The $SO(1,4)$ flux-balance laws of de Sitter at quadrupolar order

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Abstract

The linear solution for quadrupolar perturbations around de Sitter spacetime was recently constructed. In this paper, we provide the flux-balance laws for each background symmetry (dilatations, rotations, spatial translations and cosmological boosts) in terms of source moments at quadrupolar order. We write the dilatation flux-balance law in two distinct ways, which allows to contrast two distinct proposals for the negative definite energy flux. The standard Poincaré flux balance laws at future null infinity are recovered in the flat limit of the $SO(1,4)$ flux-balance laws.

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Contents

1 Introduction

According to the standard model of cosmology, our universe admits a positive cosmological constant $\Lambda \approx 3 \times 10^{-122}$ in Planck units [1, 2]. Though tiny, the effect of the cosmological constant dominates in the infrared and leads to an asymptotically de Sitter spacetime structure at late times, which overarch in particular the definition of asymptotically conserved quantities. Besides, gravitational wave observations have provided within the last decade new insights into astrophysics and cosmology [3]. These observations have led to a large theoretical effort to define gravitational waves emitted from localized sources in a cosmological background, see e.g. [4–60].

Many simple theoretical questions remain unanswered. In this article, we will concentrate our focus on two questions: What is the role of the $SO(1,4)$ de Sitter symmetry regarding gravitational radiation? What is the definition of energy in asymptotically de Sitter spacetime?

It was demonstrated in [36, 39] that even though gravitational waves change the leading components of the metric at future infinity, a boundary gauge exists such the asymptotic symmetry algebra of asymptotically de Sitter spacetime is the infinite-dimensional Λ-BMS algebra, which reduces to the BMS algebra [61, 62] in the flat limit. The structure constants of the Λ-BMS algebra generically depend upon the radiation field. However, at quadratic order in perturbation theory around de Sitter, one can isolate a universal $SO(1,4)$ algebra of background symmetries, as we will review in Section 2, simply because the fluxes are already quadratic in the fields once the symmetry generators are taken to be the background Killing vectors of de Sitter. Our first objective is to derive the set of flux-balance laws associated with the background $SO(1,4)$ symmetry and evaluate them in the physical phase space. We will perform this computation at the level of the quadrupolar truncation of the linear spectrum [57], which has been demonstrated to match [58] with the spectrum of perturbations obtained from independent derivations [55, 58].

Several formulae for the energy flux in asymptotically de Sitter spacetime have been proposed [9,12,13,21–23,29–31,34,38,41,50,54,60,63–66]. Our second objective is to identify from the dilatation flux-balance law, how one can identify a charge and a flux such that (i) the flux reduces to the standard energy flux in the flat limit, (ii) the flux is manifestly non-negative at quadratic order. We will contrast our expressions for the energy loss and angular momentum flux with part of the literature. The linear momentum and boost charge loss formulae that we will derive have not yet appeared in that form in the literature to the best of our knowledge.

The rest of the paper is organized as follows. We first define the $SO(1,4)$ flux-balance laws in perturbation theory around de Sitter spacetime from first principles in Section 2. We then evaluate them on the solution space of even and odd quadratic perturbations of de Sitter in Section 3 and we prove that the standard flat spacetime limit is obtained. We compare our expressions of the energy loss and angular momentum loss with the literature in Section 4. We finally conclude in Section 5.

2 Charges and Fluxes on de Sitter space-times

The future region $\mathscr{I}^+$ of asymptotically de Sitter spacetimes with cosmological constant $\Lambda = 3H^2$ admits a Starobinsky expansion [67] of the form

$$ds^2 = -\frac{d\tau^2}{H^2\tau^2} + \frac{1}{\tau^2}(g^{(0)}_{ab} + \tau^2 g^{(2)}_{ab} + \tau^3 g^{(3)}_{ab} + O(\tau^4))dx^a dx^b, \tag{2.1}$$

where $\tau < 0$ denotes time and x^a are the other coordinates. In terms of the conformal completion, $\tau = 0$ is the future boundary $\mathscr{I}^+$ of asymptotically de Sitter spacetimes and x^a are the coordinates on $\mathscr{I}^+$. Latin letters $a, b, \dots$ will always refer to three-dimensional indices of fields defined on $\mathscr{I}^+$. The Starobinsky expansion is related to the Fefferman-Graham expansion of asymptotically anti-de Sitter spacetimes by an analytical continuation. We call $g^{(0)}_{ab}$ the three-dimensional metric on the future boundary of asymptotically de Sitter $\mathscr{I}^+$, $g^{ab}_{(0)}$ its inverse and we will denote as $D^{(0)}_a$ its metric-compatible covariant derivative. The holographic stress-energy tensor defined as $T_{ab} = \frac{3H}{16\pi G} g^{(3)}_{ab}$ is trace-free and divergence-free with respect to the boundary metric $g^{(0)}_{ab}$ outside of sources:

$$D^{(0)}_a T^{ab} = 0, \qquad g^{ab}_{(0)} T_{ab} = 0\ , \tag{2.2}$$

where T^{ab} is defined as T_{ab} with indices raised with the inverse metric $g^{ab}_{(0)}$.

In the presence of gravitational radiation, the boundary metric $g^{(0)}_{ab}$ generically admits no symmetry. In perturbation theory where $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ (where μ, ν denote 4-dimensional indices), we can however write

$$g^{(0)}_{ab} = \bar{g}^{(0)}_{ab} + \delta g^{(0)}_{ab}, \tag{2.3}$$

where the background boundary metric $\bar{g}^{(0)}_{ab}$, which describes the future boundary of de Sitter $\mathscr{I}^+_{\text{dS}}$, admits symmetries. Motivated by the Bondi framework and by the description of spatially compact binary systems that reach future infinity $\mathscr{I}^+$ at a single point $u = +\infty$, we choose $\bar{g}^{(0)}_{ab}$ to be the metric on $\mathbb{R} \times S^2$

$$\bar{g}^{(0)}_{ab} dx^a dx^b = H^2 du^2 + \mathring{q}_{AB}(x^C) dx^A dx^B. \tag{2.4}$$

Here $\mathring{q}_{AB}$ is the unit metric on S^2 and u ranges over the real line. These coordinates do not cover the entire $\mathscr{I}^+_{\text{dS}}$ of de Sitter, which has topology S^3, but $\mathscr{I}^+_{\text{dS}}$ minus two points. The topology of $\mathscr{I}^+$ depends upon the number of massive bodies. Here, we consider at least two such points removed at $u = \pm\infty$. We also gauge fix the perturbation to so-called Λ-BMS gauge [36] such that

$$g^{(0)}_{ab} dx^a dx^b = H^2 du^2 + q_{AB}(u, x^C) dx^A dx^B, \tag{2.5}$$

and $\sqrt{q} = \sqrt{\mathring{q}}$. The inverse of q_{AB} is denoted q^{AB}. The shear is defined as $C_{AB} = H^{-2}\partial_u q_{AB}$.

The background boundary metric admits 10 conformal Killing vectors $\bar{\xi}^a(u, x^A)$ which are in one-to-one correspondence with the Killing symmetries of the de Sitter background and define the $SO(1,4)$ group. By definition they obey

$$\bar{g}_{(0)c\langle a} \bar{D}_{(0)b\rangle} \bar{\xi}^c = 0. \tag{2.6}$$

where $\bar{D}_{(0)a}$ is the covariant derivative compatible with the background boundary metric $\bar{g}_{(0)ab}$ and $\langle \cdot \rangle$ denote the symmetric tracefree part. The 10 conformal Killing vectors consist of the dilatations and rotations (which are Killing vectors of the background boundary metric) and the spatial translations and cosmological boosts (which are proper conformal Killing vectors of the background boundary metric). They are given by the following table:

Charge	name	generator
E	Dilatation	$\bar{\xi}^u = 1, \quad \bar{\xi}^A = 0$
P^i	Spatial translation	$\bar{\xi}^u_{(i)} = n_i \exp(Hu), \quad \bar{\xi}^A_{(i)} = -H\mathring{D}^A n_i \exp(Hu)$
L^i	Rotation	$\bar{\xi}^u_{(i)} = 0, \quad \bar{\xi}^A_{(i)} = -\epsilon^{AB}\mathring{D}_B n_i$
K^i	Cosmological boost	$\bar{\xi}^u_{(i)} = n_i \exp(-Hu), \quad \bar{\xi}^A_{(i)} = H\mathring{D}^A n_i \exp(-Hu)$

Here n_i is the unit normal to the round sphere embedded in $\mathbb{R}^3$, and $\mathring{D}_A$ is the covariant derivative compatible with the unit sphere metric.

