A perturbation result for the Webster scalar curvature problem on the CR sphere

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Abstract

We consider the problem of prescribing the Webster scalar curvature on the unit sphere of $\mathbb{C}^{n+1}$. Using a perturbation method, we obtain existence results for curvatures close to a positive constant and satisfying an assumption of Bahri-Coron type.

1 Introduction

Let $\mathbb{S}^{2n+1}$ be the unit sphere of $\mathbb{C}^{n+1}$ and let us denote by θ_0 the standard contact form of the CR manifold $\mathbb{S}^{2n+1}$. Given a smooth function $\bar{R}$ on $\mathbb{S}^{2n+1}$, the Webster scalar curvature problem on $\mathbb{S}^{2n+1}$ consists in finding a contact form θ conformal to θ_0 such that the corresponding Webster scalar curvature is $\bar{R}$. This problem is equivalent to solve the semilinear equation

$$b_n \Delta_{\theta_0} v(\zeta) + \bar{R}_0 v(\zeta) = \bar{R}(\zeta) v(\zeta)^{b_n - 1}, \qquad \zeta \in \mathbb{S}^{2n+1}, \tag{1}$$

where $b_n = 2 + \frac{2}{n}$, Δ_{θ_0} is the sublaplacian on $(\mathbb{S}^{2n+1}, \theta_0)$ and $\bar{R}_0 = \frac{n(n+1)}{2}$ is the Webster scalar curvature of $(\mathbb{S}^{2n+1}, \theta_0)$. If v is a positive solution to (1), then $(\mathbb{S}^{2n+1}, \theta = v^{\frac{2}{n}} \theta_0)$ has Webster scalar curvature R. Moreover, using the CR equivalence F (given by the Cayley transform, see definition (9) below) between $\mathbb{S}^{2n+1}$ minus a point and the Heisenberg group $\mathbb{H}^n$, equation (1) becomes, up to an uninfluent constant,

$$-\Delta_{\mathbb{H}^n} u(\xi) = R(\xi) u(\xi)^{\frac{Q+2}{Q-2}}, \qquad \xi \in \mathbb{H}^n. \tag{2}$$

Here $\Delta_{\mathbb{H}^n}$ is the Heisenberg sublaplacian, $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$ and R corresponds to $\bar{R}$ in the equivalence F. We refer to [25] for a more detailed presentation of the problem.

Indeed in the papers [25], [26], [27], Jerison and Lee extensively studied the Yamabe problem on CR manifolds (see also the recent papers by Gamara and Yacoub [20], [21] which treat the cases left open by Jerison and Lee). On the contrary, at the authors' knowledge, very few results have been established on the Webster scalar curvature problem (see [10], [31], [37]), in spite of the wide interest on its Riemannian analogue (see [3], [6], [12], [13], [14], [15], [24], [32], [33], [35] and references therein). In recent years there has been a growing interest on equations of the same kind of (2) and various existence and non-existence results inspired by this topic have been established by several authors (see [22], [16], [8], [7], [8], [8], [29], [30], [36], [17]). On the other hand, these results are quite different in nature from the one we shall prove in this paper and do not apply directly to the Webster scalar curvature problem, but are principally related to the study of the Dirichlet problem for semilinear equations on bounded domains of $\mathbb{H}^n$. Yet we have to say that non-existence results for (2) can be obtained using the Pohozaev-type identities of [22] under certain conditions on R. In particular it turns out that a positive solution u to (2) in the Sobolev space $S_0^1(\mathbb{H}^n)$ (with the notation of section 2) satisfies the following identity

$$\int_{\mathbb{H}^n} \langle (z, 2t), \nabla R(z,t) \rangle u(z,t)^{\frac{2Q}{Q-2}} dz dt = 0$$

provided the integral is convergent and R is bounded and smooth enough (see [22] and [37]). This implies that there are no such solutions if $\langle (z, 2t), \nabla R(z,t) \rangle$ does not change sign in $\mathbb{H}^n$ and R is not constant.

In this note we shall give an existence result for positive solutions to equation (2), under suitable assumptions on R (see Theorem 1 below). In particular we will assume that R is of the form

$$R(\xi) = 1 + \varepsilon K(\xi)$$

for a small ε and a smooth Morse function K satisfying conditions (3)-(4) below. We remark that our hypotheses on R are very different from the ones in [10], [31], [37] where R is assumed to satisfy suitable decaying conditions at infinity. In particular in [10] it is required an estimate of the type $R_1(\rho)\Delta_{\mathbb{H}^n}\rho \leq R \leq R_2(\rho)\Delta_{\mathbb{H}^n}\rho$ (ρ is the homogeneous norm on $\mathbb{H}^n$ defined in (6) below) involving the degenerate term $\Delta_{\mathbb{H}^n}\rho$, which allows to "radialise" the problem and to apply ODE methods.

In this note we give a contribution in the same direction as in the papers [3], [15], [34] concerning the Riemannian case. As a matter of fact, the main tool in our proof is an abstract perturbation method due to Ambrosetti and Badiale [1] which applies in our situation as well as in [3], [34]. In this paper we want to give a first example of application of this powerful and versatile method to a non elliptic context.

