

Estimating Quantum and Private capacities of Gaussian channels via degradable extensions

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We present upper bounds on the quantum and private capacity of single-mode, phase-insensitive Bosonic Gaussian Channels based on degradable extensions. Our findings are state-of-the-art in the following parameter regions: low temperature and high transmissivity for the thermal attenuator, low temperature for additive Gaussian noise, high temperature and intermediate amplification for the thermal amplifier.

Introduction.— Quantum Shannon theory [1, 2] provides a characterization of the maximum achievable transmission rates (capacities) for classical or quantum data through a quantum channel, as maximizations of entropic functionals. Available characterizations of most capacities cannot be computed algorithmically, since they involve a limit of an infinite sequence of optimization problems, one for any number of uses of the channel. Superadditivity of quantum entropic functionals makes such regularization necessary [8–21], and can hinder the evaluation of capacities even for simple fundamental channels. While it is hard to get past regularized expressions in the general case, it is important to improve our best understanding of the capacities of channels of physical interest. Here we focus on quantum and private capacities, which give the optimal transmission rates of quantum states and private classical information over a quantum channel [3–7] and are characterized as optimization of the coherent information and the private information, respectively. We consider realistic single-mode noise models for communication across free space or optical fiber: thermal attenuators, thermal amplifiers and additive Gaussian noise [22–25]. These channels collectively known as phase-insensitive, single-mode Bosonic Gaussian Channels, can be understood as unitary interactions with a single-mode thermal environment mediated by Hamiltonians which are quadratic in the operators fields of the model. The quantum and private capacities of these type of channels have been extensively studied [26–32], but they are still unknown. Interesting variations are Gaussian interaction with general environment [33, 34], and environmental assisted communication [35].

In this work we find new state-of-the-art upper bounds on the quantum and private capacity of thermal attenuators, amplifiers and additive Gaussian noise, using degradable extensions. For degradable channels coherent and private information are additive, and one can compute quantum and private capacities using the same single letter formula [7, 36, 37]. Therefore, an established strategy to find upper bounds for quantum and private capacity is to exploit (approximately) degradable channels which reduce to the channel of interest after pre and/or post-processing [28, 29, 38–45], and use the

capacity of the extended channels as an upper bound.

To upper bound the quantum capacity of the additive Gaussian noise, we introduce a flagged extension of the additive Gaussian noise which is degradable. A flagged extension of a convex combination of Completely Positive and Trace Preserving (CPTP) maps [1, 2] $\Lambda[\rho] = \sum_i p_i \Lambda_i[\rho]$ can be written as $\Lambda^e[\rho] = \sum_i p_i \Lambda_i[\rho] \otimes \sigma_i$, such that an auxiliary system encodes into the states σ_i information on which noise model is tampering the communication line. In discrete variable quantum information, degradable flagged extensions offer the best known upper bounds to several important channels [44, 45]. In this article we extend this approach to continuous variable systems. In particular, we are inspired by the sufficient conditions of degradability of finite-dimensional flagged channels introduced in [44]. The new upper bound is state-of-the-art in the low temperature regime. Using this upper bound and data-processing, we also bound the quantum capacity of thermal amplifier, obtaining a state-of-the-art bound in the high temperature regime, for intermediate amplification values. We also improve the construction in [28] based on weak degradability to find a noisier degradable extension of the thermal attenuator, and show that it gives the best upper bound in the low noise regime, for low temperature and high transmissivity.

In the following, we first review basic facts about quantum and private capacities and the formalism of Gaussian states and channels. Next, we present the flagged extension of the additive Gaussian noise and the extension of the thermal attenuator. In both sections, we compute their quantum and private capacities, and compare the new available upper bounds with previous results.

Preliminaries.— For a possibly infinite dimensional Hilbert space \mathcal{H} , we can define the set of all trace-class linear operators as $\mathcal{T}(\mathcal{H})$. A quantum channel $\Lambda : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ is a completely positive trace preserving (CPTP) linear map on the space of trace-class operators. Quantum states $\mathcal{G}(\mathcal{H})$ are identified with positive semi-definite operators with trace one. Any quantum channel can be represented in Stinespring representation as $\Lambda[\hat{\rho}] = \text{Tr}_E(\hat{V}\hat{\rho}\hat{V}^\dagger)$, where $\hat{V} : \mathcal{H}_A \rightarrow \mathcal{H}_{BE}$ is an isometry. A complementary channel $\tilde{\Lambda} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_E)$

is defined as following $\Lambda^c[\hat{\theta}] := \text{Tr}_A[\hat{V}\hat{\theta}\hat{V}^\dagger]$. If there exists another channel $W : \mathcal{T}(\mathcal{H}_B) \rightarrow \mathcal{T}(\mathcal{H}_E)$ satisfying $W \circ \Lambda = \Lambda^c$, Λ is said to be degradable. The quantum capacity of a quantum channel is the maximum rate at which it can transmit quantum information reliably over asymptotically many uses of the channel. It is equal to the regularized coherent information, i.e.,

$$Q(\Lambda) = \lim_{n \rightarrow \infty} Q^{(1)}(\Lambda^{\otimes n})/n, \quad (1)$$

where $Q^{(1)}(\Lambda) := \sup_{\hat{\rho} \in \mathcal{G}(\mathcal{H}_A)} S(\Lambda(\hat{\rho})) - S(\Lambda^c(\hat{\rho}))$ is the coherent information of Λ , Λ^c is a complementary channel of Λ , and $S(\hat{\rho}) := -\text{Tr}[\hat{\rho} \log_2 \hat{\rho}]$ is the von Neumann entropy of the state $\hat{\rho}$. For degradable channels the coherent information is additive [7, 36] and it holds $Q(\Lambda) = Q^{(1)}(\Lambda)$. The private capacity is characterized by a different regularized expression, but it coincides with the quantum capacity for degradable channels [37]. Thus, since the upper bounds we obtain in this work come from degradable extensions, they are immediately understood as upper bounds for the private capacity too.

