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The $\mathbb{Z}_2$ anomaly in some chiral gauge theories

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Abstract: We revisit the simplest Bars-Yankielowicz (BY) model (the $\psi\eta$ model), starting from a model with an additional Dirac pair of fermions in the fundamental representation, together with a complex color-singlet scalar ϕ coupled to them through a Yukawa interaction. This model possesses a color-flavor-locked 1-form $\mathbb{Z}_N$ symmetry, due to the intersection of the color SU(N) and two nonanomalous U(1) groups. In the bulk, the model reduces to the $\psi\eta$ model studied earlier when ϕ acquires a nonzero vacuum expectation value and the extra fermions pair up, get massive and decouple (thus we will call our extended theory as the "X-ray model"), while it provides a regularization of the $\mathbb{Z}_2$ fluxes needed to study the $\mathbb{Z}_2$ anomaly. The anomalies involving the 1-form $\mathbb{Z}_N$ symmetry reduce, for N even, exactly to the mixed $\mathbb{Z}_2$ anomaly found earlier in the $\psi\eta$ model. The present work is a first significant step to clarify the meaning of the mixed $\mathbb{Z}_2 - [\mathbb{Z}_N^{(1)}]^2$ anomaly found in the $\psi\eta$ and in other BY and Georgi-Glashow type SU(N) models with even N.

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1 Introduction

The dynamics of two wide classes of chiral SU(N) gauge theories — the so-called Bars-Yankielowicz (BY) and generalized Georgi-Glashow (GG) models [1–6] — has been re-examined recently [7–9], in the light of a gauged color-flavor locked $\mathbb{Z}_N$ 1-form symmetry[1] and of the stronger forms of 't Hooft anomaly matching constraints following from that. In particular, certain mixed anomalies involving a $\mathbb{Z}_2$ symmetry were found to imply, in a class of theories with even N,[2] that chirally symmetric confining vacua in these models, where the global symmetries in the infrared are saturated by the hypothetical massless composite fermions were inconsistent. These massless "baryons" reproduce the conventional 't Hooft anomalies but do not match the mixed $\mathbb{Z}_2 - [\mathbb{Z}_N^{(1)}]^2$ anomaly.

Dynamical Higgs vacua, characterized by color-flavor locked bifermion condensates, are instead found to be compatible with the indications coming from the tighter consistency conditions involving the $\mathbb{Z}_2$ anomaly [7–9]. An independent argument [10], following from

[1] From now on, whenever there might be confusion, we will indicate a 1-form symmetry with the apex notation, e.g. the $\mathbb{Z}_N$ 1-form symmetry as $\mathbb{Z}_N^{(1)}$.

[2] More precisely, with even N and with an even number p of Dirac pairs of fermions in the fundamental representation [7–9]. We call this class of models Type I in this note; others will be referred to as Type II.

the requirement that the so-called strong anomalies be reproduced correctly in an effective low-energy action in terms of the assumed set of infrared degrees of freedom, provides a solid support for the dynamical Higgs scenario.

The arguments based on the mixed $\mathbb{Z}_2 - [\mathbb{Z}_N^{(1)}]^2$ anomalies have been put in question in [11]. The problem boils down to the singular nature of the external "$\mathbb{Z}_2$ gauge field" A_2, introduced in [7–9] to construct the color-flavor 1-form $\mathbb{Z}_N$ symmetry which is due to the intersection[3] $\mathrm{SU}(N) \cap \{\mathbb{Z}_2 \times \mathrm{U}(1)_{\psi\eta}\}$. The $\mathbb{Z}_2$ gauge field needs to wind

$$\oint_L A_2 = \frac{2\pi m}{2}\,, \qquad m \in \mathbb{Z}\,, \tag{1.1}$$

along a closed loop L, to parametrize the holonomy

$$\psi \to -\psi\,, \qquad \eta \to -\eta\,, \tag{1.2}$$

and to give the color-flavor-locked 1-form $\mathbb{Z}_N$ symmetry.[4] Such a field contains necessarily a singularity (i.e., a singular $\mathbb{Z}_2$ vortex) [7] somewhere inside the closed 2D space Σ_2 bounded by L.

The authors of [11] show that, by choosing instead a (regular, hence legitimate) "$\mathbb{Z}_2$ gauge field" A_2 such that (cf. (1.1))

$$\int_{\Sigma_2} dA_2 = 2\pi\,\mathbb{Z}\,, \tag{1.3}$$

the flux carried by the $\mathbb{Z}_N$ gauge field $B_{\mathrm{c}}^{(2)}$ becomes

$$\int_{\Sigma_2} N\,B_{\mathrm{c}}^{(2)} = 4\pi k\,, \qquad k \in \mathbb{Z}\,, \tag{1.4}$$

twice those used in [7], and accordingly the anomalies found there would disappear. However, (1.3) means that such a background $\mathbb{Z}_2$ gauge field corresponds to the trivial holonomy

$$\psi \to \psi\,, \qquad \eta \to \eta\,, \tag{1.5}$$

i.e., no transformation (an identity element of $\mathbb{Z}_2$).

To grasp correctly the main issue it is indeed necessary to distinguish the concepts of the *global* 1-form $\mathbb{Z}_N$ symmetry from the gauged version of it. The former, a color-flavor locked $\mathbb{Z}_N$ symmetry, is a generalization of the familiar center symmetry of pure $\mathrm{SU}(N)$ Yang-Mills theory. This symmetry certainly exists in the $\psi\eta$ and other models studied in [7–9], but in itself it does not lead to any consistency condition. It is another story if one tries to *gauge* this 1-form $\mathbb{Z}_N$ symmetry, by introducing the $\mathbb{Z}_N$ gauge field $B_{\mathrm{c}}^{(2)}$ with a proper $\mathbb{Z}_N$ flux (cf. (1.4)) [12–14]

$$\int_{\Sigma_2} N\,B_{\mathrm{c}}^{(2)} = 2\pi k\,, \qquad k \in \mathbb{Z}\,. \tag{1.6}$$

[3] For definiteness, here we consider the case of the "$\psi\eta$ model" studied in [7] and in [11], and adopt the notation used there.

[4] We recall that an appropriate $\mathrm{U}(1)_{\psi\eta}$ holonomy [7] together with this $\mathbb{Z}_2$ transformation, lead to a $\mathbb{Z}_N$ transformation of the fermions fields, undoing their $\mathbb{Z}_N \subset \mathrm{SU}(N)$ gauge transformations. See section 2.1 for a more detailed discussion.

Such a gauging may encounter a topological obstruction (a 't Hooft anomaly). If it does, then there are new, nontrivial UV-IR matching conditions [15–35]. This is indeed what was found in [7–9]. The question is whether the anomalies and their consequences discussed there are to be trusted, in view of the fact that the argument made use of a singular external (non-dynamical) A_2 gauge field, (1.1).

The present work aims to clarify the sense of the anomalies found in [7–9]. We start with the simplest BY model ("$\psi\eta$" model) with an extra pair of fermions $(q, \tilde{q})$ in the fundamental representation, which acts as a sort of regulator field. When a gauge-invariant, complex scalar field coupled to them through a Yukawa potential term gets a nonvanishing vacuum expectation value (VEV), v, the fermions $q, \tilde{q}$ get mass and decouple,[5] below $\sim v$. Namely, this extended model (which we call the X-ray model) reduces, below the decoupling mass scale v, to the previously considered $\psi\eta$ model.[6]

This work is organized as follows. In section 2 we introduce the extended model and discuss its symmetries. Before taking into account the scalar VEV, the model is of type II: conventional 't Hooft anomaly matching discussion allows a chirally symmetric, confining vacuum as well as a dynamical Higgs phase characterized by certain bifermion condensates. The model reduces to the previously studied $\psi\eta$ model at mass scales below the scalar VEV, v, where the extra fermions pair in a Dirac fermion, get massive and decouple. Section 3 is dedicated to the gauging of the color-flavor locked 1-form $\mathbb{Z}_N$ symmetry and to the calculation of the consequent mixed anomalies. The generalized anomaly found in the X-ray model, which is free from the subtleties related to the singular A_2 field [7], reduces to the $\mathbb{Z}_2 - [\mathbb{Z}_N^{(1)}]^2$ anomaly [7], precisely for even N (i.e. type I) theories. In section 4 we discuss a few subtle issues related to the decoupling of the fermions $q, \tilde{q}$. The summary and conclusion are in section 5.

