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Weakly non-collapsed RCD spaces are strongly non-collapsed

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Abstract. We prove that any weakly non-collapsed RCD space is actually non-collapsed, up to a renormalization of the measure. This confirms a conjecture raised by De Philippis and the second named author in full generality. One of the auxiliary results of independent interest that we obtain is about the link between the properties

- $\mathrm{tr}(\mathrm{Hess}\, f) = \Delta f$ on $U \subseteq \mathsf{X}$ for every f sufficiently regular,
- $\mathfrak{m} = c\mathscr{H}^n$ on $U \subseteq \mathsf{X}$ for some $c > 0$,

where $U \subseteq \mathsf{X}$ is open and X is a – possibly collapsed – RCD space of essential dimension n.

1. Introduction

1.1. Main result and some comments. The modern theory of metric measure spaces with Ricci curvature bounded from below began with the seminal papers [52, 59, 60], where lower bounds on the Ricci curvature were imposed via suitable convexity properties of entropy functionals in the geometry of optimal transportation. It turned out that the resulting class $\mathrm{CD}(K, N)$ of spaces contains smooth Finslerian structures, a property that – at least for some geometric applications – is undesirable. This was one of the motivations that led the second named author to develop a research program about heat flow on CD spaces (see [3, 4, 28, 33]) that ultimately led in [30] to the definition of RCD spaces as those CD spaces for which the Sobolev space $W^{1,2}$ is Hilbert. We refer to [1,61] for an account of the theory and more detailed bibliography.

The study of (R)CD spaces has been strongly influenced by the research program carried out in the late nineties by Cheeger–Colding (see [19–22]) about the structure of Ricci limit spaces, i.e. those spaces arising as Gromov–Hausdorff limits of smooth Riemannian manifolds with constant dimension and a uniform lower bound on the Ricci curvature. One of the things that emerged by their analysis (strictly related to Colding's volume convergence theorem [23])

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is that the dimension of the limit space is always bounded from above by the dimension of the manifolds and that if equality holds, then the limit structure – that in this case is called *non-collapsed Ricci limit space* – has better regularity properties.

It is therefore natural to look for a synthetic counterpart of this concept, whose definition should not rely on properties of approximating sequences, but rather be based on intrinsic properties of the space in consideration. Inspecting the properties of non-collapsed Ricci limit spaces, in [24], the following definition has been proposed.

Definition 1.1. Let $K \in \mathbb{R}$ and $N \in [1,\infty)$. A space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is called *non-collapsed* $\mathrm{RCD}(K,N)$ space, $\mathrm{ncRCD}(K,N)$ in short, if it is an $\mathrm{RCD}(K,N)$ space and $\mathfrak{m} = \mathscr{H}^N$.

Notice that, from the structural properties of RCD spaces [36, 49, 53], it is clear that if $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is $\mathrm{ncRCD}(K,N)$, then N must be an integer. Also, subsequent analysis showed that, as expected, ncRCD spaces have better regularity properties than arbitrary RCD spaces (see for instance [13]).

The analysis carried out by Cheeger–Colding and the analogy with the study of the Bakry–Émery N-Ricci curvature tensor (see (1.3) and the subsequent discussion) suggest that in fact $\mathrm{ncRCD}(K,N)$ spaces should be identifiable among $\mathrm{RCD}(K,N)$ ones by properties seemingly weaker than $\mathfrak{m} = \mathscr{H}^N$. To be more precise, we need to introduce the N-dimensional (Bishop–Gromov) density $\theta_N[\mathsf{X},\mathsf{d},\mathfrak{m}]\colon \mathsf{X} \to [0,\infty]$ as

$$\theta_N[\mathsf{X},\mathsf{d},\mathfrak{m}](x) := \lim_{r\to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_N r^N} \quad \text{for all } x \in \mathsf{X},$$

where

$$\omega_N := \frac{\pi^{N/2}}{\int_0^\infty t^{N/2} e^{-t}\,\mathrm{d}t}$$

is, for $N \in \mathbb{N}$, the volume of the unit ball of $\mathbb{R}^N$ (notice that the existence of the limit is a consequence of the Bishop–Gromov inequality). It is worth pointing out that standard results about differentiation of measures ensure that if $\mathscr{H}^N$ is a Radon measure on X, then

$$\limsup_{r\to 0^+} \frac{\mathscr{H}^N(B_r(x))}{\omega_N r^N} \le 1 \quad \text{for } \mathscr{H}^N\text{-a.e. } x \in \mathsf{X}.$$

In particular, if X is a $\mathrm{ncRCD}(K,N)$ space, we have

$$\theta_N[\mathsf{X},\mathsf{d},\mathfrak{m}](x) \le 1 \quad \text{for all } x \in \mathsf{X}. \tag{1.1}$$

Then the following conjecture is raised in [24].

Conjecture 1.2 (De Philippis–Gigli). *If*

$$\mathfrak{m}\big(\{x \in \mathsf{X} : \theta_N[\mathsf{X},\mathsf{d},\mathfrak{m}](x) < \infty\}\big) > 0, \tag{1.2}$$

then $\mathfrak{m} = c\mathscr{H}^N$ *for some* $c \in (0,\infty)$. *In particular,* $(\mathsf{X},\mathsf{d},c^{-1}\mathfrak{m})$ *is a* $\mathrm{ncRCD}(K,N)$ *space.*

Let us make few comments about the statement of the conjecture and its validity. First of all, we remark that condition (1.2) cannot be replaced by the weaker one,

$$\{x \in \mathsf{X} : \theta_N[\mathsf{X},\mathsf{d},\mathfrak{m}](x) < \infty\} \neq \emptyset,$$

because for instance the metric measure space $([0,\pi], \mathsf{d}_{\mathbb{R}}, \sin^{N-1} t\, \mathrm{d}t)$ is an $\mathrm{RCD}(N-1,N)$ space, the density θ_N is finite on $\{0,\pi\}$, which is null with respect to the reference measure $\sin^{N-1} t\, \mathrm{d}t$, and for any $N > 1$, $\sin^{N-1} t\, \mathrm{d}t$ does not coincide with $c\mathscr{H}^N$ for any $c \in (0,\infty)$.

Moreover, let us point out that the Hausdorff dimension of any $\mathrm{CD}(K,N)$ space X is at most N (see [60]). In this sense, the assumption in Conjecture 1.2 amounts to asking for the existence of a "big" portion of the space with maximal dimension (notice for instance that if $\mathfrak{m} \ll \mathscr{H}^\alpha$ for some $\alpha < N$, then $\theta_N = +\infty$ $\mathfrak{m}$-a.e.). Such "maximality" of N in the conjecture plays an important role. To see why, consider an n-dimensional weighted Riemannian manifold $(M, g, e^{-V}\mathrm{dVol}_g)$, where $V \in C^\infty(M)$ and Vol_g denotes the Riemannian volume measure, and recall that, for $N \geq 1$, the Bakry–Émery N-Ricci curvature tensor is defined as

$$\mathrm{Ric}_N := \begin{cases} \mathrm{Ric}_g + \mathrm{Hess}_g(V) - \dfrac{\mathrm{d}V \otimes \mathrm{d}V}{N-n} & \text{if } N > n, \\ \mathrm{Ric}_g & \text{if } N = n \text{ and } V \text{ is constant}, \\ -\infty & \text{otherwise}, \end{cases} \tag{1.3}$$

where Ric_g is the standard Ricci curvature induced by the metric tensor g and defined as trace of the Riemann curvature tensor. It is then known (see [5, 14, 27]) that

$$(M, \mathsf{d}_g, e^{-V}\mathrm{Vol}_g) \text{ is an } \mathrm{RCD}(K,N) \text{ space if and only if } \mathrm{Ric}_N \geq Kg. \tag{1.4}$$

On the other hand, it is clear that $(e^{-V}\mathrm{Vol}_g)(B_r(x)) \sim r^n$ for every $x \in M$ as $r \to 0^+$; thus assumption (1.2) holds if and only if $n = N$, and this information together with $\mathrm{Ric}_N \geq Kg$ forces V to be constant by the very definition of Ric_N.

It is now time to point out that, thanks to the main result of [17] – and the aforementioned structural properties –, we now know that any $\mathrm{RCD}(K,N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ admits an *essential dimension* $n \in \mathbb{N} \cap [1,N]$, meaning in particular that $\mathfrak{m} \ll \mathscr{H}^n \ll \mathfrak{m}$ on the Borel set $\mathcal{R}_n^*$ (see (2.16) below), where $\mathfrak{m}(\mathsf{X} \setminus \mathcal{R}_n^*) = 0$. We thus see from general results about differentiation of measures that

$$\text{if (1.2) holds, then we have } \theta_N[\mathsf{X}, \mathsf{d}, \mathfrak{m}] < \infty \ \mathfrak{m}\text{-a.e.} \tag{1.5}$$

$\mathrm{RCD}(K,N)$ spaces for which $\theta_N[\mathsf{X}, \mathsf{d}, \mathfrak{m}]$ is finite $\mathfrak{m}$-a.e. have been called *weakly non-collapsed* RCD *spaces* in [24], while spaces such that $\theta_N[\mathsf{X}, \mathsf{d}, \mathfrak{m}]$ is finite for *every* point have been called "non-collapsed" in [50]. It is then clear from (1.1) that

$$\begin{aligned} \text{non-collapsed} &\implies \text{non-collapsed in the sense of [50]} \\ &\implies \text{weakly non-collapsed} \end{aligned}$$

and from (1.5) that proving Conjecture 1.2 is equivalent to proving that these three "non-collapsing conditions" are equivalent (up to multiplying the reference measure by a scalar).

It is known that the conjecture holds true in the following three cases:

(1) (X, d) has an upper bound on sectional curvature in a synthetic sense, i.e. it is a $\mathrm{CAT}(\kappa)$ space for some $\kappa > 0$ (see [47]);

(2) (X, d) is isometric to a smooth Riemannian manifold, possibly with boundary [39];

(3) (X, d) is compact [41].

Our main result is the resolution of Conjecture 1.2 in full generality.

Theorem 1.3. *Conjecture* 1.2 *holds true.*

Notice that, as a consequence of our main result, we obtain that if the Hausdorff dimension of an $\mathrm{RCD}(K,N)$ space is N, then also its topological dimension is N (we refer to [54] for the relevant definitions). Indeed, under this assumption, Theorem 2.20 and our main result imply that the space is, up to a scalar multiple of the reference measure, a $\mathrm{ncRCD}(K,N)$ space. Then, from the Reifenberg flatness around a regular point (see [20] and then [24, 48]), we see that any regular point has a neighbourhood which is homeomorphic to $\mathbb{R}^N$. This proves that the topological dimension is at least N, and since, in general, this is at most the Hausdorff one (see e.g. [40, Theorem 8.14]), our claim is proved.

1.2. Strategy of the proof. The basic strategy we adopt in proving this theorem is the one introduced in [41] by the third named author to handle the compact case. Still, moving from compact to non-compact creates additional technical complications that must be handled: one of the things we do here is to replace the approximation of the heat kernel via eigenfunctions – used in [41] – with suitable decay estimates based on Gaussian bounds. Also, in the course of the proof, we obtain (by making explicit some ideas that were implicitly used in [41]) interesting intermediate results that are new even in the smooth context; see in particular formula (1.13). Finally, on general RCD spaces X of essential dimension n and $U \subseteq \mathsf{X}$ open, we establish relevant links between the properties

- $\mathrm{tr}(\mathrm{Hess}\, f) = \Delta f$ on $U \subseteq \mathsf{X}$ for every f sufficiently regular,
- $\mathfrak{m} = c\mathscr{H}^n$ on $U \subseteq \mathsf{X}$ for some $c > 0$;

see Theorem 1.5 below for the precise statement.

With this said, let us describe the main idea by having once again a look at the case of a weighted n-dimensional Riemannian manifold $(M, g, e^{-V}\mathrm{dVol}_g)$. Let us consider the reference measure $\mathfrak{m} := e^{-V}\mathrm{Vol}_g$ and the Hausdorff measure $\mathscr{H}^n = \mathrm{Vol}_g$. Assume $\mathrm{Ric}_N \geq Kg$ for some $K \in \mathbb{R}$ and some $N \in [n, \infty)$ (namely $(M, \mathsf{d}_g, \mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space; recall (1.3) and (1.4)). Now notice that the following integration by parts formulas hold: for every $f, \varphi \in C_c^\infty(M)$, we have

$$-\int_M \langle \mathrm{d}f, \mathrm{d}\varphi\rangle\, \mathrm{d}\mathfrak{m} = \int_M \varphi\, \Delta f\, \mathrm{d}\mathfrak{m}, \tag{1.6a}$$

$$-\int_M \langle \mathrm{d}f, \mathrm{d}\varphi\rangle\, \mathrm{d}\mathscr{H}^n = \int_M \varphi\, \mathrm{tr}(\mathrm{Hess}\, f)\, \mathrm{d}\mathscr{H}^n. \tag{1.6b}$$

From these identities, it is easy to conclude that

$$\mathfrak{m} = c\mathscr{H}^n \iff \mathrm{tr}(\mathrm{Hess}\, f) = \Delta f \text{ for all } f \in C_c^\infty(M). \tag{1.7}$$

Thus, recalling (1.5), we see that the desired result will follow if we can show that

$$\theta_N[M, \mathsf{d}_g, \mathfrak{m}] < \infty \text{ a.e. implies that } \mathrm{tr}(\mathrm{Hess}\, f) = \Delta f \text{ for any smooth function } f.$$

To see this, recall that, as already noticed, having $\theta_N[M, \mathsf{d}_g, \mathfrak{m}](x) < \infty$ for some point $x \in M$ implies that M is N-dimensional (and thus in particular that N is an integer); then recall (1.4) and the definition (1.3) of the N-Ricci curvature tensor.

This establishes the claim in the smooth setting. In the general case, we follow the same general ideas, but we have to deal with severe technical complications. Start observing that the analogue of (1.6a) holds in general RCD spaces by the very definition of Δ (see [30]) and that the line below (1.7) is known. Thus, to conclude along the lines above, it is sufficient to prove that (1.6b) holds on $\mathrm{RCD}(K,N)$ spaces of essential dimension n. We do not have exactly such result, but have instead the following result which is anyway sufficient to conclude.

Theorem 1.4. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n*. Let* $U\subseteq \mathsf{X}$ *be bounded and open, and assume that*

$$\inf_{r\in(0,1),x\in U}\frac{\mathfrak{m}(B_r(x))}{r^n}>0. \tag{1.8}$$

Then, for every $\varphi\in\mathrm{Lip}(\mathsf{X},\mathsf{d})$*,* $f\in D(\Delta)$ *with* $\mathrm{supp}(\varphi),\mathrm{supp}(f)\subseteq U$*, formula* (1.6b) *holds.*

See Theorem 4.1 for a slightly sharper statement. Notice also that, by the Bishop–Gromov inequality, assumption (1.8) holds trivially with $n=N$ for any bounded subset U of a weakly non-collapsed RCD space. Also, the statement above is interesting regardless of the application we just described and valid also in possibly “collapsed” RCD spaces.

Thus everything boils down to the proof of such result, and indeed, from both the technical and conceptual perspective, this is the most important part of our paper. The basic idea for the proof is to perform a smoothing of the metric tensor via heat flow. Let us describe the procedure, introduced in [15], in the smooth setting. Consider a compact smooth Riemannian manifold (M,g,dVol_g), and for every $t>0$, let $\Phi_t\colon M\to L^2(M,\mathrm{Vol}_g)$ be defined as

$$\Phi_t(x):=(y\mapsto p(x,y,t)),$$

where p is the heat kernel. We can use this map to pull-back the flat metric g_{L^2} of $L^2(M,\mathrm{Vol}_g)$ and obtain the metric tensor $g_t:=\Phi_t^*g_{L^2}$ that is explicitly given by

$$g_t=\int_M \mathrm{d}p(\cdot,y,t)\otimes\mathrm{d}p(\cdot,y,t)\,\mathrm{dVol}_g(y)\in C^\infty((T^*)^{\otimes 2}M). \tag{1.9}$$

The interesting fact is that, after appropriate rescaling, the tensors g_t converge to the original one, g. More precisely, we have

$$\|4(8\pi)^{n/2}t^{(n+2)/2}g_t-g\|_{L^\infty}\to 0\quad\text{as } t\to 0^+, \tag{1.10}$$

where n denotes the dimension of M. In fact, in [15], more is proved, as the first order Taylor expansion of $t^{(n+2)/2}g_t$ is provided, but this is not relevant for our application. A way to read this convergence is via the stability of the heat flow under measured Gromov–Hausdorff convergence of spaces with Ricci curvature uniformly bounded from below; this observation is more recent than [15], as it has been made by the second author in [28]; still, this is the argument used in the RCD setting, so let us present this viewpoint. It is clear that, for $M=\mathbb{R}^n$, the tensor g_t is just a rescaling of the Euclidean tensor. On the other hand, denoting by M^λ the manifold M equipped with the rescaled metric tensor λg, and by p^λ the associated heat kernel, it is also clear that $p(x,y,t)=p^\lambda(x,y,\lambda^{-1}t)$. Thus the asymptotics of $p(x,y,t)$ as $t\to 0^+$ correspond to that of $p^\lambda(x,y,1)$ as $\lambda\to\infty$, and as said, these kernels converge to the Euclidean ones where the evolution of the metric tensors g_t is trivial.

