objects can encode and exemplify properties, while ordinary objects just exemplify them.

Zalta's modal and theoretical framework allows for an account of essentiality. This account is presented in [Zalta, 2006]. Encoded properties play a salient role: they form the set of essential and necessary properties of abstract objects. For example, Holmes necessarily/essentially is a detective who necessarily/essentially lives on Baker Street. They do this because, as mentioned, they are the set of properties that characterize abstract objects. By contrast, essential properties to ordinary objects are those that they exemplify in all and only the worlds at which they are concrete. For instance, Socrates is essentially human because he is human only at the worlds at which he is concrete (indeed, "being human"

entails “being concrete”); by contrast, Socrates is necessarily but not essentially self-identical because he is plausibly self-identical at all the worlds at which he exists, and not only at those at which he is concrete.

In the case of Socrates and $\{Socrates\}$, Zalta takes Socrates to be an ordinary object. By contrast, $\{Socrates\}$ is abstract. $\{Socrates\}$ encodes and exemplifies some properties. Among these properties, $\{Socrates\}$ encodes “having Socrates as a member”, and so has this property essentially. By contrast, Socrates encodes no properties, so he does not encode “being a member of $\{Socrates\}$ ”, but he does not exemplify this property either. Socrates is, as Zalta argues, “given independently of any mathematical theory”. More specifically,

“We can’t abstract out any properties of Socrates in virtue of the properties exemplified by singleton Socrates according to M [Modal set theory], or in virtue of the properties Socrates himself exemplifies according to M, or in virtue of the properties encoded by the singleton of Socrates”. [Zalta, 2006], 691.

Hence, Socrates is not essentially a member of $\{Socrates\}$.

Zalta’s solution is not uncontroversial. Many criticisms are addressed to his luxurious ontology (see, for instance, [Wildman, 2018]). In what follows, I will show that Zalta’s main flaw is elsewhere. This will also point out some advantages of the solution that I have proposed.

4.5.2 Some problems with Zalta's solution

Zalta's choice of taking Socrates as an ordinary object is coherent with Fine's idea of Urelements as physical objects. Zalta also mentions an alternative reading in which Socrates is taken to be a logical object of set theory. But he does not take it to be in good standing precisely because it is far from what Fine has in mind.

I argue, however, that Zalta's solution leads to some consequences that Fine would not accept and that are problematic on their own. The problem is that if Socrates does not exemplify the property of being a member of Socrates, and exemplification is complete, then Socrates exemplifies the negation of that property. And this is true even necessarily. Zalta explicitly states this point:

“It is consistent with our theory to claim not only that $s \in (s)M$ is false but that it is necessarily false”. [Zalta, 2006], 693.

This means to invalidate not only the unintuitive result in Fine's point 3) according to which Socrates essentially belongs to $\{Socrates\}$ but also Fine's unproblematic point 2) according to which Socrates necessarily belongs to $\{Socrates\}$.

I argue that this move is problematic. There is indeed something almost inconsistent in claiming both that $\{Socrates\}$ necessarily has Socrates as a member and that Socrates is necessarily not a member of $\{Socrates\}$. Moreover, philosophically speaking, this would discourage any study on the relations between supposedly physical objects and

set theory. In short, Zalta proposes a reading according to which impure sets are abstract objects having some ordinary objects as elements. Such a reading, nonetheless, fails to account for the way in which these two kinds of objects are supposed to be related.

By contrast, Fine's account has the philosophical advantage of speaking about the relation between concrete Urelements and sets. Following his example, Socrates takes some properties from set theory (*i. e.*, Socrates necessarily belongs to {Socrates}). And the other way around holds as well. {Socrates} inherits some necessary properties from its sole member (for instance, it is necessary for {Socrates} to contain a member who is Socrates, who is a man, and so on). In other words, contrary to Zalta's solution, Fine's allows us to consider formal theories as both informative concerning the properties of the physical objects that they contain, and receptive to those properties.

4.5.3 My solution

As largely shown in Chapter 1, Fine's adoption of the mainstream thesis is problematic. In Chapter 3, I have also pointed out that Urelements of impure sets are better understood as conceptualized abstract objects. Now, the interest here is just to show that my solution better accounts for the aforementioned problem that can be raised against Zalta's reply to Fine's counterexample, namely that Urelements of impure sets and set theory involve, so to say, completely extraneous objects.

The solution with Zalta's abstract copies presented in Chapter 3

shows indeed that abstract copies are related to ordinary objects. Urelements as conceptualized objects inherit properties from ordinary objects. Hence, there is something informative that impure set theory inherits from ordinary objects, even if not directly.

Do ordinary objects inherit something from set theory? This is a delicate point that needs to be treated carefully.