2 The model and the color-flavor-locked 1-form $\mathbb{Z}_N$ (center) symmetry

We consider the $\psi\eta$ model, in which a Dirac pair of fermions in the fundamental representation of $SU(N)_c$, q and $\tilde{q}$, are added. In other words, we start with a generalized Bars-Yankielowicz model, with Weyl fermions[7]

$$\psi^{ij}\,, \quad \eta_i^A\,, \quad \xi^i\,, \qquad (i,j = 1,2,\ldots,N\,; \quad A = 1,2,,\ldots,N+5)\,, \tag{2.1}$$

in the direct-sum representation

$$\square\!\square \oplus (N+5)\,\bar{\square} \oplus \square\,. \tag{2.2}$$

The global symmetry of the model is

$$SU(N+5) \times U(1)_{\psi\eta} \times U(1)_{\psi\xi}\,, \tag{2.3}$$

[5]Similarly the NGB, although massless, decouples as it cannot be coupled with the $\psi\eta$ model with a relevant or marginal operator.

[6]Naturally, we take v such that $v \gg \Lambda_{\psi\eta}$, where $\Lambda_{\psi\eta}$ is the dynamical scale of the $\psi\eta$ model.

[7]This model was called $\{\mathcal{S}, N, p\}$ model ($p = 1$) in the classification of [8].

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	$\mathrm{SU}(N)_{\mathrm{c}}$	$\mathrm{SU}(N+4)$	$\mathrm{U}(1)_{\psi\eta}$	$\mathrm{U}(1)_V$	$\mathrm{U}(1)_0$	$\tilde{\mathrm{U}}(1)$
ψ	$\square\!\square$	$(\cdot)$	$\frac{N+4}{2}$	0	1	$\frac{N+4}{2}$
η	$\bar{\square}$	$\square$	$-\frac{N+2}{2}$	0	-1	$-\frac{N+2}{2}$
q	$\square$	$(\cdot)$	0	1	1	$\frac{N+2}{2}$
$\tilde{q}$	$\bar{\square}$	$(\cdot)$	0	-1	1	$-\frac{N+2}{2}$
ϕ	$(\cdot)$	$(\cdot)$	0	0	-2	0

Table 1. The fields and charges of the X-ray model with respect to the nonanomalous symmetries. The last symmetry, $\tilde{\mathrm{U}}(1)$, is not linearly independent, but it is particularly useful to define it for our discussion.

where $U(1)_{\psi\eta}$ and $U(1)_{\psi\xi}$ are two anomaly-free combinations of the chiral U(1) symmetries associated with the fermions, ψ, η and χ.

We shall rename the fields as $\eta^{N+5} = \tilde{q}$ and $\xi = q$ below, so that the matter content is

$$\psi^{ij}\,, \quad \eta_i^A\,, \quad q^i, \quad \tilde{q}_i\,, \qquad (i,j = 1,2,\ldots,N\,; \quad A = 1,2,\ldots,N+4)\,. \tag{2.4}$$

We furthermore add a color-singlet complex scalar ϕ coupled to the $(q,\tilde{q})$ pair as,

$$\Delta L = g_Y \phi\, q\, \tilde{q} + \mathrm{h.c.}\,. \tag{2.5}$$

The Yukawa coupling (2.5) breaks the global symmetry as

$$\mathrm{SU}(N+4) \times \mathrm{U}(1)_{\psi\eta} \times \mathrm{U}(1)_0 \times \tilde{\mathrm{U}}(1)\,, \tag{2.6}$$

where the charges are given in table 1.

The Yukawa coupling breaks explicitly part of the global symmetry of the original model, (2.1), (2.2). The implications of the conventional 't Hooft anomaly-matching conditions [36], with respect to the unbroken global symmetry, therefore remain the same as in the original generalized Bars-Yankielowicz model, (2.1), (2.2). The model is of Type II: 't Hooft anomaly matching allows both dynamical Higgs phase (with bifermion condensates) and confining, chirally symmetric phase (with no condensate formation). See appendix A.

We assume that the potential for the ϕ field is such that it acquires a nonvanishing VEV,

$$\langle \phi \rangle = v \gg \Lambda_{\psi\eta}\,. \tag{2.7}$$

The system at mass scales μ below v

$$\mu \ll \langle \phi \rangle \tag{2.8}$$

reduces exactly to the $\psi\eta$ model, studied in [7–9], as the fermions q and $\tilde{q}$ get mass and decouple. The global $\mathrm{U}(1)_V$ and $\tilde{U}(1)$ symmetries remain unbroken, they reduce respectively to the identity $\mathbb{1}$ and to $\mathrm{U}(1)_{\psi\eta}$ when the fermions q and $\tilde{q}$ decouple. The $\mathrm{U}(1)_0$ symmetry is broken as

$$\mathrm{U}(1)_0 \to \mathbb{Z}_2\,, \tag{2.9}$$

where $\mathbb{Z}_2$ acts as

$$\psi \to -\psi\,, \qquad \eta \to -\eta\,. \tag{2.10}$$

We refer to this model as the X-ray theory.

Clearly, besides the $\psi\eta$ model, the breaking of $\mathrm{U}(1)_0$ introduces also a massless NGB. However, the NGB cannot couple to the $\psi\eta$ degrees of freedom through relevant or marginal operators:[8] in the limit $\Lambda \ll v$, the NGB sector decouples.

As $\mathrm{U}(1)_0$ and $\tilde{U}(1)$ symmetries are free of (strong) anomalies, one may introduce external regular gauge fields, A_0 and $\tilde{A}$, respectively.

2.1 Color-flavor locked 1-form $\mathbb{Z}_N$ symmetry

As the idea of color-flavor locked $\mathbb{Z}_N$ 1-form symmetry is central below, let us briefly review it. Let us consider an $\mathrm{SU}(N)$ gauge theory with a set of the massless matter Weyl fermions $\{\psi^k\}$. In general, the color $\mathbb{Z}_N^{(1)}$ symmetry is broken by the fermions (unless the fermions present are all in the adjoint representation of $\mathrm{SU}(N)$). However the situation changes if some global, nonanomalous $\mathrm{U}(1)$ symmetries, $\mathrm{U}(1)_i$, $i = 1, 2, \ldots$, are present, such that when $\mathrm{U}(1)_i$ are gauged (in the usual sense, by the introduction of external gauge fields A_i^μ), the color $\mathbb{Z}_N \subset \mathrm{SU}(N)$ and the $\mathrm{U}(1)_i$ transformations can compensate each other for the fermions. This allows to define a global color-flavor locked $\mathbb{Z}_N^{(1)}$ symmetry.

The action of a $\mathbb{Z}_N^{(1)}$ generator on Wilsons loops that stretch along the non-contractible loop L is

$$\mathrm{SU}(N):\ \mathcal{P}e^{i\oint_L a} \to e^{\frac{2\pi i}{N}}\mathcal{P}e^{i\oint_L a}\,, \qquad \mathrm{U}(1)_i:\ e^{i\oint_L A_i} \to \left(e^{\frac{2\pi i}{N}p_i}\right) e^{i\oint_L A_i}\,, \tag{2.11}$$

where $a \equiv a_\mu^A t^A\, dx^\mu$ is the $\mathrm{SU}(N)$ gauge field, A_i is the $\mathrm{U}(1)_i$ gauge field, and the integers p_i defines an embedding of $\mathbb{Z}_N \hookrightarrow \mathrm{U}(1)_i$.

As, locally, (2.11) can be realized as a gauge transformation, it can fail to be a symmetry only if it ruins the periodicity[9] of the fermion fields. To check it, one should compute the action of (2.11) on the ψ_k Wilson loop, i.e.

$$W[L]_k = \left(\mathcal{P}e^{i\oint_L R_k(a)}\right)\left(\Pi_i\, e^{i\oint_L q_i^k A_i}\right) \tag{2.12}$$

(here ψ_k transforms under $\mathrm{SU}(N)$ in the irrep R_k with N-arity $\mathcal{N}_k$, and has charge q_i^k under $\mathrm{U}(1)_i$):

$$W[L]_k \to e^{\frac{2\pi i}{N}\mathcal{N}_k} e^{\frac{2\pi i}{N}\sum_i q_i^k p_i} W[L]_k \tag{2.13}$$

If the action is trivial, i.e.

$$\frac{2\pi i}{N}\mathcal{N}_k + \frac{2\pi i}{N}\sum_i q_i^k p_i \in 2\pi\mathbb{Z} \qquad \text{for each } \psi_k\,, \tag{2.14}$$

the fermions periodicity conditions are preserved and (2.11) defines a new color-flavor locked $\mathbb{Z}_N$ 1-form symmetry.

[8] As $v \gg \mu \gg \Lambda$, the theory is perturbative, and we can trust this classical dimensional analysis.
[9] Or anti-periodicity, if L is along the thermal cycle.