For an arbitrary vector ξ^a, one has the identity

$$\int_M du d^2\Omega \sqrt{\mathring{q}} g^{ac}_{(0)} D_{(0)c}(T_{ab}\xi^b) = \int_M du d^2\Omega \sqrt{\mathring{q}} T^{ab} g_{(0)c\langle a} D_{(0)b\rangle}\xi^c \ . \tag{2.7}$$

for any region M of $\mathscr{I}^+$ outside of sources thanks to Eq. (2.2). In perturbation theory, T_{ab} admits linear and quadratic contributions in the perturbation $\delta g_{\mu\nu}$, $T_{ab} = \delta T_{ab} + \delta^2 T_{ab} + O((\delta g)^3)$. Let us now set $\xi^a = \bar{\xi}^a$ a background conformal Killing vector. Then, for any choice of spheres S_1 and S_2 such that $\partial M = S_1 \cup S_2$ (with opposite relative orientation), one has the equalities at linear and quadratic order, respectively,

$$\int_{S_1} d^2\Omega \sqrt{\mathring{q}} n^a \delta T_{ab} \bar{\xi}^b - \int_{S_2} d^2\Omega \sqrt{\mathring{q}} n^a \delta T_{ab} \bar{\xi}^b = 0, \tag{2.8}$$

$$\int_{S_1} d^2\Omega \sqrt{\mathring{q}} n^a \delta^2 T_{ab} \bar{\xi}^b - \int_{S_2} d^2\Omega \sqrt{\mathring{q}} n^a \delta^2 T_{ab} \bar{\xi}^b = \int_M du d^2\Omega \sqrt{\mathring{q}} \delta T^{ab} \mathcal{L}_{\bar{\xi}} \delta g^{(0)}_{ab}, \tag{2.9}$$

where n^a is the unit normal to S_1 and S_2 defined as $n^a \partial_a = H^{-1} \partial_u$. The right-hand side of Eq. (2.8) is zero thanks to the conformal Killing equation (2.6).

Finally, we derived that on a given section of $\mathscr{I}^+$ the charge defined from the holographic stress-energy tensor as

$$Q^T_{\bar{\xi}}(u) \equiv - \int_{S^2} d^2\Omega \sqrt{\mathring{q}} n^a T_{ab} \bar{\xi}^b \tag{2.10}$$

is a constant in linear theory, i.e. $Q_{\bar{\xi}}(u_1) = Q_{\bar{\xi}}(u_2)$ for any u_1, u_2. At quadratic order, the presence of radiation will induce a change in $Q_{\bar{\xi}}(u)$ given by the flux-balance formula

$$Q^T_{\bar{\xi}}(u_2) - Q^T_{\bar{\xi}}(u_1) = \int_{u_1}^{u_2} du \, \dot{Q}^T_{\bar{\xi}}(u), \tag{2.11}$$

$$\dot{Q}^T_{\bar{\xi}}(u) \equiv \int_S d^2\Omega \sqrt{\mathring{q}} \delta T_{ab} \bar{g}^{ac}_{(0)} \bar{g}^{bd}_{(0)} \mathcal{L}_{\bar{\xi}} \delta g^{(0)}_{cd} + O((\delta g)^3). \tag{2.12}$$

Instead of considering the background symmetry generator $\bar{\xi}^a$ we could consider an arbitrary perturbation of the vector $\xi^a = \bar{\xi}^a + \delta\xi^a$. Assuming that the Λ-BMS gauge is obeyed in the full theory, the generic infinitesimal diffeomorphism of the (4-dimensional) metric is parameterized by a vector ξ^a of the boundary metric which obeys the equations

$$\partial_u \xi^u = \frac{1}{2\sqrt{q}} \partial_A (\sqrt{q} \xi^A), \tag{2.13a}$$

$$\partial_u \xi^A = -H^2 q^{AB} \partial_B \xi^u. \tag{2.13b}$$

Let us denote as ξ^μ the 4-dimensional vector associated with ξ^a in Λ-BMS gauge. A linearized diffeomorphism of the 4-dimensional metric is simply the Lie derivative of the background metric with respect to ξ^μ, i.e. linearized gravitational fields are defined up to a gauge transformation $\delta g_{\mu\nu} \to \delta g_{\mu\nu} + \mathcal{L}_\xi \bar{g}_{\mu\nu}$. We have $\mathcal{L}_\xi \bar{g}_{\mu\nu} = \mathcal{L}_{\delta\xi} \bar{g}_{\mu\nu}$ where the equality follows from the Killing equation $\mathcal{L}_{\bar{\xi}} \bar{g}_{\mu\nu} = 0$. Therefore, at linear order in perturbation theory and in Λ-BMS gauge, a perturbed vector $\delta\xi^a$ obeys the linearization of Eq. (2.13) where q_{AB} is replaced by the background $\mathring{q}_{AB}$:

$$\partial_u \delta\xi^u = \frac{1}{2\sqrt{\mathring{q}}}\partial_A(\sqrt{\mathring{q}}\delta\xi^A), \tag{2.14a}$$

$$\partial_u \delta\xi^A = -H^2 \mathring{q}^{AB}\partial_B \delta\xi^u. \tag{2.14b}$$

For dilatations and rotations, we have $\partial_A \bar{\xi}^u = 0$ and we can fix $\xi^a = \bar{\xi}^a$ in the non-linear theory because the right-hand side of Eq. (2.13b) is zero. We can therefore define universal (i.e. field independent) time translation and rotations that form a $\mathbb{R}\times SO(3)$ subalgebra of the Λ-BMS algebra in the non-linear theory. This subalgebra is part of the larger $\mathbb{R}\times \text{ADiff}(S^2)$ universal subalgebra of the Λ-BMS algebra which consists of time translations and area-preserving diffeomorphism (with generators $\xi^u = 0$, $\xi^A = \varepsilon^{AB}\partial_B \Psi(\theta,\phi)$). The universal $\mathbb{R}\times SO(3)$ subalgebra of asymptotic symmetries was also identified in [55] in a different boundary gauge. The background spatial translations and cosmological boosts are generically deformed in the non-linear theory. Yet, in the linear theory, the deformation vector is a background $\Lambda-$BMS generator (that obeys by definition (2.14)), which we can conventionally fix to be vanishing. With this convention, all background symmetries are undeformed in the linear theory.

We define the cosmological Bondi mass aspect $M^{(\Lambda)}$, angular momentum aspect $N_A^{(\Lambda)}$ and higher Bondi aspect $J_{AB}^{(\Lambda)}$ from the decomposition of the stress-energy tensor (see Eq. (3.15) of [36])

$$T_{ab} = \frac{3H}{16\pi G}\begin{bmatrix} -\frac{4}{3}M^{(\Lambda)} & -\frac{2}{3}N_B^{(\Lambda)} \\ -\frac{2}{3}N_B^{(\Lambda)} & J_{AB}^{(\Lambda)} + \frac{2}{\Lambda}M^{(\Lambda)}q_{AB}\end{bmatrix}. \tag{2.15}$$

Here $J_{AB}^{(\Lambda)}$ is traceless: $q^{AB}J_{AB}^{(\Lambda)} = 0$. The Bondi mass aspect M and Bondi angular momentum aspect N_A defined in the flat limit $H\mapsto 0$ are related to the cosmological quantities as (see Eqs. (2.38) and (2.52) of [36])

$$M^{(\Lambda)} = M + \frac{1}{16}\partial_u(C_{CD}C^{CD}), \tag{2.16a}$$

$$N_A^{(\Lambda)} = N_A - \frac{1}{2H^2}D^B N_{AB} + \frac{9}{32}\partial_A(C_{CD}C^{CD}). \tag{2.16b}$$

The charges (for the vector ξ^a) can be rewritten as

$$Q_\xi^T = \frac{1}{4\pi G}\int_{S^2} d^2\Omega\sqrt{\mathring{q}}\left[M^{(\Lambda)}\xi^u + \frac{1}{2}N_A^{(\Lambda)}\xi^A\right]. \tag{2.17}$$

In terms of Bondi aspects, the conservation of the holographic stress-energy tensor (2.2) is equivalent to

$$\partial_u M^{(\Lambda)} + \frac{H^2}{2}D^A N_A^{(\Lambda)} + \frac{3H^4}{8}C_{AB}J^{(\Lambda)AB} = 0, \tag{2.18a}$$

$$\partial_u N_A^{(\Lambda)} - \partial_A M^{(\Lambda)} - \frac{3H^2}{2}D^B J_{AB}^{(\Lambda)} = 0. \tag{2.18b}$$

Here D_A is the covariant derivative compatible with q_{AB}. Using Eqs. (2.18) and (2.13), the charge flux is then equivalently written as

$$\dot{Q}^T_\xi = -\frac{3H^2}{32\pi G}\int d^2\Omega\sqrt{q}\left(H^2\xi^u C_{AB} + 2D_A\xi_B\right)J^{(\Lambda)AB}, \tag{2.19}$$

$$= -\frac{3H^2}{32\pi G}\int d^2\Omega\sqrt{q}\left(\xi^u\partial_u q_{AB} + \mathcal{L}_{\mathring{\xi}} q_{AB}\right)J^{(\Lambda)AB}, \tag{2.20}$$

where the Lie derivative is here understood to act on the S^2 manifold using the pullback vector $\mathring{\xi} = \xi^A\partial_A$.