The technique, see Theorem 3, consists in the reduction of the problem to the study of a finite dimensional functional. In this study, see section 3, the positive solutions $\omega_{\lambda,\xi}$ of the unperturbed problem (7) below play a prominent role. In particular, in Lemma 5 we prove some uniqueness results for the linearization of (7) at $\omega_{\lambda,\xi}$ and in section 4 we study how the bubbles $\omega_{\lambda,\xi}$ transform under a Kelvin-type inversion.

Our main result is the following Theorem.

Theorem 1 *Let $K = \bar{K} \circ F^{-1} : \mathbb{H}^n \to \mathbb{R}$ be the composition of a smooth function $\bar{K}$ on $\mathbb{S}^{2n+1}$ with the CR equivalence F^{-1} (see (15)). Suppose that K is a Morse function satisfying the non-degeneracy assumption*

$$\Delta_{\mathbb{H}^n} K(\xi) \neq 0 \text{ if } K'(\xi) = 0, \tag{3}$$

and the inequality

$$\sum_{K'(\xi)=0, \Delta_{\mathbb{H}^n} K(\xi)<0} (-1)^{m(K,\xi)} \neq -1, \tag{4}$$

where $m(K, \xi)$ denotes the Morse index of K at ξ. Assume also that the south pole is not a critical point of $\bar{K}$. Then for $|\varepsilon|$ sufficiently small and for $R(\xi) = 1 + \varepsilon K(\xi)$, problem (2) possesses a positive solution.

The condition on $\bar{K}$ at the south pole of $\mathbb{S}^{2n+1}$ can be always achieved with a unitary transformation in $\mathbb{C}^{n+1}$ and does not affect the generality of the result. We also remark that the existence result in Theorem 1 holds true for every dimension, as for the perturbative results in [15] and in [3] regarding the Riemannian case. It is expectable that, in the non-perturbative case, the hypotheses for an existence theorem should strongly depend on the dimension n.

Finally we would like to observe that the solution found in Theorem 1 satisfies the estimate (5) below, by means of the following result, whose proof is essentially contained in [29] (see the proof of Theorem 1.1 of that paper).

Proposition 2 *Let $R \in L^\infty(\mathbb{H}^n)$. If u is a non-negative solution to equation (2) in the Sobolev space $S^1_0(\mathbb{H}^n)$, then there exists a positive constant c such that*

$$u(\xi) \leq c\,\omega(\xi), \qquad \xi \in \mathbb{H}^n, \tag{5}$$

where ω denotes the solution (defined in (8)) of the unperturbed problem (7) below.

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2 Notation

We denote by $\xi = (z,t) = (x+iy,t) \equiv (x,y,t)$ the points of $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$. The group law on $\mathbb{H}^n$ is given by $\xi \circ \xi' = (z+z', t+t'+2\mathrm{Im}(z\bar{z'}))$. We also denote by $\tau_\xi(\xi') = \xi \circ \xi'$ the left translations, by $\delta_\lambda(\xi) = (\lambda z, \lambda^2 t)$, $\lambda > 0$, the natural dilations, by $Q = 2n+2$ the homogeneous dimension and by

$$\rho(\xi) = (|z|^4 + t^2)^{1/4} \tag{6}$$

the homogeneous norm on $\mathbb{H}^n$. The sublaplacian on $\mathbb{H}^n$ is the differential operator

$$\Delta_{\mathbb{H}^n} = \textstyle\sum_{j=1}^n ({X_j}^2 + {Y_j}^2),$$

where $X_j = \partial_{x_j} + 2y_j\partial_t$, $Y_j = \partial_{y_j} - 2x_j\partial_t$. We will work in the Folland-Stein Sobolev space $S_0^1(\mathbb{H}^n)$, which is defined as the completion of $C_0^\infty(\mathbb{H}^n)$ with respect to the norm

$$\|u\|^2_{S_0^1(\mathbb{H}^n)} = \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2.$$

Here $\nabla_{\mathbb{H}^n} = (X_1, ..., X_n, Y_1, ..., Y_n)$ denotes the so-called subelliptic gradient on $\mathbb{H}^n$. A remarkable result of Jerison and Lee [27] states that all the positive solutions to the problem

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = u^{Q^*-1} \\ u \in S_0^1(\mathbb{H}^n) \end{cases} \tag{7}$$

($Q^* = \frac{2Q}{Q-2}$) are in the form $u = \omega_{\lambda,\xi}$ for some $\lambda > 0$, $\xi \in \mathbb{H}^n$, where $\omega_{\lambda,\xi} = \lambda^{\frac{2-Q}{2}} \omega \circ \delta_{\lambda^{-1}} \circ \tau_{\xi^{-1}}$ and

$$\omega(z,t) = c_0(t^2 + (1+|z|^2)^2)^{(2-Q)/4}, \tag{8}$$

being c_0 a suitable positive constant. We finally give the expression of the CR equivalence $F : \mathbb{S}^{2n+1} \setminus \{(0,...,0,-1)\} \to \mathbb{H}^n$,

$$F(\zeta_1, ..., \zeta_{n+1}) = \big(\frac{\zeta_1}{1+\zeta_{n+1}}, ..., \frac{\zeta_n}{1+\zeta_{n+1}}, \mathrm{Re}(i\frac{1-\zeta_{n+1}}{1+\zeta_{n+1}})\big), \tag{9}$$

that we shall use in section 4.

3 The abstract perturbation method

In this section we recall some abstract perturbation results, variational in nature, developed in [1] (see also [2]), which we will use to obtain our existence results.