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	ψ	η	q	$\tilde{q}$
$\mathbb{Z}_N \subset \mathrm{SU}(N)$	$\frac{4\pi}{N}$	$-\frac{2\pi}{N}$	$\frac{2\pi}{N}$	$-\frac{2\pi}{N}$
$\tilde{U}(1)$	$\frac{N+4}{2}\beta$	$-\frac{N+2}{2}\beta$	$\frac{N+2}{2}\beta$	$-\frac{N+2}{2}\beta$
$\mathrm{U}(1)_0$	γ	$-\gamma$	γ	γ

Table 2. The choice $\beta = \frac{2\pi}{N}$ and $\gamma = \pm\pi$ reproduces indeed $\mathbb{Z}_N$.

As the ordinary $\mathbb{Z}_N^{(1)}$ center transformation, such a color-flavor combined $\mathbb{Z}_N^{(1)}$ center symmetry is still just a *global 1-form symmetry.*

A more powerful idea is to introduce the *gauging of this 1-form symmetry* and studying possible topological obstructions in doing so (generalized 't Hooft's anomalies) [15–35]. As in the case of conventional gauging of 0-form symmetries, the idea of gauging is that of *identifying* the field configurations connected by the given symmetry transformations, and of eliminating the double counting in the sum over field configurations. However, as one is now dealing with a 1-form symmetry, the associated gauge transformations are parametrized by a 1-form Abelian gauge function[10] $\lambda = \lambda_\mu(x)dx^\mu$, see (3.8) below.

3 Gauging 1-form $\mathbb{Z}_N$ symmetry: mixed anomalies

We consider now the gauging of the 1-form $\mathbb{Z}_N$ symmetry in the X-ray model, that arises because the subgroup (see table 2)

$$\mathbb{Z}_N = \mathrm{SU}(N)_\mathrm{c} \cap (\tilde{\mathrm{U}}(1) \times \mathrm{U}(1)_0) \tag{3.1}$$

acts trivially on any field of the theory.[11] In other words, the symmetry group that acts faithfully on the fundamental fields is

$$\frac{\mathrm{SU}(N)_\mathrm{c} \times \tilde{\mathrm{U}}(1) \times \mathrm{U}(1)_0}{\mathbb{Z}_N}\,, \tag{3.2}$$

so to get all the 't Hooft anomalies of the theory we should consider a gauge connection of (3.2) rather than by the simple product principal bundle

$$\mathrm{SU}(N) \times \tilde{\mathrm{U}}(1) \times \mathrm{U}(1)_0\,. \tag{3.3}$$

To gauge (3.3) it is enough to introduce the U(1) gauge connections $\tilde{C}$ and C_0 in addition to the dynamical color gauge SU(N) field, a. However, by doing so, one obtains only a subset of all the possible gauge connections allowed by the gauging of (3.2): gauging (3.2)

[10] Here we remember the crucial aspect of higher form symmetries: they are all Abelian. This is the reason why the color-flavor locked 1-form symmetries are possible.

[11] Also, as

$$\mathrm{U}(1)_{\psi\eta} \times \mathrm{U}(1)_V \supset \tilde{U}(1)\,; \qquad Q_{\psi\eta} + \frac{N+2}{2}Q_V = \tilde{Q}$$

it is possible to gauge the 1-form $\mathbb{Z}_N$ symmetry together with $\mathrm{U}(1)_{\psi\eta}$, $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_0$. Here we choose to proceed with gauging $\mathbb{Z}_N$ lying in the intersection (3.1).

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one can allow $\tilde{C}$, C_0 and a not to be proper gauge connection, individually, e.g. one can allow fractional Dirac quantization for $\tilde{C}$ and C_0.

A very convenient way to describe a generic gauge connection for (3.2) is by introducing a pair of fields [15–35]

$$(B_{\rm c}^{(2)}\,, B_{\rm c}^{(1)}) \tag{3.4}$$

where $B_{\rm c}^{(1)}$ is a well-defined[12] U(1) gauge connection, and $B_{\rm c}^{(2)}$ is a 2-form gauge field that satisfies

$$NB_{\rm c}^{(2)} = dB_{\rm c}^{(1)}\,, \tag{3.5}$$

thus

$$\int_{\Sigma} B_{\rm c}^{(2)} = \frac{2\pi}{N}\mathbb{Z}\,, \tag{3.6}$$

for any 2-cycle Σ.

Then we embed a, $\tilde{C}$ and C_0 into

$$\widetilde{a} = a + \frac{1}{N}B_{\rm c}^{(1)}\,, \quad A_0 = C_0 + \frac{1}{2}B_{\rm c}^{(1)} \quad \text{and} \quad \tilde{A} = \tilde{C} - \frac{1}{N}B_{\rm c}^{(1)}\,, \tag{3.7}$$

where $\widetilde{a}$ is a U(N) connection, and A_0 and $\tilde{A}$ are well-defined U(1) connections.[13] Doing so, the $\mathbb{Z}_N$ 1-form symmetry of the original group is embedded in a continuous 1-form symmetry

$$\begin{aligned} &B_{\rm c}^{(2)} \to B_{\rm c}^{(2)} + {\rm d}\lambda_{\rm c}\,, \qquad B_{\rm c}^{(1)} \to B_{\rm c}^{(1)} + N\lambda_{\rm c}\,, \\ &\widetilde{a} \to \widetilde{a} + \lambda_{\rm c}\,, \qquad \tilde{A} \to \tilde{A} - \lambda_{\rm c}\,, \qquad A_0 \to A_0 + \frac{N}{2}\lambda_{\rm c} \end{aligned} \tag{3.8}$$

parameterized by the U(1) gauge connection $\lambda_{\rm c}$, which cancel any local degrees of freedom introduced by $B_{\rm c}^{(1)}$.

Local physics is not affected by these global issues, so the fermionic Lagrangian (locally) still reads

$$\begin{aligned} &\overline{\psi}\gamma^{\mu}\left(\partial + \mathcal{R}_{\rm S}(a) + \frac{N+4}{2}\tilde{C} + C_0\right)_{\mu} P_{\rm L}\psi \\ &+ \overline{\eta}\gamma^{\mu}\left(\partial + \mathcal{R}_{{\rm F}^*}(a) - \frac{N+2}{2}\tilde{C} - C_0\right)_{\mu} P_{\rm L}\eta \\ &+ \overline{q}\gamma^{\mu}\left(\partial + \mathcal{R}_{\rm F}(a) + \frac{N+2}{2}\tilde{C} + C_0\right)_{\mu} P_{\rm L}q \\ &+ \overline{\tilde{q}}\gamma^{\mu}\left(\partial + \mathcal{R}_{{\rm F}^*}(a) - \frac{N+2}{2}\tilde{C} + C_0\right)_{\mu} P_{\rm L}\tilde{q}\,. \end{aligned} \tag{3.9}$$

[12] With well-defined U(1) connection we mean that they satisfy the usual Dirac quantization condition.

[13] In this definition, there is an ambiguity, as we could have set $A_0 = C_0 - \frac{1}{2}B_{\rm c}^{(1)}$ instead. The construction would be equivalent, but, to describe the same background, we would need to add some integer flux for A_0. The same sign ambiguity is present also for the $\psi\eta$ model. We will comment on the consequences of this sign choice on anomalies in footnote 16.

However, as the faithful symmetry group is (3.2), we can express this Lagrangian in terms of well-defined geometrical entities (well-defined gauge connection) as

$$\begin{aligned}
&\overline{\psi}\gamma^\mu\left(\partial+\mathcal{R}_{\mathrm{S}}\left(\tilde{a}_{\mathrm{c}}-\frac{1}{N}B_{\mathrm{c}}^{(1)}\right)+\frac{N+4}{2}\left(\tilde{A}+\frac{1}{N}B_{\mathrm{c}}^{(1)}\right)+\left(A_0-\frac{1}{2}B_{\mathrm{c}}^{(1)}\right)\right)_\mu P_{\mathrm{L}}\psi\\
&+\overline{\eta}\gamma^\mu\left(\partial-\left(\tilde{a}_{\mathrm{c}}-\frac{1}{N}B_{\mathrm{c}}^{(1)}\right)-\frac{N+2}{2}\left(\tilde{A}+\frac{1}{N}B_{\mathrm{c}}^{(1)}\right)-\left(A_0-\frac{1}{2}B_{\mathrm{c}}^{(1)}\right)\right)_\mu P_{\mathrm{L}}\eta\\
&+\overline{q}\gamma^\mu\left(\partial+\left(\tilde{a}_{\mathrm{c}}-\frac{1}{N}B_{\mathrm{c}}^{(1)}\right)+\frac{N+2}{2}\left(\tilde{A}+\frac{1}{N}B_{\mathrm{c}}^{(1)}\right)+\left(A_0-\frac{1}{2}B_{\mathrm{c}}^{(1)}\right)\right)_\mu P_{\mathrm{L}}q\\
&+\overline{\tilde{q}}\gamma^\mu\left(\partial-\left(\tilde{a}_{\mathrm{c}}-\frac{1}{N}B_{\mathrm{c}}^{(1)}\right)-\frac{N+2}{2}\left(\tilde{A}+\frac{1}{N}B_{\mathrm{c}}^{(1)}\right)+\left(A_0-\frac{1}{2}B_{\mathrm{c}}^{(1)}\right)\right)_\mu P_{\mathrm{L}}\tilde{q} \qquad (3.10)
\end{aligned}$$

which is explicitly invariant under the 1-form symmetry (3.8). The effective field-strength tensors acting on the fermions are accordingly:

$$\begin{aligned}
&\mathcal{R}_{\mathrm{S}}(F(\widetilde{a})-B_{\mathrm{c}}^{(2)})+\frac{N+4}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right),\\
&\mathcal{R}_{\mathrm{F}^*}(F(\widetilde{a})-B_{\mathrm{c}}^{(2)})-\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})-\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right),\\
&\mathcal{R}_{\mathrm{F}}(F(\widetilde{a})-B_{\mathrm{c}}^{(2)})+\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right),\\
&\mathcal{R}_{\mathrm{F}^*}(F(\widetilde{a})-B_{\mathrm{c}}^{(2)})-\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right). \qquad (3.11)
\end{aligned}$$

