Taming Bounded Depth with Nested Sequents

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Abstract

Bounded depth refers to a property of Kripke frames that serve as semantics for intuitionistic logic. We introduce nested sequent calculi for the intermediate logics of bounded depth. Our calculi are obtained in a modular way by adding suitable structural rules to a variant of Fitting's calculus for intuitionistic propositional logic, for which we present the first syntactic cut elimination proof. This proof modularly extends to the new nested sequent calculi introduced in this paper.

Keywords: Nested Sequents, Cut Elimination, Intermediate Logics, Bounded Depth Kripke Models.

1 Introduction

Nested sequents are a natural extension of ordinary Gentzen sequents, (re)discovered several times in different contexts [15,28,2]. Whereas Gentzen sequents are lists (or multisets) of formulas, nested sequents are trees of multisets of formulas. The tree structure makes nested sequents well-suited to handle logics having Kripke-style semantics. Indeed, nested sequents have been employed to provide internal analytic calculi for modal logics for which this was not possible before, for example, for the modal logic KB in [2]. Here, *internal* means that there are no "external" semantic constructs (like labels) in the syntax, and *analytic* means that all derivations in these calculi have the property that they only contain formulas which are subformulas of the conclusion (subformula property). This feature of analytic calculi renders them a natural starting point for proving meta-logical properties as decidability, complexity, and interpolation, and for developing automated reasoning methods.

Nested sequents have been effective for classical modal logics in the S5-cube [2], as well as for intuitionistic modal logics in the IS5-cube [29,25,17], and more generally, for all extensions of IK with Horn-Scott-Lemmon axioms [22]. Furthermore, they have been used to provide focused proof systems for classical and intuitionistic modal logics [3,4], to construct interpolation proofs [21], to implement proof search [11,27], and to "tame" modal logics with path-axioms [12], where "taming" means to have proof systems suited for proof search. In all these calculi, the tree structure of the nested sequents corresponds to the accessibilty relation between the worlds in the Kripke frame.

An analytic nested calculus has also been proposed for intuitionistic propositional logic (IPL) by Fitting [8]; there the tree structure of the nested sequents corresponds to the future relation between the worlds in the Kripke frame. His calculus, that has been obtained as a notational variant of prefixed tableaux, is not equipped with a direct cut elimination proof, which seemed hard to define.

This leads to the first contribution of this paper. We import insights from the nested sequent calculi for modal logics to introduce NIPL — a variant [1] of Fitting's calculus for IPL that allows for a direct cut elimination proof. In the spirit of Belnap's conditions for cut elimination in display calculi [1], our proof relies on the abstract conditions (N1)–(N5), which will be presented in Section 3. The rules of NIPL are all invertible, and therefore NIPL offers a purely formula-driven approach to proof search.

The next question we address is: Can we extend NIPL by additional structural rules to capture intermediate logics (i.e., logics between intuitionistic and classical logic), in a similar way as the nested systems for the modal logics K and IK are extended with additional rules corresponding to additional axioms [25,13]? Moreover, can this be done in such a way that the cut elimination property is preserved, and can this be proved in a modular way, i.e., can we reuse the existing proof and only add the cases concerning the new rules? [2]

For hypersequents—which are disjunctions of ordinary sequents—we have such results, provided the additional axioms for intermediate logics follow a certain shape [5]. This is not the case for the family of intermediate logics characterized by Kripke models of bounded depth, usually denoted by $\mathsf{BD_n}$. The logic $\mathsf{BD_1}$ is just classical logic and the logic $\mathsf{BD_2}$ is one of the seven interpolable intermediate logics [24]. Analytic hypersequent calculi for $\mathsf{BD_n}$ are indeed provably not obtainable [18] by using the methodology in [5] which extends the base hypersequent calculus for IPL with suitable structural rules corresponding to the additional axioms.

Modular analytic calculi for $\mathsf{BD_n}$ have been defined using frameworks more powerful than hypersequents (and nested sequents); however, the objects that the resulting calculi manipulate (labelled or display sequents [26,6]) cannot be translated into formulas of the logic, and hence there is no real subformula property even when the calculus is analytic; moreover the rules of these calculi are not invertible.

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[1] NIPL is a notational variant of the calculus in [23].

[2] Girard has argued in [10] that the lack in modularity is one of the main technical limitations in structural proof theory.

This brings us to our second contribution: modular nested sequent calculi for the logics $\mathsf{BD_n}$ (for all $n \geq 1$). The calculi are obtained by adding suitable structural rules to NIPL. We show that the cut elimination proof for NIPL scales to the new systems, as our conditions (N1)–(N5) are preserved. It is interesting to note that the additional rules for $\mathsf{BD_n}$, with $n \geq 2$, have more than one premise. To our knowledge, this is a novelty. So far, there are no nested sequent systems with multi-premise structural rules.

2 Preliminaries

The *formulas* of intuitionistic propositional logic (IPL), denoted by $A, B, C, \ldots$, are generated from a countable set of atoms $\{p, q, \ldots\}$ via the grammar

$$A ::= p \mid \bot \mid A \vee A \mid A \wedge A \mid A \supset A$$

We define the *degree* of a formula to be the number of connectives in it. A *nested sequent* is a finite tree of multisets of formulas. In ordinary sequents for intuitionistic logic we distinguish between the left and the right hand side of the turnstile. To make this distinction in nested sequents, we use polarities on formulas. There are two polarities, input (intuitively as if on the left of the turnstile in the conventional sequent calculus), denoted by a $\bullet$ superscript and output (intuitively as if on the right of the turnstile), denoted by a $\circ$ superscript. Now, a nested sequent can be written as:

$$\Gamma = A_1^\bullet, ..., A_m^\bullet, B_1^\circ, ..., B_n^\circ, [\Gamma_1], ..., [\Gamma_k] \tag{1}$$

where $A_1^\bullet, ..., A_m^\bullet, B_1^\circ, ..., B_n^\circ$ is the multiset of formulas at the root of the sequent tree of Γ, and where $\Gamma_1, \ldots, \Gamma_k$ are its immediate subtrees. We use $\varnothing$ to denote the *empty sequent*, i.e., where $m = n = k = 0$ in (1) above. We use capital Greek letters Γ, Δ, Σ, $\ldots$, to denote nested sequents, and we assume that the associativity and commutativity of the comma is implicit in our systems, and that $\varnothing$ acts as its unit. We write $\Gamma^\bullet$ for $A_1^\bullet, ..., A_m^\bullet$ and Γ° for $B_1^\circ, ..., B_n^\circ, [\Gamma_1], ..., [\Gamma_k]$ if Γ is as in (1) above. In other words, for every nested sequent Γ we have that $\Gamma = \Gamma^\bullet, \Gamma^\circ$. More generally, we will write $\Gamma^\bullet$, $\Delta^\bullet$, $\Sigma^\bullet$, $\ldots$, for multisets of input formulas (i.e., all formulas have $\bullet$-polarity, and there are no nestings), and we will write Γ°, Δ°, Σ°, $\ldots$, for sequents that have only $\circ$-formulas at their root nodes (i.e., there are no $\bullet$-formulas at the root, but there can be nestings with $\bullet$-formulas inside).

