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The Quadratic Quasi-Normal Modes of a Schwarzschild Black Hole

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Abstract. Black hole spectroscopy probes the spectrum of quasi-normal modes (QNMs) during the ringdown phase of a merger. Future detectors will be sensitive not only to linear QNMs, but also to non-linear excitations known as quadratic QNMs (QQNMs). These contributions must be understood for accurate waveform modelling and precision tests of gravity.

We present a perturbative framework that yields QQNM amplitudes and phases for arbitrary angular momentum, parity, and overtone number in the Schwarzschild case. The relative amplitude of a quadratic mode to its linear parents is a universal prediction of black hole perturbation theory, independent of the initial perturbation up to a known dependence on parity, and in some cases large enough to be detectable. We reproduce known results and predict new quadratic modes of potential observational interest. In the eikonal limit, the structure of these ratios simplifies, as does the well studied linear frequency spectrum.

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1 Introduction

Observations of black hole mergers provide a unique opportunity to test General Relativity (GR) in the strong-gravity regime and to search for new phenomena in the gravitational sector. In particular, the ringdown phase of the gravitational-wave signal encodes the relaxation of the remnant to a stationary black hole through the emission of a superposition of quasi-normal modes (QNMs). These damped oscillations have frequencies that are fully determined by the mass and spin of the final black hole, making their measurement a direct probe of the no-hair theorem and the foundation of the programme of black-hole spectroscopy.

As sensitivity improves, and especially with the advent of LISA, current models that restrict to purely linear perturbations may prove increasingly inadequate [1, 2, 3], and the observation of non-linear effects becomes a realistic prospect. In particular, the gravitational-wave signal acquires additional contributions, called quadratic QNMs (QQNMs), which are generated by mode couplings beyond linear perturbation theory and have already been seen in numerical simulations [4].

The aim of this work is to investigate such non-linearities in the QNM signal in the simple but realistic setting of a non-rotating Schwarzschild black hole, with the goal of extending the theoretical framework of black-hole spectroscopy beyond the linear approximation.

2 Black Hole Perturbation Theory

Black hole perturbation theory starts from an expansion of the metric around the background Schwarzschild solution. This metric ansatz is then substituted into the Einstein equations.

2.1 Einstein Equations

The metric is written as

$$\mathbf{g} = \mathbf{g}^{\text{Schw.}} + \varepsilon\,\mathbf{h}^{(1)} + \varepsilon^2\,\mathbf{h}^{(2)} + O(\varepsilon^3)\ , \tag{1}$$

where $\mathbf{h}^{(1)}$, $\mathbf{h}^{(2)}$ represent the linear and quadratic (first and second order) perturbations of the metric, while ε can either be viewed as a book-keeping parameter used to separate the order of the perturbations, or more physically as quantifying the order of magnitude of the amplitude of the perturbations.

Up to second order in ε, Einstein's equations read

$$G^{(1)}_{\mu\nu}\big[\mathbf{h}^{(1)}\big] = 0\ , \quad G^{(1)}_{\mu\nu}\big[\mathbf{h}^{(2)}\big] = -G^{(2)}_{\mu\nu}\big[\mathbf{h}^{(1)}, \mathbf{h}^{(1)}\big] \equiv S_{\mu\nu}\ , \tag{2}$$

where $G^{(i)}_{\mu\nu}$ are differential operators linear in their arguments. The first condition determines the linear QNM frequencies at which $\mathbf{h}^{(1)}$ can oscillate, while the second encapsulates how two linear perturbations are coupled in GR. We will assume that linear perturbations are known. Additionally we will focus on the coupling of two given linear QNMs with angular momenta $\ell_{1,2}$, $m_{1,2}$, overtone numbers $n_{1,2}$ and intrinsic parity $p_{1,2}$, instead of a generic superposition, exploiting the linearity of $G^{(2)}_{\mu\nu}$. These are good "quantum numbers" because of the symmetries of the background metric.