Let E be an Hilbert space, and consider a family of functionals $f_\varepsilon : E \to \mathbb{R}$ of the form

$$f_\varepsilon(u) = f_0(u) - \varepsilon\, G(u), \qquad u \in E. \tag{10}$$

Here f_0 is the so-called *unperturbed functional*, and G is the perturbation of f_0; both f_0 and G are assumed to be of class C^2 on E. Moreover the functional f_0 satisfies the following conditions:

(i) f_0 possesses a finite-dimensional manifold Z of critical points; we assume that Z is parameterized by a function $\alpha : A \to Z$, being A an open subset of $\mathbb{R}^d$, $d \geq 1$;

(ii) for all $z \in Z$, $f_0''(z)$ is of the form $Id - \mathcal{K}_z$, where $\mathcal{K}_z$ is a compact operator;

(iii) for all $z \in Z$ there holds $T_z Z = Ker f_0''(z)$.

We define $\Gamma : A \to \mathbb{R}$ as $\Gamma = G \circ \alpha$.

The following Theorem is proved in [1], see in particular also Theorem 2.1 and Remark 2.2 in [3].

Theorem 3 *Assume that properties $(i) - (iii)$ above hold true, and suppose there exists an open set $\Omega \subseteq A$ such that $\Gamma' \neq 0$ on $\partial\Omega$ and*

$$deg(\Gamma', \Omega, 0) \neq 0. \tag{11}$$

Then for $|\varepsilon|$ sufficiently small there exists a critical point u_ε of f_ε.

Remark 4 *(a) The inclusion $T_z Z \subseteq Ker f_0''(z)$ is always true: (iii) is a non-degeneracy condition which allows to apply the Implicit Function Theorem.*

(b) The solution u_ε is close *to Z in the sense that there exists $z_\varepsilon \in Z$, $\alpha^{-1}(z_\varepsilon) \in \Omega$, such that $\|u_\varepsilon - z_\varepsilon\| \leq C|\varepsilon|$, for some constant C depending on Ω, f_0 and G.*

The above abstract Theorem will be applied here to the following setting:

$$E = S_0^1(\mathbb{H}^n), \qquad f_0(u) = \frac{1}{2}\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 - \frac{1}{Q^*}\int_{\mathbb{H}^n} |u|^{Q^*}, \qquad G(u) = \frac{1}{Q^*}\int_{\mathbb{H}^n} K\,|u|^{Q^*}.$$

In the reminder of this section we will show that conditions $(i) - (iii)$ above hold true. In the next sections, under hypotheses (3)-(4), we will show that condition (11) is satisfied for some suitable set Ω.

Let Z be defined as

$$Z = \{\omega_{\lambda,\xi} \,:\, \lambda \in \mathbb{R}_+, \xi \in \mathbb{H}^n\}.$$

All the functions in Z are critical points of f_0 and Z is clearly a $(2n+2)$-dimensional manifold in E parameterized by the map $\alpha : A = \mathbb{R}_+ \times \mathbb{H}^n \to E$, $\alpha(\lambda, \xi) = \omega_{\lambda,\xi}$.

About property (ii), one has

$$(f_0''(\omega_{\lambda,\xi})[v], w) = \int_{\mathbb{H}^n} (\nabla_{\mathbb{H}^n} v, \nabla_{\mathbb{H}^n} w) - (Q^* - 1)\int_{\mathbb{H}^n} \omega_{\lambda,\xi}^{Q^*-2} v\, w, \qquad v, w \in S_0^1(\mathbb{H}^n).$$

Using the above formula, it is standard to check that $f_0''(\omega_{\lambda,\xi})$ is of the form *Identity - Compact* (as in the Euclidean case, see [3]). Hence property (ii) follows.

By the invariance with respect to translations and dilations, it is sufficient to verify *(iii)* for $(\lambda, \xi) = (1, 0)$. This is the content of the following lemma.

Lemma 5 *A function $u \in S_0^1(\mathbb{H}^n)$ is a solution of the following equation*

$$-\Delta_{\mathbb{H}^n} u = (Q^* - 1)\,\omega^{Q^*-2} u \qquad \text{in } \mathbb{H}^n \tag{12}$$

if and only if there exist some coefficients $\mu \in \mathbb{R}$ and $\nu \in \mathbb{R}^{2n+1}$ such that

$$u = \mu \frac{\partial \omega_{\lambda,\xi}}{\partial \lambda}\Big|_{(\lambda,\xi)=(1,0)} + \sum_{i=1}^{2n+1} \nu_i \frac{\partial \omega_{\lambda,\xi}}{\partial \xi_i}\Big|_{(\lambda,\xi)=(1,0)}. \tag{13}$$

PROOF. Since $T_\omega Z \subseteq Ker f_0''(\omega)$, a function as in (13) satisfies (12). Let us prove the opposite implication. It is sufficient to prove that the vector space of the solutions to (12) has dimension $2n+2$. Let us consider the linear isometry $\iota : S^1(\mathbb{S}^{2n+1}) \to S_0^1(\mathbb{H}^n)$ defined (up to some constant) by

$$\iota(v)(\xi) = \omega(\xi) v(F^{-1}(\xi)), \qquad v \in S^1(\mathbb{S}^{2n+1}), \quad \xi \in \mathbb{H}^n.$$

