J. reine angew. Math. **765** (2020), 171–203
DOI 10.1515/crelle-2019-0027

Journal für die reine und angewandte Mathematik

Regularity of Lagrangian flows over $\mathrm{RCD}^*(K, N)$ spaces

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Abstract. The aim of this note is to provide regularity results for Regular Lagrangian flows of Sobolev vector fields over compact metric measure spaces verifying the Riemannian curvature dimension condition. We first prove, borrowing some ideas already present in the literature, that flows generated by vector fields with bounded symmetric derivative are Lipschitz, providing the natural extension of the standard Cauchy–Lipschitz theorem to this setting. Then we prove a Lusin-type regularity result in the Sobolev case (under the additional assumption that the m.m.s. is Ahlfors regular) therefore extending the already known Euclidean result.

Introduction

The theory of metric measure spaces with Riemannian Ricci curvature bounded from below and dimension bounded from above ($\mathrm{RCD}^*(K, N)$ metric measure spaces for short), although being very recent, is a very rapidly increasing research area with several contributions that, apart from their own theoretical interest, have often given new insights in the understanding of more classical questions of analysis and Riemannian geometry.

The introduction of the notion of metric measure spaces with Ricci curvature bounded from below and dimension bounded from above ($\mathrm{CD}(K, N)$ m.m.s. for short) dates back to the seminal and independent works of Lott and Villani [35] and Sturm [38, 39]. Crucial properties of this theory (which is therein formulated in terms of convexity-type properties of suitable energies over the Wasserstein space) are the compatibility with the case of smooth Riemannian manifolds and the stability with respect to suitable notions of convergence of metric measure spaces.

Many geometrical and analytical properties have been proven for $\mathrm{CD}(K, N)$ metric measure spaces (see for instance [41]). However, this class turns to be too large for some purposes, since it includes for instance smooth Finsler manifolds. In order to single out spaces with a Riemannian-like behaviour from the above introduced broader class, in [7] the authors proposed a notion of m.m.s. with Riemannian Ricci curvature bounded from below, adding to

The authors acknowledge the support of the PRIN2015 MIUR Project "Calcolo delle Variazioni".

the CD condition the requirement of linearity of the heat flow (which is the gradient flow of the so-called Cheeger energy). Later on the theory was adapted in [22], [10] and [21] to the dimensional case, with the introduction of the $\mathrm{RCD}(K,N)$ condition.[1)]

This paper deals with the regularity of flows of vector fields over $\mathrm{RCD}^*(K,N)$ metric measure spaces. To better introduce the reader to the notions that will be considered in the rest of the paper we briefly recall the Euclidean side of the story, which has been considered from much more time in the literature (but still reserves challenging open problems and questions).

In the Euclidean setting the Cauchy–Lipschitz theory guarantees existence, uniqueness and Lipschitz regularity for flows of Lipschitz vector fields. It is well known instead that lowering the regularity assumptions on the vector field might lead to non-uniqueness for integral curves, moreover, if one considers vector fields that are not defined everywhere but only Lebesgue-almost everywhere there is also need to introduce a notion of flow more general than the one adopted in the smooth case.

Motivated by the study of some PDEs in kinetic theory and fluid mechanics, Di Perna and Lions introduced in [20] a suitable notion of flow of Sobolev vector field and studied the associated existence and uniqueness problem. Later on their theory was revisited and extended to the case of vector fields with BV spatial regularity by Ambrosio in [1] (see also [4]), where the notion of Regular Lagrangian Flow was introduced as a good global selection of integral curves of the vector field. Moreover, Crippa and De Lellis in [18] were able to prove a mild regularity result for Regular Lagrangian Flows of Sobolev vector fields, namely that (locally) they are Lipschitz if we neglect a subset of the domain whose measure can be made arbitrarily small (where, of course, we have to pay the price that the Lipschitz constant becomes arbitrarily large). Such a result, known in the literature as Lusin-type regularity, holds true for instance for real-valued Sobolev functions (and it is already known to be true also when the domain is a sufficiently regular metric measure space, see [3]).

Over an arbitrary metric measure space $(X,\mathsf{d},\mathfrak{m})$ vector fields can be defined both as derivations over an algebra of scalar functions (which is the interpretation adopted in [12]) and as sections of the tangent module (see [23] for the latter viewpoint and for the equivalence with the first one). Moreover, restricting the analysis to more regular metric measure spaces (such as $\mathrm{RCD}(K,\infty)$ metric measure spaces) where a second order differential calculus can be developed, one can introduce also reasonable notions of Sobolev vector fields (see again [12] and [23] for the definitions of the spaces of vector fields with symmetric derivative in L^2 and of Sobolev vector fields, respectively).

A remark concerning the discussion above is in order. On metric measure spaces we do not have a priori at our disposal a notion of Lipschitz vector field (and also in the case of smooth Riemannian manifolds this notion is less natural and more subtle, since it requires parallel transport to compare tangent vectors at different points); in addition we do not have a notion of tangent vector at a given point. With this said, when trying to develop a theory of flows of vector fields in this very abstract setting, it is more natural to look at the generalized theory of Regular Lagrangian Flows than at the Cauchy–Lipschitz theory. In [12] Ambrosio

[1)] We will work with the slightly modified $\mathrm{RCD}^*(K,N)$ condition in this paper. We avoid any further comment about the differences with respect to the $\mathrm{RCD}(K,N)$ condition in this introduction but we remark that very recently Cavalletti and Milman proved their equivalence in [16].

and Trevisan were able to prove that this theory was the right one in order to get existence and uniqueness of flows of vector fields with symmetric covariant derivative in L^2 on a large class of metric measure spaces including the one of $\mathrm{RCD}(K,\infty)$ spaces.

After having established such a result one might wonder if the flow maps have some further regularity property. At a speculative level, proving such a result for a certain class of metric measure spaces could give new insights about their geometry and their regularity. We remark that this very recent theory has already been useful in some applications (see [28], [32] and [27]).

The main result of this note is Theorem 3.11 below, where we extend the Lusin-type regularity result by Crippa and De Lellis from the Euclidean setting to that one of compact $\mathrm{RCD}^*(K,N)$ metric measure spaces n-Ahlfors regular for some $1 < n \le N < +\infty$. We remark that, up to our knowledge, this is also the first intrinsic proof of the regularity result over smooth compact Riemannian manifolds, since it avoids any use of local charts.

This paper is organized as follows: in the preliminary Section 1 we introduce the main notations and collect the basic results of the theory of RCD metric measure spaces and Regular Lagrangian flows that are needed in the rest of the work. Then, in Section 2, following some ideas already present in [32] and [40], we provide a full Lipschitz regularity result for flows of vector fields with bounded symmetric derivative. Finally, in Section 3, which is the core of this note, we prove the sought Lusin-type regularity results.

Acknowledgement. The authors would like to thank Luigi Ambrosio for suggesting them the study of this problem and for kind and numerous comments and suggestions. They are grateful to the anonymous reviewer for the detailed report which greatly improved this note.

1. Preliminaries

1.1. RCD metric measure spaces. Throughout this note by metric measure space (m.m.s. for short) we mean a triple $(X,\mathsf{d},\mathfrak{m})$, where (X,d) is a complete and separable metric space and $\mathfrak{m}$ is a probability measure defined on the Borel σ-algebra of (X,d).

We shall adopt the standard metric notation: we will indicate by $B(x,r)$ the open ball of radius r centred at $x \in X$, by $\mathrm{Lip}(X,\mathsf{d})$ the space of Lipschitz functions over (X,d), by $\mathrm{Lip}\, f$ the Lipschitz constant of $f \in \mathrm{Lip}(X,\mathsf{d})$. Moreover, we introduce the notation

$$\mathrm{lip}\, f(x) := \limsup_{y\to x} \frac{|f(x)-f(y)|}{\mathsf{d}(x,y)}$$

for the so-called slope of a function $f : X \to \mathbb{R}$.

We will denote by $L^p(X,\mathfrak{m}) = L^p(X) = L^p$ the spaces of Borel p-integrable functions over $(X,\mathfrak{m})$ for any $1 \le p \le +\infty$ and by $L^0(X,\mathfrak{m})$ the space of $\mathfrak{m}$-measurable functions over X.

Unless otherwise stated from now on we assume $(X,\mathsf{d},\mathfrak{m})$ to be a compact $\mathrm{RCD}^*(K,N)$ metric measure space for some $K \in \mathbb{R}$ (lower bound on the Ricci curvature) and $1 \le N < +\infty$ (upper bound on the dimension). Let us assume without loss of generality that $\mathfrak{m}$ is fully supported on X, this assumption is justified by the fact that if $(X,\mathsf{d},\mathfrak{m})$ is $\mathrm{RCD}^*(K,N)$, then so is $(\mathrm{supp}\,\mathfrak{m},\mathsf{d},\mathfrak{m})$.

We remark that the notion of RCD$^*(K,N)$ m.m.s. was introduced and firstly studied in [22], [10] and [21]), while the introduction of the RCD(K,∞) condition dates back to the work [7]. We just recall that those spaces can be introduced and studied both from an Eulerian point of view (based on the so-called Γ-calculus) and from a Lagrangian point of view (based on optimal transportation techniques).

Below we briefly describe the main properties of RCD$^*(K,N)$ metric measure spaces that will play a role in our work.

As a first geometric property we recall that RCD$^*(K,N)$ metric measure spaces satisfy the Bishop–Gromov inequality (which holds true more generally for any CD$^*(K,N)$ m.m.s., see [35]). Together with the compactness assumption, the Bishop–Gromov inequality implies that $(X,\mathsf{d},\mathfrak{m})$ is doubling, that is there exists $c_D>0$ such that

$$\mathfrak{m}(B(x,2r))\le c_D\,\mathfrak{m}(B(x,r))$$

for any $x\in X$ and for any $r>0$.

For any $f\in L^1(X,\mathfrak{m})$ we will denote by Mf the Hardy–Littlewood maximal function of f, which is defined by

$$Mf(x):=\sup_{r>0}\fint_{B(x,r)}|f(z)|\,\mathrm{d}\mathfrak{m}(z),$$

where

$$\fint_{B(x,r)}f(z)\,\mathrm{d}\mathfrak{m}(z):=\frac{1}{\mathfrak{m}(B(x,r))}\int_{B(x,r)}f(z)\,\mathrm{d}\mathfrak{m}(z).$$

We recall that, since $(X,\mathsf{d},\mathfrak{m})$ is a doubling m.m.s., the maximal operator M is bounded from $L^p(X,\mathfrak{m})$ into itself for any $1<p\le+\infty$.

We go on with a brief discussion about Sobolev functions and vector fields over $(X,\mathsf{d},\mathfrak{m})$ referring to [6], [7] and [23] for a more detailed discussion about this topic.

Definition 1.1. For any $1<p<+\infty$ we define $W^{1,p}(X,\mathsf{d},\mathfrak{m})(=W^{1,p}(X))$ to be the space of those $f\in L^p(X,\mathfrak{m})$ such that there exists a sequence $f_n\to f$ in $L^p(X,\mathfrak{m})$ with $f_n\in\mathrm{Lip}(X,\mathsf{d})$ for any $n\in\mathbb{N}$ and $\sup_n\|\mathrm{lip}\,f_n\|_{L^p}<+\infty$.

The definition of Sobolev space is strongly related to the introduction of the Cheeger energy $\mathsf{Ch}_p:L^p(X,\mathfrak{m})\to[0,+\infty]$ which is defined by

$$\mathsf{Ch}_p(f):=\inf\left\{\liminf_{n\to\infty}\int(\mathrm{lip}\,f_n)^p\,\mathrm{d}\mathfrak{m}:\,f_n\to f\text{ in }L^p,\ f_n\in\mathrm{Lip}(X,\mathsf{d})\right\}\tag{1.1}$$

and turns out to be a convex and lower semicontinuous functional from $L^p(X,\mathfrak{m})$ to $[0,+\infty]$ whose finiteness domain coincides with $W^{1,p}(X,\mathsf{d},\mathfrak{m})$.

By looking at the optimal approximating sequence in (1.1) one can identify a distinguished object, called minimal relaxed gradient and denoted by $|\nabla f|_p$, which provides the integral representation

$$\mathsf{Ch}_p(f)=\int_X|\nabla f|_p^p\,\mathrm{d}\mathfrak{m}$$

for any $f\in W^{1,p}(X,\mathsf{d},\mathfrak{m})$. As the notation suggests $|\nabla f|_p$ depends a priori on the integrability exponent p. The space $W^{1,p}(X,\mathsf{d},\mathfrak{m})$ is a Banach space with respect to the norm $\|f\|_{W^{1,p}}^p:=\|f\|_{L^p}^p+\mathsf{Ch}_p(f)$, moreover it holds that the inequality $|\nabla f|_p\le\mathrm{lip}\,f$ holds true $\mathfrak{m}$-a.e. on X for any $f\in\mathrm{Lip}(X,\mathsf{d})$.

We point out that to single out $\mathrm{RCD}^*(K, N)$ metric measure spaces from the broader class of $\mathrm{CD}^*(K, N)$ metric measure spaces one adds the request that $\mathsf{Ch} := \mathsf{Ch}_2$ is a quadratic form on $L^2(X, \mathfrak{m})$ to the curvature-dimension condition. In this way the space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$, which in general is only a Banach space, turns out to be a Hilbert space.

This global assumption has in turn strong consequences on the infinitesimal behaviour of the space (indeed any m.m.s whose $W^{1,2}$ space is Hilbert is called infinitesimally Hilbertian). In particular, in the smooth setting it allows to single out Riemannian manifolds in the class of Finsler manifolds.

For any $f, g \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ we define a function $\nabla f \cdot \nabla g \in L^1(X\ \mathfrak{m})$ by

$$\nabla f \cdot \nabla g(x) := \frac{1}{4}|\nabla(f+g)|^2(x) - \frac{1}{4}|\nabla(f-g)|^2(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X.$$

It is proved in [25] that under the $\mathrm{RCD}(K, \infty)$ assumption the minimal relaxed gradient $|\nabla f|_p$ does not depend on p, for this reason we will use the notation $|\nabla f|$.

In order to introduce the heat flow and its main properties we begin by recalling the notion of Laplacian.

Definition 1.2. The Laplacian $\Delta : D(\Delta) \to L^2(X, \mathfrak{m})$ is a densely defined linear operator whose domain consists of all functions $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ satisfying

$$\int hg \,\mathrm{d}\mathfrak{m} = -\int \nabla h \cdot \nabla f \,\mathrm{d}\mathfrak{m} \quad \text{for all } h \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$$

for some $g \in L^2(X, \mathfrak{m})$. The unique g with this property is denoted by Δf. We remark that the linearity of Δ follows from the quadraticity of Ch.