Note that by turning off the 1-form gauge fields ($B_{\mathrm{c}}^{(2)}=0$, $B_{\mathrm{c}}^{(1)}=0$), one goes back to the standard $\mathrm{SU}(N)\times\tilde{\mathrm{U}}(1)\times\mathrm{U}(1)_0$ gauge theory.

The anomalies are compactly expressed by a six-dimensional (6D) anomaly functional [37, 38]

$$\begin{aligned}
\mathcal{A}^{6\mathrm{D}}=\int_{\Sigma_6}\frac{2\pi}{3!(2\pi)^3}\Bigg\{&\mathrm{tr}_{\mathrm{c}}\left(\mathcal{R}_{\mathrm{S}}(\tilde{F}_{\mathrm{c}}-B_{\mathrm{c}}^{(2)})+\frac{N+4}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right)^3\\
&+\mathrm{tr}_{\mathrm{c,f}}\left(\mathcal{R}_{\mathrm{F}^*}(\tilde{F}_{\mathrm{c}}-B_{\mathrm{c}}^{(2)})-\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})-\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right)\right)^3\\
&+\mathrm{tr}_{\mathrm{c}}\left(\mathcal{R}(\tilde{F}_{\mathrm{c}}-B_{\mathrm{c}}^{(2)})+\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right)\right)^3\\
&+\mathrm{tr}_{\mathrm{c}}\left(\mathcal{R}_{\mathrm{F}^*}(\tilde{F}_{\mathrm{c}}-B_{\mathrm{c}}^{(2)})-\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right)\right)^3\Bigg\}. \qquad (3.12)
\end{aligned}$$

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Expanding the 6D anomaly functional (3.12), one finds

$$
\begin{aligned}
&\frac{2\pi}{3!(2\pi)^3}\int_{\Sigma_6}\left\{[(N+4)-(N+4)+1-1]\,\mathrm{tr}_{\mathrm{c}}(\tilde{F}_{\mathrm{c}}-B_{\mathrm{c}}^{(2)})^3\right\}\\
&+\frac{1}{8\pi^2}\int_{\Sigma_6}\mathrm{tr}_{\mathrm{c}}(\tilde{F}_{\mathrm{c}}-B_{\mathrm{c}}^{(2)})^2\bigg\{(N+2)\left[\frac{N+4}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right]\\
&\qquad+(N+4)\left[-\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})-\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right)\right]\\
&\qquad+1\cdot\left[\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right]\\
&\qquad+1\cdot\left[-\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right)\right]\bigg\}\\
&+\frac{1}{24\pi^2}\int_{\Sigma_6}\bigg\{\frac{N(N+1)}{2}\left[\frac{N+4}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right]^3\\
&\qquad+(N+4)N\left[-\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})-\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right)\right]^3\\
&\qquad+N\left[\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right)\right]^3\\
&\qquad+N\left[-\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right)\right]^3\bigg\},
\end{aligned}
\tag{3.13}
$$

by making use of the known formulas for the traces of quadratic and cubic forms in different representations. Note that the terms proportional to $\mathrm{tr}_{\mathrm{c}}(\tilde{F}_{\mathrm{c}}-B_{\mathrm{c}}^{(2)})^3$ and $\mathrm{tr}_{\mathrm{c}}(\tilde{F}_{\mathrm{c}}-B_{\mathrm{c}}^{(2)})^2$ in (3.13) cancel completely as they should. Thus the anomalies are expressed by the last four lines of (3.13) only:

$$
\begin{aligned}
\mathcal{A}^{6\mathrm{D}}=\;&\frac{1}{24\pi^2}\int_{\Sigma_6}\bigg\{\frac{N(N+1)}{2}\left[\frac{N+4}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right]^3\\
&\qquad+(N+4)N\left[-\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})-\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right)\right]^3\\
&\qquad+N\left[\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right)\right]^3\\
&\qquad+N\left[-\frac{N+2}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)})+\left(dA_0-\frac{N}{2}B_{\mathrm{c}}^{(2)}\right)\right]^3\bigg\}.
\end{aligned}
\tag{3.14}
$$

Below we are going to extract the mixed anomalies, involving the $\mathrm{U}(1)_0$ or $\tilde{U}(1)$ gauge fields, A_0, $\tilde{A}$, together with the 1-form $\mathbb{Z}_N$ gauge field, $(B_{\mathrm{c}}^{(2)}, B_{\mathrm{c}}^{(1)})$. To compute such anomalies explicitly it is useful to take as our spacetime manifold the 4-torus, $T^4=T_1^2\times T_2^2$, and

$$
\int_{T_1^2}B_c^{(2)}=\frac{2\pi}{N}\,,\qquad \int_{T_2^2}B_c^{(2)}=\frac{2\pi}{N}\,,\qquad \int_{T^4}\left(B_c^{(2)}\right)=\frac{8\pi^2}{N^2}\,. \tag{3.15}
$$

We recall again that if $(B_{\mathrm{c}}^{(2)}, B_{\mathrm{c}}^{(1)})$ is set to zero, the UV anomalies simply express the conventional 't Hooft anomaly triangles involving the $\mathrm{U}(1)_0\times\tilde{U}(1)$ background fields, and by construction those are matched by the assumed set of the massless baryons of a candidate IR theory such as the one discussed in appendix B. What we shall exhibit below is only the new, stronger anomalies introduced by the gauging of the 1-form $\mathbb{Z}_N$ symmetry. As will be discussed below (section 3.3) the consequence of these is that the confining, symmetric vacuum with just one massless baryon and no other nontrivial sectors, is not consistent.

3.1 $\tilde{A} - [B_{\rm c}^{(2)}]^2$ anomaly

To calculate the anomaly in $\tilde{U}(1)$ caused by the introduction of the 1-form $\mathbb{Z}_N$ gauge fields, let us briefly recall the procedure for calculating the anomalies in 4D theory according to the Stora-Zumino descent procedure [37–39], starting from the 6D anomaly functional, (3.14), in our case.[14] One collects the terms of the form, $(B_{\rm c}^{(2)})^2 d\tilde{A}$, integrate to get a 5D functional of the form,

$$\propto \int_{\Sigma_5} (B_{\rm c}^{(2)})^2 \tilde{A} \,. \tag{3.16}$$

Now the variation $\tilde{A} \to \tilde{A} \, + \delta \, \tilde{A}$

$$\delta \tilde{A} = d\, \delta\alpha \tag{3.17}$$

yields, by anomaly inflow, the anomalous variation in the (boundary) 4D theory

$$\delta S_{\delta\alpha} = \frac{\tilde{K}}{8\pi^2} \int_{\Sigma_4} (B_{\rm c}^{(2)})^2 \, \delta\alpha \,. \tag{3.18}$$

By collecting terms we find

$$\tilde{K} = -\frac{N^3(N+3)}{2} \neq 0 \,. \tag{3.19}$$

The $\tilde{U}(1)$ symmetry is broken (i.e., gets anomalous) by the generalized 1-form gauging of the $\mathbb{Z}_N$.

3.2 $A_0 - [B_{\rm c}^{(2)}]^2$ anomaly

An analogous calculation leads to the $\mathrm{U}(1)_0$ anomaly due to the 1-form gauging of the $\mathbb{Z}_N$ symmetry,

$$\delta S_{\delta\alpha_0} = \frac{K_0}{8\pi^2} \int_{\Sigma_4} (B_{\rm c}^{(2)})^2 \, \delta\alpha_0 \,, \qquad K_0 = N^2(N+3) \,. \tag{3.20}$$

This appears to imply that the $\mathrm{U}(1)_0$ symmetry is also broken by the 1-form gauging of the $\mathbb{Z}_N$ symmetry.