The *corresponding formula* of the sequent in (1) above is defined as

$$fm(\Gamma) = \bigwedge_{i=1}^{m} A_i \supset \left(\bigvee_{j=1}^{n} B_j \vee \bigvee_{l=1}^{k} fm(\Gamma_l) \right) \tag{2}$$

A *(sequent) context* is a nested sequent with a hole $\{\ \}$, taking the place of a formula. Contexts are denoted by $\Gamma\{\ \}$, and $\Gamma\{\Delta\}$ is the sequent obtained from $\Gamma\{\ \}$ by replacing the occurrence of $\{\ \}$ with Δ. We write $\Gamma\{\varnothing\}$ for the sequent obtained from $\Gamma\{\ \}$ by removing the $\{\ \}$ (i.e., the hole is filled with

Initial Sequents

$$\frac{}{\Gamma\{p^\bullet, \Delta\{p^\circ\}\}}\ \mathsf{ax} \qquad\qquad \frac{}{\Gamma\{\bot^\bullet\}}\ \bot^\bullet$$

Logical Rules

$$\frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}\ \wedge^\bullet \qquad\qquad \frac{\Gamma\{A^\circ\} \qquad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}\ \wedge^\circ$$

$$\frac{\Gamma\{A^\bullet\} \qquad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}}\ \vee^\bullet \qquad\qquad \frac{\Gamma\{A^\circ, B^\circ\}}{\Gamma\{A \vee B^\circ\}}\ \vee^\circ$$

$$\frac{\Gamma\{A \supset B^\bullet, \Delta\{\Sigma, A^\circ\}\} \qquad \Gamma\{A \supset B^\bullet, \Delta\{\Sigma, B^\bullet\}\}}{\Gamma\{A \supset B^\bullet, \Delta\{\Sigma\}\}}\ \supset^\bullet \qquad \frac{\Gamma\{[A^\bullet, B^\circ]\}}{\Gamma\{A \supset B^\circ\}}\ \supset^\circ$$

Figure 1. The calculus NIPL

$$\frac{\Gamma\{A^\circ\} \qquad \Gamma\{A^\bullet\}}{\Gamma\{\varnothing\}}\ \mathsf{cut}$$

Figure 2. The cut rule

$$\frac{\Gamma\{\varnothing\}}{\Gamma\{\Delta\}}\ \mathsf{w} \quad \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}}\ \mathsf{c} \quad \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}}\ \mathsf{t} \quad \frac{\Gamma\{\Sigma^\circ\}}{\Gamma\{[\Sigma^\circ]\}}\ \mathsf{4} \quad \frac{\Gamma\{[\Sigma^\bullet, \Delta]\}}{\Gamma\{\Sigma^\bullet, [\Delta]\}}\ \mathsf{lift} \quad \frac{\Gamma\{\Sigma^\circ, [\Delta]\}}{\Gamma\{[\Sigma^\circ, \Delta]\}}\ \mathsf{lower}$$

Figure 3. Admissible structural rules

nothing). The depth of a context $\Gamma\{\ \}$, denoted by $depth(\Gamma\{\ \})$, is the length of the path in the sequent tree from the root to the hole $\{\ \}$. It is defined inductively as follows: $depth(\{\ \}) = 0$ and $depth(\Gamma', \Gamma\{\ \}) = depth(\Gamma\{\ \})$ and $depth([\Gamma\{\ \}]) = depth(\Gamma\{\ \}) + 1$.

We will also use the notation $\Gamma\{\!\{\Delta\}\!\}$ as abbreviation for $\Gamma\{[\Delta]\}$.

Example 2.1 Let $\Gamma\{\ \} = A^\bullet, B^\circ, [\{\ \}, [D^\bullet, C^\circ]]$. We have that

$$\Gamma\{B^\circ\} = A^\bullet, B^\circ, [B^\circ, [D^\bullet, C^\circ]]$$

and

$$\Gamma\{\varnothing\} = A^\bullet, B^\circ, [[D^\bullet, C^\circ]].$$

Let $\Delta = F^\bullet, [G^\circ]$, then

$$\Gamma\{\Delta\} = A^\bullet, B^\circ, [F^\bullet, [G^\circ], [D^\bullet, C^\circ]]$$

and

$$\Gamma\{\!\{\Delta\}\!\} = A^\bullet, B^\circ, [[F^\bullet, [G^\circ]], [D^\bullet, C^\circ]].$$

3 Nested Sequent Calculus for IPL

An elegant nested sequent calculus for Intuitionistic Propositional Logic IPL was introduced by Fitting [8], as a notational variant of prefixed tableaux. The lack of a direct cut elimination proof in his calculus has prevented its extension to cover intermediate logics. Indeed, to the best of our knowledge there are no analytic nested calculi for any intermediate logic (other than classical and intuitionistic logic). Even methods to extract such calculi from more powerful frameworks (like the structural refinement method in [22,20] that used labelled calculi as starting point), do not seem to work for intermediate logics. [3] In order to define analytic nested calculi for intermediate logics, the nested sequent formalism has been extended in various ways, giving rise to, e.g., linear nested calculi [19], and injective nested calculi [16].

In general, proving syntactic cut elimination for nested calculi is harder than for other proof theoretic formalisms, e.g., (hyper)sequent or display calculus. Often this result is obtained by translating the nested calculus at hand to other formalisms, as e.g. in [13,8]. The few existing cut elimination proofs for nested calculi are indeed tailored to specific systems [28,2,25,29], and their proofs do not seem to be generalizable (in particular, to deal with multi-premise structural rules).

In this section we present NIPL, a variant of Fitting's calculus for IPL designed to have all invertible rules, and to admit a direct cut elimination proof. The system NIPL, whose rules are shown in Figure 1, is obtained from Fitting's calculus by using multisets instead of sets and by absorbing the rule lift into the initial sequents and the rule $\supset^\bullet$. Observe indeed that ax and $\supset^\bullet$ can be simulated in Fitting's calculus by repeated applications of lift. As an immediate consequence we obtain the soundness of NIPL with respect to IPL.

Terminology: As in standard sequent calculi, we call *context* the part left unchanged from premises to conclusions, we call *principal* the introduced formula in a logical rule, and the rest *active* part/formulas (active formulas in the initial sequents are $p^\bullet$, p°, and $\bot^\bullet$).

As we will show in the next section, NIPL satisfies the following properties that guarantee a relatively simple proof of the elimination of the cut rule depicted in Figure 2.

(N1) *All rules are height-preserving invertible.*

(N2) *Dedicated structural rules are height-preserving admissible.* These rules, displayed in Figure 3, are the usual weakening (w) and contraction (c), the lift-rule from [8], variations of the rules for the modal axioms t and 4, from [25], and the new lower-rule which can be seen as the inverse of lift.

(N3) *A cut over formulas that are not principal can be shifted upwards over its premises.* This condition is implied by Belnap's sufficient conditions (C2)–(C7) for cut elimination in display calculi [1].

(N4) *All logical rules are reductive.* This means that they allow the replacement of a cut whose cut formula is principal in the left and right premise of the cut rule by cuts on smaller formulas (possibly using the dedicated structural rules from (N2)). This property is the nested sequent formulation of Belnap's (C8) condition [1].

[3] See also [14] for the correspondence between labeled systems and nesed sequents.

(N5) *Cuts having an initial sequent as one of their premises can be removed.*

Let us mention two useful features of NIPL. The first is standard in well-designed sequent-style calculi: the general form of the ax-rule is derivable.

Lemma 3.1 (Axiom expansion) *The sequent* $\Gamma\{A^\bullet, \Pi\{A^\circ, \Delta\}\}$ *is derivable in* NIPL *for every context* Γ, Π, Δ *and every formula* A.

Proof By induction on the degree of the formula A. We detail the case in which A is of the shape $B \supset C$, the other cases being similar.

$$\dfrac{\dfrac{\Gamma\{B \supset C^\bullet, \Pi\{[B^\bullet, B^\circ, C^\circ], \Delta\}\} \qquad \Gamma\{B \supset C^\bullet, \Pi\{[B^\bullet, C^\bullet, C^\circ], \Delta\}\}}{\Gamma\{B \supset C^\bullet, \Pi\{[B^\bullet, C^\circ], \Delta\}\}}\supset^\bullet}{\Gamma\{B \supset C^\bullet, \Pi\{B \supset C^\circ, \Delta\}\}}\supset^\circ$$

The premises are derivable by induction hypothesis. □

The second feature concerns the admissibility of the necessitation rule

$$\frac{\Gamma}{[\Gamma]}\ \mathsf{nec}$$

which will be used in the heuristic for generating the structural rules for $\mathsf{BD_n}$ in Section 5. Note that unlike all other rules, nec is shallow, as it cannot be applied inside a context.