In this work we consider infinite dimensional Hilbert spaces $L^2(\mathbb{R}^n)$ of square integrable functions, corresponding to m modes of harmonic oscillators characterized by position and momentum operators which we group as $\hat{\mathbf{r}} = (\hat{x}_1, \hat{p}_1, \dots, \hat{x}_n, \hat{p}_n)^T$. They satisfy the canonical commutation relations $[\hat{r}_i, \hat{r}_j] = i\Omega_{ij}I$, where

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\oplus n}. \quad (2)$$

For $\mathbf{r} \in \mathbb{R}^{2n}$ we introduce the (unitary) displacement operators $\hat{D}_{\mathbf{r}} = e^{i\mathbf{r}^T \Omega \hat{\mathbf{r}}}$ satisfying the identity $\hat{D}_{\mathbf{r}_1 + \mathbf{r}_2} = \hat{D}_{\mathbf{r}_1} \hat{D}_{\mathbf{r}_2} e^{i\mathbf{r}_1^T \Omega \mathbf{r}_2 / 2}$, and define the characteristic function of a trace class operator $\hat{\rho}$ on $L^2(\mathbb{R}^n)$ as $\chi(\mathbf{r}) := \text{Tr}[\hat{\rho} \hat{D}_{-\mathbf{r}}]$ [24, 25]. The Gaussian states are those states such that their characteristic function is Gaussian, i.e. $\chi(\mathbf{r}) = \exp\{-\frac{1}{4}\mathbf{r}^T \Omega^T V \Omega \mathbf{r} + i\mathbf{r}^T \Omega \mathbf{m}\}$, with $\mathbf{m} := \text{Tr}[\hat{\mathbf{r}} \hat{\rho}]$ and $V := \text{Tr}[\{(\hat{\mathbf{r}} - \mathbf{m}), (\hat{\mathbf{r}} - \mathbf{m})^T\} \hat{\rho}]$ being the associated statistical mean vector and covariance matrix where $\{\hat{A}, \hat{B}\} := \hat{A}\hat{B} + \hat{B}\hat{A}$. Bosonic Gaussian Channels can now be identified with the CPTP super-operators on n modes that map the set of Gaussian states to itself [23–25] and can hence be characterized by how they transform \mathbf{m} and V . In the following we shall specifically focus on single mode ($n = 1$) thermal attenuator BGCs $\mathcal{E}_{\eta, N}$ defined by the mapping [46]

$$\mathbf{m} \xrightarrow{\mathcal{E}_{\eta, N}} \mathbf{m}' = \sqrt{\eta} \mathbf{m}, \quad (3)$$

$$V \xrightarrow{\mathcal{E}_{\eta, N}} V' = \eta V + (1 - \eta)(2N + 1)I_2, \quad (4)$$

with $0 \leq \eta \leq 1$ and $N \geq 0$ being the characteristic parameters of the model, and I_2 being the two dimensional identity. Via Stinespring representation [23–25]

these special transformations are better understood in terms of a beam splitter coupling with an extra environmental mode E initialized in a thermal (Gaussian) state. Specifically labelling with A the system mode, and indicating with $\hat{\rho}_A$ its input state we can write

$$\mathcal{E}_{\eta, N}[\hat{\rho}_A] := \text{Tr}_E[\hat{U}_\eta(\hat{\rho}_A \otimes \hat{\tau}_E)]. \quad (5)$$

In this expression Tr_E represents the partial trace over the environment while \hat{U}_η ($0 \leq \eta \leq 1$) is two-mode unitary operator that transforms $\hat{\mathbf{r}}$ according to

$$\hat{U}_\eta \hat{\mathbf{r}} \hat{U}_\eta^\dagger = \begin{pmatrix} \sqrt{\eta} I_2 & \sqrt{1-\eta} I_2 \\ -\sqrt{1-\eta} I_2 & \sqrt{\eta} I_2 \end{pmatrix} \hat{\mathbf{r}}. \quad (6)$$

The thermal state $\hat{\tau}$ entering in (5) is finally defined by $\mathbf{m}_{\hat{\tau}} = (0, 0)$ and $V_{\hat{\tau}} = (2N + 1)I_2$. Such density matrix is purified by a two-mode squeezed state $|\tau\rangle$, which has $\mathbf{m}_{|\tau\rangle} = (0, 0, 0, 0)$ and

$$V_{|\tau\rangle} = \begin{pmatrix} (2N + 1)V_{\hat{\tau}} & \sqrt{V_{\hat{\tau}}^2 - 1}\sigma_3 \\ \sqrt{V_{\hat{\tau}}^2 - 1}\sigma_3 & (2N + 1)V_{\hat{\tau}} \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (7)$$

We shall also consider single-mode thermal amplifier BGCs $\Phi_{g, N}$. In this case the input state interacts with a thermal bath through a two mode squeezing operator with parameter $g \geq 1$ inducing the mapping

$$\mathbf{m} \xrightarrow{\Phi_{g, N}} \mathbf{m}' = \sqrt{g} \mathbf{m}, \quad (8)$$

$$V \xrightarrow{\Phi_{g, N}} V' = gV + (g - 1)(2N + 1)I_2. \quad (9)$$

Finally we study the single-mode Additive Gaussian Noise Channel (AGNC) Λ_β that can be expressed as

$$\Lambda_\beta[\hat{\rho}] := \frac{\beta}{2\pi} \int_{\mathbb{R}^2} d\mathbf{r} e^{-\frac{\beta}{2}\mathbf{r}^T \mathbf{r}} \hat{D}_{\mathbf{r}} \hat{\rho} \hat{D}_{\mathbf{r}}^\dagger, \quad (10)$$

where $\beta > 0$ is the inverse temperature, that induces the mapping

$$\mathbf{m} \xrightarrow{\Lambda_\beta} \mathbf{m}' = \mathbf{m}, \quad (11)$$

$$V \xrightarrow{\Lambda_\beta} V' = V + 2I_2/\beta. \quad (12)$$

Upper bounds for the AGNC.– In the high temperature regime (i.e. for $1/\beta \geq 0.5$) the quantum capacity of the channel Λ_β is known to be exactly zero, using data-processing techniques [1]. In particular in [30], the following upper bound is computed using data-processing:

$$Q(\Lambda_\beta) \leq Q_{\text{NAJ}}(\beta) = \max\{\log_2(\beta - 1), 0\}, \quad (13)$$

and correctly gives $Q(\Lambda_\beta) = 0$ for $1/\beta \geq 0.5$. In the low temperature instead we only have a lower bound for $Q(\Lambda_\beta)$ given by the coherent information for one use of the channel, evaluated on an infinite temperature state, i.e.

$$Q(\Lambda_\beta) \geq Q_L(\Lambda_\beta) := \max\{\log_2 \beta - 1/\ln 2, 0\}, \quad (14)$$

and the inequality,

$$Q(\Lambda_\beta) \leq Q_{\text{PLOB}}(\beta) = \log_2 \beta - 1/\ln 2 + 1/(\beta \ln 2), \quad (15)$$

which follows from an upper bound for the generic two-way quantum capacity [26].