However, the scalar VEV $\langle\phi\rangle = v$ breaks spontaneously the $\mathrm{U}(1)_0$ symmetry to $\mathbb{Z}_2$. It means that, in contrast to (3.18), (3.19), the variation (3.20) cannot be used to examine the generalized UV-IR anomaly matching check. For that purpose, we can use only the nonanomalous[15] and unbroken symmetry operation, i.e., variations corresponding to a nontrivial $\mathbb{Z}_2$ transformation $\delta\alpha_0 = \pm\pi$. Taking into account the nontrivial 't Hooft flux (3.6), (3.15)),

$$\frac{1}{8\pi^2} \int_{\Sigma_4} (B_{\rm c}^{(2)})^2 = \frac{n}{N^2} \,, \qquad n \in \mathbb{Z} \,, \tag{3.21}$$

[14] As emphasized in [7] all the calculations can be done staying in 4D, à la Fujikawa. That approach will give directly (3.18), for instance, from the functional Jacobian.

[15] In the sense of the standard strong anomaly.

and the crucial coefficient of the anomaly, $K_0 = N^2(N+3)$, it is seen that the partition function changes sign for even[16] N. We reproduce exactly the $\mathbb{Z}_2$ anomaly found in [7].

3.2.1 Remarks

The anomalies found in section 3.1, section 3.2 represent the main result of the present work.

As in our earlier work [7–9], the nontrivial 't Hooft $\mathbb{Z}_N$ flux (1.6), (3.21), mean that one is considering the 4D spacetime compactified in e.g., bi-torus, $T^2 \times T^2$. See section 4 below for more remarks on $\mathbb{Z}_2$ vortices in such a spacetime, implied by (3.6).

3.3 Chirally symmetric vacuum versus dynamical Higgs phase

Now what is the implication of the mixed anomalies found in the X-ray model, (3.18), (3.21) to the physics in the infrared, that is, the phase of the $\psi\eta$ model? We consider here two particularly interesting dynamical possibilities, a confining, chirally symmetric vacuum and a dynamical Higgs phase, which are both known to be compatible with the conventional 't Hooft anomaly-matching constraints.

If we assume that the infrared system was confining, chirally symmetric one, with no bifermion condensates forming, then the conventional 't Hooft anomalies would be matched by a low-energy theory consisting just of a single color-singlet massless composite fermion, the baryon $\mathcal{B}_{11} \sim \psi\eta\eta$ (see appendix B). Knowing its quantum numbers, we can construct the infrared anomaly functional, following the same procedure used at the beginning of this section. The answer is the expression (B.3), which does not contain the 1-form gauge field $B_{\rm c}^{(2)}$: it reproduce neither of the mixed anomalies, (3.18) or (3.21). We must conclude that such a vacuum, with just $\mathcal{B}_{11} \sim \psi\eta\eta$ and nothing else, cannot represent the correct IR physics of the $\psi\eta$ model, as the X-ray model reduces to it in the infrared.

On the other hand, the dynamical Higgs phase (analyzed in appendix C) is characterized by bifermion condensates

$$\langle \psi^{ij} \eta_i^B \rangle = c_{\psi\eta}\, \Lambda^3 \delta^{jB} \neq 0\,, \qquad j, B = 1, \dots, N\,. \tag{3.22}$$

Under this assumption, both $\mathrm{U}(1)_0$ and $\tilde{U}(1)$ are broken by the condensate so, if one requires the condensate (3.22) to be everywhere non-vanishing, then, as it is charged under $\mathrm{U}(1)_0$ and $\tilde{\mathrm{U}}(1)$, one cannot allow any non-vanishing $B_{\rm c}^{(2)}$ fields. If, on the other way around, one imposes a non-vanishing $B_{\rm c}^{(2)}$ field, then $\psi^{ij}\eta_i^B$ cannot condense everywhere, and, similarly with ϕ form the X-ray to the UV, there must be vortices where the condensate (3.22) vanishes. We leave a more in-depth description of the matching in this case for subsequent work, but, disregarding the details, the matching must work as one can arrive at the same

[16] By taking the equivalent definition of C_0 in footnote 13, one obtains $K_0 = -\frac{1}{2}N^2(N+2)(N+3)$, which signals an $\mathbb{Z}_2$ anomaly only for $N = 0 \mod 4$. Exactly the same happens in the $\psi\eta$ model. One might wonder how is it possible that the two constructions lead to different anomalies. However, the puzzle is only apparent, as the system has also a $A_0(dA_0)^2$ anomaly, and, if one takes also it into account, the anomalous phase under a $\mathbb{Z}_2^F$ transformation depends only on the background and not on the sign convention chosen. Moreover, the choice of the convention is totally irrelevant to discuss the 't Hooft anomaly matching with the confining phase, as the $[\mathbb{Z}_2]^3$ anomaly is matched for every N.

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phase perturbatively, by substituting the composite operator $\psi^{ij}\eta_i^B$ by a fundamental scalar field with the same quantum number of it and a suitable potential.

This can be understood as a consistent way in which the infrared dynamics reflects the impossibility (an anomaly), (3.18), of gauging the color-flavor locked 1-form symmetry, (3.4), found in the UV theory.[17]

4 Reduction to the $\psi\eta$ model, $\mathbb{Z}_2$ vortex and the fermion zeromodes

In order to make the argument of the present work water-tight, let us discuss here a subtle question associated with the reduction of the X-ray theory to the $\psi\eta$ model in the infrared. The basic statement is that nonvanishing VEV $\langle\phi\rangle$ gives mass to the extra Dirac pair of fermions, $q, \tilde{q}$, and that the system indeed reduces in the infrared to the $\psi\eta$ model (the simplest BY model), studied in [7–9].

The point is that the generalized, mixed anomalies (3.18) and (3.21), occur in the background of the external $\tilde{U}(1)$ and $\mathrm{U}(1)_0$ gauge fields with fluxes, (3.6). In the case of the $\tilde{U}(1)$ gauge field $\tilde{A}$ this does not present a problem. On the other hand, $\mathrm{U}(1)_0$ is spontaneously broken to a $\mathbb{Z}_2$ by the ϕ VEV, see table 1. This means that the relevant background fields (A_0, ϕ) correspond to a (regular) $\mathbb{Z}_2$ vortex configuration. Again this does not present any issue in itself: there is nothing wrong in considering such a particular (and convenient) background and asking if the gauging of the color-flavor-locked 1-form $\mathbb{Z}_N$ symmetry encounters a topological obstruction (a 't Hooft anomaly). This is what is studied in section 3.1, section 3.2 and section 3.3.

A (possible) problem is that $q, \tilde{q}$ fields are massive everywhere and decouple from the system, except along the vortex core, where $\phi = 0$ and $m_{q,\tilde{q}} = 0$. As is well known, such a system develops a chiral two-dimensional $q, \tilde{q}$ zero-mode, traveling along the vortex core with light velocity. They will produce an anomaly in the $\tilde{U}(1)$ gauge symmetry in the 2D vortex worldsheet, as discussed, e.g., by Callan and Harvey [40]. To make the parallelism with the problem discussed in [40] complete, let us for the moment forget about the contribution of the fermions ψ and η in table 1. It will be taken care of later.

In a 4D system considered in [40], a Dirac fermion Ψ, with an electric charge, is coupled to a complex scalar field Φ via a Yukawa interaction,

$$\mathcal{L}_Y = g_Y \bar{\Psi}\Phi\Psi\,, \tag{4.1}$$

and Φ is assumed to get a nonvanishing VEV, $\langle\Phi\rangle = v \neq 0$. The axial $\mathrm{U}(1)_A$ is spontaneously broken by the condensate, whereas the vector (electromagnetic) symmetry $\mathrm{U}(1)_{\mathrm{em}}$ remains exact. Such a system can develop a solitonic vortex,

$$\Phi(x) = f(\rho)\, e^{i\theta}\,, \quad f(0) = 0\,, \quad f(\infty) = v \qquad x_2 + ix_3 = \rho\, e^{i\theta}\,. \tag{4.2}$$

Now the zero-mode for Ψ which develops on the string (vortex core) turns out to have a chiral nature in the vortex worldsheet (x_0, x_1). As Ψ is charged, such a massless fermion

[17] The logic of this argument is somewhat similar to the one employed in [19] in the study of the vacuum of the pure $\mathrm{SU}(N)$ Yang-Mills theory at $\theta = \pi$.

causes a 2D chiral anomaly

$$D_a J_a = \frac{1}{2\pi} \epsilon_{ab} \, \partial^a A^b \,, \qquad a, b = 0, 1 \,, \tag{4.3}$$

where A^μ and J^μ are the $\mathrm{U}(1)_{\mathrm{em}}$ gauge field and its covariant current. As $\mathrm{U}(1)_{\mathrm{em}}$ is supposed to be an exact conserved symmetry of the system, this appears to present a paradox.