On any compact $\mathrm{RCD}^*(K, N)$ m.m.s the operator $-\Delta$ is densely defined, self-adjoint and compact. We will denote by $(\lambda_i)_{i\in\mathbb{N}}$ its spectrum (where the eigenvalues are counted with multiplicity and in increasing order, $\lambda_i \geq 0$ for any i and $\lambda_i \to \infty$ as i goes to infinity) and by $(u_i)_{i\in\mathbb{N}}$ the associated eigenfunctions, normalized in such a way that $\|u_i\|_{L^2} = 1$ for any $i \in \mathbb{N}$. Recall that $(u_i)_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(X, \mathfrak{m})$. Furthermore, the sequence $(\lambda_i)_{i\in\mathbb{N}}$ has more than linear growth at infinity. A standard reference for this result in the smooth framework is [34, Chapter 10] and the arguments presented therein can be adapted to the case of our interest.

The heat flow P_t is defined as the $L^2(X, \mathfrak{m})$-gradient flow of $\frac{1}{2}\mathsf{Ch}$, whose existence and uniqueness follow from the Komura-Brezis theory. It can equivalently be characterized by saying that for any $u \in L^2(X, \mathfrak{m})$ the curve $t \mapsto P_t u \in L^2(X, \mathfrak{m})$ is locally absolutely continuous in $(0, +\infty)$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t u = \Delta P_t u \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0, +\infty).$$

Under our assumptions the heat flow provides a linear, continuous and self-adjoint contraction semigroup in $L^2(X, \mathfrak{m})$. Moreover, P_t extends to a linear, continuous and mass preserving operator, still denoted by P_t, in all the L^p spaces for $1 \leq p < +\infty$.

In [7] it is proved that for $\mathrm{RCD}(K, \infty)$ metric measure spaces the dual semigroup

$$\bar{P}_t : \mathscr{P}(X) \to \mathscr{P}(X)$$

of P_t, defined by

$$\int_X f \,\mathrm{d}\bar{P}_t\mu := \int_X P_t f \,\mathrm{d}\mu \quad \text{for all } \mu \in \mathscr{P}(X), \quad \text{for all } f \in \mathrm{Lip}_b(X),$$

for $t > 0$, maps probability measures into probability measures absolutely continuous with respect to $\mathfrak{m}$. Then, for any $t > 0$, we can introduce the so-called *heat kernel*

$$p_t : X \times X \to [0, +\infty)$$

by

$$p_t(x, \cdot)\mathfrak{m} := \bar{P}_t\delta_x.$$

From now on for any $f \in L^\infty(X, \mathfrak{m})$ we will denote by $P_t f$ the representative pointwise everywhere defined by

$$P_t f(x) = \int_X f(y) p_t(x, y) \,\mathrm{d}\mathfrak{m}(y).$$

Since compact $\mathrm{RCD}^*(K,N)$ metric measure spaces are doubling, as we have already remarked, and they satisfy a local Poincaré inequality (see [41]) the general theory of Dirichlet forms (see [37]) guarantees that we can find a locally Hölder continuous representative of p on $X \times X \times (0, +\infty)$.

We also recall from [8] the spectral identity which provides an explicit expression for the heat kernel in terms of the eigenfunctions of the Laplacian, namely

$$p_t(x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} u_i(x) u_i(y) \tag{1.2}$$

for any $t > 0$, where, by choosing the Hölder continuous representative of u_i, whose Hölder norm grows linearly with λ_i, one obtains the Hölder continuous representative of p_t (taking into account that, as we already remarked, the sequence of eigenvalues grows at least linearly at infinity).

Moreover, in [33] the following finer properties of the heat kernel have been proven: there exist constants $C_1 \geq 1$, $C_2 > 0$ and $C_3 \geq 0$ depending only on K and N such that

$$\begin{aligned}\frac{1}{C_1}\frac{1}{\mathfrak{m}(B(x,\sqrt{t}))}\exp\left\{-\frac{d^2(x,y)}{3t} - C_3 t\right\} &\leq p_t(x,y) \\ &\leq C_1 \frac{1}{\mathfrak{m}(B(x,\sqrt{t}))}\exp\left\{-\frac{d^2(x,y)}{5t} + C_3 t\right\}\end{aligned} \tag{1.3}$$

for any $x, y \in X$ and for any $t \in (0, +\infty)$ and for any $x \in X$ and $t \in (0, +\infty)$ it holds

$$|\nabla p_t(x, \cdot)|(y) \leq C_2 \frac{1}{\mathfrak{m}(B(x,\sqrt{t}))\sqrt{t}} \exp\left\{-\frac{d^2(x,y)}{5t} + C_3 t\right\} \tag{1.4}$$

for $\mathfrak{m}$-a.e. $y \in X$.

We go on by stating a few regularity properties of $\mathrm{RCD}^*(K,N)$ metric measure spaces (which hold true more generally for any $\mathrm{RCD}(K,\infty)$ m.m.s.) referring again to [7] for a more detailed discussion and the proofs of these results.

First we have the *Bakry–Émery contraction* estimate:

$$|\nabla P_t f|^2 \leq e^{-2Kt} P_t |\nabla f|^2 \quad \mathfrak{m}\text{-a.e.} \tag{1.5}$$

for any $t > 0$ and for any $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$.

Another non-trivial regularity property is the so-called $L^\infty-\mathrm{Lip}$ *regularization* of the heat flow, that is for any $f \in L^\infty(X,\mathfrak{m})$ we have that $P_t f \in \mathrm{Lip}(X)$ with

$$\sqrt{2I_{2K}(t)}\,\mathrm{Lip}(P_t f) \le \|f\|_{L^\infty} \quad \text{for any } t>0, \tag{1.6}$$

where $I_L(t) := \int_0^t e^{Lr}\,\mathrm{d}r$.

Then we have the so-called *Sobolev to Lipschitz property*: any $f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ with $|\nabla f| \in L^\infty(X,\mathfrak{m})$ admits a Lipschitz representative $\bar f$ such that $\mathrm{Lip}\,\bar f \le \|\nabla f\|_\infty$, that actually implies the equality $\mathrm{Lip}\,\bar f = \|\nabla f\|_\infty$ since, in general, $\mathrm{Lip}\, f \ge \|\mathrm{lip}\, f\|_\infty$ and $\mathrm{lip}\, f \ge |\nabla f|$ $\mathfrak{m}$-a.e. on X.

Following [23] we introduce the space of "test" functions $\mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ by

$$\begin{aligned}\mathrm{Test}(X,\mathsf{d},\mathfrak{m}) := \{f \in D(\Delta)\cap L^\infty(X,\mathfrak{m}) : |\nabla f| \in L^\infty(X) \text{ and}\\ \Delta f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})\cap L^\infty(X,\mathfrak{m})\}\end{aligned} \tag{1.7}$$

and we remark that for any $g \in L^\infty(X)$ it holds that $P_t g \in \mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ for any $t>0$, thanks to (1.5), (1.6), the fact that P_t maps $L^2(X,\mathfrak{m})$ in $D(\Delta)$ and the commutation

$$\Delta P_t f = P_t \Delta f \quad \text{for all } f \in D(\Delta).$$

Let us point out, for the sake of clarity, that in general the condition $\Delta f \in L^\infty(X,\mathfrak{m})$ is not included in the definition of the space of test functions. Still, thanks to the compactness of (X,d) and the Gaussian estimates for the heat kernel this extra condition can be added to (1.7), as it is argued in [30].

We conclude this preliminary section with some finer regularity properties which hold true under the stronger assumption that the metric measure space is Ahlfors regular.

Definition 1.3. We say that $(X,\mathsf{d},\mathfrak{m})$ is n-Ahlfors regular for some $1 \le n < +\infty$ if there exist constants $0 < c_1 \le c_2$ such that

$$c_1 r^n \le \mathfrak{m}(B(x,r)) \le c_2 r^n \tag{1.8}$$

for any $0<r<D$ and for any $x \in X$, where we denoted by D the diameter of X.

Remark 1.4. Let us observe that assumption (1.8) guarantees integrability of certain powers of the distance function, namely for any $x \in X$ and for any $\alpha < n$ we have that

$$y \mapsto \mathsf{d}(x,y)^{-\alpha}$$

is $\mathfrak{m}$-integrable. Indeed by Cavalieri's formula we have that

$$\begin{aligned}\int_X \frac{1}{\mathsf{d}(x,y)^\alpha}\,\mathrm{d}\mathfrak{m}(x) &= n\int_0^\infty \mathfrak{m}(\{y : \mathsf{d}(x,y)^{-\alpha} > \lambda\})\,\mathrm{d}\lambda\\ &= n\int_0^\infty \mathfrak{m}(B(x,\lambda^{-\frac{1}{\alpha}}))\,\mathrm{d}\lambda\\ &\le D^{-\alpha} + c_1\int_{D^{-\alpha}}^\infty \lambda^{-\frac{n}{\alpha}}\,\mathrm{d}\lambda.\end{aligned}$$

In RCD*(K,N) spaces it can be proved that eigenfunctions of the Laplacian have Lipschitz representatives. Moreover, they belong to the space Test introduced above. The result of the forthcoming Lemma 1.5 provides also quantitative estimates on their Lipschitz norms, under the additional assumption that $\mathfrak{m}$ is an Ahlfors regular probability measure.

Lemma 1.5. *Let u_i be an eigenfunction of $-\Delta$ associated to the eigenvalue λ_i. Then u_i has a Lipschitz representative. Moreover, it holds*

$$\|u_i\|_{L^\infty} \le \frac{C_1 e}{c_1}(C_3+\lambda_i)^{\frac{n}{2}}, \quad \|\nabla u_i\|_{L^\infty} \le \sqrt{\frac{\lambda_i+|K|}{2}}\|u_i\|_{L^\infty}. \tag{1.9}$$

Proof. Observe that by (1.8) and (1.3) we get

$$p_t(x,y) \le \frac{C_1 e^{C_3 t}}{c_1 t^{\frac{n}{2}}} \quad \text{for all } x,y \in X,$$

which yields the ultracontractivity property of the heat semigroup, namely

$$\|P_t u\|_{L^\infty} \le \frac{C_1 e^{C_3 t}}{c_1 t^{\frac{n}{2}}}\|u\|_{L^1} \quad \text{for all } t>0, \tag{1.10}$$

for any $u \in L^1(X,\mathfrak{m})$.

Observe that, since $-\Delta u_i = \lambda_i u_i$, it holds that $P_t u_i = e^{-\lambda_i t}u_i$ for any $t>0$.

An application of (1.10) with $t = \frac{1}{C_3+\lambda_i}$ yields now to the desired estimate

$$\|u_i\|_\infty \le \frac{C_1 e}{c_1}(C_3+\lambda_i)^{\frac{n}{2}},$$

since $\|u_i\|_{L^1} \le \|u_i\|_{L^2} = 1$.

In order to prove the second estimate in (1.9) we apply (1.6) to get

$$\|\nabla u_i\|_{L^\infty} = e^{\lambda_i t}\|\nabla P_t u_i\|_{L^\infty} = e^{\lambda_i t}\,\mathrm{Lip}(P_t u_i) \le \frac{e^{\lambda_i t}}{\sqrt{2 I_{2K}(t)}}\|u_i\|_{L^\infty}.$$

Observing now that $I_L(s) \ge s e^{-|L|s}$ and choosing $t := \frac{1}{\lambda_i+|K|}$, we obtain the desired conclusion. □

1.2. Regular Lagrangian flows. In this subsection we recall the notion of Regular Lagrangian flow (RLF for short) firstly introduced in the Euclidean setting by Ambrosio in [1] (inspired by the earlier work by Di Perna and Lions [20]).

The notion of Regular Lagrangian flow was introduced to study ordinary differential equations associated to weakly differentiable vector fields. It is indeed well known that, in general, it is not possible to define in a unique way a flow associated to a non-Lipschitz vector field since the trajectories starting from a fixed point are often not unique. Roughly speaking, the RLF is a selection of trajectories that provides a very robust notion of flow.

In order to define the concept of Regular Lagrangian flow over $\mathrm{RCD}^*(K,N)$ spaces we introduce the notion of vector field through that one of derivation that has been adopted in [12].

Definition 1.6. We say that a linear functional $b : \mathrm{Lip}(X,\mathsf{d}) \to L^0(X,\mathfrak{m})$ is a derivation if it satisfies the Leibniz rule, that is

$$b(fg) = b(f)g + f b(g)$$

for any $f,g \in \mathrm{Lip}(X,\mathsf{d})$.

Given a derivation b, we will write $|b| \in L^p$ if there exists some function $g \in L^p(X,\mathfrak{m})$ such that

$$b(f) \le g|\nabla f| \quad \mathfrak{m}\text{-a.e. on } X, \tag{1.11}$$

for any $f \in \mathrm{Lip}(X,\mathsf{d})$ and we will denote by $|b|$ the minimal (in the $\mathfrak{m}$-a.e. sense) g with such a property.

We will also use the notation $b \cdot \nabla f$ in place of $b(f)$ in the rest of the paper.

We remark that if a derivation b is in L^p, then it can be extended in a unique way to a linear functional on $W^{1,q}(X,\mathsf{d},\mathfrak{m})$ still satisfying (1.11), where q is the dual exponent of p. Moreover, any $f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ defines in a canonical way a derivation b_f in L^2 through the formula $b_f(g) = \nabla f \cdot \nabla g$, usually called the *gradient derivation* associated to f.

A notion of divergence can be introduced by integration by parts.

Definition 1.7. Let b be a derivation with $|b| \in L^1(X,\mathfrak{m})$. We say that $\operatorname{div} b \in L^p(X,\mathfrak{m})$ if there exists $g \in L^p(X,\mathfrak{m})$ such that

$$\int_X b(f)\,\mathrm{d}\mathfrak{m} = -\int_X gf\,\mathrm{d}\mathfrak{m}$$

for any $f \in \mathrm{Lip}(X,\mathsf{d})$. By a density argument it is easy to check that such a g is unique (when it exists) and we will denote it by $\operatorname{div} b$.

In the rest of the present note we will write $b \in L^p(TX)$ to denote a derivation such that $|b| \in L^p(X,\mathfrak{m})$. We refer to [23] for the introduction of the so-called tangent and cotangent modules over an arbitrary metric measure space and for the identification results between derivations and elements of the tangent module which stand behind the use of this notation.

We also introduce here the notion of time-dependent vector field.

Definition 1.8. Let us fix $T > 0$, and $p \in [1,+\infty]$. We say that a map

$$b : [0,T] \to L^p(TM)$$

is a time-dependent vector field if for every $f \in W^{1,q}(X,\mathsf{d},\mathfrak{m})$ the map

$$(t,x) \mapsto b_t \cdot \nabla f(x),$$

is measurable with respect to the product σ-algebra $\mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(X)$. We say that b is bounded if

$$\|b\|_{L^\infty} := \||b|\|_{L^\infty([0,T]\times X)} < \infty,$$

and that $b \in L^1((0,T); L^p(X,\mathfrak{m}))$ if

$$\int_0^T \||b_s|\|_{L^p}\,\mathrm{d}s < \infty.$$

In the context of $\mathrm{RCD}^*(K,N)$ spaces the definition of Regular Lagrangian flow reads as follows (see [12] and [13]).