Generalizing the procedure introduced in [44] we present an improved upper bound considering the following flagged extension of Λ_β , i.e.

$$\Lambda_\beta^e[\hat{\rho}] := \frac{\beta}{2\pi} \int_{\mathbb{R}^2} d\mathbf{r} e^{-\frac{\beta}{2}\mathbf{r}^T \mathbf{r}} \hat{D}_\mathbf{r} \hat{\rho} \hat{D}_\mathbf{r}^\dagger \otimes |\phi_\mathbf{r}\rangle \langle \phi_\mathbf{r}|, \quad (16)$$

where setting $\mathbf{r} := (x, p)$, $|\phi_\mathbf{r}\rangle$ are product of displaced squeezed states defined by

$$|\phi_\mathbf{r}\rangle := \hat{D}_{(0,-p/2)} |\beta/2\rangle \otimes \hat{D}_{(0,x/2)} |\beta/2\rangle, \quad (17)$$

with $|\beta/2\rangle$ being a single-mode squeezed vacuum with mean values $\mathbf{m} = 0$ and covariance matrix $V = \begin{pmatrix} 2/\beta & 0 \\ 0 & \beta/2 \end{pmatrix}$. As explicitly shown in [47] Λ_β^e is degradable: here we notice that the intuition for choosing the flags states as in (17) comes from a result by the same authors on finite dimensional channels [44]. Indeed there we proved that if $\{\sqrt{p_i} \hat{U}_i\}_{i=1,\dots,n}$ are unitary Kraus operators of a CPTP map \mathcal{N} and $\{|i\rangle\}_{i=1,\dots,n}$ is an orthonormal basis for flags, sufficient conditions for the degradability of $\mathcal{N}^e[\hat{\rho}] := \sum_{i=1}^n p_i \hat{U}_i \hat{\rho} \hat{U}_i^\dagger \otimes |i\rangle \langle i|$ are the following

$$\langle i' | \phi_i \rangle \sqrt{p_i} \hat{U}_i \hat{U}_i = \langle i | \phi_{i'} \rangle \sqrt{p_{i'}} \hat{U}_{i'} \hat{U}_{i'} \quad \forall i, i'. \quad (18)$$

If we can use a continuous set of flags, replacing the orthonormal basis for the flag space with the (two-mode) pseudo-eigenbasis $\{|x_1, x_2\rangle\}_{x_1, x_2 \in \mathbb{R}}$ of the position operators of the ancillary modes, applying this result to Λ_β^e we get

$$\langle \gamma x', \gamma p' | \phi_\mathbf{r} \rangle e^{-\frac{\beta}{4}\mathbf{r}^T \mathbf{r}} \hat{D}_\mathbf{r} \hat{D}_\mathbf{r} = \langle \gamma x, \gamma p | \phi_{\mathbf{r}'} \rangle e^{-\frac{\beta}{4}\mathbf{r}'^T \mathbf{r}'} \hat{D}_\mathbf{r} \hat{D}_\mathbf{r}', \quad (19)$$

for all $\mathbf{r}' = (x', p')$, $\mathbf{r} = (x, p) \in \mathbb{R}^2$ with γ a suitable rescaling factor to determine. With the choice (17) of the flags we have explicitly

$$\langle \gamma x', \gamma p' | \phi_\mathbf{r} \rangle = \sqrt{\frac{\beta}{2\pi}} e^{-\beta \frac{\gamma^2 x'^2 + \gamma^2 p'^2}{4} - \gamma \frac{i p' x - i x' p}{2}}, \quad (20)$$

which satisfies the condition (19) with $\gamma = 1$. Exploiting the degradability of Λ_β^e and the fact that it is gauge-covariant (in a generalized sense that we specify in [47]), the capacity of this map can be easily computed [1] leading to the following inequality

$$Q(\Lambda_\beta) \leq Q(\Lambda_\beta^e) = \log_2 \beta - 1/\ln 2 + 2h\left(\sqrt{1 + 1/\beta^2}\right), \quad (21)$$

with $h(x) := \frac{x+1}{2} \log_2 \left(\frac{x+1}{2}\right) - \frac{x-1}{2} \log_2 \left(\frac{x-1}{2}\right)$ (see [47] for details). As shown in Fig. 1, Eq. (21) is better than (15) where $Q(\Lambda_\beta)$ is supposed to be non-zero, i.e. for $1/\beta \leq 0.5$.

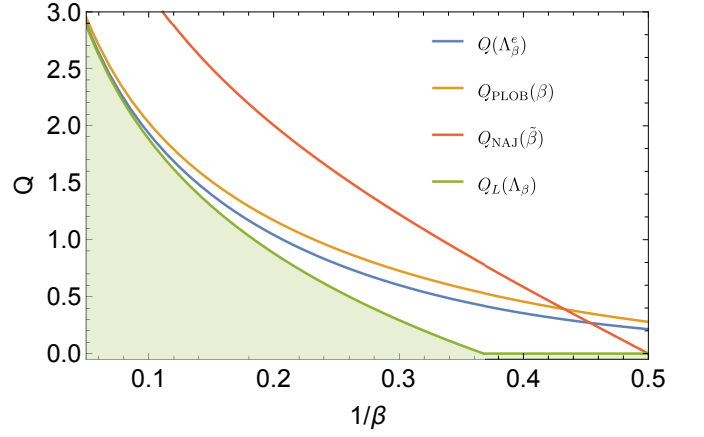


FIG. 1: Quantum capacity region for the AGNC Λ_β : comparison of the upper bound $Q(\Lambda_\beta^e)$ of Eq. (21) with Q_{PLOB} in Eq. (15) [26] and $Q_{\text{NAJ}}(\beta)$ in Eq. (13) [30]. $Q_L(\Lambda_\beta)$ is the lower bound of Eq. (14).