These symmetries can actually be leveraged to predict the quantum numbers of the quadratic QNMs that result from the coupling of two modes. QQNMs will have quantum numbers ℓ, m, p and frequency ω constrained by

$$p_1(-1)^{\ell_1} \cdot p_2(-1)^{\ell_2} = p(-1)^{\ell}\ , \tag{3}$$

$$|\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2\ ,\ m = m_1 + m_2\ , \quad \operatorname{Re}\omega = \operatorname{Re}\omega_1 \pm \operatorname{Re}\omega_2\ ,\ \operatorname{Im}\omega = \operatorname{Im}\omega_1 + \operatorname{Im}\omega_2 \tag{4}$$

where in the last equation we remembered that real sinusoids contain exponentials of both positive frequency (ordinary modes) and negative frequency (mirror modes). Rotational symmetry also fixes the dependence on m, m_1, m_2 to be fully captured by a Clebsch–Gordan coefficient. Symmetries heavily constrain QQNMs, but do not fix their amplitudes. We will see that, contrary to the linear case where the amplitudes of linear QNMs cannot be determined purely from black hole perturbation theory, quadratic amplitudes are fixed in terms of the linear ones.

2.2 Master Scalars

Out of the ten components of the metric perturbation, only two correspond to the physical polarisations of the graviton. A common way to reduce the number of variables is to fix the Regge-Wheeler gauge, removing four variables, and then solve the constraint coming from the Einstein equations to remove four more. The remaining two degrees of freedom are captured by master scalars, specific combinations of the metric perturbation components from which the full perturbation can be reconstructed. They belong to different parity sectors: odd (Regge-Wheeler) and even (Zerilli). Adopting the same definition at both linear and quadratic order, the master scalars $\psi^i_\pm$, $i = 1, 2$ satisfy the Regge–Wheeler and Zerilli (RWZ) equations at linear order; at second order the equations acquire a source term $S_\pm$, built from combinations of $S_{\mu\nu}$. For the formulas that reconstruct $h^{(i)}_{\mu\nu}$ in terms of $\psi^{(i)}_\pm$ and more details about these expressions, the reader can see [5, 6, 7]. The right hand side of the RWZ equation at second order is known and proportional to the product of the master scalars of the linear perturbations. We will now seek a solution of the second order RWZ equation that satisfies the QNM boundary conditions

$$\psi^{(2)}_\pm \underset{r_*\to+\infty}{\propto} e^{i\omega r_*}, \quad \psi^{(2)}_\pm \underset{r_*\to-\infty}{\propto} e^{-i\omega r_*} \,. \tag{5}$$

The standard strategy to solve the RWZ equation with QNM boundary conditions eq. (5) is to employ the Leaver algorithm, generalised to the case where a source appears [8]. Unfortunately the conditions in eq. (5) require the asymptotic fall-off of the source $S \underset{r\to+\infty}{\sim} \mathcal{O}\left(\frac{1}{r^2}\right)$, which is not satisfied.

The solution to this problem was pointed out in [8, 9], and amounts to a redefinition of the master scalars at second order. The logic is that the master scalars we work with are not themselves physical, and they only acquire meaning when paired with the formulas that reconstruct the full metric perturbation $h_{\mu\nu}$ in terms of them. Starting from a master scalar $\psi^{(2)}$ we can define a new master scalar $\tilde{\psi}^{(2)}$

$$\tilde{\psi}^{(2)} = \psi^{(2)} + \Delta(r)\psi^{(1)}_1\psi^{(1)}_2 \,, \tag{6}$$

where $\Delta(r)$ is some completely explicit function, and the master scalars of the two linear order perturbations $\psi^{(1)}_{1,2}$ are written explicitly so that the quadratic master scalar remains proportional to the amplitude of the linear modes. The new reconstruction formulas follow from imposing that the same metric perturbation should be recovered. The dynamics of $\tilde{\psi}$ is governed by a Regge-Wheeler/Zerilli equation with modified source term

$$\left(\partial^2_{r_*} - \partial^2_t - V_\pm(r)\right)\tilde{\psi}^{(2)}_\pm = S_\pm + \left(\partial^2_{r_*} - \partial^2_t - V_\pm(r)\right)[\Delta(r)\psi^{(1)}_1\psi^{(1)}_2] \equiv S'_\pm \,. \tag{7}$$

By examining S in the large r limit, we can choose a function $\Delta(r)$ so that $S'_\pm$ has the correct asymptotic behaviour. Lastly, we implement the Leaver algorithm and apply it to find $\tilde{\psi}^{(2)}$. We emphasise that this is the only numerical step throughout the whole procedure.

