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*On Roth type conditions, duality
and central Birkhoff sums for i.e.m.*

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ON ROTH TYPE CONDITIONS, DUALITY AND CENTRAL BIRKHOFF SUMS FOR I.E.M.

by

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Abstract. — We introduce two Diophantine conditions on rotation numbers of interval exchange maps (i.e.m.) and translation surfaces: the *absolute Roth type condition* is a weakening of the notion of Roth type i.e.m., while the *dual Roth type condition* is a condition on the *backward* rotation number of a translation surface. We show that results on the cohomological equation previously proved in [38] for restricted Roth type i.e.m. (on the solvability under finitely many obstructions and the regularity of the solutions) can be extended to restricted *absolute* Roth type i.e.m. Under the dual Roth type condition, we associate to a class of functions with *subpolynomial* deviations of ergodic averages (corresponding to relative homology classes) *distributional* limit shapes, which are constructed in a similar way to the *limit shapes* of Birkhoff sums associated in [36] to functions which correspond to positive Lyapunov exponents.

In August 2010 J.C.Y. wrote a first text (12 pages) containing the results obtained on dual Birkhoff sums with the title “On Birkhoff sums for i.e.m.” (a further version of the same text was written in March 2011). This constitutes the heart of Section 4 of this paper. Notable progress was made during a visit of S.M. and J.C.Y. to C.U. in Bristol in October 2014, when the homological interpretation emerged and J.C.Y. wrote a new version of this draft introducing the notion of KZ-hyperbolic translation surfaces (this homological interpretation is included in Section 6). Motivated by the discussions in Bristol, the notion of absolute Roth type i.e.m. emerged and was developed during further meetings in of S.M. with J.C.Y. in Paris in December 2014, March 2015 and September 2015. This notion, together with the completeness of backward rotation numbers, is the object of the text “Absolute Roth Type and Backward Rotation Number” (14 pages) written by J.C.Y. in April 2015. This text formed the basis for Section 3 and parts of it are here included as two Appendixes (Appendix A and Appendix B). The final version of this manuscript was prepared by S.M. and C.U. who are fully and solely responsible for any mistake or imprecision.

Jean-Christophe discussed publicly the results obtained in our collaboration in his talk “Problèmes de petits diviseurs pour les échanges d’intervalles” on May 20, 2015 given at the “Journée Surfaces plates” held at the Institut Galilée of the University of Paris 13 (as Carlos Matheus informed us) and also during his talk “Diophantine conditions for interval exchange map” on September 28, 2015 given in Oxford as part of the Workshop *Geometry and Dynamics on Moduli Spaces* at the 2015 Clay Research conference.

Résumé (Sur les conditions de type Roth, la dualité et les sommes de Birkhoff centrées pour les échange d'intervalles)

Nous introduisons deux conditions diophantiennes pour les nombres de rotation des échange d'intervalles (i.e.m) et des surfaces de translation: la condition absolue de type Roth est un affaiblissement de la notion de type Roth i.e.m, tandis que la condition duale de type Roth est une condition sur le nombre de rotation rétrograde d'une surface de translation. Nous montrons que les résultats sur l'équation cohomologique prouvés précédemment dans [MY] pour les i.e.m. de type Roth restreint (sur la solvabilité en supposant un nombre fini d'obstructions et la régularité des solutions) peuvent être étendues aux i.e.m. de type Roth absolu restreint. Sous la condition duale de type Roth, nous associons des formes limites (limit shapes) distributionnelles à une classe de fonctions avec des déviations sous-polynomiales des moyennes ergodiques (correspondantes aux classes d'homologie relatives), qui sont construites de manière similaire aux formes limites des sommes de Birkhoff associées dans [MMY2] aux fonctions qui correspondent aux exposants de Lyapunov positifs.

1. Introduction

Diophantine conditions play a central role in the study of the dynamics of rotations of the circle, diffeomorphisms of the circle and more in general area-preserving flows on tori. These conditions, which convey information on how well a rotation number α can be approximated by rationals (and hence on small divisors problems), are often expressed in terms of growth rates for the entries of the continued fraction expansion of α .

In the study of dynamics on surfaces, one often requires similar Diophantine conditions on interval exchange maps and linear flows. Passing from genus one to higher genus, a natural generalization of linear flows on tori is indeed provided by linear flows on *translation surfaces* (see § 2.2 for definitions); *interval exchange maps*, which will be shortened throughout this paper by *i.e.m.*, are piecewise isometries which, analogously to rotations in genus one, arise as Poincaré maps of linear flows (see § 2.1 for the definition). An algorithm which plays in this context an analogous role to the continued fraction expansion is the *Rauzy-Veech induction*, first introduced [40, 45] and used since then as an essential tool for proving many results on the ergodic and spectral properties of i.e.m., flows on surfaces and rational billiards, see for example [46, 53, 35, 25, 2, 1, 43, 44, 37, 12, 11, 41, 29].

Diophantine conditions on i.e.m. can be prescribed imposing conditions on the behaviour of the Rauzy-Veech induction matrices and the related (extended) Kontsevich-Zorich cocycle. In this spirit, in their work [35] on the cohomological equation for i.e.m., S. M. , P. Moussa and J.-C. Y., define a Diophantine condition on i.e.m. which generalize the notion of Roth type rotation and under which they show that the cohomological equation can be solved under finitely many obstructions (see Theorem 3.14 for a generalized statement). After Forni's celebrated paper [22] on the cohomological

equation associated to linear flows on surfaces of higher genus, this was the first result giving an explicit diophantine condition sufficient to guarantee the existence of a solution to the cohomological equation. The i.e.m. which satisfy this condition are called *Roth type* i.e.m. (or i.e.m. of Roth type) and have full measure (as shown in [35], see also [34] for the sketch of a simpler proof based on the results in [3]). A reformulation and a strengthening of the Roth type condition (namely, *restricted* Roth type) were then defined in [37] also for *generalized* i.e.m. and provide the Diophantine condition under which a linearization result is proved in [37].

Two new Diophantine conditions related to the Roth type condition for i.e.m. are introduced in this paper, for the applications that we explain in § 1.1 and § 1.2 below. More precisely (in § 3) we introduce the notion of *absolute Roth type* i.e.m., thus defining a class of i.e.m. which include and generalize Roth type i.e.m. but for which the results mentioned before on linearization and the cohomological equation still hold. We then define the notion of *dual Roth type* for a translation surface (or more precisely, for its suspension data or backward rotation number, see § 4). This is a Diophantine condition which is *dual* to the Roth type condition for i.e.m. in a sense which will be made precise further on (see § 4 and § 6). Let us now explain the motivation for introducing these conditions and the results which we proved assuming them, starting with the notion of dual Roth type.

1.1. Dual Roth type and distributional limit shapes. — Results on deviations of ergodic averages and ergodic integrals are a central part of the study of i.e.m. and translation flows, see for example the works by Zorich [53, 52], Forni [23], Avila-Viana [4] and Bufetov [11] among others. Deviations of ergodic averages, i.e., the oscillations of the Birkhoff sum $S_n f(x) := \sum_{k=0}^{n-1} f(T^k x)$ of a function $f : [0, 1] \rightarrow \mathbb{R}$ of zero average over the orbit of (typical) point $x \in [0, 1]$ under a i.e.m. $T : [0, 1] \rightarrow [0, 1]$ are of polynomial nature. In [53] Zorich shows for example that for a typical i.e.m. T and any mean-zero function f constant on the intervals exchanged by T , we have $S_n f(x) = O(x^\nu)$ for some power exponent $\nu < 1$; more precisely for a full measure set of T there exists $\nu = \nu(f)$ such that for all $x \in [0, 1]$,

$$(1.1) \quad \limsup_{n \rightarrow +\infty} \frac{\log S_n f(x)}{\log n} = \nu.$$

Remark also that if f is a *coboundary* for T with bounded transfer function, i.e., $f = g \circ T - T$ where the *transfer function* $g : [0, 1] \rightarrow \mathbb{R}$ is bounded, then Birkhoff sums $S_n f(x)$ are uniformly bounded (and in particular $\nu = 0$). The power exponent can be understood in terms of Lyapunov exponents of (a suitable acceleration) of the Kontsevich-Zorich cocycle associated to the Rauzy-Veech induction [54, 52]. In particular, ν is a ratio of Lyapunov exponents and depends on the position of the piecewise constant function f (identified with a vector of \mathbb{R}^d) with respect to the Oseledets filtration of the Kontsevich-Zorich cocycle. Using this interpretation, it follows from the work of Forni [23] (see also [4]) that, for a typical choice of function f , ν is positive; furthermore one also has $\nu < 1$ as an immediate consequence of the work

of Veech [48] (see also [23] for a more general result)). A powerful result of similar nature for ergodic integrals of smooth area-preserving flows was proved by Forni in [23]: the *power spectrum* of ergodic integrals is related to Lyapunov exponents of the Kontsevich-Zorich cocycle and Forni's invariant distributions.

A finer analysis of the behaviour of Birkhoff sums or integrals, beyond the *size* of oscillations, appears in the works [11, 36]. In [36], motivated by the study of wandering intervals in affine i.e.m., S. M., P. Moussa and J.C.Y. introduced an object called *limit shape* and used it to describe the *shape* of ergodic sums (see § 3.4 and § 3.7.3 in [36]). Roughly speaking these are obtained by looking at suitably rescaled Birkhoff sums, where time is renormalized according to the leading Lyapunov exponent of the Kontsevich-Zorich cocycle, whereas the range of the sum is renormalized using one of the other positive exponents, according to the choice of f . After this double rescaling one obtains a sequence of shapes exponentially converging (in the Hausdorff metric) to the graph of a Hölder function. In [11], Bufetov studies limit theorems for ergodic integrals of translation flows and describe weak limit distributions in terms of objects that he calls *Hölder cocycles* (or, in the context of Markov compacta, *finitely-additive measures*) and turn out to be *dual* to Forni's invariant distributions (see [11] for details). We remark that limit shapes and Hölder cocycles, despite having been introduced independently, are intrinsically related: limit shapes are essentially *graphs* of Hölder cocycles along flow leaves. Let us remark that similar results can also be proved for horocycle flows on negatively curved surfaces, see [10] where the existence of Hölder cocycles is proved in this context.

Both limit shapes and Hölder cocycles are associated to functions which display *truly polynomial* deviations, i.e., for which the exponent ν in (1.1) is strictly positive. More precisely, from the work of Forni [23] and Avila-Viana [4] it follows that for a typical i.e.m. T with d exchanged subintervals, the extended Kontsevich-Zorich cocycle has g positive Lyapunov exponents, g negative and $s-1$ zero ones, where $d = 2g + s - 1$ and g and s can be computed from the combinatorics of T (g is the genus and s is the number of marked points of any translation surface which suspends T , see § 2.2). For typical i.e.m. T , functions which are *coboundaries* with bounded transfer functions and hence have *bounded* Birkhoff sums, can be associated to the *stable* space of the Kontsevich-Zorich cocycle, which correspond to *negative* Lyapunov exponents.

We construct in this paper objects similar to the limit shapes introduced in [36] for functions which display *subpolynomial* deviations of ergodic averages, i.e., functions for which $\nu = \nu(f)$ in (1.1) is equal to zero, but are not coboundaries. An important example of this type of function arise when considering rotations of the circle (which correspond to i.e.m. with $d = 2$) and a mean zero function $\chi = \chi_{[0,\beta]} - \beta$, where $\chi_{[0,\beta]}$ is the characteristic function of the interval $[0, \beta]$. The function χ can be seen as a piecewise constant function on a i.e.m. with $d = 3$. It is well known that in this case $S_n \chi$ display logarithmic deviations (which can be described for example in terms of the Ostrowski expansion of β with respect to α , see for example [7]). Results on the subdiffusive behaviour of these Birkhoff sums were proved for example in [27, 14, 15] (see also [8] for a related result in the context of substitutions). Celebrated results

on limit distributions for the Birkhoff sums $S_n\chi$ under additional randomness were proved for example by Kesten in [30, 31] (where the rotation number is randomized) and by Beck in [5, 6] (where time is randomized, see also [17, 39, 9, 19] for recent extensions of Beck's temporal limit distribution results and [17, 18] for results in the context of horocycle flows).

More in general, we show that for a typical i.e.m. T (for any number of exchanged intervals $d \geq 3$) the characteristic function $\chi_{[0,\beta]}$ can be *corrected* by adding a function constant on the intervals exchanged by T in order to display subpolynomial deviations (see Proposition 5.3) and hence these seem to be natural functions for future investigations towards a generalizations of Kesten's and Beck's results. Functions with subpolynomial deviations can more generally be associated to *relative* homology classes in $H_1(M, \Sigma, \mathbb{R})$ where M is a (typical) translation surface with conical singularities Σ (see § 6).

We prove in § 4 that for a full measure set of translation surfaces, to each function with subpolynomial deviations (or more in general, in § 6, to each relative homology class with subpolynomial deviations), we can associate *limit distribution* on Hölder functions, which is constructed in a similar way to the limit shapes in [36] (see § 5.2 and § 5.4 for details). More precisely, limit shapes in [36] are Hölder functions defined as pointwise limit of a sequence of piecewise affine functions obtained by plotting suitably rescaled graphs of Birkhoff sums. Convergence exploits positivity of the Lyapunov exponent, which implies that oscillations on different scales are of exponentially smaller orders. We define similarly a sequence of piecewise affine functions in our setup, but they fail to converge pointwise since all oscillations are of the same magnitude. We prove on the other hand that there is convergence in the sense of distributions, when integrating these graphs against Hölder continuous functions (see Theorem 5.6).

The Diophantine condition that we need to ensure distributional convergence of this sequence of piecewise affine functions turns out to be a Roth type-like condition, but not for the usual Kontsevich-Zorich cocycle, but for a dual cocycle. This is the (weak) *dual Roth type* condition that we define in § 4.3. In the proof of convergence, we also need to introduce the notion of *dual special Birkhoff sums* (see § 4.2). This is based on a *duality* between horizontal and vertical flow and future and past of the Teichmüller geodesic flow. The same duality is also exploited crucially in the paper by Bufetov [11]. This duality has also a homological interpretation, which we explain in Section 6.

The construction of limit distributions can also be done at the level of homology bases, as explain in Section 6. Distributional limit objects can hence be associated to relative homology classes $\Upsilon \in H_1(M, \Sigma, \mathbb{R})$ which have non trivial image $\partial\Upsilon$ by the boundary operator $\partial : H_1(M, \Sigma, \mathbb{R}) \rightarrow H_0(M \setminus \Sigma, \mathbb{R})$ (see § . In view of the connection between the limit shapes in [36] and Bufetov's Hölder cocycles [11] and the duality of the latter and Forni's invariant distributions (explained in [11], see ADD), our distributional limit objects are presumably connected to invariant distributions in relative homology.

1.2. Absolute Roth type. — To motivate the notion of absolute Roth type i.e.m., consider the following example. Let $[0, \beta)$ be an interval contained in the domain of an i.e.m. T_0 and let χ be the (possibly corrected, as explained above) characteristic function of $[0, \beta)$. To study the Birkhoff sums $S_n\chi$ of χ under T_0 , it is convenient to *add* the discontinuity β of χ as *marked point* (or fake discontinuity), so that the function χ can be considered as function which is *constant* over the subintervals exchanged by an i.e.m. T of $d+1$ of intervals. It is natural for many questions to *fix* χ and *vary* the i.e.m. T_0 . One would hence like to exploit a Diophantine condition on T which only depends on the underlying T_0 and not on the position of the fake discontinuity β .

Let us recall that the number of exchanged intervals d_0 coincides with the dimension of the *relative* homology $H(S_0, \Sigma_0, \mathbb{R})$, where M_0 is any translation surface which *suspends* the i.e.m. T_0 (see § 2.2 for definitions) and Σ_0 is the set of conical singularities of M_0 . The *Roth type* condition for i.e.m. is defined in terms of the *Rauzy-Veech cocycle*, which acts on the relative homology $H(M_0, \Sigma_0, \mathbb{R})$. If T_0 in the above example is such that $d_0 = 2g$, where g is the genus of M_0 , i.e., d_0 equals the dimension of the *absolute* homology $H(M_0, \mathbb{R})$, the growth of $S_n\chi$, where χ is a *corrected* characteristic function, is given by the growth of a *relative* homology class in $H(M, \Sigma, \mathbb{R})$, where M suspends T and has singularities Σ . One would hence like to introduce Diophantine conditions on T which depends only on the *absolute* homology of M .

These considerations led us to define the notion of *absolute Roth type* i.e.m. (given in § 3). This is a Diophantine condition on an i.e.m. T defined in terms of the Rauzy-Veech cocycle, and hence on the action on $H(M, \Sigma, \mathbb{R})$, but which depends on how the cocycle acts on the *absolute* homology $H(M, \mathbb{R})$, which can be identified with a subspace of \mathbb{R}^d defined in terms of the combinatorial data of T (see § 3 for details). We stress that the absolute Roth type condition is *weaker* than Roth type, i.e., Roth type i.e.m. satisfy the absolute Roth type condition (see Lemma 3.6 in § 3.3).

While we were thinking of this notion, Chaika and Eskin [13] proved a result on Oseledets genericity of the Kontsevich-Zorich cocycle which, as an application, implies that given any translation surface S , for a.e. direction θ the i.e.m. obtained as Poincaré maps of the linear flow in direction θ to a segment in *good position* (see § 2.2 for definitions) satisfy the absolute Roth type condition. This motivated us to prove that several of the results on the cohomological equation for i.e.m. of (restricted) Roth type, in particular the main results in [38], are still valid under the weaker (restricted) absolute Roth type condition. These results are presented in § 3.5 (see also § 3.6 and Appendix B).

1.3. Outline of the paper. — We first recall background material, in particular definitions and notations for i.e.m. and Rauzy Veech induction (see Section 2). The reader who is already familiar with this can skip to Section 2.8 where we simply summarize the notation.

In Section 3 we first recall the Roth type condition introduced by [35] and we then define absolute Roth type. We then states the results, first proved for Roth type, that still hold under this weaker condition. We prove in this section two crucial estimates

that shows that Roth type allows to control the lengths of inducing subintervals for the positive acceleration of Rauzy-Veech induction. The proofs of the results which extend from Roth type to absolute Roth type can then be adapted with minor modifications, which, for completeness, we discuss in an appendix (see Appendix B). We then recall the result by Chaika and Eskin in [13] and get as a corollary that the results on the cohomological equation holds for every surface in a.e. direction.

In Section 4 we introduce the second variation of Roth type condition studied in this paper, namely dual Roth type. We first introduce the notion of dual Birkhoff sums, which are used to give the definition. We also prove that some classical results, in particular estimates on the growth of special Birkhoff sums of BV functions, also holds for special dual Birkhoff sums under this dual Roth type condition.

In Section 5 we study a distributional analogue of limit shapes for functions with subpolynomial deviations of ergodic averages. We first show how a natural class of functions belonging to the central space of the can be obtained by correcting indicatrix functions. We then define the affine graphs which plot the behaviour of central Birkhoff sums. Under the Roth type DC that we introduced in the previous section, we when show that these graphs converge in the sense of distributions (see Theorem 5.6).

Finally, in Section 6, we give a homological interpretation of distributional limit shapes. We define suitable dual bases of relative homology and show that the graphs which we studied in the previous section and their convergence have a homological interpretation.

The Appendix A contain the involved proof that backward rotation numbers are infinitely complete. The Appendix B, as already mentioned, contains a summary of the structure of the proof of the results on the cohomological equation with an indication of the changes one has to do to adapt them to absolute Roth type i.e.m.

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2. Background material and notations

In this section we recall basic definitions and notation for interval exchange maps and Rauzy-Veech induction. We refer to the lecture notes [49, 51, 50, 54] for further

information. The reader who is already familiar with this material might want to skip it and to refer to § 2.8 which provides a brief summary of the notation used.

2.1. Interval exchange maps. — Let $I \subset \mathbb{R}$ be a bounded open interval. An *interval exchange map* (i.e.m.) i.e.m. T on I is defined by the following data. Let \mathcal{A} be an alphabet with $d \geq 2$ symbols. Consider two partitions (modulo 0) of I into d open intervals indexed by \mathcal{A} (the *top* and *bottom* partitions):

$$I = \bigsqcup_{\alpha \in \mathcal{A}} I_{\alpha}^t = \bigsqcup_{\alpha \in \mathcal{A}} I_{\alpha}^b$$

such that $|I_{\alpha}^t| = |I_{\alpha}^b|$ for every $\alpha \in A$. The map T is defined on $\bigsqcup I_{\alpha}^t$ so that its restriction to each I_{α}^t is a translation onto the corresponding I_{α}^b . The partitions are determined by combinatorial data on one side, lengths on the other side, as follows. The combinatorial data is a pair $\pi = (\pi_t, \pi_b)$ of bijections from \mathcal{A} onto $\{1, \dots, d\}$ which indicates in which order the intervals are met in the top and in the bottom partition respectively. We always assume that the combinatorial data are *irreducible*: for $1 \leq k < d$, we have

$$(2.1) \quad \pi_t^{-1}(\{1, \dots, k\}) \neq \pi_b^{-1}(\{1, \dots, k\}).$$

The length data prescribe the lengths $(\lambda_{\alpha})_{\alpha \in \mathcal{A}}$ of the partition elements, i.e., $|I_{\alpha}^t| = |I_{\alpha}^b| = \lambda_{\alpha}$ for $\alpha \in A$. The i.e.m. $T = T_{\pi, \lambda}$ determined by these data acts on $I := (0, \sum_{\alpha \in \mathcal{A}} \lambda_{\alpha})$; the subintervals of the top partition are

$$I_{\alpha}^t = I_{\alpha}^t(T) := \left(\sum_{\pi_t \beta < \pi_t \alpha} \lambda_{\beta}, \sum_{\pi_t \beta \leq \pi_t \alpha} \lambda_{\beta} \right)$$

and those in the bottom partition are

$$I_{\alpha}^b = I_{\alpha}^b(T) := \left(\sum_{\pi_b \beta < \pi_b \alpha} \lambda_{\beta}, \sum_{\pi_b \beta \leq \pi_b \alpha} \lambda_{\beta} \right).$$

We also write simply I_{α} for I_{α}^t , $\alpha \in A$ and refer to them as the *intervals exchanged by T* (or simply *exchanged intervals*). The i.e.m. T sends I_{α}^t on I_{α}^b by the translation $x \mapsto x + \delta_{\alpha}$, where the *translation vector* $(\delta_{\alpha})_{\alpha \in \mathcal{A}}$ is given by

$$\delta_{\alpha} = \sum_{\beta} \Omega_{\alpha\beta} \lambda_{\beta}$$

and $\Omega_{\pi} = (\Omega_{\alpha, \beta})_{\alpha, \beta \in \mathcal{A}}$ is the antisymmetric matrix associated by the combinatorial data π defined by

$$(2.2) \quad \Omega_{\alpha\beta} = \begin{cases} +1 & \text{if } \pi_t(\alpha) < \pi_t(\beta), \pi_b(\alpha) > \pi_b(\beta), \\ -1 & \text{if } \pi_t(\alpha) > \pi_t(\beta), \pi_b(\alpha) < \pi_b(\beta), \\ 0 & \text{otherwise.} \end{cases}$$

The points $u_1^t < \dots < u_{d-1}^t$ separating the I_α^t are called the *singularities* of T . The points $u_1^b < \dots < u_{d-1}^b$ separating the I_α^b are called the singularities of T^{-1} . We also define u_0, u_d so that $I = (u_0, u_d)$. We will also write $u_i^t(T)$ and $u_i^b(T)$ when we want to stress the dependence on T . The action of T on $\bigsqcup I_\alpha^t \subset I$ can be extended to I by right-continuity. A *connection* is a triple (u_i^t, u_j^b, m) , where m is a nonnegative integer, such that $T^m(u_j^b) = u_i^t$.

2.2. Translation surfaces and suspension data. — Let M be a compact connected orientable topological surface of genus $g \geq 1$ and let $\Sigma = \{A_1, \dots, A_s\}$ be a non-empty finite subset of M . A *structure of translation surface* on (M, Σ) is a maximal atlas ζ for $M - \Sigma$ of charts by open sets of $\mathbb{C} \simeq \mathbb{R}^2$ which satisfies two properties: any coordinate change between two charts is locally a translation of \mathbb{R}^2 and for every $1 \leq i \leq s$, there exists an integer $\kappa_i \geq 1$ and a ramified covering $\pi : (V_i, A_i) \rightarrow (W_i, 0)$ of degree κ_i such that every injective restriction of π is a chart of ζ . Here V_i and $W_i \subset \mathbb{R}^2$ are neighborhoods respectively of A_i and 0 .

A translation surface structure naturally provides a complex structure and an holomorphic 1-form ω which does not vanish on $M - \Sigma$ and has at each point A_i a zero of order $\kappa_i - 1$. It also provides a *flat metric* on $M - \Sigma$: the metric exhibits a (true) singularity at each A_i such that $\kappa_i > 1$ and the total angle around each $A_i \in \Sigma$ is $2\pi\kappa_i$. The genus g of the surface and the order of the zeros of ω are related by the Riemann-Roch formula $2g - 2 = \sum_{i=1}^s (\kappa_i - 1)$.