Coming back to the RCD setting, we recall that the heat kernel is well-defined in this context [4], and a differential calculus is available in this framework [32]. Thus the same definition as in (1.9) can be given, and one can wonder whether the same convergence result as in (1.10) holds. Interestingly, in this case, one has

$$\|t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t - c_n g\|_{L^p_{\mathrm{loc}}} \to 0 \quad \text{as } t \to 0^+ \quad \text{for all } p \in [1,\infty), \tag{1.11}$$

for some constant c_n depending only on the essential dimension of X (this has been proved in [8] for compact $\mathrm{RCD}(K,N)$ spaces and is generalized here to the non-compact setting). Notice that the loss from convergence in L^∞ to convergence in L^p_{loc} is unavoidable, but unharmful for our purposes. It is important to remark that the factor in front of g_t is now not constant anymore: this has to do with Gaussian gradient estimates for the heat kernel. Now let $U \subseteq \mathsf{X}$ be open and bounded, and assume that $\mathscr{H}^n$ is a Radon measure on U (this is always the case if (1.8) holds). In this case, by standard results about differentiations of measures, we have

$$\lim_{t\to 0^+} \frac{t\mathfrak{m}(B_{\sqrt{t}}(\cdot))}{t^{(n+2)/2}} = c'_n \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}\mathscr{H}^n} \quad \mathfrak{m}\text{-a.e. on } U.$$

Thus if (1.8) holds, from (1.11), we deduce that

$$\Big\| t^{(n+2)/2} g_t - c''_n \frac{\mathrm{d}\mathscr{H}^n}{\mathrm{d}\mathfrak{m}} g \Big\|_{L^p(U)} \to 0 \quad \text{as } t \to 0^+. \tag{1.12}$$

We couple this information with the following explicit computation of the adjoint ∇^* of the covariant derivative of g_t:

$$\nabla^* g_t(x) = -\frac{1}{4}\mathrm{d}_x \Delta_x p(x,x,2t). \tag{1.13}$$

This formula was obtained in [41] in the compact setting by expanding the heat kernel via eigenfunctions of the Laplacian. This approach does not work in our current framework, and we will rather proceed via a somehow more direct approach based on "local" Bochner integration (see Section 3.1).

We are almost done: by explicit computations based on Gaussian estimates, one can see that

$$t^{(n+2)/2}\mathrm{d}_x \Delta_x p(x,x,2t) \rightharpoonup 0 \quad \text{as } t \to 0^+$$

in a suitable sense; thus, coupling this information with (1.13), (1.12) and the closure of ∇^*, we conclude that

$$\nabla^*\Big(\frac{\mathrm{d}\mathscr{H}^n}{\mathrm{d}\mathfrak{m}} g\Big) = 0 \quad \text{in } U.$$

This latter equation is a restatement of (1.6b) for f, φ with support in U, i.e. this argument gives Theorem 1.4, as desired.

We conclude emphasizing that our proof also yields the following result, which is of independent interest and will play a prominent role in the proof of Theorem 1.3.

Theorem 1.5. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n*, and let* U *be a connected open subset of* X *with*

$$\inf_{r\in(0,1),\, x\in A} \frac{\mathfrak{m}(B_r(x))}{r^n} > 0 \tag{1.14}$$

for any compact subset $A \subseteq U$*.*

Then the following two conditions are equivalent:

(1) *for every* $f \in D(\Delta)$*,*

$$\Delta f = \mathrm{tr}(\mathrm{Hess}\, f) \quad \mathfrak{m}\text{-}a.e.\ on\ U;$$

(2) *for some* $c \in (0, \infty)$*,*

$$\mathfrak{m} \llcorner U = c\mathscr{H}^n \llcorner U.$$

Notice that this has nothing to do with non-collapsing properties, and in particular, it can very well be that assumption (1.14) holds for $U = \mathsf{X}$. Moreover, items (1) and (2) may hold only on some $U \subsetneq \mathsf{X}$: just consider the case of a weighted Riemannian manifold as before with V constant on U but non-constant outside U.

1.3. Applications. The following applications seem to be already known to experts if Theorem 1.3 is established (for instance [43, 48]). However, for readers' convenience, let us give them precisely. Roughly speaking, they are based on a fact that the space of weakly non-collapsed spaces is open in the space of $\mathrm{RCD}(K, N)$ spaces because of the lower semicontinuity of the essential dimensions with respect to pointed measured Gromov–Hausdorff convergence proved in [51] (Theorem 2.16).

It is known that pointed Gromov–Hausdorff (pGH) and pointed measured Gromov–Hausdorff (pmGH) convergences are metrizable (see for instance in [34]). Thus "ϵ-pGH close" and "ϵ-pmGH close" make sense as in the following theorem. Note that, as the sequential compactness of $\mathrm{RCD}(K, N)$ spaces is known (Theorem 2.8), any such metric determines the same compact topology.

The first application is stated as follows.

Theorem 1.6. *For any* $K \in \mathbb{R}$*, any* $N \in \mathbb{N}$*, any* $\delta \in (0, \infty)$ *and any* $v \in (0, \infty)$*, there exists* $\epsilon := \epsilon(K, N, \delta, v) \in (0, 1)$ *such that if a pointed* $\mathrm{RCD}(K, N)$ *space* $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, x)$ *is so that* $(\mathsf{X}, \mathsf{d}, x)$ *is* ϵ-pGH *close to* $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, y)$ *for some non-collapsed* $\mathrm{RCD}(K, N)$ *space* $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathscr{H}^N)$ *with*

$$\mathscr{H}^N(B_1(y)) \geq v, \tag{1.15}$$

then $\mathfrak{m} = c\mathscr{H}^N$ *for some* $c \in (0, \infty)$*, and moreover,* $|\mathscr{H}^N(B_1(x)) - \mathscr{H}^N(B_1(y))| < \delta$*.*

The next application shows that the non-collapsed condition can be recognized from an infinitesimal point of view.

Theorem 1.7. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. If the essential dimension of some tangent cone* $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathfrak{m}_Y, y)$ *at some point* $x \in \mathsf{X}$ *is equal to* N*, then* $\mathfrak{m} = c\mathscr{H}^N$ *for some* $c \in (0, \infty)$*.*

Note that the converse implication also holds in Theorem 1.7, namely if $(\mathsf{X}, \mathsf{d}, \mathscr{H}^N)$ is a non-collapsed $\mathrm{RCD}(K, N)$ space, then any tangent cone at any point is also a pointed non-collapsed $\mathrm{RCD}(0, N)$ space (see Theorem 2.18).

The following final application shows that the non-collapsed condition can be also recognized from the asymptotical point of view. Note that the LHS of (1.17) exists by the Bishop–Gromov inequality and does not depend on the choice of $x \in \mathsf{X}$.

Theorem 1.8. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(0, N)$ *space, and assume that*

$$\sup_{x \in \mathsf{X}} \mathfrak{m}(B_1(x)) < \infty \tag{1.16}$$

and that, for some (hence all) $x \in \mathsf{X}$,

$$\lim_{r \to \infty} \frac{\mathfrak{m}(B_r(x))}{r^N} > 0. \tag{1.17}$$

Then $\mathfrak{m} = c\mathscr{H}^N$ *for some* $c \in (0, \infty)$.

Notice that assumption (1.16) is essential, as this simple example shows: just consider the $\mathrm{RCD}(0, N)$ space $([0, \infty), \mathsf{d}_{\mathbb{R}}, x^{N-1}\mathscr{H}^1)$, which satisfies (1.17) but is clearly not non-collapsed. Conversely, any non-collapsed $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathscr{H}^N)$ satisfies (1.16), as a consequence of the Bishop–Gromov inequality and (1.1).

2. Preliminaries

Throughout the paper,

- by *metric measure space* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, we always intend a complete and separable metric space equipped with a non-negative Borel measure finite on bounded sets such that $\mathrm{supp}(\mathfrak{m}) = \mathsf{X}$;
- C denotes a positive constant that may vary from step to step; occasionally, we may emphasize the parameters on which the constant depends so that, say, $C(K, N)$ denotes a positive constant depending only on K and N;
- $\mathrm{Lip}(\mathsf{X}, \mathsf{d})$, $\mathrm{Lip}_{\mathrm{b}}(\mathsf{X}, \mathsf{d})$ and $\mathrm{Lip}_{\mathrm{bs}}(\mathsf{X}, \mathsf{d})$ denote the set of all Lipschitz functions, bounded Lipschitz functions and Lipschitz functions with bounded support, respectively, on a metric space (X, d);
- we denote by $\mathrm{lip}\, f : \mathsf{X} \to [0, \infty]$ the *local Lipschitz constant* of the function $f : \mathsf{X} \to \mathbb{R}$ defined by
$$\mathrm{lip}\, f(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{\mathsf{d}(y, x)}$$
if x is not isolated, and it has to be understood as 0 if x is isolated;
- L^p_{loc} means that the restriction (for functions, tensors, and so on) to any compact subset of the domain is L^p.

2.1. Definition and heat flow. Fix a metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. The Cheeger energy $\mathsf{Ch}\colon L^2(\mathsf{X}, \mathfrak{m}) \to [0, \infty]$ is defined by

$$\mathsf{Ch}(f) := \inf_{\|f_i - f\|_{L^2} \to 0} \left\{ \liminf_{i \to \infty} \int_{\mathsf{X}} (\mathrm{lip}\, f_i)^2 \, \mathrm{d}\mathfrak{m} : f_i \in \mathrm{Lip}_b(\mathsf{X}, \mathsf{d}) \cap L^2(\mathsf{X}, \mathfrak{m}) \right\}. \tag{2.1}$$

Then the Sobolev space $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is defined as the finiteness domain of Ch. By looking at the optimal sequence in (2.1), one can identify a canonical object $|\mathrm{D} f|$, called the minimal relaxed slope, which is local on Borel sets (i.e. $|\mathrm{D} f_1| = |\mathrm{D} f_2|$ $\mathfrak{m}$-a.e. on $\{f_1 = f_2\}$) and provides integral representation to Ch, namely

$$\mathsf{Ch}(f) = \int_{\mathsf{X}} |\mathrm{D} f|^2 \,\mathrm{d}\mathfrak{m} \quad \text{for all } f \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}).$$

We are now in a position to introduce the definition of $\mathrm{RCD}(K,N)$ spaces (the equivalence of the following definition with the one proposed in [30] is in [5, 27]; see also [11]).

Definition 2.1 ($\mathrm{RCD}(K,N)$ space). For any $K \in \mathbb{R}$ and any $N \in [1,\infty]$, a metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is said to be an $\mathrm{RCD}(K,N)$ space if the following four conditions are satisfied.

(1) There exist $x \in X$ and $C > 1$ such that $\mathfrak{m}(B_r(x)) \le Ce^{Cr^2}$ holds for any $r > 0$.

(2) Ch is a quadratic form. In this case, for $f_i \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ $(i = 1, 2)$, we put

$$\langle \nabla f_1, \nabla f_2 \rangle := \lim_{\epsilon \to 0} \frac{|\mathrm{D}(f_1 + \epsilon f_2)|^2 - |\mathrm{D} f_1|^2}{2\epsilon} \in L^1(\mathsf{X}, \mathfrak{m}).$$

(3) Any $f \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ with $|\mathrm{D} f| \le 1$ $\mathfrak{m}$-a.e. has a 1-Lipschitz representative.

(4) For any $f \in D(\Delta)$ with $\Delta f \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, we have

$$\frac{1}{2} \int_{\mathsf{X}} |\mathrm{D} f|^2 \Delta\varphi \,\mathrm{d}\mathfrak{m} \ge \int_{\mathsf{X}} \varphi \Big(\frac{(\Delta f)^2}{N} + \langle \nabla \Delta f, \nabla f \rangle + K |\mathrm{D} f|^2 \Big) \,\mathrm{d}\mathfrak{m} \tag{2.2}$$

for any $\varphi \in D(\Delta) \cap L^\infty(\mathsf{X}, \mathfrak{m})$ with $\varphi \ge 0$, $\Delta\varphi \in L^\infty(\mathsf{X}, \mathfrak{m})$, where

$$D(\Delta) := \Big\{ f \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}) : \text{there exists } h \in L^2(\mathsf{X}, \mathfrak{m}) \text{ such that } \int_{\mathsf{X}} \langle \nabla f, \nabla \varphi \rangle \,\mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} h\varphi \,\mathrm{d}\mathfrak{m} \text{ for all } \varphi \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}) \Big\}$$

and $\Delta f := h$ for any $f \in D(\Delta)$.

We point out that, unless otherwise specified, when we write $\mathrm{RCD}(K,N)$, we implicitly assume $N < \infty$. Notice that, by the very definition, if $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space, then $(\mathsf{X}, a\mathsf{d}, b\mathfrak{m})$ is an $\mathrm{RCD}(a^{-2}K, N)$ space for any $a, b \in (0,\infty)$, for $N \in [1,\infty]$.

In the rest of this subsection, let us fix an $\mathrm{RCD}(K,\infty)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ and let us introduce the fundamental properties, except for the *second order differential calculus* developed in [32] which will be treated in Subsection 2.2.

First let us recall the *heat flow* associated with Ch,

$$\mathrm{h}_t \colon L^2(\mathsf{X}, \mathfrak{m}) \to L^2(\mathsf{X}, \mathfrak{m}).$$

This family of maps is characterized by the properties $\mathrm{h}_t f \to f$ in $L^2(\mathsf{X}, \mathfrak{m})$ as $t \to 0^+$, $\mathrm{h}_t f \in D(\Delta)$ for any $f \in L^2$, $t > 0$, and for any $t > 0$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{h}_t f = \Delta \mathrm{h}_t f \quad \text{in } L^2(\mathsf{X}, \mathfrak{m}). \tag{2.3}$$

It will be useful to keep in mind the following a-priori estimates [35, Remark 5.2.11]:

$$\||\mathrm{Dh}_t f|\|_{L^2} \le \frac{\|f\|_{L^2}}{\sqrt{t}}, \quad \|\Delta \mathrm{h}_t f\|_{L^2} \le \frac{\|f\|_{L^2}}{t} \quad \text{for all } f \in L^2(\mathsf{X}, \mathfrak{m}) \text{ and all } t > 0 \tag{2.4}$$

as well as the fact that

$$t \mapsto \|\mathrm{h}_t f\|_{L^2} \text{ is non-increasing for every } f \in L^2(\mathsf{X}, \mathfrak{m}). \tag{2.5}$$

Then the 1-*Bakry–Émery* estimate proved in [56, Corollary 4.3] is stated as, for any $f \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$,

$$|\mathrm{Dh}_t f|(x) \le e^{-Kt}|\mathrm{D}f|(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathsf{X},$$

which in particular implies

$$\mathrm{h}_t f \to f \quad \text{in } H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}).$$

It is also worth pointing out that the heat flow h_t also acts on $L^p(\mathsf{X}, \mathfrak{m})$ for any $p \in [1, \infty]$ with

$$\|\mathrm{h}_t f\|_{L^p} \le \|f\|_{L^p} \quad \text{for all } f \in L^p(\mathsf{X}, \mathfrak{m}).$$

Finally, let us recall that the following (1, 1)-Poincaré inequality is satisfied:

$$\int_{B_r(x)} \left| f - \frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} f \, \mathrm{d}\mathfrak{m} \right| \mathrm{d}\mathfrak{m} \le 4e^{|K|r^2} r \int_{B_{3r}(x)} |\mathrm{D}f| \, \mathrm{d}\mathfrak{m} \quad \text{for all } f \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}) \text{ and all } r > 0, \tag{2.6}$$

which is also valid for larger class, $\mathrm{CD}(K, \infty)$ spaces. See [55] for the detail.

2.2. Calculus on RCD(K, ∞) spaces. Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. We assume that the readers are familiar with the notion of normed module, introduced in [32], inspired by the theory developed in [62]. Here we just recall a few basic definitions, mostly to fix the notation. Unless otherwise stated, the material comes from [31, 32].

An L^0-normed module is a topological vector space $\mathscr{M}$ that is also a module over the commutative ring with unity $L^0(\mathsf{X}, \mathfrak{m})$, possessing a *pointwise norm*, i.e. a map

$$|\cdot| : \mathscr{M} \to L^0(\mathsf{X}, \mathfrak{m})$$

such that

$$|fv + gw| \le |f||v| + |g||w| \quad \mathfrak{m}\text{-a.e., for all } v, w \in \mathscr{M} \text{ and all } f, g \in L^0(\mathsf{X}, \mathfrak{m}),$$

and such that the distance

$$\mathsf{d}_{\mathscr{M}}(v, w) := \int_{\mathsf{X}} 1 \wedge |v - w| \, \mathrm{d}\mathfrak{m}' \tag{2.7}$$

is complete and induces the topology of $\mathscr{M}$, where here $\mathfrak{m}'$ is a Borel probability measure such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$ (the actual choice of $\mathfrak{m}'$ affects the distance but neither the topology nor the completeness).

Moreover, $\mathscr{M}$ is said to be a Hilbert module provided

$$|v+w|^2+|v-w|^2=2(|v|^2+|w|^2)\quad \mathfrak{m}\text{-a.e., for all } v,w\in\mathscr{M},$$

and in this case, by polarization, we can define a pointwise scalar product as

$$\langle v,w\rangle := \frac{1}{2}(|v+w|^2-|v|^2-|w|^2)\quad \mathfrak{m}\text{-a.e., for all } v,w\in\mathscr{M},$$

which turns out to be L^0-bilinear and continuous. The tensor product of two Hilbert modules $\mathscr{M}_1,\mathscr{M}_2$ is defined as the completion of the algebraic tensor product as L^0-modules with respect to the distance induced by the pointwise norm that in turn is induced by the pointwise scalar product characterized by

$$\langle v_1\otimes w_1, v_2\otimes w_2\rangle_{\mathrm{HS}} := \langle v_1,v_2\rangle_1\langle w_1,w_2\rangle_2.$$

The pointwise norm and scalar product on a tensor product will often be denoted with the subscript HS, standing for *Hilbert–Schmidt*. The dual $\mathscr{M}^*$ of $\mathscr{M}$ is defined as the collection of L^0-linear and continuous maps $L\colon\mathscr{M}\to L^0(\mathsf{X},\mathfrak{m})$, is equipped with the natural multiplication by L^0 functions ($f\cdot L(v):=L(fv)$) and the pointwise norm

$$|L|_* := \operatorname*{ess\,sup}_{v:|v|\le 1\ \mathfrak{m}\text{-a.e.}} L(v).$$

It is then easy to check that $\mathscr{M}^*$ equipped with the topology induced by the distance defined as in (2.7) is an L^0-normed module. If $\mathscr{M}$ is Hilbert, then so is $\mathscr{M}^*$, and the map sending $v\in\mathscr{M}$ to $(w\mapsto\langle v,w\rangle)\in\mathscr{M}^*$ is an isomorphism of L^0-modules, called Riesz isomorphism.