The solution to this puzzle [40] is the following. As the system suffers from the ABJ anomaly for the axial $\mathrm{U}(1)_A$ symmetry ($\mathrm{U}(1)_A - [\mathrm{U}(1)_{\mathrm{em}}]^2$ triangle), the spontaneous breaking of the $\mathrm{U}(1)_A(1)$ means that the low-energy ($\mu \ll v$) 4D effective action has an axion-like (or better, $\pi_0 - 2\gamma$ like) term,

$$\mathcal{L}_{\pi_0\gamma\gamma} = \frac{e^2}{32\pi^2} \int d^4x \, \pi(x) \, \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \,, \tag{4.4}$$

where $\pi(x)$ is the pion field,

$$\Phi(x) = v \, e^{i\pi(x)/v} \,. \tag{4.5}$$

Now, in the presence of the soliton vortex, the pion field $\pi(x)$ is ill-defined as one goes around the vortex string, see (4.2). As a result, the $\mathrm{U}(1)_{\mathrm{em}}$ variation $\delta A_\mu = \partial_\mu \omega$ in $\mathcal{L}_{\pi_0\gamma\gamma}$ turns out to be nonvanishing. The nontrivial vorticity in $\pi(x) \sim \theta(x)$

$$\partial^\mu \partial^\nu \theta(x) = -2\pi \epsilon^{\mu\nu} \delta(x_2) \delta(x_3) \,, \qquad \mu, \nu = 2, 3 \tag{4.6}$$

indeed gives rise [40] to ("the anomaly-inflow") $\delta\mathcal{L}_{\pi_0\gamma\gamma}$ in the vortex worldsheet (x_0, x_1), which precisely cancels the 2D chiral anomaly (4.3) generated by the fermion zeromode.

The Callan-Harvey argument exactly applies to our model, upon identifying (see table 1),

$$\Psi \equiv \begin{pmatrix} q \\ \tilde{q}^c \end{pmatrix} \,, \qquad \mathrm{U}(1)_{\mathrm{em}} \equiv \tilde{U}(1) \,, \qquad \mathrm{U}(1)_A \equiv \mathrm{U}(1)_0 \,, \tag{4.7}$$

as long as the effects of the other fermions ψ and η are not considered.

In our model $q^i - \bar{q}_i$ form, in the bulk, a Dirac fermion fundamental of $\mathrm{SU}(N)_{\mathrm{c}}$, meaning that also the 2D world-sheet fermion is fundamental under $\mathrm{SU}(N)_{\mathrm{c}}$. Because of that the same mechanism (a local 2D anomaly, canceled by a bulk inflow) happens also for $\mathrm{SU}(N)_{\mathrm{c}}$, without any significant difference.

More interestingly, the fact that the world-sheet fermions are coupled with the bulk gauge field means that, as we continue to follow the RG-flow and approach $\mu \sim \Lambda$, something should happen. In this work, we do not prescribe in detail what happens: we assume that what remains of the vortices in IR does not contribute to the 't Hooft anomaly matching of the anomalies found above.[18]

As was recalled at the end of section 3.2, the 't Hooft fluxes (3.6), (3.21) mean that one is working in a bi-torus, $T_1 \times T_2$ spacetime. The associated fractional flux A_0 (3.6) hence the $\mathbb{Z}_2$ vortex, must accordingly be considered both in T_1 and in T_2. The Callan-Harvey solution of an apparent puzzle associated with the vortex (a point on T_1) and the fermion

[18] If we lift this hypothesis some other interesting possibilities might arise. We will discuss them in a future work.

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zeromodes propagating in the vortex worldsheet T_2, has been adapted to our problem as explained above. Exactly the same argument eliminates any issue concerning the second vortex punctuating T_2 and the chiral fermion zero-mode generating an anomaly in T_1. The details will appear elsewhere.

As a final remark, we note that the questions (the fermion zeromodes traveling along the vortex core, etc.) discussed here concern perturbative, infinitesimal $\tilde{U}(1)$ variations of the system. Regarding the $\mathbb{Z}_2 - [\mathbb{Z}_N^{(1)}]^2$ discussed in subsection 3.2, apparently, the analysis might be more involved, and the 2D chiral fermions might, in principle, contribute to this anomaly. However, this is not the case: by explicit calculation both in the X-ray model (as shown in subsection 3.2) and in the $\psi\eta$ model (as shown in ref. [35]) we have found a nontrivial $\mathbb{Z}_2 - [\mathbb{Z}_N^{(1)}]^2$ anomaly, thus, being them $\mathbb{Z}_2$ anomalies, they must agree, and the overall contribution of the vortex physics must vanish.

5 Discussion and summary

All Bars-Yankielowicz (BY) and generalized Georgi-Glashow (GG) models [1–6] possess a nonanomalous fermion parity symmetry $(\mathbb{Z}_2)_F$,[19]

$$\psi_i \to -\psi_i \tag{5.1}$$

where i labels the fermions present in the model. In the standard quantization, the instanton analysis tells us that (5.1) is a nonanomalous symmetry of the quantum theory. However, in some cases with even N (models of type I[20]), this statement holds because its anomaly is given by

$$\Delta S = \sum_i b_i \times \frac{1}{8\pi^2} \int_{\Sigma_4} \mathrm{tr}_F \left[F(a)^2\right] \times (\pm\pi) = 2\pi\mathbb{Z}\,, \tag{5.2}$$

with

$$\sum_i b_i = \text{even integer} \neq 0\,, \tag{5.3}$$

whereas $\frac{1}{8\pi^2}\int_{\Sigma_4} \mathrm{tr}_F\left[F(a)^2\right]$ is the standard integer instanton number. It is essential to realize that the $(\mathbb{Z}_2)_F$ anomaly is absent because the sum of the anomaly coefficients $\sum_i b_i$ is a nonzero even number, *not* because it vanishes.

For the $\psi\eta$ model,

$$G = \frac{\mathrm{SU}(N)_{\mathrm{c}} \times \mathrm{SU}(N+4) \times \mathrm{U}(1)_{\psi\eta} \times (\mathbb{Z}_2)_F}{\mathbb{Z}_N \times \mathbb{Z}_{N+4}} = \frac{\tilde{G}}{\mathbb{Z}_N \times \mathbb{Z}_{N+4}}\,. \tag{5.4}$$

The group $\tilde{G}$ is doubly-connected ($\Pi_0(\tilde{G}) = \mathbb{Z}_2$) [8]. This always happens in models of type I. Instead, in type II models, where $(\mathbb{Z}_2)_F$ is a subset of a continuous $\tilde{G}$.

[19] $(\mathbb{Z}_2)_F$ is equivalent to a subgroup of the proper Lorentz group. The point is whether or not in the non-trivial 2-form gauge background, $B_{\mathrm{c}}^{(2)}$, the symmetry is broken by a ('t Hooft) anomaly.

[20] Among the generalized SU(N) BY and GG models with p Dirac pairs of fermions in the fundamental representation, the models of type I are those with N and p both even. Other models are called type II in this note.

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In general, in a type I theory, the gauging of the 1-form $\mathbb{Z}_N$ symmetry leads to the $(\mathbb{Z}_2)_F$ anomaly, given by a master formula [9][21]

$$\Delta S^{(\text{Mixed anomaly})} = (\pm\pi) \cdot \sum_i c_i \left(d(R_i)\mathcal{N}(R_i)^2 - N \cdot D(R_i) \right) \frac{1}{8\pi^2} \int_{\Sigma_4} (B_{\rm c}^{(2)})^2 \,. \tag{5.5}$$

The calculation gives

$$\sum_i c_i \left(d(R_i)\mathcal{N}(R_i)^2 - N \cdot D(R_i) \right) = N^2 \,, \tag{5.6}$$

but (see (3.15))

$$\frac{1}{8\pi^2} \int_{\Sigma_4} (B_{\rm c}^{(2)})^2 = \frac{1}{N^2} \,, \tag{5.7}$$

therefore

$$\Delta S^{(\text{Mixed anomaly})} = \pm\pi \,. \tag{5.8}$$

The partition function changes sign under $(\mathbb{Z}_2)_F$, in the $\psi\eta$ model with N even, and in all other type I models: the mixed $(\mathbb{Z}_2)_F - [\mathbb{Z}_N]^2$ anomaly.

As the candidate massless baryons do not support this generalized anomaly (see (B.3) in the simplest, $\psi\eta$ model), such a confining vacuum cannot represent a correct phase in type I models.