The geodesic flow of the flat metric on $M - \Sigma$ gives rise to a 1-parameter family of constant unitary directional flows on $M - \Sigma$, called *linear flows*, containing in particular a *vertical flow* $\partial/\partial y$ and a *horizontal flow* $\partial/\partial x$. An orbit of the vertical (horizontal) flow which ends (resp. starts) at a point of Σ is called an *ingoing* (resp. *outgoing*) *vertical* (*horizontal*) *separatrix*. A *vertical connection* (resp. *horizontal connection*) is an orbit of the vertical (resp. horizontal) flow which both starts and ends at a point of Σ .

The group $GL(2, \mathbb{R})$ acts naturally on translation surfaces: given the structure of translation surface ζ on (M, Σ) and $g \in GL(2, \mathbb{R})$, one defines a new structure $g \cdot \zeta$ by postcomposing the charts of ζ by g . There are two distinguished one-parameter subgroups of $GL(2, \mathbb{R})$ whose actions will turn out of be important (for example in §3.6): the Teichmüller geodesic flow $(g_t)_{t \in \mathbb{R}}$ and the group of rotations $(r_\theta)_{\theta \in S^1}$, which are given respectively by

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad t \in \mathbb{R}; \quad r_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in S^1.$$

The rotation r_θ has the effect of rotating the flat surface (M, Σ, ζ) by the angle θ .

Given a translation surface (M, Σ, ζ) , an open bounded horizontal segment I is in *good position* if

1. I meets every vertical connexion;

2. the endpoints of I are distinct and each of them either belongs to Σ or is connected to a point of Σ by a vertical segment not meeting I .

If there is no vertical connexion, or no horizontal connexion, then such segments always exist. One may even ask that the left endpoint of I is in Σ ([Yo4, Proposition 5.7, p.16]) In particular, one can always find $g \in GL(2, \mathbb{R})$, preserving the vertical direction, and a segment in good position which is horizontal for $g.\zeta$. The Poincaré first return map of the vertical flow on a horizontal segment in good position is a i.e.m. with minimal possible number of exchanged intervals.

Conversely, starting from an interval exchange map T , one can construct, following Veech [46], a translation surface M_T for which T appears as a return map on a standard interval and the vertical flow on M becomes a suspension flow on T_I . Furthermore, if M is a translation surface with no horizontal and no vertical saddle connections and I is a horizontal interval in good position, the return map T_I of the vertical flow of M on I is an i.e.m. of minimal number of exchanged intervals and the translation surface $(M, \Sigma, \kappa, \zeta)$ can be recovered from T_I and appropriate *suspension data* via Veech's *zippered rectangles construction* ([46], see e.g., [50], Section 4). We will limit ourselves here to explain a simple version of this construction.

Let $\pi = (\pi_t, \pi_b)$ be a combinatorial data. A vector $\tau \in \mathbb{R}^{\mathcal{A}}$ is a *suspension vector* (for π) if it satisfies the following inequalities

$$(2.3) \quad \sum_{\pi_t(\alpha) < k} \tau_\alpha > 0, \quad \sum_{\pi_b(\alpha) < k} \tau_\alpha < 0 \quad \text{for all } 1 < k \leq d.$$

We denote by Θ_π the cone of all suspension vectors for π . This cone is non-empty exactly when π is irreducible. In this case indeed, one can see that θ_π contains the canonical suspension vector

$$(2.4) \quad \tau_\alpha^{\text{can}} = \pi_b(\alpha) - \pi_t(\alpha), \quad \alpha \in \mathcal{A}.$$

Let T be an i.e.m. with irreducible combinatorial data $\pi = (\pi_t, \pi_b)$ acting on an interval $I(T) = (u_0(T), u_d(T))$ and let $\tau \in \mathbb{R}^{\mathcal{A}}$ be a suspension vector. We identify as usual \mathbb{R}^2 with \mathbb{C} and set, for $\alpha \in \mathcal{A}$,

$$(2.5) \quad \lambda_\alpha = |I_\alpha^t| = |I_\alpha^b|, \quad \zeta_\alpha := \lambda_\alpha + \sqrt{-1} \tau_\alpha,$$

and then, for $0 \leq i \leq d$

$$U_i = u_0 + \sum_{\pi_t(\alpha) \leq i} \zeta_\alpha, \quad V_i = u_0 + \sum_{\pi_b(\alpha) \leq i} \zeta_\alpha.$$

Consider the *top* polygonal line connecting the points U_0, U_1, \dots, U_d and the *bottom* polygonal line connecting the points V_0, V_1, \dots, V_d . Both lines have the same endpoints, i.e., $U_0 = V_0 = u_0(T)$, $U_d = V_d$, and that, from the suspension condition (2.3), all intermediary points in the top (resp. bottom) line lie in the upper (resp. lower) half-plane, i.e., $\Im U_i > 0 > \Im V_i$ for $0 < i < d$.

Assume that *the two lines do not intersect except from their endpoints*. This is in particular the case for $\tau = \tau^{\text{can}}$. Then we can construct a translation surface $M(\lambda, \pi, \tau)$ considering the closed polygon bounded by the two lines and identifying

for each $\alpha \in \mathcal{A}$ the ζ_α side of the top line with the ζ_α side of the bottom line through the appropriate translation. This produces a translation surface, whose singularity set Σ is by definition the image of the vertices of the polygon.

The above non-intersection assumption, unfortunately, is not valid for all values of the data, but one can still associate to data (λ, π, τ) as above a translation surface $M(\lambda, \pi, \tau)$. Consider the vector $q = -\Omega_\pi \tau$ and observe that, for all $\alpha \in \mathcal{A}$, since

$$q_\alpha = \sum_{\pi_t(\beta) < \pi_t(\alpha)} \tau_\beta - \sum_{\pi_b(\beta) < \pi_b(\alpha)} \tau_\beta,$$

it follows from the suspension data condition (2.3) that $q_\alpha > 0$ for each $\alpha \in \mathcal{A}$. The surface $M(\lambda, \pi, \tau)$ the general case is then built by indentifying the boundaries of the d rectangles $R_\alpha := I_\alpha^t(T) \times [0, q_\alpha]$, $\alpha \in \mathcal{A}$, where $I_\alpha^t(T) \subset I := I(T)$ are the intervals exchanged by $T = T_{\pi, \lambda}$. The top horizontal side of R_α is identified by a translation to $I_\alpha^b(T) \subset I$. The vertical sides are then identified so that the images of the points U_0, U_1, \dots, U_d defined above are exactly the singularities of the identified surface. We refer the reader to [50], Section 4 for details. This construction is known as Veech’s *zippered rectangles construction*. Notice that T is the Poincaré map first return of the vertical flow on $M(\lambda, \pi, \tau)$ to I and q_α is the return time of any point $x \in I_\alpha = I_\alpha^t(T)$.

We will denote by M_T the surface $M(\lambda, \pi, \tau^{\text{can}})$ obtained as suspension from the canonical suspension data τ^{can} . For $M = M_T$ or more in general $M = M(\pi, \lambda, \tau)$, one can see that the cardinality s of the singularity set Σ , the genus g of M and the number d of intervals are related by $d = 2g + s - 1$. Both g and s only depend on π , the genus g can be computed from the antisymmetric matrix Ω defined in (2.2) above: the rank of Ω is $2g$. Another way to compute s (and thus g) consists in going around the marked points, a procedure that we briefly recall in §2.7.

Remark 2.1. — One can show that $\{\zeta_\alpha, \alpha \in \mathcal{A}\}$ provide a basis of the relative homology group $H_1(M, \Sigma, \mathbb{R})$ and hence one can identify $\mathbb{R}^{\mathcal{A}}$ with $H_1(M, \Sigma, \mathbb{R})$ by the map $(x_\alpha)_\alpha \mapsto [\sum_\alpha x_\alpha \zeta_\alpha]$ (see [49],[50]).

2.3. A step of Rauzy–Veech algorithm, Rauzy Veech diagrams and Rauzy-Veech matrices. — Let T be an i.e.m. with no connection. We have then $u_{d-1}^t \neq u_{d-1}^b$. Set $\hat{u}_d := \max(u_{d-1}^t, u_{d-1}^b)$, $\hat{I} := (u_0, \hat{u}_d)$, and denote by \hat{T} the first return map of T in \hat{I} . The return time is 1 or 2. One checks that \hat{T} is an i.e.m. on \hat{I} whose combinatorial data $\hat{\pi}$ are canonically labeled by the same alphabet \mathcal{A} than π (cf. [35], p. 829). Moreover \hat{T} has no connection; this allows to iterate the algorithm. We say that \hat{T} is deduced from T by an elementary step of the Rauzy–Veech algorithm. We say that the step is of *top* (resp. *bottom*) *type* if $u_{d-1}^t < u_{d-1}^b$ (resp. $u_{d-1}^t > u_{d-1}^b$). One then writes $\hat{\pi} = R_t(\pi)$ (resp. $\hat{\pi} = R_b(\pi)$).

A *Rauzy class* on the alphabet \mathcal{A} is a nonempty set of irreducible combinatorial data which is invariant under R_t, R_b and minimal with respect to this property. A *Rauzy diagram* is a graph whose vertices are the elements of a Rauzy class and whose arrows connect a vertex π to its images $R_t(\pi)$ and $R_b(\pi)$. Each vertex is therefore

the origin of two arrows. As R_t, R_b are invertible, each vertex is also the endpoint of two arrows. An arrow connecting π to $R_t(\pi)$ (respectively $R_b(\pi)$) is said to be of *top type* (resp. *bottom type*). The *winner* of an arrow of top (resp. bottom) type starting at $\pi = (\pi_t, \pi_b)$ with $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$ is the letter α_t (resp. α_b) while the *loser* is α_b (resp. α_t).

To an arrow γ of a Rauzy diagram \mathcal{D} starting at π of top (resp. bottom) type, is associated the matrix $B_\gamma \in SL(\mathbb{Z}^{\mathcal{A}})$ defined by

$$B_\gamma = \mathbb{I} + E_{\alpha_b \alpha_t} \quad (\text{resp. } B_\gamma = \mathbb{I} + E_{\alpha_t \alpha_b}),$$

where $E_{\alpha\beta}$ is the elementary matrix whose only nonzero coefficient is 1 in position $\alpha\beta$.

A path γ in a Rauzy diagram is *complete* if each letter in \mathcal{A} is the winner of at least one arrow in γ ; it is *k-complete* if γ is the concatenation of k complete paths. An infinite path is ∞ -*complete* if it is the concatenation of infinitely many complete paths.

For a path γ in \mathcal{D} consisting of successive arrows $\gamma_1, \dots, \gamma_n$, we set

$$B_\gamma = B_{\gamma_n} \cdots B_{\gamma_1}.$$

It belongs to $SL(\mathbb{Z}^{\mathcal{A}})$ and has nonnegative coefficients. Writing π for the origin of γ and π' for the endpoint of γ , one has

$$(2.6) \quad B_\gamma \Omega_\pi {}^t B_\gamma = \Omega_{\pi'}.$$

Notice that from this relation it follows that

$$B_\gamma \text{Im } \Omega_\pi = \text{Im } \Omega_{\pi'}, \quad {}^t B_\gamma^{-1} \ker \Omega_\pi = \ker \Omega_{\pi'}.$$

Setting, for $v, w \in \text{Im } \Omega_\pi$,

$$\langle \Omega_\pi v, \Omega_\pi w \rangle := {}^t v \Omega_\pi w$$

defines a symplectic structure on $\text{Im } \Omega_\pi$, and similarly on $\text{Im } \Omega_{\pi'}$. Thus, (2.6) shows that the cocycle B_γ is indeed symplectic relative to the (degenerate) symplectic structure induced by the matrices Ω_π and $\Omega_{\pi'}$.

2.4. Iterations of the Rauzy-Veech map and the Zorich algorithm. — Let $T = T^{(0)}$ be an i.e.m. with no connection. We denote by \mathcal{A} the alphabet for the combinatorial data $\pi^{(0)}$ of $T^{(0)}$ and by \mathcal{D} the Rauzy diagram on \mathcal{A} having $\pi^{(0)}$ as a vertex.

The i.e.m. $T^{(1)}$, with combinatorial data $\pi^{(1)}$, deduced from $T^{(0)}$ by the elementary step of the Rauzy-Veech algorithm has also no connection. It is therefore possible to iterate this elementary step indefinitely and get a sequence $T^{(n)}$ of i.e.m. with combinatorial data $\pi^{(n)}$, acting on a decreasing sequence $I^{(n)}$ of intervals. Remark that for $m, n \in \mathbb{Z}$ with $n \leq m$, $I^{(m)}$ is a subinterval of $I^{(n)}$ with the same left endpoint, and $T^{(m)}$ is the first return map into $I^{(m)}$ under iteration of $T^{(n)}$.

We also define a sequence γ_n , $n \in \mathbb{N}$, of arrows in \mathcal{D} associated to the successive steps of the algorithm, so that γ_n is the arrow from $\pi^{(n-1)}$ to $\pi^{(n)}$ corresponding to the n -th step. For $n < m$, we also write $\gamma(n, m)$ for the path from $\pi^{(n)}$ to $\pi^{(m)}$ composed by

$$\gamma(n, m) = \gamma_{n+1} \star \cdots \star \gamma_{m-1} \star \gamma_m,$$

where \star denotes the juxtaposition of paths. In particular $\gamma(n-1, n)$ coincides with the n -th arrow γ_n .

We write $\underline{\gamma} = \gamma(T)$ for the infinite path $\gamma_1 \star \gamma_2 \star \dots$ starting from $\pi^{(0)}$ formed by the γ_n , $n \geq 1$. It is ∞ -complete (cf. [35], p. 832). We call $\gamma(T)$ the *rotation number* of T . Conversely, an ∞ -complete path is equal to $\underline{\gamma}(T)$ for some i.e.m. with no connection.

Given T with no connection, let $\gamma(T)$ be its rotation number. For any integers $m > n$, let

$$B(n, m) := B_{\gamma(n, m)} = B_{\gamma_m} B_{\gamma_{m-1}} \cdots B_{\gamma_{n+1}}$$

be the matrix associated to the path $\gamma(n, m) = \gamma_{n+1} \star \dots \star \gamma_{m-1} \star \gamma_m$ from $\pi^{(n)}$ to $\pi^{(m)}$. In particular, for $n \geq 1$, $B(n-1, n)$ is the elementary matrix associated to the n -th arrow γ_n of $\gamma(T)$ connecting $\pi^{(n-1)}$ to $\pi^{(n)}$. For any $p \leq n \leq m$ the following cocycle relation holds:

$$(2.7) \quad B(p, m) = B(n, m)B(p, n).$$

We also denote by

$$(2.8) \quad B_\alpha(n, m) := \sum_{\beta \in \mathcal{A}} B(n, m)_{\alpha\beta}, \quad m > n, \quad \alpha, \beta \in \mathcal{A}.$$

The entries of the matrix $B(n, m)$, $m > n$, have the following dynamical interpretation. For any $\alpha, \beta \in \mathcal{A}$, the entry $B(n, m)_{\alpha\beta}$ gives the number of times in the orbit $x, T^{(n)}(x), \dots$, of any point $x \in I_\alpha^{(m)}$ under $T^{(n)}$ visits $I_\beta^{(n)}$ up to the first return time of x to $I^{(m)}$. Hence, $B_\alpha(n, m)$ gives the first return time to $I^{(m)}$ of any point $x \in I_\alpha^{(m)}$ under $T^{(n)}$.

Following Zorich [52], it is often convenient to group together in a single Zorich step successive elementary steps of the Rauzy–Veech algorithm whose corresponding arrows have the same type (or equivalently the same winner); we therefore introduce a sequence $0 = n_0 < n_1 < \dots$ such that for each k all arrows in $\gamma(n_k, n_{k+1})$ have the same type and this type is alternatively top and bottom. For $n \geq 0$, the integer k such that $n_k \leq n < n_{k+1}$ is called the *Zorich time* and denoted by $Z(n)$.

2.5. Dynamics of the continued fraction algorithms. — Let \mathcal{R} be a Rauzy class on an alphabet \mathcal{A} . The elementary step of the Rauzy–Veech algorithm,

$$(\pi, \lambda) \mapsto (\hat{\pi}, \hat{\lambda}),$$

considered up to rescaling, defines a map from $\mathcal{R} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}})$ to itself, denoted by Q_{RV} . There exists a unique absolutely continuous measure m_{RV} invariant under these dynamics ([46]); it is conservative and ergodic but has infinite total mass, which does not allow all ergodic–theoretic machinery to apply. Replacing a Rauzy–Veech elementary step by a Zorich step gives a new map Q_Z on $\mathcal{R} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}})$. This map has now a *finite* absolutely continuous invariant measure m_Z , which is ergodic ([52]).

It is also useful to consider the *natural extensions* of the maps Q_{RV} and Q_Z , defined through the suspension data which serve to construct translation surfaces from i.e.m. For $\pi \in \mathcal{R}$, let Θ_π be the convex open cone of suspension vectors in $\mathbb{R}^{\mathcal{A}}$, see (2.3). Define also

$$\Theta_\pi^t = \left\{ \tau \in \Theta_\pi, \sum_\alpha \tau_\alpha < 0 \right\}, \quad \Theta_\pi^b = \left\{ \tau \in \Theta_\pi, \sum_\alpha \tau_\alpha > 0 \right\}.$$

Let $\gamma : \pi \rightarrow \hat{\pi}$ be an arrow in the Rauzy diagram \mathcal{D} associated to \mathcal{R} . Then ${}^t B_\gamma^{-1}$ sends Θ_π isomorphically onto $\Theta_{\hat{\pi}}^t$ (resp. $\Theta_{\hat{\pi}}^b$) when γ is of top type (resp. bottom type).

The natural extension \hat{Q}_{RV} is defined on $\bigsqcup_{\pi \in \mathcal{R}} \{\pi\} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}}) \times \mathbb{P}(\Theta_\pi)$ by

$$(\pi, \lambda, \tau) \mapsto (\hat{\pi}, {}^t B_\gamma^{-1} \lambda, {}^t B_\gamma^{-1} \tau)$$

where γ is the arrow starting at π , associated to the map Q_{RV} at (π, λ) . The map \hat{Q}_{RV} has again a unique absolutely continuous invariant measure \hat{m}_{RV} ; it is ergodic, conservative but infinite. One defines similarly a natural extension \hat{Q}_Z for Q_Z ; it has a unique absolutely continuous invariant measure \hat{m}_Z , which is finite and ergodic.

The sequences $(\pi^{(n)})_{n \in \mathbb{Z}}$, $(\lambda^{(n)})_{n \in \mathbb{Z}}$, $(\tau^{(n)})_{n \in \mathbb{Z}}$ defined by the Rauzy-Veech algorithm satisfy, for $m \leq n$

$$\lambda_\alpha^{(n)} = \sum_\beta B(m, n)_{\alpha\beta}^* \lambda_\beta^{(m)}, \quad \tau_\alpha^{(n)} = \sum_\beta B(m, n)_{\alpha\beta}^* \tau_\beta^{(m)}, \quad \text{where } B^* = {}^t B(m, n)^{-1}.$$

The associated sequences $\delta^{(n)} = \Omega_{\pi^{(n)}} \lambda^{(n)}$ and $q^{(n)} = \Omega_{\pi^{(n)}} \tau^{(n)}$, for $n \geq 0$, satisfy

$$\delta_\alpha^{(n)} = \sum_\beta B(m, n)_{\alpha\beta} \delta_\beta^{(m)}, \quad q_\alpha^{(n)} = \sum_\beta B_{\alpha\beta}.$$

2.6. Special Birkhoff sums and the extended Kontsevich-Zorich cocycle. — For a real number $r \geq 0$ we denote by $C^r(\bigsqcup I_\alpha)$ the product $\prod_{\alpha \in \mathcal{A}} C^r(I_\alpha)$. If $r = 0$ we sometimes drop the exponent in $C^r(\bigsqcup I_\alpha)$.

Let $(T^{(n)})_{n \in \mathbb{Z}}$ be a sequence of i.e.m. acting on intervals $I^{(n)}$ related by the Rauzy-Veech algorithm as in §2.8. Let $m \leq n$. For a function $\varphi \in C(\bigsqcup I_\alpha^{(m)})$, we define the *special Birkhoff sum* $S(m, n)\varphi \in C(\bigsqcup I_\alpha^{(n)})$ by

$$(2.9) \quad S(m, n)\varphi(x) = \sum_{0 \leq j < r(x)} \varphi((T^{(m)})^j(x)), \quad \forall \alpha \in \mathcal{A}, \forall x \in I_\alpha^{(n)},$$

where $r(x)$ is the return time of x into $I^{(n)}$ under $T^{(m)}$. One has

$$r(x) = \sum_{\beta \in \mathcal{A}} B_{\alpha\beta}, \quad \forall \alpha \in \mathcal{A}, \forall x \in I_\alpha^{(n)},$$

with $B = B(m, n)$.

For $n \in \mathbb{Z}$, $\Gamma(n) := \Gamma(T^{(n)})$ is the d -dimensional subset of $C(\bigsqcup I_\alpha^{t,(n)})$ consisting of functions which are constant on each $I_\alpha^{t,(n)}$. The space $\Gamma(T^{(n)})$ is canonically isomorphic to $\mathbb{R}^{\mathcal{A}}$. One can check that the operator $S(m, n)$ sends $\Gamma(T^{(m)})$ onto $\Gamma(T^{(n)})$, and the matrix of this restriction is $B(m, n)$.

This defines a d -dimensional cocycle over Q_{RV} (and also above \hat{Q}_{RV} , Q_Z and \hat{Q}_Z) called the (extended) Kontsevich-Zorich cocycle.

Special Birkhoff sums operators satisfy the following cocycle relation: for any $n \leq n' \leq n''$, one has

$$S(n, n'') = S(n', n'') \circ S(n, n').$$

Observe also that, if ψ is integrable on $I^{(n)}$, then $S(n, n')\psi$ is integrable on $I^{(n')}$ for any $n' \geq n$ and we have

$$\int_{I^{(n')}} S(n, n')\psi = \int_{I^{(n)}} \psi.$$

2.7. The boundary operator. — Let π be irreducible combinatorial data over the alphabet \mathcal{A} . Define a $2d$ -element set \mathcal{S}° by

$$\mathcal{S}^\circ = \{U_0 = V_0, U_1, V_1, \dots, U_{d-1}, V_{d-1}, U_d = V_d\}.$$

These symbols correspond to the vertices of the polygon produced by the suspension of an i.e.m. T with combinatorial data π (cf. § 2.8).

Going anticlockwise around the vertices (taking the gluing into account) produces a permutation ⁽¹⁾ σ of \mathcal{S}° :

$$\begin{aligned} \sigma(U_i) &= V_j & \text{with } \pi_b^{-1}(j+1) &= \pi_t^{-1}(i+1) & \text{for } 0 \leq i < d \\ \sigma(V_{j'}) &= U_{i'} & \text{with } \pi_t^{-1}(i') &= \pi_b^{-1}(j') & \text{for } 0 < j' \leq d. \end{aligned}$$

The cycles of σ in \mathcal{S}° are canonically associated to the marked points on the translation surface M_T obtained by suspension of T . We will denote by Σ the set of cycles of σ , by s the cardinality of Σ (cf. § 2.8).

Let $\varphi \in C^0(\bigsqcup I_\alpha)$. We write $\varphi(u_i^t - 0)$ (resp. $\varphi(u_i^t + 0)$) for its value at u_i^t considered as a point in $[u_{i-1}^t, u_i^t]$ (resp. $[u_i^t, u_{i+1}^t]$). We will use the convention that $\varphi(u_0 - 0) = \varphi(u_d + 0) = 0$. Let T be an i.e.m. with combinatorial data π , and let Σ be the set of cycles of the associated permutation σ of \mathcal{S}° . The boundary operator ∂ is the linear operator from $C^0(\bigsqcup I_\alpha^t)$ to \mathbb{R}^Σ defined by

$$(2.10) \quad (\partial\varphi)_C := \sum_{0 \leq i \leq d, U_i \in C} (\varphi(u_i^t - 0) - \varphi(u_i^t + 0)),$$

for any $\varphi \in C^0(\bigsqcup I_\alpha)$, $C \in \Sigma$.

Remark 2.2. — The name boundary operator is due to the following homological interpretation. The space $\Gamma(T) \subset C^0(\bigsqcup I_\alpha)$ is naturally isomorphic to the first relative homology group $H_1(M_T, \Sigma, \mathbb{R})$ of the translation surface M_T : the characteristic

⁽¹⁾ We remark that the presentation of the permutation σ is different from [37].

function of $I_\alpha^t(T)$ corresponds to the homology class $[\zeta_\alpha]$ of the side ζ_α (oriented rightwards) of the polygon giving rise to M_T (see § 2.2). Through this identification, the restriction of the boundary operator to $\Gamma(T)$ is the usual boundary operator

$$\partial : H_1(M_T, \Sigma, \mathbb{R}) \longrightarrow H_0(\Sigma, \mathbb{R}) \simeq \mathbb{R}^\Sigma.$$

We will denote by $\Gamma_\partial(T)$ the kernel of the restriction of ∂ to $\Gamma(T)$. We note in the following proposition several useful properties of the cycles and of the boundary operator, for which we refer to [37] (and in particular to Proposition 3.2, p. 1597).