The kind of differential calculus on metric measure spaces we are going to use in this manuscript is based around the following result that defines both the *cotangent module* and the *differential* of Sobolev functions.

Theorem 2.2. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Then there is a unique, up to unique isomorphism, couple $(L^0(T^*(\mathsf{X},\mathsf{d},\mathfrak{m})),\mathrm{d})$ such that $L^0(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ is an L^0-normed module,* $\mathrm{d}\colon H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})\to L^0(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ *is linear and such that*

(1) $|\mathrm{d}f|=|Df|$ *$\mathfrak{m}$-a.e. for every $f\in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$,*

(2) *L^0-linear combinations of elements of the form* $\mathrm{d}f$ *for $f\in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ are dense in $L^0(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$.*

The dual of $L^0(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ is denoted $L^0(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$ and called *tangent module*. Elements of $L^0(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ are called 1-forms and elements of $L^0(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$ are called vector fields on X.

In this case, we shall denote by $\nabla f\in L^0(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$ the image of $\mathrm{d}f$ under the Riesz isomorphism.

The tensor product of $L^0(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$ with itself will be denoted $L^0(T^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$, similarly for $L^0(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$. Notice that, rather trivially,

$$L^0(T^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))\quad\text{and}\quad L^0((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$$

are the dual of each other, in a natural way.

For $p \in [1,\infty]$, the collection of 1-forms ω with $|\omega| \in L^p(\mathsf{X},\mathfrak{m})$, $L^p_{\mathrm{loc}}(\mathsf{X},\mathfrak{m})$ will be denoted $L^p(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$, $L^p_{\mathrm{loc}}(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$, respectively. Similarly for vector fields and other tensors. Convergence in the spaces $L^p(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ and $L^p_{\mathrm{loc}}(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$, respectively, is defined in the obvious way.

All this for general metric measure spaces. In the $\mathrm{RCD}(K,\infty)$ case, we now recall the definition of the set of *test* functions (introduced in [56])

$$\mathrm{Test}\,F(\mathsf{X},\mathsf{d},\mathfrak{m}) := \{f \in \mathrm{Lip}(\mathsf{X},\mathsf{d}) \cap D(\Delta) \cap L^\infty(\mathsf{X},\mathfrak{m}) : \Delta f \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})\},$$

which is an algebra. It is known [56] that $|\nabla f|^2 \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ for any $f \in \mathrm{Test}\,F(\mathsf{X},\mathsf{d},\mathfrak{m})$, that $\mathrm{Test}\,F(\mathsf{X},\mathsf{d},\mathfrak{m})$ is dense in $(D(\Delta),\|\cdot\|_D)$, where $\|f\|_D^2 := \|f\|^2_{H^{1,2}} + \|\Delta f\|^2_{L^2}$, and that if $f \in L^2 \cap L^\infty(\mathsf{X},\mathfrak{m})$, then $\mathrm{h}_t f \in \mathrm{Test}\,F(\mathsf{X},\mathsf{d},\mathfrak{m})$ for any $t > 0$. The following result is proved in [32].

Theorem 2.3. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space. For any* $f \in \mathrm{Test}\,F(\mathsf{X},\mathsf{d},\mathfrak{m})$*, there exists a unique* $T \in L^2((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ *such that, for all* $f_i \in \mathrm{Test}\,F(\mathsf{X},\mathsf{d},\mathfrak{m})$*,*

$$T(\nabla f_1,\nabla f_2) = \frac{1}{2}\big(\langle\nabla f_1,\nabla\langle\nabla f_2,\nabla f\rangle\rangle + \langle\nabla f_2,\nabla\langle\nabla f_1,\nabla f\rangle\rangle - \langle f,\nabla\langle\nabla f_1,\nabla f_2\rangle\rangle\big) \tag{2.8}$$

holds for $\mathfrak{m}$*-a.e.* $x \in \mathsf{X}$*. Since* T *is unique, we denote it by* $\mathrm{Hess}\,f$ *and call it the* Hessian *of* f*. Moreover, for any* $f \in \mathrm{Test}\,F(\mathsf{X},\mathsf{d},\mathfrak{m})$ *and any* $\varphi \in D(\Delta) \cap L^\infty(\mathsf{X},\mathfrak{m})$ *with* $\Delta\varphi \in L^\infty(\mathsf{X},\mathfrak{m})$ *and* $\varphi \geq 0$*, we have*

$$\int_{\mathsf{X}} \varphi|\mathrm{Hess}\,f|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} \leq \int_{\mathsf{X}} \frac{1}{2}\Delta\varphi\cdot|\nabla f|^2 - \varphi\langle\nabla\Delta f,\nabla f\rangle - K\varphi|\nabla f|^2\,\mathrm{d}\mathfrak{m}, \tag{2.9}$$

$$\int_{\mathsf{X}} |\mathrm{Hess}\,f|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} \leq \int_{\mathsf{X}} (\Delta f)^2 - K|\nabla f|^2\,\mathrm{d}\mathfrak{m}. \tag{2.10}$$

Thanks to (2.9) with the density of $\mathrm{Test}\,F(\mathsf{X},\mathsf{d},\mathfrak{m})$ in $D(\Delta)$, for any $f \in D(\Delta)$, we can also define

$$\mathrm{Hess}\,f \in L^2((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$$

with equality (2.8), where $\langle\nabla f,\nabla f_i\rangle \in H^{1,1}(\mathsf{X},\mathsf{d},\mathfrak{m})$.

Definition 2.4 (Divergence div). Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space. Denote by $D(\mathrm{div})$, $D_{\mathrm{loc}}(\mathrm{div})$ the set of all $V \in L^2(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$ and $V \in L^2_{\mathrm{loc}}(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$ for which there exists $f \in L^2(\mathsf{X},\mathfrak{m})$ and $f \in L^2_{\mathrm{loc}}(\mathsf{X},\mathfrak{m})$, respectively, such that

$$\int_{\mathsf{X}} \langle V,\nabla h\rangle\,\mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} fh\,\mathrm{d}\mathfrak{m} \quad \text{for all } h \in \mathrm{Lip}_{\mathrm{bs}}(\mathsf{X},\mathsf{d}).$$

Since f is unique (because $\mathrm{Lip}_{\mathrm{bs}}(\mathsf{X},\mathsf{d})$ is dense in $L^2(\mathsf{X},\mathfrak{m})$), we define $\mathrm{div}\,V := f$.

Note that, for any $f \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$, $f \in D(\Delta)$ if and only if $\nabla f \in D(\mathrm{div})$. Moreover, if $f \in D(\Delta)$, then for any $\varphi \in \mathrm{Lip}_{\mathrm{b}}(\mathsf{X},\mathsf{d})$, we have $\varphi\nabla f \in D(\mathrm{div})$ with

$$\mathrm{div}(\varphi\nabla f) = \langle\nabla\varphi,\nabla f\rangle + \varphi\Delta f.$$

Recalling that the covariant derivative of $f\,\mathrm{d}h$ is given by $\mathrm{d}f \otimes \mathrm{d}h + f\,\mathrm{Hess}\,h$, the following definition is justified.

Definition 2.5 (Adjoint operator ∇^*). Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space. Denote by $D(\nabla^*)$, $D_{\mathrm{loc}}(\nabla^*)$ the set of all $T\in L^2((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ and $T\in L^2_{\mathrm{loc}}((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ for which there exists $\eta\in L^2(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ and $\eta\in L^2_{\mathrm{loc}}(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$, respectively, such that

$$\int_{\mathsf{X}}\langle T,\mathrm{d}f\otimes\mathrm{d}h+f\,\mathrm{Hess}\,h\rangle_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m}=\int_{\mathsf{X}}\langle\eta,f\,\mathrm{d}h\rangle\,\mathrm{d}\mathfrak{m}$$

$$\text{for all } f\in\mathrm{Lip}_{\mathrm{bs}}(\mathsf{X},\mathsf{d}) \text{ and all } h\in D(\Delta).$$

Since η is unique (because objects of the form $f\,\mathrm{d}h$ generate $L^2(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$), we denote it by ∇^*T.

It follows from a direct calculation that the following holds. See [42, Proposition 2.18] for the proof.

Proposition 2.6. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space and let* $f\in\mathrm{Test}\,F(\mathsf{X},\mathsf{d},\mathfrak{m})$. *Then we have* $\mathrm{d}f\otimes\mathrm{d}f\in D(\nabla^*)$ *with*

$$\nabla^*(\mathrm{d}f\otimes\mathrm{d}f)=-\Delta f\,\mathrm{d}f-\frac{1}{2}\mathrm{d}|\mathrm{d}f|^2.$$

2.3. Structure of RCD(K,N) spaces and convergence. Assume that $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space for some $K\in\mathbb{R}$ and some $N\in[1,\infty)$. The main purpose of this subsection is to provide a more detailed metric measure structure theory of $(\mathsf{X},\mathsf{d},\mathfrak{m})$, which we will need later. For our purpose, it is enough to discuss the case when $K<0$.

First let us recall the Bishop–Gromov inequality (which is also valid for larger class, so-called $\mathrm{CD}(K,N)$ spaces, see [52, Theorem 5.31], [60, Theorem 2.3])

$$\frac{\mathfrak{m}(B_R(x))}{\mathfrak{m}(B_r(x))}\le\frac{\int_0^R\sinh\Big(t\sqrt{\frac{-K}{N-1}}\Big)^{N-1}\mathrm{d}t}{\int_0^r\sinh\Big(t\sqrt{\frac{-K}{N-1}}\Big)^{N-1}\mathrm{d}t}\quad\text{for all } x\in\mathsf{X}\text{ and all } r<R,\tag{2.11}$$

where, in the case $N=1$,

$$\sinh\Big(t\sqrt{\frac{-K}{N-1}}\Big)^{N-1}$$

has to be interpreted as 1. It then follows from (2.11) that

$$\frac{\mathfrak{m}(B_R(x))}{\mathfrak{m}(B_r(x))}\le C(K,N)\exp\Big(C(K,N)\frac{R}{r}\Big)\quad\text{for all } x\in\mathsf{X}\text{ and all } r<R,\tag{2.12}$$

$$\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_r(y))}\le C(K,N)\exp\Big(C(K,N)\frac{\mathsf{d}(x,y)}{r}\Big)\quad\text{for all } x,y\in\mathsf{X}\text{ and all } r>0\tag{2.13}$$

are satisfied. It is well known that, from the Bishop–Gromov inequality, it follows that the metric structure (X,d) is proper, hence geodesic, being (X,d) a length space. The length space property of RCD spaces follows quite easily from the so-called *Sobolev to Lipschitz* property, namely item (3) of Definition 2.1 (e.g. [5, Theorem 3.10] and references therein).

The following elementary lemma will play a role later.

Lemma 2.7. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Then, for any* $t \in (0, 1]$, *any* $\alpha \in \mathbb{R}$, *any* $\beta \in (0, \infty)$ *and any* $x \in \mathsf{X}$, *we have*

$$\int_{\mathsf{X}} \mathfrak{m}(B_{\sqrt{t}}(y))^{\alpha} \exp\Big(-\frac{\beta \mathsf{d}^2(x, y)}{t}\Big) \, d\mathfrak{m}(y) \le C(K, N, \alpha, \beta) \mathfrak{m}(B_{\sqrt{t}}(x))^{\alpha+1}. \tag{2.14}$$

Proof. Considering a rescaling $\sqrt{\beta/t} \cdot \mathsf{d}$ with (2.11), it is enough to prove (2.14) assuming $\beta = t = 1$. Then, by (2.12) and (2.13),

$$\begin{aligned}
&\int_{\mathsf{X}} \mathfrak{m}(B_1(y))^{\alpha} \exp(-\mathsf{d}^2(x, y)) \, d\mathfrak{m}(y) \\
&\quad = \sum_{j=-\infty}^{\infty} \int_{B_{2^{j+1}}(x) \setminus B_{2^j}(x)} \mathfrak{m}(B_1(y))^{\alpha} \exp(-\mathsf{d}^2(x, y)) d\mathfrak{m}(y) \\
&\quad \le C(K, N) \mathfrak{m}(B_1(x))^{\alpha} \sum_{j=-\infty}^{\infty} \int_{B_{2^{j+1}}(x) \setminus B_{2^j}(x)} \exp(C(\alpha, K, N) 2^{j+1} - 2^{2j}) \, d\mathfrak{m}(y) \\
&\quad = C(K, N) \mathfrak{m}(B_1(x))^{\alpha} \sum_{j=-\infty}^{\infty} \mathfrak{m}(B_{2^{j+1}}(x) \setminus B_{2^j}(x)) \exp(C(\alpha, K, N) 2^{j+1} - 2^{2j}) \\
&\quad \le C(K, N) \mathfrak{m}(B_1(x))^{\alpha} \sum_{j=-\infty}^{\infty} \mathfrak{m}(B_1(x)) \exp(C(K, N) 2^{j}) \\
&\qquad\qquad \cdot \exp(C(\alpha, K, N) 2^{j+1} - 2^{2j}) \\
&\quad \le C(\alpha, K, N) \mathfrak{m}(B_1(x))^{\alpha+1}.
\end{aligned}$$

□

For the definition of *pointed measured Gromov–Hausdorff convergence* and the following compactness result, we refer, for instance, to [34, Section 3].

Theorem 2.8. *If a sequence of pointed* $\mathrm{RCD}(K, N)$ *spaces* $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ *satisfies*

$$0 < \liminf_{i \to \infty} \mathfrak{m}_i(B_1(x_i)) \le \limsup_{i \to \infty} \mathfrak{m}_i(B_1(x_i)) < \infty,$$

then it has a subsequence $(\mathsf{X}_{i_j}, \mathsf{d}_{i_j}, \mathfrak{m}_{i_j}, x_{i_j})$ pmGH *converging to a pointed* $\mathrm{RCD}(K, N)$ *space* $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, x)$.

Next we introduce the notion of *tangent cones*.

Definition 2.9 (Tangent cones). Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. For $x \in \mathsf{X}$, we denote by $\mathrm{Tan}(\mathsf{X}, \mathsf{d}, \mathfrak{m}, x)$ the set of tangent cones to $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ at x: the collection of all isomorphism classes of pointed metric measure spaces $(\mathsf{Y}, \mathsf{d}_{\mathsf{Y}}, \mathfrak{m}_{\mathsf{Y}}, y)$ such that, as $i \to \infty$, one has

$$\Big(\mathsf{X}, \frac{1}{r_i}\mathsf{d}, \frac{1}{\mathfrak{m}(B_{r_i}(x))}\mathfrak{m}, x\Big) \xrightarrow{\mathrm{pmGH}} (\mathsf{Y}, \mathsf{d}_{\mathsf{Y}}, \mathfrak{m}_{\mathsf{Y}}, y) \tag{2.15}$$

for some $r_i \to 0^+$.

Note that Theorem 2.8 proves $\mathrm{Tan}(\mathsf{X}, \mathsf{d}, \mathfrak{m}, x) \neq \emptyset$ for any $x \in \mathsf{X}$. We are now in a position to introduce the key notions of *regular sets* and the *essential dimension* as follows.

Definition 2.10 (Regular set $\mathcal{R}_k$). Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. For any $k \geq 1$, we denote by $\mathcal{R}_k$ the k-dimensional regular set of $(X, \mathsf{d}, \mathfrak{m})$, namely the set of points $x \in \mathsf{X}$ such that $\mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, x) = \{(\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, (\omega_k)^{-1}\mathscr{H}^k, 0_k)\}$, where ω_k is the k-dimensional volume of the unit ball in $\mathbb{R}^k$ with respect to the k-dimensional Hausdorff measure $\mathscr{H}^k$.

The following result is proved in [17, Theorem 0.1].

Theorem 2.11 (Essential dimension). *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Then there exists a unique integer* $n \in [1, N]$, *called the* essential dimension of $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, *denoted by* $\mathrm{ess\,dim}(\mathsf{X})$, *such that* $\mathfrak{m}(X \setminus \mathcal{R}_n) = 0$.

Remark 2.12. The essential dimension is a purely metric concept; actually, it is equal to the maximal number $n \in \mathbb{N}$ satisfying

$$\left(\mathsf{X}, \frac{1}{r_i}\mathsf{d}, x\right) \xrightarrow{\mathrm{pGH}} (\mathbb{R}^n, \mathsf{d}_{\mathbb{R}^n}, 0_n)$$

for some $x \in \mathsf{X}$ and some $r_i \to 0^+$ because of the splitting theorem [29, Theorem 1.4] and the phenomenon of *propagation of regularity*. See [51, Remark 4.3], [43, Proposition 2.4] and [16]. □

Next let us introduce a relationship between $\mathfrak{m}$ and the Hausdorff measure of the essential dimension. See [9, 25, 36, 49] for the detail.

Theorem 2.13. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and let* n *be its essential dimension. Then* $\mathfrak{m} \ll \mathscr{H}^n \llcorner \mathcal{R}_n$. *Also, letting* $\mathfrak{m} = \theta \mathscr{H}^n \llcorner \mathcal{R}_n$ *and*

$$\mathcal{R}_n^* := \left\{x \in \mathcal{R}_n : \text{there exists } \lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_n r^n} \in (0, \infty)\right\}, \tag{2.16}$$

we have that $\mathfrak{m}(\mathcal{R}_n \setminus \mathcal{R}_n^*) = 0$, $\mathfrak{m} \llcorner \mathcal{R}_n^*$ *and* $\mathscr{H}^n \llcorner \mathcal{R}_n^*$ *are mutually absolutely continuous and*

$$\lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_n r^n} = \theta(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathcal{R}_n^*.$$

Moreover, $\mathscr{H}^n(\mathcal{R}_n \setminus \mathcal{R}_n^*) = 0$ *if* $n = N$.