Upper bounds for the thermal amplifier.— Invoking data-processing inequality [2], we can use Eq. (21) to upper bound the quantum capacity thermal amplifier $\Phi_{g,N}$ (a similar argument could also be invoked for the thermal attenuator $\mathcal{E}_{\eta,N}$ but the result we get is worse than the bounds reported in the next section). As a matter of fact any thermal amplifier can be written as a composition of a zero temperature amplifier and an AGNC i.e. $\Phi_{g,N} = \Lambda_{\tilde{\beta}} \circ \Phi_{g,0}$ with $\tilde{\beta} = \frac{1}{(g-1)N}$. Therefore we get

$$Q(\Phi_{g,N}) \leq Q(\Lambda_{\tilde{\beta}}) \leq Q(\Lambda_{\tilde{\beta}}^e), \quad (22)$$

$$Q(\Phi_{g,N}) \leq Q(\Lambda_{\tilde{\beta}}) \leq Q_{\text{NAJ}}(\tilde{\beta}), \quad (23)$$

The bound of Eq. (23) comes directly to the data-processing decompositions, but to our knowledge it was not explicitly pointed out previously, being implicit in Proposition 12.65 of [1]. It is the best bound at high g for any N , and it gives zero quantum capacity for the lowest g . The new bound in Eq. (22), in the high temperature regime $N > 5$ and for intermediate values of g is probably better than the previous best bound reported in [26]:

$$Q(\Phi_{g,N}) \leq Q_{\text{AmPLOB}}(g, N) := \log_2 \left(\frac{g^{N+1}}{g-1} \right) - h(2N+1). \quad (24)$$

A comparison between these function is reported in Fig. 2, together with the lower bound given by the coherent information for one use of the channel, evaluated on an infinite temperature state,

$$Q(\Phi_{g,N}) \geq Q_L(\Phi_{g,N}) := \max\left\{ \log_2 \left(\frac{g}{g-1} \right) - h(2N+1), 0 \right\}. \quad (25)$$

Upper bounds for the thermal attenuator.— To deal with the quantum capacity of the channel $\mathcal{E}_{\eta,N}$ of Eq. (5) we construct a degradable extension of such map. We first define the passive unitary operator

$$\hat{W}_\eta := \hat{U}_{\eta AE} \otimes \hat{U}_{\eta A' E'}, \quad (26)$$

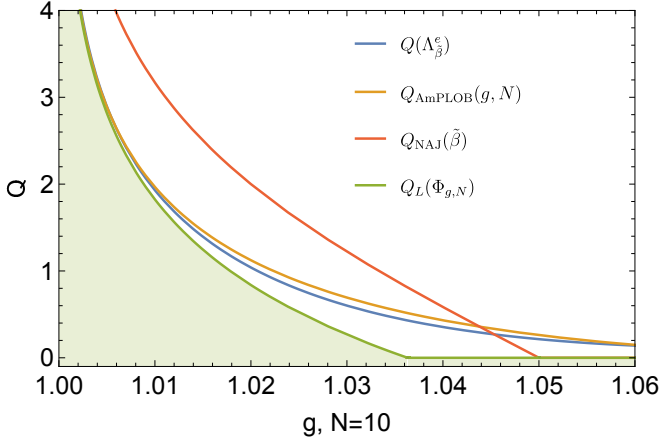


FIG. 2: Quantum capacity region for the thermal amplifier channel $\Phi_{g,N}$: comparison between the upper bound $Q(\Lambda_{\beta}^e)$ of Eq. (22) with $Q_{\text{AmPLOB}}(g, N)$ of Eq. (24) [26] and $Q_{\text{NAJ}}(\beta)$ in Eq. (13) [30] for $N = 10$. $Q_L(\Phi_{g,N})$ is the lower bound of Eq (25)

where $\hat{U}_{\eta_{AE}}$ and $\hat{U}_{\eta_{A'E'}}$ are beam splitters transformations acting respectively on the pair of modes A, E and A', E' . We introduce hence the channel

$$\mathcal{F}_{\eta,N}[\rho_{AA'}] := \text{Tr}_{EE'}[\hat{W}_{\eta}(\rho_{AA'} \otimes |\tau\rangle\langle\tau|_{EE'})\hat{W}_{\eta}^{\dagger}], \quad (27)$$

and define the extension of $\mathcal{E}_{\eta,N}$ as

$$\mathcal{E}_{\eta,N}^e[\rho_A] := \mathcal{F}_{\eta,N}(\rho_A \otimes |0\rangle\langle 0|_{A'}). \quad (28)$$

The map $\mathcal{F}_{\eta,N}$ is manifestly Gaussian, and its action on the first and second moments is

$$\mathbf{m} \xrightarrow{\mathcal{F}_{\eta,N}} \mathbf{m}' = \sqrt{\eta} \mathbf{m}, \quad (29)$$

$$V \xrightarrow{\mathcal{F}_{\eta,N}} V' = \eta V + (1 - \eta)V_{|\tau}, \quad (30)$$

With the Stinespring representation in Eq. (27) the complementary channel can now be computed as $\mathcal{F}_{\eta,N}^c = \mathcal{F}_{1-\eta,N}$. Simple algebra shows that if $\eta > 1/2$ then

$$\mathcal{F}_{\eta,N}^c = \mathcal{F}_{1-\eta,N} = \mathcal{F}_{(1-\eta)/\eta,N} \circ \mathcal{F}_{\eta,N}, \quad (31)$$

implying that in such regime $\mathcal{F}_{\eta,N}$ (and thus $\mathcal{E}_{\eta,N}^e$) is degradable. The quantum capacity of $\mathcal{E}_{\eta,N}^e$ can be thus calculated by evaluating the coherent information of the channel leading to

$$Q(\mathcal{E}_{\eta,N}) \leq Q(\mathcal{E}_{\eta,N}^e) = \log_2\left(\frac{\eta}{1-\eta}\right) + h((1-\eta)(2N+1) + \eta) - h(\eta(2N+1) + 1 - \eta). \quad (32)$$