Proposition 2.3. — *Let T be an i.e.m. with combinatorial data π :*

1. Let $\psi \in C^0([u_0, u_d])$. One has $\partial\psi = \partial(\psi \circ T)$.
2. If $\varphi \in C^0(\bigsqcup I_\alpha)$ satisfies $\varphi(u_i^t + 0) = \varphi(u_i^t - 0)$ for $0 \leq i \leq d$ then $\partial\varphi = 0$.
3. The boundary operator $\partial : C^0(\bigsqcup I_\alpha) \rightarrow \mathbb{R}^\Sigma$ is onto.
4. The restriction of ∂ to $\Gamma(T)$ has as kernel $\Gamma_\partial(T)$ the image of Ω_π .
5. The image of the restriction of ∂ to $\Gamma(T)$ is $\mathbb{R}_0^\Sigma := \{x \in \mathbb{R}^\Sigma, \sum_C x_C = 0\}$.
6. Let $n \geq 0$, let $T^{(n)}$ be the i.e.m. obtained from T by n steps of the Rauzy–Veech algorithm. For $\varphi \in C^0(\bigsqcup I_\alpha)$ one has

$$\partial(S(0, n)\varphi) = \partial\varphi,$$

where the left-hand side boundary operator is defined using the combinatorial data $\pi^{(n)}$ of $T^{(n)}$.

2.8. Summary of notation. — For convenience of the reader, in particular the reader who is already familiar with Rauzy–Veech induction, we summarize in this section the notation used in the rest of the paper.

Interval exchange maps

- \mathcal{A} is the alphabet used to index the intervals.
- $d := \#\mathcal{A}$.
- $\pi = (\pi_t, \pi_b)$ are the combinatorial data of an i.e.m. T , acting on an interval I .
- $I_\alpha := I_\alpha^t = I_\alpha^t(T)$, $I_\alpha^b = I_\alpha^b(T)$, for $\alpha \in \mathcal{A}$ are the subintervals in the top and bottom subdivisions of I .
- $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$ are the length data, i.e., $\lambda_\alpha = |I_\alpha^t| = |I_\alpha^b|$ for all $\alpha \in \mathcal{A}$.
- $\delta = (\delta_\alpha)_{\alpha \in \mathcal{A}}$ is the translation vector of T , i.e., $T : I_\alpha^t \rightarrow I_\alpha^b$ is given by $x \mapsto x + \delta_\alpha$.
- $\Omega_\pi = (\Omega_{\alpha, \beta})_{\alpha, \beta \in \mathcal{A}}$ is the antisymmetric matrix associated by the combinatorial data π , given by (2.2) (the length vector λ and the translation vector δ are related by $\delta = \Omega\lambda$).
- $u_0 = u_0(T)$, $u_1 = u_1(T)$ endpoints of $I = I(T) = (u_0, u_d)$.
- $u_1^t < \dots < u_{d-1}^t$ singularities of T . We also write $u_i^t(T) = u_i^t(T)$, $1 \leq i \leq d-1$.
- $u_1^b < \dots < u_{d-1}^b$ singularities of T^{-1} . We also write $u_i^b(T) = u_i^b(T)$, $1 \leq i \leq d-1$.
- T has *no connection* if there is no m nonnegative integer, such that $T^m(u_j^b) = u_i^t$ for some $1 \leq i, j < d$.

Translation surfaces and suspension data

- $\tau = (\tau_\alpha)_{\alpha \in \mathcal{A}}$ is a *suspension vector* for the combinatorial data π .
- $M(\pi, \lambda, \tau)$ translation surface obtained from the zippered rectangle construction with data (π, λ, τ) .
- $M_T = M(\pi, \lambda, \tau^{\text{can}})$ canonical suspension associated to the canonical suspension vector τ^{can} in (2.4).
- $\zeta_\alpha := \lambda_\alpha + i\tau_\alpha$, for $\alpha \in \mathcal{A}$ are the periods of $M(\pi, \lambda, \tau)$.
- ω is the canonical holomorphic 1-form on M .
- $g = g(\pi)$ is the genus of M ; it is also the half-rank of Ω .
- Σ is the set of marked points on M .
- $s := \#\Sigma$ depends only on π . One has $d = 2g + s - 1$.
- $\mathcal{S} := \{U_0 = V_0, U_1, V_1, \dots, U_{d-1}, V_{d-1}, U_d = V_d\}$ vertices of the suspension (singularities of $M(\pi, \lambda, \tau)$).
- T is the return map to I for the upwards vertical flow of M .
- $q = (q_\alpha)_{\alpha \in \mathcal{A}}$ given by $q = -\Omega\tau$ is the vector of heights of the rectangles in the zippered rectangle construction of $M(\pi, \lambda, \tau)$; q_α is the return time of I_α^t to I under the vertical flow on $M(\pi, \lambda, \tau)$.

The Rauzy-Veech renormalization algorithm

- \mathcal{R} is the *Rauzy class* of the combinatorial data π of T .
- \mathcal{D} is the associated *Rauzy diagram*. The set of vertices of \mathcal{D} is \mathcal{R} .
- α_t is the element of \mathcal{A} such that $\pi_t(\alpha_t) = d$; α_b is the element of \mathcal{A} such that $\pi_b(\alpha_b) = d$.
- Each arrow γ in \mathcal{D} has a *type top* or *bottom* and is associated to two elements of \mathcal{A} , called *winner* and *loser*; if γ is of type *type top* (resp. *bottom*), the winner is α_t and the loser α_b (resp. the winner is α_b and the loser α_t).
- $E_{\alpha\beta}$ is the elementary matrix whose only nonzero coefficient is 1 in position $\alpha\beta$.
- B_γ , where γ is an arrow starting at π of top (resp. bottom) type, is the matrix $B_\gamma = \mathbb{I} + E_{\alpha_b\alpha_t}$ (resp. $B_\gamma = \mathbb{I} + E_{\alpha_t\alpha_b}$).
- $B_\gamma = B_{\gamma_n} \cdots B_{\gamma_1}$ if γ is a path γ in \mathcal{D} consisting of successive arrows $\gamma_1, \dots, \gamma_n$.
- $T^{(n)} := (\lambda^{(n)}, \pi^{(n)})$ for $n \in \mathbb{N}$ is the sequence of (non-renormalized) i.e.m. obtained by applying the Rauzy-Veech algorithm starting from $T^{(0)} := (\lambda^{(0)}, \pi^{(0)})$ with no connection.
- $(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$ for $n \in \mathbb{Z}$ is the sequence produced by the Rauzy-Veech induction from $(\pi^{(0)}, \lambda^{(0)}, \tau^{(0)}) := (\pi, \lambda, \tau)$ s.t. $M(\pi, \lambda, \tau)$ has *neither horizontal nor vertical connections*.
- the interval $I^{(n)}$ is the domain of the i.e.m. $T^{(n)} = (\pi^{(n)}, \lambda^{(n)})$.
- $I_\alpha^{(n)} = I_\alpha^t(T^{(n)})$, $\alpha \in \mathcal{A}$, are the subintervals exchanged by $T^{(n)}$.
- $\gamma_n = \gamma(n-1, n)$ for $n \in \mathbb{N}$ (or $n \in \mathbb{Z}$) is the arrow of \mathcal{D} from $\pi^{(n-1)}$ to $\pi^{(n)}$.
- $\gamma(n, m)$, for $n < m$, is the path in \mathcal{R} from $\pi^{(n)}$ to $\pi^{(m)}$ given by $\gamma_{n+1} \star \cdots \star \gamma_{m-1} \star \gamma_m$.

- $\gamma(T)$ for $T = (\pi, \lambda)$ with no connection, is the *rotation number* of T ; $\gamma(T)$ is the juxtaposition $\gamma_1 * \gamma_2 *$ of the arrows $\gamma_1, \gamma_2, \dots$,
- $B(m, n)$, where $m < n$ is the matrix product $B(m, n) := B_{\gamma}(m, n) = B_{\gamma_n} B_{\gamma_{n-1}} \cdots B_{\gamma_{m+1}}$.
- $B_{\alpha}(m, n) := \sum_{\beta \in \mathcal{A}} B(m, n)_{\alpha\beta}$ is the return time of any $x \in I_{\alpha}^{(n)}$ into $I^{(n)}$ under $T^{(m)}$.
- $Q_{\text{RV}} : \mathcal{R} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}}) \rightarrow \mathcal{R} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}})$ *Rauzy-Veech map* (rescaled step of Rauzy-Veech algorithm).
- $Q_{\text{Z}} : \mathcal{R} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}}) \rightarrow \mathcal{R} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}})$ *Zorich map*, acceleration of the Rauzy-Veech map;
- $Z(n)$ *Zorich time*, i.e., the integer k such that $n_k \leq n < n_{k+1}$ where n_k are such that $Q_{\text{Z}}^k(T) = Q_{\text{RV}}^{n_k}(T)$.
- m_{RV} (resp. m_{Z}) unique absolutely continuous measure invariant under Q_{RV} (resp. Q_{Z}).
- \hat{Q}_{RV} (resp. \hat{Q}_{Z}), defined on $\mathcal{R} \times \mathbb{P}((\mathbb{R}^+)^{\mathcal{A}}) \times \mathbb{P}(\Theta_{\pi})$: *natural extension* of Q_{RV} (resp. Q_{Z}).
- \hat{m}_{Q} (resp. \hat{m}_{Z}) invariant measures for \hat{Q}_{RV} (resp. \hat{Q}_{Z});

Functional spaces, special Birkhoff sums and boundary operator

- $C(\bigsqcup I_{\alpha})$ space of functions on $\bigsqcup I_{\alpha}$ which belong to $C(\overline{I_{\alpha}})$ for every $\alpha \in \mathcal{A}$.
- $C^r(\bigsqcup I_{\alpha})$ for $r \geq 1$ space of functions on $\bigsqcup I_{\alpha}$ which belong to $C^r(\overline{I_{\alpha}})$ for every $\alpha \in \mathcal{A}$.
- $\Gamma(T) \subset C(\bigsqcup I_{\alpha})$ d -dimensional space of functions which are constant on each $I_{\alpha}^{(n)}$.
- Given T with no connection, $\Gamma(n) := \Gamma(T^{(n)})$, where $T^{(n)} = \mathcal{R}^n(T)$, for $n \in \mathbb{Z}$.
- $S(m, n) : C(\bigsqcup I_{\alpha}^{(n)}) \rightarrow C(\bigsqcup I_{\alpha}^{(m)})$ *special Birkhoff sums operator*; for $\varphi \in C(\bigsqcup I_{\alpha}^{(n)})$, for $\alpha \in \mathcal{A}$ and $x \in I_{\alpha}^{(n)}$, $S(m, n)\varphi(x) = \sum_{0 \leq j < B(m, n)} \varphi((T^{(m)})^j(x))$;
- the restriction $S(m, n) : \Gamma(m) \rightarrow \Gamma(n)$ is given by $B(m, n) = B_{\gamma(m, n)}$.
- $\sigma : \mathcal{S} \rightarrow \mathcal{S}$ permutation which identifies the vertices of \mathcal{S} going clockwise around singularities.
- Σ set of cardinality s identified with *cycles* of σ .
- $\varphi(u_i^t \pm 0)$ value of $\varphi \in C^0(\bigsqcup I_{\alpha})$ at u_i^t considered as a point in $[u_i^t, u_{i+1}^t]$ or $[u_{i-1}^t, u_i^t]$ respectively.
- $\partial : C^0(\bigsqcup I_{\alpha}) \rightarrow \mathbb{R}^{\Sigma}$ *boundary operator*, defined for any $\varphi \in C^0(\bigsqcup I_{\alpha})$ and $C \in \Sigma$ by $(\partial\varphi)_C := \sum_{0 \leq i \leq d, u_i \in C} (\varphi(u_i^t - 0) - \varphi(u_i^t + 0))$.

3. Absolute Roth type and applications

In this section we define the absolute Roth type condition for i.e.m. We first recall the definition of (restricted) Roth type i.e.m. which was give in [35, 37]. The adjective *absolute* refers here to *absolute homology*, since as we will see this condition is

expressed in terms of cones which are isomorphic to the absolute homology of the suspended surface; in contrast, the usual Roth type condition concerns *relative* homology (we could have called it in this paper *relative Roth type condition*, but we chose not to do so for consistency with the literature). We then state the extension of the main results from [38] on the cohomological equation to i.e.m. of (restricted) absolute Roth type (see Theorem 3.14 below, as well as Theorem B.1 and Theorem B.4 in the Appendix B). In § 3.4 we prove two crucial estimates which are needed to adapt the corresponding proofs to deal with absolute Roth type i.e.m.

3.1. Roth Type i.e.m. — We first recall the Roth type diophantine condition on an i.e.m. introduced in [35] and then slightly modified in [37].

For a matrix $B = (B_{\alpha\beta})_{\alpha,\beta \in \mathcal{A}}$, we define

$$(3.1) \quad \|B\| := \max_{\alpha} \sum_{\beta} |B_{\alpha\beta}|,$$

which is the operator norm for the ℓ_{∞} norm on $\mathbb{R}^{\mathcal{A}}$.

Let T be an i.e.m., with combinatorial data π . In the space $\Gamma(T) \simeq \mathbb{R}^{\mathcal{A}}$ of piecewise constant functions on $I(T)$, we define the hyperplane

$$\Gamma_0(T) := \{\chi \in \Gamma(T), \int \chi(x) dx = 0\}.$$

Assuming that T has no connection, we also define the *stable subspace*

$$\Gamma_s(T) := \{\chi \in \Gamma(T), \exists \sigma > 0, \|B(0, n)\chi\| = \mathcal{O}(\|B(0, n)\|^{-\sigma})\}.$$

As $\Gamma_s(T)$ is finite-dimensional, there exists an exponent $\sigma > 0$ which works for every $\chi \in \Gamma_s(T)$. We fix such an exponent in the rest of the paper. The subspace $\Gamma_s(T)$ is contained in $\text{Im } \Omega(\pi)$, and is an isotropic subspace of this symplectic space. We have

$$\Gamma_s(n) := \Gamma_s(T^{(n)}) = B(0, n)\Gamma_s(T), \quad \text{for } n \geq 0.$$

Let us first define a suitable *acceleration* of the Rauzy-Veech algorithm. Consider a *rotation number* $\underline{\gamma} = \gamma_0 \star \gamma_1 \star \dots$, i.e., an infinite ∞ -complete path in \mathcal{D} obtained concatenating successive arrows $\gamma_0, \gamma_1, \dots$. Define $\hat{n}_0 = 0$. Define inductively \hat{n}_k for $k \geq 1$ as the smallest integer $n > \hat{n}_{k-1}$ such that the path $\gamma_{\hat{n}_{k-1}+1} \star \dots \star \gamma_{\hat{n}}$ is complete (see 2.3).

Following [38], *Roth type i.e.m.* are those i.e.m. whose rotation number satisfy three conditions (a) (*matrix growth*), (b) (*spectral gap*), (c) (*coherence*). Furthermore, i.e.m. which in addition satisfy also a fourth condition (d) (*hyperbolicity*), are called of *restricted Roth type*. We now recall the definition of these four conditions. The sequence (\hat{n}_k) that appears in Condition (a) is the sequence of accelerated times defined just above.

Definition 3.1. — An i.e.m. T with no connection is of *restricted Roth type* if the following four conditions are satisfied:

(a) (*matrix growth*)

$$\forall \tau > 0, \exists C_\tau > 0, \forall k > 0, \|B(\hat{n}_{k-1}, \hat{n}_k)\| \leq C_\tau \|B(\hat{n}_0, \hat{n}_{k-1})\|^\tau.$$

(b) (*spectral gap*) There exists $\theta > 0$ such that

$$\|B(0, n)|_{\Gamma_0(T)}\| = \mathcal{O}(\|B(0, n)\|^{1-\theta}).$$

(c) (*coherence*) For $0 \leq m \leq n$, let

$$B_s(m, n) : \Gamma_s(T^{(m)}) \rightarrow \Gamma_s(T^{(n)}),$$

$$B_b(m, n) : \Gamma(T^{(m)})/\Gamma_s(T^{(m)}) \rightarrow \Gamma(T^{(n)})/\Gamma_s(T^{(n)})$$

be the operators induced by $B(m, n)$. We ask that, for all $\tau > 0$,

$$\|B_s(m, n)\| = \mathcal{O}(\|B(0, n)\|^\tau), \quad \|(B_b(m, n))^{-1}\| = \mathcal{O}(\|B(0, n)\|^\tau).$$

(d) (*hyperbolicity*) $\dim \Gamma_s(T) = g$.

The i.e.m. with no connection that satisfy conditions (a), (b) and (c) are called *Roth type* i.e.m. Finally, we will say that i.e.m. with no connection that satisfy conditions (a) and (b) only are i.e.m. of *weak* Roth type. We will sometimes in this paper refer to these conditions as *direct* Roth type conditions to distinguish it from the *dual* Roth type condition which we will introduce later.

In [35], condition (a) had a slightly different (but equivalent) formulation, as explained in the remark below.

Remark 3.2. — Condition (a) can be rephrased in the following way. Define $\tilde{n}_0 = 0$. Define inductively \tilde{n}_k for $k > 0$ as the smallest integer $n > \tilde{n}_{k-1}$ such that the matrix $B(\tilde{n}_{k-1}, n)$ has positive coefficients. Then condition (a) is equivalent to

$$(3.2) \quad \forall \tau > 0, \exists C_\tau > 0, \forall k > 0, \|B(\tilde{n}_{k-1}, \tilde{n}_k)\| \leq C_\tau \|B(\tilde{n}_0, \tilde{n}_{k-1})\|^\tau.$$

Indeed, let γ be a finite path in \mathcal{D} . If γ is not complete, at least one of the coefficients of B_γ is equal to 0. Conversely, if γ is $(2d-3)$ -complete (or, when $d = 2$, 2-complete), all coefficients of B_γ are positive, see [35] (more precisely, see the lemma in §1.2.2, page 833). These two facts imply the equivalence of the two formulations, namely of condition (a) and condition (3.2). It is the condition above which was used in [38].

Let us remark that an i.e.m. T with no connection satisfying condition (b) is uniquely ergodic. Let us also recall that for any combinatorial data, the set of i.e.m. of relative restricted Roth type has *full measure*. First, it is obvious that almost all i.e.m. have no connection. That condition (c) is almost surely satisfied is a consequence of Oseledets theorem applied to the Kontsevich-Zorich cocycle. Condition (d) follows from the hyperbolicity of the (restricted) Kontsevich-Zorich cocycle, proved by Forni ([23]). A proof that condition (a) has full measure is provided in [35], but much better diophantine estimates were later obtained in [3]. A simpler proof where full measure is deduced from the estimates in [3] is sketched in [34]. Finally, the fact that condition (b) has full measure is a consequence from the fact that the larger Lyapunov exponent of the Kontsevich-Zorich cocycle is simple (see [47]).

3.2. The cones $\mathcal{C}(\pi)$. — We give in this section some preliminary definitions which we need in order to define *absolute* Roth type i.e.m.

We denote by \mathcal{C} the positive open cone of $\mathbb{R}^{\mathcal{J}}$. Remark first that for $m < n$, the matrix $B(m, n)$ has positive coefficients iff

$$\overline{B(m, n)(\mathcal{C})} \subset \mathcal{C} \cup \{0\}.$$

For $\pi \in \mathcal{R}$, let Ω_π be the antisymmetric matrix associated to π (see §2.1) and let $H(\pi) \subset \mathbb{R}^{\mathcal{J}}$ the image of Ω_π . The dimension of $H(\pi)$ is $2g$ and the codimension of $H(\pi)$ is $s - 1$, where g is the genus of the translation surfaces associated to \mathcal{R} and s is the number of marked points (see §2.2). We are interested in the case $s > 1$ where $H(\pi) \subsetneq \mathbb{R}^{\mathcal{J}}$.

We denote by $\mathcal{C}(\pi)$ the intersection $\mathcal{C} \cap H(\pi)$. Let us recall that, from the properties of the extended KZ-cocycle, for any oriented path $\gamma : \pi \rightarrow \pi'$ in \mathcal{D} , one has

$$(3.3) \quad B_\gamma(H(\pi)) = H(\pi'), \quad B_\gamma(\mathcal{C}(\pi)) \subset \mathcal{C}(\pi').$$

Let us explain the connection between $H(\pi)$ and absolute homology. Let M be a translation surface with combinatorial data π , namely a surface of the form $M = M(\pi, \lambda, \tau)$ where $\lambda \in (\mathbb{R}^+)^{\mathcal{J}}$ and $\tau \in \Theta_\pi$. Denote by Σ the set of marked points of M . Recall from §2.2 that the homology classes $[\zeta_\alpha]$, $\alpha \in \mathcal{A}$ of the vectors ζ_α defined in (2.5) provide a base of relative homology $H_1(M, \Sigma, \mathbb{Z})$ (see Remark 2.1). When we identify $\mathbb{R}^{\mathcal{J}}$ to $H_1(M, \Sigma, \mathbb{R})$ as explained in Remark 2.1, the subspace $H(\pi)$ is identified with the absolute homology group $H_1(M, \mathbb{R}) \subset H_1(M, \Sigma, \mathbb{R})$, see for example [51]. This homological interpretation will be discussed further in Section 6.

The following result is well known, we recall its proof for convenience.

Proposition 3.3. — *For any $\pi \in \mathcal{R}$, the open cone $\mathcal{C}(\pi) \subset H(\pi)$ is not empty.*

Proof. — Recall that an element $\pi = (\pi_t, \pi_b)$ is *standard* if there exist letters $\alpha_t, \alpha_b \in \mathcal{J}$ such that $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$ and $\pi_t(\alpha_b) = \pi_b(\alpha_t) = 1$. Every Rauzy class contains at least one standard vertex. By (3.3), it is sufficient to prove that $\mathcal{C}(\pi)$ is not empty when π is standard.

Consider the (absolute) homology class $\zeta \in H_1(M, \mathbb{Z})$ associated to the loop obtained by concatenation of paths in the polygon joining U_0 to U_d and U_1 to U_{d-1} (this is a loop because π is standard). One has

$$\zeta = 2 \sum_{\alpha \in \mathcal{A}} \zeta_\alpha - \zeta_{\alpha_t} - \zeta_{\alpha_b} =: \sum x_\alpha \zeta_\alpha.$$

As ζ is an absolute homology class, one has $x \in H(\pi)$. Obviously, x belongs also to \mathcal{C} . □

For later reference, we also state the

Proposition 3.4. — *There exists a constant $C = C(\mathcal{R}) > 1$ such that, for any path $\gamma : \pi \rightarrow \pi'$ in \mathcal{D} , one has, for $B := B_\gamma$*

$$\|B|_{H(\pi)}\| \leq \|B\| \leq C \|B|_{H(\pi)}\|.$$

Proof. — Recall that we choose as norm the operator norm for the ℓ^∞ -norm on $\mathbb{R}^{\mathcal{A}}$, see (3.1). The first inequality is trivial. The second is a consequence of Proposition 3.3. Indeed, fix a vector v in $\mathcal{C}(\pi)$; one has $\|B\| = \max_\alpha \sum_\beta B_{\alpha\beta}$. Hence, writing $w = Bv$,

$$\min_\beta v_\beta \|B\| \leq \max_\alpha \sum_\beta B_{\alpha\beta} v_\beta = \max_\alpha w_\alpha \leq \|B|_{H(\pi)}\| \max_\beta v_\beta,$$

which gives the required estimate with $C = \max_\pi \left(\frac{\max_\beta v_\beta}{\min_\beta v_\beta} \right)$. \square

3.3. Absolute Roth type i.e.m. — To define the (restricted) *absolute* Roth type Diophantine condition for i.e.m. we will modify the definition of restricted Roth type recalled in the previous section. We will not change the last three conditions (b), (c) and (d), but replace Condition (a) by a weaker Condition (a)'.

Let γ be the rotation number of an i.e.m. with no connection and let $\pi^{(n)}$, $n \geq 0$ be the vertices of the paths and $B(m, n)$, $n > m$, the associated Rauzy-Veech matrices (see § 2.3). To introduce condition (a)', we set $n_0 = 0$ and define n_k for $k > 0$ as the smallest integer $n > n_{k-1}$ such that the matrix $B(n_{k-1}, n)$ satisfies

$$\overline{B(n_{k-1}, n)(\mathcal{C}(\pi^{(n_{k-1})}))} \subset \mathcal{C}(\pi^{(n)}) \cup \{0\}.$$

Then we ask that

$$(a)' \quad \forall \tau > 0, \exists C_\tau > 0, \forall k > 0, \|B(n_{k-1}, n_k)\| \leq C_\tau \|B(n_0, n_{k-1})\|^\tau.$$

Definition 3.5. — An i.e.m. with no connection is of *absolute Roth type* if its rotation number satisfies conditions (a)', (b), (c), (d).