A more general and classical result concerning densities, that we shall use later on, is the following (see e.g. [12, Theorem 2.4.3] for a proof).

Lemma 2.14. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space,* $\alpha \geq 0$ *and* $A \subseteq \mathsf{X}$ *a Borel subset such that*

$$\limsup_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{r^\alpha} > 0 \quad \text{for all } x \in A.$$

Then $\mathscr{H}^\alpha \llcorner A$ *is a Radon measure absolutely continuous with respect to* $\mathfrak{m}$.

The fact that $L^0(T(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ is a Hilbert module is an indication of the existence of some (weak) Riemannian metric on X. This statement can easily be made more explicit by building upon the fact that such module has local dimension equal to the essential dimension of X (see [37]).

Proposition 2.15. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n*. Then there is a unique* $g \in L^0((T^*)^{\otimes 2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ *such that*

$$g(V_1 \otimes V_2) = \langle V_1, V_2 \rangle \quad \mathfrak{m}\text{-a.e., for all } V_1, V_2 \in L^0(T(\mathsf{X}, \mathsf{d}, \mathfrak{m})).$$

Moreover, g satisfies

$$|g|_{\mathrm{HS}} = \sqrt{n}, \quad \mathfrak{m}\text{-a.e.} \tag{2.17}$$

We can use this "metric tensor" to define the *trace* of any $T \in L^0((T^*)^{\otimes 2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ by

$$\mathrm{tr}(T) := \langle T, g \rangle_{\mathrm{HS}} \in L^0(\mathsf{X}, \mathfrak{m}).$$

Notice that, by (2.17) and the Cauchy–Schwarz inequality, it follows that if

$$T \in L^p_{\mathrm{loc}}((T^*)^{\otimes 2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})),$$

then $\mathrm{tr}(T) \in L^p_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$.

Finally, let us end this subsection by recalling the lower semicontinuity of the essential dimensions with respect to pmGH convergence proved in [51, Theorem 1.5], where this is also understood as a consequence of L^2_{loc}-weak convergence of Riemannian metrics (see [8, Remark 5.20]), and an alternative proof of the theorem below can be based on Remark 2.12.

Theorem 2.16. *Let*

$$(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, x_i) \xrightarrow{\mathrm{pmGH}} (\mathsf{X}, \mathsf{d}, \mathfrak{m}, x)$$

be a pmGH *convergent sequence of pointed* $\mathrm{RCD}(K, N)$ *spaces. Then*

$$\liminf_{i \to \infty} \mathrm{ess\,dim}(\mathsf{X}_i) \geq \mathrm{ess\,dim}(\mathsf{X}).$$

2.4. Non-collapsed RCD(K, N) spaces.

Let us start recalling the following.

Definition 2.17 (Non-collapsed $\mathrm{RCD}(K, N)$ space). An $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is said to be *non-collapsed* if $\mathfrak{m} = \mathscr{H}^N$.

This definition was introduced in [24, Definition 1.1] as a synthetic counterpart of non-collapsed Ricci limit spaces. As explained in the introduction, non-collapsed $\mathrm{RCD}(K, N)$ spaces have finer properties than general $\mathrm{RCD}(K, N)$ spaces already introduced in Subsection 2.2. Let us give one of the properties as follows (see [24, Theorem 1.2]).

Theorem 2.18 (From pGH to pmGH). *Let* $K \in \mathbb{R}$, $N \in \mathbb{N}$, *and let* $(\mathsf{X}_i, \mathsf{d}_i, \mathscr{H}^N, x_i)$ *be a sequence of pointed non-collapsed* $\mathrm{RCD}(K, N)$ *spaces. Then, after passing to a subsequence, there exists a pointed proper geodesic space* $(\mathsf{X}, \mathsf{d}, x)$ *such that*

$$(\mathsf{X}_i, \mathsf{d}_i, x_i) \xrightarrow{\mathrm{pGH}} (\mathsf{X}, \mathsf{d}, x).$$

Moreover, if $\inf_i \mathscr{H}^N(B_1(x_i)) > 0$, *then it follows that* $(\mathsf{X}, \mathsf{d}, \mathscr{H}^N, x)$ *is also a pointed non-collapsed* $\mathrm{RCD}(K, N)$ *space and the convergence of* $(\mathsf{X}_i, \mathsf{d}_i, \mathscr{H}^N, x_i)$ *to such space is in the* pmGH *topology.*

We remark that the above theorem is tightly related to the following continuity result, which is the generalization to the RCD class of the classical statement by Colding about volume convergence under lower Ricci bounds [23] (see [24, Theorem 1.3]).

Theorem 2.19 (Continuity of $\mathscr{H}^N$). *For $K \in \mathbb{R}$, $N \in [1, \infty)$, let $\mathbb{B}(K, N)$ be the collection of (isometry classes of) open unit balls on* RCD(K, N) *spaces. Equip $\mathbb{B}(K, N)$ with the Gromov–Hausdorff distance. Then the map $\mathbb{B}(K, N) \ni B \mapsto \mathscr{H}^N(B) \in \mathbb{R}$ is continuous.*

For our main purpose, we need a notion weaker than the non-collapsed one. In order to give the precise definition, let us recall the following result which is just a combination from previous known ones.

Theorem 2.20. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an* RCD(K, N) *space. Then the following five conditions are equivalent.*

(1) *The essential dimension of X is N.*

(2) $\mathfrak{m}$ *is absolutely continuous with respect to $\mathscr{H}^N$.*

(3) *Inequality* (1.2) *holds.*

(4) *$N \in \mathbb{N}$ and the Hausdorff dimension of (X, d) is greater than $N - 1$.*

(5) *The Hausdorff dimension of (X, d) is N.*

Proof. The equivalence between item (1) and item (2) is proved in [24, Theorem 1.12]. Since the implication from item (2) to item (3) is a direct consequence of Theorem 2.13, let us check the implication from item (3) to item (1) as follows. The positivity (1.2) with Theorems 2.11 and 2.13 yields $\mathscr{H}^N(\mathcal{R}_n^*) > 0$, where n denotes the essential dimension. In particular, $N \le n$. Since the converse inequality is always satisfied by Theorem 2.13, we have item (1).

Notice that item (2) implies item (4); we show now that item (4) implies item (1). To see this, notice that the proof of [24, Theorem 1.4] shows that if item (4) holds, then there is an iterated tangent space isomorphic to $\mathbb{R}^N$. Since the essential dimension of the N-dimensional Euclidean space is N, the conclusion follows from Theorem 2.16.

If we assume item (5), then, since the Hausdorff dimension of (X, d) is at most the integer part of N (by [24, Corollary 1.5]), we see that N is an integer so that item (4) holds. Finally, if item (2) holds, then the Hausdorff dimension of (X, d) is at least N so that we conclude by [24, Corollary 1.5] again. □

We are now in a position to introduce the notion of weakly non-collapsed RCD(K, N) spaces (our definition is trivially equivalent to the one in [24]).

Definition 2.21 (Weakly non-collapsed RCD(K, N) space). An RCD(K, N) space is said to be *weakly non-collapsed* if one (and thus any) of the items in Theorem 2.20 is satisfied.

Note that any non-collapsed RCD(K, N) space is a weakly non-collapsed RCD(K, N) space.

We conclude the section recalling – see e.g. the introduction – that one expects the notion of non-collapsed space to be related to the fact that the trace of the Hessian is the Laplacian.

A first instance of this behaviour is contained in the following result that is basically extracted from [38, Proposition 3.2] (notice that Definition 2.1 tells that if the stated inequality (2.18) holds without restrictions on the support of φ, then the space is an $\mathrm{RCD}(K,n)$ space, and thus, since n is assumed to be the essential dimension, the space is weakly non-collapsed).

Theorem 2.22. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension n and let* $U\subseteq\mathsf{X}$ *be open. Then the following two conditions are equivalent.*

(1) *For any* $f\in\mathrm{Test}\,F(\mathsf{X},\mathsf{d},\mathfrak{m})$ *and any* $\varphi\in D(\Delta)$ *non-negative with* $\mathrm{supp}(\varphi)\subseteq U$ *and* $\Delta\varphi\in L^\infty(\mathsf{X},\mathfrak{m})$*, we have*

$$\frac{1}{2}\int_U \Delta\varphi|\nabla f|^2\,\mathrm{d}\mathfrak{m}\geq\int_U\varphi\Big(\frac{(\Delta f)^2}{n}+\langle\nabla\Delta f,\nabla f\rangle+K|\nabla f|^2\Big)\,\mathrm{d}\mathfrak{m}. \tag{2.18}$$

(2) *For any* $f\in D(\Delta)$*, we have*

$$\Delta f=\mathrm{tr}(\mathrm{Hess}\,f)\quad \mathfrak{m}\text{-}a.e.\ in\ U. \tag{2.19}$$

Proof. It is easy to see the implication from item (2) to item (1) is trivial because we know

$$|\mathrm{tr}(\mathrm{Hess}\,f)|=|\langle\mathrm{Hess}\,f,g\rangle_{\mathrm{HS}}|\leq|\mathrm{Hess}\,f|_{\mathrm{HS}}|g|_{\mathrm{HS}}=|\mathrm{Hess}\,f|_{\mathrm{HS}}\cdot\sqrt{n}.$$

Thus item (2) gives $|\mathrm{Hess}\,f|^2_{\mathrm{HS}}\geq(\Delta f)^2/n$, and therefore item (1) follows directly from (2.9).

For the reverse implication, we closely follow the proof of [38, Proposition 3.2] keeping in mind (2.18) and the existence, for any $A\subseteq U$ with A compact and U open, of a test function identically 1 on A and with support in U (see e.g. [10] or [35, Lemma 6.2.15]). In this way, we easily obtain that (2.19) holds for any $f\in\mathrm{Test}\,F(\mathsf{X},\mathsf{d},\mathfrak{m})$. Then, by the density of $\mathrm{Test}\,F(\mathsf{X},\mathsf{d},\mathfrak{m})$ in $D(\Delta)$ (see for example [41, Lemma 2.2]), (2.19) holds for $f\in D(\Delta)$. □

3. Smoothing of the Riemannian metric by the heat kernel

3.1. Local Hille's theorem. In this section, we collect some basic results about local (differentiation) operators: the main result we have in mind is the version of Hille's theorem stated in Lemma 3.3 below. We shall apply the notions presented here to the operators d, Δ, ∇^*, but in order to highlight the similarities among the various approaches, we shall give a rather abstract presentation.

Thus let us fix a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ and two L^0-normed modules $\mathscr{M},\mathscr{N}$. For $p\in[1,\infty]$, we shall denote by $L^p(\mathscr{M})$ and $L^p_{\mathrm{loc}}(\mathscr{M})$ the collection of those $v\in\mathscr{M}$ with $|v|\in L^p(\mathsf{X},\mathfrak{m})$ and $|v|\in L^p_{\mathrm{loc}}(\mathsf{X},\mathfrak{m})$, respectively. Similarly for $\mathscr{N}$.

Definition 3.1 (Weakly local operators). Let $p\in[1,\infty]$, and let

$$L\colon D(L)\subseteq L^p(\mathscr{M})\to L^p(\mathscr{N})$$

be a linear operator. We say that L is *weakly local* provided

$$L(v)=L(w)\quad \mathfrak{m}\text{-a.e. on the essential interior of } \{v=w\} \text{ for any } v,w\in D(L).$$

In other words, L is weakly local provided, for any $v,w\in D(L)$ and $U\subseteq\mathsf{X}$ open such that $v=w$ $\mathfrak{m}$-a.e. on U, we have $L(v)=L(w)$ $\mathfrak{m}$-a.e. on U.

Weakly local operators can naturally be extended as follows (variants of this definition are possible, but for us, the following is sufficient).

Definition 3.2 (Extension of weakly local operators). Let $p \in [1, \infty]$, and let

$$L \colon D(L) \subseteq L^p(\mathscr{M}) \to L^p(\mathscr{N})$$

be a weakly local operator. We then define $D_{\mathrm{loc}}(L) \subseteq L^p_{\mathrm{loc}}(\mathscr{M})$ as the collection of those v's such that, for every $U \subseteq \mathsf{X}$ bounded and open, there is $v_U \in D(L) \subseteq L^p(\mathscr{M})$ with $v_U = v$ $\mathfrak{m}$-a.e. on U.

For $v \in D_{\mathrm{loc}}(L)$, we define $L(v) \in L^p_{\mathrm{loc}}(\mathscr{N})$ via

$$L(v) = L(v_U) \quad \mathfrak{m}\text{-a.e. on } U \text{ for all } U \subseteq \mathsf{X} \text{ open and bounded,}$$

where v_U is as above.

It is clear from the definition that $L \colon D_{\mathrm{loc}}(L) \subseteq L^p_{\mathrm{loc}}(\mathscr{M}) \to L^p_{\mathrm{loc}}(\mathscr{N})$ is well-posed and that the resulting operator is linear. We are interested in a version of Hille's theorem for this kind of operators, and to this aim, we need first to introduce the notion of integrable function with values in $L^p(\mathscr{M})$.

For the standard notion of Bochner integration of Banach valued maps, we refer to [26]. Given a metric measure space $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mu)$ (the topology here is not really relevant, but in our applications, we shall mostly have $\mathsf{Y} = \mathsf{X}$), we shall denote by $L^1(\mathsf{Y}, \mu; L^p_{\mathrm{loc}}(\mathscr{M}))$ the collection of (equivalence classes up to μ-a.e. equality of) maps $y \mapsto v_y \in L^p_{\mathrm{loc}}(\mathscr{M})$ such that, for any $A \subseteq \mathsf{X}$ Borel and bounded, the map $y \mapsto \chi_A v_y$ is in $L^1(\mathsf{Y}, \mu; L^p(\mathscr{M}))$ (here we are endowing $L^p(\mathscr{M})$ with its natural Banach structure).

With these definitions, the following result is rather trivial (but nevertheless useful).

Lemma 3.3 (Local Hille's theorem – abstract version). *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *and* $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mu)$ *be metric measure spaces,* $\mathscr{M}, \mathscr{N}$ *two* L^0*-normed modules,* $p \in [1, \infty]$*, and let*

$$L \colon D(L) \subseteq L^p(\mathscr{M}) \to L^p(\mathscr{N})$$

be a weakly local and closed linear operator. Also, let

$$y \mapsto v_y \in L^p_{\mathrm{loc}}(\mathscr{M})$$

be in $L^1(\mathsf{Y}, \mu; L^p_{\mathrm{loc}}(\mathscr{M}))$*. Assume that*

(i) $v_y \in D_{\mathrm{loc}}(L)$ *for* μ*-a.e.* y*;*

(ii) *there exists a "cut-off" operator* T*: for every pair* $V \subseteq U \subseteq \mathsf{X}$ *bounded and open with* $\mathsf{d}(V, \mathsf{X} \setminus U) > 0$*, there is a linear map* $T \colon L^p_{\mathrm{loc}}(\mathscr{M}) \to L^p(\mathscr{M})$ *such that*

$$\begin{aligned} T(v) &= v && \mathfrak{m}\text{-a.e. on } V, \\ T(v) &= T(\chi_U v) && \mathfrak{m}\text{-a.e.}, \\ \|T(v)\|_{L^p(\mathscr{M})} &\le C \|\chi_U v\|_{L^p(\mathscr{M})} \end{aligned} \tag{3.1}$$

for every $v \in L^p_{\mathrm{loc}}(\mathscr{M})$ *and some* $C > 0$ *independent of* v*;*

(iii) *L has the following "stability under cut-off by T" property: for any V, U as above and T given by item* (ii)*, we have $T(v_y) \in D(L)$ for μ-a.e. y, and the map $y \mapsto L(T(v_y))$ is in $L^1(\mathsf{Y}, \mu; L^p(\mathscr{N}))$.*

Then $\int_{\mathsf{Y}} v_y \, \mathrm{d}\mathfrak{m}(y) \in D_{\mathrm{loc}}(L)$, the map $y \mapsto L(v_y)$ is in $L^1(\mathsf{Y}, \mu; L^p_{\mathrm{loc}}(\mathscr{M}))$ and

$$L\biggl(\int_{\mathsf{Y}} v_y \, \mathrm{d}\mathfrak{m}(y)\biggr) = \int_{\mathsf{Y}} L(v_y) \, \mathrm{d}\mathfrak{m}(y).$$

Proof. Fix $V \subseteq \mathsf{X}$ open and bounded, and then let $U \supseteq V$ be open and bounded with $\mathsf{d}(V, \mathsf{X} \setminus U) > 0$. Let $T\colon L^p_{\mathrm{loc}}(\mathscr{M}) \to L^p(\mathscr{M})$ be given by item (ii). By assumption (i), we know that $y \mapsto \chi_U v_y \in L^p(\mathscr{M})$ is in $L^1(\mathsf{X}, \mathfrak{m}; L^p(\mathscr{M}))$, and the third formula in (3.1) gives that T is continuous as a map from $L^p(\mathscr{M})$ to itself. It follows that $y \mapsto T(\chi_U v_y) = T(v_y)$ is in $L^1(\mathsf{Y}, \mu; L^p(\mathscr{M}))$. Then assumption (iii) and the classical theorem by Hille ensure that

$$\int_{\mathsf{Y}} T(v_y) \, \mathrm{d}\mu(y) \in D(L) \quad \text{and} \quad L\biggl(\int_{\mathsf{Y}} T(v_y) \, \mathrm{d}\mu(y)\biggr) = \int_{\mathsf{Y}} L(T(v_y)) \, \mathrm{d}\mu(y). \tag{3.2}$$

Now notice that the first formula in (3.1) gives that $T(v_y) = v_y$ on V for every y; thus the weak locality of L also gives that $L(T(v_y)) = L(v_y)$ on V for every y. It follows that $y \mapsto \chi_V L(v_y)$ is in $L^1(\mathsf{Y}, \mu; L^p(\mathscr{N}))$ and that

$$\int_{\mathsf{Y}} \chi_V v_y \, \mathrm{d}\mu(y) = \int_{\mathsf{Y}} \chi_V T(v_y) \, \mathrm{d}\mu(y), \quad \int_{\mathsf{Y}} \chi_V L(v_y) \, \mathrm{d}\mu(y) = \int_{\mathsf{Y}} \chi_V L(T(v_y)) \, \mathrm{d}\mu(y).$$

Thus, using again the weak locality of L, it follows that

$$\begin{aligned}\chi_V L\biggl(\int_{\mathsf{Y}} v_y \, \mathrm{d}\mu(y)\biggr) &= \chi_V L\biggl(\int_{\mathsf{Y}} T(v_y) \, \mathrm{d}\mu(y)\biggr) \overset{(3.2)}{=} \chi_V \int_{\mathsf{Y}} L(T(v_y)) \, \mathrm{d}\mu(y) \\ &= \int_{\mathsf{Y}} \chi_V L(T(v_y)) \, \mathrm{d}\mu(y) = \int_{\mathsf{Y}} \chi_V L(v_y) \, \mathrm{d}\mu(y).\end{aligned}$$

Since V was arbitrary, this is the conclusion. □

We now see how to apply this general statement to the concrete cases of $L = \mathrm{d}, \Delta, \nabla^*$. The idea is to use, as map T, the multiplication with a Lipschitz cut-off function φ with support in U and identically 1 on V. For the case of the Laplacian, this does not really work, as one would need to multiply by a Lipschitz function with bounded Laplacian in order to remain in the domain of the operator. The problem is that, on general $\mathrm{RCD}(K, \infty)$ spaces, it is not clear whether this sort of cut-off functions exists (but see [10] or [35, Lemma 6.2.15] for the case of proper RCD spaces). This issue is, however, easily dealt with by recalling that the Laplacian is the divergence of the gradient and applying the above theorem twice (this amounts to localizing Δf by looking at $\mathrm{div}(\varphi \nabla(\varphi f))$).