Once more this upper bound should be compared with previous upper bounds. At low noise, that is at low N and high η , the best upper bound available is once more an upper bound for the generic two-way quantum capacity [26]:

$$Q(\mathcal{E}_{\eta,N}) \leq Q_{\text{PLOB}}(\eta, N) = -\log_2((1-\eta)\eta^N) - h(2N+1). \quad (33)$$

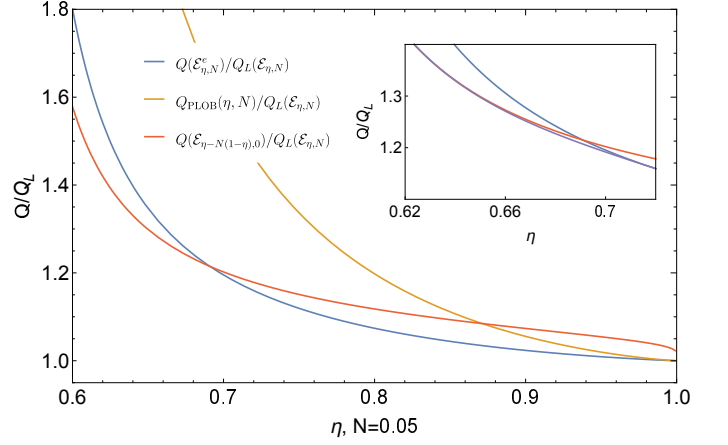


FIG. 3: Thermal attenuator: ratio between the upper bounds $Q(\mathcal{E}_{\eta-N(1-\eta),0})$ [28], $Q_{\text{PLOB}}(\eta, N)$ [26], $Q(\mathcal{E}_{\eta,N}^e)$ (this work), and $Q_L(\mathcal{E}_{\eta,N})$ for $N = 0.05$. In the inset we plot a close-up in the region where $Q(\mathcal{E}_{\eta-N(1-\eta),0})$ and $Q(\mathcal{E}_{\eta,N}^e)$ intersect. The purple line is the improved bound using the argument of Eq. (36).

Other bounds come from data processing [28–30], the best in the low noise regime being:

$$Q(\mathcal{E}_{\eta,N}) \leq Q(\mathcal{E}_{\eta-N(1-\eta),0}) = \log_2 \frac{\eta-N(1-\eta)}{(N+1)(1-\eta)}, \quad (34)$$

A lower bound on $Q(\mathcal{E}_{\eta,N})$ is given by the coherent information for one use of the channel, evaluated on an infinite temperature state,

$$Q(\mathcal{E}_{\eta,N}) \geq Q_L(\mathcal{E}_{\eta,N}) := \max\{\log_2\left(\frac{\eta}{1-\eta}\right) - h(2N+1), 0\}. \quad (35)$$

A comparison between all these curves is reported in Fig. 3, showing that while our inequality (32) performs worse than (34) for low η , it gives an improvement with respect to (33) for high η . We finally remark that in our construction the choice of $|0\rangle\langle 0|_{A'}$ in the definition Eq. (28) of the extended attenuator is not necessarily optimal. Other Gaussian states could be chosen and the analysis could be done in the same way. In particular, the extension $\mathcal{F}_{\eta,N} \otimes \mathcal{I}[\rho_A \otimes |\tau'\rangle\langle\tau'|_{A'B}]$ gives a slightly better upper bound (optimizing over the single parameter in $|\tau'\rangle$ states), not noticeable on the plot in Fig. 3.

Conclusion and Remarks.– The new bounds contained in this paper complement the bounds in [26, 28–30]. They have been determined using degradable extensions and data-processing. This contribution extends the applicability of this technique, which now gives the best upper bounds at low noise for a large collection of channels of physical interest. In particular, the flagged extension of the AGNC nicely generalizes the construction of flagged extensions to infinite dimensional channels, improving the upper bound on the quantum capacity by a considerable margin.

The various bounds available are not directly comparable on the whole parameter region. However, taking into

account all the possible data-processing decompositions

$$\mathcal{E}_{\eta,N} = \mathcal{E}_{\eta_a,N_{a,1}} \circ \Phi_{g_a,N_{a,2}} = \Phi_{g'_a,N'_{a,1}} \circ \mathcal{E}_{\eta'_a,N'_{a,2}}, \quad (36)$$

$$\Phi_{g,N} = \mathcal{E}_{\eta_b,N_{b,1}} \circ \Phi_{g_b,N_{b,2}} = \Phi_{g'_b,N'_{b,1}} \circ \mathcal{E}_{\eta'_b,N'_{b,2}}, \quad (37)$$

using available direct bounds on the channels appearing in the decompositions, and minimizing over the decompositions, one can combine the bounds from the previous papers and the present ones, obtaining even better upper bounds. An example is seen in the inset of Fig. 3.

We mention also that the results presented here could be easily applied also for the case of energy constrained quantum capacity.

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Proof of Gaussianity of flagged additive Gaussian noise

In this section we review the class of BGCs known as classical mixing channels, then show that flagged additive Gaussian noise is a classical mixing channel.