The following lemma shows that the absolute Roth type condition is weaker than the Roth type condition.

Lemma 3.6. — *A rotation number which satisfies condition (a) also satisfies condition (a)'. Thus, (restricted) Roth type i.e.m. are (restricted) absolute Roth type i.e.m.*

Proof. — Observe first that, if $B(m, n)$, where $0 \leq m < n$, has positive coefficients, we have

$$(3.4) \quad \overline{B(m, n)(\mathcal{C}(\pi^{(m)}))} \subset \mathcal{C}(\pi^{(n)}) \cup \{0\}.$$

Let $(n_k)_{k \geq 0}$ be the subsequence of times used to defined absolute Roth type condition (a)' and let $(\tilde{n}_k)_{k \geq 0}$ be the subsequence of times defined in Remark 3.2 so that the matrix $B(\tilde{n}_{k-1}, \tilde{n}_k)$ has positive coefficients. Fix some $k \geq 0$. Let $j := j(k)$ be the smallest l such that $\tilde{n}_l \geq n_k$. We claim that we have the inequalities

$$\tilde{n}_{j-1} < n_k \leq \tilde{n}_j, \quad n_{k+1} \leq \tilde{n}_{j+1}.$$

The first set of inequalities follow from the definition of $j = j(k)$; to prove the last inequality, notice that, since $B(\tilde{n}_j, \tilde{n}_{j+1}) > 0$, also $B(n_k, \tilde{n}_{j+1}) > 0$ and thus by (3.4) we have that

$$\overline{B(n_k, \tilde{n}_{j+1})(\mathcal{C}(\pi^{(n_k)}))} \subset \mathcal{C}(\pi^{(\tilde{n}_{j+1})}) \cup \{0\}.$$

This, by definition of the sequence $(n_k)_{k \geq 0}$, implies that $n_{k+1} \leq \tilde{n}_{j+1}$.

Recall that by Remark 3.2, condition (a) is equivalent to asking that for all $\tau > 0$, there exists $C_\tau > 0$ such that $\|B(\tilde{n}_{k-1}, \tilde{n}_k)\| \leq C_\tau \|B(\tilde{n}_0, \tilde{n}_{k-1})\|^\tau$ for all positive integers k . Thus, using the inequalities among indexes and the cocycle identity, we get

$$\begin{aligned} \|B(n_k, n_{k+1})\| &\leq \|B(\tilde{n}_{j-1}, \tilde{n}_{j+1})\| \leq C_\tau^2 \|B(0, \tilde{n}_{j-1})\|^\tau \|B(0, \tilde{n}_j)\|^\tau \\ &\leq C_\tau^2 \|B(0, \tilde{n}_{j-1})\|^\tau \|B(0, n_k)\|^\tau \|B(n_k, \tilde{n}_j)\|^\tau \\ &\leq C_\tau^2 \|B(0, n_k)\|^{2\tau} \|B(\tilde{n}_{j-1}, \tilde{n}_j)\|^\tau \leq C' \|B(0, n_k)\|^{2\tau + \tau^2}. \end{aligned}$$

for some $C' = C'(\tau)$. Since $\tau > 0$ was arbitrary, this shows that (a)' holds in this case. \square

Remark 3.7. — On the other hand, the converse of Lemma 3.6 is in general not true (for $s > 1$), namely there exists i.e.m. which are of absolute Roth type but *not* of Roth type. For instance when $d = 3$, one can construct i.e.m. which are obtained from a Roth type rotation number by marking a point and there is a set of measure zero but full Hausdorff dimension of marked points for which the 3 i.e.m. fails to be Roth type.

Finally, let us remark that since (restricted) Roth type i.e.m. have full measure, from Lemma 3.6 it also follows that (restricted) absolute Roth type i.e.m. also have full measure.

3.4. Two crucial estimates. — We state and prove in this section two crucial estimates on the lengths of induced subintervals (see Corollary 3.11 and Proposition 3.13) which are needed in particular to prove results on the cohomological equation, but are also of independent interest (see for example [32, 33]). These estimates were proved for i.e.m. of Roth type in [35]. They require a different proof for i.e.m. of *absolute* Roth type, which is given in this section. Once these estimates are proved, one can easily generalize all the main results on the cohomological equation proved in [35, 38] for (restricted) Roth type i.e.m. to (restricted) absolute Roth type i.e.m. (see § 3.5 and Appendix B). We present the proof of these estimates this section, since they show how the absolute Roth type condition is used.

Let T be an i.e.m. with combinatorial data in \mathcal{R} and no connection. Let $\underline{\gamma}$ be the rotation number of T . Let $(T^{(n)})_{n \geq 0}$ be the sequence of i.e.m. obtained from $T =: T^{(0)}$ by the Rauzy-Veech algorithm. Recall that, for $n \geq 1$, $T^{(n)}$ acts on $I^{(n)} \subset I^{(n-1)}$ exchanging the subintervals $I_\alpha^{(n)} := I_\alpha^{(n)}(T^{(n)})$, $\alpha \in \mathcal{A}$. Recall from [35] (see also Lemma 3.14 of [38]) that condition (a) in the Roth type definition implies the following estimates on the lengths of induced subintervals.

Proposition 3.8. — *Assume that $\underline{\gamma}$ satisfies condition (a). Then, for any $\tau > 0$, there exists $C_\tau > 0$ such that, for any $\alpha \in \mathcal{A}$, any $n \geq 0$, one has*

$$C_\tau^{-1} |I^{(0)}| \|B(0, n)\|^{-1-\tau} \leq |I_\alpha^{(n)}| \leq C_\tau |I^{(0)}| \|B(0, n)\|^{-1+\tau}.$$

Remark 3.9. — One can also see (cf. [37] Appendix C, Proposition C1, p. 1639) that condition (a) of Roth type is equivalent to the following: for all $\tau > 0$, there exists $C > 0$ such that for all $n \in \mathbb{N}$, we have

$$\max_{\alpha \in \mathcal{A}} |I_\alpha^{(n)}| \leq C \left(\min_{\alpha \in \mathcal{A}} |I_\alpha^{(n)}| \right)^{1-\tau}.$$

We will find analogous estimates to those provided by Prop. 3.8 when $\underline{\gamma}$ satisfies only the weaker condition (a'). We deal first with the upper bound, which is the easiest one (and the same than in Proposition 3.8). The upper bound will follow from the following proposition (see Corollary 3.11 below).

Proposition 3.10. — Assume that $\underline{\gamma}$ satisfies condition (a'). Then, for any $\tau > 0$, there exists $C_\tau > 0$ such that, for any $\alpha \in \mathcal{A}$, any $n \geq 0$, one has

$$\sum_{\beta \in \mathcal{A}} B_{\alpha\beta}(0, n) \geq C_\tau^{-1} \|B(0, n)\|^{1-\tau}.$$

Let us state and prove the corollary giving the upper bound before giving the proof of Proposition 3.10.

Corollary 3.11. — For any $\tau > 0$, there exists $C_\tau > 0$ such that, for any $\alpha \in \mathcal{A}$, any $n \geq 0$, one has

$$|I_\alpha^{(n)}| \leq C_\tau |I^{(0)}| \|B(0, n)\|^{-1+\tau}.$$

Proof. — The upper bound follows immediately from Proposition 3.10 combined with the simple observation that

$$\sum_{\alpha, \beta \in \mathcal{A}} B_{\alpha\beta}(0, n) |I_\alpha^{(n)}| = |I^{(0)}|. \quad \square$$

For the proof of Proposition 3.10, we need the following lemma.

Lemma 3.12. — There exists a constant $C = C(\mathcal{R})$ with the following property. Let $\gamma : \pi \rightarrow \pi'$ be a finite path in \mathcal{D} such that

$$\overline{B_\gamma(\mathcal{L}(\pi))} \subset \mathcal{L}(\pi') \cup \{0\}.$$

For any vector $w \in \overline{B_\gamma(\mathcal{L}(\pi))}$, one has

$$\max_\alpha w_\alpha \leq C \|B_\gamma\| \min_\alpha w_\alpha.$$

Proof. — Fix $\pi \in \mathcal{R}$. As the subspace $H(\pi)$ is rational, one can find primitive integral vectors $v^i \in H(\pi)$ which span the extremal rays of $\mathcal{L}(\pi)$. Let $C(\pi)$ be the supremum over i of the $\|v^i\|$. Let $\gamma : \pi \rightarrow \pi'$ be as in the lemma, and let $w^i := B_\gamma v^i$. Each w^i is an integral vector with positive coordinates. We have therefore

$$\max_\alpha w_\alpha^i \leq \|B_\gamma\| \|v^i\| \leq C(\pi) \|B_\gamma\| \leq C(\pi) \|B_\gamma\| \min_\alpha w_\alpha^i.$$

For $v \in \overline{\mathcal{C}(\pi)}$, $w = B_\gamma v$, we write $v = \sum_i t_i v^i$ with $t_i \geq 0$. For $\alpha, \beta \in \mathcal{A}$, we obtain

$$w_\alpha = \sum_i t_i w_\alpha^i \leq C(\pi) \|B_\gamma\| \sum_i t_i w_\beta^i = C(\pi) \|B_\gamma\| w_\beta.$$

Therefore the conclusion of the lemma holds with $C(\mathcal{R}) := \max_{\pi \in \mathcal{R}} C(\pi)$. □

Proof of Proposition 3.10. — We first prove the inequality of the proposition when $n = n_k$. Fix $u \in \mathcal{C}(\pi^{(n_0)})$; let

$$v = B(n_0, n_{k-1})u, \quad w = B(n_0, n_k)u = B(n_{k-1}, n_k)v.$$

For $\alpha, \alpha' \in \mathcal{A}$, using Lemma 3.12, condition (a)' and that $\|B(0, n)\|$ is non decreasing in n , one has that

$$\begin{aligned} \sum_{\beta \in \mathcal{A}} B_{\alpha\beta}(0, n_k) &\leq (\min_\beta u_\beta)^{-1} \sum_{\beta \in \mathcal{A}} B_{\alpha\beta}(0, n_k) u_\beta = (\min_\beta u_\beta)^{-1} w_\alpha \\ &\leq C \|B(n_{k-1}, n_k)\| (\min_\beta u_\beta)^{-1} w_{\alpha'} \\ &\leq CC'_\tau \|B(0, n_{k-1})\|^\tau (\min_\beta u_\beta)^{-1} \sum_{\beta \in \mathcal{A}} B_{\alpha'\beta}(0, n_k) u_\beta \\ &\leq CC'_\tau \|B(0, n_k)\|^\tau \frac{\max_\beta u_\beta}{\min_\beta u_\beta} \sum_{\beta \in \mathcal{A}} B_{\alpha'\beta}(0, n_k). \end{aligned}$$

As $\sum_{\alpha, \beta} B_{\alpha\beta}(0, n_k)$ is of the order of $\|B(0, n_k)\|$, we obtain the inequality of the proposition for $n = n_k$.

When $n_k < n < n_{k+1}$, we use that each $B_{\alpha, \beta}(0, n)$ is a non-decreasing function of n . For $\alpha \in \mathcal{A}$, we have

$$\sum_{\beta \in \mathcal{A}} B_{\alpha\beta}(0, n) \geq \sum_{\beta \in \mathcal{A}} B_{\alpha\beta}(0, n_k) \geq C_\tau^{-1} \|B(0, n_k)\|^{1-\tau}.$$

On the other hand, we have

$$\|B(0, n)\| \leq \|B(0, n_k)\| \|B(n_k, n_{k+1})\| \leq \tilde{C}_\tau \|B(0, n_k)\|^{1+\tau}.$$

As $\tau > 0$ is arbitrary, we obtain the estimate of the proposition. □

The estimate of $|I_\alpha^{(n)}|$ from below, as stated in Proposition 3.8, is *not* a consequence of condition (a') (even for $d = 3$). The true statement is slightly more sophisticated:

Proposition 3.13. — *Assume that $\underline{\gamma}$ satisfies condition (a'). Then, for any $\tau > 0$, there exists $C_\tau > 0$ such that, for any $\alpha \in \mathcal{A}$, any $\ell \geq 0$, one has*

$$|I_\alpha^{(n_\ell)}| \geq C_\tau^{-1} |I^{(0)}| \|B(0, n_k)\|^{-\tau-1}$$

where k is the largest integer greater or equal to ℓ such that $\sum_\beta B_{\beta\alpha}(n_\ell, n_k) < s$.

Proof. — Fix $\alpha \in \mathcal{A}$. Let n_k be as in the proposition. We have $\sum_{\beta} B_{\beta\alpha}(n_{\ell}, n_{k+1}) \geq s$. It means that the interval $I_{\alpha}^{(n_{\ell})}$ decomposes into at least s subintervals of level n_{k+1} . Therefore there are at least $s+1$ points (including the endpoints of $I_{\alpha}^{(n_{\ell})} = I_{\alpha}^t(T^{(n_{\ell})})$) bounding these subintervals and two of them must correspond to the *same* marked point. The (horizontal) segment inside the transversal which has these two points as endpoints hence give an *absolute* homology class ζ , which we express in the base $\zeta_{\alpha}^{(n_{k+1})}$, $\alpha \in \mathcal{A}$, as

$$\zeta := \sum_{\beta} x_{\beta} \zeta_{\beta}^{(n_{k+1})}.$$

The coefficients x_{β} are integral, non-negative and the vector $x = (x_{\beta})$ belongs to $H(\pi^{(n_{k+1})})$.

Let us also represent ζ at level n_{k+2} as

$$\zeta := \sum_{\beta} X_{\beta} \zeta_{\beta}^{(n_{k+2})}.$$

The vectors x and X are related by

$$X = B(n_{k+1}, n_{k+2})x.$$

As $x \in \overline{\mathcal{O}(\pi^{(n_{k+1})})}$ and the coefficients X_{β} are integers, we get from the definition of n_{k+2} that $X_{\beta} \geq 1$ for all $\beta \in A$. We therefore have

$$|I_{\alpha}^{(n_{\ell})}| \geq \int_{\zeta} \operatorname{Re} \omega \geq \sum_{\beta} \int_{\zeta_{\beta}^{(n_{k+2})}} \operatorname{Re} \omega = \sum_{\beta} \lambda_{\beta}^{(n_{k+2})} \geq |I^{(n_{k+2})}|.$$

One has also

$$|I_{\alpha}^{(0)}| = \sum_{\beta, \mu} B_{\beta\mu}(0, n_{k+2}) |I_{\beta}^{(n_{k+2})}| \leq |I^{(n_{k+2})}| \sum_{\beta, \mu} B_{\beta\mu}(0, n_{k+2}),$$

and we conclude from condition (a'),

$$|I_{\alpha}^{(n_{\ell})}| \geq C^{-1} |I^{(0)}| \|B(0, n_{k+2})\|^{-1} \geq C_{\tau}^{-1} |I^{(0)}| \|B(0, n_k)\|^{-\tau-1}. \quad \square$$

3.5. Results on the cohomological equation for absolute Roth type i.e.m. — We show in this paper that the main results on the cohomological equations for i.e.m. which were proved in [38] under the (restricted) Roth type diophantine condition, actually hold under the weaker (restricted) *absolute Roth type* condition. One of the motivations for this weakening is shown in §3.6.

As recalled in the introduction, motivated by Forni's celebrated paper [22] on the cohomological equation for area preserving flows on surfaces of higher genus, in [35] Roth type i.e.m. were introduced in order to provide an *explicit* diophantine condition sufficient to guarantee the existence of a solution to the cohomological equation up to finitely many obstructions. More precisely, in [35] it is proven that, given a sufficiently regular datum (namely, for functions which are of absolutely continuous on each of the intervals I_{α}^t with derivative of bounded variation and of zero mean on I), after subtracting (to the datum) a correction function which is constant on each I_{α}^t , the

cohomological equation has a bounded solution. In [37] this result was used for proving a linearization theorem for generalized interval exchange maps with a rotation number of restricted Roth type. More recently, for restricted Roth type i.e.m., in [38] a more refined regularity result was proved, namely the Hölder regularity of the solutions of the cohomological equation with C^r , $r > 1$ data.

In this section we limit ourselves to the statement of the generalization of the main theorem of [38] (Theorem 3.10, p.127). In Appendix B we will give the proof of this theorem, as well as the statement and the proof of the generalizations of other two results of [38]: the growth estimate for special Birkhoff sums with C^1 data (Theorem 3.11, p. 128) and the regularity result for the solutions of the cohomological equation in higher differentiability (Theorem 3.22, p. 138).

Let T be an i.e.m. and let $r \geq 0$ be a real number. The boundary operator ∂ which appears in the following result is the operator defined in in §2.7. We denote by $C_0^r(\bigsqcup I_\alpha)$ the kernel of the boundary operator ∂ in $C^r(\bigsqcup I_\alpha)$. We recall from §2.7 that the coboundary of a continuous function belongs to the kernel of the boundary operator (it follows from properties (1) and (2) of Proposition 2.3).

The following result generalizes Theorem 3.10 of [38] to *absolute* restricted Roth type i.e.m.

Theorem 3.14 (Hölder solutions to the cohomological equation)

Let T be an i.e.m. of absolute restricted Roth type. Let $\Gamma_u(T)$ be a subspace of $\Gamma_\partial(T)$ which is supplementing $\Gamma_s(T)$ in $\Gamma_\partial(T)$. Let $r > 1$ be a real number. There exist $\bar{\delta} > 0$ and bounded linear operators $L_0 : \varphi \mapsto \psi$ from $C_0^r(\bigsqcup I_\alpha)$ to the space $C^{\bar{\delta}}([u_0, u_d])$ of Hölder continuous functions of exponent $\bar{\delta}$ and $L_1 : \varphi \mapsto \chi$ from $C_0^r(\bigsqcup I_\alpha)$ to $\Gamma_u(T)$ such that any $\varphi \in C_0^r(\bigsqcup I_\alpha)$ satisfies

$$\varphi = \chi + \psi \circ T - \psi, \quad \int_{u_0}^{u_d} \psi(x) dx = 0.$$

The operators L_0 and L_1 are uniquely defined by the conclusions of the theorem.

Remark 3.15. — The exponent $\bar{\delta}$ depends only on r and the constants θ, σ appearing in §3.1. Thus, since in the difference of definition between absolute Roth type and Roth type does not concern the two constants θ (spectral gap) e σ (in the definition of stable space), the exponent $\bar{\delta}$ in this Theorem 3.14 is the same than the exponent in Theorem 3.10 of [38]. One can show that $\bar{\delta}$ tends to 0 as $r > 1$ tends to 1 and conjecture that one can take any $\bar{\delta} > 0$ for any fixed $r > 1$. This is indeed the case in the Sobolev scale, as proved by Forni in [25]. Very recently, this optimal loss of derivatives was also proved in the Hölder class in the pseudo-Anosov case (namely in the case of periodic Rauzy-Veech induction), see [20].

The proofs of these theorems for i.e.m. of restricted absolute Roth type follow essentially the same proofs given in the respective papers for restricted Roth type. The main difference is in the proof of two crucial estimates on lengths of subintervals of i.e.m. in a Rauzy-Veech orbit, presented above in §3.4. Once these estimates are

proved, the proofs of the theorems for restricted Roth type i.e.m. can be followed and adapted with minor modifications to the case of restricted absolute Roth type i.e.m. For completeness, in the Appendix B we follow the original proof and indicate, for convenience of the interested reader, where and which modifications are needed.

3.6. Results on the cohomological equation for translation surfaces in a.e. direction. —

The notion of (absolute) Roth type i.e.m. naturally leads to define (absolute) Roth type translation surfaces: the following definition of Roth type translation surfaces was given in [37] and we now extend it to absolute Roth type.

Definition 3.16. — Let (M, Σ, ζ) be a translation surface. It is of *absolute Roth type* if there exists *some* open bounded horizontal segment I in good position such that the return map T_I of the vertical flow on I is an i.e.m. of *absolute Roth type*. Absolute restricted Roth type, restricted Roth type and Roth type translation surfaces are defined analogously.

In Appendix C of [37] it is proved that for a (restricted) Roth type translation surface, T_I will actually be of (restricted) Roth type for *any* horizontal segment I in good position. Analogously, one can show that for an absolute (restricted) Roth type translation surface, T_I will be of absolute (restricted) Roth type for *any* horizontal segment I in good position.

In [13] Chaika and Eskin prove that, for all translation surfaces (M, Σ, ζ) and for almost every angle θ , the translation surface $(M, \Sigma, r_\theta \cdot \zeta)$ obtained by rotating by r_θ the translation structure ζ is generic for the Teichmüller geodesic flow and Oseledets generic for the Kontsevich-Zorich cocycle. As an application of this result, they prove the following lemma, which we now state using the definition of absolute Roth type translation surface just given above.⁽²⁾

Lemma 3.17 ([13]). — *For all translation surfaces (M, Σ, ζ) , for almost every angle θ , the translation surface $(M, \Sigma, r_\theta \cdot \zeta)$ is of absolute Roth type.*

Remark 3.18. — Furthermore, it also follows from [13] that⁽³⁾ if the orbit closure $\mathcal{M} := \overline{SL(2, \mathbb{R}) \cdot (M, \Sigma, \zeta)}$ under the linear action of $SL(2, \mathbb{R})$ is such that $\lambda_g > 0$, where $\pm\lambda_i$, for $1 \leq i \leq g$ here denote the Lyapunov exponents of the Kontsevich-Zorich cocycle restricted to the locus \mathcal{M} , then, for almost every angle θ , the translation surface $(M, \Sigma, r_\theta \cdot \zeta)$ is of *restricted absolute Roth type*.

⁽²⁾ In Section 1.2.2 of [13], the authors define a diophantine condition on the vertical flow of the translation surface, called in the paper *Roth type* and consisting of three conditions (a), (b) and (c) (see Definition 1.13, [13]). Their definition is given in a geometric language, but one can check that it is equivalent to our notion of *absolute Roth type* translation surface (Definition 3.16 above), and not to the definition of Roth types surface given in [35]. Lemma 1.16 in [13] shows that Condition (a) holds for a.e. θ , while the proof that (b) and (c) also holds for a.e. θ is given at the end of Section 1.2.2.

⁽³⁾ More precisely, it follows from Lemma 1.16 and Thm 1.4 of [13], see the comment at the very end of § 1.2.2 in [13].

Combining this remark with the Theorem 3.14 proved in this paper, we get the following corollary ⁽⁴⁾.

Corollary 3.19. — *Assume that (M, Σ, ζ) is such that the orbit closure $SL(2, \mathbb{R}) \cdot (M, \Sigma, \zeta)$ has $\lambda_g > 0$. Then, for almost every angle θ , any i.e.m. T obtained as first return map of the linear flow in direction θ on (M, Σ, ζ) on a segment in good position satisfies the conclusion of Theorem 3.14. Thus, for any $r > 1$ there exists $\bar{\delta} > 0$ such that for any $\varphi \in C_0^r(\bigsqcup I_\alpha^t(T))$, there exists a Hölder continuous function $\psi \in C^{\bar{\delta}}([u_0(T), u_d(T)])$ with zero mean and a piecewise constant correction $\chi \in \Gamma_u(T)$ such that $\varphi = \chi + \psi \circ T - \psi$.*

Remark 3.20. — We remark that results on the cohomological equation for almost all directions on a fixed translation surface were first proved by Forni in [22, 25]. The result in this corollary, compared with the latter results, improves on the loss of derivatives. However, a stronger assumption is assumed, namely hyperbolicity of the KZ cocycle ⁽⁵⁾. While this paper was under review, two new results on the cohomological equation have appeared: in [26] the Diophantine condition from [35] is weakened to allow the presence of zero Lyapunov exponents, but the loss of derivatives is worst than in our (or [35]) results; in [20], the optimal loss of derivatives without hyperbolicity assumptions is proved, but only in the pseudo-Anosov (periodic Rauzy-Veech induction) special case (a measure zero class).

4. Dual Roth Type

We define in this section another Diophantine condition for translation surfaces which we call *dual Roth type*. In order to define it, we need first to introduce *dual special Birkhoff sums* (see § 4.2). This is a notion which we believe is of independent interest, which is based on a form of *duality* for Rauzy-Veech induction (which heuristically correspond to considering backward time in the Teichmüller geodesic flow and flipping the role of horizontal and vertical flow on translation surfaces). It should be noticed that this type of duality was exploited also in the work of Bufetov [11], who discovered a duality between invariant distributions for the horizontal (vertical) flow and finitely additive functionals for the vertical (horizontal) flow. An analogous duality was later found by Bufetov and Forni for horocycle flows on hyperbolic surfaces in

⁽⁴⁾ In [13], a result on the solvability of the cohomological equation (under finitely many obstructions) for all translation surfaces in a.e. direction is stated (see the paragraph after Theorem 1.14 in [13]) and deduced from Lemma 3.17 and [35], but since the Definition 1.13 in [13] is *not* equivalent to Roth type (but to absolute Roth type, see previous footnote), the results from [35] cannot be directly applied, but require the strengthening proved in this paper. In private communications, though, Chaika and Eskin told us that their proof can be modified to actually show that all translation surfaces are of Roth type for a.e. θ .