3 Physical results

To quantify radiation at null infinity, the Newman-Penrose scalar Ψ_4 should be computed. Its most simple expression is

$$\Psi_4 = \mathfrak{h}_+ - i\mathfrak{h}_\times \,, \tag{8}$$

where $\mathfrak{h}_+$, $\mathfrak{h}_\times$ are the physical graviton degrees of freedom, most easily extracted in asymptotically transverse traceless gauge. The infinitesimal diffeomorphism necessary to bring the metric to this gauge is discussed in detail in [7]. The Newman-Penrose scalar can be computed for both linear and quadratic perturbations, and it decomposes as

$$\Psi_4 = \frac{M}{r}\sum_{\ell m\mathcal{N}} \mathcal{A}_{\ell m\mathcal{N}} e^{i\omega_{\ell\mathcal{N}}(r_*-t)} \;\; {}_{-2}Y^{\ell m}(\theta,\phi) \tag{9}$$

for both linear and quadratic perturbations. The variable $\mathcal{N}$ encodes additional quantum numbers: at linear order, the overtone number n and $\mathfrak{m} = \pm 1$, distinguishing ordinary from mirror modes; at quadratic order, it retains the quantum numbers of the parent modes. The amplitudes $\mathcal{A}_{\ell m\mathcal{N}}$ of the modes are independent of the black hole mass M and the distance r where detection occurs.

Once the quantum numbers of both parent modes and of the resulting quadratic mode are fixed, the ratio of the quadratic amplitudes over the product of the linear amplitudes

$$\mathcal{R}^{\ell m\mathcal{N}}_{(\ell_1 m_1\mathcal{N}_1)\times(\ell_2 m_2\mathcal{N}_2)} \equiv \frac{\mathcal{A}^{(2)}_{\ell m\mathcal{N}}}{\mathcal{A}^{(1)}_{1,\,\ell_1 m_1\mathcal{N}_1}\mathcal{A}^{(1)}_{2,\,\ell_2 m_2\mathcal{N}_2}} \tag{10}$$

becomes independent of the amplitude of the linear modes (the only free parameters left) and thus represents a completely universal quantity characterising black holes, similar to the frequency spectrum at linear order.

3.1 Accounting for parity

However, as noticed also by [10], one of the assumptions for universality is that the parity of the linear modes is fixed. While the decomposition in eq. (9) neatly separates the remaining quantum numbers, the definition in eq. (8) shows that each $\mathcal{A}_{\ell m \mathcal{N}}$ mixes contributions from both even and odd parity modes. The parity p of the QQNM is fixed in terms of the QQNM angular momentum ℓ, together with the parity p_i and angular momenta ℓ_i of the linear modes, by the selection rule in eq. (3). As a consequence, eq. (10) depends on the parity content of the linear modes and reduces to four independent values, corresponding to the parity combinations $(+,+)$, $(+,-)$, $(-,+)$, $(-,-)$.