The flux is quadratic in the metric perturbation. Upon expanding $\xi^a = \bar{\xi}^a + \delta\xi^a + O(\delta^2)$ (as well as its 2-dimensional pullback $\mathring{\xi}^A = \bar{\xi}^A + \delta\xi^A + O(\delta^2)$), $q_{AB} = \mathring{q}_{AB} + \delta q_{AB} + O(\delta^2)$, $J^{(\Lambda)}_{AB} = \delta J^{(\Lambda)}_{AB} + \delta^2 J^{(\Lambda)}_{AB} + O(\delta^3)$ we have

$$\delta J^{(\Lambda)}_{AB}\mathring{q}^{AB} = 0, \tag{2.21}$$

$$\delta^2 J^{(\Lambda)}_{AB}\mathring{q}^{AB} = \delta J^{(\Lambda)}_{AB}\mathring{q}^{AC}\mathring{q}^{BD}\delta q_{CD}, \tag{2.22}$$

as a consequence of the tracelessness of $J^{(\Lambda)AB}$, as well as

$$\begin{aligned} 2\delta(D^A\bar{\xi}^B)\delta J^{(\Lambda)}_{AB} + 2\delta^2 J^{(\Lambda)AB}\mathring{D}_B\bar{\xi}_A &= (-\mathcal{L}_{\mathring{\bar{\xi}}}\delta q^{AB} + \mathring{D}_C\bar{\xi}^C\mathring{q}^{AC}\mathring{q}^{BD}\delta q_{CD})\delta J^{(\Lambda)}_{AB} \\ &= (\mathcal{L}_{\mathring{\bar{\xi}}}\delta q_{AB} - \mathring{D}_C\bar{\xi}^C\delta q_{AB})\mathring{q}^{AC}\mathring{q}^{BD}\delta J^{(\Lambda)}_{CD}. \end{aligned} \tag{2.23}$$

The flux-formula therefore reads at quadratic level as

$$\dot{Q}^T_\xi = -\frac{3H^2}{32\pi G}\int d^2\Omega\sqrt{\mathring{q}}\mathring{q}^{AC}\mathring{q}^{BD}\left(\bar{\xi}^u\partial_u\delta q_{AB} + \mathcal{L}_{\mathring{\bar{\xi}}}\delta q_{AB} + \mathcal{L}_{\delta\xi}\mathring{q}_{AB} - \mathring{D}_E\bar{\xi}^E\delta q_{AB}\right)\delta J^{(\Lambda)}_{CD} + O(\delta^3)\,. \tag{2.24}$$

Here $\delta\xi^a$ obeys to Eqs. (2.14). We choose the solution $\delta\xi^a = 0$, i.e. we do not act with any further $\Lambda-$BMS symmetries but only with the background symmetry. In that sense, there are flux-balance laws associated with $SO(1,4)$ symmetry at quadratic order in perturbation theory.

In the flat limit, using the conventions of [68], the generators associated with time translations, spatial translations and rotations are simply the limit $H \mapsto 0$ of E, P^i and L^i, respectively. Lorenz boosts are associated with the limit $H \mapsto 0$ of the generator $\frac{1}{2H}(K^i - P^i)$ with components $\bar{\xi}^u_{(i)} = -\frac{1}{H}n_i\sinh(Hu)$, $\bar{\xi}^A_{(i)} = \mathring{D}^A n_i\cosh(Hu)$. Using these relationships we notice that the charges Q^T_ξ do not have the standard flat limit, see Eq. (3.2) of [69]. In order to recover the standard flat limit, we will define instead the charges as

$$Q_\xi(u) = Q^T_\xi(u) + \Delta Q_\xi(u), \tag{2.25}$$

where the shifting term $\Delta Q_\xi = O((\delta g)^2)$ will be defined in the next section. There is an ambiguity in defining this shift. We will partially fix this ambiguity by requiring that the energy is positive definite.

3 The $SO(1,4)$ flux-balance laws

In this section we will analyse the flux formulae at quadratic order in the quadrupolar approximation. We will use the quadrupolar solution derived in [57]. In Bondi coordinates and Λ-BMS gauge, the metric reads as

$$
\begin{aligned}
ds^2 = &\left(\frac{\Lambda r^2}{3} - \left(\frac{1}{2}R[q] + \frac{\Lambda}{12}C_{AB}C^{AB}\right) + \frac{2M}{r} + O(r^{-2})\right)du^2 \\
&+ 2\Bigg(-1 + \frac{1}{r^2}\frac{C_{AB}C^{AB}}{16} + \frac{1}{r^4}\frac{3}{16}\left(C^{AB}E_{AB} - \frac{3}{32}(C_{AB}C^{AB})^2\right) + O(r^{-6})\Bigg)dudr \\
&+ 2\left(\frac{1}{2}D^B C_{BA} + \frac{1}{r}\left(\frac{2}{3}N_A + \frac{1}{6}C_A^F D^E C_{EF}\right) + O(r^{-2})\right)dx^A du \\
&+ \left(r^2 q_{AB} + rC_{AB} + \frac{1}{4}q_{AB}C_{CD}C^{CD} + \frac{E_{AB}}{r} + O(r^{-2})\right)dx^A dx^B.
\end{aligned}
\tag{3.1}
$$

At linear order, we can expand the boundary metric q_{AB}, the shear C_{AB} and the higher Bondi moment E_{AB} as

$$
q_{AB} = \mathring{q}_{AB} + \delta q_{AB}, \qquad C_{AB} = \delta C_{AB}, \qquad E_{AB} = \delta E_{AB}. \tag{3.2}
$$

For convenience we introduce the notations

$$
\delta q_{AB} \equiv e^i_{\langle A}e^j_{B\rangle}\delta q_{ij}, \qquad \delta C_{AB} \equiv e^i_{\langle A}e^j_{B\rangle}\delta C_{ij}, \qquad \delta E_{AB} \equiv e^i_{\langle A}e^j_{B\rangle}\delta E_{ij}. \tag{3.3}
$$

Here r is the radial coordinate and $\theta^A = (\theta, \phi)$ are the angular coordinates. The unit normal vector is denoted as $n^i = x^i/r$. We employ the natural basis on the unit 2-sphere $e_A = \frac{\partial}{\partial \theta^A}$ embedded in $\mathbb{R}^3$ with components $e^i_A = \partial n^i/\partial\theta^A$. Given the unit sphere metric $\mathring{q}_{AB} = \mathrm{diag}(1, \sin^2\theta)$, we have $n^i e^i_A = 0$, $\mathring{q}_{AB} = \delta_{ij}e^i_A e^j_B$, and $\mathring{q}^{AB}e^i_A e^j_B = \perp^{ij}$, where $\perp^{ij} = \delta^{ij} - n^i n^j$ is the projector operator. Using the covariant derivative $\mathring{D}_A$ compatible with the unit sphere metric, $\mathring{D}_A \mathring{q}_{BC} = 0$, we have $\mathring{D}_A e^i_B = \mathring{D}_B e^i_A = \mathring{D}_A\mathring{D}_B n^i = -\mathring{q}_{AB}n^i$. The transverse-traceless projector is denoted $\perp^{ijkl}_{\mathrm{tt}} = \perp^{k(i}\perp^{j)l} - \frac{1}{2}\perp^{ij}\perp^{kl}$. The tt projected part of any symmetric tensor T_{ij} will be defined as $T^{\mathrm{tt}}_{ij} = \perp^{ijkl}_{\mathrm{tt}} T_{kl}$. We also use the notation $e^i_{\langle A}e^j_{B\rangle} = e^i_{(A}e^j_{B)} - \frac{1}{2}\mathring{q}_{AB}\perp^{ij}$ for the tracefree product of the basis vectors.

At quadrupolar order in the perturbation we can express the Bondi fields in terms of the even and odd source quadrupolar moments $Q^{(\rho+p)}_{ij}$ and J_{ij} as [57]

$$
G^{-1}\delta q_{ij} = \partial_u\zeta_{ij} + 2H^2\partial_u Q^{(\rho+p)}_{ij} + 2H^2 n_k\epsilon_{kl(i}\left(K_{j)l} + H\int^u du' K_{j)l}(u')\right), \tag{3.4}
$$

$$
G^{-1}\delta C_{ij} = 3\zeta_{ij} + 2(\partial_u^2 - H^2)Q^{(\rho+p)}_{ij} + 2n_k\epsilon_{kl(i}(\partial_u + H)K_{j)l}, \tag{3.5}
$$

$$
G^{-1}\delta E_{ij} = 2Q^{(\rho+p)}_{ij} + 2n_k\epsilon_{kl(i}J_{j)l}. \tag{3.6}
$$

Here $K_{ij} = -(\partial_u - H)J_{ij}$ and ζ_{ij} obeys the differential equation

$$
\partial_u^2\zeta_{ij} - 3H^2\zeta_{ij} = -2GH^4 Q^{(\rho+p)}_{ij}. \tag{3.7}
$$

We further have

$$G^{-1}\delta M = Q^{(\rho)} - HP_{i|i} - 3n_i(P_i - HQ_i^{(\rho)} - H^2 P_{i|kk}) + (3n_i n_j - \delta_{ij})(\partial_u^2 Q_{ij}^{(\rho+p)} - H^2 Q_{ij}^{(\rho+p)}), \tag{3.8}$$

$$G^{-1}\delta N_i = Q_i^{(\rho)} + HP_{i|kk} + n^j(\epsilon_{ijk}J_k + 2\partial_u Q_{ij}^{(\rho+p)}) - 2\epsilon_{ijk}n_j n_l(K_{kl} - HJ_{kl}), \tag{3.9}$$

where the linear Bondi mass aspect has been simplified as Eq. (5.35) of [58].