⁽⁵⁾ We already recalled that the Masur-Veech measure is hyperbolic, as proved in [23]. Results on the presence (or absence) of zero Lyapunov exponents and hyperbolicity for other $SL(2, \mathbb{R})$ invariant measures appear in several works in the literature, see for example [24, 21].

their work [10]. Our motivation for introducing a dual special Birkhoff sums operator and dual Roth type conditions in the combinatorial set up of Rauzy-Veech induction came from the results proved in Section 5.

4.1. Backward Rauzy-Veech induction and backward rotation numbers. — Let $M := M(\pi, \lambda, \tau)$ be a translation surface constructed from combinatorial data $\pi \in \mathcal{R}$, length data $\lambda \in \mathcal{L}$ and suspension data $\tau \in \Theta_\pi$. We assume in this section that M has *no horizontal saddle connections*. Then, starting from $(\pi^{(0)}, \lambda^{(0)}, \tau^{(0)}) := (\pi, \lambda, \tau)$, we can iterate indefinitely the Rauzy-Veech *backward* algorithm and (using the notation recalled in §2) construct the sequence as $\zeta_\alpha^{(n)} = \lambda_\alpha^{(n)} + i\tau_\alpha^{(n)}$ for $n \leq 0$ follows:

- when $\sum_\alpha \tau_\alpha^{(n)} < 0$, we perform the inverse of an elementary *top* step of the Rauzy-Veech algorithm

$$\begin{aligned}\pi^{(n-1)} &= R_t^{-1} \left(\pi^{(n)} \right) \\ \zeta_{\alpha_w}^{(n-1)} &= \zeta_{\alpha_w}^{(n)} + \zeta_{\alpha_\ell}^{(n)}, \\ \zeta_\alpha^{(n-1)} &= \zeta_\alpha^{(n)}, \quad \forall \alpha \neq \alpha_w,\end{aligned}$$

where α_w, α_ℓ are the letters such that

$$\pi_t^{(n)}(\alpha_w) = d, \quad \pi_b^{(n)}(\alpha_\ell) = \pi_b^{(n)}(\alpha_w) + 1;$$

- when $\sum_\alpha \tau_\alpha^{(n)} > 0$, we perform the inverse of an elementary *bottom* step of the Rauzy-Veech algorithm

$$\begin{aligned}\pi^{(n-1)} &= R_b^{-1} \left(\pi^{(n)} \right) \\ \zeta_{\alpha_w}^{(n-1)} &= \zeta_{\alpha_w}^{(n)} + \zeta_{\alpha_\ell}^{(n)}, \\ \zeta_\alpha^{(n-1)} &= \zeta_\alpha^{(n)}, \quad \forall \alpha \neq \alpha_w,\end{aligned}$$

where α_w, α_ℓ are the letters such that

$$\pi_b^{(n)}(\alpha_w) = d, \quad \pi_t^{(n)}(\alpha_\ell) = \pi_t^{(n)}(\alpha_w) + 1.$$

The case $\sum_\alpha \tau_\alpha^{(n)} = 0$ never occurs because there are no horizontal saddle connections.

We obtain in this way a *backward rotation number*, i.e., an infinite path $\underline{\gamma}$ in \mathcal{D} with terminal point $\pi^{(0)}$ of the form

$$\underline{\gamma} = \cdots \star \gamma_{n-1} \star \gamma_n \star \cdots \star \gamma_{-1} \star \gamma_0,$$

where $\gamma_n, n \geq 0$, is the arrow from $\pi^{(n-1)}$ to $\pi^{(n)}$.

Remark 4.1. — Remark that the equations for $\zeta_\alpha^{(n)} = \lambda_\alpha^{(n)} + i\tau_\alpha^{(n)}$ split into independent sets of equations for $\lambda^{(n)}$ and $\tau^{(n)}$. Since the type of backward move (top or bottom) depends only on $\tau^{(n)}$, it follows that the backward rotation number $\underline{\gamma}$ only depends on (π, τ) .

Remark 4.2. — The backward rotation number is defined as soon as the preferred rightwards separatrix in M (issued from the leftmost vertex O of the polygon) is not a saddle connection. Other horizontal connections are apparently not detected by the algorithm (i.e the algorithm does not stop in this case).

In particular, if τ is such that its components τ_α , $\alpha \in \mathcal{A}$, are irrationally independent over \mathbb{Q} , then the backward Rauzy algorithm does not stop. Thus, to any π and a.e. $\tau \in \Theta_\pi$, one can associate to the pair (π, τ) a backward rotation number.

Recall that a finite path in \mathcal{D} is *complete* if every letter in \mathcal{A} is the winner of at least one of its arrows and that an infinite path is ∞ -*complete* if it is the concatenation of infinitely many finite complete paths.

Proposition 4.3 (Completeness of backward rotation numbers)

When $M = M(\pi, \lambda, \tau)$ has no horizontal connection, the backward rotation number $\underline{\gamma}$ associated to (π, τ) is ∞ -complete.

We give the proof of this proposition in the Appendix A, since the proof is quite long and rather involved combinatorially. It would be interesting to know whether the converse to the above proposition is true (see Remark 4.2 above).

4.2. Dual special Birkhoff sums. — Let $M(\pi, \lambda, \tau)$ be a translation surface as above that has *no horizontal saddle connections*. Thus, we have well defined data $(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$ for any $n \in \mathbb{Z}$. For any $n \in \mathbb{Z}$ we denote by $T^{(n)}$ the i.e.m. with data $(\pi^{(n)}, \lambda^{(n)})$. Let $I^{(n)} = I(T^{(n)})$ be the sequence of intervals on which $T^{(n)}$ acts for $n \in \mathbb{Z}$.

Let us recall that special Birkhoff sums (see (2.9)) are operators $S(m, n)$, where $m \leq n$, which map from functions defined on $\bigsqcup I_\alpha^{(m)}$ to functions defined on $\bigsqcup I_\alpha^{(n)}$ and that, when acting on the space of piecewise constant functions, $S(m, n) : \Gamma(m) \rightarrow \Gamma(n)$ is represented by the KZ cocycle matrix $B(m, n)$ (see § 2.6).

We will now define a dual operator, which we call *dual special Birkhoff sums*. It is useful to remark that this operator plays for the *horizontal* linear flow on a translation surface the role of the special Birkhoff sums play for the *vertical* flow.

Let $q^{(n)}$ be the vectors given by $q^{(n)} = -\Omega_{\pi^{(n)}} \tau^{(n)}$, so that the translation surface $M(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$ can be represented as a zippered rectangle over $T^{(n)}$ with heights $q^{(n)}$ (see § 2.2). For $n \in \mathbb{Z}$, $\alpha \in \mathcal{A}$, let

$$L_\alpha^{(n)} := \left(0, q_\alpha^{(n)}(\pi, \tau)\right), \quad L^{(n)} := \bigsqcup_{\alpha \in \mathcal{A}} L_\alpha^{(n)}.$$

The disjoint union $L^{(n)}$ will be the domain of the functions on which the n -th special Birkhoff sum operator will act. Geometrically, one should think of the intervals $L_\alpha^{(n)}$, $\alpha \in \mathcal{A}$, as *vertical* intervals, one for each zippered rectangle representation with heights $q^{(n)}$ of the translation surface $M(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$. Remark that, for any $n \leq 0$, $q_\alpha^{(n)}$ depends only on π and τ (see Remark 4.1).

For $n' \leq n$, $\alpha \in \mathcal{A}$, we can write the interval $L_\alpha^{(n)}$ as a disjoint union (modulo 0) of a number of translated copies of the $L_\beta^{(n')}$ as follows (see also Figure 1).

Let us recall that $B_\alpha(n', n) := \sum_{\beta \in \mathcal{A}} B(n', n)_{\alpha\beta}$ gives the return time in $I^{(n)}$ of $I_\alpha^{(n)}$ under $T^{(n')}$ (see § refsec:algorithms). For $0 \leq j < B_\alpha(n', n)$, let $\beta(\alpha, j)$ be the letter in \mathcal{A} such that

$$(T^{(n')})^j(I_\alpha^{(n')}) \subset I_{\beta(\alpha, j)}^{(n')}.$$

Then, for any $\alpha \in \mathcal{A}$, we can write

$$q_\alpha^{(n)} = \sum_{0 \leq j < B_\alpha(n', n)} q_{\beta(\alpha, j)}^{(n')}.$$

Correspondingly, one can write the interval $L_\alpha^{(n)}$, which has length $q_\alpha^{(n)}$, as union of intervals of lengths $q_{\beta(\alpha, l)}^{(n')}$, each of which is translated copy of the corresponding interval $L_{\beta(\alpha, l)}^{(n')}$, i.e.,

$$L_\alpha^{(n)} = \bigcup_{0 \leq j < B_\alpha(n', n)} \left(L_{\beta(\alpha, j)}^{(n')}(\pi, \tau) + \text{Sq}(j) \right), \quad \text{where } \text{Sq}(j) := \sum_{0 \leq i < j} q_{\beta(\alpha, i)}^{(n')}(\pi, \tau).$$

Thus, collecting together all j such that $\beta(\alpha, j) = \beta$ for a fixed $\beta \in \mathcal{A}$ (there are $B(n', n)_{\alpha, \beta}$ of them) and then denoting by $j_{\ell, \alpha, \beta}^{(n', n)}(L_\beta^{(n')})$ the ℓ -th shifted copy of $L_\beta^{(n')}$, for $0 \leq \ell < B(n', n)_{\alpha, \beta}$, we can write

$$(4.1) \quad L_\alpha^{(n)} = \bigcup_{\substack{\beta \in \mathcal{A} \\ \text{s. t. } \beta(\alpha, j) = \beta}} \bigcup_{0 \leq j < B_\alpha(n', n)} \left(L_{\beta(\alpha, j)}^{(n')} + \text{Sq}(j) \right) = \bigcup_{\beta \in \mathcal{A}} \bigcup_{0 \leq \ell < B(n', n)_{\alpha, \beta}} j_{\ell, \alpha, \beta}^{(n', n)}(L_\beta^{(n')}).$$

We define for every $n' \leq n$, an operator $S^\sharp(n, n')$ that sends functions on $L^{(n)}$ to functions on $L^{(n')}$. Let ψ be a function defined on $L^{(n)}$. Then the image $S^\sharp(n, n')\psi$ is the function $\Psi(x)$ on $L^{(n')}$ defined as follows:

$$(4.2) \quad \begin{aligned} S^\sharp(n, n')\psi &= \Psi(x) := \sum_{\alpha \in \mathcal{A}} \sum_{\substack{0 \leq j < B_\alpha(n', n) \\ \text{s. t. } \beta(\alpha, j) = \beta}} \left(L_{\beta(\alpha, j)}^{(n')} + \text{Sq}(j) \right) \\ &= \sum_{\alpha \in \mathcal{A}} \sum_{0 \leq \ell < B(n', n)_{\alpha, \beta}} \psi(j_{\ell, \alpha, \beta}^{(n', n)}(x)), \quad \text{if } x \in L_\beta^{(n)}. \end{aligned}$$

We will call $S^\sharp(n, n')\psi$ the *dual special Birkhoff sums* and $S^\sharp(n, n')$ *dual special Birkhoff sum operator*.

Let $\Gamma^\sharp(n)$ be the space of functions on $L^{(n)}$ which are constant on each $L_\alpha^{(n)}$; it is canonically isomorphic to $\mathbb{R}^{\mathcal{A}}$. The operator $S^\sharp(n, n')$ sends $\Gamma^\sharp(n)$ to $\Gamma^\sharp(n')$ and the matrix w.r.t. the canonical bases is the transpose matrix ${}^t B(n', n)$ (notice that here the order of n, n' is reverted). This explains the name *dual Birkhoff sum operator* for $S^\sharp(n, n')$.

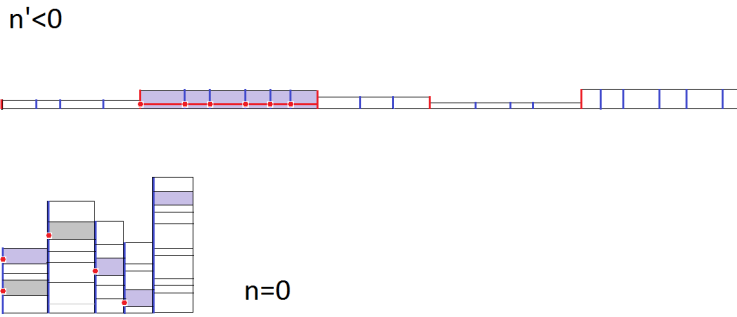


FIGURE 1. Geometric interpretations of the *dual* special Birkhoff sums $S^\sharp(n, n')$, $n' < n$.

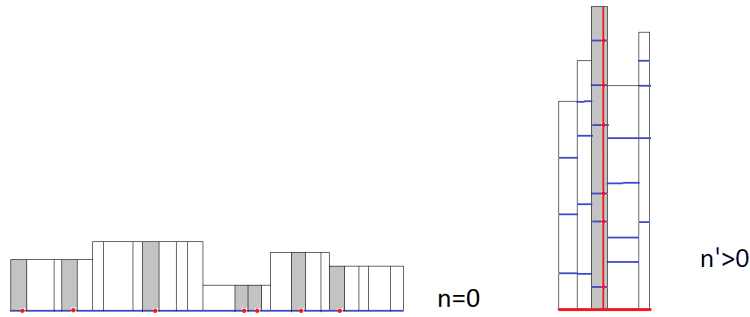


FIGURE 2. Geometric interpretations of the *direct* special Birkhoff sums $S(n, n')$, $n' > n$.

Remark 4.4. — One can interpret this definition as a *special Birkhoff sum* for a Poincaré map of the *horizontal* translation flow as follows. For each $n' \leq n$, $\alpha \in \mathcal{A}$, identify each interval $L_\alpha^{(n')}$ with the left vertical side of the rectangle $R_\alpha^{(n')} = I_\alpha^{(n')} \times [0, q_\alpha^{(n')}]$ of the zippered rectangle presentation of $M(\pi^{(n')}, \lambda^{(n')}, \tau^{(n')})$. For $x \in L_\beta^{(n')}$, consider the leaf of the horizontal flow that starts at x , up to the first time it hits again the union $\bigcup_{\alpha \in \mathcal{A}} L_\alpha^{(n')}$ (see the top part of Figure 1, the horizontal leaf is contained in the colored column). This piece of horizontal leaf hits the union $L^{(n)}$ (on which ψ is defined) in exactly $\sum_\alpha B(n', n)_{\alpha, \beta}$ points (also shown in top part of Figure 1, as dots along the horizontal leaf). The dual special Birkhoff sum $S^\sharp(n, n')\psi(x)$ is obtained by adding the values of the function ψ at these hitting points.

To highlight the duality with (direct) special Birkhoff sums, let us remark that the special Birkhoff sums can be defined similarly as follows (refer to Figure 2). Given

$n'' > n$ and a function ϕ defined on $I^{(n)}$, $S^\sharp(n, n'')\phi(x)$ is obtained by considering a leaf of the *vertical* flow starting from $x \in I_\beta^{(n'')}$ up to its first return to $I^{(n'')}$ and summing up the values of the function ϕ at the hitting points of this piece of a leaf with the interval $I^{(n)}$ (as shown in the right picture of Figure 2). One can also give an equivalent definition of (direct) special Birkhoff sum similar (and dual to) the Definition (4.2). Namely, the interval $I_\alpha^{(n)}$ can be written as a union of translated copies of the intervals $I_\beta^{(n'')}$ (as one can see in the left picture of Figure 2)). Let $j_{\ell, \alpha, \beta}^{(n, n'')}(I_\beta^{(n'')})$ be the ℓ 's copy of the interval $I_\beta^{(n'')}$ inside $I_\alpha^{(n)}$, where $0 \leq \ell < B(n, n'')_{\alpha, \beta}$. Then

$$S(n, n'')\phi(x) = \sum_{\alpha \in \mathcal{A}} \sum_{0 \leq \ell < B(n, n'')_{\alpha, \beta}} j_{\ell, \alpha, \beta}^{(n, n'')}(x).$$

Conversely, in order to give a definition of dual special Birkhoff sum formulated in a similar way to the classical definition of (direct) special Birkhoff sums (i.e., as a sum over an orbit of the induced i.e.m. $T^{(n)}$ up to its first return to $I^{(n'')}$), one would need to describe the applications that arise as first return maps of the horizontal linear flow on the unions $L^{(m)}$, $m \leq 0$ of vertical intervals. One can describe these as piecewise isometries of a special type and the algorithm that produces the induced map on $L^{(m-1)}$ given the one on $L^{(m)}$ turns out to be related to the da Rocha algorithm [16]. We refer to [28] in which the authors prove that a variant of da Rocha induction is *dual* (in the sense of Scheweiger [42]) to Rauzy-Veech induction for i.e.m.

The dual Birkhoff sums operators $S^\sharp(n, n')$, $n' < n$, enjoy similar properties to the (direct) Birkhoff sums operators $S(n, n'')$, $n'' > n$ (recalled in §2.6). In particular, for $n'' \leq n' \leq n$, one has:

$$S^\sharp(n, n'') = S^\sharp(n', n'') \circ S^\sharp(n, n').$$

Observe moreover that, if ψ is integrable on $L^{(n)}$, then $S^\sharp(n, n')\psi$ is integrable on $L^{(n')}$ and we have

$$\int_{L^{(n')}} S^\sharp(n, n')\psi = \int_{L^{(n)}} \psi.$$

We will denote by ${}^t B_0(n', n)$ the restriction of $S^\sharp(n, n')$ to the hyperplane $\Gamma_0^\sharp(n) \subset \Gamma^\sharp(n)$ of functions of zero mean value.

4.3. Dual Roth type condition. — We can now define the *dual Roth type* Diophantine condition.

Let $\underline{\gamma}^* = \dots \star \gamma_{-1}^* \star \gamma_0^*$ be a backward rotation number, i.e., an infinite path in the Rauzy diagram \mathcal{D} ending in π . (π, τ) , where $\tau \in \Theta_\pi$, by the Rauzy-Veech algorithm. Let us assume that $\underline{\gamma}^*$ is infinitely complete. Define $\hat{n}_0 = 0$. Given \hat{n}_k for some $k \leq 0$, define inductively $\hat{n}_{k-1} \leq 0$ as the largest (negative) integer $n < n_k$ such that the path $\gamma(n, n_k) = \gamma_{n+1} \star \dots \star \gamma_{n_k}$ is ∞ -complete (see §2.3).

As we saw in Proposition 4.3, $\underline{\gamma}^*$ is infinitely complete as long as $\underline{\gamma}^*$ is the backward rotation number of $M(\pi, \lambda, \tau)$ with no horizontal connection. In particular, since having no connection is a full measure condition and horizontal connections depend on

the suspension datum only, for any combinatorial data π , for a.e. $\tau \in \Theta_\pi$ (with respect to the induced Lebesgue measure on Θ_π), we can associate to (π, τ) an infinitely complete backward rotation number $\underline{\gamma}^*$.

We can hence give the following definition, where the sequence (\hat{n}_k) that appears in Condition (a) is the sequence of accelerated times defined above.

Definition 4.5. — A pair of suspension data (π, τ) , where $\tau \in \Theta_\pi$, is of *weak dual Roth type* (or, often in this paper, simply *dual Roth type*) if one can associate to (π, τ) an infinitely complete backward rotation number $\underline{\gamma}^*$ and the following conditions are satisfied:

- (a) (*dual matrices growth*) for every $\varepsilon > 0$, there exists $C_0 = C_0(\varepsilon) > 0$ such that, for all $k \leq 0$

$$\|B(\hat{n}_{k-1}, \hat{n}_k)\| \leq C_0 \|B(\hat{n}_k, 0)\|^\varepsilon;$$

- (b) (*dual spectral gap*) there exists $\theta > 0$ and $C_1 > 0$ such that, for all $n \leq 0$

$$\|{}^t B_0(n, 0)\| \leq C_1 \|{}^t B(n, 0)\|^{1-\theta},$$

where ${}^t B_0(n, 0)$ is the restriction of ${}^t B(n, 0)$ to the hyperplane $\Gamma^\#(0)$.

These two conditions are dual to conditions (a) and (b) in the (direct) Roth type condition. For the results in this paper, we do not require a third condition dual to the coherence condition (i.e., condition (c)) which appears in the (direct) Roth type condition, see Definition 3.1 in § 3.1). Hence we define this dual Diophantine condition with only (a) and (b), but we call it *weak dual Roth type condition* (and in parallel we call *weak Roth type* the direct condition with (a) and (b) only) to leave room to the possibility of defining also a dual Roth type (or restricted dual Roth type) with additional conditions.

Remark 4.6. — A stronger condition is to require that (π, τ) has *bounded type*, i.e., that the matrices $B(n_k, n_{k-1})$ for $k < 0$ above stay uniformly bounded (equivalently, these matrices take their values in a finite set). This clearly implies condition (a). Condition (b) follows from the uniform contraction of the Hilbert metric in the positive cone of $\mathbb{R}^{\mathcal{Z}}$.

Remark 4.7. — As in the case of Condition (a) of the (direct) Roth type condition (see Remark 3.2), also Condition (a) of Definition 4.5 can be rephrased equivalently using different sequences of times, for example using the sequence $(\tilde{n}_k)_{k \leq 0}$ defined setting $\tilde{n}_0 = 0$ and, for $k < 0$ defining inductively \tilde{n}_{k-1} as the largest integer $n < \tilde{n}_k$ such that the matrix $B(n, \tilde{n}_{k-1})$ has positive coefficients.

Definition 4.5 clearly does not depend on the choice of the norm. It is convenient for the following sections to take as the norm of a matrix the sum of all coefficients in absolute value, namely

$$\|B\| := \sum_{\alpha, \beta \in \mathcal{Z}} |B_{\alpha, \beta}|.$$

The absolute values will not appear since the matrices that we consider here all have nonnegative entries. In particular, notice that with this choice of norm we have that $\|B\| = \|{}^t B\|$. We will use this observation freely throughout the next sections.

4.4. Dual lengths control. — The following lemma shows that Condition (a) in the dual Roth type condition imply a control on lengths of the intervals $L_\beta^{(m)}$, $m < 0$, which is analogous to the control of the lengths of the subintervals $I_\alpha^{(n)}$, $n > 0$ given by Condition (a) of the Roth type condition or Condition (a) of the absolute Roth type condition (see § 3.4).

Proposition 4.8. — *For all $\varepsilon > 0$ there exists $C_0 = C_0(\varepsilon) > 0$ and $C_1 = C_1(\varepsilon) > 0$ such that, for all $\beta \in \mathcal{A}$,*

$$C_0 \|B(m, 0)\|^{-1-\varepsilon} \leq |L_\beta^{(m)}| \leq C_1 \|B(m, 0)\|^{-1+\varepsilon}.$$

We remark that the estimate from below is not needed for our immediate purposes, but useful in a more general context. We conclude this section by proving this proposition. The proof using the following auxiliary lemma.