As mentioned, I aim to stay neutral concerning M and E. I have shown that even embracing E, Fine's counterexample can be neutralized by using my account for impure sets. It has to be noticed that friends of E can nonetheless raise a revenge problem. This problem is raised by Correia (2021, personal communication), and concerns precisely some properties that Urelements inherit from set-theoretical objects. Consider that $x \downarrow y$ iff y is the "urelementisation" of the abstract copy of x . If so, we have that Socrates ordinary object \downarrow UR(Cs), i. e., UR(Cs) is the urelementisation of the abstract copy of Socrates ordinary object. Then, it is natural to hold that necessarily, if Socrates ordinary object exists, then Socrates ordinary object \downarrow UR(Cs). By M, this implies that Socrates is essentially \downarrow -related to UR(Cs). But, if this holds, we relapse into the unintuitive case of Socrates ordinary object having essentially some relations to some set-theoretical objects.

This is a huge problem not only because it points out a difficulty in replying to Fine's counterexample, but also because it raises the more general issue of the relation between abstract and concrete objects evoked by my account. Roughly speaking, the revenge problem suggests that the concrete supposedly unconceptualized content is rather structured, and

has this (set-theoretical) structure by its nature.

I think however that my framework describes a situation in which set-theoretical properties derive from operations of conceptualization on the concrete. The concrete as such does not involve set-theoretical properties but inherits them indirectly by conceptual objects. I have argued that this framework is coherent with our way of thinking about impure set theory, and this is independent of the question about E or M.

4.6 Conclusions

In this chapter, I have presented the application of my account of Urelements to the Finean case of Socrates and {Socrates}.

In Section 1, I have presented the counterexample and Fine's way of dealing with it.

After that, I have shown, in Section 2, that the validity of the counterexample rests on some philosophical assumptions, and that my account may neutralize it.

In Section 3, I have pointed out that this result also suggests that the relation of ontological dependence between sets and Urelements is symmetric.

In Section 4, I have presented Wildman's solution to Fine's counterexample which also suggests this symmetry. I have argued, nonetheless, that his framework is problematic in that it does not invalidate Fine's suggestion according to which Urelements of impure sets are concrete physical objects.

The last section is dedicated to Zalta's solution to Fine's counterexample. More specifically, I have argued that Zalta's account for essentiality does not adequately describe the relation between Urelements and impure sets. Finally, I have tried to point out how this problem is partially solved by my framework.

Chapter 5

Conclusion

In this dissertation, I have discussed the nature of impure sets. The interest of this work lies in the fact that, despite their large use in philosophical and scientific speeches, no systematic study is available. The common definition that can be extrapolated from these speeches claims that impure sets are sets containing concrete members or elements in their transitive closure, where concreteness is understood in terms of spatiotemporal location. In this dissertation, I have questioned this definition and shown that impure sets are better understood as fully abstract objects, *i.e.* objects that are not spatiotemporally located having members or elements in their transitive closure that are not spatiotemporally located.

To do this, I have discussed and rejected two main ideas about impure sets. The first is that impure sets are abstract objects having concrete members or elements in their transitive closure. I have called this idea “the mainstream thesis” because it has been adopted by many authors,

such as Cowling, Falguera, Fine, Lowe, and Parsons. Chapter 1 is dedicated to the mainstream thesis and, more specifically, to [Parsons, 2008]', and [Fine, 1992], [Fine, 2010]'s specific formulations. Parsons takes impure sets as quasi-concrete objects, *i. e.* abstract objects intrinsically related to some concrete representations; Fine describes impure sets as abstract objects having concrete elements as their parts. The arguments that I have raised against Parsons' idea address his criterion of identity and his notion of quasi-concrete objects. Concerning Fine's account, I have argued against the possibility of abstract objects composed of concrete ones. The arguments that I have proposed address Parsons' and Fine's specific accounts but also apply to all other frameworks involving the mainstream thesis plus membership as composition or representation.

The second idea that I have discussed and rejected is that impure sets are fully concrete objects, namely concrete objects having concrete members or elements in their transitive closure. To do this, in Chapter 2, I have focused on [Maddy, 1991]'s account which is divided into two different but related claims: the ontological claim according to which impure sets are spatiotemporally located where their members or elements in the transitive closure are located, and the epistemic claim according to which we perceive impure sets. I have shown that both claims are problematic. Concerning the epistemic claim, I have argued that perception is either too weak or too strong to account for all the impure sets that we need in impure set theory. Then, I have pointed out that the ontological claim can lead to unintuitive consequences.

Given that impure sets are neither abstract objects having concrete

members or elements, nor fully concrete objects, and that the idea that impure sets are concrete objects having abstract members or elements seems highly unintuitive, I have concluded that impure are fully abstract objects (*i. e.*, objects that are not spatiotemporally located having objects that are not spatiotemporally located as members or elements in their transitive closure). This means that the classic definition of impure sets has to be revised.