Definition 1.9. Let us fix a time-dependent vector field b_t (see Definition 1.8). We say that a map $\boldsymbol{X} : [0,T] \times X \to X$ is a Regular Lagrangian flow associated to b_t if the following conditions hold true:

(1) $\boldsymbol{X}(0,x) = x$ and $\boldsymbol{X}(\cdot,x) \in C([0,T];X)$ for every $x \in X$,

(2) there exists a positive constant L, called compressibility constant, such that

$$\boldsymbol{X}(t,\cdot)_{\#}\mathfrak{m} \leq L\mathfrak{m}$$

for every $t \in [0,T]$,

(3) for every $f \in \mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ the map $t \mapsto f(\boldsymbol{X}(t,x))$ is absolutely continuous on $[0,T]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{X}(t,x)) = b_t \cdot \nabla f(\boldsymbol{X}(t,x)) \quad \text{for a.e. } t \in (0,T), \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X.$$

The selection of "good" trajectories is encoded in condition (2), which is added to ensure that the trajectories of the flow do not concentrate too much with respect to $\mathfrak{m}$.

In the definition we are assuming that $\boldsymbol{X}$ is defined in every point $x \in X$. Actually the notion of RLF is stable under modification in a negligible set of initial conditions, but we prefer to work with a pointwise defined map in order to avoid technical issues.

The theory of Regular Lagrangian flows in the context of metric measure spaces was developed by Ambrosio and Trevisan in [12]. The authors work with a very weak notion of symmetric derivative for a vector field.

Definition 1.10. Let $b \in L^2(TX)$ be given with $\operatorname{div} b \in L^2(X,\mathfrak{m})$. We will say that $|\nabla_{\mathrm{sym}} b| \in L^2(X,\mathfrak{m})$ if there exists a constant $c \geq 0$ satisfying

$$\left| \int_X \nabla_{\mathrm{sym}} b(\nabla f, \nabla g)\,\mathrm{d}\mathfrak{m} \right| \leq c \| |\nabla f| \|_{L^4} \| |\nabla g| \|_{L^4} \tag{1.12}$$

for every $f, g \in \mathrm{Test}(X,\mathrm{d},\mathfrak{m})$, where

$$\int_X \nabla_{\mathrm{sym}} b(\nabla f, \nabla g)\,\mathrm{d}\mathfrak{m} := -\frac{1}{2}\int_X \{ b \cdot \nabla f \Delta g + b \cdot \nabla g \Delta f - (\operatorname{div} b) \nabla f \cdot \nabla g \}\,\mathrm{d}\mathfrak{m}.$$

We let $\|\nabla_{\mathrm{sym}} b\|_{L^2}$ be the smallest admissible c in (1.12).

The results of [12] guarantee in particular existence and uniqueness of the RLF associated to a bounded vector field b, with symmetric derivative (in the sense of Definition 1.10) in L^2 and bounded divergence, in the context of $\mathrm{RCD}^*(K,N)$ spaces, which is the one we will be interested in for the rest of this note.

We conclude this section recalling a deep relation between the Regular Lagrangian flow of b_t and solutions of the continuity equation induced by b_t (see [12]).

Proposition 1.11. *Let $\boldsymbol{X}_t$ be a Regular Lagrangian flow of the vector field b_t. Then, for any $\mu_0 \in \mathscr{P}(X)$ absolutely continuous with respect to $\mathfrak{m}$ and with density bounded from above, defining $\mu_t := (\boldsymbol{X}_t)_{\#}\mu_0$, we have that μ_t solves the continuity equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t + \operatorname{div}(b_t \mu_t) = 0 \tag{1.13}$$

in the distributional sense, that is the function $t \mapsto \int \phi \, \mathrm{d}\mu_t$ *is in* $W^{1,1}((0,T))$ *and it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \phi \, \mathrm{d}\mu_t = \int b_t \cdot \nabla\phi \, \mathrm{d}\mu_t \tag{1.14}$$

for $\mathscr{L}^1$*-a.e.* $t \in (0,T)$, *for any* $\phi \in \mathrm{Lip}(X, \mathsf{d})$.

Remark 1.12. We remark that it is possible to find a common $\mathscr{L}^1$-negligible set $\mathcal{N} \subset (0,T)$ in such a way that (1.14) is satisfied for any $t \in (0,T) \setminus \mathcal{N}$ for any $\phi \in \mathrm{Lip}(X, \mathsf{d})$ (see [24, Proposition 3.7])

2. Regularity in the "Lipschitz" case

The aim of this section is to provide a full Lipschitz regularity result for flows of (possibly time-dependent) regular vector fields with bounded symmetric covariant derivative (where the right notions of symmetric covariant derivative and "regular" are introduced in Definition 2.3).

We have to remark that, while the regularity assumption seems to be not too restrictive in view of the possible applications of this result, the assumption that the symmetric covariant derivative is bounded is very restrictive and it could happen that, for a general $\mathrm{RCD}^*(K,N)$ metric measure space, there are no vector fields satisfying this constraint. Nevertheless, we find it interesting to present this result, both to better introduce the reader to the study of flows of vector fields over non-smooth spaces, and since techniques very similar to the one we are going to present have already been proven to be useful in the study of some rigidity problems such as in [28] and [27].

The proof of the sought regularity result follows the strategy of the Lipschitz regularity of flows of Lipschitz vector fields in the Euclidean case, based on the differentiation of the distance between two flow lines of the vector field. To rule out the possible non-smoothness of the space, we work at the level of curves of absolutely continuous measures (exploiting the result of Proposition 1.11) following some ideas taken from [40] and the very recent [32]. Let us remark, for sake of correctness, that the strategy we implement, based on the application of the second order differentiation formula along W_2-geodesics, was already suggested in [32] (see in particular Remark 3.18 and Remark 3.19 therein).

We assume the reader to be familiar with the basic notions and notations of optimal transportation (referring for instance to [5] for their introduction).

Below we state two preliminary results that play a key role in the proof of Theorem 2.7. For the moment we do not need to add any extra regularity assumption to the time-dependent vector field $(b_t)_{t\in[0,T]}$ apart from measurability with respect to time.

Lemma 2.1. *Let* $(\mu_t)_{t\in[0,T]}$ *be a solution of the continuity equation* (1.13) *such that* $\mu_t \le C\,\mathfrak{m}$ *for any* $t \in [0,T]$ *for some* $C > 0$ *and fix any* $\nu \in \mathscr{P}(X)$. *Then it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2} W_2^2(\mu_t, \nu) = \int b_t \cdot \nabla\phi_t \, \mathrm{d}\mu_t \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0,T),$$

where ϕ_t *is any optimal Kantorovich potential for the transport problem between* μ_t *and* ν.

Proof. The result is stated and proved in a much more general framework in [24, Proposition 3.11]. □

Corollary 2.2. *Let $(\mu_t)_{t\in[0,T]}$ and $(\nu_t)_{t\in[0,T]}$ be solutions with uniformly bounded densities to the continuity equation induced by b_t starting from μ_0 and ν_0 respectively as in* (1.14)*. Then it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu_t)=\int b_t\cdot\nabla\phi_t\,\mathrm{d}\mu_t+\int b_t\cdot\nabla\psi_t\,\mathrm{d}\nu_t\quad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,T),$$

where (ϕ_t,ψ_t) is any couple of optimal Kantorovich potentials between μ_t and ν_t.

Proof. The desired conclusion can be obtained with almost the same proof of Lemma 2.1 above. We report here a few more details for sake of completeness.

The results of [24] guarantee that the curves $(\mu_t)_{t\in[0,T]}$ and $(\nu_t)_{t\in[0,T]}$ are absolutely continuous with values in $(\mathscr{P}(X),W_2)$. It easily follows that the curve $t\mapsto W_2^2(\mu_t,\nu_t)$ is absolutely continuous, hence it is differentiable $\mathscr{L}^1$-a.e. over $(0,T)$. It follows from Remark 1.12 and what we just observed that we can find a full $\mathscr{L}^1$-measure set $\mathcal{C}\subset(0,T)$ such that, for any $t\in\mathcal{C}$, $s\mapsto W_2^2(\mu_s,\nu_s)$ is differentiable at $s=t$ and for any $\phi,\psi\in\mathrm{Lip}(X,\mathsf{d})$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}\int\phi\,\mathrm{d}\mu_s=\int b_t\cdot\nabla\phi\,\mathrm{d}\mu_t,\quad \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}\int\psi\,\mathrm{d}\nu_s=\int b_t\cdot\nabla\psi\,\mathrm{d}\nu_t. \tag{2.1}$$

Let now (ϕ_t,ψ_t) be any couple of optimal Kantorovich potentials between μ_t and ν_t. It follows by the duality results for the optimal transport problem that for any $h>0$ sufficiently small it holds

$$\begin{aligned}&\frac{1}{2}W_2^2(\mu_{t+h},\nu_{t+h})-\frac{1}{2}W_2^2(\mu_t,\nu_t)\\ &\quad\geq\int\phi_t\,\mathrm{d}\mu_{t+h}+\int\psi_t\,\mathrm{d}\nu_{t+h}-\int\phi_t\,\mathrm{d}\mu_t-\int\psi_t\,\mathrm{d}\nu_t.\end{aligned}\tag{2.2}$$

The desired conclusion follows from (2.2) dividing by h, taking the limit as $h\to0$ at both sides and taking into account (2.1). □

Before going on we introduce, following [23], the notion of Sobolev vector field with symmetric covariant derivative in L^2 over an arbitrary $\mathrm{RCD}(K,\infty)$ m.m.s. $(X,\mathsf{d},\mathfrak{m})$ and the associated space $W^{1,2}_{C,s}(X,\mathsf{d},\mathfrak{m})$.

We refer to [23] for the construction of the module $L^2(T^{\otimes2}X)$ (starting from the tangent module $L^2(TX)$) and for the introduction of the space $W^{1,2}_C(X,\mathsf{d},\mathfrak{m})$ of vector fields with full covariant derivative in L^2.

Definition 2.3. The Sobolev space $W^{1,2}_{C,s}(TX)\subset L^2(TX)$ is the space of all maps $b\in L^2(TX)$ for which there exists a tensor $S\in L^2(T^{\otimes2}X)$ such that for any choice of test functions $h,g_1,g_2\in\mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ it holds

$$\begin{aligned}\int hS(\nabla g_1,\nabla g_2)\,\mathrm{d}\mathfrak{m}=\frac{1}{2}\int\{&-b(g_2)\,\mathrm{div}(h\nabla g_1)-b(g_1)\,\mathrm{div}(h\nabla g_2)\\ &+\mathrm{div}(hb)\nabla g_1\cdot\nabla g_2\}\,\mathrm{d}\mathfrak{m}.\end{aligned}\tag{2.3}$$

In this case we shall call the tensor S symmetric covariant derivative of b and we will denote it by $\nabla_{\mathrm{sym}}b$. We endow the space $W^{1,2}_{C,s}(TX)$ with the norm $\|\cdot\|_{W^{1,2}_{C,s}(TX)}$ defined by

$$\|b\|^2_{W^{1,2}_{C,s}(TX)}:=\|b\|^2_{L^2(TX)}+\|\nabla_{\mathrm{sym}}b\|^2_{L^2(T^{\otimes2}TX)}.$$

Remark 2.4. It easily follows from the definition that the symmetric covariant derivative is actually symmetric. Moreover, for any $b \in W^{1,2}_C(TX)$ it holds that $b \in W^{1,2}_{C,s}(TX)$ and $\nabla_{\mathrm{sym}} b$ is the symmetric part of ∇b (we refer to [23, Section 3.4] for the definition of the covariant derivative).

In order to compare the notion of Sobolev vector field introduced above with that one introduced in Definition 1.10, we observe that the first one is easily seen to be stronger than the second one. Indeed it is sufficient to take $h = 1$ in equation (2.3) and to apply the Young inequality with exponents $2, 4$ and 4 to get the claimed conclusion.

We define the space $\mathrm{TestV}(X, \mathsf{d}, \mathfrak{m}) \subset L^2(TX)$ of test vector fields to be the set of linear combinations of the form

$$\sum_{i=1}^{n} g_i \nabla f_i,$$

where $g_i, f_i \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ for any $i = 1, \dots, n$. We recall that $\mathrm{TestV}(X) \subset W^{1,2}_C(X)$ and we will denote by $H^{1,2}_{C,s}(X)$ the closure of $\mathrm{TestV}(X)$ in $W^{1,2}_{C,s}(X)$ with respect to the $W^{1,2}_{C,s}$-norm.

Below we quote from [29] a useful differentiation formula for $H^{1,2}_{C,s}$-vector fields along Wasserstein geodesics obtained by the authors as a corollary of the second order differentiation formula for $H^{2,2}$-Sobolev functions proven in [30].

Theorem 2.5. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}^*(K,N)$ m.m.s. for some $K \in \mathbb{R}$ and N with $1 \le N < +\infty$. Let $(\eta_s)_{s\in[0,1]}$ be a W_2-geodesic connecting probability measures η_0 and η_1 absolutely continuous with respect to $\mathfrak{m}$, with bounded densities and bounded supports and assume that $b \in H^{1,2}_{C,s}(X, \mathsf{d}, \mathfrak{m})$. Then the curve*

$$s \mapsto \int b \cdot \nabla\phi_s \, \mathrm{d}\eta_s$$

is C^1 on $[0,1]$, where ϕ_s is any function such that for some $r \in [0,1]$ with $s \neq r$ it holds that $-(r-s)\phi_s$ is an optimal Kantorovich potential from η_s to η_r. Moreover, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}s} \int b \cdot \nabla\phi_s \, \mathrm{d}\eta_s = \int \nabla_{\mathrm{sym}} b(\nabla\phi_s, \nabla\phi_s) \, \mathrm{d}\eta_s$$

for any $s \in [0,1]$.

Remark 2.6. We remark that Theorem 2.5 is actually stated in [29] only for vector fields in $H^{1,2}_C(X)$. However, since the strategy of the proof goes via approximation through elements of $\mathrm{TestV}(X)$ and the statement just involves the symmetric part of the covariant derivative, it easily extends to $H^{1,2}_{C,s}(X)$.

We remark that it makes sense to say that an element of the space $L^2(T^{\otimes 2}X)$ belongs to $L^\infty(T^{\otimes 2}X)$ and to consider its L^∞-norm. With this said, we will denote by

$$L := \sup_{t\in(0,T)} \|\nabla_{\mathrm{sym}} b_t\|_{L^\infty}$$

and from now on to the basic assumptions of Section 1.2 we add the assumption that $L < +\infty$.