Let us start recalling that the differential

$$\mathrm{d}\colon H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}) \subseteq L^2(\mathsf{X}, \mathfrak{m}) \to L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$$

is weakly local (in fact even more, as there is locality on Borel sets and not just on open ones) by [32, Theorem 2.2.3]. The same holds for the divergence operator

$$\mathrm{div}\colon D(\mathrm{div}) \subseteq L^2(T(\mathsf{X}, \mathsf{d}, \mathfrak{m})) \to L^2(\mathsf{X}, \mathfrak{m}).$$

Indeed, for $v, w \in D(\mathrm{div})$ equal on some open set U, we have

$$\int_X \varphi \,\mathrm{div}\, v \,\mathrm{d}\mathfrak{m} = -\int_X \mathrm{d}\varphi(v)\,\mathrm{d}\mathfrak{m} \overset{(*)}{=} -\int_X \mathrm{d}\varphi(w)\,\mathrm{d}\mathfrak{m} = \int_X \varphi \,\mathrm{div}\, w \,\mathrm{d}\mathfrak{m}$$

for any $\varphi \in \mathrm{Lip}(\mathsf{X}, \mathsf{d})$ with $\mathrm{supp}(\varphi) \subseteq U$, having used the locality of the differential and the assumption $v = w$ on U in the starred equality $(*)$. This is sufficient to prove the claim. Similarly, starting from the locality of the covariant derivative (see [32, Proposition 3.4.9]), the weak locality of ∇^* follows. Finally, the weak locality of the Laplacian follows from that of the differential and of the divergence.

Below, for the domain $D_{\mathrm{loc}}(\mathrm{d}) \subseteq L^2_{\mathrm{loc}}(\mathsf{X}, \mathfrak{m})$, we shall use the more standard notation $H^{1,2}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. We then have the following.

Proposition 3.4 (Local Hille's theorem – concrete version). *Assume that* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K, \infty)$ *space and* $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mu)$ *a metric measure space. Let*

$$(f_y) \in L^1(\mathsf{Y}, \mu; L^2_{\mathrm{loc}}(\mathsf{X}, \mathfrak{m})),$$
$$(f_y) \in L^1(\mathsf{Y}, \mu; L^2_{\mathrm{loc}}(\mathsf{X}, \mathfrak{m})),$$
$$(A_y) \in L^1(\mathsf{Y}, \mu; L^2_{\mathrm{loc}}(T^{\otimes 2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$$

be with $f_y \in H^{1,2}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, $f_y \in D_{\mathrm{loc}}(\Delta)$ *and* $A_y \in D_{\mathrm{loc}}(\nabla^*)$, *respectively, for* μ*-a.e.* $y \in \mathsf{Y}$. *Then, for every* $U \subseteq \mathsf{X}$ *open and bounded,* $y \mapsto \chi_U \mathrm{d} f_y$, $y \mapsto \chi_U \Delta f_y$ *and* $y \mapsto \chi_U \nabla^* A_y$, *respectively, are – the equivalence class up to* μ*-a.e. equality of – a strongly Borel function (i.e. Borel and essentially separably valued).*

Now assume also that, for every $U \subseteq \mathsf{X}$ *open and bounded, we have, respectively,*

$$\int_\mathsf{Y} \|\chi_U |\mathrm{d} f_y|\|_{L^2}\,\mathrm{d}\mu(y) < \infty,$$
$$\int_\mathsf{Y} \|\chi_U \Delta f_y\|_{L^2}\,\mathrm{d}\mu(y) < \infty,$$
$$\int_\mathsf{Y} \|\chi_U |\nabla^* A_y|\|_{L^2}\,\mathrm{d}\mu(y) < \infty.$$

Then

$$\int_\mathsf{Y} f_y \,\mathrm{d}\mu(y) \in H^{1,2}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}, \mathfrak{m}),$$
$$\int_\mathsf{Y} f_y \,\mathrm{d}\mu(y) \in D_{\mathrm{loc}}(\Delta),$$
$$\int_\mathsf{Y} A_y \,\mathrm{d}\mu(y) \in D_{\mathrm{loc}}(\nabla^*)$$

with

$$\mathrm{d}\int_\mathsf{Y} f_y \,\mathrm{d}\mu(y) = \int_\mathsf{Y} \mathrm{d} f_y \,\mathrm{d}\mu(y),$$
$$\Delta \int_\mathsf{Y} f_y \,\mathrm{d}\mu(y) = \int_\mathsf{Y} \Delta f_y \,\mathrm{d}\mu(y),$$
$$\nabla^* \int_\mathsf{Y} A_y \,\mathrm{d}\mu(y) = \int_\mathsf{Y} \nabla^* A_y \,\mathrm{d}\mu(y),$$

respectively.

Proof. We start with the case of differential. We have already noticed that d is weakly local, and we know from [32, Theorem 2.2.9] that it is a closed operator. Let us check that the other assumptions in Lemma 3.3 are satisfied. Part (i) holds by our assumption; thus we pass to (ii). Let U, V be as in the statement and let $\varphi \in \mathrm{Lip}(\mathsf{X}, \mathsf{d})$ be identically 1 on V and with support in U (the hypothesis $\mathsf{d}(V, \mathsf{X} \setminus U) > 0$ grants that such φ exists). We define $T(f) := \varphi f$ and notice that the properties in (3.1) are trivial. We pass to (iii) and notice that, by the very definition of extension of d from $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ to $H^{1,2}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, it follows that the Leibniz rule holds even in $H^{1,2}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. It is then clear that we have $\varphi f \in H^{1,2}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ with $\mathrm{d}(\varphi f) = \varphi\, \mathrm{d} f + f\, \mathrm{d}\varphi$. Thus

$$|\mathrm{d}(\varphi f)| \leq \chi_U |\mathrm{d} f| \sup|\varphi| + \chi_U C|f| \in L^2(\mathsf{X}, \mathfrak{m}), \tag{3.3}$$

where C denotes the Lipschitz constant of φ. Therefore, φf is actually in $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$.

With this said, we verify the first claim. Fix $U \subseteq \mathsf{X}$ open and bounded, let $\varphi \in \mathrm{Lip}_{\mathrm{bs}}(\mathsf{X}, \mathsf{d})$ be identically 1 on U and notice that, replacing f_y with φf_y, it is sufficient to prove that if $y \mapsto f_y \in L^2(\mathsf{X}, \mathfrak{m})$ is Borel and $f_y \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ for every $y \in \mathsf{Y}$, then

$$y \mapsto \mathrm{d} f_y \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$$

is strongly Borel. Since $L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ is separable (see [2] and [32, Proposition 2.2.5]), it is enough to check Borel regularity. Also, since $\mathrm{d}\colon H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}) \to L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ is continuous, it suffices to prove that $y \mapsto f_y \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is Borel. To see this, it is sufficient to show that the unit ball in $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ belongs to the σ-algebra $\mathcal{A}$ generated by L^2-open sets in $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. But this is obvious because the lower semicontinuity of the $H^{1,2}$-norm with respect to L^2-convergence ensures that closed $H^{1,2}$-balls are also L^2-closed and thus are in $\mathcal{A}$. Since open balls are countable unions of closed balls, the first claim follows.

For the second, we now observe that what just proved, our assumption

$$\int_{\mathsf{Y}} \| \chi_U |\mathrm{d} f_y| \|_{L^2}\, \mathrm{d}\mu(y) < \infty$$

and (3.3) ensure that $y \mapsto \mathrm{d}(\varphi f_y)$ is in $L^1(\mathsf{Y}, \mu; L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$, i.e. (iii) of Lemma 3.3 holds, and the conclusion follows from such lemma.

The same line of thought gives the conclusion for ∇^*. For the Laplacian, we start noticing that, for V, U and φ as above, we have

$$\begin{aligned}\int_{\mathsf{X}} |\varphi|^2 |\mathrm{d} f|^2\, \mathrm{d}\mathfrak{m} &= -\int_{\mathsf{X}} 2 f \varphi \langle \mathrm{d} f, \mathrm{d}\varphi \rangle + \varphi^2 f \Delta f\, \mathrm{d}\mathfrak{m} \\ &\leq \int_{\mathsf{X}} \frac{1}{2} |\varphi|^2 |\mathrm{d} f|^2 + 2|\mathrm{d}\varphi|^2 |f|^2 + \frac{1}{2}|\varphi|^2 (|f|^2 + |\Delta f|^2)\, \mathrm{d}\mathfrak{m},\end{aligned}$$

i.e.

$$\frac{1}{2}\int_{\mathsf{X}} |\varphi|^2 |\mathrm{d} f|^2\, \mathrm{d}\mathfrak{m} \leq C \int_U |f|^2 + |\Delta f|^2\, \mathrm{d}\mathfrak{m}.$$

This proves that if $f \in D_{\mathrm{loc}}(\Delta) \subseteq L^2_{\mathrm{loc}}(\mathsf{X}, \mathfrak{m})$, then $f \in H^{1,2}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ as well. Hence what we previously proved tells that, for $y \mapsto f_y \in L^2_{\mathrm{loc}}$ Borel with $f_y \in D_{\mathrm{loc}}(\Delta)$ for μ-a.e. y, we have that $y \mapsto \chi_U \mathrm{d} f_y \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ is strongly Borel for any $U \subseteq \mathsf{X}$ open and bounded. Now we want to prove that the same assumptions ensure that $y \mapsto \chi_U \Delta f_y \in L^2(\mathsf{X}, \mathfrak{m})$ is Borel as well. Since the σ-algebra generated by the strong topology coincides with that generated by the weak topology (because the closed unit ball can be realized as countable intersection of weakly closed halfspaces so that closed balls are weakly Borel and thus the same holds for open balls since they are countable union of closed balls), by approximation, to get the desired Borel regularity, it is sufficient to prove that $y \mapsto \int_{\mathsf{X}} \psi \xi \Delta f_y \,\mathrm{d}\mathfrak{m}$ is Borel for any $\psi \in \mathrm{Lip}_{\mathrm{bs}}(\mathsf{X}, \mathsf{d})$ and ξ varying in a countable dense subset of $L^2(\mathsf{X}, \mathfrak{m})$. We pick $\xi \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ and notice that

$$\int_{\mathsf{X}} \psi \xi \Delta f_y \,\mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} \langle \nabla(\psi\xi), \nabla f_y \rangle \,\mathrm{d}\mathfrak{m} = -\int_U \langle \nabla(\psi\xi), \nabla f_y \rangle \,\mathrm{d}\mathfrak{m}$$

for any $U \subseteq \mathsf{X}$ open, bounded and containing the support of ψ. By what we already proved, we see that the RHS is a Borel function of y; hence the desired Borel regularity follows.

With this said, the conclusion for the Laplacian follows by first applying the result to the differential and then to the divergence (the study of the divergence operator closely follows that of ∇^* that in turn, as said, is largely based on that of d). □

Remark 3.5. The above version of Hille's theorem is compatible with the more general one recently discussed in [18, Section 3.3]. However, the presentation here being substantially simpler, we prefer giving a direct proof rather than linking the terminology to that in [18]. □

3.2. Gaussian estimates and their consequences. Thanks to (2.6) and (2.11), it follows from [58, Proposition 3.1] that there exists a unique (locally Hölder) continuous function $p\colon \mathsf{X} \times \mathsf{X} \times (0,\infty) \to (0,\infty)$, called the *heat kernel* of $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, such that the following holds:

$$\mathrm{h}_t f(x) = \int_{\mathsf{X}} p(x,y,t) f(y) \,\mathrm{d}\mathfrak{m}(y) \quad \text{for all } f \in L^2(\mathsf{X}, \mathfrak{m}) \text{ and all } x \in \mathsf{X}. \tag{3.4}$$

Let us denote $p_{y,t}(x) = p(x,y,t)$ when we consider p as a function on X for fixed $y \in \mathsf{X}$ and $t > 0$.

Let us recall the Gaussian estimates for the heat kernel p proved in [46], where we are going to use them only specialized to the case $\epsilon = 1$: for any $\epsilon \in (0,1]$, there exists a positive constant $C := C(K, N, \epsilon)$ depending only on K, N and ϵ such that, for any $x, y \in \mathsf{X}$ and any $0 < t < 1$,

$$\begin{aligned}\frac{C}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\Big(-\frac{\mathsf{d}(x,y)^2}{(4-\epsilon)t} - Ct\Big) &\le p(x,y,t) \\ &\le \frac{C}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\Big(-\frac{\mathsf{d}(x,y)^2}{(4+\epsilon)t} + Ct\Big),\end{aligned} \tag{3.5}$$

and for every $y \in \mathsf{X}$ and $t > 0$, we have

$$|\mathrm{d}p_{y,t}|(x) \le \frac{C}{\sqrt{t}\,\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\Big(-\frac{\mathsf{d}(x,y)^2}{(4+\epsilon)t} + Ct\Big) \quad \mathfrak{m}\text{-a.e. } x \in \mathsf{X}. \tag{3.6}$$

Notice that (3.5) and Lemma 2.7 ensure that $p(\cdot, y, t) \in L^2(\mathsf{X}, \mathfrak{m})$ for every $y \in \mathsf{X}$, $t > 0$; therefore, from (3.4), we deduce the Chapman–Kolmogorov equation

$$p(x,y,t+s) = \mathrm{h}_t p(\cdot, y, s)(x) = \int_{\mathsf{X}} p(x,z,t) p(z,y,s) \,\mathrm{d}\mathfrak{m}(z) \quad \text{for all } t, s > 0 \text{ and all } x, y \in \mathsf{X}. \tag{3.7}$$

Also, from (2.3), [57, Corollary 2.7] and (3.5), we deduce the estimate

$$|\Delta p(\cdot, y, t)|(x) \le \frac{C}{t\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\Big(-\frac{\mathsf{d}(x, y)^2}{(4+\epsilon)t} + Ct\Big) \quad \mathfrak{m}\text{-a.e. } x \in \mathsf{X}, \tag{3.8}$$

for every $y \in \mathsf{X}$, $t > 0$. Notice that the above discussion and estimates easily imply that

$$p_{y,t} \in \mathrm{Test}\, F(\mathsf{X}, \mathsf{d}, \mathfrak{m})$$

for every $y \in \mathsf{X}$ and $t > 0$. We shall frequently use this fact. For future reference, we also notice that (3.7) and estimate (3.5) together with (2.4), (2.11) and Lemma 2.7 give

$$\|p_{y,t}\|_{H^{1,2}} + \|\Delta p_{y,t}\|_{H^{1,2}} \le C(K, N, t)\mathfrak{m}(B_{\sqrt{t}}(y))^{-1/2}. \tag{3.9}$$

We also notice that the identity $\partial_t p(x, y, t) = \partial_t p_{y,t}(x) = \Delta p_{y,t}(x) = \Delta_x p(x, y, t)$, valid for any $t > 0$, $y \in \mathsf{X}$ and a.e. x, together with the symmetry in x, y of the heat kernel – and thus of the LHS – give

$$\Delta_x p(x, y, t) = \Delta_y p(x, y, t) \quad (\mathfrak{m} \times \mathfrak{m})\text{-a.e. } (x, y), \text{ for all } t > 0. \tag{3.10}$$

We conclude pointing out that the continuity of the heat kernel and estimate (3.5) ensure that, for any $t > 0$, the map $y \mapsto p_{y,t} \in L^2(\mathsf{X}, \mathfrak{m})$ is continuous. Thus, by the first claim in Proposition 3.4, we deduce that $y \mapsto \mathrm{d}p_{y,t} \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ is strongly Borel. Similarly for $y \mapsto \Delta p_{y,t}$ and $y \mapsto \mathrm{d}\Delta p_{y,t}$.