The aim of the present work was to cure the defect of the original analysis [7], i.e., the use of a singular $(\mathbb{Z}_2)_F$ gauge field. In a theory with a regulator Dirac pair of fields $q, \tilde{q}$ (the X-ray theory), the singular $\mathbb{Z}_2$ vortex background needed in [7] is replaced by a regular $\mathbb{Z}_2$ vortex, without affecting the crucial holonomy, (1.1). The 1-form $\mathbb{Z}_N$ symmetry lies now in the intersection between $\mathrm{SU}(N)$ and two nonanomalous $\mathrm{U}(1)$ symmetries, (3.1). In other words, the model is described by a well-defined principal bundle, (3.2). The generalized cocycle condition is met exactly as in [25].

In the X-ray theory the new anomalies are of the type, $\tilde{A} - [B_{\rm c}^{(2)}]^2$ and $A_0 - [B_{\rm c}^{(2)}]^2$. In particular, the $\tilde{U}(1) - [\mathbb{Z}_N^{(1)}]^2$ mixed anomaly (3.18) and its UV-IR mismatch occur both for even and odd N (of the $\mathrm{SU}(N)$ color group). Therefore the statement in the X-ray model is somewhat stronger than in the $\psi\eta$ model.[22] As for the $\mathrm{U}(1)_0 - [\mathbb{Z}_N^{(1)}]^2$ anomaly, (3.20), $\mathrm{U}(1)_0$ is spontaneously broken by the scalar VEV, therefore only the variations $\mathbb{Z}_2 \subset \mathrm{U}(1)_0$ can be used in the UV-IR anomaly matching algorithm. For N even, the anomaly found here reduces to the $\mathbb{Z}_2$ anomaly found in [7].

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[21] c_i is the $\mathbb{Z}_2$ charge, R is the fermion representation, $\mathcal{N}(R)$, $d(R)$, $D(R)$ are the associated N-ality, the dimension, and the Dynkin index, respectively.

[22] The argument based on the strong anomaly [10] which also favors the color-flavor locked dynamical Higgs phase, is equally valid for both even and odd N, too.

	$\mathrm{SU}(N)_{\mathrm{c}}$	$\mathrm{SU}(N+5)$	$U(1)_{\psi\eta}$	$U(1)_{\psi\xi}$
ψ	$\square\square$	$\frac{N(N+1)}{2}\cdot(\cdot)$	$N+5$	1
η	$(N+5)\cdot\bar{\square}$	$N\cdot\square$	$-(N+2)$	0
ξ	$\square$	$N\cdot(\cdot)$	0	$-(N+2)$

Table 3. The multiplicity, charges, and representation are shown for each set of fermions in the BY model, (2.1)–(2.4). $(\cdot)$ stands for a singlet representation.

	$\mathrm{SU}(N)_{\mathrm{c}}$	$\mathrm{SU}(N+5)$	$U(1)_{\psi\eta}$	$U(1)_{\psi\xi}$
$\mathcal{B}_1$	$\frac{(N+5)(N+4)}{2}\cdot(\cdot)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$-N+1$	1
$\mathcal{B}_2$	$(N+5)\cdot(\cdot)$	$\bar{\square}$	-3	$-(N+3)$
$\mathcal{B}_3$	$(\cdot)$	$(\cdot)$	$N+5$	$2N+5$

Table 4. Massless baryons in the hypothetical chirally symmetric phase of the extended BY model, (2.1)–(2.4).

A The confining, symmetric vacua in the extended BY model

The generalized Bars-Yankielovicz model was studied earlier [1–6] by adopting the conventional 't Hooft anomaly matching conditions as criteria for possible infrared phases. An interesting possibility discussed in the past is that the system confines but with no condensates forming. The global chiral symmetry of the models would be fully present in the infrared, saturated by certain massless composite fermions, "baryons". In the model, (2.1), (2.2), (2.4), all the anomalies associated with the global symmetries $\mathrm{SU}(N+5)\times U(1)_{\psi\eta}\times U(1)_{\psi\xi}$ (see table 3) can be matched by gauge-invariant (candidate) massless composite fermions,

$$(\mathcal{B}_1)^{[AB]}=\psi^{ij}\eta_i^A\eta_j^B\,,\qquad (\mathcal{B}_2)_A=\bar{\psi}_{ij}\bar{\eta}_A^i\xi^j\,,\qquad (\mathcal{B}_3)=\psi^{ij}\bar{\xi}_i\bar{\xi}_j\,,\tag{A.1}$$

the first is anti-symmetric in $A\leftrightarrow B$; their charges are listed in table 4. The anomaly matching can be verified straightforwardly via a comparison between table 3 and table 4 (see [8] for explicit checks).

Note that this model is an extended BY model with $p=1$ (one additional Dirac pair of fermions in the fundamental representation): it is a Type II model. The $\mathbb{Z}_2$ is not a genuine independent symmetry. The gauging of a color-flavor locked $\mathbb{Z}_N$ symmetry by introducing $(B_{\mathrm{c}}^{(2)}, B_{\mathrm{c}}^{(1)})$ gauge fields does not lead to any new constraints as compared with the conventional 't Hooft anomaly matching.

The situation is the same when a scalar field ϕ is introduced with the Yukawa coupling to the $(q,\tilde{q})$ pair, but *without* taking into account the scalar VEV and the consequent decoupling of $(q,\tilde{q})$. The Yukawa term simply reduces the symmetry as

$$G_F'=\mathrm{SU}(N+4)\times\mathrm{U}(1)_{\psi\eta}\times\mathrm{U}(1)_V\times\mathrm{U}(1)_0\times\tilde{\mathrm{U}}(1)\,,\tag{A.2}$$

of which three of the U(1) symmetries are independent. The decomposition of the UV fermions as a sum of the irreducible representations of the reduced symmetry group is given

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baryons		$\mathrm{SU}(N)_{\mathrm{c}}$	$\mathrm{SU}(N+4)$	$\mathrm{U}(1)_{\psi\eta}$	$\mathrm{U}(1)_V$	$\mathrm{U}(1)_0$	$\tilde{\mathrm{U}}(1)$
$\mathcal{B}_{11}$	$\psi\eta\eta$	$(\cdot)$	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	$-\frac{N}{2}$	0	-1	$-\frac{N}{2}$
$\mathcal{B}_{12}$	$\psi\eta\tilde{q}$	$(\cdot)$	$\square$	1	-1	1	$-\frac{N}{2}$
$\mathcal{B}_{21}$	$\bar{\psi}\bar{\eta}q$	$(\cdot)$	$\bar{\square}$	-1	1	1	$\frac{N}{2}$
$\mathcal{B}_{22}$	$\bar{\psi}\bar{\tilde{q}}q$	$(\cdot)$	$(\cdot)$	$-\frac{N+4}{2}$	2	-1	$\frac{N}{2}$
$\mathcal{B}_{31}$	$\psi\tilde{q}\tilde{q}$	$(\cdot)$	$(\cdot)$	$\frac{N+4}{2}$	-2	-1	$-\frac{N}{2}$

Table 5. Decomposition of the baryons in table 4 as a direct sum of the irreps of the unbroken symmetry group G_F'.

in table 1 in the main text. The decomposition of the "massless baryons" (table 4) in the direct sum of the irreps of G_F' is in table 5.

Since this model (with or without the Yukawa coupling, but without the scalar VEV, v) is of type II, the massless baryons in table 4 or table 5 reproduce all the conventional 't Hooft anomalies with respect to unbroken global symmetries, and *automatically*, also anomalies involving the $\mathbb{Z}_2$ which is a subgroup of a continuous nonanomalous symmetry group. Consideration of the gauged color-flavor locked 1-form $\mathbb{Z}_N$ symmetry does not give any new information as compared to the conventional 't Hooft anomaly-matching constraints.[23] Thus, as in any other type II models, here, the hypothetical confining phase with massless baryons table 4 or table 5 cannot be excluded by the anomaly-matching arguments only.

As we recalled several times in the text, the topology of the symmetry group space changes discontinuously in going from Type II to Type I models. What is studied in this work is precisely a realization of such a transition, by giving a mass to the extra Dirac pair fermion, $(q, \tilde{q})$, and letting it to ∞ (or equivalently, by going to energy scales much less than the scalar VEV, v.) At the decoupling mass scale, which we take as

$$\langle\phi\rangle = v \gg \Lambda_{\psi\eta}\,, \tag{A.3}$$

the $\mathrm{SU}(N)$ interactions are still weaky coupled. No "baryons" in table 4 or table 5 are yet formed. In other words, the correct degrees of freedom needed in discussing the decoupling phenomenon are the original UV fermions,

$$\psi^{ij}\,, \quad \eta_i^A\,, \quad q^i, \quad \tilde{q}_i\,, \qquad (i,j = 1,2,\ldots,N\,; \quad A = 1,2,,\ldots,N+4)\,, \tag{A.4}$$

listed in (2.4). The discussions given in section 4 appropriately take care of possible subtleties associated with the presence of the vortex backgrounds, the fermion zeromodes, and the decoupling of the fermions $(q, \tilde{q})$, below the mass scale v.