Below we state and prove the key result of this section, that will allow us to obtain both uniqueness and Lipschitz regularity for Regular Lagrangian Flows.

Theorem 2.7. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}^*(K,N)$ m.m.s. and $(b_t)_{t\in[0,T]}$ a time-dependent vector field verifying the above discussed assumptions. Let $(\mu_t)_{t\in[0,T]}$ and $(\nu_t)_{t\in[0,T]}$ denote solutions of the continuity equation induced by b_t, absolutely continuous, with uniformly bounded densities and uniformly bounded supports. Then it holds that*

$$W_2(\mu_t,\nu_t)\le e^{Lt}W_2(\mu_0,\nu_0)$$

for any $t\in[0,T]$.

Proof. Applying first Corollary 2.2 and then Theorem 2.5, denoting by $\mathcal{Q}_t$ the Hopf–Lax semigroup, we obtain that, for $\mathscr{L}^1$-a.e. $t\in(0,T)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu_t)&=\int b_t\cdot\nabla\mathcal{Q}_1(-\phi_t)\,\mathrm{d}\nu_t-\int b.\nabla(-\phi_t)\,\mathrm{d}\mu_t\\&=\int_0^1\int\nabla_{\mathrm{sym}}b_t(\nabla\phi_t^s,\nabla\phi_t^s)\,\mathrm{d}\eta_t^s\,\mathrm{d}s,\end{aligned}\tag{2.4}$$

where we denoted by ϕ_t an optimal Kantorovich potential from μ_t to ν_t, $(\eta_t^s)_{s\in[0,1]}$ the W_2-geodesic joining μ_t with ν_t and by ϕ_t^s the intermediate time potential such that

$$\frac{\mathrm{d}}{\mathrm{d}s}\eta_t^s+\operatorname{div}(\nabla\phi_t^s\eta_t^s)=0.$$

Observe that we are in a position to apply Theorem 2.5 since μ_t and ν_t are by assumption absolutely continuous with bounded densities for any $t\in[0,T]$.

Recalling that, as a consequence of the metric Brenier theorem (see [7, Proposition 3.5]),

$$\int|\nabla\phi_t^s|^2\,\mathrm{d}\eta_t^s=W_2^2(\mu_t,\nu_t)$$

for $\mathscr{L}^1$-a.e. $s\in(0,1)$ and for any $t\in[0,T]$, we can conclude from (2.4) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu_t)\le\|\nabla_{\mathrm{sym}}b_t\|_{L^\infty}W_2^2(\mu_t,\nu_t)\le LW_2^2(\mu_t,\nu_t)$$

for a.e. $t\in(0,T)$.

An application of Grönwall's lemma yields now the desired conclusion, namely that

$$W_2^2(\mu_t,\nu_t)\le e^{2Lt}W_2^2(\mu_0,\nu_0)$$

for any $t\in[0,T]$. □

As a corollary of Theorem 2.7, under our more restrictive assumptions about the regularity of the vector field, we can prove uniqueness (avoiding the theory of renormalized solutions) and Lipschitz regularity of Regular Lagrangian flows.

Theorem 2.8. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}^*(K,N)$ m.m.s. and let $(b_t)_{t\in[0,T]}$ satisfy the assumptions of Theorem 2.7. Then there exist a unique Regular Lagrangian flow $(\boldsymbol{X}_t)_{t\in[0,T]}$ of $(b_t)_{t\in[0,T]}$.*

Proof. We do not give a complete proof of this statement. We just say here that Theorem 2.7 gives uniqueness of solutions to the continuity equation induced by (b_t) in the class of probability measures a.c. with respect to $\mathfrak{m}$ and with bounded density. Thus we are in a position to proceed as in the proof of [13, Theorem 7.7] to obtain uniqueness of the RLF. □

Theorem 2.9. *Let $(X,\mathsf{d},\mathfrak{m})$ and $(b_t)_{t\in[0,T]}$ be as before. Then for any $t\in[0,T]$ we can find a representative of the RLF $\boldsymbol{X}_t$ satisfying the Lipschitz estimate*

$$\mathsf{d}(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y))\le e^{Lt}\mathsf{d}(x,y)$$

for any $x,y\in X$.

Proof. As for the proof of Theorem 2.8 above we do not give all the details. We just say here that the Lipschitz estimate for trajectories (which can be thought as solutions to the continuity equation starting from Dirac deltas) follows from Theorem 2.7 from an approximation procedure whose details can be found for instance in the proof of [32, Theorem 3.14] or in [40]. □

3. Regularity in the Sobolev case

In this section we prove a regularity property of regular Lagrangian flows associated to Sobolev vector fields in the context of (compact) Ahlfors regular $\mathrm{RCD}^*(K,N)$ metric measure spaces (see Definition 1.3). In order to better present the result and the main ideas of the proof we begin from the Euclidean setting, that is our starting point (even though non-compactness requires some modification with respect to the strategy that we will adopt in the core of this section).

In $(\mathbb{R}^d,|\cdot|,\mathscr{L}^d)$ the theory was developed by Crippa and De Lellis in [18] (implementing some ideas that were already present in [9]), the main regularity result proved therein is the following one.

Theorem 3.1. *Let $\boldsymbol{X}_t$ be a Regular Lagrangian flow associated to a time-dependent vector field $b_t\in L^1((0,T);W^{1,p}(\mathbb{R}^d;\mathbb{R}^d))\cap L^\infty((0,T);L^\infty(\mathbb{R}^n;\mathbb{R}^n))$ with $p>1$, and fix $R>0$. For every $\varepsilon>0$ there exists a compact set $K\subset B(0,R)$ such that $\mathscr{L}^d(B(0,R)\setminus K)<\varepsilon$ and*

$$\mathrm{Lip}(\boldsymbol{X}_t|_K)\le\exp\left\{\frac{C\left(1+\int_0^T\|\nabla b_t\|_{L^p(B(0,\tilde R))}\,\mathrm{d}t\right)}{\varepsilon^{\frac{1}{p}}}\right\}$$

for any $t\in[0,T]$, where $\tilde R:=R+T\|b\|_{L^\infty}$ and C depends only on d, R, p and L.

The technique adopted in [18] is based on a priori estimates of the functionals

$$Q_{t,r}(x):=\fint_{B(x,r)}\log\left(1+\left(\frac{|\boldsymbol{X}_t(x)-\boldsymbol{X}_t(y)|}{r}\right)\right)\mathrm{d}y,$$

which represent a sort of non-convex, discrete Cheeger energies associated to $\boldsymbol{X}$.

Any L^p bound of the function $x\mapsto\sup_{r>0}Q_{t,r}(x)$, depending only on the Sobolev norm of b, can be seen to imply a Lusin-type regularity property for $\boldsymbol{X}_t$ similar to the one in Theorem 3.1.

In order to find bounds for $Q_{t,r}$ one starts differentiating with respect to the time variable:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Q_{t,r}(x)&=\fint_{B(x,r)}\frac{\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{X}_t(x)-\boldsymbol{X}_t(y)|}{r+|\boldsymbol{X}_t(x)-\boldsymbol{X}_t(y)|}\,\mathrm{d}y\\&\le\fint_{B(x,r)}\frac{|b_t(\boldsymbol{X}_t(x))-b_t(\boldsymbol{X}_t(y))|}{r+|\boldsymbol{X}_t(x)-\boldsymbol{X}_t(y)|}\,\mathrm{d}y.\end{aligned}$$

To go on, it suffices to recall the maximal estimate

$$\frac{|b_t(\boldsymbol{X}_t(x)) - b_t(\boldsymbol{X}_t(y))|}{|\boldsymbol{X}_t(x) - \boldsymbol{X}_t(y)|} \le C(M|\nabla b|(\boldsymbol{X}_t(x)) + M|\nabla b|(\boldsymbol{X}_t(y))). \tag{3.1}$$

Now, using the assumption that $\boldsymbol{X}$ has bounded compression and the L^p integrability of the maximal operator for $p > 1$, it is simple to find an L^p bound of $x \mapsto \sup_{r>0} Q_{t,r}(x)$ depending only on the Sobolev norm of b.

Inequality (3.1) in the Euclidean case follows applying the well-known Lusin-approximation property of scalar Sobolev functions with Lipschitz functions to the components of the vector field. Even though the scalar Lusin-approximation property is a very robust result (it holds true in every doubling metric measure space [3] and in a rich class of non-doubling spaces [2]), it is non-trivial to extend a similar property to vector fields out of the Euclidean setting.

We are now ready to state the main result of this section. We refer to Theorem 3.11 below for a more quantitative version of this statement.

Theorem 3.2. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a compact* $\mathrm{RCD}^*(K,N)$ *m.m.s. and assume that it is n-Ahlfors regular for some* $1 < n \le N$ *(see Definition* 1.3*). Let* $(b_t)_{t\in[0,T]}$ *be a bounded time-dependent vector field with* $|\nabla_{\mathrm{sym}} b_t|$, $\operatorname{div} b_t \in L^1((0,T); L^2(X,\mathfrak{m}))$. *Let* $\boldsymbol{X}_t$ *be a Regular Lagrangian flow associated to* b_t *with compressibility constant* L. *Then for every* $\varepsilon > 0$ *there exists a Borel set* $E \subset X$ *such that* $\mathfrak{m}(X \setminus E) < \varepsilon$ *and for every* $x, y \in E$,

$$\mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)) \le C_{n,D} \exp\left\{ C_{n,T} \frac{L \int_0^t \| |\nabla_{\mathrm{sym}} b_s| + |\operatorname{div} b_s| \|_{L^2} \, \mathrm{d}s + 1}{\sqrt{\varepsilon}} \right\} \mathsf{d}(x,y)$$

for every $t \in [0,T]$.

The Ahlfors regularity property, crucial in our proof, is a non-trivial assumption. However, the class of spaces we are able to treat is not poor, since it includes, for instance, Alexandrov spaces and non-collapsed $\mathrm{RCD}^*(K,N)$ metric measure spaces (see [19]).

We conclude this preliminary discussion describing the main ideas in the proof of our result. Trying to perform the Crippa–De Lellis scheme the biggest difficulty to overcome comes from the study of the quantity

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)). \tag{3.2}$$

Indeed, in the metric context, setting $\mathsf{d}_x(y) := \mathsf{d}(x,y)$, it is not clear up to now how to obtain a useful estimate of the quantity

$$b \cdot \nabla \mathsf{d}_x(y) + b \cdot \nabla \mathsf{d}_y(x) \tag{3.3}$$

in terms of the covariant derivative of the vector field b (a part from the case of bounded symmetric derivative which, however, seems to be too specific for the applications).

Our strategy instead consists in considering a suitable power of the Green function $G(x,y) = G_x(y)$ instead of the distance function in (3.2) (we have been inspired by the survey [17]).

It is a well-known fact that on certain classes of Riemannian manifolds the Green function is equivalent to a negative power of the distance function; we extend this result to Ahlfors regular $\mathrm{RCD}^*(K,N)$ spaces.

Instead of (3.3), we need now to estimate

$$b \cdot \nabla G_x(y) + b \cdot \nabla G_y(x);$$

assuming for simplicity that $\operatorname{div} b = 0$ and thanks to the fundamental property of the Green's function

$$-\Delta G_x = \delta_x$$

(actually $-\Delta G_x = \delta_x - \mathfrak{m}$ in the case of compact manifolds), we formally compute

$$\begin{aligned} b \cdot \nabla G_x(y) + b \cdot \nabla G_y(x) &= \int_X [b(z) \cdot \nabla G_x(z) \Delta G_y(z) + b(z) \cdot \nabla G_y(z) \Delta G_x(z)] \,\mathrm{d}\mathfrak{m}(z) \\ &= -2 \int_X \nabla_{\mathrm{sym}} b(\nabla G_x, \nabla G_y) \,\mathrm{d}\mathfrak{m}, \end{aligned}$$

that, with a little bit of work, provides a maximal estimate that plays the same role of (3.1) in the Euclidean setting.

Throughout this section we assume $(X, \mathsf{d}, \mathfrak{m})$ to be an $\mathrm{RCD}^*(K, N)$ m.m.s., where $\mathfrak{m}$ is an n-Ahlfors regular probability measure for some $1 < n \leq N$.

The main technical ingredients are developed in Section 3.1, where we prove that, for the class of spaces we are interested in, the Green function of the Laplace operator is comparable with a negative power of the distance function (extending a well-known result of Riemannian geometry, see [14]). In Section 3.2 we turn the assumption on the Sobolev regularity of a vector field into a pointwise information obtaining a crucial maximal-type estimate. Throughout these two subsections we make the additional assumption that $n > 2$, needed for technical reasons related to the different behaviour of the Green function in dimension two. Finally, in Section 3.4 we propose a short argument to extend the main result to the missing case $n = 2$. We remark that, due to the results of [36], the Ahlfors regularity assumption forces n to be an integer between 1 and N. Therefore the only remaining case would be that of $n = 1$, that can be considered by iterating twice the procedure described in Section 3.4.

In order to let the notation be shorter we adopt the following convention: every positive constant that depends only on the "structural" coefficients of the space, i.e. K, N, n, D, $(\lambda_i)_{i \in \mathbb{N}}$, C_1, C_2, C_3, c_1, c_2 and on universal numerical constants, will be denoted by C.

3.1. The Green function. Let us introduce now a key object for the rest of this note, namely the Green function

$$G(x, y) := \int_0^\infty (p_t(x, y) - 1) \,\mathrm{d}t \quad \text{for all } x, y \in X. \tag{3.4}$$

In Proposition 3.3 below we prove that G is well defined and we collect some important properties, extending to the case of our interest some known estimates in Riemannian geometry (see [14] and [31]).

Recall that we are assuming $n > 2$.

Proposition 3.3. *The Green function G in* (3.4) *is well defined and finite for every $x \neq y \in X$. For every $f \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ it holds*

$$\int_X G(x, y) \Delta f(y) \,\mathrm{d}\mathfrak{m}(y) = \int_X f \,\mathrm{d}\mathfrak{m} - f(x) \tag{3.5}$$

for every $x \in X$. Moreover, G is equivalent to the function $\mathsf{d}(x,y)^{-n+2}$ up to a constant, i.e. there exist $A \geq 1$ and $\bar{A} > 0$, depending only on $(X, \mathsf{d}, \mathfrak{m})$, such that

$$|G(x,y)| \leq \frac{A}{\mathsf{d}(x,y)^{n-2}} \quad \text{for all } x, y \in X \tag{3.6}$$

and

$$G(x,y) \geq \frac{1}{A\mathsf{d}(x,y)^{n-2}} - \bar{A} \quad \text{for all } x, y \in X. \tag{3.7}$$

Finally, setting $G_x(y) := G(x,y)$, there exists $C > 0$ such that

$$\operatorname{lip} G_x(y) \leq \frac{C}{\mathsf{d}(x,y)^{n-1}} \quad \text{for all } x \neq y \in X, \tag{3.8}$$

in particular $G_x, \operatorname{lip} G_x \in L^p(X, \mathfrak{m})$ for every $p \in [1, \frac{n}{n-1})$.