3.3. Smoothing metrics g_t and computation of $\nabla^* g_t$. In order to introduce the main tool in this paper, i.e. the *smoothing metrics* g_t, let us start this subsection by observing the smooth case as follows.

For an n-dimensional weighted complete Riemannian manifold $(M, g, e^{-V}\mathrm{dVol}_g)$ satisfying $\mathrm{Ric}_N \ge Kg$ for some $K \in \mathbb{R}$ and some $N \in [n, \infty)$ (namely $(M, \mathsf{d}_g, e^{-V}\mathrm{Vol}_g)$ is an $\mathrm{RCD}(K, N)$ space; recall (1.3) and (1.4)), for any $t > 0$, define the map

$$\Phi_t \colon M \to L^2(M, e^{-V}\mathrm{Vol}_g)$$

by

$$\Phi_t(x) := (y \mapsto p(x, y, t)) \in L^2(M, e^{-V}\mathrm{Vol}_g).$$

Then the pull-back $g_t := (\Phi_t)^* g_{L^2}$ is well-defined as a smooth tensor of type $(0, 2)$, and it satisfies

$$g_t(x) = \int_M \mathrm{d}_x p_{y,t}(x) \otimes \mathrm{d}_x p_{y,t}(x) e^{-V(y)}\, \mathrm{dVol}_g(y) \quad \text{for all } x \in M,$$

where it is emphasized that the RHS of the above makes sense as Bochner integral for *any* $x \in M$ because of (3.6). In particular, thanks to Fubini's theorem, for all smooth vector fields V_i $(i = 1, 2)$ on M with bounded supports, we have

$$\begin{aligned}\int_M & g_t(V_1, V_2) e^{-V}\, \mathrm{dVol}_g \\ &= \int_M \int_M \mathrm{d}_x p_{y,t}(V_1)(x)\, \mathrm{d}_x p_{y,t}(V_2)(x) e^{-V(x)-V(y)}\, \mathrm{dVol}_g(x)\, \mathrm{dVol}_g(y),\end{aligned}$$

and it is easy to see that this equation also characterizes g_t.

Let us generalize this observation to an $\mathrm{RCD}(K,N)$ space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ as follows. Start noticing that the identity $|\mathrm{d}p_{y,t}\otimes\mathrm{d}p_{y,t}|_{\mathrm{HS}}=|\mathrm{d}p_{y,t}|^2$, (3.15) below and Lemma 2.7 ensure that, for every $t>0$, the map $y\mapsto\mathrm{d}p_{y,t}\otimes\mathrm{d}p_{y,t}$ is in $L^1(\mathsf{X},\mathfrak{m};L^2_{\mathrm{loc}}((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m})))$, i.e., for a bounded $A\subseteq\mathsf{X}$ and a fixed $\bar{x}\in A$,

$$\int_{\mathsf{X}}\sqrt{\int_A|\mathrm{d}p_{y,t}|^2\,\mathrm{d}\mathfrak{m}(x)}\,\mathrm{d}\mathfrak{m}(y)\le C\sqrt{\mathfrak{m}(A)}\int_{\mathsf{X}}e^{-\mathsf{d}^2(\bar{x},y)/(5t)}\,\mathrm{d}\mathfrak{m}(y)<\infty.$$

Hence the following definition is well-posed.

Definition 3.6 (Smoothing metrics g_t). Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space. We define the $(0,2)$ tensor $g_t\in L^0((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ on X as

$$g_t:=\int_{\mathsf{X}}\mathrm{d}_x p_{y,t}\otimes\mathrm{d}_x p_{y,t}\,\mathrm{d}\mathfrak{m}(y).$$

Notice that the basic properties of Bochner integration (Hille's Theorem) ensure that, for $V_1,V_2\in L^2(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$ with bounded support, we have

$$\int_{\mathsf{X}}g_t(V_1,V_2)\,\mathrm{d}\mathfrak{m}=\int_{\mathsf{X}}\int_{\mathsf{X}}\mathrm{d}p_{y,t}(V_1)\,\mathrm{d}p_{y,t}(V_2)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}\mathfrak{m}(y).$$

After a normalization of g_t as follows, the smoothing metrics are uniformly bounded in L^∞.

Proposition 3.7. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space. Then we have*

$$t\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t\le C(K,N)g\quad\mathfrak{m}\text{-a.e., for all } t\in(0,1],\tag{3.11}$$

in the sense of symmetric tensors. In particular, we have $g_t\in L^\infty_{\mathrm{loc}}((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ and moreover $t\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t\in L^\infty((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$.

Proof. Fix $V\in L^0(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$, and notice that, for $\mathfrak{m}$-a.e. x, we have

$$\begin{aligned}t\,\mathfrak{m}(B_{\sqrt{t}}(x))g_t(V,V)(x)&\le t\,\mathfrak{m}(B_{\sqrt{t}}(x))|V|^2(x)\int_{\mathsf{X}}|\mathrm{d}p_{y,t}|^2(x)\,\mathrm{d}\mathfrak{m}(y)\\&\overset{(3.6)}{\le}\frac{C|V|^2(x)}{\mathfrak{m}(B_{\sqrt{t}}(x))}\int_{\mathsf{X}}\exp\Big(-\frac{\mathsf{d}(x,y)^2}{5t}+Ct\Big)\,\mathrm{d}\mathfrak{m}(y).\end{aligned}$$

The conclusion follows from Lemma 2.7 (with $\alpha=0$). □

We now turn to the computation of $\nabla^* g_t$. To this aim, it is convenient to introduce the following function:

$$p_t(x):=p(x,x,t)\overset{(3.7)}{=}\int_{\mathsf{X}}p^2_{y,t/2}(x)\,\mathrm{d}\mathfrak{m}(y).$$

Notice that, thanks to bound (3.5), it is easy to see that, for every $t>0$, the map $y\mapsto p^2_{y,t/2}$ is in $L^1(\mathsf{X},\mathfrak{m};L^2(\mathsf{X},\mathfrak{m}))$. It is then clear that the identity $p_t=\int_{\mathsf{X}}p^2_{y,t/2}\,\mathrm{d}\mathfrak{m}(y)$ holds also in the sense of Bochner integrals.

Let us start collecting some estimates for this function.

Lemma 3.8. *Assume that* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K, N)$ *space. Then, for any* $t > 0$*, we have* $p_{2t}(x) \in D_{\mathrm{loc}}(\Delta)$ *with*

$$\text{(3.12)} \qquad \begin{aligned} \mathrm{d}p_t &= 2\int_{\mathsf{X}} p_{y,t/2}\mathrm{d}p_{y,t/2}\,\mathrm{d}\mathfrak{m}(y), \\ |\mathrm{d}p_t| &\leq \frac{C(K,N)}{\sqrt{t}\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))} \quad \mathfrak{m}\text{-a.e.}, \end{aligned}$$

and

$$\text{(3.13)} \qquad \begin{aligned} \Delta p_t &= 2\int_{\mathsf{X}} p_{y,t/2}\Delta p_{y,t/2} + |\mathrm{d}p_{y,t}|^2\,\mathrm{d}\mathfrak{m}(y), \\ |\Delta p_t| &\leq \frac{C(K,N)}{t\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))} \quad \mathfrak{m}\text{-a.e.} \end{aligned}$$

Finally, we also have $\Delta p_t \in H^{1,2}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *with*

$$\text{(3.14)} \qquad \mathrm{d}\Delta p_t = 2\int_{\mathsf{X}} \mathrm{d}p_{y,t/2}\Delta p_{y,t/2} + p_{y,t/2}\mathrm{d}\Delta p_{y,t/2} + \mathrm{d}|\mathrm{d}p_{y,t}|^2\,\mathrm{d}\mathfrak{m}(y).$$

Part of the claim is the fact that the integrands in (3.12) *and* (3.14) *belong to the space* $L^1(\mathsf{X}, \mathfrak{m}; L^2_{\mathrm{loc}}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$*, and the one in* (3.13) *belongs to the space* $L^1(\mathsf{X}, \mathfrak{m}; L^2_{\mathrm{loc}}(\mathsf{X}, \mathfrak{m}))$.

Proof. Using (3.5) and (3.6), we get

$$\begin{aligned} \int_{\mathsf{X}} |\mathrm{d}(p^2_{y,t/2})|\,\mathrm{d}\mathfrak{m}(y) &\leq 2\int_{\mathsf{X}} p_{y,t/2}|\mathrm{d}p_{y,t/2}|\,\mathrm{d}\mathfrak{m}(y) \\ &\leq \frac{C}{\sqrt{t}\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))^2}\int_{\mathsf{X}} \exp\Big(-\frac{2\mathsf{d}^2(\cdot,y)}{5t} + Ct\Big)\,\mathrm{d}\mathfrak{m}(y). \end{aligned}$$

Thus, from Lemma 2.7 and Proposition 3.4, we deduce that $p_{2t} \in H^{1,2}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ and that (3.12) holds. Similarly, starting from

$$\int_{\mathsf{X}} |\Delta(p^2_{y,t/2})|\,\mathrm{d}\mathfrak{m}(y) \leq 2\int_{\mathsf{X}} p_{y,t/2}|\Delta p_{y,t/2}| + |\mathrm{d}p_{y,t/2}|^2\,\mathrm{d}\mathfrak{m}(y)$$

and using estimates (3.5), (3.6) and (3.8) and then again Lemma 2.7 and Proposition 3.4, we conclude that $p_{2t}(x) \in D_{\mathrm{loc}}(\Delta)$ and that (3.13) holds.

For the last claim, we recall that $p_{y,t} \in \mathrm{Test}\,F(\mathsf{X}, \mathsf{d}, \mathfrak{m})$; thus the Leibniz rule for the Laplacian and the basic properties of test functions give

$$p_{y,t/2}\Delta p_{y,t/2} + |\mathrm{d}p_{y,t}|^2 \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$$

with

$$\mathrm{d}(p_{y,t/2}\Delta p_{y,t/2} + |\mathrm{d}p_{y,t}|^2) = \mathrm{d}p_{y,t/2}\Delta p_{y,t/2} + p_{y,t/2}\mathrm{d}\Delta p_{y,t/2} + 2\,\mathrm{Hess}\,p_{y,t}(\nabla p_{y,t}, \cdot).$$

The fact that the first two addends in the RHS are in $L^1(\mathsf{X}, \mathfrak{m}; L^2_{\mathrm{loc}}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$ can be proved as before. For the last one, we let $A \subseteq \mathsf{X}$ be Borel and bounded and $\bar{x} \in \mathsf{X}$. Then we have $\mathsf{d}(x, y) \geq \mathsf{d}(y, \bar{x}) - R$ for any $x \in A$, $y \in \mathsf{X}$ and some $R > 0$ independent of x, y. Hence (3.6) implies that

$$\||\mathrm{d}p_{y,t}|\|_{L^\infty(A)} \leq Ce^{-\mathsf{d}^2(y,\bar{x})/(5t)} \quad \text{for some } C = C(t, K, N, A, \bar{x}),$$

and thus

$$\begin{aligned}\int_{\mathsf{X}} \sqrt{\int_A |\mathrm{Hess}\, p_{y,t}(\nabla p_{y,t}, \cdot)|^2 \,\mathrm{d}\mathfrak{m}(x)}\,\mathrm{d}\mathfrak{m}(y) &\leq \int_{\mathsf{X}} \||\mathrm{Hess}\, p_{y,t}|_{\mathrm{HS}}\|_{L^2} \||\mathrm{d}p_{y,t}|\|_{L^\infty(A)}\,\mathrm{d}\mathfrak{m}(y)\\ &\overset{(2.10)}{\leq} C \int_{\mathsf{X}} (\|\Delta p_{y,t}\|_{L^2} + \||\mathrm{d}p_{y,t}|\|_{L^2}) e^{-\mathsf{d}^2(y,\bar{x})/(5t)}\,\mathrm{d}\mathfrak{m}(y)\\ &\overset{(3.9)}{\leq} C \int_{\mathsf{X}} \mathfrak{m}(B_{\sqrt{t}}(y))^{-1/2} e^{-\mathsf{d}^2(y,\bar{x})/(5t)}\,\mathrm{d}\mathfrak{m}(y) < \infty,\end{aligned}$$

where the last inequality comes from Lemma 2.7. The conclusion follows. □

To further analyse the link between g_t and p_t, the following result will be crucial.

Lemma 3.9. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Then, for every* $t > 0$, *we have that* $y \mapsto \Delta p_{y,t} \mathrm{d}p_{y,t}$ *and* $y \mapsto p_{y,t} \mathrm{d}\Delta p_{y,t}$ *are both in* $L^1(\mathsf{X}, \mathfrak{m}; L^2_{\mathrm{loc}}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$ *and*

$$\int_{\mathsf{X}} \Delta p_{y,t} \mathrm{d}p_{y,t}\,\mathrm{d}\mathfrak{m}(y) = \int_{\mathsf{X}} p_{y,t} \mathrm{d}\Delta p_{y,t}\,\mathrm{d}\mathfrak{m}(y).$$

Proof. For the first part of the claim, we argue as in the proof of Lemma 3.8 above: let $A \subseteq \mathsf{X}$ be Borel and bounded and $\bar{x} \in \mathsf{X}$. Then (3.6) implies that, for some $C = C(t, K, N, A, \bar{x})$,

$$\||\mathrm{d}p_{y,t}|\|_{L^\infty(A)} \leq \|C e^{-\mathsf{d}^2(y,\cdot)/((4+1/2)t)}\|_{L^\infty(A)} \leq C e^{-\mathsf{d}^2(y,\bar{x})/(5t)} \tag{3.15}$$

so that

$$\int_{\mathsf{X}} \sqrt{\int_A |\Delta p_{y,t} \mathrm{d}p_{y,t}|^2 \,\mathrm{d}\mathfrak{m}(x)}\,\mathrm{d}\mathfrak{m}(y) \leq C \int_{\mathsf{X}} \|\Delta p_{y,t}\|_{L^2} e^{-\mathsf{d}^2(y,\bar{x})/(5t)}\,\mathrm{d}\mathfrak{m}(y) < \infty,$$

having used bound (3.9) and Lemma 2.7 in the last step. This proves that $y \mapsto \Delta p_{y,t} \mathrm{d}p_{y,t}$ is in $L^1(\mathsf{X}, \mathfrak{m}; L^2_{\mathrm{loc}}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$, and an analogous argument gives the same for $y \mapsto p_{y,t} \mathrm{d}\Delta p_{y,t}$.

Now write the Chapman–Kolmogorov equation (3.7) as

$$\int_{\mathsf{X}} p(y, z, s) p_{z,t}\,\mathrm{d}\mathfrak{m}(z) = p_{y,t+s},$$

and observe that estimates (3.5), (3.6) and the same arguments just used ensure that, for any $y \in \mathsf{X}$, the maps $z \mapsto p(y, z, s) p_{z,t}$ and $z \mapsto p(y, z, s) \mathrm{d}p_{z,t}$ are in $L^1(\mathsf{X}, \mathfrak{m}; L^2_{\mathrm{loc}}(\mathsf{X}, \mathfrak{m}))$ and $L^1(\mathsf{X}, \mathfrak{m}; L^2_{\mathrm{loc}}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$, respectively. Thus Proposition 3.4 gives

$$\int_{\mathsf{X}} p(y, z, s)\,\mathrm{d}p_{z,t}\,\mathrm{d}\mathfrak{m}(z) = \mathrm{d}p_{y,t+s}.$$

Multiplying both sides by $p_{y,t}$, integrating in y and using Fubini's theorem, we obtain

$$\begin{aligned}\int_{\mathsf{X}} p_{z,t+s} \mathrm{d}p_{z,t}\,\mathrm{d}\mathfrak{m}(z) &\overset{(3.7)}{=} \int_{\mathsf{X}} \int_{\mathsf{X}} p_{y,t}\, p(y, z, s)\,\mathrm{d}p_{z,t}\,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}\mathfrak{m}(y)\\ &= \int_{\mathsf{X}} p_{y,t} \mathrm{d}p_{y,t+s}\,\mathrm{d}\mathfrak{m}(y).\end{aligned}$$

Thus, to conclude, it is sufficient to prove that, as $s \to 0^+$, we have

$$\begin{aligned} \int_{\mathsf{X}} \frac{p_{y,t+s}-p_{y,t}}{s} \, \mathrm{d}p_{y,t} \, \mathrm{d}\mathfrak{m}(y) &\to \int_{\mathsf{X}} \Delta p_{y,t} \, \mathrm{d}p_{y,t} \, \mathrm{d}\mathfrak{m}(y), \\ \int_{\mathsf{X}} p_{y,t} \mathrm{d}\Big(\frac{p_{y,t+s}-p_{y,t}}{s}\Big) \, \mathrm{d}\mathfrak{m}(y) &\to \int_{\mathsf{X}} p_{y,t} \, \mathrm{d}\Delta p_{y,t} \, \mathrm{d}\mathfrak{m}(y) \end{aligned} \tag{3.16}$$

in $L^2_{\mathrm{loc}}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$. We start noticing that, from (3.7), we have

$$\begin{aligned} F_{y,t} := \frac{p_{y,t+s}-p_{y,t}}{s} - \Delta p_{y,t} &= \int_0^1 \Delta(p_{y,t+rs} - p_{y,t}) \, \mathrm{d}r \\ &= s \int_0^1 r\Delta\Big(\int_0^1 \Delta p_{y,t+rsh} \, \mathrm{d}h\Big) \, \mathrm{d}r \\ &= s \int_0^1 r\Delta \mathrm{h}_{t/3}\Big(\int_0^1 \Delta \mathrm{h}_{t/3} p_{y,t/3+rsh} \, \mathrm{d}h\Big) \, \mathrm{d}r, \end{aligned}$$

and therefore, using twice (2.4), we obtain

$$\begin{aligned} \|F_{y,t}\|_{L^2} &\le sC(t) \int_0^1 \int_0^1 \|p_{y,t/3+rsh}\|_{L^2} \, \mathrm{d}h \, \mathrm{d}r \\ &\overset{(2.5)}{\le} sC(t)\|p_{y,t/3}\|_{L^2} \\ &\overset{(3.9)}{\le} sC(K,N,t)\mathfrak{m}(B_{\sqrt{t/3}}(y))^{-1/2}. \end{aligned} \tag{3.17}$$

Thus, for $A \subseteq \mathsf{X}$ Borel and bounded, we have

$$\begin{aligned} \int_{\mathsf{X}} \sqrt{\int_A |F_{y,t} \mathrm{d}p_{y,t}|^2 \, \mathrm{d}\mathfrak{m}} \, \mathrm{d}\mathfrak{m}(y) &\overset{(3.15)}{\le} C \int_{\mathsf{X}} \|F_{y,t}\|_{L^2} e^{-\mathsf{d}^2(y,\bar{x})/(5t)} \, \mathrm{d}\mathfrak{m}(y) \\ &\overset{(3.17)}{\le} sC \int_{\mathsf{X}} \frac{e^{-\mathsf{d}^2(y,\bar{x})/(5t)}}{\mathfrak{m}(B_{\sqrt{t/3}}(y))^{1/2}} \, \mathrm{d}\mathfrak{m}(y) \end{aligned}$$

for some $C = C(K, N, t, A, \bar{x})$. Since the last integral is finite by Lemma 2.7, the LHS goes to 0 as $s \to 0^+$. This proves the first formula in (3.16). The second follows along very similar lines; we omit the details. □

We are now in a position to prove the main result of this subsection.