Proposition 3.2 *If a sequent* Γ *is derivable, then so is* $[\Gamma]$.

Proof The proof is by induction on the height n of the derivation of Γ. If Γ is an initial sequent, so is $[\Gamma]$. If $n > 0$, then apply the induction hypothesis to the premise(s) of the rule and then rule again. □

4 Cut elimination for NIPL

We are going to show that NIPL satisfies conditions (N1)–(N5) and how these conditions entail the cut elimination theorem.

The preservation of the height of a derivation is crucial for all our arguments. Formally, the *height* of a derivation is the length of the longest path in the tree from its root to one of its leaves. A inference rule with premises $\Gamma_1, \ldots, \Gamma_n$ and conclusion Γ is *height-preserving invertible*, if for every derivation of Γ, there are derivations of each of $\Gamma_1, \ldots, \Gamma_n$ with at most the same height. The rule is *height-preserving admissible* if, whenever the premises are derivable, the conclusion has a derivation whose height is not bigger than any derivation of a premise.

Lemma 4.1 *The weakening rule* w *is height-preserving admissible in* NIPL.

Proof By induction on the height n of the derivation of $\Gamma\{\varnothing\}$. If $n = 0$, then $\Gamma\{\varnothing\}$ is an initial sequent and so is $\Gamma\{\Delta\}$. If $n > 0$, we apply the induction hypothesis to the premise(s) of the last rule applied and then the rule again. □

Lemma 4.2 *Every rule in* NIPL *is height-preserving invertible.*

Proof By induction on the height n of the derivation of the conclusion of each rule. The proofs for conjunction and disjunction are standard. The rule $\supset^\bullet$ is height-preserving invertible by the height-preserving admissibility of the rule of weakening. We discuss the rule $\supset^\circ$. If $\Gamma\{A \supset B^\circ\}$ is an initial sequent, then $\Gamma\{[A^\bullet, B^\circ]\}$ is an initial sequent too. If $n > 0$, then we apply the induction hypothesis to each of the premise(s) and then we apply the rule again. $\square$

Lemma 4.3 *The contraction rule* c *is height-preserving admissible in* NIPL*.*

Proof By induction on the height n of the derivation. If $\Gamma\{\Delta, \Delta\}$ is an initial sequent, the conclusion easily follows. If $n > 0$ and the principal formula is not in Δ, we apply the induction hypothesis to each of the premises and then the rule again. If $n > 0$ and the principal formula is in Δ we exploit the height-preserving invertibility of the logical rules as shown below, where ρ stands for an arbitrary rule instance and $\bar{\rho}$ stands for its inversion (which does not count for the overall height, as ρ is heigt-preserving invertible):

$$\dfrac{\dfrac{\Gamma\{\Delta', \Delta\}}{\Gamma\{\Delta, \Delta\}}\,\rho}{\Gamma\{\Delta\}}\,\mathsf{c} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\Gamma\{\Delta', \Delta\}}{\Gamma\{\Delta', \Delta'\}}\,\bar{\rho}}{\Gamma\{\Delta'\}}\,\mathsf{c}}{\Gamma\{\Delta\}}\,\rho$$

The application of c is removed invoking the induction hypothesis. The case where ρ is a binary rule is analogous and we omit the details. $\square$

The way we formulated the rules in NIPL allows us to establish the admissibility of the lift-rule. A variant of this rule was instead explicitly present in Fitting's system. Its absence (in combination with w and c) permits the use of the additive version of cut, which simplifies the cut elimination argument.

Lemma 4.4 *The* lift*-rule is height-preserving admissible in* NIPL*.*

Proof Proceed by induction on the height n of the derivation of the premise $\Gamma\{[\Sigma^\bullet, \Delta]\}$ of the rule. If $n = 0$ and no formula in $\Sigma^\bullet$ is active, then we can remove it. Otherwise, $\Gamma\{\Sigma^\bullet, [\Delta]\}$ is again an instance of ax. If $n > 0$ and no formula in Σ is principal, we apply the induction hypothesis to the premise(s) of the rule and then the rule again.
If a formula $A^\bullet$ in $\Sigma^\bullet$ is principal in $\wedge^\bullet$ or $\vee^\bullet$, we apply the induction hypothesis (possibly twice). E.g.,

$$\dfrac{\Gamma\{[\Sigma'^\bullet, A^\bullet, B^\bullet, \Delta]\}}{\Gamma\{[\Sigma'^\bullet, A \wedge B^\bullet, \Delta]\}}\,\wedge^\bullet \quad\rightsquigarrow\quad \dfrac{\dfrac{\Gamma\{[\Sigma'^\bullet, A^\bullet, B^\bullet, \Delta]\}}{\Gamma\{\Sigma'^\bullet, A^\bullet, B^\bullet, [\Delta]\}}\,\mathsf{lift}}{\Gamma\{\Sigma'^\bullet, A \wedge B^\bullet, [\Delta]\}}\,\wedge^\bullet$$

If a formula $A^\bullet$ in $\Sigma^\bullet$ is principal in $\supset^\bullet$ as in

$$\dfrac{\Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, \Delta\{\Pi, A^\circ\}]\} \qquad \Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, \Delta\{\Pi, B^\bullet\}]\}}{\Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, \Delta\{\Pi\}]\}}\,\supset^\bullet$$

we apply the induction hypothesis and the rule $\supset^\bullet$, as in

$$\dfrac{\dfrac{\Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, \Delta\{\Pi, A^\circ\}]\}}{\Gamma\{\Sigma'^\bullet, A \supset B^\bullet, [\Delta\{\Pi, A^\circ\}]\}}\,\mathsf{lift} \qquad \dfrac{\Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, \Delta\{\Pi, B^\bullet\}]\}}{\Gamma\{\Sigma'^\bullet, A \supset B^\bullet, [\Delta\{\Pi, B^\bullet\}]\}}\,\mathsf{lift}}{\Gamma\{\Sigma'^\bullet, A \supset B^\bullet, [\Delta\{\Pi\}]\}}\,\supset^\bullet$$

□

Note that with the admissibility of the lift-rule we immediately obtain completeness of NIPL with respect to IPL via Fitting's system [8].

Lemma 4.5 *The* $\mathsf{4}$*-rule is height-preserving admissible in* NIPL*.*

Proof By induction on the height n of the derivation of the rule premise. If $\Gamma\{\Sigma^\circ\}$ is an initial sequent, then so is $\Gamma\{[\Sigma^\circ]\}$. If $n > 0$ we assume that a formula in Σ° is principal, otherwise the proof is trivial. We apply the induction hypothesis to the premise(s) of the rule and then the rule again. For example, if the last rule applied is $\supset^\circ$, we have:

$$\dfrac{\Gamma\{\Delta, [A^\bullet, B^\circ]\}}{\Gamma\{\Delta, A \supset B^\circ\}}\ {\supset^\circ} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\Gamma\{\Delta, [A^\bullet, B^\circ]\}}{\Gamma\{[\Delta, [A^\bullet, B^\circ]]\}}\ \mathsf{4}}{\Gamma\{[\Delta, A \supset B^\circ]\}}\ {\supset^\circ}$$

□

Lemma 4.6 *The* lower*-rule is height-preserving admissible in* NIPL*.*

Proof The lower-rule is derivable with the following height-preserving steps:

$$\dfrac{\dfrac{\dfrac{\Gamma\{\Sigma^\circ, [\Delta]\}}{\Gamma\{[\Sigma^\circ], [\Delta]\}}\ \mathsf{4}}{\Gamma\{[\Sigma^\circ, \Delta], [\Sigma^\circ, \Delta]\}}\ \mathsf{w}}{\Gamma\{[\Sigma^\circ, \Delta]\}}\ \mathsf{c}$$