Here $S^1(\mathbb{S}^{2n+1})$ denotes the completion of $C^\infty(\mathbb{S}^{2n+1})$ with respect to the norm

$$\|v\|^2_{S^1(\mathbb{S}^{2n+1})} = \int_{\mathbb{S}^{2n+1}} (b_n |dv|^2_{\theta_0} + \bar{R}_0 v^2) \theta_0 \wedge d\theta_0^n$$

(see [25]). By means of this isometry, a function $u \in S_0^1(\mathbb{H}^n)$ is a solution to (12) if and only if the function $v = \iota^{-1} u$ solves the linear equation

$$-\Delta_\theta v = \mu_n v, \qquad \text{in } \mathbb{S}^{2n+1}, \tag{14}$$

for a suitable eigenvalue μ_n. The study of the eigenvalues of the operator $-\Delta_\theta$ on $\mathbb{S}^{2n+1}$ has been performed by G. B. Folland in [18]. In particular, it turns out that the eigenspace corresponding to the eigenvalue μ_n is $(2n+2)$-dimensional and is spanned by the functions $\{\mathrm{Re}\,\zeta_j, \mathrm{Im}\,\zeta_j\}_{j=1,\dots,n+1}$ restricted to $\mathbb{S}^{2n+1}$. Indeed a direct computation shows that (up to some constant)

$$\iota(\mathrm{Re}\,\zeta_j) = \frac{\partial \omega_{\lambda,\xi}}{\partial x_j}|_{(\lambda,\xi)=(1,0)}, \qquad \iota(\mathrm{Im}\,\zeta_j) = \frac{\partial \omega_{\lambda,\xi}}{\partial y_j}|_{(\lambda,\xi)=(1,0)}, \qquad j = 1, \dots n;$$

$$\iota(\mathrm{Re}\,\zeta_{n+1}) = \frac{\partial \omega_{\lambda,\xi}}{\partial \lambda}|_{(\lambda,\xi)=(1,0)}, \qquad \iota(\mathrm{Im}\,\zeta_{n+1}) = \frac{\partial \omega_{\lambda,\xi}}{\partial t}|_{(\lambda,\xi)=(1,0)}$$

(recall that $\xi = (x, y, t)$). This concludes the proof. ■

4 The Kelvin transform in $\mathbb{H}^n$

In this section we study some features of a Kelvin-type inversion on $\mathbb{H}^n$ (see [28]). We first recall the expression of the inverse F^{-1} of the CR equivalence F defined in (9):

$$F^{-1}(\xi) = F^{-1}(z,t) = \left(\frac{2iz}{t + i(1+|z|^2)}, \frac{-t + i(1-|z|^2)}{t + i(1+|z|^2)} \right). \tag{15}$$

We now define the Kelvin transform $\mathcal{K} : \mathbb{H}^n \setminus \{0\} \to \mathbb{H}^n \setminus \{0\}$ to be the map conjugate to the inversion in $\mathbb{C}^{n+1}$ of $\mathbb{S}^{2n+1}$ through the equivalence F:

$$\mathcal{K} = F \circ (-Id_{\mathbb{C}^{n+1}}) \circ F^{-1}. \tag{16}$$

With some simple computations, one finds that $\mathcal{K}$ has the following expression

$$\mathcal{K}(z,t) = \left(\frac{-iz}{t + i|z|^2}, -\frac{t}{\rho^4} \right). \tag{17}$$

Hereafter we simply denote $\rho = \rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$. Our aim is to study the transformation of the bubbles $\omega_{\lambda,\xi}$ under this Kelvin-type transform. First of all, let us compute the Jacobian determinant $J\mathcal{K}$ of the function $\mathcal{K}$. Since $\mathcal{K}$ is obtained as the composition (16), the determinant of $\mathcal{K}$ is the product of the determinants of the above three maps. The first one is given by $(\omega(z,t))^{Q^*}$, the second one is just 1, while the third is $(\omega(\mathcal{K}(z,t)))^{-Q^*}$. In conclusion we get

$$
\begin{aligned}
(18)\qquad J\mathcal{K}(z,t) &= \left(\frac{\left(1+\frac{|z|^2}{t^2+|z|^4}\right)^2+\frac{t^2}{\rho^8}}{(1+|z|^2)^2+t^2}\right)^{\frac{Q}{2}} = \left(\frac{\frac{1}{\rho^8}\left[(\rho^4+|z|^2)^2+t^2\right]}{(1+|z|^2)^2+t^2}\right)^{\frac{Q}{2}}\\
&= \left(\frac{\frac{1}{\rho^4}(\rho^4+2|z|^2+1)}{(1+|z|^2)^2+t^2}\right)^{\frac{Q}{2}} = \frac{1}{\rho^{2Q}}.
\end{aligned}
$$

Hence the natural transformation induced by $\mathcal{K}$ of a function $u:\mathbb{H}^n\to\mathbb{R}$ is given by

$$
u\mapsto\mathcal{A}(u);\qquad\qquad \mathcal{A}(u)(z,t)=\frac{1}{\rho^{Q-2}}u(\mathcal{K}(z,t)).
$$

Through the isometry ι, $\mathcal{A}$ is conjugated to the application

$$
(19)\qquad S^1(\mathbb{S}^{2n+1})\ni v\mapsto(\iota^{-1}\circ\mathcal{A}\circ\iota)(v)=v(-\cdot).
$$