Proof. Let us prove that the integral $\int_0^\infty (p_t(x,y) - 1)\,\mathrm{d}t$ is absolutely convergent. We assume for simplicity that the diameter D of the space is equal to 1. Let us fix $x \neq y \in X$, using estimates (1.3) for the heat kernel, identity (1.2), the Ahlfors regularity (1.8) and (1.9), we have

$$\begin{aligned}
\int_0^\infty |p_t(x,y) - 1|\,\mathrm{d}t &= \int_0^1 |p_t(x,y) - 1|\,\mathrm{d}t + \int_1^\infty |p_t(x,y) - 1|\,\mathrm{d}t \\
&\leq 1 + \int_0^1 \frac{C_1}{\mathfrak{m}(B(x,\sqrt{t}))} e^{-\frac{\mathsf{d}^2(x,y)}{5t} + C_3 t}\,\mathrm{d}t \\
&\quad + \int_1^\infty \sum_{i \geq 1} e^{-\lambda_i t} |u_i(x)||u_i(y)|\,\mathrm{d}t \\
&\leq 1 + \int_0^1 \frac{C}{t^{\frac{n}{2}}} e^{\frac{-\mathsf{d}^2(x,y)}{5t}}\,\mathrm{d}t + \sum_{i \geq 1} \frac{Ce^{-\lambda_i}}{\lambda_i} (C_3 + \lambda_i)^n.
\end{aligned}$$

Observe now that the series $\sum_{i \geq 1} \frac{Ce^{-\lambda_i}}{\lambda_i}(C_3 + \lambda_i)^n$ is convergent (since the eigenvalues have more than linear growth) and that

$$\int_0^1 \frac{C}{t^{\frac{n}{2}}} e^{-\frac{\mathsf{d}^2(x,y)}{5t}}\,\mathrm{d}t \leq \frac{1}{\mathsf{d}(x,y)^{n-2}} \int_0^\infty \frac{e^{-\frac{1}{5t}}}{t^{\frac{n}{2}}}\,\mathrm{d}t \leq \frac{C}{\mathsf{d}(x,y)^{n-2}},$$

where in the last inequality the assumption that $n > 2$ enters into play. All in all we have

$$\left| \int_0^\infty (p_t(x,y) - 1)\,\mathrm{d}t \right| \leq \frac{C}{\mathsf{d}(x,y)^{n-2}} + C \leq \frac{C}{\mathsf{d}(x,y)^{n-2}},$$

that provides the good definition of G and (3.6).

In order to prove (3.7) observe that

$$\int_0^\infty (p_t(x,y) - 1)\,\mathrm{d}t = \int_0^1 (p_t(x,y) - 1)\,\mathrm{d}t + \int_1^\infty \sum_{i \geq 1} e^{-\lambda_i t} u_i(x) u_i(y)\,\mathrm{d}t.$$

Using again (1.3) and (1.8), we obtain

$$\int_0^1 (p_t(x,y) - 1)\,\mathrm{d}t \geq \frac{C}{\mathsf{d}(x,y)^{n-2}} - 1.$$

Recalling that

$$\left|\int_1^\infty \sum_{i\geq 1} e^{-\lambda_i t} u_i(x)u_i(y)\,\mathrm{d}t\right| \leq \sum_{i\geq 1}\frac{Ce^{-\lambda_i}}{\lambda_i}(C_3+\lambda_i)^n < \infty$$

we conclude the proof of (3.7).

Let us estimate now the slope of $G_x(\cdot)$ at $y\in X$, $y\neq x$. Let us fix a parameter ε with $0<\varepsilon<\mathsf{d}(x,y)$, and a point $z\in B(y,\frac{\varepsilon}{2})$; observe that $\mathsf{d}(x,z)>\frac{\varepsilon}{2}$. We wish to estimate the incremental ratio

$$\begin{aligned}\frac{|G_x(z)-G_x(y)|}{\mathsf{d}(z,y)} \leq & \int_0^1 \frac{|p_t(x,z)-p_t(x,y)|}{\mathsf{d}(z,y)}\,\mathrm{d}t \\ & + \int_1^\infty \sum_{i\geq 1} e^{-\lambda_i t}|u_i(x)|\frac{|u_i(z)-u_i(y)|}{\mathsf{d}(y,z)}\,\mathrm{d}t.\end{aligned}$$

Calling I and II the first and the second term appearing at the right-hand side in the above written equation, in order to estimate I we observe that the slope of $p_t(x,\cdot)$ is bounded in $X\setminus B(x,\frac{\varepsilon}{2})$ uniformly in time and that a geodesic from y to z does not intersect $B(x,\frac{\varepsilon}{2})$. Thus, using the fact that the slope of a Lipschitz function is an upper gradient we obtain that the family $\frac{|p_t(x,z)-p_t(x,y)|}{\mathsf{d}(z,y)}$ is uniformly bounded when $z\in B(y,\frac{\varepsilon}{2})$. By Fatou's lemma and (1.4) we obtain

$$\begin{aligned}\limsup_{z\to y}\int_0^1 \frac{|p_t(x,z)-p_t(x,y)|}{\mathsf{d}(z,y)}\,\mathrm{d}t &\leq \int_0^1 |\nabla p_t(x,\cdot)|(y)\,\mathrm{d}t \\ &\leq \int_0^1 \frac{C}{\mathfrak{m}(B(x,\sqrt{t}))\sqrt{t}} e^{-\frac{\mathsf{d}^2(x,y)}{5t}}\,\mathrm{d}t \\ &\leq \frac{C}{\mathsf{d}(x,y)^{n-1}}.\end{aligned}$$

The estimate of II is simple. Indeed, from (1.9) we obtain

$$\begin{aligned}&\limsup_{z\to y}\int_1^\infty \sum_{i\geq 1} e^{-\lambda_i t}|u_i(x)|\frac{|u_i(z)-u_i(y)|}{\mathsf{d}(y,z)}\,\mathrm{d}t \\ &\leq \int_1^\infty \sum_{i\geq 1} e^{-\lambda_i t} C(C_3+\lambda_i)^n(|K|+\lambda_i)^{\frac{1}{2}}\,\mathrm{d}t\end{aligned}$$

and the last term is finite under our assumptions. Putting all together, we conclude

$$\operatorname{lip} G_x(y) \leq \frac{C}{\mathsf{d}(x,y)^{n-1}} + C \leq \frac{C}{\mathsf{d}(x,y)^{n-1}}.$$

By Remark 1.4 it easily follows that $G_x, \operatorname{lip} G_x \in L^p(X,\mathfrak{m})$ for every $p\in[1,\frac{n}{n-1})$.

Finally, we prove (3.5). Let us fix $f\in \mathrm{Test}(X,\mathsf{d},\mathfrak{m})$. We first observe that

$$\int_X G(\cdot,y)\Delta f(y)\,\mathrm{d}\mathfrak{m}(y) \in L^\infty(X,\mathfrak{m}), \tag{3.9}$$

as a consequence of $\Delta f\in L^\infty(X,\mathfrak{m})$ and Remark 1.4. Fix any $\varphi\in L^2(X,\mathfrak{m})$, applying Fubini's theorem, we get

$$\begin{aligned}
&\int_X \varphi(x) \int_X G(x,y)\Delta f(y)\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}\mathfrak{m}(x)\\
&\quad = n\int_0^\infty \int_X \varphi(x)\int_X (p_t(x,y)-1)\Delta f(y)\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}t\\
&\quad = n\int_0^\infty \int_X \varphi(x) P_t\Delta f(x)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}t\\
&\quad = n\int_0^\infty \frac{d}{dt}\int_X P_t f(x)\varphi(x)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}t\\
&\quad = n\int_X \Big(\int_X f\,\mathrm{d}\mathfrak{m} - f(x)\Big)\varphi(x)\,\mathrm{d}\mathfrak{m}(x),
\end{aligned}$$

where all the integrals are well defined thanks to (3.9). □

Let us introduce a "regularized" version G^ε, $\varepsilon > 0$, of G setting

$$G^\varepsilon(x,y) := \int_\varepsilon^\infty (p_t(x,y)-1)\,\mathrm{d}t \quad \text{for all } x,y\in X.$$

We will often write $G_x^\varepsilon(y) = G^\varepsilon(x,y)$. Observe that G^ε is well defined and finite for every $x, y \in X$. Estimates (3.6) and (3.8) still hold true for G^ε, namely

$$|G^\varepsilon(x,y)| \le \frac{C}{\mathsf{d}(x,y)^{n-2}}, \quad \operatorname{lip} G_x^\varepsilon(y) \le \frac{C}{\mathsf{d}(x,y)^{n-1}} \tag{3.10}$$

for every $x, y \in X$, and they can be proved with the same strategy described above. In Lemma 3.4 below we state an important regularity property of G_x^ε.

Lemma 3.4. *For every $x \in X$ it holds that $G_x^\varepsilon \in \mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ and*

$$\Delta G_x^\varepsilon(y) = 1 - p_\varepsilon(x,y) \tag{3.11}$$

for $\mathfrak{m}$-a.e. $y \in X$.

Proof. Arguing as in the proof of (3.6) we easily obtain $G_x^\varepsilon \in L^\infty(X,\mathfrak{m})$ and, with a simple application of the Fubini–Tonelli theorem, we get

$$G_x^\varepsilon = P_{\frac{\varepsilon}{2}} G_x^{\frac{\varepsilon}{2}}.$$

Taking into account the regularizing properties of the heat flow that we remarked after (1.7), we obtain $G_x^\varepsilon \in \mathrm{Test}(X,\mathsf{d},\mathfrak{m})$.

Identity (3.11) follows arguing as in the proof of (3.5). □

Finally, we observe that, for every $x \in X$, the family of functions $(G^\varepsilon(x))_{\varepsilon>0}$ is equibounded in $W^{1,p}(X,\mathsf{d},\mathfrak{m})$ for some $p > 1$, and strongly convergent as $\varepsilon \to 0$ to G_x, details can be found in Lemma 3.5 below.

Lemma 3.5. *For every $x \in X$ and for every $p \in [1, \frac{n}{n-1})$ it holds that G_x and G_x^ε belong to $W^{1,p}(X,\mathsf{d},\mathfrak{m})$ and*

$$\lim_{\varepsilon\to 0} G_x^\varepsilon = G_x \quad \textit{strongly in } W^{1,p}.$$

Proof. From (3.6), (3.10) and Remark 1.4 it immediately follows that G_x and G_x^ε belong to $L^p(X, \mathfrak{m})$ for every $p \in [1, \frac{n}{n-2})$, with a bound on L^p norms independent of ε. Moreover, G_x^ε is a Lipschitz function, thus

$$|\nabla G_x^\varepsilon| \le \operatorname{lip} G_x^\varepsilon \le \frac{C}{\mathsf{d}(x,y)^{n-1}}, \quad \mathfrak{m}\text{-a.e. on } X.$$

Using Remark 1.4 again, we conclude that $\sup_{\varepsilon>0} \|G_x^\varepsilon\|_{W^{1,p}} < \infty$ for all $p \in [1, \frac{n}{n-1})$. It is simple to check that $G_x^\varepsilon(y) \to G_x(y)$ for any $y \neq x$, when $\varepsilon \to 0$ and by (3.10) and the dominated convergence theorem, we get

$$G_x^\varepsilon \to G_x \quad \text{in } L^p \quad \text{for all } p \in [1, \tfrac{n}{n-2}). \tag{3.12}$$

Let us fix $p \in [1, \frac{n}{n-1})$. It is now obvious that $G_x \in W^{1,p}(X, \mathsf{d}, \mathfrak{m})$, since by the reflexivity of $W^{1,p}$ for $p > 1$ (see [3]) and (3.12) we deduce that $(G_x^\varepsilon)_{\varepsilon>0}$ weakly converges to G_x. It remains to prove that the convergence of $G_x^\varepsilon \to G_x$ is actually strong in $W^{1,p}$. To this end, it is enough to show

$$\lim_{\varepsilon\to 0} \int_X |\nabla G_x^\varepsilon|^p \,\mathrm{d}\mathfrak{m} = \int_X |\nabla G_x|^p \,\mathrm{d}\mathfrak{m},$$

since $G_x^\varepsilon \to G_x$ weakly in $W^{1,p}$ and the space $W^{1,p}(X, \mathsf{d}, \mathfrak{m})$ is equivalent to a uniformly convex space (see [3, Theorem 7.4]). Using a p-version of (1.5) (see [25, Proposition 3.1]) and the identity $P_\varepsilon G_x = G_x^\varepsilon$ (where the action of the semigroup is understood in $L^p(X, \mathfrak{m})$), we have that

$$\int_X |\nabla G_x^\varepsilon|^p(y) \,\mathrm{d}\mathfrak{m}(y) \le e^{-pK\varepsilon} \int_X |\nabla G_x|^p(y) \,\mathrm{d}\mathfrak{m}(y).$$

Therefore

$$\limsup_{\varepsilon\to 0} \int_X |\nabla G_x^\varepsilon|^p(y) \,\mathrm{d}\mathfrak{m}(y) \le \int_X |\nabla G_x|^p(y) \,\mathrm{d}\mathfrak{m}(y).$$

Using the lower semicontinuity of the Sobolev norm with respect to the weak topology, we conclude the proof. □

3.2. Maximal estimates for vector fields. In this subsection we state and prove a maximal estimate which turns out to be crucial in the sequel.

Proposition 3.6. *There exists a positive constant $C = C(X, \mathsf{d}, \mathfrak{m})$ such that, for every $b \in L^2(TX)$ with $\operatorname{div} b, |\nabla_{\mathrm{sym}} b| \in L^2(X, \mathfrak{m})$, it holds*

$$\begin{aligned} |b \cdot \nabla G_x(y) + b \cdot \nabla G_y(x)| \le \frac{C}{\mathsf{d}(x,y)^{n-2}} &[M(|\nabla_{\mathrm{sym}} b| + |\operatorname{div} b|)(x) \\ &+ M(|\nabla_{\mathrm{sym}} b| + |\operatorname{div} b|)(y)] \end{aligned}$$

for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x, y) \in X \times X$.

In some sense the result of Proposition 3.6 could be seen as a quantitative Lusin-type approximation property for the vector-valued case. Indeed it plays in our proof the role played by (3.1) in the original Crippa–De Lellis' scheme (see [18]).

The notion of symmetric covariant derivative we are adopting in Proposition 3.6, is the following one.