We can make this manifest by observing from eq. (9) that both at linear and quadratic order the amplitudes $\mathcal{A}$ combine parity-definite amplitudes $\tilde{\mathcal{A}}_\pm$ as $\mathcal{A}_{\ell m \mathcal{N}} \propto \tilde{\mathcal{A}}_{+,\ell m \mathcal{N}} - i\tilde{\mathcal{A}}_{-,\ell m \mathcal{N}}$. We can recover $\tilde{\mathcal{A}}_\pm$ in terms of $\mathcal{A}_{\ell m \mathcal{N}}$ and $\mathcal{A}^*_{\ell(-m)\bar{\mathcal{N}}}$, where $\bar{\mathcal{N}}$ exchanges ordinary and mirror modes, and flips the sign of m. Finally $\mathfrak{p}_{\ell m \mathcal{N}} = (-1)^m \frac{\mathcal{A}^*_{\ell(-m)\bar{\mathcal{N}}}}{\mathcal{A}_{\ell m \mathcal{N}}}$ allows us to express the ratio in eq. (10) very cleanly. $\mathfrak{p} = \pm 1$ corresponds to a purely even/odd mode. When the dust settles, $\mathcal{A}_1^{(1)}$, $\mathcal{A}_2^{(1)}$ simplify and we get (omitting all the other quantum numbers for brevity)

$$\mathcal{R} = \frac{1}{4} \sum_{P_1, P_2 = \pm 1} \mathcal{R}_{P_1,P_2} \big(1 + P_1 \mathfrak{p}_1^{(1)}\big)\big(1 + P_2 \mathfrak{p}_2^{(1)}\big) \,. \tag{11}$$

When both linear modes are purely even $\mathfrak{p}_1 = +1$, $\mathfrak{p}_2 = +1$, $\mathcal{R} \equiv \mathcal{R}_{+,+}$ as we expect. Similar checks hold for the other parity combinations. Besides releasing all the code relevant for the computation [11], we provide a simple script that outputs $\mathcal{R}$ given all the relevant parameters. Under the simplifying assumption of equatorial symmetry, $\mathfrak{p}_{1,2}$ become fixed and the resulting ratios are reported in [12]. Many ratios are $\mathcal{O}(10^{-1})$, making them interesting observational targets.

3.2 Eikonal limit

Our methods also allow us to probe the eikonal regime, i.e. large angular momentum. This is a natural direction, since the analogous problem for linear QNM frequencies was solved long ago and yielded deep insights. In [13], we took first steps toward this goal by numerically exploring large ℓ with our codes, and used the results to check recent attempts [14, 15] at extracting the eikonal limit via the Penrose limit.

We restrict our attention to $\ell = \ell_1 + \ell_2$. The dependence on the angular momenta along the z axis is fully captured by a Clebsch–Gordan coefficient, so we can restrict to $m_i = \ell_i$ as well without losing generality. We chose to either send the angular momenta of both linear modes to infinity, or to keep one of them fixed.

$$\mathcal{R}_{\ell\times\ell\to 2\ell} \simeq 0.2, \quad \mathcal{R}_{2\times\ell\to 2+\ell} \simeq 0.11\ell \,, \tag{12}$$

We learn that some amplitude ratios grow linearly with angular momentum, while others saturate to a constant. Exploring more angular momentum configurations, we found this to be quite general behaviour. These results are in tension with [14], but the authors subsequently improved their analysis [15], finding better agreement with eq. (12).

4 Outlook

Several ratios between quadratic and linear mode amplitudes are $\mathcal{O}(10^{-1})$. This implies that quadratic QNMs could reach amplitudes within the projected sensitivity of future detectors.

We confirmed earlier results previously extracted from other methods (e.g. numerical relativity), such as the $2 \times 2 \to 4$ (both with ordinary and mirror mode), and $2 \times 3 \to 5$ ratios, and also identified new quadratic modes with appreciable amplitudes relative to their linear parent modes.

Given that the number of linear QNMs relevant for waveform modelling is already large, and that quadratic modes proliferate even further, our results provide a way to systematically incorporate quadratic QNMs into templates reducing overfitting: GR itself fixes the quadratic amplitudes in terms of the linear ones, which remain the only free parameters.

This predictive power can be used in an alternative way. Besides improving fitting models that assume GR, our results can be used for tests of GR: by fitting the ringdown signal using only linear QNMs and then comparing data to the quadratic QNM prediction of GR, one could search for deviations.

A natural future direction is the extension of these methods to rotating (Kerr) black holes, where the phenomenology of quadratic QNMs is less explored but of direct astrophysical relevance.

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