We are interested in the transformation under $\mathcal{A}$ of the function ω_{λ,ξ_0}, which has the following expression

$$
(20)\qquad \omega_{\lambda,\xi_0}(\xi)=c_0\left[\lambda^2\left(\lambda^{-4}(t-t_0+2\mathrm{Im}(z\overline{z}_0))^2+(1+\lambda^{-2}|z-z_0|^2)^2\right)\right]^{-\frac{Q-2}{4}}.
$$

Applying the Kelvin transformation we find

$$
\begin{aligned}
(21)\qquad \mathcal{A}(\omega_{\lambda,\xi_0})(\xi)=c_0\Bigg[\lambda^2\rho^4\Bigg(&\lambda^{-4}\left(-\frac{t}{\rho^4}-t_0+2\mathrm{Im}\left(\frac{-iz}{t+i|z|^2}\overline{z}_0\right)\right)^2\\
&+\left(1+\lambda^{-2}\left|\frac{-iz}{t+i|z|^2}-z_0\right|^2\right)^2\Bigg)\Bigg]^{-\frac{Q-2}{4}}.
\end{aligned}
$$

We have

$$
\mathrm{Im}\left(\frac{-iz}{t+i|z|^2}\overline{z}_0\right)=-\frac{t(xx_0+yy_0)+|z|^2(x_0y-y_0x)}{\rho^4}
$$

and

$$
\frac{iz}{t+i|z|^2}+z_0=\frac{(x_0t^2+x_0|z|^4-yt+x|z|^2)+i(y_0t^2+y_0|z|^4+xt+y|z|^2)}{\rho^4}.
$$

Using the last two equations and some elementary computations, the expression of $\mathcal{A}(\omega_{\lambda,\xi_0})$ in (21) becomes

$$
\begin{aligned}
(22)\qquad \mathcal{A}(\omega_{\lambda,\xi_0})(z,t)=c_0\Bigg[&\lambda^{-2}\frac{\left[t+t_0\rho^4+2t(xx_0+yy_0)+2|z|^2(x_0y-y_0x)\right]^2}{\rho^4}\\
&+\lambda^2\frac{\left(\rho^4+\lambda^{-2}\left[|z|^2+\rho^4|z_0|^2+2t(xy_0-yx_0)+2|z|^2(xx_0+yy_0)\right]\right)^2}{\rho^4}\Bigg]^{-\frac{Q-2}{4}}.
\end{aligned}
$$

By means of the uniqueness result of Jerison and Lee [27], the above function must be of the form

$$
(23)\qquad \mathcal{A}(\omega_{\lambda,\xi_0})(z,t)=\omega_{\tilde{\lambda},\tilde{\xi_0}}(z,t)=c_0\left[\tilde{\lambda}^2\left(\tilde{\lambda}^{-4}(t-\tilde{t}_0+2\mathrm{Im}(z\overline{\tilde{z}}_0))^2+(1+\tilde{\lambda}^{-2}|z-\tilde{z}_0|^2)^2\right)\right]^{-\frac{Q-2}{4}},
$$

for some suitable $\tilde{\lambda}$, $\tilde{\xi_0}$. Indeed a function $u \in S^1_0(\mathbb{H}^n)$ is a solution to problem (7) iff $v = \iota^{-1}(u)$ (or equivalently $v(-\cdot)$) is a solution to $\Delta_{\theta_0} v + c_1 v = c_2 v^{Q^*-1}$ in $\mathbb{S}^{2n+1}$ (for suitable constants c_1, c_2). Recalling (19), we get that u is a solution to (7) iff $\mathcal{A}(u)$ is.

In order to find the explicit expression of $\tilde{\lambda}$, $\tilde{t}_0$ and $\tilde{z}_0$, we proceed in the following way. Choosing $z = 0$ and letting $t \to 0$ in the expressions (22)-(23) above we get

$$\frac{1}{\tilde{\lambda}^2}\left[\tilde{t}_0^2 + (\tilde{\lambda}^2 + |\tilde{z}_0|^2)^2\right] = \frac{1}{\lambda^2},$$

which is

$$\lambda^2 = \frac{\tilde{\lambda}^2}{\tilde{t}_0^2 + (\tilde{\lambda}^2 + |\tilde{z}_0|^2)^2}.$$

Since the transformation is involutive, we have also

$$\tilde{\lambda}^2 = \frac{\lambda^2}{t_0^2 + (\lambda^2 + |z_0|^2)^2}.$$

The expressions in the square brackets in (22)-(23) are forth order polynomials in ξ (we consider t of degree 2). Comparing the coefficients of t, ty and tx we find, with simple computations,

$$-\frac{2}{\tilde{\lambda}^2}\tilde{t}_0 = \frac{2}{\lambda^2}t_0; \qquad \frac{\tilde{x}_0}{\tilde{\lambda}^2} = \frac{t_0 y_0 - (\lambda^2 + |z_0|^2)x_0}{\lambda^2}; \qquad \frac{\tilde{y}_0}{\tilde{\lambda}^2} = -\frac{t_0 x_0 + (\lambda^2 + |z_0|^2)y_0}{\lambda^2}.$$

From the last four equations we deduce

$$\tilde{\lambda} = \frac{\lambda}{\sqrt{t_0^2 + (\lambda^2 + |z_0|^2)^2}}; \qquad \tilde{t}_0 = -\frac{t_0}{t_0^2 + (\lambda^2 + |z_0|^2)^2}; \tag{24}$$

$$\tilde{x}_0 = \frac{t_0 y_0 - (\lambda^2 + |z_0|^2)x_0}{t_0^2 + (\lambda^2 + |z_0|^2)^2}; \qquad \tilde{y}_0 = -\frac{t_0 x_0 + (\lambda^2 + |z_0|^2)y_0}{t_0^2 + (\lambda^2 + |z_0|^2)^2}. \tag{25}$$

In this way we have proved the following Lemma.