3.1 Energy loss

From Eq. (2.24), the flux of the charge associated with dilatation reads at quadratic order in the perturbations as

$$\dot{Q}_D^T = -\frac{3H^4}{8G}\oint_S \delta C_{MN}\delta J_{AB}^{(\Lambda)}\,\mathring{q}^{AM}\mathring{q}^{BN}. \tag{3.10}$$

Here we defined $\oint_S = \frac{1}{4\pi}\int_{S^2} d^2\Omega\sqrt{\mathring{q}}$. The expression for $J_{AB}^{(\Lambda)}$ reads to linear order as [36]

$$3H^4\delta J_{AB}^{(\Lambda)} = -\partial_u \delta N_{AB} - H^2\left(\mathring{D}_{(A}\mathring{D}^C\delta C_{B)C} - \frac{1}{2}\mathring{q}_{AB}\mathring{D}^C\mathring{D}^D\delta C_{CD} - \delta C_{AB}\right) - 3H^4\delta E_{AB}. \tag{3.11}$$

Using the substitution (2.16), we can rewrite the flux-balance law of the dilatation (3.10) as

$$\begin{aligned}\partial_u \oint_S \delta^2 M \;&= -\tfrac{1}{8}\oint_S \Big[\delta N_{AB}\delta N_{CD} + H^2\delta C_{AB}(\delta C_{CD} - \mathring{D}_C\mathring{D}^E\delta C_{DE}) \\ &\qquad -3H^4\delta C_{AB}\delta E_{CD}\Big]\mathring{q}^{AC}\mathring{q}^{BD}.\end{aligned} \tag{3.12}$$

This formula also matches with the quadratic part of Eq. (2.52) of [36] after integrating over 2-sphere. Here we used that the linear energy $\oint_S \delta M$ is separately conserved, $\partial_u \oint_S \delta M = 0$. From Eq. (3.8), the terms proportional to n_i and $3n_i n_j - \delta_{ij}$ integrate to zero over the sphere while the term independent of n_i is conserved as a consequence of the conservation of the stress-energy tensor, $\partial_u Q^{(\rho)} = -HQ^{(p)} = -HS_{ii} = H\partial_u P_{i|i}$ (see also Eq. (5.37) of [58]). Therefore, the energy loss only involves the quadratic perturbation $\delta^2 M$.

3.1.1 Even parity energy loss

We now ready to evaluate the flux-balance law of dilatations for quadratic modes. Let us first consider the even parity modes. The shear C_{ij} is then independent of the angles. We can derive

$$C_{AB}C^{AB} = \perp_{\text{tt}}^{ijkl} C_{ij}C_{kl} = C_{ij}^{\text{tt}}C_{ij}, \tag{3.13}$$

$$\mathring{D}_A\mathring{D}^C C_{BC} = -2(e^i_{(A}e^j_{B)} - n^i n^j \mathring{q}_{AB})C_{ij}, \tag{3.14}$$

$$C^{AB}\mathring{D}_A\mathring{D}^C C_{BC} = -2C_{ij}^{\text{tt}}C_{ij}. \tag{3.15}$$

Equation (3.11) simplifies to

$$\delta J^{(\Lambda)}_{AB} = -\frac{1}{3H^4}\partial_u \delta N_{AB} + \frac{1}{H^2}\delta C_{AB} - \delta E_{AB}. \tag{3.16}$$

Using the identities (3.13), (3.14), (3.15), the energy loss formula (3.12) becomes

$$\partial_u \oint_S \delta^2 M = -\frac{1}{8G}\oint_S (\partial_u \delta C^{\text{tt}}_{ij}\partial_u \delta C_{ij} + 3H^2 \delta C^{\text{tt}}_{ij}\delta C_{ij} - 3H^4 \delta C^{\text{tt}}_{ij}\delta E_{ij}). \tag{3.17}$$

Using the even-parity moments from (3.5), (3.6), the integrand of (3.17) can be expanded as

$$\begin{aligned} G^{-2}\left(\partial_u \delta C^{\text{tt}}_{ij}\partial_u \delta C_{ij} + 3H^2 \delta C^{\text{tt}}_{ij}\delta C_{ij} - 3H^4 \delta C^{\text{tt}}_{ij}\delta E_{ij}\right) &= \partial_u \Big(9\zeta^{\text{tt}}_{ij}\dot{\zeta}_{ij} + 12\ddot{Q}^{(\rho+p)}_{ij}\zeta^{\text{tt}}_{ij} \\ &- 12H^2 Q^{(\rho+p)}_{ij}\dot{\zeta}^{\text{tt}}_{ij} - 8H^2\ddot{Q}^{(\rho+p)}_{ij}\dot{Q}^{(\rho+p)\text{tt}}_{ij} - 12H^4 Q^{(\rho+p)}_{ij}\dot{Q}^{(\rho+p)\text{tt}}_{ij}\Big) \\ &+ 4\Big(\dddot{Q}^{(\rho+p)}_{ij}\dddot{Q}^{(\rho+p)\text{tt}}_{ij} + 5H^2\ddot{Q}^{(\rho+p)}_{ij}\ddot{Q}^{(\rho+p)\text{tt}}_{ij} + 4H^4\dot{Q}^{(\rho+p)}_{ij}\dot{Q}^{(\rho+p)\text{tt}}_{ij}\Big). \end{aligned}$$

We can finally rewrite the flux-balance law of dilatations as

$$\partial_u Q^{\text{e}}_D(u) = -\frac{G}{2}\oint_S \left(\dddot{Q}^{(\rho+p)}_{ij}\dddot{Q}^{(\rho+p)\text{tt}}_{ij} + 5H^2\ddot{Q}^{(\rho+p)}_{ij}\ddot{Q}^{(\rho+p)\text{tt}}_{ij} + 4H^4\dot{Q}^{(\rho+p)}_{ij}\dot{Q}^{(\rho+p)\text{tt}}_{ij}\right), \tag{3.18}$$

where the dilatation charge in the even sector is identified as

$$\begin{aligned} Q^{\text{e}}_D = G\oint_S \Big(&\frac{1}{G}\delta M + \frac{1}{G}\delta^2 M + \frac{9}{8}\zeta^{\text{tt}}_{ij}\dot{\zeta}_{ij} + \frac{3}{2}\ddot{Q}^{(\rho+p)}_{ij}\zeta^{\text{tt}}_{ij} - \frac{3}{2}H^2 Q^{(\rho+p)}_{ij}\dot{\zeta}^{\text{tt}}_{ij} - H^2\ddot{Q}^{(\rho+p)}_{ij}\dot{Q}^{(\rho+p)\text{tt}}_{ij} \\ &- \frac{3}{2}H^4 Q^{(\rho+p)}_{ij}\dot{Q}^{(\rho+p)\text{tt}}_{ij}\Big). \end{aligned} \tag{3.19}$$

The flux of energy is negative definite as it should. Using the identity

$$\oint \perp^{ijkl}_{\text{tt}} = \frac{2}{15}(3\delta^{k(i}\delta^{j)l} - \delta^{ij}\delta^{kl}), \tag{3.20}$$

the flux-balance law in (3.18) becomes

$$\begin{aligned} \partial_u Q^{\text{e}}_D(u) &= -\frac{G}{5}\Big((\dddot{Q}^{(\rho+p)}_{ij} - \frac{1}{3}\delta_{ij}\dddot{Q}^{(\rho+p)})^2 + 5H^2(\ddot{Q}^{(\rho+p)}_{ij} - \frac{1}{3}\delta_{ij}\ddot{Q}^{(\rho+p)})^2 \\ &+ 4H^4(\dot{Q}^{(\rho+p)}_{ij} - \frac{1}{3}\delta_{ij}\dot{Q}^{(\rho+p)})^2\Big). \end{aligned} \tag{3.21}$$

3.1.2 Odd parity energy loss

Let us now consider the odd parity modes. Denoting $C_{AB} = e^i_{\langle A}e^j_{B\rangle}C_{ij}$ with $C_{ij} = \epsilon_{ikl}n_k T_{jl}$, we now have

$$C_{AB}C^{AB} = \perp^{ijkl}_{\text{tt}} C_{ij}C_{kl} = C^{\text{tt}}_{ij}C_{ij}, \tag{3.22}$$

$$\mathring{D}_A\mathring{D}^C C_{BC} = (4n^{(i}e^{j)}_{[A}e^k_{B]} - 2n_k e^i_{(A}e^j_{B)})\epsilon_{ikl}T_{jl}, \tag{3.23}$$

$$\mathring{D}_{(A}\mathring{D}^C C_{B)C} = -2C_{AB}, \tag{3.24}$$

$$C^{AB}\mathring{D}_A\mathring{D}^C C_{BC} = -2C^{\text{tt}}_{ij}C_{ij}. \tag{3.25}$$