□

Lemma 4.7 *The* t*-rule is height-preserving admissible in* NIPL*.*

Proof By induction on the height n of the premise $\Gamma\{[\Delta]\}$. If $n = 0$, then $\Gamma\{[\Delta]\}$ is an initial sequent and so is $\Gamma\{\Delta\}$. If $n > 0$, we apply the induction hypothesis to the premise(s) and then the rule again. As an example, consider the case in which the last rule applied is $\supset^\bullet$ and formulas are introduced (bottom-up) in $[\Delta]$. We have:

$$\dfrac{\Gamma\{A \supset B^\bullet, [\Delta, A^\circ]\} \qquad \Gamma\{A \supset B^\bullet, [\Delta, B^\bullet]\}}{\Gamma\{A \supset B^\bullet, [\Delta]\}}\ {\supset^\bullet}$$

We construct the following derivation:

$$\dfrac{\dfrac{\Gamma\{A \supset B^\bullet, [\Delta, A^\circ]\}}{\Gamma\{A \supset B^\bullet, \Delta, A^\circ\}}\ \mathsf{t} \qquad \dfrac{\Gamma\{A \supset B^\bullet, [\Delta, B^\bullet]\}}{\Gamma\{A \supset B^\bullet, \Delta, B^\bullet\}}\ \mathsf{t}}{\Gamma\{A \supset B^\bullet, \Delta\}}\ {\supset^\bullet}$$

where the applications of t are removed by induction hypothesis. □

This completes the proof of the properties (N1) and (N2). To eliminate cut, we also need (N3)–(N5), which will be shown below.

Theorem 4.8 (Cut elimination) *The* cut*-rule is admissible for* NIPL*.*

Proof We consider a uppermost cut and proceed by induction on the lexicographically ordered pair (c, n) where c is the degree of its cut formula and n is the height of the derivation of $\Gamma\{A^\bullet\}$.[4]

(N5) If $n = 0$, then $\Gamma\{A^\bullet\}$ is an initial sequent. If $A^\bullet$ is not active, $\Gamma\{\varnothing\}$ is an initial sequent too. If $A^\bullet$ is active in ax, we have $A^\bullet = p^\bullet$ for some p, and

$$\dfrac{\Gamma\{p^\circ, \Delta\{p^\circ\}\} \qquad \dfrac{}{\Gamma\{p^\bullet, \Delta\{p^\circ\}\}}\,\mathsf{ax}}{\Gamma\{\Delta\{p^\circ\}\}}\,\mathsf{cut}$$

The cut is eliminated as follows:

$$\dfrac{\dfrac{\Gamma\{p^\circ, \Delta\{p^\circ\}\}}{\Gamma\{\Delta\{p^\circ, p^\circ\}\}}\,\mathsf{lower}}{\Gamma\{\Delta\{p^\circ\}\}}\,\mathsf{c}$$

The case of the axiom $\bot^\bullet$ is handled similarly, noticing that from the derivability in NIPL of $\Gamma\{\bot^\circ\}$ follows the derivability of $\Gamma\{\varnothing\}$.

(N3) If $n > 0$ and $A^\bullet$ is not principal, we apply the invertibility of the corresponding rule to $\Gamma\{A^\circ\}$, permute the cut upwards, and remove it by secondary induction hypothesis. For example, consider the following derivation, where ρ is some binary rule from NIPL:

$$\dfrac{\Gamma\{A^\circ\} \qquad \dfrac{\Gamma'\{A^\bullet\} \qquad \Gamma''\{A^\bullet\}}{\Gamma\{A^\bullet\}}\,\rho}{\Gamma\{\varnothing\}}\,\mathsf{cut}$$

We construct the following derivation, where we again use the height-preserving invertibility of ρ:

$$\dfrac{\dfrac{\dfrac{\Gamma\{A^\circ\}}{\Gamma'\{A^\circ\}}\,\bar{\rho} \qquad \Gamma'\{A^\bullet\}}{\Gamma'\{\varnothing\}}\,\mathsf{cut} \qquad \dfrac{\dfrac{\Gamma\{A^\circ\}}{\Gamma''\{A^\circ\}}\,\bar{\rho} \qquad \Gamma''\{A^\bullet\}}{\Gamma''\{\varnothing\}}\,\mathsf{cut}}{\Gamma\{\varnothing\}}\,\rho$$

(N4) If $A^\bullet$ is principal in $\wedge$ or $\vee$, the case is handled in the usual way using the invertibility of the rules. For example

$$\dfrac{\Gamma\{B \vee C^\circ\} \qquad \dfrac{\Gamma\{B^\bullet\} \qquad \Gamma\{C^\bullet\}}{\Gamma\{B \vee C^\bullet\}}\,\vee^\bullet}{\Gamma\{\varnothing\}}\,\mathsf{cut}$$

is eliminated as follows (where $\bar{\vee}^\circ$ is the inversion of $\vee^\circ$ and each cut is on a formula of lesser degree):

$$\dfrac{\dfrac{\dfrac{\Gamma\{B \vee C^\circ\}}{\Gamma\{B^\circ, C^\circ\}}\,\bar{\vee}^\circ \qquad \dfrac{\Gamma\{B^\bullet\}}{\Gamma\{B^\bullet, C^\circ\}}\,\mathsf{w}}{\Gamma\{C^\circ\}}\,\mathsf{cut} \qquad \Gamma\{C^\bullet\}}{\Gamma\{\varnothing\}}\,\mathsf{cut}$$

[4] It is enough to consider only the height of the left premise as every right rule is invertible.

The case below in which $A^\bullet$ is principal in $\supset^\bullet$:

$$
\dfrac{\Gamma\{B \supset C^\circ, \Pi\{\Sigma\}\} \qquad \dfrac{\Gamma\{B \supset C^\bullet, \Pi\{B^\circ, \Sigma\}\} \qquad \Gamma\{B \supset C^\bullet, \Pi\{C^\bullet, \Sigma\}\}}{\Gamma\{B \supset C^\bullet, \Pi\{\Sigma\}\}}\ \supset^\bullet}{\Gamma\{\Pi\{\Sigma\}\}}\ \mathsf{cut}
$$

is handled using some of the structural rules from (N2). We first construct a derivation of $\Gamma\{\Pi\{B^\circ, \Sigma\}\}$:

$$
\dfrac{\dfrac{\Gamma\{B \supset C^\circ, \Pi\{\Sigma\}\}}{\Gamma\{B \supset C^\circ, \Pi\{B^\circ, \Sigma\}\}}\ \mathsf{w} \qquad \Gamma\{B \supset C^\bullet, \Pi\{B^\circ, \Sigma\}\}}{\Gamma\{\Pi\{B^\circ, \Sigma\}\}}\ \mathsf{cut}
$$

The cut is removed by secondary induction hypothesis. A symmetrical derivation yields $\Gamma\{\Pi\{C^\bullet, \Sigma\}\}$, and the reduction is completed as follows:

$$
\dfrac{\dfrac{\dfrac{\Gamma\{\Pi\{B^\circ, \Sigma\}\}}{\Gamma\{\Pi\{B^\circ, C^\circ, \Sigma\}\}}\ \mathsf{w} \qquad \dfrac{\dfrac{\dfrac{\Gamma\{B \supset C^\circ, \Pi\{\Sigma\}\}}{\Gamma\{\Pi\{B \supset C^\circ, \Sigma\}\}}\ \mathsf{lower}}{\Gamma\{\Pi\{[B^\bullet, C^\circ], \Sigma\}\}}\ \bar{\supset}^\circ}{\Gamma\{\Pi\{B^\bullet, C^\circ, \Sigma\}\}}\ \mathsf{t}}{\Gamma\{\Pi\{C^\circ, \Sigma\}\}}\ \mathsf{cut} \qquad \Gamma\{\Pi\{C^\bullet, \Sigma\}\}}{\Gamma\{\Pi\{\Sigma\}\}}\ \mathsf{cut}
$$

□

We can now show completeness independently from Fitting's calculus:

Corollary 4.9 NIPL *is complete with respect to* IPL.