Definition 3.7. Take any $b \in L^2(TX)$ with $\operatorname{div} b \in L^2$. We say that b has symmetric derivative in L^2 if there exists a non-negative function $G \in L^2(X, \mathfrak{m})$ such that

$$\frac{1}{2}\left| \int_X \{b \cdot \nabla g \Delta f + b \cdot \nabla f \Delta g - \operatorname{div} b (\nabla f \cdot \nabla g)\} \, \mathrm{d}\mathfrak{m} \right| \leq \int_X G |\nabla f| |\nabla g| \, \mathrm{d}\mathfrak{m}$$

for every $f, g \in \mathrm{Test}(X)$. We call $|\nabla_{\mathrm{sym}} b|$ the $G \in L^2(X, \mathfrak{m})$ with minimal norm.[2)]

This definition appears as intermediate between the notion adopted by Ambrosio and Trevisan in [12] (see Definition 1.10) and the one proposed by Gigli in [23] (see Definition 2.3). Indeed, it follows from the very definitions that if a vector field admits a symmetric covariant derivative in L^2 according to Definition 3.7, then it also admits a covariant derivative in L^2 according to Ambrosio-Trevisan. On the other hand if a vector field belongs to $W^{1,2}_{C,s}$ (see Definition 2.3), then it has symmetric covariant derivative in L^2 according to Definition 3.7. We chose to work with this intermediate notion of symmetric derivative since it is the assumption we really need for our purposes.

We start with a technical lemma.

Lemma 3.8. *There exists a positive constant $C = C(X, \mathsf{d}, \mathfrak{m})$ such that for every non-negative function $f \in L^1(X, \mathfrak{m})$ it holds*

$$\int_X f(z) \frac{1}{\mathsf{d}(x,z)^{n-1}} \frac{1}{\mathsf{d}(y,z)^{n-1}} \, \mathrm{d}\mathfrak{m}(z) \leq \frac{C}{\mathsf{d}(x,y)^{n-2}} (Mf(x) + Mf(y))$$

for all $x, y \in X$.

Proof. Set $r := \frac{1}{2}\mathsf{d}(x,y)$. We split the integral

$$\int_X f(z) \frac{1}{\mathsf{d}(x,z)^{n-1}} \frac{1}{\mathsf{d}(y,z)^{n-1}} \, \mathrm{d}\mathfrak{m}(z) = I + II + III,$$

where

$$\begin{aligned}
I &:= \int_{B(x,r)} f(z) \frac{1}{\mathsf{d}(x,z)^{n-1}} \frac{1}{\mathsf{d}(y,z)^{n-1}} \, \mathrm{d}\mathfrak{m}(z), \\
II &:= \int_{B(y,r)} f(z) \frac{1}{\mathsf{d}(x,z)^{n-1}} \frac{1}{\mathsf{d}(y,z)^{n-1}} \, \mathrm{d}\mathfrak{m}(z), \\
III &:= \int_{\{\mathsf{d}(x,z) \geq r,\ \mathsf{d}(y,z) \geq r\}} f(z) \frac{1}{\mathsf{d}(x,z)^{n-1}} \frac{1}{\mathsf{d}(y,z)^{n-1}} \, \mathrm{d}\mathfrak{m}(z).
\end{aligned}$$

In order to estimate I, we observe that

$$\mathsf{d}(y,z) \geq \mathsf{d}(y,x) - \mathsf{d}(x,z) \geq \frac{1}{2}\mathsf{d}(x,y)$$

for all $z \in B(x,r)$, thus

$$\int_{B(x,r)} f(z) \frac{1}{\mathsf{d}(x,z)^{n-1}} \frac{1}{\mathsf{d}(y,z)^{n-1}} \, \mathrm{d}\mathfrak{m}(z) \leq \frac{2^{n-1}}{\mathsf{d}(x,y)^{n-1}} \int_{B(x,r)} f(z) \frac{1}{\mathsf{d}(x,z)^{n-1}} \, \mathrm{d}\mathfrak{m}(z).$$

2) Up to now we do not know if $|\nabla_{\mathrm{sym}} b|$ has to be the minimal object also in the pointwise $\mathfrak{m}$-a.e. sense.

Arguing exactly as in the proof of (3.9), we find

$$\int_{B(x,r)} f(z)\frac{1}{\mathsf{d}(x,z)^{n-1}}\,\mathrm{d}\mathfrak{m}(z) \le C\mathsf{d}(x,y)Mf(x),$$

and we conclude

$$I \le \frac{C}{\mathsf{d}(x,y)^{n-2}}Mf(x). \tag{3.13}$$

The estimate

$$II \le \frac{C}{\mathsf{d}(x,y)^{n-2}}Mf(y) \tag{3.14}$$

follows by the same reasoning.

We are left with the estimate of III. By Young's inequality we have

$$III \le \frac{1}{2}\int_{\{\mathsf{d}(x,z)\ge r\}} f(z)\frac{1}{\mathsf{d}(x,z)^{2n-2}}\,\mathrm{d}\mathfrak{m}(z) + \frac{1}{2}\int_{\{\mathsf{d}(y,z)\ge r\}} f(z)\frac{1}{\mathsf{d}(y,z)^{2n-2}}\,\mathrm{d}\mathfrak{m}(z).$$

Using the Ahlfors regularity (1.8), for every $w\in X$, we obtain

$$\begin{aligned}
&\int_{\mathsf{d}(w,z)\ge r} f(z)\frac{1}{\mathsf{d}(w,z)^{2n-2}}\,\mathrm{d}\mathfrak{m}(z)\\
&= \sum_{k=0}^{\log_2(\frac{D}{2r})}\int_{B(w,r2^{k+1})\setminus B(w,r2^k)} \frac{f(z)}{\mathsf{d}(w,z)^{2n-2}}\,\mathrm{d}\mathfrak{m}(z)\\
&\le \sum_{k=0}^{\log_2(\frac{D}{2r})}\frac{\mathfrak{m}(B(w,r2^{k+1}))}{(2^kr)^{2n-2}}\frac{1}{\mathfrak{m}(B(w,r2^{k+1}))}\int_{B(w,r2^{k+1})} f(z)\,\mathrm{d}\mathfrak{m}(z)\\
&\le C\sum_{k=0}^{\log_2(\frac{D}{2r})}\frac{(r2^{k+1})^n}{(2^kr)^{2n-2}}Mf(w)\\
&\le \frac{C}{\mathsf{d}(w,y)^{n-2}}Mf(w)\sum_{k=0}^{\log_2(\frac{D}{2r})}2^{-k(n-2)}.
\end{aligned}$$

Putting this last estimate, applied with $w=x$ and $w=y$, together with (3.13) and (3.14), we obtain the desired conclusion. □

Proof of Proposition 3.6. First of all we remark that $|b\cdot\nabla G_x(y)+b\cdot\nabla G_y(x)|$ is well defined $\mathfrak{m}\times\mathfrak{m}$-a.e., since b is a bounded vector field and $G_x, G_y\in W^{1,p}$ for some $p>1$. As a first step we prove the following

Claim. *For every $\varepsilon>0$ it holds that*

$$\begin{aligned}
|P_\varepsilon(b\cdot\nabla G_x^\varepsilon)(y)+P_\varepsilon(b\cdot\nabla G_y^\varepsilon)(x)| \le \frac{C}{\mathsf{d}(x,y)^{n-2}}&[M(|\nabla_{\mathrm{sym}}b|+|\mathrm{div}\, b|)(x)\\
&+M(|\nabla_{\mathrm{sym}}b|+|\mathrm{div}\, b|)(y)]
\end{aligned}$$

for every $x,y\in X$.

Recalling the result of Lemma 3.4, we have

$$
\begin{aligned}
&|P_\varepsilon(b\cdot\nabla G_x^\varepsilon)(y)+P_\varepsilon(b\cdot\nabla G_y^\varepsilon)(x)| \\
&\quad = n\left|\int_X b\cdot\nabla G_x^\varepsilon(z)\,p_\varepsilon(y,z)\,\mathrm{d}\mathfrak{m}(z)+\int_X b\cdot\nabla G_y^\varepsilon(z)\,p_\varepsilon(x,z)\,\mathrm{d}\mathfrak{m}(z)\right| \\
&\quad = n\left|-\int_X [b\cdot\nabla G_x^\varepsilon\Delta G_y^\varepsilon+b\cdot\nabla G_y^\varepsilon\Delta G_x^\varepsilon+\operatorname{div} b(G_x^\varepsilon+G_y^\varepsilon)]\,\mathrm{d}\mathfrak{m}(z)\right| \\
&\quad \le \left|-\int_X [b\cdot\nabla G_x^\varepsilon\Delta G_y^\varepsilon+b\cdot\nabla G_y^\varepsilon\Delta G_x^\varepsilon-\operatorname{div} b(\nabla G_x^\varepsilon\cdot\nabla G_y^\varepsilon)]\,\mathrm{d}\mathfrak{m}(z)\right| \\
&\qquad\qquad +\left|\int_X \operatorname{div} b(G_x^\varepsilon+G_y^\varepsilon+\nabla G_x^\varepsilon\cdot\nabla G_y^\varepsilon)\,\mathrm{d}\mathfrak{m}(z)\right| \\
&\quad = n2\left|\int_X \nabla_{\mathrm{sym}} b(\nabla G_x^\varepsilon,\nabla G_y^\varepsilon)\,\mathrm{d}\mathfrak{m}(z)\right| \\
&\qquad\qquad +\left|\int_X \operatorname{div} b(G_x^\varepsilon+G_y^\varepsilon+\nabla G_x^\varepsilon\cdot\nabla G_y^\varepsilon)\,\mathrm{d}\mathfrak{m}(z)\right|.
\end{aligned}
$$

Now using (3.6) and (3.8), we get

$$
\begin{aligned}
&|P_\varepsilon(b\cdot\nabla G_x^\varepsilon)(y)+P_\varepsilon(b\cdot\nabla G_y^\varepsilon)(x)| \\
&\quad \le C\left|\int_X (|\nabla_{\mathrm{sym}} b|(z)+|\operatorname{div} b(z)|)\frac{1}{\mathsf{d}(x,z)^{n-1}}\frac{1}{\mathsf{d}(y,z)^{n-1}}\,\mathrm{d}\mathfrak{m}(z)\right|,
\end{aligned}
$$

where we have implicitly exploited the inequality

$$
\frac{1}{\mathsf{d}(\cdot\,,z)^{n-2}}\le\frac{D}{\mathsf{d}(\cdot\,,z)^{n-1}}.
$$

Applying Lemma 3.8 with $f:=|\nabla_{\mathrm{sym}} b|+|\operatorname{div} b|$, we conclude the proof of the claim.

We want to prove now that

$$
P_\varepsilon(b\cdot\nabla G_x^\varepsilon)(y)+P_\varepsilon(b\cdot\nabla G_y^\varepsilon)(x)\to b\cdot\nabla G_x(y)+b\cdot\nabla G_y(x)
$$

strongly in $L^p(X\times X,\mathfrak{m}\times\mathfrak{m})$ when $\varepsilon\to 0$; this convergence result together with the uniform estimate we proved above will yield the desired conclusion (by considering a sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ such that $\varepsilon_i\to 0$ and the above considered convergence holds in the $\mathfrak{m}\times\mathfrak{m}$-a.e. sense).

Entering into the details, we are going to prove that

$$
\int_X\int_X |P_\varepsilon(b\cdot\nabla G_x^\varepsilon)(y)-b\cdot\nabla G_x(y)|^p\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\to 0\quad(\varepsilon\to 0).
$$

Recalling the L^p-norm contractivity property of the semigroup P_t, we have that for any fixed $x\in X$ it holds

$$
\begin{aligned}
\|P_\varepsilon(b\cdot\nabla G_x^\varepsilon)-b\cdot\nabla G_x\|_p &\le \|P_\varepsilon(b\cdot\nabla G_x^\varepsilon)-P_\varepsilon(b\cdot\nabla G_x)\|_p+\|P_\varepsilon(b\cdot\nabla G_x)-b\cdot\nabla G_x\|_p \\
&\le \||b|\|_\infty\|\nabla(G_x^\varepsilon-G_x)\|_p+\|P_\varepsilon(b\cdot\nabla G_x)-b\cdot\nabla G_x\|_p.
\end{aligned}
$$

The last two terms go to zero when $\varepsilon\to 0$, moreover they are uniformly bounded in x, thus

$$
\begin{aligned}
&\int_X\int_X |P_\varepsilon(b\cdot\nabla G_x^\varepsilon)(y)-b\cdot\nabla G_x(y)|^p\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}\mathfrak{m}(x) \\
&\quad\le\int_X \|b\|_{L^\infty}\|\nabla(G_x^\varepsilon-G_x)\|_p\,\mathrm{d}\mathfrak{m}(x)+\int_X\|P_\varepsilon(b\cdot\nabla G_x)-b\cdot\nabla G_x\|_p\,\mathrm{d}\mathfrak{m}(x)
\end{aligned}
$$

goes to zero by the dominated convergence theorem. □

3.3. A Lusin-type regularity result. Throughout this subsection the time-dependent vector field b_t and the Regular Lagrangian flow $\boldsymbol{X}_t$ associated to b_t, with compressibility constant L, are fixed. Our aim is to implement a strategy very similar to the one developed in [18] in order to prove our main regularity result Theorem 3.2.

We begin by observing that the results of Proposition 3.3 ensure that, possibly increasing the constant A and setting $\bar{G}(x, y) := G(x, y) + \bar{A}$, $\bar{G}_x(y) := \bar{G}(x, y)$, we have

$$\frac{1}{A\mathsf{d}(x, y)^{n-2}} \leq \bar{G}(x, y) \leq \frac{A}{\mathsf{d}(x, y)^{n-2}} \tag{3.15}$$

for any $x, y \in X$ such that $x \neq y$. It follows in particular that $\bar{G}(x, y) > \alpha > 0$ for any $x, y \in X$ such that $x \neq y$.

Observe that, in terms of the function $\bar{G}$, the statement of Proposition 3.6 can be rewritten as

$$|b \cdot \nabla \bar{G}_x(y) + b \cdot \nabla \bar{G}_y(x)| \leq C\bar{G}(x, y)[M(|\nabla_{\mathrm{sym}} b| + |\mathrm{div}\, b|)(x) + M(|\nabla_{\mathrm{sym}} b| + |\mathrm{div}\, b|)(y)] \tag{3.16}$$

for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x, y) \in X \times X$.

We introduce, for any $t \in [0, T]$ and for any $0 < r \leq D$, the functional

$$Q_{t,r}(x) := \fint_{B(x,r)} \log\left(1 + \frac{1}{A}\left(\frac{\mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y))}{r}\right)^{n-2}\right) \mathrm{d}\mathfrak{m}(y),$$

where A is the constant introduced in (3.15). Moreover, we set

$$Q^*(x) := \sup_{0 \leq t \leq T} \sup_{0 < r \leq D} Q_{t,r}(x). \tag{3.17}$$

With the aim of finding bounds on Q^*, we first state and prove a technical lemma.