Theorem 3.10. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Then, for every* $t > 0$, *we have* $g_t \in D_{\mathrm{loc}}(\nabla^*)$ *with*

$$\nabla^* g_t = -\frac{1}{4}\mathrm{d}\Delta p_{2t}. \tag{3.18}$$

Proof. For any $y \in \mathsf{X}$, Proposition 2.6 tells

$$\nabla^*(\mathrm{d}p_{y,t} \otimes \mathrm{d}p_{y,t}) = -\Delta p_{y,t}\mathrm{d}p_{y,t} - \frac{1}{2}\mathrm{d}|\nabla p_{y,t}|^2.$$

Also, arguing as in Lemma 3.8, it is easy to see that

$$y \mapsto -\Delta p_{y,t}\mathrm{d}p_{y,t} - \frac{1}{2}\mathrm{d}|\nabla p_{y,t}|^2$$

belongs to the space $L^1(\mathsf{X}, \mathfrak{m}; L^2_{\mathrm{loc}}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$. Thus, taking into account Lemma 3.9, we obtain

$$\begin{aligned}
&\int_{\mathsf{X}} \nabla^*(\mathrm{d}p_{y,t} \otimes \mathrm{d}p_{y,t})\,\mathrm{d}\mathfrak{m}(y) \\
&\qquad = -\frac{1}{2}\int_{\mathsf{X}} \Delta p_{y,t}\mathrm{d}p_{y,t} + p_{y,t}\mathrm{d}\Delta p_{y,t} + \mathrm{d}|\mathrm{d}p_{y,t}|^2\,\mathrm{d}\mathfrak{m}(y) \\
&\qquad \overset{(3.14)}{=} -\frac{1}{4}\mathrm{d}\Delta p_{2t}.
\end{aligned}$$

The conclusion comes from the very definition of g_t and Proposition 3.4. □

3.4. Asymptotic behaviour as $t \to 0^+$. The goal of this subsection is to study the behaviour of g_t and $\nabla^* g_t$ as $t \to 0^+$.

We start with the following result, which generalizes the analogous statement [8, Theorem 5.10] to the non-compact setting.

Theorem 3.11. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an* RCD(K, N) *space of essential dimension n. Then $t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t \to c_n g$ strongly in L^p_{loc} for any $p \in [1, \infty)$, where c_n is a positive constant depending only on n.*

Proof. Since the proof is essentially the same as that of [8, Theorem 5.10] after replacing L^p by L^p_{loc} (recall that [8, Theorem 5.10] discussed only on the case when (X, d) is compact), we shall only give a sketch of the proof.

Fix $V \in L^\infty(T(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ with bounded support. First let us discuss the asymptotic behaviour of the following as $t \to 0^+$ for fixed $y \in \mathsf{X}$ and $L > 0$:

$$\begin{aligned}
&\int_{\mathsf{X}} t\mathfrak{m}(B_{\sqrt{t}}(x))|\mathrm{d}p_{y,t}(V)|^2(x)\,\mathrm{d}\mathfrak{m}(x) \\
&\qquad = \int_{B_{L\sqrt{t}}(y)} t\mathfrak{m}(B_{\sqrt{t}}(\cdot))|\mathrm{d}p_{y,t}(V)|^2\,\mathrm{d}\mathfrak{m} \\
&\qquad\quad + \int_{\mathsf{X}\setminus B_{L\sqrt{t}}(y)} t\mathfrak{m}(B_{\sqrt{t}}(\cdot))|\mathrm{d}p_{y,t}(V)|^2\,\mathrm{d}\mathfrak{m}.
\end{aligned} \tag{3.19}$$

The key idea to control each term in the RHS of (3.19) is to apply *blow-up arguments* (i.e. we discuss the behaviour of the rescaled spaces $(\mathsf{X}, \sqrt{t}^{-1}\mathsf{d}, \mathfrak{m}(B_{\sqrt{t}}(y))^{-1}\mathfrak{m}, y)$ with respect to the pointed measured Gromov–Hausdorff convergence as $t \to 0^+$) in conjunction with the stability of the heat flow first observed in [28]. More precisely, we use the stability results proved in [7, Corollary 5.5, Theorem 5.7, Lemma 5.8], [6, Theorem 4.4], [9, Theorem 3.3] (with [8, Theorem 2.19]), [34, Theorem 6.8] with Theorem 2.13 and (3.6). Combining these, letting $t \to 0^+$ and then letting $L \to \infty$ in the RHS of (3.19), the following hold for $\mathfrak{m}$-a.e. $y \in \mathsf{X}$.

(1) The first term of the RHS of (3.19) converges to $c_n|V|^2(y)$.

(2) The second term of the RHS of (3.19) converges to 0.

Thus, as $t \to 0^+$, we obtain

$$\int_{\mathsf{X}} t\mathfrak{m}(B_{\sqrt{t}}(x))|\mathrm{d}p_{y,t}(V)|^2(x)\,\mathrm{d}\mathfrak{m}(x) \to c_n|V|^2(y) \quad \mathfrak{m}\text{-a.e. } y \in \mathsf{X}.$$

Thus, combining this with (3.11) and the dominated convergence theorem, we get

$$\int_{\mathsf{X}} t\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t(V,V)\,\mathrm{d}\mathfrak{m} \to c_n \int_{\mathsf{X}} |V|^2\,\mathrm{d}\mathfrak{m},$$

which proves that $t\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t$ L^p-weakly converge to $c_n g$ on any bounded subset A of X because g_t is symmetric and V is arbitrary.

In order to get the L^p_{loc}-strong convergence, it suffices to check

$$\lim_{t\to 0^+}\int_{\mathsf{X}} \varphi|t\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} = c_n^2\int_{\mathsf{X}}\varphi|g|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} = c_n^2 n\int_{\mathsf{X}}\varphi\,\mathrm{d}\mathfrak{m} \tag{3.20}$$

for every $\varphi \in \mathrm{Lip}_{\mathrm{bs}}(\mathsf{X},\mathsf{d})$ because this implies the L^2_{loc}-strong convergence, and the improvement to the L^p_{loc}-strong one comes from (3.11). Let us check (3.20) as follows.

For any $z \in \mathcal{R}_n$, applying blow-up arguments as explained above, again allows us to deduce

$$F(z,t) := \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(z))}\int_{B_{\sqrt{t}}(z)} |t\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} \to c_n^2 n,$$

and thus (recalling (3.11) to use the dominate convergence theorem), for $\varphi \in \mathrm{Lip}_{\mathrm{bs}}(\mathsf{X},\mathsf{d})$, we have

$$\lim_{t\to 0^+}\int_{\mathsf{X}}\varphi(z)F(z,t)\,\mathrm{d}\mathfrak{m}(z) = c_n^2 n\int_{\mathsf{X}}\varphi\,\mathrm{d}\mathfrak{m}. \tag{3.21}$$

On the other hand, we have

$$\begin{aligned}\int_{\mathsf{X}}\varphi(z)F(z,t)\,\mathrm{d}\mathfrak{m}(z) \\ = \int_{\mathsf{X}}|t\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t|^2_{\mathrm{HS}}(x)\underbrace{\int_{B_{\sqrt{t}}(x)}\frac{\varphi(z)}{\mathfrak{m}(B_{\sqrt{t}}(z))}\mathrm{d}\mathfrak{m}(z)}_{=:G(x,t)}\,\mathrm{d}\mathfrak{m}(x).\end{aligned} \tag{3.22}$$

Now notice that $\sup_{t,x} G(x,t) < \infty$ (because of (2.13)) and $\lim_{t\to 0^+} G(x,t) = \varphi(x)$ for $\mathfrak{m}$-a.e. x (because of the convergence of the blow-ups to the Euclidean space). It follows (again using (3.11) to use the dominate convergence theorem) that

$$\lim_{t\to 0^+}\int_{\mathsf{X}}|t\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t|^2_{\mathrm{HS}}(x)\left|\int_{B_{\sqrt{t}}(x)}\frac{\varphi(z)}{\mathfrak{m}(B_{\sqrt{t}}(z))}\mathrm{d}\mathfrak{m}(z) - \varphi(x)\right|\mathrm{d}\mathfrak{m}(x) = 0,$$

which together with (3.21) and (3.22) gives (3.20) and the conclusion. □

Remark 3.12. In Theorem 3.11, the conclusion cannot be improved to the case when $p = \infty$ in general. For example, the RCD(0, 1) space $([0,\pi], \mathsf{d}_{\mathbb{R}}, \mathscr{H}^1)$ satisfies

$$\liminf_{t\to 0^+}\|t\mathscr{H}^1(B_{\sqrt{t}}(\cdot))g_t - c_1 g_{\mathbb{R}}\|_{L^\infty} > 0.$$

See [8, Remark 5.11] for details. It is worth pointing out that the verification of

$$\|t\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t - c_n g\|_{L^\infty_{\mathrm{loc}}} \to 0 \tag{3.23}$$

is closely related to the nonexistence of singular points (actually, the singular points are $\{0,\pi\}$ in this example). See also [44, Theorem 1.1].

In connection with this pointing out, if $(M, g, e^{-V}\mathrm{dVol}_g)$ is any weighted complete n-dimensional Riemannian manifold with $\mathrm{Ric}_N \geq Kg$ for some $K \in \mathbb{R}$ and some $N \in [n, \infty)$, applying a construction of the heat kernel by *parametrix*, we can actually prove that (3.23) holds. More precisely, we have, as $t \to 0^+$,

$$4(8\pi)^{n/2} t^{(n+2)/2} g_t = e^V g - e^V \Big(\frac{2}{3}\Big(\mathrm{Ric}_g - \frac{1}{2}\mathrm{Scal}_g g\Big) - \mathrm{d}V \otimes \mathrm{d}V - \Delta^g V g + \frac{|\nabla^g V|^2}{2} g\Big) t + O(t^2),$$

which is uniform on any bounded set. See [15, Theorem 5] and [45, Theorem 3.5] for details. □

Corollary 3.13. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n. *Let* A *be a bounded Borel subset of* X *with*

$$\inf_{r \in (0,1),\, x \in A} \frac{\mathfrak{m}(B_r(x))}{r^n} > 0. \tag{3.24}$$

Then $\mathscr{H}^n \llcorner A$ *is a Radon measure absolutely continuous with respect to* $\mathfrak{m}$ *and*

$$\chi_A \omega_n t^{(n+2)/2} g_t \to \chi_A c_n \frac{\mathrm{d}\mathscr{H}^n \llcorner A}{\mathrm{d}\mathfrak{m}} g \quad \text{in } L^p((T^*)^{\otimes 2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})) \text{ for all } p \in [1, \infty).$$

Proof. The first part of the claim follows from Lemma 2.14 and (3.24). Then Theorem 2.13 ensures that, as $r \to 0^+$,

$$\frac{\omega r^n}{\mathfrak{m}(B_r(x))} \to \frac{\mathrm{d}\mathscr{H}^n \llcorner A}{\mathrm{d}\mathfrak{m}} \quad \mathfrak{m}\text{-a.e. } x \in A.$$

Thus (3.24), the dominated convergence theorem and Theorem 3.11 give the conclusion. □

We now turn to the asymptotic of Δp_{2t}.

Proposition 3.14. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space,* $K \in \mathbb{R}$*,* $N \in [1, \infty)$*. Then, as* $t \to 0^+$*, we have*

$$t\,\mathfrak{m}(B_{\sqrt{t}}(\cdot))\Delta p_{2t}(\cdot) \to 0 \quad \text{in } L^p_{\mathrm{loc}}(\mathsf{X}, \mathfrak{m}) \text{ for all } p \in [1, \infty).$$

Proof. The proof is based on blow-up arguments, which is similar to that of Theorem 3.11. Therefore, we give only a sketch of the proof (see also [8]).

Let us first prove that, for any $z \in \mathcal{R}_n$, as $t \to 0^+$,

$$\frac{1}{\mathfrak{m}(B_{\sqrt{t}}(z))} \int_{B_{\sqrt{t}}(z)} t\,\mathfrak{m}(B_{\sqrt{t}}(x))|\Delta p_{2t}(x)| \,\mathrm{d}\mathfrak{m}(x) \to 0. \tag{3.25}$$

In order to prove this, consider the pointed measured Gromov–Hausdorff convergent sequence of the rescaled space,

$$(\mathsf{X}^{t,z}, \mathsf{d}^{t,z}, \mathfrak{m}^{t,z}, z) := \Big(\mathsf{X}, \frac{1}{\sqrt{t}}\mathsf{d}, \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(z))}\mathfrak{m}, z\Big) \xrightarrow{\text{pmGH}} \Big(, \mathbb{R}^n, \mathsf{d}_{\mathbb{R}^n}, \frac{1}{\omega_n}\mathscr{H}^n, 0_n\Big), \tag{3.26}$$

and denote by $p^{t,z}$ and $\Delta^{t,z}$ the heat kernel and the Laplacian of $(\mathsf{X}^{t,z}, \mathsf{d}^{t,z}, \mathfrak{m}^{t,z})$, respectively, namely

$$p^{t,z}(x,y,s) = \mathfrak{m}(B_{\sqrt{t}}(z))p(x,y,ts), \quad \Delta^{t,z} f = t\Delta f.$$

Thus the LHS of (3.25) is equal to

$$\int_{B_1^{\mathsf{d}^{t,z}}(z)} \mathfrak{m}^{t,z}(B_1^{\mathsf{d}^{t,z}}(x))|\Delta^{t,z} p_2^{t,z}(x)|\mathrm{d}\mathfrak{m}^{t,z}(x). \tag{3.27}$$

Applying the stability results already used in the proof of Theorem 3.11 shows that $\Delta^{t,z} p_2^{t,z}$ L^2_{loc}-strongly converge to $\Delta^{g_{\mathbb{R}^n}}(\omega_n \widetilde{p}_2)$ with respect to (3.26), where $\widetilde{p}(x)$ denotes the heat kernel of the n-dimensional Euclidean space evaluated at (x,x). Since $\Delta^{g_{\mathbb{R}^n}}(\omega_n \widetilde{p}_2) = 0$ because $\widetilde{p}_2$ is constant, (3.27) converges to

$$\int_{B_1(0_n)} |\Delta^{g_{\mathbb{R}^n}}(\omega_n \widetilde{p}_2)|\mathrm{d}\Big(\frac{1}{\omega_n}\mathscr{H}^n\Big) = 0$$

as $t \to 0^+$, which proves (3.25).

Fix a $\varphi \in \mathrm{Lip}_{\mathrm{bs}}(\mathsf{X},\mathsf{d})$. Applying (3.25) with (3.13), the dominated convergence theorem yields

$$\int_{\mathsf{X}} \frac{\varphi(z)}{\mathfrak{m}(B_{\sqrt{t}}(z))} \int_{B_{\sqrt{t}}(z)} t\,\mathfrak{m}(B_{\sqrt{t}}(x))|\Delta p_{2t}(x)|\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(z) \to 0.$$

On the other hand, (3.13) and dominated convergence (recall (2.13)) imply

$$\begin{aligned}\Bigg|\int_{\mathsf{X}} \frac{\varphi(z)}{\mathfrak{m}(B_{\sqrt{t}}(z))} \int_{B_{\sqrt{t}}(z)} & t\,\mathfrak{m}(B_{\sqrt{t}}(x))|\Delta p_{2t}(x)|\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(z) \\ &- \int_{\mathsf{X}} \varphi(z) t\,\mathfrak{m}(B_{\sqrt{t}}(z))|\Delta p_{2t}(z)|\,\mathrm{d}\mathfrak{m}(z)\Bigg| \\ \le C(K,N) & \int_{\mathsf{X}} \Bigg|\varphi(z) - \int_{B_{\sqrt{t}}(z)} \frac{\varphi(x)}{\mathfrak{m}(B_{\sqrt{t}}(x))}\mathrm{d}\mathfrak{m}(x)\Bigg|\mathrm{d}\mathfrak{m}(z) \to 0.\end{aligned}$$

Thus

$$\int_{\mathsf{X}} \varphi(x) t\,\mathfrak{m}(B_{\sqrt{t}}(x))|\Delta p_{2t}(x)|\,\mathrm{d}\mathfrak{m}(x) \to 0. \tag{3.28}$$

The desired L^p_{loc}-strong convergence comes from (3.28) and (3.13). □

Corollary 3.15. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space. Also, let A be a bounded Borel subset of X, and let $n \in \mathbb{N}$ be such that*

$$\inf_{r\in(0,1),\, x\in A} \frac{\mathfrak{m}(B_r(x))}{r^n} > 0.$$

Then, as $t \to 0^+$,

$$t^{(n+2)/2}\Delta p_{2t} \to 0 \quad \text{in } L^2(A,\mathfrak{m}).$$

Proof. This is a direct consequence of Proposition 3.14. □

Remark 3.16. Although the above convergence results are stated for the strong convergence in order to get our best knowledge, their weak convergences are enough to justify our main results as easily seen in the next section. □

4. Proof of the main results

From both the technical and conceptual points of view, the following is the crucial result in this paper. Its proof is basically a combination of the convergence results established in Corollaries 3.13, 3.15 together with formula (3.18).