Lemma 6 *For $\xi_0 \in \mathbb{H}^n$ and $\lambda > 0$ there holds*

$$\mathcal{A}(\omega_{\lambda,\xi_0}) = \omega_{\tilde{\lambda},\tilde{\xi_0}},$$

where $\tilde{\lambda}$ and $\tilde{\xi_0}$ are given in (24), (25).

5 Study of the function Γ

By our definition in Section 3, the explicit expression of $\Gamma(\lambda, \xi)$ is the following

$$\Gamma(\lambda, \xi) = \frac{1}{Q^*}\int_{\mathbb{H}^n} K\omega_{\lambda,\xi}^{Q^*}, \qquad \lambda > 0,\ \xi \in \mathbb{H}^n. \tag{26}$$

Remark 7 *We recall that K is taken to be equal to $\bar{K} \circ F^{-1}$ for some $\bar{K} \in C^\infty(\mathbb{S}^{2n+1})$. Hence K turns out to be bounded on $\mathbb{H}^n$ together with any derivative.*

The main features in the study of the function Γ are given in the following Proposition.

Proposition 8 *The function Γ is of class C^2 in A and can be extended with C^1 regularity to the hyperplane $\{\lambda = 0\}$ by setting*

$$\Gamma(0,\xi) = c_0\, K(\xi), \tag{27}$$

where $c_0 = \frac{1}{Q^*}\int_{\mathbb{H}^n} \omega^{Q^*}$. Moreover, for any compact set $\Sigma \subseteq \mathbb{H}^n$, there exists a positive constant C_Σ such that

$$\left|\frac{\partial\Gamma}{\partial\lambda}(\lambda,\xi) - c_1 \Delta_{\mathbb{H}^n} K(\xi)\,\lambda\right| \le C_\Sigma \lambda^2 \qquad \textit{for all } \xi\in\Sigma \textit{ and for all } \lambda > 0, \tag{28}$$

where $c_1 = \frac{1}{2nQ^*}\int_{\mathbb{H}^n} |z|^2 \omega(z,t)^{Q^*}\, dz\, dt$. In addition it turns out that

$$\nabla^2_\xi \Gamma(\lambda,\xi) \to c_0 \nabla^2 K(\xi) \quad \textit{as } \lambda \to 0 \quad \textit{uniformly on the compact sets of } \mathbb{H}^n. \tag{29}$$

Proof. The first and the last part of the statement can be proved using the dominated convergence theorem observing that, using a change of variables,

$$\Gamma(\lambda,\xi_0) = \frac{1}{Q^*}\int_{\mathbb{H}^n} K(\xi_0 \circ \delta_\lambda \xi)\, \omega(\xi)^{Q^*}\, d\xi \tag{30}$$

and recalling Remark 7. Let us now prove (28). Let us fix $\xi_0 \in \Sigma$ and let K_{ξ_0} denote the function $K \circ \tau_{\xi_0}$. Differentiating (30) with respect to λ we get

$$\frac{\partial\Gamma}{\partial\lambda}(\lambda,\xi_0) = \frac{1}{Q^*}\int_{\mathbb{H}^n} \langle (K_{\xi_0})'(\delta_\lambda\xi), (z, 2\,\lambda\, t)\rangle \omega(\xi)^{Q^*}\, d\xi. \tag{31}$$

Using the smoothness assumption on K and the boundedness of K', we deduce from a Taylor expansion

$$\left|\frac{\partial K_{\xi_0}}{\partial \xi_j}(\eta) - \frac{\partial K_{\xi_0}}{\partial \xi_j}(0) - \sum_{l=1}^{2n} \frac{\partial^2 K_{\xi_0}}{\partial\xi_j\,\partial\xi_l}(0)\,\eta_l\right| \le C_\Sigma\, \rho(\eta)^2, \qquad j = 1,\dots,2n; \tag{32}$$

$$\left|\frac{\partial K_{\xi_0}}{\partial \xi_{2n+1}}(\eta) - \frac{\partial K_{\xi_0}}{\partial \xi_{2n+1}}(0)\right| \le C_\Sigma\, \rho(\eta), \tag{33}$$

for all $\eta \in \mathbb{H}^n$ and independently of $\xi_0 \in \Sigma$.