Lemma 4.9. — *For every $\varepsilon > 0$, there exists $C' = C'(\varepsilon) > 0$ such that, for all $m \leq 0$, all $\alpha, \beta \in \mathcal{A}$*

$$\max(1, B_{\alpha, \beta}(m, 0)) \geq C' \|B(m, 0)\|^{1-\varepsilon}.$$

Proof. — Let $(\hat{n}_k)_{k \geq 0}$ be the subsequence of induction times defined at the beginning of § 4.3. Without loss of generality, we can assume that $m \leq \hat{n}_{-4d+5}$, since if $\hat{n}_{-4d+5} < m \leq 0$, the conclusion is true as soon as we take $C' \leq \|B(\hat{n}_{-4d+5}, 0)\|^{-1}$. Therefore, if k the unique integer such that $\hat{n}_{k-1} < m \leq \hat{n}_k$, we can assume that $k \leq -4d + 5$. By [35] (see lemma in § 1.2.2 of [35]), all coefficients of the matrix $X := B(\hat{n}_k, \hat{n}_{k+2d-3})$ are strictly positive, hence at least equal to 1; the same is true for the coefficients of the matrix $Y := B(\hat{n}_{-2d+3}, 0)$. Moreover, Setting $V := B(\hat{n}_{k+2d-3}, \hat{n}_{-2d+3})$ (which is well defined since $k < -4d + 6$), we can write (recalling the cocycle relation (2.7))

$$U := B(\hat{n}_k, 0) = B(\hat{n}_{-2d+3}, 0)B(\hat{n}_{k+2d-3}, \hat{n}_{-2d+3})B(\hat{n}_k, \hat{n}_{k+2d-3}) = YVX.$$

Thus, we have that, for all $\alpha, \beta, \alpha', \beta' \in \mathcal{A}$ (using also that $B(n, 0)_{\alpha, \beta}$ is a non decreasing function of n and the definition of k),

$$B_{\alpha, \beta}(m, 0) \geq U_{\alpha, \beta} \geq Y_{\alpha, \alpha'} V_{\alpha', \beta'} X_{\beta', \beta} \geq V_{\alpha', \beta'}.$$

Choosing α', β' such that $V_{\alpha', \beta'} = \max_{\alpha, \beta} V_{\alpha, \beta}$, since $\max_{\alpha, \beta} V_{\alpha, \beta}$ is a norm on $\mathbb{R}^{\mathcal{A}}$ and all norms on \mathbb{R}^d are equivalent, we get that there exists $c > 0$ such that

$$B_{\alpha, \beta}(m, 0) \geq c \|V\|.$$

On the other hand, from condition (a) in the Definition 4.5 of dual Roth type and the choice of k , we have that for any $\varepsilon > 0$ there exists $\tilde{C}_0 = \tilde{C}_0(\varepsilon)$ and $\tilde{C}_1 = \tilde{C}_1(\varepsilon)$ such that

$$\|B(m, 0)\| \leq \|B(\hat{n}_{k-1}, 0)\| \leq \|YV\| \|B(\hat{n}_{k-1}, \hat{n}_{k+2d-3})\| \leq \tilde{C}_0 \|YV\|^{1+\varepsilon} \leq \tilde{C}_1 \|V\|^{1+\varepsilon},$$

giving the estimate of the lemma. \square

Proof of Proposition 4.8. — It follows from the geometric decomposition (4.1) that, for every $\alpha \in \mathcal{J}$, for every $n' < n''$

$$(4.3) \quad |L_\alpha^{(n'')}| = \sum_{\beta \in \mathcal{J}} B_{\alpha, \beta}(n', n'') |L_\beta^{(n')}|,$$

which gives that for every $\beta \in \mathcal{J}$, for every $n' < n''$

$$(4.4) \quad |L_\beta^{(n')}| \leq \max_{\alpha \in \mathcal{J}} \left((B_{\alpha, \beta}(n', n''))^{-1} |L_\alpha^{(n'')}| \right) \leq \max_{\alpha \in \mathcal{J}} (B_{\alpha, \beta}(n', n''))^{-1} \max_{\alpha \in \mathcal{J}} |L_\alpha^{(n'')}|.$$

The estimate from above follows from Lemma 4.9 and the inequality (4.4) above for $n' := m$ and $n'' := 0$.

To prove the estimate from below, we define $k \leq 0$ as in the proof of the previous lemma, i.e., so that $\hat{n}_{k-1} < m \leq \hat{n}_k$. Writing (recalling the cocycle Equation (2.7))

$$V =: B(\hat{n}_{k-2d+2}, 0) = B(m, 0)B(\hat{n}_{k-2d+2}, m) = B(m, 0)X, \text{ for } X := B(\hat{n}_{k-2d+2}, m),$$

we know again (from $m > \hat{n}_{k-1}$, $k-1 - (k-2d+2) = 2d-3$ and the lemma in § 1.2.2 of [35]) that all coefficients of X are at least equal to 1. We have then that for all $\beta \in \mathcal{J}$ (by (4.3) for $n' := \hat{n}_{k-2d+2}$, $n'' := m$)

$$|L_\beta^{(m)}| \geq \max_{\alpha \in \mathcal{J}} |L_\alpha^{(\hat{n}_{k-2d+2})}|.$$

From (4.3) for $n' := \hat{n}_{k-2d+2}$, $n'' := 0$ and the definition of $\|\cdot\|$, we get that there exists $c > 0$ such that

$$\max_{\alpha \in \mathcal{J}} |L_\alpha^{(\hat{n}_{k-2d+2})}| \geq c \|B(\hat{n}_{k-2d+2}, 0)\|^{-1}.$$

Finally, using repeatedly condition (a) of the dual Roth type Definition 4.5 (and then $m \leq \hat{n}_k$ and monotonicity of $n \mapsto \|B(n, 0)\|$), we have that for every ε there exists $C(\varepsilon)$ such that

$$\begin{aligned} \|B(\hat{n}_{k-2d+2}, 0)\| &\leq \|B(m, 0)\| \|B(\hat{n}_{k-2d+2}, \hat{n}_k)\| \\ &\leq C(\varepsilon) \|B(\hat{n}_k, 0)\|^{1+\varepsilon} \leq C(\varepsilon) \|B(m, 0)\|^{1+\varepsilon}. \end{aligned}$$

The three previous inequalities imply the required estimate. \square

4.5. Estimates on dual special Birkhoff sums of Hölder functions. — The main result in this section is a bound on the growth of dual special Birkhoff sums for Hölder continuous functions under the dual Roth type condition. The main result proved in this section is the following proposition. The analogous estimate for growth of (direct) special Birkhoff sums for Hölder continuous was given in [35] (see the proposition at the end of § 2.3 in [35]).

Proposition 4.10. — Assume that (π, τ) is of dual Roth type. Then, for every function ψ on $L^{(0)}$ which has mean value 0 and is Hölder continuous for some exponent $\eta \in (0, 1)$, the dual Birkhoff sums $S^\sharp(0, n)\psi$ satisfy

$$\|S^\sharp(0, n)\psi\|_{C^0} \leq C(\eta)\|\psi\|_{C^\eta} \|B(n, 0)\|^{1-\frac{1}{20}\theta\eta},$$

where θ is the exponent that appears in the definition of dual Roth type (see (b) in Definition 4.5).

The rest of the section is devoted to the proof of this proposition. We obtain an estimate on the C^0 -norm of $S^\sharp(0, n)\psi$ by the same method used for (direct) Birkhoff sums. Let us remark that following the same methods in [35], the Proposition 4.10 can also be proved for functions of bounded variation.

Proof of Proposition 4.10. — Let (π, τ) be of dual Roth type and let ψ be a function on $L^{(0)}$ as in the assumptions of the proposition. We first write $\psi = \psi_0 + \nu_0$, where ν_0 belongs to the space $\Gamma_0^\sharp(0)$ (defined in § 4.2) and ψ_0 has mean value 0 on each interval $L_\alpha^{(0)}$. For $0 \geq m > n$, we write

$$S^\sharp(m, m-1)\psi_m = \psi_{m-1} + \nu_{m-1},$$

where $\nu_{m-1} \in \Gamma_0^\sharp(m-1)$ and ψ_{m-1} has mean value 0 on each interval $L_\alpha^{(m-1)}$. We have then

$$(4.5) \quad S^\sharp(0, n)\psi = \psi_n + \sum_{n \leq m \leq 0} S^\sharp(m, n)\nu_m.$$

To estimate the terms in this sum, we first observe that, for $n \leq m \leq 0$ and each $\beta \in \mathcal{A}$ and $x, y \in L_\beta^{(m)}$, using that ψ is Hölder-continuous, we have that

$$|\psi_m(x) - \psi_m(y)| = |S^\sharp(0, m)\psi(x) - S^\sharp(0, m)\psi(y)| \leq \|\psi\|_{C^\eta} |L_\beta^{(m)}|^\eta \sum_{\alpha \in \mathcal{A}} B_{\alpha, \beta}(m, 0).$$

As ψ_m has mean 0 in each $L_\beta^{(m)}$, this gives

$$\|\psi_m\|_{C^0} \leq \|\psi\|_{C^\eta} |L_\beta^{(m)}|^\eta \sum_{\alpha \in \mathcal{A}} B_{\alpha, \beta}(m, 0).$$

The estimate from above in Lemma 4.8 gives, for all $\varepsilon > 0$

$$\|\psi_m\|_{C^0} \leq C(\varepsilon)\|\psi\|_{C^\eta} \|B(m, 0)\|^{1-\eta(1-\varepsilon)},$$

and thus also, as $S^\sharp(m, m-1)\psi_m = \psi_{m-1} + \nu_{m-1}$,

$$\|\nu_{m-1}\|_{C^0} \leq C(\varepsilon)\|\psi\|_{C^\eta} \|B(m, 0)\|^{1-\eta(1-\varepsilon)}.$$

From the Formula (4.5) above, we get, taking $\varepsilon = \frac{1}{2}$

$$\|S^\sharp(0, n)\psi\|_{C^0} \leq C\|\psi\|_{C^\eta} \sum_{m=0}^n \|B_0(n, m)\| \|B(m, 0)\|^{1-\frac{1}{2}\eta}.$$

In this sum, we distinguish two cases:

Case 1. — When $\|B(m, 0)\| \leq \|B(n, 0)\|^{\frac{\theta}{3}}$, we write $B_0(n, m) = B_0(m, 0)^{-1}B_0(n, 0)$ and obtain, from condition (b) and the symplecticity of $B(m, 0)$

$$\begin{aligned} \|{}^t B_0(n, m)\| \|B(m, 0)\|^{1-\frac{1}{2}\eta} &\leq \|B_0(m, 0)^{-1}\| \|B_0(n, 0)\| \|B(n, 0)\|^{\frac{\theta}{3}} \\ &\leq \|B(m, 0)^{-1}\| \|B_0(n, 0)\| \|B(n, 0)\|^{\frac{\theta}{3}} \\ &\leq C \|B(m, 0)\| \|B(n, 0)\|^{1-\theta} \|B(n, 0)\|^{\frac{\theta}{3}} \\ &\leq C \|B(n, 0)\|^{1-\frac{\theta}{3}}. \end{aligned}$$

Case 2. — When $\|B(m, 0)\| > \|B(n, 0)\|^{\frac{\theta}{3}}$, we use the trivial bound of $\|{}^t B_0(n, m)\|$ by $\|B(n, m)\|$ and the following inequality, which is proved below: for all $\varepsilon > 0$

$$\|B(n, m)\| \|B(m, 0)\| \leq C(\varepsilon) \|B(n, 0)\|^{1+\varepsilon}.$$

We then obtain, taking $\varepsilon = \frac{1}{12}\theta\eta$

$$\begin{aligned} \|{}^t B_0(n, m)\| \|B(m, 0)\|^{1-\frac{1}{2}\eta} &\leq \|B(n, m)\| \|B(m, 0)\|^{1-\frac{1}{2}\eta} \\ &\leq C \|B(n, 0)\|^{1+\frac{1}{12}\theta\eta} \|B(n, 0)\|^{-\frac{1}{6}\theta\eta} \\ &\leq C \|B(n, 0)\|^{1-\frac{1}{12}\theta\eta}. \end{aligned}$$

To prove that $\|B(n, m)\| \|B(m, 0)\| \leq C(\varepsilon) \|B(n, 0)\|^{1+\varepsilon}$, we may assume that $n \leq \hat{n}_{-2d+4}$. Then, there exist integers k, l with $n \leq \hat{n}_k \leq m \leq \hat{n}_l$ such that $l = k + 2d - 3$ so that all coefficients of $B(\hat{n}_k, \hat{n}_l)$ are positive (by the lemma in § 2.3 of [35]) and hence at least equal to 1. Furthermore, by condition (a), there exists $C_1 = C_1(\varepsilon)$ such that $\|B(\hat{n}_k, \hat{n}_l)\| \leq C_1 \|B(n, 0)\|^{\frac{\varepsilon}{2}}$. This gives

$$\begin{aligned} \|B(n, m)\| \|B(m, 0)\| &\leq C(\varepsilon)^2 \|B(n, \hat{n}_k)\| \|B(\hat{n}_l, 0)\| \|B(n, 0)\|^\varepsilon \\ &\leq C(\varepsilon)^2 \|B(n, 0)\|^{1+\varepsilon}, \end{aligned}$$

as required.

We conclude that

$$\|S^{\sharp}(0, n)\psi\|_{C^0} \leq C \|\psi\|_{C^\eta} n \|B(n, 0)\|^{1-\frac{1}{12}\theta\eta}.$$

Since we also know that $\|B(n, 0)\|$ grows exponentially fast with relation to Zorich time, this is sufficient to conclude that $n \|B(n, 0)\|^{-\varepsilon}$ converges to 0 for any $\varepsilon > 0$. This concludes the proof of the proposition. □

5. Distributional limit shapes

Let \mathcal{R} be a Rauzy class on an alphabet \mathcal{A} of cardinality d , \mathcal{D} be the associated diagram. Let g be the genus of the associated translation surfaces and s the cardinality of the marked set on such surfaces. The rank of the matrices Ω_π , $\pi \in \mathcal{R}$ is equal to $2g$ and we have $d = 2g + s - 1$ (see § 2.2). We assume throughout this section that $s > 1$.

Consider a point $\underline{T} = (\pi, \lambda, \tau)$ in $\mathcal{R} \times (\mathbb{R}^+)^{\mathcal{A}} \times \Theta_\pi$ which is *typical* for the dynamics of the natural extension \hat{Q}_{RV} (defined in 2.4) and typical (in the sense of Oseledets genericity) for the Kontsevich-Zorich cocycle. Let $T := T_{\pi, \lambda}$ be the associated i.e.m.

acting on the interval I . For $n \in \mathbb{Z}$, let $\underline{T}^{(n)} = (\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$ be the image of \underline{T} under \hat{Q}_{RV}^n , and let $T^{(n)}$ be the associated i.e.m. acting on the interval $I^{(n)}$. For any integers $n, n' \in \mathbb{Z}$ with $n \geq n'$, $T^{(n')}$ is the first return map of $T^{(n)}$ in $I^{(n')}$.

For each $n \in \mathbb{Z}$, we denote respectively by $\Gamma_s^{(n)}$, $\Gamma_u^{(n)}$, $\Gamma_c^{(n)}$ the stable, unstable and central subspaces of the KZ-cocycle at $(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$ (see §3). By hyperbolicity of the Kontsevich-Zorich cocycle (proved in [23]), we have

$$\dim(\Gamma_s^{(n)}) = \dim(\Gamma_u^{(n)}) = g, \quad \dim(\Gamma_c^{(n)}) = s - 1,$$

$$\Gamma_s^{(n)} \oplus \Gamma_u^{(n)} = \text{Im}(\Omega_{\pi^{(n)}}).$$

We will say that $\underline{\chi} := (\chi^{(n)})_{n \in \mathbb{Z}}$ is a sequence of *central functions* if $\chi^{(0)} = \chi \in \Gamma_c^{(0)}$ and

$$\chi^{(n)} = S(0, n)\chi, \quad \text{for } n > 0,$$

$$\chi^{(n)} = S(n, 0)^{-1}\chi, \quad \text{for } n < 0.$$

A particularly interesting example of central functions can be constructed from characteristic functions as shown in §5.1.

5.1. Corrected characteristic functions. — Assume only for this subsection that $s = 1$. Fix a point $\xi^{(0)} \in I^{(0)}$.

Definition 5.1. — A sequence $\underline{\xi} := (\xi^{(n)})_{n \in \mathbb{Z}}$ is *compatible* with $\xi^{(0)} \in I^{(0)}$ if:

- for each $n \in \mathbb{Z}$, $\xi^{(n)}$ is a point of $I^{(n)}$;
- for $n \geq n'$, $\xi^{(n)}$ is the first entry point in $I^{(n)}$ when iterating $(T^{(n')})^{-1}$ from $\xi^{(n')}$.

We will say in this case that $\underline{\xi}$ is a compatible sequence.

Remark 5.2. — It is clear that $\xi^{(n_0)}$ determines $\xi^{(n)}$ for all $n \geq n_0$. On the other hand, given $\xi^{(n)}$ for $n \geq n_0$, one can for instance choose $\xi^{(n)} = \xi^{(n_0)}$ for $n \leq n_0$.

In particular, by Remark 5.2, sequences compatible with $\xi^{(0)}$ do exist. For any such sequence, for any $n \in \mathbb{Z}$, let $\tilde{\chi}^{(n)}$ be the function defined on $I^{(n)}$ and equal to the characteristic function of the interval $(0, \xi^{(n)})$.

Let $\Gamma^{(n)}$ be the d -dimensional vector space (canonically isomorphic to $\mathbb{R}^{\mathcal{C}}$) of functions defined on $I^{(n)}$ and constant on each $I_\alpha^{(n)}$. Finally, let $\Gamma^{(n)}(\xi)$ be the d -dimensional affine space given by

$$\Gamma^{(n)}(\xi) := \tilde{\chi}^{(n)} + \Gamma^{(n)}.$$

The following proposition shows that any sequence $(\tilde{\chi}^{(n)})_{n \in \mathbb{Z}}$ of characteristic functions as above can be *corrected* by adding a function constant on the subintervals exchanged by $T^{(0)}$, to produce a sequence $(\chi^{(n)})_{n \in \mathbb{Z}}$ of central functions:

Proposition 5.3 (Projection to central sequences). — *For any sequence $\underline{\xi} := (\xi^{(n)})_{n \in \mathbb{Z}}$ compatible with $\xi^{(0)} \in I^{(0)}$, for any integers n, n' with $n \geq n'$, the special Birkhoff sum operator $S(n', n)$ (cf. §2.6) is an isomorphism from $\Gamma^{(n')}$ onto $\Gamma^{(n)}$ and from $\Gamma^{(n')}(\xi)$ onto $\Gamma^{(n)}(\xi)$.*

Furthermore, there exists a unique sequence $(\chi^{(n)})_{n \in \mathbb{Z}}$ such that $\chi^{(n)}$ belongs to $\Gamma^{(n)}(\xi)$ for any $n \in \mathbb{Z}$,

$$S(n', n)(\chi^{(n')}) = \chi^{(n)}$$

for all $n, n' \in \mathbb{Z}$ with $n \geq n'$ and

$$(5.1) \quad \lim_{n \rightarrow \pm\infty} \frac{1}{Z(n)} \log \|\chi^{(n)}\|_\infty = 0,$$

where $Z(n)$ is the Zorich time (see § 2.3).

Proof. — The first part of the assertion is already known (see [35]). For the second part, let $n \geq n'$ and consider $S(n', n)\tilde{\chi}^{(n')}$. It is locally constant on $I^{(n)}$, except at $\xi^{(n)}$ and at the singularities of $T^{(n)}$. Moreover, the difference between the left and right limits at $\xi^{(n)}$ is equal to 1. Therefore $S(n', n)\tilde{\chi}^{(n')}$ belongs to $\Gamma^{(n)}(\xi)$ and the first part of the proposition follows.

Let $\Gamma^{(n)} = \Gamma_s^{(n)} \oplus \Gamma_u^{(n)}$ be the Oseledets decomposition into stable and unstable subspaces. Write then for $n \in \mathbb{Z}$

$$S(n-1, n)\tilde{\chi}^{(n-1)} = \tilde{\chi}^{(n)} + \Delta\tilde{\chi}_s^{(n)} + \Delta\tilde{\chi}_u^{(n)}$$

with $\Delta\tilde{\chi}_s^{(n)} \in \Gamma_s^{(n)}$ and $\Delta\tilde{\chi}_u^{(n)} \in \Gamma_u^{(n)}$. Observe that, since $\tilde{\chi}^{(n)}$ is bounded by 1 and n here indexes Rauzy-Veech elementary steps, we have that $\|S(n-1, n)\tilde{\chi}^{(n-1)}\|_\infty \leq 2$.

We look for $\chi^{(n)} = \tilde{\chi}^{(n)} + \Delta\chi_s^{(n)} + \Delta\chi_u^{(n)}$ satisfying the requirements of the proposition. In particular, we must have

$$\begin{aligned} \Delta\chi_s^{(n)} &= \Delta\tilde{\chi}_s^{(n)} + S(n-1, n)\Delta\chi_s^{(n-1)}, \\ \Delta\chi_u^{(n)} &= \Delta\tilde{\chi}_u^{(n)} + S(n-1, n)\Delta\chi_u^{(n-1)}. \end{aligned}$$

This leads to the formulas

$$\begin{aligned} \Delta\chi_s^{(n)} &= \sum_{n' \leq n} S(n', n)\Delta\tilde{\chi}_s^{(n')}, \\ \Delta\chi_u^{(n)} &= - \sum_{n' > n} (S(n, n'))^{-1}\Delta\tilde{\chi}_u^{(n')}. \end{aligned}$$

We will now show that the convergence of these series follows from the hyperbolicity of the Kontsevich-Zorich cocycle and this will conclude the proof. Let us first remark that, as $\underline{T} = (\pi, \lambda, \tau)$ is typical and, as already observed, $\|S(n-1, n)\tilde{\chi}^{(n-1)}\|_\infty \leq 2$, there is, for every $\varepsilon > 0$, a constant $C := C(\varepsilon, \underline{T})$ such that, for all $n \in \mathbb{Z}$

$$\|\Delta\tilde{\chi}_s^{(n)}\|_\infty + \|\Delta\tilde{\chi}_u^{(n)}\|_\infty \leq C \exp(\varepsilon|Z(n)|).$$

As the cocycle is hyperbolic, there exists a constant $C' := C'(\varepsilon, \underline{T})$ such that, for all $n \geq m$, all v_s in $\Gamma_s^{(m)}$, all v_u in $\Gamma_u^{(n)}$, denoting by θ_g the smallest positive exponent of the KZ-cocycle w.r.t the Zorich dynamics \hat{Q}_Z , we have

$$\begin{aligned} \|S(m, n)v_s\|_\infty &\leq C' \exp[(Z(m) - Z(n))\theta_g + (|Z(m)| + |Z(n)|)\varepsilon]\|v_s\|_\infty, \\ \|(S(m, n))^{-1}v_u\|_\infty &\leq C' \exp[(Z(m) - Z(n))\theta_g + (|Z(m)| + |Z(n)|)\varepsilon]\|v_u\|_\infty. \end{aligned}$$

These equations show that the series above converge.

As ε is arbitrary, the sequence $\chi^{(n)}$ is well-defined and satisfies the required properties.

The uniqueness of the sequence with the required properties follows from the hyperbolicity of the KZ-cocycle: the difference between two solutions would grow subexponentially (in Zorich time) and thus must be equal to 0. \square

Remark 5.4. — This proposition uses the assumption, assumed throughout this section, that $\underline{T} = (\pi, \lambda, \tau)$ is *typical* for the dynamics of \hat{Q}_{RV} and Oseledets generic for the Kontsevich-Zorich cocycle. In § 6 we will prove a similar result in the homological context under weaker assumptions on (π, λ, τ) than Oseledets genericity (see in particular the notion of KZ hyperbolic introduced in § 6.3). Furthermore, one can prove the existence of a sequence $\Gamma^{(n)}(\xi)$ as in the conclusion of Proposition 5.3 but satisfying (5.1) only for $n \rightarrow +\infty$ (resp. $n \rightarrow -\infty$) assuming only negative (resp. positive) KZ hyperbolicity (see, in § 6.3, Propositions 6.8 and 6.9 respectively). In this case though uniqueness only holds modulo $\Gamma_s^{(n)}$ (resp. $\Gamma_u^{(n)}$).

5.2. The functions $\Omega_\alpha^{(n)}(\pi, \tau, \chi)$. — We will now define an object analogous to the *limit shapes* defined in [36], but suited to study Birkhoff sums of central functions $\chi \in \Gamma_c^{(0)}$.

The largest Lyapunov exponent of the Kontsevich-Zorich cocycle is simple and the associated 1-dimensional eigenspace $F_1(\pi, \tau)$ does not depend on the λ variable. Moreover, $F_1(\pi, \tau)$ is generated by a vector $q(\pi, \tau)$ contained in the positive cone $(\mathbb{R}^+)^{\mathcal{A}}$. We normalize $q(\pi, \tau)$ by asking that its ℓ^2 -norm is equal to 1. For $n \leq 0$, we write

$$(S(n, 0))^{-1}q(\pi, \tau) =: q^{(n)}(\pi, \tau) = \Theta_1^{(n)}q(\pi^{(n)}, \tau^{(n)})$$

with $\Theta_1^{(n)} \in \mathbb{R}^+$ satisfying

$$\lim_{n \rightarrow -\infty} \frac{1}{Z(n)} \log \Theta_1^{(n)} = \theta_1.$$

(θ_1 is the largest Lyapunov exponent of the KZ-cocycle.)

Let $\chi \in \Gamma_c^{(0)}$. Write $\chi^{(n)} := (S(n, 0))^{-1}(\chi)$ for $n \leq 0$. For any $\alpha \in \mathcal{A}$ and any $n \leq 0$, we define a function $\Omega_\alpha^{(n)} = \Omega_\alpha^{(n)}(\pi, \tau, \chi)$ on the interval $[0, q_\alpha(\pi, \tau)]$ by the following requirements:

1. recall that $B_\alpha(n, 0)$ (defined in (2.8)) is the return time in $I^{(0)}$ of $I_\alpha^{(0)}$ under $T^{(n)}$; for $0 \leq j < B_\alpha(n, 0)$, let $\beta(\alpha, j)$ be the letter in \mathcal{A} such that $(T^{(n)})^j(I_\alpha^{(0)}) \subset I_{\beta(\alpha, j)}^{(n)}$; for $0 \leq \ell \leq B_\alpha(n, 0)$, set

$$\text{Sq}(\ell) = \sum_{0 \leq j < \ell} q_{\beta(\alpha, j)}^{(n)}(\pi, \tau).$$

Observe that $\text{Sq}(0) = 0$ and $\text{Sq}(B_\alpha(n, 0)) = q_\alpha(\pi, \tau)$. The function $\Omega_\alpha^{(n)}$ is continuous on $[0, q_\alpha(\pi, \tau)]$ and its restriction to each interval $[\text{Sq}(\ell), \text{Sq}(\ell + 1)]$ is affine.