Eq. (3.11) again simplifies to Eq. (3.16) and the energy loss formula (3.12) becomes again (3.17). Using the odd-parity moments from (3.5), (3.6), the integrand of (3.17) becomes,

$$G^{-2}\left(\partial_u\delta C^{\rm tt}_{ij}\partial_u\delta C_{ij}+3H^2\delta C^{\rm tt}_{ij}\delta C_{ij}-3H^4\delta C^{\rm tt}_{ij}\delta E_{ij}\right)=-8H^2\partial_u(\ddot{J}^{\rm tt}_{ij}\dot{J}_{ij})-12H^4\partial_u(J^{\rm tt}_{ij}\dot{J}_{ij})$$
$$+4\left(\dddot{J}^{\rm tt}_{ij}\dddot{J}_{ij}+5H^2\ddot{J}^{\rm tt}_{ij}\ddot{J}_{ij}+4H^4\dot{J}^{\rm tt}_{ij}\dot{J}_{ij}\right).$$

We can finally rewrite the flux-balance law of dilatations in the odd sector as

$$\partial_u Q^{\rm o}_D(u)=-\frac{G}{2}\oint_S\left(\dddot{J}^{\rm tt}_{ij}\dddot{J}_{ij}+5H^2\ddot{J}^{\rm tt}_{ij}\ddot{J}_{ij}+4H^4\dot{J}^{\rm tt}_{ij}\dot{J}_{ij}\right),\tag{3.26}$$

where the dilatation charge is identified as

$$Q^{\rm o}_D=\oint_S\left(\frac{1}{G}\delta^2M-GH^2\ddot{J}^{\rm tt}_{ij}\dot{J}_{ij}-\frac{3GH^4}{2}J^{\rm tt}_{ij}\dot{J}_{ij}\right).\tag{3.27}$$

Finally, the total dilation charge Q_D is the sum of the even and odd sectors:

$$Q_D=G\oint_S\left(\frac{1}{G}\delta M+\frac{1}{G}\delta^2M+\frac{9}{8}\zeta^{\rm tt}_{ij}\dot{\zeta}_{ij}+\frac{3}{2}\ddot{Q}^{(\rho+p)}_{ij}\zeta^{\rm tt}_{ij}-\frac{3}{2}H^2Q^{(\rho+p)}_{ij}\dot{\zeta}^{\rm tt}_{ij}-H^2\ddot{Q}^{(\rho+p)}_{ij}\dot{Q}^{(\rho+p){\rm tt}}_{ij}\right.$$
$$\left.-\frac{3}{2}H^4Q^{(\rho+p)}_{ij}\dot{Q}^{(\rho+p){\rm tt}}_{ij}-GH^2\ddot{J}^{\rm tt}_{ij}\dot{J}_{ij}-\frac{3GH^4}{2}J^{\rm tt}_{ij}\dot{J}_{ij}\right).\tag{3.28}$$

It obeys the dilatation flux balance law

$$\dot{Q}_D=\partial_uQ^{\rm e}_D+\partial_uQ^{\rm o}_D,\tag{3.29}$$

where the even and odd energy fluxes are given in Eqs. (3.18), (3.26).

3.2 Angular momentum loss

From (2.24), the quadratic part of the angular momentum loss formula becomes,

$$\dot{Q}^T_{L^i}=-\frac{3H^2}{8G}\oint_S\mathring{q}^{AM}\mathring{q}^{BN}\mathcal{L}_{\mathring{\xi}_{(i)}}\delta q_{MN}\,\delta J^{(\Lambda)}_{AB}.\tag{3.30}$$

Using Eq. (3.16), the angular momentum loss formula can be written as

$$\partial_u\left(Q^T_{L^i}-\frac{1}{8GH^2}\oint_S\left(\mathcal{L}_{\mathring{\xi}_{(i)}}\delta q_{MN}\right)\partial_u\delta C_{AB}\;\mathring{q}^{AM}\mathring{q}^{BN}\right)=-\frac{1}{8G}\oint_S\left[\left(\mathcal{L}_{\mathring{\xi}_{(i)}}\delta C_{MN}\right)\partial_u\delta C_{AB}\right.$$
$$\left.+3\left(\mathcal{L}_{\mathring{\xi}_{(i)}}\delta q_{MN}\right)\delta C_{AB}-3H^2\left(\mathcal{L}_{\mathring{\xi}_{(i)}}\delta q_{MN}\right)\delta E_{AB}\right]\mathring{q}^{AM}\mathring{q}^{BN}.\tag{3.31}$$

Let us first discuss the even parity modes. We note the identities $\epsilon^{AB}=e^A_ie^B_jn_k\epsilon_{ijk}$, $\oint_S 3n_in_j=\delta_{ij}$ and $\oint_S n_in_jn_kn_l=\frac{1}{15}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$. After performing the integral over the sphere, the right-hand side of Eq. (3.31) becomes

$$-\frac{1}{10G}\epsilon_i{}^{mn}\left(\delta C_{mk}\partial_u\delta C_{kn}+3\delta q_{mk}\delta C_{kn}-3H^2\delta q_{mk}\delta E_{kn}\right).\tag{3.32}$$

After some algebra we can rewrite the latter expression as

$$
\begin{aligned}
&-\tfrac{G}{10}\epsilon_i{}^{mn}\partial_u\Big(6\zeta_{km}\ddot{Q}^{(\rho+p)}_{kn}-6H^2\zeta_{km}Q^{(\rho+p)}_{kn}+6\dot{\zeta}_{kn}\dot{Q}^{(\rho+p)}_{km}-4H^2Q_{km}\ddot{Q}^{(\rho+p)}_{kn}\Big)\\
&\quad-\tfrac{2G}{5}\epsilon_i{}^{mn}\Big(\ddot{Q}^{(\rho+p)}_{km}\dddot{Q}^{(\rho+p)}_{kn}+5H^2\dot{Q}^{(\rho+p)}_{km}\ddot{Q}^{(\rho+p)}_{kn}+4H^4Q^{(\rho+p)}_{km}\dot{Q}^{(\rho+p)}_{kn}\Big).
\end{aligned}
\tag{3.33}
$$

Let us now consider the odd parity modes. After performing the integral over the sphere and reorganizing the terms, the right-hand side of Eq. (3.31) becomes

$$
\begin{aligned}
-\tfrac{G}{10}\epsilon_i{}^{mn}\Big[&\partial_u\Big(-4H^2J_{km}\ddot{J}_{kn}-12H^4\textstyle\int^u J_{mk}\dot{J}_{kn}\Big)\\
&+4\Big(\ddot{J}_{km}\dddot{J}_{kn}+5H^2\dot{J}_{km}\ddot{J}_{kn}+4H^4J_{km}\dot{J}_{kn}\Big)\Big].
\end{aligned}
\tag{3.34}
$$

Combining both even and odd parity sectors, the flux-balance law of angular momentum finally reads as

$$
\begin{aligned}
\partial_u Q_{L^i}=-\frac{2G}{5}\epsilon_{imn}\Big(&\ddot{Q}^{(\rho+p)}_{km}\dddot{Q}^{(\rho+p)}_{kn}+5H^2\dot{Q}^{(\rho+p)}_{km}\ddot{Q}^{(\rho+p)}_{kn}+4H^4Q^{(\rho+p)}_{km}\dot{Q}^{(\rho+p)}_{kn}\\
&+\ddot{J}_{km}\dddot{J}_{kn}+5H^2\dot{J}_{km}\ddot{J}_{kn}+4H^4J_{km}\dot{J}_{kn}\Big),
\end{aligned}
\tag{3.35}
$$

where the final angular momentum charge (in both parity sectors) is

$$
\begin{aligned}
Q_{L_i}=\,&Q^T_{L_i}-\frac{1}{8GH^2}\oint_S\Big(\mathcal{L}_{\mathring{\bar{\xi}}_{(i)}}\delta q_{MN}\Big)\,\partial_u\delta C_{AB}\;\mathring{q}^{AM}\mathring{q}^{BN}\\
&+\frac{G}{10}\epsilon_{imn}\Big(6\zeta_{km}\ddot{Q}^{(\rho+p)}_{kn}-6H^2\zeta_{km}Q^{(\rho+p)}_{kn}+6\dot{\zeta}_{kn}\dot{Q}^{(\rho+p)}_{km}-4H^2Q_{km}\ddot{Q}^{(\rho+p)}_{kn}\\
&-4H^2J_{km}\ddot{J}_{kn}-12H^4\int^u J_{km}\dot{J}_{kn}\Big).
\end{aligned}
\tag{3.36}
$$

3.3 Linear momentum and boost charge losses

From Eq. (2.24) with $\delta\xi^a=0$, the quadratic part of the linear momentum loss and cosmological boost formula becomes,

$$
\dot{Q}^T_{\bar{\xi}_{(i)}}=-\frac{3H^2}{8G}\oint_S\Big(H^2\bar{\xi}^u_{(i)}\delta C_{MN}+\mathcal{L}_{\mathring{\bar{\xi}}}\delta q_{MN}-\mathring{D}_C\bar{\xi}^C_{(i)}\delta q_{MN}\Big)\delta J^{(\Lambda)}_{AB}\mathring{q}^{AM}\mathring{q}^{BN},
\tag{3.37}
$$

where $\bar{\xi}^u_{(i)}=n_i\exp(\varepsilon Hu)$; $\bar{\xi}^A_{(i)}=-\varepsilon H\exp(\varepsilon Hu)\mathring{D}_A n_i$. For linear momenta $\varepsilon=1$ while for cosmological boosts $\varepsilon=-1$. We simplify the expression as

$$
\dot{Q}^T_{\bar{\xi}_{(i)}}=-\frac{3H^2}{8G}\oint_S\exp(\varepsilon Hu)\Big(H^2n_i\delta C_{MN}-\varepsilon H(\mathring{D}^C n_i)(\mathring{D}_C\delta q_{MN})\Big)\delta J^{(\Lambda)}_{AB}\mathring{q}^{AM}\mathring{q}^{BN}.
\tag{3.38}
$$