Proof It is easy to check that every axiom of IPL can be proved in NIPL and modus ponens can be simulated by cut. The claim follows by Theorem 4.8. □

5 Intermediate Logics of Bounded Depth

We introduce nested calculi for the propositional intermediate logics $\mathsf{BD_n}$ (Bounded Depth n) which are semantically characterized by intuitionistic Kripke frames in which every chain is of length less or equal to n. Our calculi are defined in a modular way by extending NIPL with suitable structural rules, which preserve conditions (N1)–(N5) and therefore cut elimination.

Definition 5.1 A *bounded depth n Kripke frame* is a pair $\langle P, \leq \rangle$, where P is a non empty set of *worlds*, denoted by $x, y, z, \ldots$, and $\leq$ is a partial order on P with

$$
\forall x_0 \ldots x_n. \left(\bigwedge_{0 \leq i \leq n-1} x_i \leq x_{i+1} \supset \bigvee_{0 \leq i \leq n-1} x_{i+1} \leq x_i \right). \tag{3}
$$

A *bounded depth Kripke model* is a triple $\langle P, \leq, v \rangle$, where $\langle P, \leq \rangle$ is a bounded depth Kripke frame and v a function which maps propositional atoms to subsets of P, such that for all worlds x and y, if $x \leq y$ and $x \in v(p)$ then

$y \in v(p)$ (monotonicity condition). Truth conditions for a formula in a world are defined as usual, see, e.g. [9]. A formula is *true in a model* if it is true in every world of the model. A formula is *valid* if it is true in every bounded depth Kripke model. A sequent is true in a world (resp. true in a model, resp. valid) if the corresponding formula is.

The logics $\mathsf{BD_n}$ are axiomatized by the schemata

$$\mathbf{bd_1} := p_1 \vee (p_1 \to \bot) \qquad \text{and} \qquad \mathbf{bd_{n+1}} := p_{n+1} \vee (p_{n+1} \to \mathbf{bd}_n)$$

where with an abuse of notation here and below we use $p_{(i)}, q$ as metavariables, to be substituted by arbitrary formulas. For each n, the logic $\mathsf{BD_n}$ is sound and complete with respect to the class of bounded depth n Kripke frames [9]. $\mathsf{BD_1}$ turns out to be classical logic, and $\mathsf{BD_2}$ one of the seven interpolable intermediate logics [24].

Before presenting the peculiar structural rule capturing $\mathsf{BD_n}$, we sketch its genesis, using $\mathsf{BD_2}$ as case study. The starting point for the rule's definition is (an adaptation to nested sequents of) the algorithm for transforming Hilbert axioms into structural hypersequent rules given in [5], and into display rules in [6,7]. Indeed, we start with the axiom schema $\mathbf{bd_2} := p \vee (p \supset (q \vee \neg q))$ characterizing $\mathsf{BD_2}$. By using the invertible rules of NIPL, the axiom is decomposed into the equivalent nested sequent in which all connectives are removed:

$$p^\circ, [p^\bullet, q^\circ, [q^\bullet]] \tag{4}$$

This sequent can be transformed into the equivalent (interderivable) rule

$$\frac{\Delta_0, [\Delta_0, \Delta_1, \Sigma, [\Delta_2]] \qquad \Delta_0, [\Delta_1, \Sigma, [\Delta_2, \Sigma]]}{\Delta_0, [\Delta_1, \Sigma, [\Delta_2]]}\; bd_2'$$

following the steps below:

- First, move p° and q° to the premise, as shown below left, and then $p^\bullet$ and $q^\bullet$, as in the rule below right (Δ_i, Σ are *fresh* metavariables for sequents):

$$\frac{\Sigma, q^\bullet \quad \Delta_0, p^\bullet}{\Delta_0, [p^\bullet, \Sigma, [q^\bullet]]} \qquad \frac{\Sigma, q^\bullet \quad \Delta_0, p^\bullet \quad \Delta_0, [\Delta_1, p^\circ, \Sigma, [\Delta_2]] \quad \Delta_0, [\Delta_1, \Sigma, [\Delta_2, q^\circ]]}{\Delta_0, [\Delta_1, \Sigma, [\Delta_2]]}\; r_2$$

  It is easy to see that each of these rules derives the sequent (4) in NIPL (instantiate $\Delta_0 := p^\circ$ and $\Delta_1 := p^\bullet$ and $\Sigma := q^\circ$ and $\Delta_2 := q^\bullet$). The converse direction follows by nec (Proposition 3.2), w, c and cut with the sequent in (4).

- The rule (bd_2') is obtained from the (r_2) rule above by cutting the premises of the latter in all possible ways (using first nec and w). This ensures that (bd_2') derives (r_2) in NIPL. For the converse direction we show that from any instance of the premisses of (bd_2') we can derive an instance of the premises of (r_2) for suitable p and q. Take $p := fm(\Delta_0^\circ)$, and $q := fm(\Sigma^\circ)$.

As shown below, the (bd_2') rule allows the derivation of the axiom $\mathbf{bd_2}$, and therefore its addition to NIPL result in a complete calculus for the $\mathsf{BD_2}$ logic.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^\circ, [A^\circ, A^\bullet, B^\circ, [B^\bullet]] \qquad A^\circ, [A^\bullet, B^\circ, [B^\bullet, B^\circ]]}{A^\circ, [A^\bullet, B^\circ, [B^\bullet]]}\, bd'_2}{A^\circ, [A^\bullet, B^\circ, \neg B^\circ]}\supset^\circ}{A^\circ, [A^\bullet, B \vee \neg B^\circ]}\vee^\circ}{A^\circ, A \supset (B \vee \neg B)^\circ}\supset^\circ}{A \vee (A \supset (B \vee \neg B))^\circ}\vee^\circ$$

However, (bd'_2) does not preserve cut elimination. Prior identifying Δ_1 and Σ, we modify (bd'_2) from shallow to *deep* (using the terminology in [13]), to permit its application on structures in any arbitrary node in the tree. This leads to:

$$\dfrac{\Gamma\{\Delta_0\{\!\!\{\Delta_0\{\varnothing\}, \Delta_1\{\!\!\{\Delta_2\}\!\!\}\}\!\!\}\} \qquad \Gamma\{\Delta_0\{\!\!\{\Delta_1\{\!\!\{\Delta_1\{\varnothing\}, \Delta_2\}\!\!\}\}\!\!\}\}}{\Gamma\{\Delta_0\{\!\!\{\Delta_1\{\!\!\{\Delta_2\}\!\!\}\}\!\!\}\}}\,\mathsf{bd_2}$$

The argument applies also to $\mathsf{BD_n}$ $(n \geq 1)$ resulting in a rule having n premises.

Definition 5.2 $\mathsf{NIPLBD_n}$ is the calculus NIPL extended with the rule $\mathsf{bd_n}$:

$$\dfrac{\Gamma\{\Delta_0\{\!\!\{\ldots\{\!\!\{\Delta_{i-1}\{\varnothing\}, \Delta_i\{\!\!\{\ldots\{\!\!\{\Delta_n\}\!\!\}\ldots\}\!\!\}\}\!\!\}\ldots\}\!\!\}\} \quad \Big| \quad 1 \leqslant i \leqslant n}{\Gamma\{\Delta_0\{\!\!\{\Delta_1\{\!\!\{\ldots\Delta_{n-1}\{\!\!\{\Delta_n\}\!\!\}\ldots\}\!\!\}\}\!\!\}\}}\,\mathsf{bd_n}$$

Proposition 5.3 *For every n, the rule $\mathsf{bd_n}$ is sound with respect to intuitionistic Kripke frames with depth bounded by n.*

Proof By contradiction. We assume that $\Gamma\{\Delta_0\{\!\!\{\ldots\{\!\!\{\Delta_n\}\!\!\}\ldots\}\!\!\}\}$ is not valid. Hence there are worlds $x_0, \ldots, x_n$ such that:

- $x_i \leqslant x_{i+1}$, with $0 \leqslant i \leqslant n-1$.
- $x_i \Vdash \bigwedge \Delta_i^\bullet$ and $x_i \nVdash \bigvee \Delta_i^\circ$.

where $\Delta_i^\bullet$,(Δ_i°) are the input (output) formulas (formulas and boxed sequents) in Δ_i. By (3) we have $x_{i+1} \leqslant x_i$ for some $i \in \{0, ..., n-1\}$. In each case, by monotonicity (w.r.t. compound formulas) we get $x_{i+1} \Vdash \bigwedge \Delta_i^\bullet$ and thus $x_{i+1} \Vdash \bigvee \Delta_i^\circ$, which (again by monotonicity) yields $x_i \Vdash \bigvee \Delta_i^\circ$, which is a contradiction. □

6 Cut Elimination for $\mathsf{NIPLBD_n}$

We show that cut elimination holds for $\mathsf{NIPLBD_n}$. The proof extends that for NIPL in a modular way: only the cases concerning the new rules need to be considered. We start by showing properties (N1) and (N2): invertibility of all rules, and height-preserving admissibility of the dedicated rules in Fig. 3.