For any $n \times n$ square matrix $Y \geq 0$ with eigenvalues $\lambda_1, \dots, \lambda_n$, let us indicate the support of Y as $S(Y)$, $\det_+ Y = \prod_{i:\lambda_i > 0} \lambda_i$, and pseudoinverse of Y as $Y^{\ominus 1}$. Classical mixing channels have the form:

$$\Lambda_Y[\hat{\rho}] := \int_{S(Y)} d\mathbf{r} \frac{e^{-\mathbf{r}^\top Y^{\ominus 1} \mathbf{r}}}{\sqrt{\pi}^{\dim S(Y)} \sqrt{\det_+ Y}} \hat{D}_{\mathbf{r}} \hat{\rho} \hat{D}_{\mathbf{r}}^\dagger. \quad (38)$$

Channels of this type are Gaussian and the action on the first and second moments is

$$\mathbf{m} \xrightarrow{\Lambda_Y} \mathbf{m}' = \mathbf{m}, \quad V \xrightarrow{\Lambda_Y} V' = V + Y. \quad (39)$$

By direct comparison with Eq. (16) of the main text it follows that the flagged additive Gaussian noise of Λ_β^e is a classical mixing channel applied to the state $\hat{\rho} \otimes |\beta/2\rangle\langle\beta/2| \otimes |\beta/2\rangle\langle\beta/2|$ with the matrix Y equal to

$$Y = \begin{pmatrix} \frac{2}{\beta} & 0 & 0 & 0 & 0 & -\frac{1}{\beta} \\ 0 & \frac{2}{\beta} & 0 & \frac{1}{\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 & \frac{1}{2\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\beta} & 0 & 0 & 0 & 0 & \frac{1}{2\beta} \end{pmatrix}, \quad (40)$$

and thus Gaussian.

Computing the Coherent information of extended channels

In this section, we review how to compute the von Neumann entropy of Gaussian states, then we calculate the coherent information of the flagged additive Gaussian noise Λ_β^e (see Eq. (16) of the main text) and the extended thermal attenuator $\mathcal{E}_{\eta,N}^e$ (see Eq. (27) of the main text). The von Neumann entropy of a Gaussian state can be computed from its covariance matrix V . In particular, we need to compute the symplectic eigenvalues d_1, \dots, d_n of V , such that $V = SDS^T$ for some symplectic matrix S and D diagonal with elements $d_1, d_1, d_2, d_2, \dots, d_n, d_n$. In particular, these eigenvalues can be obtained from the eigenvalues of the matrix $i\Omega V$, which correspond to $d_1, -d_1, \dots, d_n, -d_n$. The Von Neumann entropy of a state $\hat{\rho}$ with covariance matrix V is then

$$S(\hat{\rho}) = \sum_{i=1}^n h(d_i), \quad (41)$$

with

$$h(x) := \frac{x+1}{2} \log_2 \left(\frac{x+1}{2} \right) - \frac{x-1}{2} \log_2 \left(\frac{x-1}{2} \right). \quad (42)$$

For flagged additive Gaussian noise channel Λ_β^e and extended thermal attenuator $\mathcal{E}_{\eta,N}^e$, gaussian states maximize the coherent information since the channels are degradable and admit a Gaussian degrading map, satisfying the conditions of Theorem 12.40 of [1]. The degrading map is explicitly Gaussian for the extended thermal attenuator; in the case of the flagged additive noise the degrading map we present in the next section is not explicitly Gaussian, but its complementary is Gaussian. Since any Gaussian channel admit a Stinespring representation with a Gaussian unitary, there exists a Gaussian dilation of the complementary of the degrading map. By the properties of the Stinespring representation, the Stinespring dilation of the degrading map is isometric to a Gaussian isometry, with the connecting isometry acting trivially on the systems associated with the output of the complementary of the degrading map. This is enough to apply Theorem 12.40 of [1]. An explicit Gaussian degrading map will be presented elsewhere.

If in addition the channel is gauge-covariant, the maximization can be restricted to gauge-invariant states [1]. In our case, both the thermal attenuator and the flagged additive noise satisfy a generalized gauge-covariance property. Defining the Gaussian unitary on one mode $\hat{R}(\theta)$ acting on $\hat{\mathbf{r}} = (\hat{x}, \hat{p})$ as the rotation matrix $R(\theta) := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$

$$\hat{R}(\theta)\hat{\mathbf{r}}\hat{R}(\theta)^\dagger = R(\theta)\hat{\mathbf{r}}, \quad (43)$$

we have

$$\Lambda_\beta^e[\hat{R}(\theta)\hat{\rho}\hat{R}(\theta)^\dagger] = \hat{R}'(\theta)\Lambda_\beta^e[\hat{\rho}]\hat{R}'(\theta)^\dagger, \quad (44)$$

with $\hat{R}'(\theta)$ being a three mode Gaussian unitary acting on $\hat{\mathbf{r}} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \hat{x}_3, \hat{p}_3)$ as

$$\hat{R}'(\theta)\hat{\mathbf{r}}\hat{R}'(\theta)^\dagger = R'(\theta)\hat{\mathbf{r}}, \quad (45)$$

with

$$R'(\theta) := \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & 0 & \sin\theta & 0 \\ 0 & 0 & 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}. \quad (46)$$

In a similar way,

$$\mathcal{E}_{\eta,N}^e[\hat{R}(\theta)\hat{\rho}\hat{R}(\theta)^\dagger] = \hat{R}''(\theta)\mathcal{E}_{\eta,N}^e[\hat{\rho}]\hat{R}''(\theta)^\dagger, \quad (47)$$

with $\hat{R}''(\theta)$ being a two mode Gaussian unitary acting on $\hat{\mathbf{r}} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2)$ as

$$\hat{R}''(\theta)\hat{\mathbf{r}}\hat{R}''(\theta)^\dagger = R''(\theta)\hat{\mathbf{r}}, \quad (48)$$

with

$$R''(\theta) := \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}. \quad (49)$$

Adapting the argument in [1] for gauge-covariant channels, Eq. (44),(47), together with the concavity of the coherent information of degradable channels, imply that the maximum of the coherent information is attained on gauge-invariant Gaussian states, which coincides with thermal states for channels with one mode as input. Moreover, since $\Lambda_\beta^e \circ \hat{D}_s = \hat{D}_s \circ \Lambda_\beta^e$ and $\mathcal{E}_{\eta,N}^e \hat{D}_s = \hat{D}_{\sqrt{\eta}s} \circ \mathcal{E}_{\eta,N}^e$, by concavity and unitarily invariance of the coherent information we have that higher energy thermal states have higher coherent information.