In order to perform the angular integrals, we note that for any symmetric tracefree tensors V_{AB}, W_{AB} that can be decomposed into even and odd parts as

$$V_{AB} = e^i_{\langle A} e^j_{B\rangle} (V^e_{ij} + \epsilon_{ikl} n_k V^o_{lj}), \tag{3.39}$$

$$W_{AB} = e^i_{\langle A} e^j_{B\rangle} (W^e_{ij} + \epsilon_{ikl} n_k W^o_{lj}), \tag{3.40}$$

we have the integrals

$$\oint_S n^i V_{AB} W^{AB} = -\frac{4}{15} \epsilon_{imt} \left(V^e_{ml} W^o_{tl} + W^e_{ml} V^o_{tl} \right), \tag{3.41}$$

$$\oint_S e^C_i \mathring{D}_C V_{AB} W^{AB} = -\frac{4}{15} \epsilon_{imt} \left(V^e_{ml} W^o_{tl} + W^e_{ml} V^o_{tl} \right). \tag{3.42}$$

Therefore,

$$\dot{Q}^T_{\bar{\xi}_{(i)}} = \frac{H^2}{10G} \exp(\varepsilon H u) \epsilon_{imt} \left[(H^2 \delta C^e_{ml} - \varepsilon H \delta q^e_{ml}) \delta J^{(\Lambda)o}_{tl} + (H^2 \delta C^o_{tl} - \varepsilon H \delta q^o_{tl}) \delta J^{(\Lambda)e}_{ml} \right]. \tag{3.43}$$

We now substitute $\delta J^{(\Lambda)}_{ij}$ using Eq. (3.16) and integrate by parts to obtain

$$\begin{aligned} \dot{Q}^{\text{int}}_{\bar{\xi}_{(i)}} = \frac{\exp(\varepsilon H u)}{10G} \epsilon_{imt} \Bigg[&\frac{1}{3} \left(2\delta N^e_{ml} \delta N^o_{tl} + 6H^2 \delta C^e_{ml} \delta C^o_{tl} - 3H^4 \delta C^e_{ml} \delta E^o_{tl} - 3H^4 \delta E^e_{ml} \delta C^o_{tl} \right) \\ &- \frac{\varepsilon H}{3} \delta q^e_{ml} \left(3\delta C^o_{lt} - 3H^2 \delta E^o_{tl} \right) - \frac{\varepsilon H}{3} \delta q^o_{lt} \left(3\delta C^e_{ml} - 3H^2 \delta E^e_{ml} \right) \\ &- \frac{1}{3} \left(\delta q^e_{ml} \delta N^o_{tl} + \delta N^e_{ml} \delta q^o_{lt} \right) \Bigg], \end{aligned} \tag{3.44}$$

where the charge (defined as an intermediate quantity for now, hence the superscript) is

$$Q^{\text{int}}_{\bar{\xi}_{(i)}} = Q^T_{\bar{\xi}_{(i)}} + \frac{1}{30G} \exp(\varepsilon H u) \epsilon_{imt} \left[\delta C^e_{ml} \delta N^o_{tl} + \delta N^e_{ml} \delta C^o_{tl} - \frac{\varepsilon}{H} (\delta q^e_{ml} \delta N^o_{lt} + \delta N^e_{ml} \delta q^o_{lt}) \right]. \tag{3.45}$$

Substituting all Bondi fields in terms of multipolar moments, we rewrite the flux-balance law as

$$\dot{Q}_{\bar{\xi}_{(i)}} = -\frac{4G e^{\varepsilon H u}}{15} \epsilon_{imt} (3\varepsilon H^3 \zeta_{ml} \dot{J}_{lt} + 2H^6 Q_{ml} J_{lt} + 6H^4 \dot{Q}_{ml} \dot{J}_{lt} + 6H^2 \ddot{Q}_{ml} \ddot{J}_{lt} + \dddot{Q}_{ml} \dddot{J}_{lt}), \tag{3.46}$$

where the final charge is defined as

$$\begin{aligned} Q_{\bar{\xi}_{(i)}} = Q^{\text{int}}_{\bar{\xi}_{(i)}} + \frac{e^{\varepsilon H u}}{15} \epsilon_{imt} \Bigg[&6H^2 \partial_u (\dot{Q}_{ml} \dot{J}_{lt}) + 8H^4 \partial_u (Q_{ml} J_{lt}) - 4\varepsilon H^5 \partial_u \left(Q_{ml} \int^u J_{lt} \right) \\ - 5\dot{\zeta}_{ml} \ddot{J}_{lt} + 3H^2 (\dot{\zeta}_{ml} J_{lt} - \zeta_{ml} \dot{J}_{lt}) &+ 8\varepsilon H \dot{\zeta}_{ml} \dot{J}_{lt} - \left(2H^4 \ddot{Q}_{ml} - 6H^6 Q_{ml} - 3\varepsilon H^3 \dot{\zeta}_{ml} \right) \int^u J_{lt} \Bigg]. \end{aligned} \tag{3.47}$$

3.4 Flat spacetime limit

Let us now prove that we recover the standard Poincaré flux-balance laws in the quadrupolar approximation upon taking the flat spacetime limit. In the quadrupolar truncation, these laws read as [70][4] (original derivations can be found in [71–75])

$$\partial_u \mathcal{E} = -\frac{G}{5} \dddot{I}^{\text{PM}}_{ij} \dddot{I}^{\text{PM}}_{ij} - \frac{16G}{45} \dddot{J}^{\text{PM}}_{ij} \dddot{J}^{\text{PM}}_{ij}, \tag{3.48}$$

$$\partial_u \mathcal{P}_i = -\frac{16G}{45} \epsilon_{ijk} \dddot{I}^{\text{PM}}_{jl} \dddot{J}^{\text{PM}}_{kl}, \tag{3.49}$$

$$\partial_u \mathcal{J}_i = -\frac{2G}{5} \epsilon_{ijk} \ddot{I}^{\text{PM}}_{jl} \dddot{I}^{\text{PM}}_{kl} - \frac{32G}{45} \epsilon_{ijk} \ddot{J}^{\text{PM}}_{jl} \dddot{J}^{\text{PM}}_{kl}, \tag{3.50}$$

$$\partial_u \mathcal{N}_i = \mathcal{P}_i. \tag{3.51}$$

where the flat tensors I^{PM}_{ij}, I^{PM}_{ij} are symmetric and tracefree. The charges are expressed as the flat limit of the $SO(1,4)$ generators as $\mathcal{E} = \mathcal{Q}_D$, $\mathcal{P}_i = Q_{P^i}$, $\mathcal{J}_i = Q_{L^i}$ and $\mathcal{K}_i = \lim_{H\mapsto 0} \frac{1}{2H}(Q_{K^i} - Q_{P^i})$ with $\mathcal{N}_i = \mathcal{K}_i + u\mathcal{P}_i$.

The flat limit of Eqs. (3.21)-(3.26)-(3.35) reproduce the flat flux-balance laws of energy and angular momentum with the dictionary

$$I^{\text{PM}}_{ij} = Q^{(\rho+p)}_{ij} - \frac{1}{3}\delta_{ij} Q^{(\rho+p)}_{kk}, \tag{3.52}$$

$$J^{\text{PM}}_{ij} = \frac{3}{4} J_{ij}. \tag{3.53}$$

The flat limit of Eq. (3.46) reproduces the flat flux-balance law for the momentum. For Lorentz boosts we have to extract the $O(H)$ term in Eq. (3.44). Note that $\delta q_{ab} = O(H^2)$, therefore, only the $\exp(\varepsilon H u)\delta N^e_{ml} \delta N^o_{tl}$ term in Eq. (3.44) will contribute to the boost charge. From Eq. (3.46) we obtain

$$\dot{\mathcal{K}}_i = \lim_{H\mapsto 0} \frac{1}{2H}(\dot{Q}_{K^i} - \dot{Q}_{P^i}) = \frac{4Gu}{15} \epsilon_{iml} \dddot{Q}_{mj} \dddot{J}_{jl}. \tag{3.54}$$

We can finally shift the charge as

$$\mathcal{N}_i = \mathcal{K}_i - \frac{4Gu}{15} \int^u \epsilon_{iml} (\dddot{Q}_{mj} \dddot{J}_{jl}) \tag{3.55}$$

to obtain the correct flux-balance law (3.51).