[23]This was shown explicitly in section 4 of [7]. It is a trivial exercise to write explicitly the low-energy effective anomaly functionals as in appendix B, but keeping the contributions of all the baryons in 4 or 5 and to check that the consideration of the gauged color-flavor locked 1-form $\mathbb{Z}_N$ symmetry does not yield any new constraints as compared to the old 't Hooft matching conditions.

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B A confining chirally symmetric phase in the X-ray model — $\psi\eta$ model

In the X-ray model the scalar field gets a VEV, $\langle\phi\rangle = v \gg \Lambda_{\psi\eta}$, where $\Lambda_{\psi\eta}$ is the RG invariant mass scale of the $\psi\eta$ model. The fermions q and $\tilde{q}$ become massive and decouple before the SU(N) interactions become strong.

A possible confining, symmetric phase (with no bifermion condensation) of the $\psi\eta$ system has been discussed earlier in [6, 7]: the candidate massless composite fermion is just $\mathcal{B}_{11}$ of table 5.

The global $\mathrm{U}(1)_V$ and $\tilde{U}(1)$ symmetries reduce respectively to the identity $\mathbb{1}$ and to $\mathrm{U}(1)_{\psi\eta}$. The $\mathrm{U}(1)_0$ symmetry is broken as

$$\mathrm{U}(1)_0 \to \mathbb{Z}_2\,, \tag{B.1}$$

where $\mathbb{Z}_2$ acts as

$$\psi \to -\psi\,, \qquad \eta \to -\eta\,. \tag{B.2}$$

The anomaly functional in IR can be found by introducing

1. $\tilde{A}$: $\tilde{U}(1)$ 1-form gauge field,
2. A: $\mathrm{U}(1)_0$ 1-form gauge field,
3. $B_{\mathrm{c}}^{(2)}$: $\mathbb{Z}_N$ 2-form gauge field,

(the dynamical color gauge SU(N) field, a, does not appear in the infrared effective theory). It is given solely by the contribution of $\mathcal{B}_{11}$,

$$\begin{aligned}\mathcal{A}^{6\mathrm{D}} &= \frac{1}{24\pi^2}\int_{\Sigma_6}\Big\{\frac{(N+4)(N+3)}{2}\left(-\frac{N}{2}(d\tilde{A}+B_{\mathrm{c}}^{(2)}) - \left(dA_0 - \frac{N}{2}B_{\mathrm{c}}^{(2)}\right)\right)^3 \\ &= \frac{1}{24\pi^2}\frac{(N+4)(N+3)}{2}\int_{\Sigma_6}\left(-\frac{N}{2}d\tilde{A} - dA_0\right)^3\Big\}\,,\end{aligned} \tag{B.3}$$

which does not contain the 1-form gauge field $B_{\mathrm{c}}^{(2)}$. Thus the mixed anomalies found in the UV in section 3.1 and section 3.2, cannot be reproduced in confining, chirally symmetric vacuum in the X-ray model, i.e., in the $\psi\eta$ model.

There is a subtle point to appreciate in the relation between what we discussed in appendix A and the inconsistency of the model with only $\mathcal{B}_{11}$. The model of table 5 reproduces all the anomalies when $v = 0$. When the scalar acquires an expectation value $v \neq 0$ the baryons $\mathcal{B}_{12}$, $\mathcal{B}_{21}$, $\mathcal{B}_{22}$, $\mathcal{B}_{31}$ all get mass and decouple below the mass scale v, thus leaving the theory with just $\mathcal{B}_{11}$ that does not reproduce new anomalies involving $B_{\mathrm{c}}^{(2)}$ in UV. How is it possible that a vectorial sector decouples and the anomaly matching is changed? To answer this question note that the pairs $(\mathcal{B}_{12}, \mathcal{B}_{21})$ and $(\mathcal{B}_{22}, \mathcal{B}_{31})$ are vectorlike with respect to $\mathrm{SU}(N+4) \times \tilde{\mathrm{U}}(1)$, but not with respect to $\mathrm{U}(1)_0$, which is however broken to $\mathbb{Z}_2$. Possible operators that mimic the mechanism that gives mass to these baryons are the Yukawa couplings with the scalar

$$\phi\,\mathcal{B}_{12}\mathcal{B}_{21} + \mathrm{h.c.} \qquad \text{and} \qquad \phi^*\mathcal{B}_{22}\mathcal{B}_{31} + \mathrm{h.c.} \tag{B.4}$$

	$\mathrm{SU}(N)_{\mathrm{cf}_\eta}$	$\mathrm{SU}(4)_\eta$	$\mathrm{U}(1)'_{\psi\eta}$
ψ	$\square\square$	$\frac{N(N+1)}{2}\cdot(\cdot)$	2
η_1	$\overline{\square\square} \oplus \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$N^2\cdot(\cdot)$	-2
η_2	$4\cdot\bar{\square}$	$N\cdot\square$	-1
$\tilde{q}$	$\bar{\square}$	$N\cdot(\cdot)$	0
q	$\square$	$N\cdot(\cdot)$	0

Table 6. UV fields in the model, table 3, are decomposed as a direct sum of the representations of the unbroken group G'_F of (C.1).

	$\mathrm{SU}(N)_{\mathrm{cf}_\eta}$	$\mathrm{SU}(4)_\eta$	$\mathrm{U}(1)'_{\psi\eta}$
$\mathcal{B}_1$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$\frac{N(N-1)}{2}\cdot(\cdot)$	-2
$\mathcal{B}_2$	$4\cdot\bar{\square}$	$N\cdot\square$	-1

Table 7. IR fermion fields in the Higgs vacuum of our model, (2.1)–(2.4), which are a subset of the baryons $\mathcal{B}_{11}$ in table 4. More precisely, $\mathcal{B}_1 \sim \psi\eta_1\eta_1$; $\mathcal{B}_2 \sim \psi\eta_1\eta_2$.

Giving mass to $\mathcal{B}_{12}$, $\mathcal{B}_{21}$, $\mathcal{B}_{22}$, $\mathcal{B}_{31}$ in this way does not leave just $\mathcal{B}_{11}$ in the IR, but also fermion zero modes localized on vortices where $\phi = 0$. This confining symmetric theory with $\mathcal{B}_{11}$ in the bulk plus degrees of freedom localized on vortices will require further investigations in the future.

C The dynamical Higgs phase

It was noted in [6–9] that in all the BY and GG models another possible phase is a dynamical (color-flavor-locked) Higgs vacuum, in which the color SU(N) is completely broken and the global symmetry is partially realized in the Nambu-Goldstone mode. In the X-ray model considered in this work, (2.1)–(2.5), the proposed bifermionic condensates, (3.22), together with the scalar condensate $\langle\phi\rangle$, break the global symmetries as

$$\begin{aligned} G_F &= \mathrm{SU}(N+4) \times \mathrm{U}(1)_{\psi\eta} \times \mathrm{U}(1)_0 \times \tilde{\mathrm{U}}(1) \\ \longrightarrow G'_F &= \mathrm{SU}(N)_{\mathrm{cf}} \times \mathrm{SU}(4)_\eta \times \mathrm{U}(1)'_{\psi\eta}\,, \end{aligned} \tag{C.1}$$

where $\mathrm{U}(1)'_{\psi\eta}$ is generated by an appropriate linear combination of the SU($N+4$) generator, $\begin{pmatrix} 4\,\mathbf{1}_N & \\ & -N\,\mathbf{1}_4 \end{pmatrix}$ and that of $\mathrm{U}(1)_{\psi\eta}$. The fermions in the UV are decomposed into the sum of irreducible representations of the unbroken group, in table 6. The baryons which remain massless among those in table 4 are listed in table 7.

Finally, we note that both $\mathrm{U}(1)_0$ and $\tilde{U}(1)$ of the X ray model, (2.1)–(2.7), and hence the color-flavor locked $\mathbb{Z}_N \subset \mathrm{SU}(N)_{\mathrm{c}} \times (\tilde{\mathrm{U}}(1) \times \mathrm{U}(1)_0)$ itself, are spontaneously broken by the bifermion condensates, (3.22). It follows that the mixed anomalies found in the X-ray

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model in section 3, are perfectly consistent with the physics of the dynamical Higgs phase, in contrast to the case of the confining, chirally symmetric phase discussed in appendix. B.

Now, unlike the somewhat mysterious matching equations in the hypothetical confining phase (as those fully exposed in [8]), the conventional, 't Hooft anomaly matching constraints with respect to the unbroken group G'_F in the dynamical Higgs phase are trivially satisfied, as can be seen by inspection of table 6 and table 7.[24]

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