Lemma 3.9. *Assume that b is a bounded vector field on $(0, T) \times X$. Then, for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x, y) \in X \times X$ the map $t \mapsto G(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y))$ belongs to $W^{1,1}((0, T))$ and its derivative is given by the formula*

$$\frac{d}{dt} G(\boldsymbol{X}_t(y), \boldsymbol{X}_t(x)) = b_t \cdot \nabla G_{\boldsymbol{X}_t(x)}(\boldsymbol{X}_t(y)) + b_t \cdot \nabla G_{\boldsymbol{X}_t(y)}(\boldsymbol{X}_t(x))$$

for a.e. $t \in (0, T)$.

Proof. From (1.2) and the definition of G^ε we find the pointwise identity

$$G^\varepsilon(x, y) = \sum_{i=1}^{\infty} \frac{e^{-\lambda_i \varepsilon}}{\lambda_i} u_i(x) u_i(y)$$

for every $x, y \in X$. Setting

$$G^{\varepsilon,N}(x, y) := \sum_{i=1}^{N} \frac{e^{-\lambda_i \varepsilon}}{\lambda_i} u_i(x) u_i(y),$$

we have that, for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x, y) \in X \times X$, the map $t \mapsto G^{\varepsilon,N}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y))$ is absolutely continuous for every $N \in \mathbb{N}$. Setting analogously to the previous definition

$$G_x^{\varepsilon,N}(y) := G^{\varepsilon,N}(x, y),$$

for a.e. $t \in (0, T)$ it holds

$$\begin{aligned}(3.18)\quad \frac{d}{dt} G^{\varepsilon,N}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)) &= \sum_{i=1}^{N} \frac{e^{-\varepsilon\lambda_i}}{\lambda_i}(b_t \cdot \nabla u_i(\boldsymbol{X}_t(x)) u_i(\boldsymbol{X}_t(y)) + u_i(\boldsymbol{X}_t(x)) b_t \cdot \nabla u_i(\boldsymbol{X}_t(y))) \\ &= n b_t \cdot \nabla G^{\varepsilon,N}_{\boldsymbol{X}_t(x)}(\boldsymbol{X}_t(y)) + b_t \cdot \nabla G^{\varepsilon,N}_{\boldsymbol{X}_t(y)}(\boldsymbol{X}_t(x)),\end{aligned}$$

since $u_i \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ for any $i \in \mathbb{N}$. Our aim is to pass to the limit in (3.18), first letting $N \to \infty$ and then $\varepsilon \to 0$. Observe that, for every $\varepsilon > 0$, when $N \to \infty$ we have $G^{\varepsilon,N} \to G^{\varepsilon}$ in $W^{1,2}(X \times X)$ (moreover it holds $G^{\varepsilon,N}_x \to G^{\varepsilon}_x$ strongly in $W^{1,2}(X)$, uniformly in $x \in X$) and also, when $\varepsilon \to 0$, $G^{\varepsilon} \to G$ in $L^1(X \times X)$, and $G^{\varepsilon}_x \to G_x$ in $W^{1,p}(X)$ for every $p \in [1, \frac{n}{n-1})$, uniformly in $x \in X$ (see Lemma 3.5). Moreover, $G^{\varepsilon} \in \mathrm{Test}(X \times X)$ (it can be proved arguing as in Lemma 3.4).

With this said, in order to conclude the proof, it suffices to show the following technical result: let $(F^n(x, y))_{n \in \mathbb{N}}$ be a sequence of symmetric functions belonging to $\mathrm{Test}(X \times X)$, assume that F^n satisfies (3.18) for every $n \in \mathbb{N}$. If $F^n \to F$ in $L^1(X \times X)$ and there exists $p > 1$ such that, for every $x \in X$, $F^n(x, \cdot) := F^n_x(\cdot) \to F_x(\cdot)$ in $W^{1,p}(X)$, uniformly with respect to $x \in X$, then the function $t \mapsto F(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y))$ belongs to the space $W^{1,1}((0, T))$ for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x, y) \in X \times X$ and satisfies (3.18).

Let us fix $t \in [0, T]$, starting from the $\mathfrak{m} \times \mathfrak{m}$-a.e. equality

$$F^n(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)) - F^n(x, y) = \int_0^t \{b_s \cdot \nabla F^n_{\boldsymbol{X}_s(x)}(\boldsymbol{X}_s(y)) + b_s \cdot \nabla F^n_{\boldsymbol{X}_s(y)}(\boldsymbol{X}_s(x)))\}\,\mathrm{d}s,$$

we wish to pass to the limit for $n \to \infty$. We observe that the left-hand side converges to $F(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)) - F(x, y)$ in $L^1(X \times X)$ (here the compressibility property of $\boldsymbol{X}_t$ plays a role), it remains only to prove that the right-hand side converges to

$$\int_0^t \{b_s \cdot \nabla F_{\boldsymbol{X}_s(x)}(\boldsymbol{X}_s(y)) + b_s \cdot \nabla F_{\boldsymbol{X}_s(y)}(\boldsymbol{X}_s(x))\}\,\mathrm{d}s \quad \text{in } L^1(X \times X).$$

Using again the compressibility property of $\boldsymbol{X}_t$, we have

$$\begin{aligned}&\int_{X \times X} \left| \int_0^t b_s \cdot \nabla F^n_{\boldsymbol{X}_s(x)}(\boldsymbol{X}_s(y))\,\mathrm{d}s - \int_0^t b_s \cdot \nabla F_{\boldsymbol{X}_s(x)}(\boldsymbol{X}_s(y))\,\mathrm{d}s \right|^p \mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) \\ &\le t^{p-1} \|b\|_{L^\infty} \int_0^t \int_X \int_X |\nabla(F^n_{\boldsymbol{X}_s(x)} - F_{\boldsymbol{X}_s(x)})|^p(\boldsymbol{X}_s(y))\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}s \\ &\le L^2 t^p \|b\|_{L^\infty} \int_X \|\nabla(F^n_x - F_x)\|^p_{L^p}\,\mathrm{d}\mathfrak{m}(x),\end{aligned}$$

that, under our assumptions on the sequence F^n, goes to zero when $n \to \infty$. Arguing similarly for the term $\int_0^t b_s \cdot \nabla F^n_{\boldsymbol{X}_s(y)}(\boldsymbol{X}_s(x))\,\mathrm{d}s$, we conclude the proof. □

Theorem 3.10. *Let b be a time-dependent bounded vector field, assume that $|\nabla_{\mathrm{sym}} b|$ and $\operatorname{div} b$ belong to $L^1((0, T); L^2(X, \mathfrak{m}))$. Let $\boldsymbol{X}_t$ be a Regular Lagrangian flow associated to b, with compressibility constant L. Then, with the above introduced notation, we have*

$$\|Q^*\|_{L^2} \le C\left[L \int_0^T \||\nabla_{\mathrm{sym}} b_s| + |\operatorname{div} b_s|\|_{L^2}\,\mathrm{d}s + 1\right],$$

where $C = C(T, X, \mathsf{d}, \mathfrak{m})$.

Proof. As a first step we introduce the functional

$$\Phi_{t,r}(x) := \fint_{B(x,r)} \log\left(1 + \frac{1}{r^{n-2}\bar{G}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)))}\right) \mathrm{d}\mathfrak{m}(y)$$

for $r \in (0, D)$ and $t \in [0, T]$.

We observe that the monotonicity of the logarithm and our construction guarantee that $Q_{t,r} \le \Phi_{t,r}$ pointwise for any $t \in [0, T]$ and for any $0 < r \le D$.

What we just proved tells that it suffices to bound $\Phi_{t,r}$ (which in some sense is a more "regular" functional) in order to bound $Q_{t,r}$. To this end, we fix $r > 0$ and $t \in [0, T]$. By Lemma 3.9 we have that $t \to \Phi_{t,r}(x)$ belongs to $W^{1,1}((0, T))$ for $\mathfrak{m}$-a.e. $x \in X$ (actually it is absolutely continuous since it is continuous) and it holds

$$\begin{aligned}
\Phi_{t,r}(x) &= \Phi_{0,r}(x) + \int_0^t \frac{d}{ds}\Phi_{s,r}(x)\,\mathrm{d}s \\
&\le \Phi_{0,r}(x) + \int_0^t \fint_{B(x,r)} \frac{|\frac{d}{ds}\bar{G}(\boldsymbol{X}_s(x), \boldsymbol{X}_s(y))|}{\bar{G}(\boldsymbol{X}_s(x), \boldsymbol{X}_s(y))} \\
&\qquad\qquad \cdot \frac{1}{\bar{G}(\boldsymbol{X}_s(x), \boldsymbol{X}_s(y))r^{n-2} + 1}\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}s \\
&\le \Phi_{0,r}(x) + \int_0^t \fint_{B(x,r)} \frac{|b_s \cdot \nabla\bar{G}_{\boldsymbol{X}_s(x)}(\boldsymbol{X}_s(y)) + b_s \cdot \nabla\bar{G}_{\boldsymbol{X}_s(y)}(\boldsymbol{X}_s(x))|}{\bar{G}(\boldsymbol{X}_s(x), \boldsymbol{X}_s(y))}\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}s.
\end{aligned}$$

Setting $g_s := |\nabla_{\mathrm{sym}} b_s| + |\mathrm{div}\, b_s|$, from Proposition 3.6 and (3.16) we obtain

$$\begin{aligned}
\Phi_{t,r}(x) &\le \int_0^t \fint_{B(x,r)} \frac{|b_s \cdot \nabla\bar{G}_{\boldsymbol{X}_s(x)}(\boldsymbol{X}_s(y)) + b_s \cdot \nabla\bar{G}_{\boldsymbol{X}_s(y)}(\boldsymbol{X}_s(x))|}{\bar{G}(\boldsymbol{X}_s(x), \boldsymbol{X}_s(y))}\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}s + \Phi_{0,r} \\
&\le C\int_0^t \fint_{B(x,r)} (Mg_s(\boldsymbol{X}_s(x)) + Mg_s(\boldsymbol{X}_s(y)))\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}s + \Phi_{0,r} \\
&\le C\int_0^t [Mg_s(\boldsymbol{X}_s(x)) + M(Mg_s\boldsymbol{X}_s(\cdot))(x)]\,\mathrm{d}s + \frac{1}{A}
\end{aligned}$$

for $\mathfrak{m}$-a.e. $x \in X$, and the negligible set does not depend on r. From (3.7) we get

$$\begin{aligned}
\sup_{0\le t\le T}\ \sup_{0<r\le D} Q_{t,r}(x) &\le \sup_{0\le t\le T}\ \sup_{0<r\le D} \Phi_{t,r}(x) \\
&\le C\int_0^T [Mg_s(\boldsymbol{X}_s(x)) + M(Mg_s(\boldsymbol{X}_s(\cdot))(x)]\,\mathrm{d}s + \frac{1}{A}
\end{aligned}$$

for $\mathfrak{m}$-a.e. $x \in X$. Taking the L^2-norms (here the assumption that the RLF has compressibility constant $L < +\infty$ enters into play once more), we obtain the sought estimate. □

Below we state and prove the main regularity result for Regular Lagrangian flows.

Theorem 3.11. *Let b be a time-dependent vector field and $\boldsymbol{X}$ a Regular Lagrangian flow associated to b with compressibility constant L. For any $x, y \in X$ and for any $t \in [0, T]$ it holds*

$$\mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)) \le Ce^{C(Q^*(x)+Q^*(y))}\mathsf{d}(x, y), \tag{3.19}$$

where Q^* has been defined in (3.17)*, and $C = C(X, \mathsf{d}, \mathfrak{m})$.*

Moreover, when b *is bounded and* $|\nabla_{\mathrm{sym}} b|, \operatorname{div} b \in L^1((0,T); L^2(X,\mathfrak{m}))$, *for every* $\varepsilon > 0$ *there exists a Borel set* $E \subset X$ *such that* $\mathfrak{m}(X \setminus E) < \varepsilon$ *and for every* $x, y \in E$,

$$\mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)) \le C \exp\left(2C \frac{\|Q^*\|_{L^2}}{\sqrt{\varepsilon}}\right) \mathsf{d}(x,y) \tag{3.20}$$

for every $t \in [0,T]$, *where we remark that this last statement makes sense since, under our regularity assumptions on* b, *Theorem* 3.10 *guarantees that* $\|Q^*\|_{L^2} < +\infty$.

Proof. Fix any $x, y \in X$ such that $x \neq y$ and set $r := \mathsf{d}(x,y)$. Exploiting the inequality $(a+b)^m \le 2^{m-1}(a^m + b^m) \le 2^m(a^m + b^m)$, the triangular inequality and the subadditivity and monotonicity of $t \mapsto \log(1+t)$, we obtain that

$$\log\left(1 + \frac{1}{A}\left(\frac{\mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y))}{2r}\right)^{n-2}\right) \le \log\left(1 + \frac{1}{A}\left(\frac{\mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(z))}{r}\right)^{n-2}\right) + \log\left(1 + \frac{1}{A}\left(\frac{\mathsf{d}(\boldsymbol{X}_t(z), \boldsymbol{X}_t(y))}{r}\right)^{n-2}\right)$$

for any $z \in X$. Let us fix $w \in X$ such that $\mathsf{d}(x,w) = \mathsf{d}(y,w) = \frac{r}{2}$ (observe that such a point exists, since (X,d) is a geodesic metric space) and we take the mean value of the above written inequality (with respect to the z variable) over $B(w,\frac{r}{2})$ obtaining

$$\begin{aligned}
&\log\left(1 + \frac{1}{A}\left(\frac{\mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y))}{2r}\right)^{n-2}\right) \\
&\le \fint_{B(w,\frac{r}{2})} \log\left(1 + \frac{1}{A}\left(\frac{\mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(z))}{r}\right)^{n-2}\right) \mathrm{d}\mathfrak{m}(z) \\
&\qquad + \fint_{B(w,r/2)} \log\left(1 + \frac{1}{A}\left(\frac{\mathsf{d}(\boldsymbol{X}_t(z), \boldsymbol{X}_t(y))}{r}\right)^{n-2}\right) \mathrm{d}\mathfrak{m}(z) \\
&\le C \fint_{B(x,r)} \log\left(1 + \frac{1}{A}\left(\frac{\mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(z))}{r}\right)^{n-2}\right) \mathrm{d}\mathfrak{m}(z) \\
&\qquad + C \fint_{B(y,r)} \log\left(1 + \frac{1}{A}\left(\frac{\mathsf{d}(\boldsymbol{X}_t(z), \boldsymbol{X}_t(y))}{r}\right)^{n-2}\right) \mathrm{d}\mathfrak{m}(z),
\end{aligned}$$

where in the second inequality we enlarge the domain of integration and control the ratios between volumes of balls with radii $\frac{r}{2}$ and r in a uniform way, thanks to the Ahlfors regularity assumption.