Thus, to compute the coherent information of the flagged additive noise we have to find the covariance matrix V_M of $\Lambda_\beta^e[\hat{\rho}_M]$ and the covariance matrix V'_M of $(\Lambda_\beta^e \otimes \mathcal{I})[|\rho_M\rangle\rangle\langle\langle\rho_M|]$ where $\hat{\rho}_M$ is the thermal state with average photon number M and $|\rho_M\rangle\rangle$ is its purification, which can be taken to be the two-mode squeezed state $|\tau\rangle$.

We obtain

$$V_M = \begin{pmatrix} 2M+1+\frac{2}{\beta} & 0 & 0 & 0 & 0 & -\frac{1}{\beta} \\ 0 & 2M+1+\frac{2}{\beta} & 0 & \frac{1}{\beta} & 0 & 0 \\ 0 & 0 & \frac{2}{\beta} & 0 & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 & \frac{\beta}{2} + \frac{1}{2\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{\beta} & 0 \\ -\frac{1}{\beta} & 0 & 0 & 0 & 0 & \frac{\beta}{2} + \frac{1}{2\beta} \end{pmatrix} \quad (50)$$

$$V'_M = \begin{pmatrix} 2M+1+\frac{2}{\beta} & 0 & 0 & 0 & 0 & -\frac{1}{\beta} & 2\sqrt{M(M+1)} & 0 \\ 0 & 2M+1+\frac{2}{\beta} & 0 & \frac{1}{\beta} & 0 & 0 & 0 & -2\sqrt{M(M+1)} \\ 0 & 0 & \frac{2}{\beta} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 & \frac{\beta}{2} + \frac{1}{2\beta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{\beta} & 0 & 0 & 0 \\ -\frac{1}{\beta} & 0 & 0 & 0 & 0 & \frac{\beta}{2} + \frac{1}{2\beta} & 0 & 0 \\ 2\sqrt{M(M+1)} & 0 & 0 & 0 & 0 & 0 & 2M+1 & 0 \\ 0 & -2\sqrt{M(M+1)} & 0 & 0 & 0 & 0 & 0 & 2M+1 \end{pmatrix}. \quad (51)$$

The eigenvalues of $i\Omega V_M$ are

$$\pm 2M + O(1), \quad \pm \frac{\sqrt{1+\beta^2}}{\beta} + O(1/M), \quad \pm \frac{\sqrt{1+\beta^2}}{\beta} + O(1/M), \quad (52)$$

while the eigenvalues if $i\Omega V'_M$ are

$$\pm 2\frac{1}{\beta^{1/2}}\sqrt{M} + O(1), \quad \pm 2\frac{1}{\beta^{1/2}}\sqrt{M} + O(1), \quad \pm 1, \quad \pm 1. \quad (53)$$

Therefore we have

$$Q(\Lambda_\beta^e) = \lim_{M \rightarrow \infty} S(\Lambda_\beta^e[\rho_M]) - S((\Lambda_\beta^e \otimes \mathcal{I})[|\tau\rangle\langle\tau|_M]) = \log_2 \beta - 1/\log 2 + 2h\left(\frac{\sqrt{1+\beta^2}}{\beta}\right), \quad (54)$$

as indicated in Eq. (21) of the main text.

To compute the coherent information of the extended thermal attenuator $\mathcal{E}_{\eta,N}^e$ we have to find the covariance matrix V_M of $\mathcal{E}_{\eta,N}^e[\hat{\rho}_M]$ and the covariance matrix V'_M of the complementary channel $\mathcal{E}_{\eta,N}^{e,c}[\hat{\rho}_M] = \mathcal{E}_{1-\eta,N}^e[\hat{\rho}_M]$ where $\hat{\rho}_M$ is again the thermal state with average photon number M . We obtain

$$V_M = \begin{pmatrix} \eta(2M+1) + (1-\eta)\eta(2N+1) & 0 & (1-\eta)2\sqrt{N(N+1)} & 0 \\ 0 & \eta(2M+1) + (1-\eta)(2N+1) & 0 & -(1-\eta)2\sqrt{N(N+1)} \\ (1-\eta)2\sqrt{N(N+1)} & 0 & \eta + (1-\eta)(2N+1) & 0 \\ 0 & -(1-\eta)2\sqrt{N(N+1)} & 0 & \eta + (1-\eta)(2N+1) \end{pmatrix}. \quad (55)$$

while V'_M is obtained from the above expression by exchanging $\eta \rightarrow 1-\eta$. The eigenvalues if $i\Omega V_M$ are hence

$$\pm \eta M + O(1), \quad \pm(\eta + (1-\eta)(2N+1)) + O(1/M), \quad (56)$$

while the eigenvalues if $i\Omega V'_M$ are

$$\pm (1-\eta)M + O(1), \quad \pm((1-\eta) + \eta(2N+1)) + O(1/M). \quad (57)$$

Therefore we have

$$\begin{aligned} Q(\mathcal{E}_{\eta,N}^e) &= \lim_{M \rightarrow \infty} S(\mathcal{E}_{\eta,N}^e[\hat{\rho}_M]) - S(\mathcal{E}_{1-\eta,N}^e[\hat{\rho}_M]) \\ &= -\log_2\left(\frac{\eta}{1-\eta}\right) + h(\eta + (1-\eta)(2N+1)) - h((1-\eta) + \eta(2N+1)), \end{aligned}$$

as indicated in Eq. (32) of the main text.