From the invariance properties of the operator $\Delta_{\mathbb{H}^n}$ with respect to the left-translations, and using its explicit expression in the origin, we obtain

$$\Delta_{\mathbb{H}^n} K(\xi_0) = \Delta_{\mathbb{H}^n} K_{\xi_0}(0) = \sum_{l=1}^{2n} \frac{\partial^2 K_{\xi_0}}{\partial \xi_l^2}(0).$$

By oddness, we have

$$\int_{\mathbb{H}^n} \xi_j \omega^{Q^*}(\xi) d\xi = 0, \qquad j = 1,\dots,2n+1; \tag{34}$$

and

$$\int_{\mathbb{H}^n} \xi_j\, \xi_l \omega^{Q^*}(\xi) d\xi = 0, \qquad j,l = 1,\dots,2n, j \ne l. \tag{35}$$

We explicitly remark that the functions $\rho(\cdot)^k\, \omega(\cdot)^{Q^*}$ belong to $L^1(\mathbb{H}^n)$ for $k < Q$, and hence the above integrals are well-defined.

From (34) and (35) we deduce that

(36)
$$c_1\Delta_{\mathbb{H}^n}K(\xi_0)\,\lambda=\frac{1}{Q^*}\int_{\mathbb{H}^n}\left[\sum_{j=1}^{2n}\left(\frac{\partial K_{\xi_0}}{\partial\xi_j}(0)+\sum_{l=1}^{2n}\frac{\partial^2K_{\xi_0}}{\partial\xi_j\,\partial\xi_l}(0)\lambda\,\xi_l\right)\xi_j+2\,\lambda\,\xi_{2n+1}\frac{\partial K_{\xi_0}}{\partial\xi_{2n+1}}(0)\right]\omega^{Q^*}(\xi)d\xi.$$

Subtracting (36) from (31), using the estimates (32), (33) for $\eta=\delta_\lambda\xi$, and taking into account that $\rho(\delta_\lambda\xi)=\lambda\,\rho(\xi)$, we obtain

$$\begin{aligned}\left|\frac{\partial\Gamma}{\partial\lambda}(\lambda,\xi_0)-c_1\Delta_{\mathbb{H}^n}K(\xi_0)\,\lambda\right| &\le C_\Sigma\,\lambda^2\int_{\mathbb{H}^n}\left(\sum_{j=1}^{2n}|\xi_j|\,\rho(\xi)^2+|\xi_{2n+1}|\,\rho(\xi)\right)\omega^{Q^*}(\xi)d\xi\\ &\le C_\Sigma\lambda^2\int_{\mathbb{H}^n}\rho(\xi)^3\omega^{Q^*}(\xi)d\xi\le C_\Sigma\,\lambda^2.\end{aligned}$$

This concludes the proof. ■

Using Proposition 8 and some properties of the Kelvin transform introduced in section 4, we can have information about the degree of Γ' on some suitably *large* subsets of A.

Proposition 9 *Suppose that the function K satisfies the hypotheses of Theorem 1. Then there exists an open set $\Omega\subseteq A$ such that $\Gamma'\neq 0$ on $\partial\Omega$ and*

$$deg(\Gamma',\Omega,0)=\sum_{K'(\xi)=0,\Delta_{\mathbb{H}^n}K(\xi)<0}(-1)^{m(K,\xi)}+1.$$

PROOF. Let $\mathcal{B}_s\subseteq\mathbb{R}_+\times\mathbb{H}^n$ denote the Euclidean ball

$$\mathcal{B}_s=\left\{(\lambda,\xi)\,:\,|(\lambda,\xi)-(s,0)|^2<\left(s-\frac{1}{s}\right)^2\right\}$$

with center $(s,0)$ and radius $s-\frac{1}{s}$. We note that $\{\mathcal{B}_s\}_s$ invade the whole $\mathbb{H}^n\times\mathbb{R}_+$ as $s\to+\infty$. We claim that choosing $\Omega=\mathcal{B}_s$, with s sufficiently large, the properties in Proposition 9 will be satisfied. We define also the Euclidean ball in $\mathbb{H}^n$ to be

$$A_r=\{\xi\in\mathbb{H}^n\,:\,|\xi|\le r\}\,.$$

Since we have assumed that the south pole of $\mathbb{S}^{2n+1}$ is not a critical point of $\bar K=K\circ F$, if $r>0$ is chosen large enough, the function K does not possess critical points in $\mathbb{H}^n\setminus A_r$. We are going now to apply Proposition 8 with $\Sigma=A_{2r}$.

Let $\xi_1,\dots,\xi_l$ denote the critical points of K. By the non-degeneracy assumption (3) there exist small numbers $\sigma\in(0,r)$ and $\delta>0$ such that

(37)
$$\Delta_{\mathbb{H}^n}K(\xi)>\delta,\qquad\text{for all }\xi\in\cup_{i=1}^l B_\sigma(\xi_i),$$

where $B_\sigma(\xi_i)$ is the Euclidean ball of radius σ and center ξ_i. Fixing this number σ and applying Proposition 8 we find, for s sufficiently large

(38)
$$\Gamma'\neq 0\text{ on }\partial\mathcal{B}_s\cap\left((0,\sigma)\times\cup_{i=1}^l B_\sigma(\xi_i)\right).$$

On the other hand, Γ can be extended to $\{\lambda=0\}$ as a function of class C^1 by (27), and moreover $K'\neq 0$ on $A_{2r}\setminus\{\xi_1,...,\xi_l\}$. As a consequence we find

(39)
$$\Gamma'\neq 0\text{ on }\partial\mathcal{B}_s\cap\left((0,\sigma)\times(A_{2r}\setminus\cup_{i=1}^l B_\sigma(\xi_i))\right).$$

From equations (38) and (39) we deduce

$$\Gamma' \neq 0 \text{ on } \partial\mathcal{B}_s \cap ((0,\sigma) \times A_{2r}), \tag{40}$$

provided r and s are sufficiently large. Let us prove now that $\Gamma' \neq 0$ on the reminder of $\partial\mathcal{B}_s$. In order to do this, it is convenient to use the Kelvin transform introduced above, see formula (17). We recall that the determinant of $\mathcal{K}$ is given by (18).