It follows by the definition of Q^* that for any $x, y \in X$ such that $x \neq y$ and for any $t \in [0,T]$ it holds

$$\log\left(1 + \frac{1}{A}\left(\frac{\mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y))}{2\mathsf{d}(x,y)}\right)^{n-2}\right) \le C(Q^*(x) + Q^*(y)),$$

which easily yields (3.19).

Let us define

$$E := \left\{x \in X : Q^*(x) \le \frac{\|Q^*\|_{L^2}}{\sqrt{\varepsilon}}\right\},$$

by Chebyshev inequality we deduce that $\mathfrak{m}(X \setminus E) < \varepsilon$. The conclusion of (3.20) now directly follows from (3.19). □

3.4. The case $n = 2$. In order to conclude the proof of our result Theorem 3.2, we have to deal with the case $n = 2$.

In order to reduce this case to an application of the result, we proved for $n > 2$ we "add a dimension" to the given space by considering its product with the standard $\mathbb{S}^1$.

To this end, given a 2-Ahlfors regular RCD$^*(K, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ we define $(\bar{X}, \bar{\mathsf{d}}, \bar{\mathfrak{m}})$ by

(1) $\bar{X} := X \times \mathbb{S}^1$,

(2) $\bar{\mathsf{d}}^2((x, s), (x', s')) := \mathsf{d}_X^2(x, x') + \mathsf{d}_{\mathbb{S}^1}^2(s, s')$ for every $x, x' \in X$ and $s, s' \in \mathbb{S}^1$,

(3) $\bar{\mathfrak{m}} := \mathfrak{m} \times \mathrm{d}s$, where $\mathrm{d}s$ is the (normalized) volume measure of $\mathbb{S}^1$.

Then $(\bar{X}, \bar{\mathsf{d}}, \bar{\mathfrak{m}})$ is an RCD$^*(K, N + 1)$ m.m.s. (see [7, Secttion 6] and [15]) and it is 3-Ahlfors regular, as an elementary application of Fubini's theorem shows.

We will denote by π_1 and π_2 the canonical projections from $\bar{X}$ to X and $\mathbb{S}^1$, respectively. This being said we introduce the so-called algebra of tensor products by

$$\mathcal{A} := \left\{ \sum_{j=1}^{n} g_j \circ \pi_1 h_j \circ \pi_2 : n \in \mathbb{N},\ g_j \in W^{1,2} \cap L^\infty(X) \text{ and } h_j \in W^{1,2} \cap L^\infty(\mathbb{S}^1) \right\}.$$

A crucial property for the rest of the discussion in this section is the strong form of density of the product algebra $\mathcal{A}$ (the terminology is borrowed from [28]), namely it holds that for any $f \in W^{1,2}(\bar{X}) \cap L^\infty(X)$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in \mathcal{A}$ uniformly bounded and converging to f in $W^{1,2}(\bar{X})$. This property can be proved with minor modifications of the strategy developed in [26], where the case of products with the Euclidean line or intervals is considered. We also remark that a more direct approach to the proof of this density result can be obtained exploiting the result of [11, Theorem B.1]. Indeed, knowing that the algebra generated by distances from points is dense, the observation that the distance squared from a point in the product belongs to the product algebra (actually $\bar{\mathsf{d}}^2((x, s), \cdot) = \mathsf{d}_X^2(x, \cdot) + \mathsf{d}_{\mathbb{S}^1}^2(s, \cdot)$) together with an approximation procedure (which is needed to recover the distance from the distance squared) yields the desired conclusion.

In order to be able to apply Theorem 3.2 in the space $\bar{X}$, we are going to lift the given vector field b_t on X to a vector field $\bar{b}_t$ on $\bar{X}$.

In [28] the study of tangent and cotangent modules of product spaces is performed in great generality. We just observe here that, in the case of our interest, we are in a position to lift b_t in a trivial way by saying that, for any $f \in W^{1,2}(\bar{X})$,

$$\bar{b}_t \cdot \nabla f(x, s) = b_t \cdot \nabla f_s(x)$$

for $\bar{\mathfrak{m}}$-a.e. $(x, s) \in \bar{X}$, where $f_s(x) := f(x, s)$ and we are implicitly exploiting the tensorization property of the Cheeger energy (see [7]). Observe that if b_t belongs to $L^\infty((0, T); L^\infty(TX))$, then $\bar{b}_t \in L^\infty((0, T); L^\infty(T\bar{X}))$ and the norms are actually preserved.

Given a Regular Lagrangian flow $\boldsymbol{X}_t$ of b_t on $(X, \mathsf{d}, \mathfrak{m})$, we go on by setting

$$\bar{\boldsymbol{X}}_t(x, s) := (\boldsymbol{X}_t(x), s) \quad \text{for all } x \in X, \quad \text{for all } s \in \mathbb{S}^1$$

for every $t \in [0, T]$.

In Lemmas 3.12 and 3.13 below we prove that $\bar{\boldsymbol{X}}_t$ is a Regular Lagrangian flow of $\bar{b}_t$ and that $\bar{b}_t$ inherits the Sobolev regularity from b_t. These remarks will put us in a position to apply Theorem 3.11.

Lemma 3.12. *Under the previous assumptions $(\bar{X}_t)_{t\in[0,T]}$ is a Regular Lagrangian flow of $\bar{b}_t$.*

Proof. We begin by observing that, if X_t has compression bounded by L, then $\bar{X}_t$ has compression bounded by L itself, as a simple application of the change of variables formula shows.

This being said it remains to check that condition (3) in Definition 1.9 is satisfied (since the trajectories of $\bar{X}_t$ inherit the continuity from the trajectories of X_t). To this end, let $\tilde{\mathcal{A}}$ be the algebra generated by functions of the form $f\circ\pi_1 g\circ\pi_2$, where $g\in\mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ and $f\in\mathrm{Test}(\mathbb{S}^1)$. Observe that, for $f\in\tilde{\mathcal{A}}$, the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}f(\bar{X}_t(x,s))=\bar{b}_t\cdot\nabla f(\bar{X}_t(x,s)) \tag{3.21}$$

for $\bar{\mathfrak{m}}$-a.e. (x,s) and for $\mathscr{L}^1$-a.e. $t\in(0,T)$ directly follows from the assumption that X_t is a Regular Lagrangian flow of b_t on X and from the definition of $\bar{b}_t$. To conclude that (3.21) holds true for any $f\in\mathrm{Test}(\bar{X})$, it is now sufficient to take into account the density of the product algebra in the strong form. □

Lemma 3.13. *Assume that $b\in L^2(TX)$ has divergence in $L^2(X,\mathfrak{m})$. Then $\bar{b}$ has divergence in $L^2(\bar{X},\bar{\mathfrak{m}})$ and it holds $\operatorname{div}\bar{b}(x,s)=\operatorname{div}b(x)$ for $\bar{\mathfrak{m}}$-a.e. $(x,s)\in\bar{X}$.*

Moreover, if b has symmetric derivative in $L^2(X,\mathfrak{m})$ according to Definition 3.7, then $\bar{b}$ has symmetric derivative in $L^2(\bar{X},\bar{\mathfrak{m}})$ itself and it holds $\|\nabla_{\mathrm{sym}}\bar{b}\|_{L^2(\bar{\mathfrak{m}})}\le\|\nabla_{\mathrm{sym}}b\|_{L^2(\mathfrak{m})}$, where we omitted the implicit dependence on the space of the divergence and the symmetric covariant derivative.

Proof. The proof of the first conclusion can be found in [28, Proposition 3.15].

We pass to the proof of the second statement, which is strongly inspired by the proof of an analogous result concerning the Hessian that can be found in [28, Appendix A].

Recall that we defined $|\nabla_{\mathrm{sym}}\bar{b}|$ to be the function $h\in L^2(\bar{\mathfrak{m}})$ with smallest L^2-norm such that

$$\frac{1}{2}\left|\int_{\bar{X}}\bar{b}\cdot\nabla g\Delta f+\bar{b}\cdot\nabla f\Delta g-\operatorname{div}\bar{b}(\nabla f\cdot\nabla g)\,\mathrm{d}\bar{\mathfrak{m}}\right|\le\int_{\bar{X}}h|\nabla f||\nabla g|\,\mathrm{d}\bar{\mathfrak{m}} \tag{3.22}$$

for any $f,g\in\mathrm{Test}(\bar{X})$. Therefore to prove the desired conclusion, it suffices to show that $h(x,s):=|\nabla_{\mathrm{sym}}b|(x)$ is an admissible function in (3.22). Moreover, thanks to the strong density of the algebra $\mathcal{A}$ and to the approximation result of [28, Lemma A.3], it is sufficient to verify (3.22) in the case where $f,g\in\tilde{\mathcal{A}}$, where the algebra $\tilde{\mathcal{A}}$ was introduced in the proof of Lemma 3.12.

Denoting by Δ_X and $\Delta_{\mathbb{S}^1}$ the Laplacians on X and $\mathbb{S}^1$, respectively, we recall from [7, p. 52] that (with a slight abuse of notation) it holds $\Delta_{\bar{X}}=\Delta_X+\Delta_{\mathbb{S}^1}$. Then we compute

$$\begin{aligned}
&\frac{1}{2}\int_{\bar{X}}\bar{b}\cdot\nabla f\Delta g+\bar{b}\cdot\nabla g\Delta f-\operatorname{div}\bar{b}\nabla f\cdot\nabla g\,\mathrm{d}\bar{\mathfrak{m}}\\
&\quad=n\frac{1}{2}\int_{\mathbb{S}^1}\left[\int_X b\cdot\nabla_X f\Delta_X g+b\cdot\nabla_X g\Delta_X f-\operatorname{div}b\;\nabla_X f\cdot\nabla_X g\,\mathrm{d}\mathfrak{m}\right]\mathrm{d}s\\
&\qquad\quad+\frac{1}{2}\int_{\mathbb{S}^1}\left[\int_X b\cdot\nabla_X f\Delta_{\mathbb{S}^1}g+b\cdot\nabla_X g\Delta_{\mathbb{S}^1}f-\operatorname{div}b\;\nabla_{\mathbb{S}^1}f\cdot\nabla_{\mathbb{S}^1}g\,\mathrm{d}\mathfrak{m}\right]\mathrm{d}s,
\end{aligned}$$

where we exploited the definition of $\bar{b}$, the previously proven identity $\operatorname{div}\bar{b} = \operatorname{div} b \circ \pi_1$ and the tensorization of the Cheeger energy again. The first of the two terms appearing above is bounded by

$$\int_{\bar{X}} h|\nabla_X f||\nabla_Y g| \,\mathrm{d}\bar{\mathfrak{m}} \le \int_{\bar{X}} h|\nabla f||\nabla g| \,\mathrm{d}\bar{\mathfrak{m}},$$

since b has symmetric derivative in L^2.

To conclude, we are going to prove that

$$R := \int_{\mathbb{S}^1}\left[\int_X b\cdot\nabla_X f \Delta_{\mathbb{S}^1} g + b\cdot\nabla_X g \Delta_{\mathbb{S}^1} f - \operatorname{div} b\, \nabla_{\mathbb{S}^1} f\cdot\nabla_{\mathbb{S}^1} g\right]\mathrm{d}\mathfrak{m}\,\mathrm{d}s = 0.$$

Observe that applying the Leibniz rule for the divergence and integrating by parts, we obtain

$$\begin{aligned}
&\int_{\mathbb{S}^1}\int_X b\cdot\nabla_X f \Delta_{\mathbb{S}^1} g \,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \\
&\quad = n - \int_{\mathbb{S}^1}\int_X \operatorname{div} b\ f\ \Delta_{\mathbb{S}^1} g \,\mathrm{d}\mathfrak{m}\,\mathrm{d}s + \int_{\mathbb{S}^1}\int_X \operatorname{div}(bf)\Delta_{\mathbb{S}^1} g\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \\
&\quad = n\int_X \operatorname{div} b \int_{\mathbb{S}^1} \nabla_{\mathbb{S}^1} f\cdot\nabla_{\mathbb{S}^1} g\,\mathrm{d}s\,\mathrm{d}\mathfrak{m} - \int_{\mathbb{S}^1}\int_X f\ b\cdot\nabla_X(\Delta_{\mathbb{S}^1} g)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \\
&\quad = n\int_{\mathbb{S}^1}\int_X \operatorname{div} b\ \nabla_{\mathbb{S}^1} f\cdot\nabla_{\mathbb{S}^1} g\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s - \int_{\mathbb{S}^1}\int_X f\ b\cdot\nabla_X(\Delta_{\mathbb{S}^1} g)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s,
\end{aligned}$$

which yields

$$R = \int_{\mathbb{S}^1}\int_X f[\Delta_{\mathbb{S}^1}(b\cdot\nabla_X g) - b\cdot\nabla_X(\Delta_{\mathbb{S}^1} g)]\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s.$$

To get the desired conclusion, we just observe that, for any $g\in\tilde{\mathcal{A}}$, it holds

$$\Delta_{\mathbb{S}^1}(b\cdot\nabla_X g) = b\cdot\nabla_X(\Delta_{\mathbb{S}^1} g). \qquad \square$$

As we anticipated the results of Lemmas 3.12 and 3.13 put us in a position to apply Theorem 3.2 to $\bar{\boldsymbol{X}}_t$.

It follows that there exist a function $Q^*(x,s)$ and a constant $C = C(X,\mathsf{d},\mathfrak{m})$ such that

$$\bar{\mathsf{d}}(\bar{\boldsymbol{X}}_t(x,s),\bar{\boldsymbol{X}}_t(x',s')) \le Ce^{C(Q^*(x,s)+Q^*(x',s'))}\bar{\mathsf{d}}((x,s),(x',s'))$$

for every $x,x'\in X$ and $s,s'\in\mathbb{S}^1$. Choosing $s=s'$ and setting $Q^*(x) := \sup_{s\in\mathbb{S}^1} Q^*(x,s)$, we obtain that

$$\mathsf{d}(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y)) \le Ce^{C(Q^*(x)+Q^*(y))}\mathsf{d}(x,y)$$

for any $x,y\in X$. Moreover, it follows from the proof of Theorem 3.10 and from the result (and the proof) of Lemma 3.13 that

$$\sup_{s\in S^1} Q^*(x,s) \le C\int_0^T [Mg_t(\boldsymbol{X}_t(x)) + M(Mg_t(\boldsymbol{X}_t(\cdot)))(x)]\,\mathrm{d}s + C,$$

where $g := |\nabla_{\mathrm{sym}} b_t|(x) + |\operatorname{div} b_t|(x)$. Thus there exists $C = C(T,X,\mathsf{d},\mathfrak{m})$ such that

$$\|Q^*\|_{L^2} \le C\left[L\int_0^T \||\nabla_{\mathrm{sym}} b_t| + |\operatorname{div} b_t|\|_{L^2}\,\mathrm{d}t + 1\right].$$

The regularity result is now proved.

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Eingegangen 13. April 2018, in revidierter Fassung 22. Mai 2019