Lemma 6.1 *The weakening rule is height-preserving admissible in* $\mathsf{NIPLBD_n}$.

Proof Proceeds as the proof of Lemma 4.1. □

Lemma 6.2 *Every rule is height-preserving invertible in* $\mathsf{NIPLBD_n}$.

Proof By induction on the height of the derivation for every rule of the system. The structural rule $\mathsf{bd_n}$ is height-preserving invertible by using Lemma 6.1. The addition of the rule $\mathsf{bd_n}$ preserves the invertibility of the other rules. The strategy consists in applying (possibly twice, due to the repetition of the contexts) the induction hypothesis to each premise of the rule. $\square$

Lemma 6.3 *The contraction rule is height-preserving admissible in* $\mathsf{NIPLBD_n}$.

Proof By induction on the height of the derivation. The only additional case concerns the rule $\mathsf{bd_n}$. Since the principal formulas are repeated in each premise of $\mathsf{bd_n}$ we just apply the induction hypothesis and then the rule again. We give a concrete example of this qualitative analysis; to improve the readability we consider the particular case of the $\mathsf{bd_2}$ rule.

$$\dfrac{\Gamma\{\Delta_0\{[\Delta_0\{\varnothing\}, \Delta_1\llbracket\Delta_2\rrbracket], [\Delta_1\llbracket\Delta_2\rrbracket]\}\} \qquad \Gamma\{\Delta_0\{[\Delta_1\llbracket\Delta_1\{\varnothing\}, \Delta_2\rrbracket], [\Delta_1\llbracket\Delta_2\rrbracket]\}\}}{\Gamma\{\Delta_0\{[\Delta_1\llbracket\Delta_2\rrbracket], [\Delta_1\llbracket\Delta_2\rrbracket]\}\}}\;\mathsf{bd_2}$$

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma\{\Delta_0\{[\Delta_0\{\varnothing\}, \Delta_1\llbracket\Delta_2\rrbracket], [\Delta_1\llbracket\Delta_2\rrbracket]\}\}}{\Gamma\{\Delta_0\{[\Delta_0\{\varnothing\}, \Delta_1\llbracket\Delta_2\rrbracket], [\Delta_0\{\varnothing\}, \Delta_1\llbracket\Delta_2\rrbracket]\}\}}\;\mathsf{w}}{\Gamma\{\Delta_0\{[\Delta_0\{\varnothing\}, \Delta_1\llbracket\Delta_2\rrbracket]\}\}}\;\mathsf{c} \qquad \dfrac{\dfrac{\Gamma\{\Delta_0\{[\Delta_1\llbracket\Delta_1\{\varnothing\}, \Delta_2\rrbracket], [\Delta_1\llbracket\Delta_2\rrbracket]\}\}}{\Gamma\{\Delta_0\{[\Delta_1\llbracket\Delta_1\{\varnothing\}, \Delta_2\rrbracket], [\Delta_1\llbracket\Delta_1\{\varnothing\}, \Delta_2\rrbracket]\}\}}\;\mathsf{w}}{\Gamma\{\Delta_0\{[\Delta_1\llbracket\Delta_1\{\varnothing\}, \Delta_2\rrbracket]\}\}}\;\mathsf{c}}{\Gamma\{\Delta_0\{[\Delta_1\llbracket\Delta_2\rrbracket]\}\}}\;\mathsf{bd_2}$$

$\square$

Lemma 6.4 *The rule* lift *is height-preserving admissible in* $\mathsf{NIPLBD_n}$.

Proof By induction on the height of the derivation. To simplify the notation we consider only $\mathsf{bd_2}$; the generalization to $\mathsf{bd_n}$ is immediate.

Let $\Gamma\{[\Sigma^\bullet, \Delta]\}$ be the conclusion of $\mathsf{bd_2}$, we need to consider two subcases. Either $[\Sigma^\bullet, \Delta]$ is moved by the rule or not. In the latter case we simply apply the induction hypothesis to the premises of the rule and then the rule again. In the former case we need to distinguish two further subcases. Either $[\Sigma^\bullet, \Delta]$ is Δ_i for $i \in \{0, 1, 2\}$ or not. In the latter case we apply the induction hypothesis to the premises and then the rule again. In the former case assume $[\Delta_1\{\ \}] = [\Sigma^\bullet, \Delta\{\ \}]$. We have:

$$\dfrac{\Gamma\{\Delta_0\llbracket\Delta_0\{\varnothing\}, \Sigma^\bullet, \Delta\llbracket\Delta_2\rrbracket\rrbracket\} \qquad \Gamma\{\Delta_0\llbracket\Sigma^\bullet, \Delta\llbracket\Sigma^\bullet, \Delta\{\varnothing\}, \Delta_2\rrbracket\rrbracket\}}{\Gamma\{\Delta_0\llbracket\Sigma^\bullet, \Delta\llbracket\Delta_2\rrbracket\rrbracket\}}\;\mathsf{bd_2}$$

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma\{\Delta_0\llbracket\Delta_0\{\varnothing\}, \Sigma^\bullet, \Delta\llbracket\Delta_2\rrbracket\rrbracket\}}{\Gamma\{\Delta_0\{\Sigma^\bullet, [\Delta_0\{\varnothing\}, \Delta\llbracket\Delta_2\rrbracket]\}\}}\;\mathsf{lift}}{\Gamma\{\Delta_0\{\Sigma^\bullet, [\Delta_0\{\Sigma^\bullet\}, \Delta\llbracket\Delta_2\rrbracket]\}\}}\;\mathsf{w} \qquad \dfrac{\dfrac{\Gamma\{\Delta_0\llbracket\Sigma^\bullet, \Delta\llbracket\Sigma^\bullet, \Delta\{\varnothing\}, \Delta_2\rrbracket\rrbracket\}}{\Gamma\{\Delta_0\{\Sigma^\bullet, \Sigma^\bullet, [\Delta\llbracket\Delta\{\varnothing\}, \Delta_2\rrbracket]\}\}}\;\text{several } \mathsf{lift}}{\Gamma\{\Delta_0\{\Sigma^\bullet, [\Delta\llbracket\Delta\{\varnothing\}, \Delta_2\rrbracket]\}\}}\;\mathsf{c}}{\Gamma\{\Delta_0\{\Sigma^\bullet, [\Delta\llbracket\Delta_2\rrbracket]\}\}}\;\mathsf{bd_2}$$

$\square$

Lemma 6.5 *The rule* 4 *is height-preserving admissible in* $\mathsf{NIPLBD_n}$.