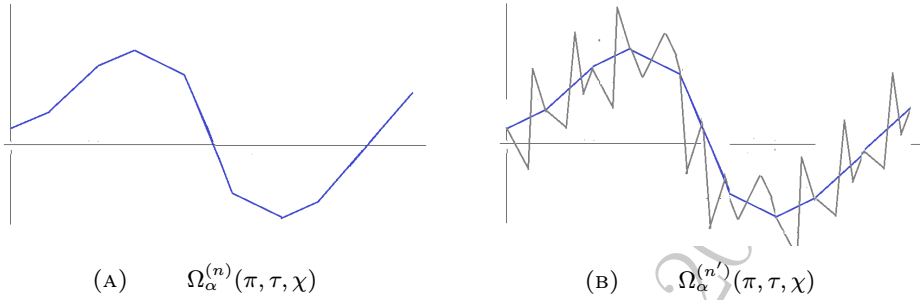


FIGURE 3. For $n' \leq n \leq 0$, $\Omega_\alpha^{(n)}(\pi, \tau, \chi)$ refines $\Omega_\alpha^{(n')}(\pi, \tau, \chi)$.

2. For $0 \leq \ell < B_\alpha(n, 0)$, one has

$$\Omega_\alpha^{(n)}(\text{Sq}(\ell + 1)) - \Omega_\alpha^{(n)}(\text{Sq}(\ell)) = \chi^{(n)}((T^{(n)})^\ell(I_\alpha^{(0)})).$$

3. The function $\Omega_\alpha^{(n)}$ is defined by the first two conditions up to the addition of a constant function. This constant is now determined by the requirement that $\Omega_\alpha^{(n)}$ should have mean 0 on $[0, q_\alpha(\pi, \tau)]$.

5.3. Comparing the functions $\Omega_\alpha^{(n)}(\pi, \tau, \chi)$ and $\Omega_\alpha^{(n')}(\pi, \tau, \chi)$. — Let $n' \leq n \leq 0$. We will compare the functions $\Omega_\alpha^{(n)}(\pi, \tau, \chi)$ and $\Omega_\alpha^{(n')}(\pi, \tau, \chi)$, which are defined on the same interval $[0, q_\alpha(\pi, \tau)]$.

Lemma 5.5. — For every $\alpha \in \mathcal{A}$ and every $0 \leq \ell \leq B_\alpha(n, 0)$, we have

$$\Omega_\alpha^{(n')}(\text{Sq}(\ell)) - \Omega_\alpha^{(n)}(\text{Sq}(\ell)) = \Omega_\alpha^{(n')}(0) - \Omega_\alpha^{(n)}(0).$$

Moreover, for every $\alpha \in \mathcal{A}$ and every $0 \leq \ell < B_\alpha(n, 0)$, the restriction of $\Omega_\alpha^{(n')}$ to $[\text{Sq}(\ell), \text{Sq}(\ell + 1)]$ for

$$0 \leq x \leq \text{Sq}(\ell + 1) - \text{Sq}(\ell) = q_{\beta(\alpha, \ell)}^{(n)}(\pi, \tau) = \Theta_1^{(n)} q_{\beta(\alpha, \ell)}(\pi^{(n)}, \tau^{(n)})$$

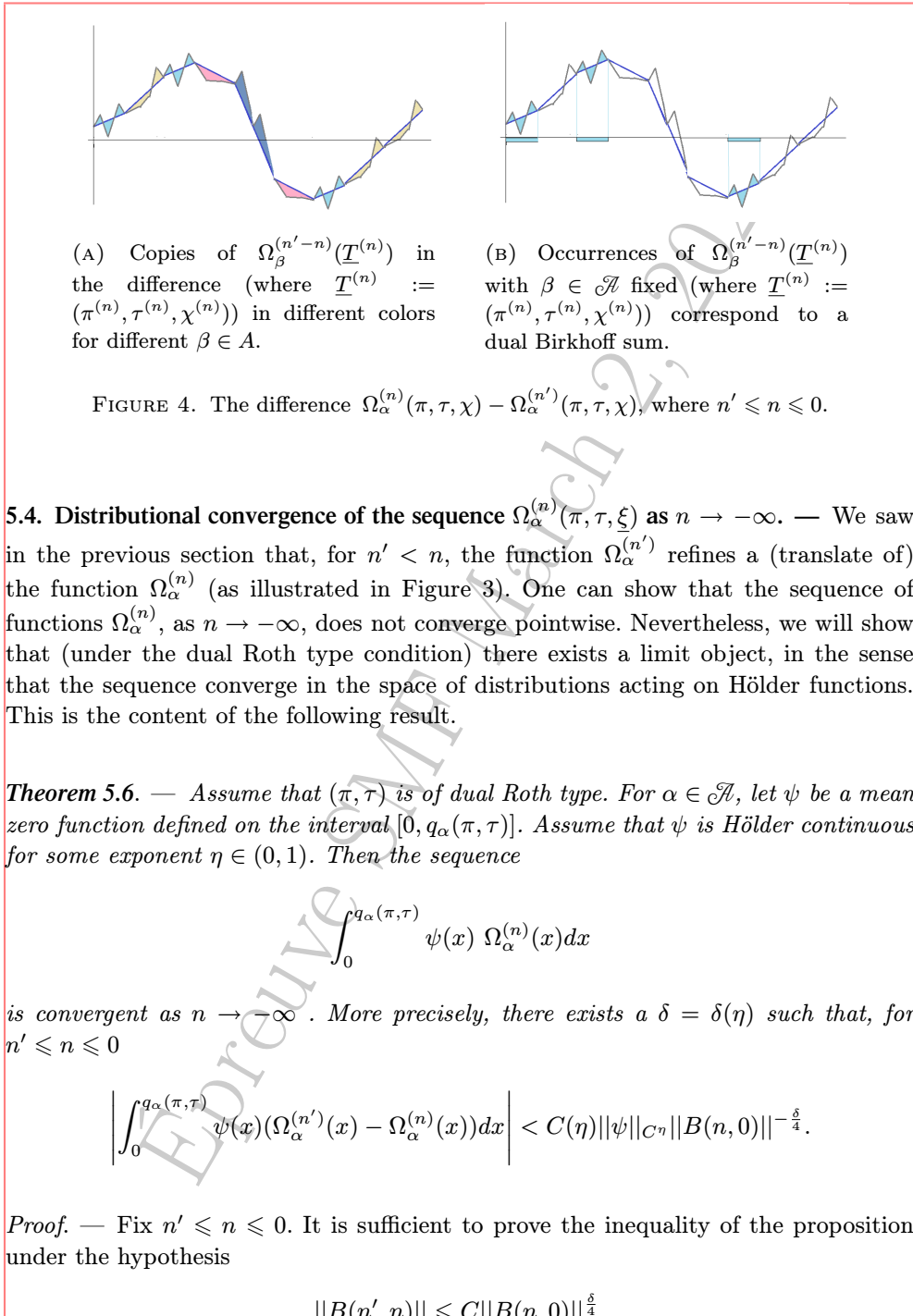
satisfies

$$\Omega_\alpha^{(n')}(\text{Sq}(\ell) + x) = c(\ell) + \Omega_\alpha^{(n'-n)}(\pi^{(n)}, \tau^{(n)}, \chi^{(n)}) \left(\frac{x}{\Theta_1^{(n)}} \right),$$

where $c(\ell)$ is the mean value of $\Omega_\alpha^{(n')}$ on $[\text{Sq}(\ell), \text{Sq}(\ell + 1)]$.

Proof. — The statements of the proposition follow directly from the definition of the functions $\Omega_\alpha^{(n')}$, $\Omega_\alpha^{(n)}$ and from the relations

$$S(n', n)q^{(n')}(\pi, \tau) = q^{(n)}(\pi, \tau) = \Theta_1^{(n)}q(\pi^{(n)}, \tau^{(n)}), \quad S(n', n)\chi^{(n')} = \chi^{(n)}. \quad \square$$



By decomposing the interval of integration into intervals of the form $[\text{Sq}(\ell), \text{Sq}(\ell + 1)]$ and

$$\begin{aligned} \int_0^{q_\alpha(\pi, \tau)} \psi(x)(\Omega_\alpha^{(n')}(x) - \Omega_\alpha^{(n)}(x))dx &= \sum_{0 \leq \ell < B_\alpha(n, 0)} \int_{\text{Sq}(\ell)}^{\text{Sq}(\ell+1)} \psi(x)(\Omega_\alpha^{(n')}(x) - \Omega_\alpha^{(n)}(x))dx \\ &= \sum_{\beta \in \mathcal{J}} \sum_{0 \leq \ell < Q_\alpha^{(n)}, \bar{\beta}(\alpha, \ell) = \bar{\beta}} \int_{\text{Sq}(\ell)}^{\text{Sq}(\ell+1)} \psi(x)(\Omega_\alpha^{(n')}(x) - \Omega_\alpha^{(n)}(x))dx \\ &= \sum_{\beta \in \mathcal{J}} \int_0^{q_{\bar{\beta}}^{(n)}(\pi, \tau)} \tilde{\Omega}_{\bar{\beta}}^* \left(\frac{x}{\Theta_1^{(n)}} \right) \sum_{0 \leq \ell < Q_\alpha^{(n)}, \bar{\beta}(\alpha, \ell) = \bar{\beta}} \psi(x + \text{Sq}(\ell))dx. \end{aligned}$$

Here, $\tilde{\Omega}_{\bar{\beta}}^*$ is according to Lemma 5.5, the function on $[0, q_{\bar{\beta}}(\pi^{(n)}, \tau^{(n)})]$ which differs from $\Omega_{\bar{\beta}}^{(n'-n)}(\pi^{(n)}, \tau^{(n)}, \xi^{(n)})$ by an affine function and satisfies

$$\tilde{\Omega}_{\bar{\beta}}^*(0) = \tilde{\Omega}_{\bar{\beta}}^*(q_{\bar{\beta}}(\pi^{(n)}, \tau^{(n)})) = \Omega_\alpha^{(n')}(0) - \Omega_\alpha^{(n)}(0).$$

Let us consider ψ , which is defined on $L_\alpha^{(0)} = [0, q_\alpha]$, as a function on $L^{(0)} = \bigsqcup_{\beta \in \mathcal{J}} L_\beta^{(0)}$ which is zero on each $L_\beta^{(0)}$ for $\beta \neq \alpha$ and remark that ψ is Hölder on the disjoint union $L^{(n)}$. Since we are assuming that (π, τ) is of dual Roth type, by the estimates of dual Birkhoff sums given in Proposition 4.10, we hence have that, for $x \in L_\alpha^{(0)}$,

$$\begin{aligned} \left| \sum_{0 \leq \ell < Q_\alpha^{(n)}, \bar{\beta}(\alpha, \ell) = \bar{\beta}} \psi(x + \text{Sq}(\ell)) \right| &= \left| \sum_{\alpha \in \mathcal{J}} \sum_{0 \leq \ell < Q_\alpha^{(n)}, \bar{\beta}(\alpha, \ell) = \bar{\beta}} \psi(x + \text{Sq}(\ell)) \right| = |S^\sharp(0, n)\psi(x)| \\ &\leq C(\eta) \|\psi\|_{C^\eta} \|B(n, 0)\|^{1-\delta}, \end{aligned}$$

for some $\delta > 0$ depending on η . It follows that

$$\begin{aligned} (5.2) \quad \left| \int_0^{q_\alpha(\pi, \tau)} \psi(x)(\Omega_\alpha^{(n')}(x) - \Omega_\alpha^{(n)}(x))dx \right| \\ \leq C(\eta) \|\psi\|_{C^\eta} \|B(n, 0)\|^{1-\delta} \sum_{\beta \in \mathcal{J}} \int_0^{q_{\bar{\beta}}^{(n)}(\pi, \tau)} \left| \tilde{\Omega}_{\bar{\beta}}^* \left(\frac{x}{\Theta_1^{(n)}} \right) \right| dx. \end{aligned}$$

Changing integration variable, we have

$$\int_0^{q_{\bar{\beta}}^{(n)}(\pi, \tau)} \left| \tilde{\Omega}_{\bar{\beta}}^* \left(\frac{x}{\Theta_1^{(n)}} \right) \right| dx = \Theta_1^{(n)} \int_0^{q_{\bar{\beta}}(\pi^{(n)}, \tau^{(n)})} |\tilde{\Omega}_{\bar{\beta}}^*(x)| dx.$$

Since we also have that

$$\Theta_1^{(n)} \leq C \|B(n, 0)\|^{-1 + \frac{1}{4}\delta},$$

and also that

$$\left| \Omega_{\bar{\beta}}^{(n'-n)}(\pi^{(n)}, \tau^{(n)}, \xi^{(n)}) \right|_{C^0} \leq C \|B(n', n)\| \|\chi^{(n')}\|_\infty,$$

with $\|B(n', n)\| \leq C\|B(n, 0)\|^{\frac{\delta}{4}}$ by hypothesis and

$$\|\chi^{(n')}\|_{\infty} \leq C\|B(n', 0)\|^{\frac{\delta}{8}} \leq C(\|B(n', n)\| \|B(n, 0)\|)^{\frac{\delta}{8}} \leq C\|B(n, 0)\|^{\frac{\delta}{4}},$$

this gives that

$$\sum_{\beta \in \mathcal{I}} \int_0^{q_{\beta}^{(n)}(\pi, \tau)} \left| \tilde{\Omega}_{\beta}^* \left(\frac{x}{\Theta_1^{(n)}} \right) \right| dx \leq C\|B(n, 0)\|^{-1 + \frac{3}{4}\delta}.$$

Combining this with (5.2) gives the required inequality. \square

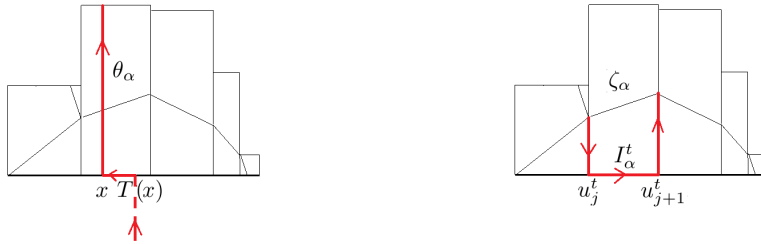
6. Homological interpretation

In this section we give a different interpretation of the distributional limit shapes introduced in the previous section and we relate them to (relative) homology classes and sections of the boundary operator. In §6.1 we recall some homology classes in $H_1(M, \Sigma, \mathbb{Z})$ and $H_1(M \setminus \Sigma, \mathbb{Z})$ associated to the Rauzy-Veech algorithm. In §6.4 we describe in the homological context a construction which is very similar to the construction of central functions $(\xi^{(n)})_{n \in \mathbb{Z}}$ correcting the characteristic functions $(\tilde{\xi}^{(n)})_{n \in \mathbb{Z}}$ done in §5.1. There are however several differences (besides the context). In particular, we use weaker assumptions than the typical Oseledets hyperbolic behavior for the KZ-cocycle assumed in §5.1. In particular, we make separate assumptions on the backward and forward rotation numbers, called respectively positive KZ-hyperbolicity (for future behavior) and negative KZ-hyperbolicity (for past behavior). One of the points of introducing such assumptions is that there are explicit cases where it might be possible to check these assumptions while Oseledets hyperbolic behavior might be false or very difficult to check. Finally, in §6.7 and §6.4 we construct the distributional analogue of limit shapes in this homological context.

6.1. Bases of homology associated to the Rauzy-Veech algorithm. — Let us define bases for the punctured surface homology $H_1(M \setminus \Sigma, \mathbb{Z})$ and the relative homology $H_1(M, \Sigma, \mathbb{Z})$ naturally induced by the zippered rectangle presentation of a translation surface. The setting and notations here used are those of §2.8. We also refer the reader to the lecture notes [50] (see in particular Section 4.5 in [50]) for more details on the homological interpretation of the bases induced by zippered rectangles.

The classes $\theta_{\alpha} \in H_1(M \setminus \Sigma, \mathbb{Z})$. — Let $\alpha \in \mathcal{A}$, $x \in I_{\alpha}^t$. Consider the simple loop in $M \setminus \Sigma$ consisting of the upwards vertical segment (of length q_{α}) joining x to $T(x)$ followed by the horizontal segment contained in I joining $T(x)$ to x (as shown in Figure 5a). The homology class in $H_1(M \setminus \Sigma, \mathbb{Z})$ of this loop does not depend on x and is denoted by θ_{α} . The classes $(\theta_{\alpha})_{\alpha \in \mathcal{A}}$ form a basis of $H_1(M \setminus \Sigma, \mathbb{Z})$. One has

$$\int_{\theta_{\alpha}} \omega = -\delta_{\alpha} + iq_{\alpha}.$$



(A) The classes $\theta_\alpha \in H_1(M \setminus \Sigma, \mathbb{Z})$. (B) The class $\zeta_\alpha \in H_1(M, \Sigma, \mathbb{Z})$.

FIGURE 5. The bases for $H_1(M \setminus \Sigma, \mathbb{Z})$ and $H_1(M, \Sigma, \mathbb{Z})$ described in § 6.1.

The classes $\zeta_\alpha \in H_1(M, \Sigma, \mathbb{Z})$. — Let $\alpha \in \mathcal{A}$. Write $I_\alpha^t = (u_j^t, u_{j+1}^t)$, with the convention that $u_0^t = u_0^b = u_0$, $u_d^t = u_d^b = u_d$. Consider the following path in M (shown in Figure 5b) with endpoints in Σ : the first part is a vertical separatrix segment from a point of Σ to u_j^t (this part is reduced to a point if $j = 0$), the second part is the horizontal segment $I_\alpha^t \subset I$ from u_j^t to u_{j+1}^t , the third part is a vertical separatrix segment from u_{j+1}^t to a point of Σ . The homology class of this path in $H_1(M, \Sigma, \mathbb{Z})$ is denoted by ζ_α . The classes $(\zeta_\alpha)_{\alpha \in \mathcal{A}}$ form a basis of $H_1(M, \Sigma, \mathbb{Z})$. One has

$$\int_{\zeta_\alpha} \omega = \lambda_\alpha + i\tau_\alpha.$$

Write $I_\alpha^b = (u_\ell^b, u_{\ell+1}^b)$. An alternative path representing ζ_α is made of a vertical separatrix segment connecting a point of Σ to u_ℓ^b , the horizontal segment I_α^b connecting u_ℓ^b to $u_{\ell+1}^b$, and a vertical separatrix segment connecting $u_{\ell+1}^b$ to a point of Σ .

Duality and relation with Ω_π . — The intersection form $i(\cdot, \cdot)$ defines a non degenerate bilinear pairing between $H_1(M, \Sigma, \mathbb{Z})$ and $H_1(M \setminus \Sigma, \mathbb{Z})$ for which the bases $(\zeta_\alpha)_{\alpha \in \mathcal{A}}$ and $(\theta_\alpha)_{\alpha \in \mathcal{A}}$ are dual: one has

$$i(\zeta_\alpha, \theta_\beta) = \delta_{\alpha\beta},$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol.

Consider the homomorphism ι from $H_1(M \setminus \Sigma, \mathbb{Z})$ to $H_1(M, \Sigma, \mathbb{Z})$ obtained by composing the natural homomorphisms from $H_1(M \setminus \Sigma, \mathbb{Z})$ to $H_1(M, \mathbb{Z})$ and from $H_1(M, \mathbb{Z})$ to $H_1(M, \Sigma, \mathbb{Z})$. For $\alpha, \beta \in \mathcal{A}$, one has

$$\iota(\theta_\alpha) = - \sum_{\beta} \Omega_{\alpha\beta} \zeta_\beta.$$

6.2. Change of bases associated to the Rauzy-Veech algorithm. — Let $(T^{(n)})_{n \in \mathbb{Z}}$ be the sequence of i.e.m. deduced from (π, λ, τ) by the Rauzy-Veech algorithm. Notice that all translation surfaces $M(\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$ are canonically isomorphic to $M := M^{(0)}$. Therefore, for each $n \in \mathbb{Z}$, the previous constructions define a basis $(\theta_\alpha^{(n)})_{\alpha \in \mathcal{A}}$ of $H_1(M \setminus \Sigma, \mathbb{Z})$ and a basis $(\zeta_\alpha^{(n)})_{\alpha \in \mathcal{A}}$ of $H_1(M, \Sigma, \mathbb{Z})$.

These bases are related, for $m \leq n$, by

$$\begin{aligned}\theta_\alpha^{(n)} &= \sum_\beta B_{\alpha\beta} \theta_\beta^{(m)}, & \text{where } B &= B(m, n) \\ \zeta_\alpha^{(n)} &= \sum_\beta B_{\alpha\beta}^* \zeta_\beta^{(m)}, & \text{where } B &= B^* = {}^t B(m, n)^{-1}.\end{aligned}$$

Let Υ be a homology class in $H_1(M, \Sigma, \mathbb{R})$. For $n \in \mathbb{Z}$, write

$$\Upsilon = \sum_\alpha \chi_\alpha^{(n)} \zeta_\alpha^{(n)}.$$

The coefficients $\chi^{(n)} = (\chi_\alpha^{(n)})_{\alpha \in \mathcal{A}}$ are related, for $m \leq n$, by

$$\chi^{(n)} = B(m, n) \chi^{(m)},$$

which is the usual homological interpretation of the Kontzevich-Zorich cocycle. Notice that

$$\chi_\alpha^{(n)} = i(\Upsilon, \theta_\alpha^{(n)}).$$

6.3. KZ-hyperbolic translation surfaces. — Consider the homology exact sequence

$$0 \longrightarrow H_1(M, \mathbb{Z}) \longrightarrow H_1(M, \Sigma, \mathbb{Z}) \longrightarrow H_0(\Sigma, \mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

and the corresponding sequence with real coefficients

$$0 \longrightarrow H_1(M, \mathbb{R}) \longrightarrow H_1(M, \Sigma, \mathbb{R}) \longrightarrow H_0(\Sigma, \mathbb{R}) \longrightarrow \mathbb{R} \longrightarrow 0.$$

We identify $H_0(\Sigma, \mathbb{R})$ with \mathbb{R}^Σ . Then the image of $H_1(M, \Sigma, \mathbb{R})$ is the hyperplane \mathbb{R}_0^Σ consisting of vectors with vanishing coordinate sum. Under an appropriate hypothesis on (π, λ, τ) , we will construct a preferred section $\nu : \mathbb{R}_0^\Sigma \rightarrow H_1(M, \Sigma, \mathbb{R})$ of the boundary map $\partial : H_1(M, \Sigma, \mathbb{R}) \rightarrow H_0(\Sigma, \mathbb{R})$.

In the following definition, we use the notation of the end of § 6.1.

Definition 6.1. — Assume that M has no vertical connection. The *stable subspace* of $H_1(M, \Sigma, \mathbb{R})$, denoted by Γ^s , is

$$\Gamma^s = \{\Upsilon \in H_1(M, \Sigma, \mathbb{R}) \mid \exists \sigma > 0, C > 0, \|\chi^{(n)}\| \leq C \|B(0, n)\|^{-\sigma} \|\chi^{(0)}\|, \forall n > 0\}.$$

Assume that M has no horizontal connection. The *unstable subspace* of $H_1(M, \Sigma, \mathbb{R})$, denoted by Γ^u , is

$$\Gamma^u = \{\Upsilon \in H_1(M, \Sigma, \mathbb{R}) \mid \exists \sigma > 0, C > 0, \|\chi^{(n)}\| \leq C \|B(n, 0)\|^{-\sigma} \|\chi^{(0)}\|, \forall n < 0\}.$$

Notice that both Γ^s and Γ^u are subspaces of $H_1(M, \mathbb{R})$ (more precisely of the image of $H_1(M, \mathbb{R})$ in $H_1(M, \Sigma, \mathbb{R})$). Moreover, when $H_1(M, \mathbb{R})$ is equipped with the symplectic intersection form, both Γ^s and Γ^u are isotropic. Notice also that, as $H_1(M, \Sigma, \mathbb{R})$ is finite-dimensional, one can choose an exponent $\sigma > 0$ which works for all vectors in Γ^s and all vectors in Γ^u .

When we identify $H_1(M, \Sigma, \mathbb{R})$ with $\mathbb{R}^{\mathcal{A}}$ through the basis $(\zeta_\alpha^{(0)})$, The subspace Γ^s only depend on (π, λ) and the subspace Γ^u on (π, τ) .