4 Comparison with the literature

The first proposed quadrupolar formula for the energy flux for even parity modes was derived in [7], see also [17]. In our earlier work [57], we pointed out that there is an inconsistent

[4] Note that $\mathcal{G}_i$ in [70] is $\mathcal{N}_i$ in our conventions, see Section 9 of [68].

quadrupolar truncation in [7,17], which leads to incorrect terms proportional to $Q_{ij}^{(p)}$ even in the even parity sector. Performing the quadrupolar truncation using the methods of [57], let us now upgrade the flux formula of [7] to include the $Q_{ij}^{(p)}$ terms. We first note the algebraic identity

$$\dddot{Q}_{ij}^{(\rho+p)}\dddot{Q}_{ij}^{\text{tt}(\rho+p)} + 5H^2\ddot{Q}_{ij}^{(\rho+p)}\ddot{Q}_{ij}^{\text{tt}(\rho+p)} + 4H^4\dot{Q}_{ij}^{(\rho+p)}\dot{Q}_{ij}^{\text{tt}(\rho+p)} = \mathcal{R}_{ij}^{\text{e}}\mathcal{R}_{ij}^{\text{e tt}} - \partial_u \delta E^{\text{e}},$$

where we defined

$$\mathcal{R}_{ij}^{\text{e}} = \dddot{Q}_{ij}^{(\rho+p)} - 3H\ddot{Q}_{ij}^{(\rho+p)} + 2H^2\dot{Q}_{ij}^{(\rho+p)}, \tag{4.1}$$

$$\delta E^{\text{e}} = -3H\ddot{Q}_{ij}^{(\rho+p)}\ddot{Q}_{ij}^{\text{tt}(\rho+p)} + 4H^2\dot{Q}_{ij}^{(\rho+p)}\ddot{Q}_{ij}^{\text{tt}(\rho+p)} - 6H^3\dot{Q}_{ij}^{(\rho+p)}\dot{Q}_{ij}^{\text{tt}(\rho+p)}. \tag{4.2}$$

The same identity holds in the odd sector with $Q_{ij}^{(\rho+p)}$ replaced by J_{ij}, namely,

$$\dddot{J}_{ij}\dddot{J}_{ij}^{\text{tt}} + 5H^2\ddot{J}_{ij}\ddot{J}_{ij}^{\text{tt}} + 4H^4\dot{J}_{ij}\dot{J}_{ij}^{\text{tt}} = \mathcal{R}_{ij}^{\text{o}}\mathcal{R}_{ij}^{\text{o tt}} - \partial_u \delta E^{\text{o}},$$

where we defined

$$\mathcal{R}_{ij}^{\text{o}} = \dddot{J}_{ij} - 3H\ddot{J}_{ij} + 2H^2\dot{J}_{ij}, \tag{4.3}$$

$$\delta E^{\text{o}} = -3H\ddot{J}_{ij}\ddot{J}_{ij}^{\text{tt}} + 4H^2\dot{J}_{ij}\ddot{J}_{ij}^{\text{tt}} - 6H^3\dot{J}_{ij}\dot{J}_{ij}^{\text{tt}}. \tag{4.4}$$

The energy flux balance law (3.29) can therefore be equivalently defined as

$$\begin{aligned}\partial_u\left(Q_D + \frac{G}{2}\oint_S \delta(E^{\text{e}} + E^{\text{o}})\right) &= -\frac{G}{2}\oint_S\left(\mathcal{R}_{ij}^{\text{e}}\mathcal{R}_{ij}^{\text{e tt}} + \mathcal{R}_{ij}^{\text{o}}\mathcal{R}_{ij}^{\text{o tt}}\right)\\ &= -\frac{G}{5}\left(\mathcal{R}_{ij}^{\text{e}}\mathcal{R}_{ij}^{\text{e}} + \mathcal{R}_{ij}^{\text{o}}\mathcal{R}_{ij}^{\text{o}}\right).\end{aligned} \tag{4.5}$$

The right-hand side is negative definite. It depends upon the combination $Q_{ij}^{(\rho+p)}$, consistently with the Bondi fields (3.4), (3.5), (3.6), (3.8). The energy flux (4.5) is our upgrade of the formula proposed in [7][5].

We can also upgrade the angular momentum flux proposed in [7]. Again we note the identity

$$\ddot{Q}_{km}^{(\rho+p)}\dddot{Q}_{kn}^{(\rho+p)} + 5H^2\dot{Q}_{km}^{(\rho+p)}\ddot{Q}_{kn}^{(\rho+p)} + 4H^4 Q_{km}^{(\rho+p)}\dot{Q}_{kn}^{(\rho+p)} = \mathcal{L}_{km}^{\text{e}}\mathcal{R}_{kn}^{\text{e}} - \partial_u \delta L^{\text{e}}, \tag{4.6}$$

[5]Note that the power radiated quadrupolar formula in [7] was given in terms of the transverse-traceless (TT) part of $\mathcal{R}_{ij}$ field. Let us recall that any symmetric rank-2 tensor can be decomposed as $Q_{ij} = \frac{1}{3}\delta_{ij}\delta^{kl}Q_{kl} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)B + \partial_i B_j^T + \partial_j B_i^T + Q_{ij}^{TT}$ where Q_{ij}^{TT} is the TT part of Q_{ij}, which satisfies $\partial^i Q_{ij}^{TT} = 0 = \delta^{ij}Q_{ij}^{TT}$. The two notions, tt projection and TT part of a rank-2 symmetric tensor are therefore generically distinct. However, this distinction disappears after integrating over the two sphere at infinity [60,76]. Note in addition that in [7], $\dot{Q}_{ij}$ denotes the Lie derivative with respect to dilatation Killing vector field (T) in conformal coordinates of de Sitter, i.e. $\dot{Q}_{ij} = \mathcal{L}_T Q_{ij}$. After conversion to Bondi frame, it becomes $\mathcal{L}_T Q_{ij} = (\partial_u - 2H)Q_{ij}$.

where we defined

$$\begin{aligned}\mathcal{L}^{\rm e}_{km} &= \ddot{Q}^{(\rho+p)}_{km} - 3H\dot{Q}^{(\rho+p)}_{km} + 2H^2 Q^{(\rho+p)}_{km}, & (4.7)\\ \delta L^{\rm e} &= -3H\dot{Q}^{(\rho+p)}_{km}\ddot{Q}^{(\rho+p)}_{kn} + 2H^2 Q^{(\rho+p)}_{km}\ddot{Q}^{(\rho+p)}_{kn} + 2H^2\dot{Q}^{(\rho+p)}_{km}\dot{Q}^{(\rho+p)}_{kn} - 6H^3 Q^{(\rho+p)}_{km}\dot{Q}^{(\rho+p)}_{kn}. & (4.8)\end{aligned}$$

The same identity holds in the odd sector with $Q^{(\rho+p)}_{ij}$ replaced by J_{ij},

$$\ddot{J}_{km}\dddot{J}_{kn} + 5H^2\dot{J}_{km}\ddot{J}_{kn} + 4H^4 J_{km}\dot{J}_{kn} = \mathcal{L}^{\rm o}_{km}\mathcal{R}^{\rm o}_{kn} - \partial_u \delta L^{\rm o}_T, \tag{4.9}$$

where we defined

$$\begin{aligned}\mathcal{L}^{\rm o}_{km} &= \ddot{J}_{km} - 3H\dot{J}_{km} + 2H^2 J_{km}, & (4.10)\\ \delta L^{\rm o} &= -3H\dot{J}_{km}\ddot{J}_{kn} + 2H^2 J_{km}\ddot{J}_{kn} + 2H^2\dot{J}_{km}\dot{J}_{kn} - 6H^3 J_{km}\dot{J}_{kn}. & (4.11)\end{aligned}$$

Hence the angular momentum flux balance law (3.35) can also be equivalently defined as

$$\partial_u\left(Q_{L^i} - \frac{2G}{5}\epsilon_{imn}(\delta L^{\rm e} + \delta L^{\rm o}_T)\right) = -\frac{2G}{5}\epsilon_{imn}(\mathcal{L}^{\rm e}_{km}\mathcal{R}^{\rm e}_{kn} + \mathcal{L}^{o}_{km}\mathcal{R}^{\rm o}_{kn}). \tag{4.12}$$

This is our upgrade of the angular momentum flux proposed in [7].

In anti-de Sitter spacetime with Dirichlet boundary conditions, the $SO(3,2)$ conserved quantities are defined using the holographic stress-energy tensor [77]. More precisely, they are defined from the analytic continuation to negative cosmological constant of the charge formula (2.10). We started from this charge formula as initial ansatz in the case of positive cosmological constant in this paper. When rewritten in Bondi variables, the charge can be expressed in terms of the Λ-corrected Bondi mass aspects $M^{(\Lambda)}$ and Bondi angular momentum aspects $N^{(\Lambda)}_A$ while the flux is proportional to the field $J^{(\Lambda)}_{AB}$ [36, 39], see Eq. (2.19). However, for leaky boundary conditions, where the boundary metric is allowed to vary, as it is the case here, the holographic stress-tensor does not allow by itself to define the quasi-conserved quantities. Indeed, computing the flux of $M^{(\Lambda)}$ (2.16) using Eq. (3.17) shows that the "would be energy flux" associated with the Bondi mass is not manifestly negative definite. It cannot therefore lead to a proposal for a definition of energy in the presence of a positive cosmological constant.