Proof By induction on the height of the derivation. Assume the last applied rule is $\mathsf{bd_n}$. We observe that the general form of Σ° is $A_1^\circ, \ldots, A_n^\circ, [\Sigma_1], \ldots, [\Sigma_m]$. We distinguish three cases: either Σ° does not move, or it entirely moves to another boxed sequent, or different components of Σ° move to different sequents. In the first and second case we simply apply the induction hypothesis to the premises and then the rule again. In the latter case we have that Σ° is of shape $\Sigma^\circ = \Sigma'\{\!\!\{\Sigma''\{\Sigma'''\}\}\!\!\}$ and we have

$$\dfrac{\dfrac{\cdots \qquad \Gamma\{\Delta_0\{\!\!\{\ldots\{\!\!\{\Delta_i', \Sigma'\{\!\!\{\Delta_i', \Sigma'\{\varnothing\}, \Sigma''\{\ldots\{\!\!\{\Delta_n\}\!\!\}\ldots\}\}\!\!\}\ldots\}\!\!\}\}\!\!\}\} \qquad \cdots}{\Gamma\{\Delta_0\{\!\!\{\ldots\{\!\!\{\Delta_i', \Sigma'\{\!\!\{\Sigma''\{\ldots\{\!\!\{\Delta_n\}\!\!\}\ldots\}\}\!\!\}\ldots\}\!\!\}\}\!\!\}\}}\,\mathsf{bd_n}}{\Gamma\{\Delta_0\{\!\!\{\ldots\{\!\!\{\Delta_i', [\Sigma'\{\!\!\{\Sigma''\{\ldots\{\!\!\{\Delta_n\}\!\!\}\ldots\}\}\!\!\}]\ldots\}\!\!\}\}\!\!\}\}}\,\mathsf{4}$$

and we can proceed as follows:

$$\dfrac{\cdots \qquad \dfrac{\dfrac{\Gamma\{\Delta_0\{\!\!\{\ldots\{\!\!\{\Delta_i', \Sigma'\{\!\!\{\Delta_i', \Sigma'\{\varnothing\}, \Sigma''\{\ldots\{\!\!\{\Delta_n\}\!\!\}\ldots\}\}\!\!\}\ldots\}\!\!\}\}\!\!\}\}}{\Gamma\{\Delta_0\{\!\!\{\ldots\{\!\!\{\Delta_i', [\Sigma'\{\!\!\{\Delta_i', \Sigma'\{\varnothing\}, \Sigma''\{\ldots\{\!\!\{\Delta_n\}\!\!\}\ldots\}\}\!\!\}]\ldots\}\!\!\}\}\!\!\}\}}\,\mathsf{4}}{\Gamma\{\Delta_0\{\!\!\{\ldots\{\!\!\{\Delta_i', [\Sigma'\{\!\!\{\Delta_i', [\Sigma'\{\varnothing\}], \Sigma''\{\ldots\{\!\!\{\Delta_n\}\!\!\}\ldots\}\}\!\!\}]\ldots\}\!\!\}\}\!\!\}\}}\,\mathsf{4} \qquad \cdots}{\Gamma\{\Delta_0\{\!\!\{\ldots\{\!\!\{\Delta_i', [\Sigma'\{\!\!\{\Sigma''\{\ldots\{\!\!\{\Delta_n\}\!\!\}\ldots\}\}\!\!\}]\ldots\}\!\!\}\}\!\!\}\}}\,\mathsf{bd_n}$$

□

Lemma 6.6 *The rule* lower *is height-preserving admissible in* $\mathsf{NIPLBD_n}$.

Proof This proof is literally the same as for NIPL, applying 4, w and c. □

We now need to prove the admissibility of the t rule, which removes boxes. To simplify the proof in the presence of $\mathsf{bd_n}$, we consider the auxiliary rule lift^* below.

Lemma 6.7 *The following rule is derivable with* $\{\mathsf{t}, \mathsf{c}, \mathsf{4}, \mathsf{lift}, \mathsf{w}\}$.

$$\dfrac{\Gamma\{\Sigma\{\!\!\{\Delta, \Sigma\{\Delta\}\}\!\!\}\}}{\Gamma\{\Sigma\{\!\!\{\Delta, \Sigma\{\varnothing\}\}\!\!\}\}}\,\mathsf{lift}^*$$

Proof For better readability we only show the case where the depth of $\Sigma\{\ \}$ is 1, i.e., $\Sigma\{\ \}= \Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, \{\ \}]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma\{\Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, [\Delta, \Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, \Delta]]]\}}{\Gamma\{\Sigma_1, [{\Sigma_2}^\bullet, {\Sigma_2}^\circ, [\Delta, \Sigma_1, {\Sigma_2}^\bullet, {\Sigma_2}^\circ, \Delta]]\}}\,\mathsf{t}}{\Gamma\{\Sigma_1, [{\Sigma_2}^\bullet, {\Sigma_2}^\circ, [\Delta, \Sigma_1, {\Sigma_2}^\bullet, {\Sigma_2}^\circ]]\}}\,\mathsf{c}}{\Gamma\{\Sigma_1, [{\Sigma_2}^\bullet, {\Sigma_2}^\bullet, {\Sigma_2}^\circ, [\Delta, \Sigma_1, {\Sigma_2}^\circ]]\}}\,\mathsf{lift}}{\Gamma\{\Sigma_1, [{\Sigma_2}^\bullet, {\Sigma_2}^\circ, [\Delta, \Sigma_1, {\Sigma_2}^\circ]]\}}\,\mathsf{c}}{\Gamma\{\Sigma_1, [{\Sigma_2}^\bullet, {\Sigma_2}^\circ, [\Delta, \Sigma_1, [{\Sigma_2}^\circ]]]\}}\,\mathsf{4}}{\Gamma\{\Sigma_1, [{\Sigma_2}^\bullet, {\Sigma_2}^\circ, [\Delta, \Sigma_1, [{\Sigma_2}^\bullet, {\Sigma_2}^\circ]]]\}}\,\mathsf{w}$$

If the depth of $\Sigma\{\ \}$ is n, all steps in this derivation have to repeated n times.□

Lemma 6.8 *The rule* t *is height-preserving admissible in* $\mathsf{NIPLBD_n}$.

Proof The only new case to consider is the $\mathsf{bd_n}$-rule. Again, for the sake of clarity, we discuss $\mathsf{bd_2}$. We distinguish the following subcases: either we apply t to $[\Delta_i]$ for some i$\in \{1, 2\}$, or to some other boxed sequent. In the latter case, we apply the induction hypothesis (possibly twice) to the premises and then the rule again. In the former case we assume w.l.o.g. that we apply t to Δ_1:

$$\dfrac{\dfrac{\Gamma\{\Delta_0\{\!\{\Delta_0\{\varnothing\},\Delta_1\{\!\{\Delta_2\}\!\}\}\!\}\} \qquad \Gamma\{\Delta_0\{\!\{\Delta_1\{\!\{\Delta_1\{\varnothing\},\Delta_2\}\!\}\}\!\}\}}{\Gamma\{\Delta_0\{\!\{\Delta_1\{\!\{\Delta_2\}\!\}\}\!\}\}}\,\mathsf{bd_2}}{\Gamma\{\Delta_0\{\Delta_1\{\!\{\Delta_2\}\!\}\}\}}\,\mathsf{t}$$

We have two subcases to distinguish here:

- $[\Delta_1]$ is an immediate child of Δ_0 in the nested sequent tree. We have:

$$\dfrac{\dfrac{\Gamma\{\Delta_0,[\Delta_0,\Delta_1\{\!\{\Delta_2\}\!\}]\}}{\Gamma\{\Delta_0,\Delta_0,\Delta_1\{\!\{\Delta_2\}\!\}\}}\,\mathsf{t}}{\Gamma\{\Delta_0,\Delta_1\{\!\{\Delta_2\}\!\}\}}\,\mathsf{c}$$

  by applying the induction hypothesis to the left subproof, discarding the right one.