Theorem 4.1 (Integration-by-parts formula). *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *space of essential dimension n. Let also $U \subseteq \mathsf{X}$ be open, and assume that*

$$\inf_{r\in(0,1),\, x\in A} \frac{\mathfrak{m}(B_r(x))}{r^n} > 0$$

for every compact subset A of U. Then, for any $\varphi \in \mathrm{Lip}_{\mathrm{bs}}(\mathsf{X}, \mathsf{d})$ with $\mathrm{supp}(\varphi) \subseteq U$, $f \in D(\Delta)$, *it holds that*

$$\int_{\mathsf{X}} \langle \mathrm{d}\varphi, \mathrm{d}f\rangle \, \mathrm{d}\mathscr{H}^n = -\int_{\mathsf{X}} \varphi \, \mathrm{tr}(\mathrm{Hess}\, f) \, \mathrm{d}\mathscr{H}^n.$$

Proof. The assumptions on φ, f ensure that $\varphi \, \mathrm{d}f$ is in the domain of the covariant derivative with $\nabla(\varphi \, \mathrm{d}f) = \mathrm{d}\varphi \otimes \mathrm{d}f + \varphi \, \mathrm{Hess}\, f$ (see [32, Theorem 3.4.2, Proposition 3.4.5]), with identifications under the Riesz isomorphisms. Thus (3.18) gives

$$\int_{\mathsf{X}} \langle t^{(n+2)/2} g_t, \nabla(\varphi \, \mathrm{d}f)\rangle_{\mathrm{HS}} \, \mathrm{d}\mathfrak{m} = -\frac{1}{4}\int_{\mathsf{X}} \langle \nabla\Delta(t^{(n+2)/2} p_{2t}), \varphi\nabla f\rangle \, \mathrm{d}\mathfrak{m} \tag{4.1}$$
$$= \frac{1}{4}\int_{\mathsf{X}} \Delta(t^{(n+2)/2} p_{2t}) \, \mathrm{div}(\varphi\nabla f) \, \mathrm{d}\mathfrak{m}.$$

Let us take the limit $t \to 0^+$ in (4.1). The RHS converges to 0 because of Corollary 3.15 applied with $A := \mathrm{supp}(\varphi)$. On the other hand, by Corollary 3.13 applied with $A := \mathrm{supp}(\varphi)$, the LHS of (4.1) converges to, up to multiplying by a constant,

$$\int_{\mathsf{X}} \langle g, \nabla(\varphi \, \mathrm{d}f)\rangle_{\mathrm{HS}} \, \mathrm{d}\mathscr{H}^n = \int_{\mathsf{X}} \langle \mathrm{d}\varphi, \mathrm{d}f\rangle \, \mathrm{d}\mathscr{H}^n + \int_{\mathsf{X}} \varphi \, \mathrm{tr}(\mathrm{Hess}\, f) \, \mathrm{d}\mathscr{H}^n.$$

This completes the proof. □

To deduce from the above the equivalence of the "weak" and "strong" non-collapsed conditions, we shall use the following simple result.

Lemma 4.2. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *space. Also, let $U \subseteq \mathsf{X}$ be an open connected set and let $\xi \in L^\infty_{\mathrm{loc}}(U, \mathfrak{m})$. Assume that, for every $\psi \in \mathrm{Lip}_{\mathrm{bs}}(\mathsf{X}, \mathsf{d})$ with support in U and $f \in D(\Delta)$, it holds*

$$\int_{\mathsf{X}} \xi \langle \nabla\psi, \nabla f\rangle \, \mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} \xi\psi\Delta f \, \mathrm{d}\mathfrak{m}. \tag{4.2}$$

Then ξ is constant on U.

Proof. It suffices to check that ξ is locally constant on U because U is connected. Let $z \in \mathsf{X}$ and $r \in (0, \frac{1}{6})$ with $B_{3r}(z) \subseteq U$, and let $\psi \in \mathrm{Lip}(\mathsf{X}, \mathsf{d})$ be identically 1 on $B_{2r}(z)$ and with support in $B_{3r}(z)$. Also, set $\xi_t := \mathrm{h}_t(\chi_{B_{2r}(z)}\xi) \in D(\Delta)$, namely

$$\xi_t(y) = \int_{B_{2r}(z)} p(x, y, t)\xi(x) \, \mathrm{d}\mathfrak{m}(x)$$

for $\mathfrak{m}$-a.e. $y \in \mathsf{X}$, and notice that Hille's theorem (see also Proposition 3.4) gives

$$\Delta\xi_t(y) = \int_{B_{2r}(z)} \Delta_y p(x, y, t)\xi(x)\, \mathrm{d}\mathfrak{m}(x) \overset{(3.10)}{=} \int_{B_{2r}(z)} \xi \Delta p_{y,t}\, \mathrm{d}\mathfrak{m}.$$

This identity and assumption (4.2) (with $f = p_{y,t}$) give

$$\Delta\xi_t(y) = \int_{\mathsf{X}} (\chi_{B_{2r}(z)} - \psi)\xi\Delta p_{y,t}\, \mathrm{d}\mathfrak{m} - \int_{\mathsf{X}} \xi\langle\nabla\psi, \nabla p_{y,t}\rangle\, \mathrm{d}\mathfrak{m}$$

for $\mathfrak{m}$-a.e. $y \in \mathsf{X}$. Therefore, the assumption $\xi \in L^\infty_{\mathrm{loc}}(U, \mathfrak{m})$ tells that, for $y \in B_r(z)$, we have

$$\begin{aligned} |\Delta\xi_t|(y) &\le C \int_{B_{3r}(z)\setminus B_{2r}(z)} |\Delta p_{y,t}|\, \mathrm{d}\mathfrak{m} + C \int_{B_{3r}(z)\setminus B_{2r}(z)} |\nabla p_{y,t}|\, \mathrm{d}\mathfrak{m} \\ \text{(by (3.6), (3.8))} \quad &\le C(t^{-1} + t^{-1/2}) \exp\Big(-\frac{r^2}{5t}\Big) \int_{B_{3r}(z)} \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(x))}\, \mathrm{d}\mathfrak{m}(x), \end{aligned}$$

where C is a positive constant which is independent of t and y. Now notice that (2.11) and the assumption $r \in (0, \frac{1}{6})$ ensure that

$$\frac{1}{\mathfrak{m}(B_{\sqrt{t}}(x))} \le \frac{C(K, N)}{\mathfrak{m}(B_1(z))} t^{-N/2}$$

for every $t \in (0, 1)$ and $x \in B_{3r}(z)$. It then follows that $\Delta\xi_t$ uniformly converges to 0 on $B_r(z)$.

Let now $\varphi \in \mathrm{Lip}(\mathsf{X}, \mathsf{d})$ be with support in $B_r(z)$, and notice that

$$\begin{aligned} \int_{\mathsf{X}} |\mathrm{d}(\varphi\xi_t)|^2\, \mathrm{d}\mathfrak{m} &= \int_{\mathsf{X}} |\xi_t|^2|\mathrm{d}\varphi|^2 + 2\xi_t\varphi\langle \mathrm{d}\xi_t, \mathrm{d}\varphi\rangle + |\varphi|^2|\mathrm{d}\xi_t|^2\, \mathrm{d}\mathfrak{m} \\ &= \int_{\mathsf{X}} |\xi_t|^2|\mathrm{d}\varphi|^2 - |\varphi|^2\xi_t\Delta\xi_t\, \mathrm{d}\mathfrak{m}. \end{aligned}$$

By what we proved, we see that the RHS is bounded as $t \to 0^+$; hence the lower semicontinuity of the Cheeger energy ensures that $\varphi\xi \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. Now choose $\varphi \in \mathrm{Lip}(\mathsf{X}, \mathsf{d})$ identically 1 on $B_{r/2}(z)$ and with support in $B_r(z)$, and let $\eta \in \mathrm{Lip}(\mathsf{X}, \mathsf{d})$ be arbitrary with support in $B_{r/2}(z)$. Since $\mathrm{supp}(\eta) \subseteq \{\varphi = 1\}$, from (4.2), it follows that

$$\int_{\mathsf{X}} \varphi\xi\langle\nabla\eta, \nabla f\rangle\, \mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} \eta\xi\varphi\Delta f\, \mathrm{d}\mathfrak{m} \tag{4.3}$$

for any $f \in D(\Delta)$. Moreover, by what we just proved, the following computations are justified:

$$-\int_{\mathsf{X}} \varphi\xi\eta\Delta f\, \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \langle\nabla(\varphi\xi\eta), \nabla f\rangle\, \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \varphi\xi\langle\nabla\eta, \nabla f\rangle + \eta\langle\nabla(\varphi\xi), \nabla f\rangle\, \mathrm{d}\mathfrak{m}.$$

This and (4.3) imply that $\int_{\mathsf{X}} \eta\langle\nabla\xi, \nabla f\rangle\, \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \eta\langle\nabla(\varphi\xi), \nabla f\rangle\, \mathrm{d}\mathfrak{m} = 0$. The arbitrariness of η then gives $\langle\nabla(\varphi\xi), \nabla f\rangle = 0$ $\mathfrak{m}$-a.e. on $B_{r/2}(z)$. Then the density of $D(\Delta)$ in $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ gives $\nabla(\varphi\xi) = 0$ $\mathfrak{m}$-a.e. on $B_{r/2}(z)$. In turn, this implies (e.g. from the Sobolev to Lipschitz property) that $\varphi\xi$, and thus ξ, has a representative which is constant in $B_{r/2}(z)$, which is sufficient to conclude. □

We have now all the ingredients to prove the main equivalence result of this manuscript.

Proof of Theorem 1.5. Under (1.14), we can apply Theorem 4.1 and deduce the integration-by-parts formula

$$\int_{\mathsf{X}} \langle \mathrm{d}\varphi, \mathrm{d}f\rangle \frac{\mathrm{d}\mathscr{H}^n}{\mathrm{d}\mathfrak{m}} \mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} \varphi \operatorname{tr}(\operatorname{Hess} f) \frac{\mathrm{d}\mathscr{H}^n}{\mathrm{d}\mathfrak{m}} \mathrm{d}\mathfrak{m},$$

valid for any $\varphi \in \operatorname{Lip}(\mathsf{X}, \mathsf{d})$ with support in U and any $f \in D(\Delta)$. Now notice that (1.14) together with Theorem 2.13 imply that $\frac{\mathrm{d}\mathscr{H}^n}{\mathrm{d}\mathfrak{m}} \in L^\infty_{\mathrm{loc}}(U, \mathfrak{m})$. Hence if item (1) holds, we can apply Lemma 4.2 with $\xi = \chi_U \frac{\mathrm{d}\mathscr{H}^n}{\mathrm{d}\mathfrak{m}}$ to deduce that item (2) holds as well.

Conversely, if item (2) holds, for all φ and f as above, we have

$$-\int_{\mathsf{X}} \varphi \Delta f \, \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \langle \mathrm{d}\varphi, \mathrm{d}f\rangle \, \mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} \varphi \operatorname{tr}(\operatorname{Hess} f) \, \mathrm{d}\mathfrak{m},$$

having used item (2) and the integration-by-parts formula in the last step. By the arbitrariness of φ, this proves item (1). □

Proof of Theorem 1.3. From the Bishop–Gromov inequality (2.11), it easily follows that, for any bounded set A of X, we have

$$\inf_{r \in (0,1),\, x \in A} \frac{\mathfrak{m}(B_r(x))}{r^N} > 0. \tag{4.4}$$

On the other hand, Theorem 2.20 gives that the essential dimension of X is N; thus Theorem 2.22 with (2.2) shows

$$\Delta f = \operatorname{tr}(\operatorname{Hess} f) \quad \text{for all } f \in D(\Delta). \tag{4.5}$$

Then the conclusion follows from (4.4), (4.5) and Theorem 1.5. □

Proof of Theorem 1.6. From the continuity of $\mathscr{H}^N$ in the compact (as a consequence of Theorem 2.8) space of unit balls in $\mathrm{RCD}(K, N)$ spaces stated in Theorem 2.19, we see that, picking ϵ sufficiently small, the conclusion $|\mathscr{H}^N(B_1(x)) - \mathscr{H}^N(B_1(y))| < \delta$ holds true. Thus we concentrate on the first part of the claim.

The proof is done by contradiction. If not, there exist a sequence $\epsilon_i \to 0^+$, a sequence of pointed $\mathrm{RCD}(K, N)$ spaces $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ and a sequence of non-collapsed $\mathrm{RCD}(K, N)$ spaces $(\mathsf{Y}_i, \mathsf{d}_{\mathsf{Y}_i}, \mathscr{H}^N, y_i)$, where $\mathscr{H}^N(B_1(y_i)) \geq v$ such that $(\mathsf{X}_i, \mathsf{d}_i, x_i)$ is ϵ_i-pGH close to $(\mathsf{Y}_i, \mathsf{d}_{\mathsf{Y}_i}, y_i)$ and so that $\mathfrak{m}_i$ is not proportional to $\mathscr{H}^N$.

Thanks to Theorem 2.8, after passing to a non-relabelled subsequence, there exists a pointed $\mathrm{RCD}(K, N)$ space $(\mathsf{Z}, \mathsf{d}_{\mathsf{Z}}, \mathfrak{m}_{\mathsf{Z}}, z)$ such that

$$\Big(\mathsf{X}_i, \mathsf{d}_i, \frac{1}{\mathfrak{m}_i(B_1(x_i))}\mathfrak{m}_i, x_i\Big) \xrightarrow{\text{pmGH}} (\mathsf{Z}, \mathsf{d}_{\mathsf{Z}}, \mathfrak{m}_{\mathsf{Z}}, z), \quad (\mathsf{Y}_i, \mathsf{d}_{\mathsf{Y}_i}, y_i) \xrightarrow{\text{pGH}} (\mathsf{Z}, \mathsf{d}_{\mathsf{Z}}, z).$$

Thanks to Theorem 2.18, with (1.15), we have

$$(\mathsf{Y}_i, \mathsf{d}_{\mathsf{Y}_i}, \mathscr{H}^N, y_i) \xrightarrow{\text{pmGH}} (\mathsf{Z}, \mathsf{d}_{\mathsf{Z}}, \mathscr{H}^N, z),$$

with $\mathscr{H}^N(B_1(z)) \geq v$. Recalling Theorem 2.20, we see that $(\mathsf{Z}, \mathsf{d}_{\mathsf{Z}}, \mathfrak{m}_{\mathsf{Z}})$ is weakly non-collapsed, in particular, has essential dimension N. Then the lower semicontinuity statement given by Theorem 2.16 gives

$$N \geq \liminf_{i \to \infty} \operatorname{ess\,dim}(\mathsf{X}_i) \geq \operatorname{ess\,dim}(\mathsf{Z}) = N.$$

It follows that $\operatorname{ess\,dim}(\mathsf{X}_i) = N$ for any sufficiently large i. Thus, from the characterization of weakly non-collapsed spaces in Theorem 2.20 and our main result, Theorem 1.3, it follows that $\mathfrak{m}_i = c_i \mathscr{H}^N$ for every i sufficiently large. This provides the desired contradiction. □

Proof of Theorem 1.7. Let us take $r_i \to 0^+$ with (2.15), according to the assumption $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathfrak{m}_\mathsf{Y}, y) \in \operatorname{Tan}(\mathsf{X}, \mathsf{d}, \mathfrak{m}, x)$. As the essential dimension does not change under rescaling as in the LHS of (2.15), we see, by Theorem 2.16 and the assumption $\operatorname{ess\,dim}(\mathsf{Y}) = N$, that $\operatorname{ess\,dim}(\mathsf{X}) = N$. Thus we conclude by our main result, Theorem 1.3, taking into account also Theorem 2.20. □

Proof of Theorem 1.8. Let us take $r_i \to \infty$ and a sequence of rescaled spaces as in the LHS of (2.15); by Theorem 2.8 (here we use the fact that the space is an $\mathrm{RCD}(K, N)$ space with $K = 0$), we can extract a non-relabelled subsequence of $\{r_i\}_i$ such that these rescaled spaces converge to the $\mathrm{RCD}(0, N)$ space $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathfrak{m}_\mathsf{Y}, y)$ in the pmGH topology. Therefore, if $z \in \mathsf{Y}$, we take a sequence $\{y_i\}_i \subseteq \mathsf{X}$ that converges to z under this pmGH convergence,

$$\begin{aligned}\frac{\mathfrak{m}_\mathsf{Y}(B_r(z))}{r^N} &= \lim_i \frac{\mathfrak{m}(B_{rr_i}(y_i))}{r^N \mathfrak{m}(B_{r_i}(x))} = \lim_i \frac{\mathfrak{m}(B_{rr_i}(y_i))}{(rr_i)^N} \frac{r_i^N}{\mathfrak{m}(B_{r_i}(x))} \\ &\le \limsup_i \mathfrak{m}(B_1(y_i)) \frac{r_i^N}{\mathfrak{m}(B_{r_i}(x))} \le C,\end{aligned}$$

where C is independent of r. Here we have used the Bishop–Gromov inequality (2.11) for the first inequality and our assumptions for the last inequality. Therefore, using Theorem 2.20, we see that $\operatorname{ess\,dim}(\mathsf{Y}) = N$ so that we can conclude as in the proof of Theorem 1.7. □

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