In Chapter 3, I have stated that the most compelling strategy to argue that members or elements of impure sets are abstract objects can be found by exploring the relations between impure sets and the notions of concept, conceptualization, and conceptualized objects. More specifically, I have argued that Urelements of impure sets can be considered as cases of conceptualized objects. I have also suggested that these conceptualized objects can be modeled using [Zalta, 1983]’s notion of individual concepts (or abstract copies) formulated in his Object Theory plus the logical notion of Urelement.

Incidentally, this new definition says something about the general notion of set. Sets are indeed fully abstract objects. Also, they are intimately related to concepts, conceptualization, and conceptualized objects. Their identity is given by Extensionality which is a principle reached by conceptual tools. More specifically, sets are fully abstract objects containing other sets and/or Urelement-functions some of which are saturated by abstract objects or abstract copies of ordinary objects.

Finally, in Chapter 4, I have discussed one application of my account of impure sets to [Fine, 1994]’s famous counterexample involving

Socrates and $\{\text{Socrates}\}$ and shown that my solution is superior to that of [Wildman, 2013] and [Zalta, 2006].

As largely mentioned, my account is only sketched and needs to be further developed and clarified. To conclude, I will mention some directions in which it can be extended.

In this dissertation, I have focused on the relation between Urelements and impure sets to which they belong because my strategy aims to, so to say, “make set-theoretical” certain objects which are not set-theoretical *prima facie*. In other words, sets are, of course, already set-theoretical, and, for this reason, they naturally display some set-theoretical properties such as “possibly belonging to other sets”. By contrast, the case of Urelements such as Socrates, Plato, and so on, needed more explanations. The account generalizes, nonetheless, to higher-order impure sets.

Cantor’s definition of sets involves that they possibly belong to other sets (with the exception of the very special case of Von Neumann-Bernays-Gödel’s proper class, if any). Then, to know to which sets they can belong I propose the same strategy I have used for the Urelements, this time applied to sets. Consider the case of $\{\text{Socrates}\}$ belonging to $\{\{\text{Socrates}\}\}$. Since $\{\text{Socrates}\}$ is an abstract object exemplifying and encoding the property of “being identical to itself”, then it belongs to the set $[x:x=\{\text{Socrates}\}]$ and to the set $]x:x=\{\text{Socrates}\}[$, *i. e.*, to the [singleton of the singleton Socrates] and to the]singleton of the singleton Socrates[. To formalize:

$\{\text{Socrates}\} (= \{\text{Socrates}\}) \rightarrow \{\text{Socrates}\} \in [x: x = \{\text{Socrates}\}]$

Ergo $\text{Socrates} \in [\text{Socrates}]$

And

$(= \{\text{Socrates}\}) \{\text{Socrates}\} \rightarrow \{\text{Socrates}\} \in]x: x = \{\text{Socrates}\}[$

Ergo $\{\text{Socrates}\} \in]\{\text{Socrates}\}[$

Moreover, the account can be applied to cases that do not involve sets or set-theoretical tools. As mentioned, the relations that appear in impure set theory just represent a way in which an abstract object can refer to abstract and concrete contents: modal thoughts, states of affairs, situations and worlds, objective and cognitive content of natural language, and fictional objects are other examples. The account can also be extended to treat these kinds of objects.

As I have shown, abstract objects of Zalta's Object Theory model Husserl's *Noemata*. As largely mentioned, OT also applies to Fregean Senses, Leibnizian Concepts, and Platonic Forms. In what follows, I will mention some other unexplored situations to which I think that the account successfully applies. I will nonetheless limit myself to briefly explaining the problem leaving the details of the solution for another time.

Many of these situations are suggested by Correia (2021, personal

communication) and concern, again, some counterexamples to the modal account of essence and, more specifically, cases of supposedly asymmetric essences.

One of these cases is the asymmetry between Socrates and its existence. Roughly speaking, we may take Socrates' existence to be a state of affairs that, of necessity, exists just in case Socrates does. But we do not want to say that what Socrates is depends upon the state of his existence. Here, OT could shed light on the nature of these objects and on how they are related.

Another case concerns the notion of identity. It might indeed be maintained that the property of being identical to Socrates depends upon Socrates and also that the property essentially, and hence necessarily, exists. Then, one can hardly maintain that the property is something that, by its nature, cannot exist without Socrates. For, in that case, Socrates would likewise necessarily exist, which is unintuitive. Here, OT can analyze objects such as properties in terms of abstract objects, and investigate their relations with the ordinary objects with which they are associated.

Another case concerns tropes and individuals. Consider Socrates and the trope human being of Socrates. Plausibly, Socrates cannot exist without instantiating the trope human being of Socrates. Equally plausibly, the trope human being of Socrates cannot have Socrates as an instance without existing. Then, Socrates rigidly necessitates the trope human being of Socrates. Here OT can be used to clarify the nature of tropes and their relations with their instances.

Here we are at the end of our journey on impure sets. I hope that my research will be extended and further improved. For the time being, I would like to thank the patient reader hoping that they have taken some pleasure from this early stage study.

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