- $[\Delta_1]$ is not an immediate child of Δ_0 in the nested sequent tree, so we can assume that Δ_0 contains at least one nesting, thus we have:

$$\Delta_0\{\ \}=\Delta_0'\{\!\{\Delta_0''\{\ \}\}\!\}$$

  for some contexts $\Delta_0'\{\ \}$ and $\Delta_0''\{\ \}$. We construct the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma\{\Delta_0'\{\!\{\Delta_0'',[\Delta_0'\{\!\{\Delta_0''\{\varnothing\}\}\!\},\Delta_1\{\!\{\Delta_2\}\!\}]\}\!\}\}}{\Gamma\{\Delta_0'\{\!\{\Delta_0'',\Delta_0'\{\!\{\Delta_0''\{\varnothing\}\}\!\},\Delta_1\{\!\{\Delta_2\}\!\}\}\!\}\}}\,\mathsf{t}}{\Gamma\{\Delta_0'\{\!\{\Delta_0'',\Delta_0'\{\!\{\Delta_0''\{\varnothing\}\}\!\},\Delta_1\{\!\{\Delta_2\}\!\}\}\!\}\}}\,\mathsf{t}}{\Gamma\{\Delta_0'\{\!\{\Delta_0'',\Delta_0'\{\varnothing\},\Delta_1\{\!\{\Delta_2\}\!\}\}\!\}\}}\,\mathsf{lift}^* \qquad \dfrac{\dfrac{\Gamma\{\Delta_0'\{\!\{\Delta_0'',[\Delta_1\{\!\{\Delta_1\{\varnothing\},\Delta_2\}\!\}]\}\!\}\}}{\Gamma\{\Delta_0'\{\!\{\Delta_0'',\Delta_1\{\!\{\Delta_1\{\varnothing\},\Delta_2\}\!\}\}\!\}\}}\,\mathsf{t}}{\Gamma\{\Delta_0'\{\!\{\Delta_0'',\Delta_1\{\!\{\Delta_0'',\Delta_1\{\varnothing\},\Delta_2\}\!\}\}\!\}\}}\,\mathsf{w}}{\Gamma\{\Delta_0'\{\!\{\Delta_0''\{\Delta_1\{\!\{\Delta_2\}\!\}\}\}\!\}\}}\,\mathsf{bd_2}$$

  which yields the desired conclusion.

□

This completes the proofs of conditions (N1) and (N2).

Theorem 6.9 (Cut elimination) *The cut rule is eliminable in* $\mathsf{NIPLBD_n}$.

Proof The proof proceeds as for NIPL. We only need to check that the cut rule can be shifted up over $\mathsf{bd_n}$. That means we only have to verify (N3), as (N4) and (N5) are not affected by adding new structural rules to the system (and (N1) and (N2) still hold).

If the cut formula $A^\bullet$ is not active in $\mathsf{bd_n}$, we can permute the cut upwards and remove it by induction hypothesis. If $A^\bullet$ is active, assume w.l.o.g. that the cut formula is in $[\Delta_i]$. We distinguish two cases according to its position.

- If $A^\bullet$ is in the branch from Δ_0 to Δ_n, we have:

$$\dfrac{\Gamma\{\Delta_0\{\!\{A^\circ,\Delta_1\{\!\{\Delta_2\ldots\{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\} \qquad \dfrac{\ldots \quad \Gamma\{\Delta_0\{\!\{A^\bullet,\Delta_1\{\!\{A^\bullet,\Delta_1\{\varnothing\},\Delta_2\ldots\{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\} \quad \ldots}{\Gamma\{\Delta_0\{\!\{A^\bullet,\Delta_1\{\!\{\Delta_2\ldots\{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}\,\mathsf{bd_n}}{\Gamma\{\Delta_0\{\!\{\Delta_1\{\!\{\Delta_2\ldots\{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}\,\mathsf{cut}$$

  (we display the premise in which the position of $A^\bullet$ changes). For each premise different from $\Gamma\{\Delta_0\{\!\{A^\bullet,\Delta_1\{\!\{A^\bullet,\Delta_1\{\varnothing\},\Delta_2\ldots\{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}$, we proceed by height-preserving admissibility of w and cross-cuts which are removed by induction hypothesis. Then we construct the following derivation:

$$\dfrac{\dfrac{\Gamma\{\Delta_0\{\!\{A^\circ, \Delta_1\{\!\{\Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}{\Gamma\{\Delta_0\{\!\{A^\circ, \Delta_1\{\!\{\Delta_1\{\varnothing\}, \Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}\,\mathsf{w} \qquad \dfrac{\Gamma\{\Delta_0\{\!\{A^\bullet, \Delta_1\{\!\{A^\bullet, \Delta_1\{\varnothing\}, \Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}{\Gamma\{\Delta_0\{\!\{A^\bullet, \Delta_1\{\!\{\Delta_1\{\varnothing\}, \Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}\,\mathsf{lift}}{\Gamma\{\Delta_0\{\!\{\Delta_1\{\!\{\Delta_1\{\varnothing\}, \Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}\,\mathsf{cut}$$

to which we can now apply the rule $\mathsf{bd_n}$ and obtain the desired conclusion.

- If $A^\bullet$ is not in the branch from Δ_0 to Δ_n, then $A^\bullet$ must be inside a bracket in this branch, as in:

$$\dfrac{\Gamma\{\Delta_0\{\!\{[\Delta\{A^\circ\}], \Delta_1\{\!\{\Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\} \qquad \dfrac{\ldots \quad \Gamma\{\Delta_0\{\!\{[\Delta\{A^\bullet\}], \Delta_1\{\!\{[\Delta\{A^\bullet\}], \Delta_1\{\varnothing\}, \Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\} \quad \ldots}{\Gamma\{\Delta_0\{\!\{[\Delta\{A^\bullet\}], \Delta_1\{\!\{\Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}\,\mathsf{bd_n}}{\Gamma\{\Delta_0\{\!\{[\Delta\{\varnothing\}], \Delta_1\{\!\{\Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}\,\mathsf{cut}$$

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma\{\Delta_0\{\!\{[\Delta\{A^\circ\}], \Delta_1\{\!\{\Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}{\Gamma\{\Delta_0\{\!\{\Delta_1\{\!\{[\Delta\{A^\circ\}], \Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}\,\mathsf{lower}}{\Gamma\{\Delta_0\{\!\{\Delta_1\{\!\{[\Delta\{A^\circ\}], \Delta_1\{\varnothing\}, \Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}\,\mathsf{w} \qquad \dfrac{\Gamma\{\Delta_0\{\!\{[\Delta\{A^\bullet\}], \Delta_1\{\!\{[\Delta\{A^\bullet\}], \Delta_1\{\varnothing\}, \Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}{\Gamma\{\Delta_0\{\!\{\Delta_1\{\!\{[\Delta\{A^\bullet\}], \Delta_1\{\varnothing\}, \Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}\,\mathsf{lower,\ c}}{\Gamma\{\Delta_0\{\!\{\Delta_1\{\!\{[\Delta\{\varnothing\}], \Delta_1\{\varnothing\}, \Delta_2 \ldots \{\!\{\Delta_n\}\!\}\}\!\}\}\!\}\}}\,\mathsf{cut}$$

The cut is removed invoking the secondary induction hypothesis. With respect to the other premises we permute the cut upwards and we remove it applying the secondary induction hypothesis. The desired sequent then follows by an application of $\mathsf{bd_n}$. $\square$

Corollary 6.10 $\mathsf{NIPLBD_n}$ *is complete with respect to* $\mathsf{BD_n}$.

Proof Follows from Corollary 4.9, the derivability of the axiom $\mathbf{bd_n}$, and the cut elimination theorem. $\square$

Concluding Remark

The cut elimination proof contained in this paper makes use of the auxiliary rules t and $\mathsf{4}$, originally introduced in the modal logics context [25]. Our results can be seen as "transfer of knowledge" from (the proof theory of) modal logics to intermediate logics. We consider it an important aspect or our future research to also initiate a transfer back to modal logics. In fact, notice that our cut elimination proof holds for NIPL extended by any structural rule that preserves properties (N1)–(N5). This paves the way for the definition of an algorithm, along the line of that in [5,6], to introduce analytic nested calculi for a large class of intermediate logics starting from their axiomatizations. Many such intermediate logics have indeed modal counterparts that have not yet been investigated from a proof theoretical point of view. We plan to do so using our work on nested sequents.

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