Bonga, Bunster, and Pérez (BBP) [55] studied several aspects of gravitational waves in de Sitter for both linear and the non-linear theory. For the study of linearised solutions they focused on solving the homogeneous Einstein equations in a partial wave expansion to the gravitational field. One of the results of their paper is the derivation of an energy flux formula for $l = 2$ mode of linearized perturbations around de Sitter. Starting from the energy loss formula in a different boundary gauge fixing introduced in [38, 44], they obtained a quadratic energy flux formula for linearized pertubations in de Sitter background. In [58], an energy flux formula for de Sitter Teukolsky waves has also been obtained. With the correct identification of the variables, the energy flux formula for BBP quadrupolar waves exactly matches with that of de Sitter Teukolsky waves. In this section, we wish to compare our quadrupole formula with the energy flux formula for BBP quadrupolar waves in de Sitter [55].

For $l = 2$ even parity linearized solutions of [55], the energy flux is given by

$$E^{(E)} = -\frac{3}{8\pi G}\sum_{m=-2}^{m=+2}\int_{u=-\infty}^{u=+\infty}\left[|\dddot{a}_m|^2 + 5H^2|\ddot{a}_m|^2 + 4H^4|\dot{a}_m|^2\right]du, \tag{4.13}$$

where $a_m = a_m(u)$ is a function of retarded time u. The function $a_m(u)$ can also be understood as the coefficient of the $l = 2$ even parity solution of homogeneous linearized Einstein's equation. The relationship between the variables a_m used in [55] and our even parity quadrupole moments are obtained from [58], where a relation between the BBP wave solutions and $l = 2$ mode quadrupolar truncated solutions [57] have been achieved via de Sitter Teukolsky waves. Using Eqs. (3.55), (5.79) of [58], we obtain

$$a_m = \frac{H^2}{2}q_m - \frac{1}{6}\ddot{q}_m, \tag{4.14}$$

where q_m can be written in terms of $l = 2$ spherical harmonics $Y^{2m}(\theta,\phi) = \mathcal{Y}^{2m}_{ij}n_i n_j$,

$$q_m = \frac{1}{H^4}\frac{24\pi}{15}(\mathcal{Y}^{2m}_{ij})^*\zeta_{ij}(u). \tag{4.15}$$

An explicit expression for $\mathcal{Y}^{2m}_{ij}$ can be found in Eq. (3.218) of [78]. Therefore using (4.14), and the identity (3.7), we have

$$a_m = \frac{8\pi G}{15}(\mathcal{Y}^{2m}_{ij})^* Q^{(\rho+p)}_{ij}. \tag{4.16}$$

Therefore,

$$\sum_{m=-2}^{m=+2}|\dddot{a}_m|^2 = \frac{8\pi G^2}{15}\left(\dddot{Q}^{(\rho+p)}_{ij}\dddot{Q}^{(\rho+p)}_{ij} - \frac{1}{3}\dddot{Q}^{(\rho+p)}_{ij}\dddot{Q}^{(\rho+p)}_{kl}\delta_{ij}\delta_{kl}\right), \tag{4.17}$$

$$= \frac{8\pi G^2}{15}\left(\dddot{Q}^{(\rho+p)}_{ij} - \frac{1}{3}\dddot{Q}^{(\rho+p)}\delta_{ij}\right)^2, \tag{4.18}$$

where we have used the identity,

$$\sum_{m=-2}^{m=+2}(\mathcal{Y}^{2m}_{ij})^*(\mathcal{Y}^{2m}_{kl}) = \frac{15}{16\pi}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}). \tag{4.19}$$

Hence the energy flux in terms of even-parity quadrupole moments becomes

$$E^{(E)} = -\frac{G}{5}\int_{-\infty}^{+\infty}du\Big((\dddot{Q}^{(\rho+p)}_{ij} - \frac{1}{3}\dddot{Q}^{(\rho+p)}\delta_{ij})^2 + 5H^2(\ddot{Q}^{(\rho+p)}_{ij} - \frac{1}{3}\ddot{Q}^{(\rho+p)}\delta_{ij})^2 + 4H^4(\dot{Q}^{(\rho+p)}_{ij} - \frac{1}{3}\dot{Q}^{(\rho+p)}\delta_{ij})^2\Big). \tag{4.20}$$

This can also be written as

$$E^{(E)} = -\frac{G}{8\pi}\int_{-\infty}^{+\infty} du \int_{S^2} d\Omega \bigg(\dddot{Q}_{ij}^{(\rho+p)} \dddot{Q}_{ij}^{(\rho+p)\text{tt}} + 5H^2 \ddot{Q}_{ij}^{(\rho+p)} \ddot{Q}_{ij}^{(\rho+p)\text{tt}} + 4H^4 \dot{Q}_{ij}^{(\rho+p)} \dot{Q}_{ij}^{(\rho+p)\text{tt}} \bigg). \tag{4.21}$$

Considering now the $l = 2$ odd parity linearized solution, the energy flux is given by

$$E^{(B)} = -\frac{3}{8\pi G}\sum_{m=-2}^{m=+2}\int_{u=-\infty}^{u=+\infty}\Big[|\dddot{b}_m|^2 + 5H^2|\ddot{b}_m|^2 + 4H^4|\dot{b}_m|^2\Big] du, \tag{4.22}$$

where $b_m = b_m(u)$ is a function of retarded time u. The relationship between the variable b_m used in [55] and our odd parity $l = 2$ mode quadrupole moments are given by [58],

$$b_m = -\frac{8\pi G}{15}(\mathcal{Y}_{ij}^{2m})^* J_{ij}. \tag{4.23}$$

Therefore using (4.19), we have

$$\sum_{m=-2}^{m=+2} |\dddot{b}_m|^2 \;=\; \frac{8\pi G^2}{15}\dddot{J}_{ij}\dddot{J}_{ij}. \tag{4.24}$$

Note that by construction J_{ij} is symmetric and traceless. Hence the energy flux in terms of odd-parity quadrupole moments becomes

$$E^{(B)} = -\frac{G}{5}\int_{-\infty}^{+\infty} du \bigg(\dddot{J}_{ij}\dddot{J}_{ij} + 5H^2 \ddot{J}_{ij}\ddot{J}_{ij} + 4H^4 \dot{J}_{ij}\dot{J}_{ij} \bigg). \tag{4.25}$$

This can also be written as

$$E^{(B)} = -\frac{G}{8\pi}\int_{-\infty}^{+\infty} du \int_{S^2} d\Omega \bigg(\dddot{J}_{ij}\dddot{J}_{ij}^{\text{tt}} + 5H^2 \ddot{J}_{ij}\ddot{J}_{ij}^{\text{tt}} + 4H^4 \dot{J}_{ij}\dot{J}_{ij}^{\text{tt}} \bigg). \tag{4.26}$$

We have therefore demonstrated that the energy flux formulae (3.18), (3.26) exactly coincide with the ones derived in [55]. This is remarkable because these formulae have been obtained in different boundary gauges. In [55], the angular components g_{AB} of the boundary metric were kept fixed, while here we kept fixed the determinant of g_{AB} and the mixed components g_{uA}. This does not prove that the energy flux formulae are gauge invariant, but they are at least invariant under an interesting class of gauge transformations.

Given that the quadratic energy loss formula derived in [38] and [41] agree with the one derived in [55] as shown in [55] we also proved the equivalence of the energy flux formulae (3.18), (3.26) with [38, 41].

5 Conclusion

We derived the full set of flux-balance laws for linear (even and odd) quadrupolar perturbations associated with background $SO(1,4)$ symmetries. These laws reduce to the standard

Poincaré flux-balance laws in the flat limit. Even though the flux-balance laws have been determined, their split into the time derivative of the charge (left-hand side) and the flux (right-hand side) is not unique. We warn the reader that even if the symmetry generators form a $SO(1,4)$ algebra, we did not prove that the charges form a $SO(1,4)$ algebra under a suitable bracket. As proven in [39], the charges Q^T_ξ defined from the holographic stress-energy tensor represent the Λ-BMS algebra under the adjusted bracket [69]. However, since the charges are here defined with a shift (2.25), one would need to study the impact of this shift on the representation theorem. This remains an open question.

After analysis, we obtained two distinct proposals for the energy loss. Both proposals admit the standard flat limit and are negative definite at quadratic level. They differ by terms proportional to the Hubble radius. The first proposal is given in Eq. (4.5). It corresponds to an upgrade of the formula first proposed in [7] after correcting the even parity sector following [57], and after including the odd parity contribution. The second proposal is given as Eq. (3.29). After integrating over $\mathscr{I}^+$, it matches with the total energy loss formula derived in [55,58]. We further complemented the previous literature [55,58] by providing the corresponding definition of energy and energy flux on a fixed cut (i.e. fixed u) of $\mathscr{I}^+$ in terms of multipolar moments. The existence of two proposals for the energy loss indicates that fundamental requirements are missing to uniquely single out one expression. One such requirement is gauge invariance. The Bondi framework is based upon one gauge choice and even though it was successful in the asymptotically flat regime, we expect that a gauge-independent framework is required to settle this issue. The comparison of proposed definitions of energy loss based on Weyl scalars and Killing spinors [9,13,23] with the current framework is left for future work.

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