Degradability of flagged additive noise channel

Here we prove the degradability of the channel Λ_β^e . We define the unitary operator $\hat{U}^{(x)} : L_2(\mathbb{R}^2) \rightarrow L_2(\mathbb{R}^2)$

$$\hat{U}^{(x)} : \psi(x_1, x_2) \rightarrow \psi(x_1, x_2 + x_1), \quad (58)$$

and $\hat{U}^{(p)} : L_2(\mathbb{R}^2) \rightarrow L_2(\mathbb{R}^2)$

$$\hat{U}^{(p)} : \psi(x_1, x_2) \rightarrow \psi(x_1, x_2) e^{-ix_1 x_2}. \quad (59)$$

We define the pure states of two modes $|C_1\rangle$ and $|C_2\rangle$ with wave functions respectively

$$\begin{aligned} \psi_{x,\beta}(x_1, x_2) &= \sqrt{\frac{\beta}{2\pi}} e^{-\beta \frac{x_1^2}{4} - i \frac{x_1 x_2}{2} - \beta \frac{x_2^2}{4}}, \\ \psi_{p,\beta}(x_1, x_2) &= \sqrt{\frac{\beta}{2\pi}} e^{-\beta \frac{x_1^2}{4} + i \frac{x_1 x_2}{2} - \beta \frac{x_2^2}{4}}. \end{aligned} \quad (60)$$

We also note that the wave function of a displaced squeezed state $\hat{D}_{(0,x)} |\beta\rangle$ with $x \in \mathbb{R}$ is

$$\psi_{\beta,x}(x') = \sqrt{\frac{\beta}{\pi}} e^{-\beta \frac{x'^2}{2} - ix' x}. \quad (61)$$

Note hence that

$$\begin{aligned} &\langle \psi|_A \otimes \langle C_1|_{XP'} \otimes \langle C_2|_{PX'} \hat{U}_{XA}^{(x)\dagger} \hat{U}_{PA}^{(p)\dagger} [f(x_X, x_P)] = \\ &= \sqrt{\frac{\beta}{2\pi}} \int_{\mathbb{R}^2} dx_{X'} dx_{P'} \langle \psi| \hat{D}_{(x_X,0)}^\dagger \hat{D}_{(0,x_P)}^\dagger \langle \beta/2|_{P'} \hat{D}_{(0,x_X/2)}^\dagger \langle \beta/2|_{X'} \hat{D}_{(0,-x_P/2)}^\dagger e^{-\frac{\beta}{4}(x_X^2 + x_P^2)} f(x_X, x_P). \end{aligned} \quad (62)$$

It follows that

$$\begin{aligned} \Lambda_\beta^e[|\psi\rangle\langle\psi|] &:= \text{Tr}_{XP} [\hat{U}_{PA}^{(p)} \hat{U}_{XA}^{(x)} |\psi\rangle\langle\psi|_A \otimes |C_1\rangle\langle C_1|_{XP'} \otimes |C_2\rangle\langle C_2|_{PX'} U_{XA}^{(x)\dagger} U_{PA}^{(p)\dagger}] \\ &= \frac{\beta}{2\pi} \int_{\mathbb{R}^2} dx_X dx_P e^{-\frac{\beta}{2}(x_X^2 + x_P^2)} \hat{D}_{(x_X, x_P)} \hat{\rho} \hat{D}_{(x_X, x_P)}^\dagger \\ &\quad \otimes \hat{D}_{(0, -x_P/2)} |\beta/2\rangle\langle\beta/2|_{X'} \hat{D}_{(0, -x_P/2)}^\dagger \otimes \hat{D}_{(0, x_X/2)} |\beta/2\rangle\langle\beta/2|_{P'} \hat{D}_{(0, x_X/2)}^\dagger. \end{aligned}$$

The wave function of $U_{P'A}^{(p)} U_{X'A}^{(x)} U_{PA}^{(p)} U_{XA}^{(x)} |\psi\rangle_A \otimes |C_1\rangle_{XP'} \otimes |C_2\rangle_{PX'}$ is

$$\sqrt{\frac{\beta}{2\pi}} e^{-\beta \frac{x_X^2}{4} - i \frac{x_X x_{P'}}{2} - \beta \frac{x_{P'}^2}{4}} e^{-\beta \frac{x_P^2}{4} + i \frac{x_P x_{X'}}{2} - \beta \frac{x_{X'}^2}{4}} \psi(x_A + x_X + x_{X'}) e^{i(x_A - x_{X'})x_P + ix_A x_{P'}} \quad (63)$$

$$= \sqrt{\frac{\beta}{2\pi}} e^{-\beta \frac{x_X^2}{4} - i \frac{x_X x_{P'}}{2} - \beta \frac{x_{X'}^2}{4}} e^{-\beta \frac{x_P^2}{4} - i \frac{x_P x_{X'}}{2} - \beta \frac{x_{X'}^2}{4}} \psi(x_A + x_X + x_{X'}) e^{ix_A(x_P + x_{P'})}. \quad (64)$$

Since this wave function is symmetric under exchange $X \leftrightarrow X'$, $P \leftrightarrow P'$, defining the map

$$W_{X'P'A \rightarrow X'P'}[\hat{\rho}] := \text{Tr}_A [\hat{U}_{P'A}^{(p)} \hat{U}_{X'A}^{(x)} \hat{\rho}_{X'P'A} \hat{U}_{X'A}^{(x)\dagger} \hat{U}_{P'A}^{(p)\dagger}], \quad (65)$$

we have that

$$\begin{aligned} W_{X'P'A \rightarrow X'P'} \circ \Lambda_{\beta_{A \rightarrow X'P'A}}[\rho] &= \text{Tr}_{XPA} [\hat{U}_{P'A}^{(p)} \hat{U}_{X'A}^{(x)} \hat{U}_{P'A}^{(p)} \hat{U}_{X'A}^{(x)} \hat{\rho}_A \otimes |C_1\rangle\langle C_1|_{XP'} \otimes |C_2\rangle\langle C_2|_{PX'} (\hat{U}_{P'A}^{(p)} \hat{U}_{X'A}^{(x)} \hat{U}_{P'A}^{(p)} \hat{U}_{X'A}^{(x)})^\dagger] \\ &= \text{Tr}_{X'P'A} [\hat{U}_{P'A}^{(p)} \hat{U}_{X'A}^{(x)} \hat{U}_{P'A}^{(p)} \hat{U}_{X'A}^{(x)} \hat{\rho}_A \otimes |C_1\rangle\langle C_1|_{XP'} \otimes |C_2\rangle\langle C_2|_{PX'} (\hat{U}_{P'A}^{(p)} \hat{U}_{X'A}^{(x)} \hat{U}_{P'A}^{(p)} \hat{U}_{X'A}^{(x)})^\dagger] = \Lambda_{\beta_{A \rightarrow XP}}^{e,c}[\hat{\rho}_A]. \end{aligned} \quad (66)$$

Thus $\Lambda_\beta^{e,c}$ is degradable.