Let $\tilde{K} : \mathbb{H}^n \to \mathbb{R}$ be defined by $\tilde{K} = \bar{K} \circ (-Id_{\mathbb{C}^{n+1}}) \circ F^{-1}$. Clearly $\tilde{K}_{|\mathbb{H}^n \setminus \{0\}} = K \circ \mathcal{K}$. Since we are assuming that the south pole of $\mathbb{S}^{2n+1}$ is not critical for $\bar{K}$, the north pole is not critical for $\bar{K} \circ (-Id_{\mathbb{C}^{n+1}})$, and hence $0 \in \mathbb{H}^n$ is also not critical for $\tilde{K}$. Using the change of variables induced by the transformation $\mathcal{K}$ and taking into account Lemma 6 we get

$$\Gamma(\lambda,\xi) = \frac{1}{Q^*}\int_{\mathbb{H}^n} K\omega_{\lambda,\xi}^{Q^*} = \frac{1}{Q^*}\int_{\mathbb{H}^n} \tilde{K}\mathcal{A}(\omega_{\lambda,\xi})^{Q^*} = \frac{1}{Q^*}\int_{\mathbb{H}^n} \tilde{K}\omega_{\tilde{\lambda},\tilde{\xi}}^{Q^*} =: \tilde{\Gamma}(\tilde{\lambda},\tilde{\xi}), \tag{41}$$

where the parameters $\tilde{\lambda}$ and $\tilde{\xi}$ are given in (24) and (25), and where $\tilde{\Gamma}$ denotes the counterpart of Γ when considering $\tilde{K}$ instead of K. We note in particular that, since the map $(\lambda,\xi) \to (\tilde{\lambda},\tilde{\xi})$ is a diffeomorphism, it is $\Gamma'(\lambda,\xi) = 0$ if and only if $\tilde{\Gamma}'(\tilde{\lambda},\tilde{\xi}) = 0$. Since $0 \in \mathbb{H}^n$ is not critical for $\tilde{K}$, and since also $\tilde{\Gamma}$ is of class C^1 up to the boundary of A (see Proposition 8), $\tilde{\Gamma}' \neq 0$ in some neighborhood of $(0,0) \in [0,+\infty[\times\mathbb{H}^n$. This implies that $\Gamma' \neq 0$ in some *neighborhood of infinity* (see (24)-(25)) and hence

$$\Gamma' \neq 0 \text{ on } \partial\mathcal{B}_s \setminus \big((0,\sigma) \times A_{2r}\big), \tag{42}$$

provided r and s are chosen to be sufficiently large. Hence, choosing $\Omega = \mathcal{B}_s$ with s sufficiently large, (40) and (42) imply $\Gamma' \neq 0$ on $\partial\Omega$.

The computation of the degree is standard. From property (29) and from the fact that K is a Morse function, it follows that for s large $\Gamma|_{\partial\mathcal{B}_s}$ possesses only non-degenerate critical points, which are in one-to-one correspondence with those of K. Moreover from (28), the critical points of $\Gamma|_{\partial\mathcal{B}_s}$ with entering gradient are in correspondence with the critical points of K with $\Delta_{\mathbb{H}^n} K > 0$. Then the degree formula follows from [23]. This concludes the proof. ■

Proof of Theorem 1. To prove the existence of a weak solution u_ε, it is sufficient to apply Theorem 3 and Proposition 9. By Remark 4 and by the positivity of the functions $\omega_{\lambda,\xi}$, it follows that $\|u_\varepsilon^-\|_{Q^*} \to 0$ as $\varepsilon \to 0$, where u_ε^- denotes the negative part of u_ε. On the other hand, testing equation (2) on u_ε^- and using the Sobolev-type embedding $S_0^1(\mathbb{H}^n) \hookrightarrow L^{Q^*}(\mathbb{H}^n)$ (see e.g. [27]), one finds

$$\|u_\varepsilon^-\|_{Q^*}^2 \leq C\|u_\varepsilon^-\|_{S_0^1(\mathbb{H}^n)}^2 = C\int_{\mathbb{H}^n} R|u_\varepsilon^-|^{Q^*} \leq C\|u_\varepsilon^-\|_{Q^*}^{Q^*}.$$

Hence, since $\|u_\varepsilon^-\|_{Q^*}$ is small, it follows that $u_\varepsilon^- \equiv 0$ for $|\varepsilon|$ small. Therefore $u_\varepsilon \geq 0$ and $u_\varepsilon \not\equiv 0$, again by Remark 4. Moreover, using a standard regularization technique, based on the results of Folland and Stein [19], it is easy to recognize that u_ε is also smooth. Then the positivity of u_ε follows from the strong minimum principle for the solutions of $\Delta_{\mathbb{H}^n} u - Cu \leq 0$, see e.g. [9]. ■

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