Definition 6.2. — The translation surface $M = M(\pi, \lambda, \tau)$ is *positively KZ-hyperbolic* if M has no vertical connection and the following properties hold

1. for any $\tau > 0$, one has

$$\sum_{n>0} \|B(0, n)\|^{-\tau} < +\infty;$$

2. the dimension of Γ^s is equal to g ;
3. (coherence) For any $\tau > 0$, there exists $C_\tau > 0$ such that, for any $\mathcal{Y} \in \Gamma^s$, any $0 \leq m \leq n$, one has

$$\|\chi^{(n)}\| \leq C_\tau \|B(0, n)\|^\tau \|\chi^{(m)}\|.$$

Definition 6.3. — The translation surface $M = M(\pi, \lambda, \tau)$ is *negatively KZ-hyperbolic* if M has no horizontal connection and the following properties hold

1. for any $\tau > 0$, one has

$$\sum_{n<0} \|B(0, n)\|^{-\tau} < +\infty;$$

2. the dimension of Γ^u is equal to g ;
3. (coherence) For any $\tau > 0$, there exists $C_\tau > 0$ such that, for any $\mathcal{Y} \in \Gamma^u$, any $0 \geq m \geq n$, one has

$$\|\chi^{(n)}\| \leq C_\tau \|B(0, n)\|^\tau \|\chi^{(m)}\|.$$

Definition 6.4. — The translation surface $M = M(\pi, \lambda, \tau)$ is *KZ-hyperbolic* if it is both positively and negatively KZ-hyperbolic and one has

$$H_1(M, \mathbb{R}) = \Gamma^s \oplus \Gamma^u.$$

Remark 6.5. — By Forni’s results in [23], for any combinatorial data π , almost all data (λ, τ) produce KZ-hyperbolic translation surfaces.

Remark 6.6. — Assume that M is positively KZ-hyperbolic. For $\mathcal{Y} \in H_1(M, \Sigma, \mathbb{R})$ and $n \geq 0$, define

$$\|\mathcal{Y}\|_{n, \star s} := \min\{\|\chi\|_\infty \mid \mathcal{Y} - \sum \chi_\alpha \zeta_\alpha^{(n)} \in \Gamma^s\}.$$

It follows from the symplecticity of the restricted KZ-cocycle that one has, for $\mathcal{Y} \in H_1(M, \mathbb{R})$, $n \geq m \geq 0$, and any $\tau > 0$

$$\begin{aligned} \|\mathcal{Y}\|_{0, \star s} &\leq C \|B(0, n)\|^{-\sigma} \|\mathcal{Y}\|_{n, \star s}; \\ \|\mathcal{Y}\|_{m, \star s} &\leq C_\tau \|B(0, n)\|^\tau \|\mathcal{Y}\|_{n, \star s}. \end{aligned}$$

The proof is an adaptation of the results discussed in Section 3.2 of [38].

Remark 6.7. — Similarly, assume that M is negatively KZ-hyperbolic. For $\Upsilon \in H_1(M, \Sigma, \mathbb{R})$ and $n \leq 0$, define

$$\|\Upsilon\|_{n, \star u} := \min\{\|\chi\|_\infty \mid \Upsilon - \sum \chi_\alpha \zeta_\alpha^{(n)} \in \Gamma^u\}.$$

Then one has, for $\Upsilon \in H_1(M, \mathbb{R})$, $n \leq m \leq 0$, and any $\tau > 0$

$$\begin{aligned} \|\Upsilon\|_{0, \star u} &\leq C \|B(n, 0)\|^{-\sigma} \|\Upsilon\|_{n, \star u}; \\ \|\Upsilon\|_{m, \star u} &\leq C_\tau \|B(n, 0)\|^\tau \|\Upsilon\|_{n, \star u}. \end{aligned}$$

Proposition 6.8. — Assume that M is positively KZ-hyperbolic. Then, for every $v \in \mathbb{R}_0^\Sigma$, there exists a class $\Upsilon = \sum \chi_\alpha^{(n)} \zeta_\alpha^{(n)} \in H_1(M, \Sigma, \mathbb{R})$ such that $\partial\Upsilon = v$ and

$$\overline{\lim}_{n \rightarrow +\infty} \frac{\log(\|\chi^{(n)}\|/\|\chi^{(0)}\|)}{\log \|B(0, n)\|} = 0.$$

Moreover, two such classes differ by an element in Γ^s . Therefore the proposition defines a linear section from \mathbb{R}_0^Σ to $H_1(M, \Sigma, \mathbb{R})/\Gamma^s$.

The proof of this proposition is given below. Remark that the content of the proposition is non-empty only when $s \geq 2$.

We also have the following symmetric proposition.

Proposition 6.9. — Assume that M is negatively KZ-hyperbolic. Then, for every $v \in \mathbb{R}_0^\Sigma$, there exists a class $\Upsilon = \sum \chi_\alpha^{(n)} \zeta_\alpha^{(n)} \in H_1(M, \Sigma, \mathbb{R})$ such that $\partial\Upsilon = v$ and

$$\overline{\lim}_{n \rightarrow -\infty} \frac{\log(\|\chi^{(n)}\|/\|\chi^{(0)}\|)}{\log \|B(n, 0)\|} = 0.$$

Moreover, two such classes differ by an element in Γ^u . Therefore the proposition defines a linear section from \mathbb{R}_0^Σ to $H_1(M, \Sigma, \mathbb{R})/\Gamma^u$.

We omit the proof of Proposition 6.9 since it is analogous to the proof of Proposition 6.8 given below.

Finally, combining the two propositions, we have the following corollary.

Corollary 6.10. — Assume that M is KZ-hyperbolic. Then, for every $v \in \mathbb{R}_0^\Sigma$, there exists a unique class $\Upsilon = \sum \chi_\alpha^{(n)} \zeta_\alpha^{(n)} \in H_1(M, \Sigma, \mathbb{R})$ such that $\partial\Upsilon = v$ and

$$\overline{\lim}_{n \rightarrow +\infty} \frac{\log(\|\chi^{(n)}\|/\|\chi^{(0)}\|)}{\log \|B(0, n)\|} = \overline{\lim}_{n \rightarrow -\infty} \frac{\log(\|\chi^{(n)}\|/\|\chi^{(0)}\|)}{\log \|B(n, 0)\|} = 0.$$

The section $v \mapsto \Upsilon$ from \mathbb{R}_0^Σ to $H_1(M, \Sigma, \mathbb{R})$ is linear.

Proof of Proposition 6.8. — Uniqueness modulo Γ^s is clear from the first inequality in Remark 6.6. To prove existence, we choose, for each $n \geq 0$, a class $\Xi(n) = \sum_\alpha \hat{\chi}_\alpha^{(n)} \zeta_\alpha^{(n)} \in H_1(M, \Sigma, \mathbb{R})$ such that $\partial\Xi(n) = v$ and

$$\|\hat{\chi}^{(n)}\| \leq C = C(v).$$

We have therefore (as the coefficients of $B(n-1, n)$ are equal to 0 or 1)

$$\|\Xi(n) - \Xi(n-1)\|_{n, *s} \leq C.$$

As $\Xi(n) - \Xi(n-1)$ belongs to $H_1(M, \mathbb{R})$, we deduce from the first inequality in Remark 6.6 that

$$\|\Xi(n) - \Xi(n-1)\|_{0, *s} \leq C\|B(0, n)\|^{-\sigma}.$$

We can therefore choose a class $\Xi(n-1, n) = \sum x_\alpha(n)\zeta_\alpha^{(0)} \in H_1(M, \mathbb{R})$ such that $\Xi(n) - \Xi(n-1) - \Xi(n-1, n) \in \Gamma^s$ and

$$\|x(n)\|_\infty \leq C\|B(0, n)\|^{-\sigma}.$$

Therefore the series $\sum_{n>0} \Xi(n-1, n)$ is convergent. We will see that

$$\Upsilon := \Xi(0) + \sum_{n>0} \Xi(n-1, n)$$

satisfies the conclusions of the proposition. Clearly we have $\partial\Upsilon = v$.

Next we estimate $\|\Upsilon\|_{m, *s}$. One has

$$\Upsilon = \Xi(m) + \sum_{n>m} \Xi(n-1, n) \pmod{\Gamma^s}.$$

From the choice of $\Xi(m)$, one has $\|\Xi(m)\|_{m, *s} \leq C$. To estimate $\|\Xi(n-1, n)\|_{m, *s}$, there are two cases

— If $\|B(0, n)\|^{\sigma/2} \leq \|B(0, m)\|$, we use the second inequality in Remark 6.6 to get, for any $\tau > 0$

$$\|\Xi(n-1, n)\|_{m, *s} \leq C_\tau \|B(0, m)\|^\tau.$$

— If $\|B(0, n)\|^{\sigma/2} \geq \|B(0, m)\|$, we have

$$\begin{aligned} \|\Xi(n-1, n)\|_{m, *s} &\leq \|B(0, m)\| \|\Xi(n-1, n)\|_{0, *s} \\ &\leq C\|B(0, m)\| \|B(0, n)\|^{-\sigma} \\ &\leq C\|B(0, n)\|^{-\sigma/2}. \end{aligned}$$

From the first condition in Definition 6.2, we have

$$\sum \|B(0, n)\|^{-\sigma/2} \leq C,$$

and

$$\sum_{\|B(0, n)\|^{\sigma/2} \leq \|B(0, m)\|} \|B(0, m)\|^\tau \leq C_\tau \|B(0, m)\|^{2\tau}.$$

After renaming τ , we get $\|\Upsilon\|_{m, *s} \leq C_\tau \|B(0, m)\|^\tau$.

Thus, for each $m \geq 0$, we can write

$$\Upsilon = \sum y_\alpha(m)\zeta_\alpha^{(m)} + \Upsilon(m),$$

with $\|y(m)\|_\infty \leq C_\tau \|B(0, m)\|^\tau$ and $\Upsilon(m) \in \Gamma^s$. We can also take $\Upsilon(0) = 0$. As the coefficients of $B(m-1, m)$ are equal to 0 or 1, we have, for $m > 0$

$$\Upsilon(m) - \Upsilon(m-1) = \sum \tilde{y}_\alpha(m)\zeta_\alpha^{(m)}, \quad \|\tilde{y}(m)\|_\infty \leq C_\tau \|B(0, m)\|^\tau.$$

From the coherence property in Definition 6.2, we finally obtain $\mathcal{T}(m) = \sum \widehat{y}_\alpha(m) \zeta_\alpha^{(m)}$ with

$$\|\widehat{y}(m)\|_\infty \leq C'_\tau \|B(0, m)\|^\tau \sum_{\ell \leq m} \|B(0, \ell)\|^\tau \leq C''_\tau \|B(0, m)\|^{3\tau}.$$

This concludes the proof of the proposition. □

6.4. Piecewise-affine paths in $H_1(M \setminus \Sigma, \mathbb{R})$. — We assume that M has no horizontal connection. Let $n < 0$; denote by α_w the winner of the arrow $\gamma(n, n + 1)$, by α_ℓ its loser. Recall that we have $\theta_\alpha^{(n+1)} = \theta_\alpha^{(n)}$ for $\alpha \neq \alpha_\ell$ and

$$\theta_{\alpha_\ell}^{(n+1)} = \theta_{\alpha_\ell}^{(n)} + \theta_{\alpha_w}^{(n)}.$$

To the arrow $\gamma = \gamma(n, n + 1)$, we associate a substitution transformation ς_γ on the alphabet \mathcal{A} defined as follows:

— If γ is of top type, ς_γ is defined by

$$\alpha_\ell \mapsto \alpha_\ell \alpha_w, \quad \alpha \mapsto \alpha, \quad \forall \alpha \neq \alpha_\ell.$$

— If γ is of bottom type, ς_γ is defined by

$$\alpha_\ell \mapsto \alpha_w \alpha_\ell, \quad \alpha \mapsto \alpha, \quad \forall \alpha \neq \alpha_\ell.$$

For $\alpha \in \mathcal{A}$, we then define inductively a sequence of words $(W_\alpha(n))_{n \leq 0}$ by

$$W_\alpha(0) = \alpha, \quad W_\alpha(n) = \varsigma_\gamma(n, n+1)(W_\alpha(n + 1)), \quad \forall n < 0.$$

One can see that for any $n \leq 0$ and $\alpha, \beta \in \mathcal{A}$, the number of occurrences of β in the word $W_\alpha(n)$ is equal to $(B(n, 0))_{\alpha\beta}$.

The relation to the Rauzy-Veech algorithm is given by the following proposition.

Proposition 6.11. — *Write $W_\alpha(n) = \beta_0 \cdots \beta_{N-1}$. Then we have, for $0 \leq j < N$*

$$(T^{(n)})^j(I_\alpha^{(0)}) \subset I_{\beta_j}^{(n)}.$$

The proof is clear from the definition of the Rauzy-Veech algorithm.

Remark 6.12. — Equivalently, for any $n < 0$ and $\alpha, \beta \in \mathcal{A}$, we also have the following geometric interpretation of the word $W_\alpha(n) = \beta_0 \cdots \beta_{N-1}$. The rectangle $R_\alpha^{(0)}$ can be seen as a union of rectangles (of the same width, but shorter height) each fully contained in a rectangle $R_\beta^{(b)}$ for some $\beta \in A$ (refer to the bottom part of Figure 1). These rectangles, in the order from the bottom to the top of $R_\alpha^{(0)}$, are contained in $R_{\beta_\ell}^{(n)}$ for $\ell = 0, 1, \dots, N - 1$. In other words, if one considers a leaf of the vertical flow from a point $x \in I_\alpha^{(0)}$ of length $q_\alpha^{(0)}$, it crosses $I^{(n)}$ exactly N times, and the intersections belong, in order, to $I_{\beta_0}^{(n)}, I_{\beta_1}^{(n)}, \dots, I_{\beta_{N-1}}^{(n)}$.

In other words (cf. Remark 6.12), if one considers a leaf of the horizontal flow starting from a point $x \in L_\alpha^{(0)}$ and crossing all the rectangle $R_\alpha^{(0)}$ horizontally (see for example the top part of Figure 1), then its intersections with $L^{(n)}$ will be contained, in order, in the intervals $L_{\beta_\ell}^{(n)}$, for $\ell = 0, 1, \dots, N - 1$.

To the word $W_\alpha(n) = \beta_0 \cdots \beta_{N-1}$, we associate a broken line $\mathcal{W}_\alpha(n)$ in $H_1(M \setminus \Sigma, \mathbb{R})$ as follows:

- for $0 \leq j \leq N$, let $\mathcal{W}_{\alpha,j}(n) := \sum_{0 \leq i < j} \theta_{\beta_i}^{(n)} \in H_1(M \setminus \Sigma, \mathbb{Z})$;
- set $\mathcal{W}_\alpha(n)$ to be the concatenation of the segments from $\mathcal{W}_{\alpha,j}(n)$ to $\mathcal{W}_{\alpha,j+1}(n)$ for $0 \leq j < N$.

The points $\mathcal{W}_{\alpha,j}(n)$, for $0 \leq j \leq N$, are called the *vertices* of $\mathcal{W}_\alpha(n)$. The following properties hold.

- For $n \leq 0$, any vertex of $\mathcal{W}_\alpha(n)$ is also a vertex of $\mathcal{W}_\alpha(n - 1)$.
- The first vertex $\mathcal{W}_{\alpha,0}(n)$ is equal to 0.
- The last vertex $\mathcal{W}_{\alpha,N}(n)$ is equal to $\theta_\alpha^{(0)}$.

Let $p_v : H_1(M \setminus \Sigma, \mathbb{R}) \rightarrow \mathbb{R}$ be the linear form determined by integration of $\mathfrak{S}\omega$. The restriction of p_v to $\mathcal{W}_\alpha(n)$ is a piecewise-affine homeomorphism onto the interval $[0, q_\alpha^{(0)}]$. The image of the segment from $\mathcal{W}_{\alpha,j}(n)$ to $\mathcal{W}_{\alpha,j+1}(n)$ is a subinterval of length $q_{\beta_j}^{(n)}$.

6.5. The functions $\Omega_\alpha^{(n)}(\mathcal{Y})$. — In this subsection, we assume that M is negatively KZ-hyperbolic of *dual Roth type*. Let $v \in \mathbb{R}_0^\Sigma$, and let $\mathcal{Y} \in H_1(M, \Sigma, \mathbb{R})$ be a class satisfying the conclusion of Proposition 6.9. Let $p_{\mathcal{Y}} : H_1(M \setminus \Sigma, \mathbb{R}) \rightarrow \mathbb{R}^2$ be the linear map defined by

$$p_{\mathcal{Y}}(\theta) = (p_v(\theta), i(\mathcal{Y}, \theta)).$$

From the above considerations (cf. § 6.4), the image $p_{\mathcal{Y}}(\mathcal{W}_\alpha(n))$ is the graph of a piecewise-affine function $\Omega_\alpha^{(n)}(\mathcal{Y}) : [0, q_\alpha^{(0)}] \rightarrow \mathbb{R}$. This construction is analogous in the homological context to the construction which was done in § 5.2 (see the functions $\Omega_\alpha^{(n)}(\pi, \tau, \chi)$ associated in § 5.2 to a sequence $\chi = (\chi^{(n)})_n$ of central functions). Following the same arguments as in § 5.3 and § 5.4, one hence has the following result (analogous in this context to Theorem 5.6):

Proposition 6.13. — *Assume that (π, τ) is of dual Roth type. For any $\alpha \in \mathcal{A}$, $\Omega_\alpha^{(n)}(\mathcal{Y}) : [0, q_\alpha^{(0)}] \rightarrow \mathbb{R}$ converge as $n \rightarrow -\infty$ as a distribution on Hölder functions, i.e., for any function ψ on the interval $[0, q_\alpha(\pi, \tau)]$ which is mean zero and Hölder continuous for some exponent $\eta \in (0, 1)$, the sequence*

$$\int_0^{q_\alpha(\pi, \tau)} \psi(x) \Omega_\alpha^{(n)}(\mathcal{Y})(x) dx$$

is convergent as $n \rightarrow -\infty$.

Since in this section we only made assumptions on the backward rotation number (π, τ) , the homology class $\mathcal{Y} \in H_1(M, \Sigma, \mathbb{R})$ (and hence the limiting distribution) is only well-determined by its image under the boundary operator modulo the unstable subspace Γ^u (cf. Proposition 6.9). However, we also have the following result.

Proposition 6.14. — Let \mathcal{Y}' be another class in $H_1(M, \Sigma, \mathbb{R})$ such that $\mathcal{Y}' - \mathcal{Y}$ belongs to Γ^u . For any $n \leq 0$, any $\alpha \in \mathcal{A}$, the difference $\Omega_\alpha^{(n)}(\mathcal{Y}') - \Omega_\alpha^{(n)}(\mathcal{Y})$ is Hölder continuous, with a Hölder exponent and a Hölder constant independent of n and α . Moreover, the sequence $\Omega_\alpha^{(n)}(\mathcal{Y}') - \Omega_\alpha^{(n)}(\mathcal{Y})$ converges uniformly when n goes to $-\infty$ to a Hölder continuous function on $[0, q_\alpha^{(0)}]$.

Proof. — We have an estimate

$$\max_\alpha q_\alpha^{(n)} \leq C^{-1} \|B(n, 0)\|^{-1}, \quad \forall n \leq 0.$$

As (π, τ) is of dual Roth type, we also have, for all $\tau > 0$

$$\min_\alpha q_\alpha^{(n)} \leq C_\tau^{-1} \|B(n, 0)\|^{-1-\tau}.$$

On the other hand, for two consecutive vertices $\mathcal{W}_{\alpha,j}(n), \mathcal{W}_{\alpha,j+1}(n)$ of $\mathcal{W}_\alpha(n)$, one has

$$\begin{aligned} |i(\mathcal{Y}' - \mathcal{Y}, \mathcal{W}_{\alpha,j+1}(n)) - i(\mathcal{Y}' - \mathcal{Y}, \mathcal{W}_{\alpha,j}(n))| &= i(\mathcal{Y}' - \mathcal{Y}, \theta_{\beta_j}^{(n)}) \\ &\leq C \|B(n, 0)\|^{-\sigma}. \end{aligned}$$

Using the space decomposition of [38] (see Section 3.8 in [38]), one gets the conclusions of the theorem from these estimates. \square

Since by Proposition 6.13, for any $\alpha \in \mathcal{A}$ the sequence $\Omega_\alpha^{(n)}(\mathcal{Y})$ converge to a distribution on Hölder continuous functions, Proposition 6.14 shows that the limit distribution is uniquely determined, up to the addition of Hölder functions, by the projection of $\mathcal{Y} \in H_1(M, \Sigma, \mathbb{R})$ onto the stable space Γ^s (along the unstable space).

6.6. Piecewise-affine paths in $H_1(M, \Sigma, \mathbb{R})$. — In this section, we exploit the duality between horizontal and vertical flows and between $H_1(M \setminus \Sigma, \mathbb{R})$ and $H_1(M, \Sigma, \mathbb{R})$ in order to describe a construction and results which are *dual* to the one in the previous two sections, namely §6.4 and §6.7. In particular we describe how to construct distributional limit shapes for relative homology classes with subpolynomial deviations for the *horizontal* flow. This dual result requires the KZ *negative* hyperbolicity assumption, together with the *direct* Roth type condition (and is based on the study of special Birkhoff sums instead than dual ones).

Let us first construct piecewise-affine paths in $H_1(M, \Sigma, \mathbb{R})$, in a dual way to the definition of piecewise-affine paths in $H_1(M \setminus \Sigma, \mathbb{R})$ in §6.4.

We assume that M has no *vertical* connection. Let $n > 0$. To the arrow $\gamma = \gamma(n, n+1)$, we associate a substitution transformation ς_γ on the alphabet \mathcal{A} defined as follows. Denote by α_w the winner of arrow γ , by α_ℓ its loser.

— If γ is of top type, ς_γ is defined by

$$\alpha_w \mapsto \alpha_w \alpha_\ell, \quad \alpha \mapsto \alpha, \quad \forall \alpha \neq \alpha_\ell.$$

— If γ is of bottom type, ς_γ is defined by

$$\alpha_w \mapsto \alpha_w \alpha_\ell, \quad \alpha_\ell \mapsto \alpha_w \quad \alpha \mapsto \alpha, \quad \forall \alpha \notin \{\alpha_\ell, \alpha_w\}.$$

For $\alpha \in \mathcal{A}$, we then define inductively a sequence of words $(W_\alpha^*(n))_{n \geq 0}$ by

$$W_\alpha^*(0) = \alpha, \quad W_\alpha^*(n+1) = \varsigma_{\gamma(n,n+1)}(W_\alpha^*(n)), \quad \forall n \geq 0.$$

One can see that for any $n \geq 0$ and $\alpha, \beta \in \mathcal{A}$, the number of occurrences of β in the word $W_\alpha^*(n)$ is equal to $(B(0, n))_{\alpha\beta}$.

The relation to the Rauzy-Veech algorithm is the following. For any $n \geq 0$ and $\alpha \in \mathcal{A}$ the rectangle $R_\alpha^{(0)}$ of base $I_\alpha^{(0)}$ can be seen as union of rectangles (of same height but shorter width) fully contained in the rectangles $R_\beta^{(0)}$, $\beta \in A$ (refer to the right hand side of Figure 2). In particular, the interval $I_\alpha^{(0)}$ can be seen as a union of translates of the intervals $I_\beta^{(n)}$. If we write

$$W_\alpha^*(n) = \beta_0 \cdots \beta_{N-1},$$

then we have that $I_\alpha^{(0)}$ is union, from left to right, of translates of the intervals $I_{\beta_0}^{(n)}$, $I_{\beta_1}^{(n)}$, \dots , $I_{\beta_{N-1}}^{(n)}$, or, more precisely

$$I_\alpha^{(0)} = \bigcup_{0 \leq \ell \leq N-1} j_{\alpha, \ell}^{(0, n)}(I_{\beta_\ell}^{(n)}), \quad \text{where } j_{\alpha, \ell}^{(0, n)}(x) = x + \sum_{0 \leq i < \ell} \lambda_{\beta_i}^{(n)}.$$

In other words (cf. Remark 6.12), if one considers a leaf of the horizontal flow starting from a point $x \in L_\alpha^{(0)}$ and crossing all the rectangle $R_\alpha^{(0)}$ horizontally (see for example the top part of Figure 1), then its intersections with $L^{(n)}$ will be contained, in order, in the intervals $L_{\beta_\ell}^{(n)}$, for $\ell = 0, 1, \dots, N-1$.

To the word $W_\alpha^*(n) = \beta_0 \cdots \beta_{N-1}$, we associate a broken line $\mathcal{W}_\alpha^*(n)$ in $H_1(M, \Sigma, \mathbb{R})$ as follows:

- for $0 \leq j \leq N$, let $\mathcal{W}_{\alpha, j}^*(n) := \sum_{0 \leq i < j} \zeta_{\beta_i}^{(n)} \in H_1(M, \Sigma, \mathbb{Z})$;
- set $\mathcal{W}_\alpha^*(n)$ to be the concatenation of the segments from $\mathcal{W}_{\alpha, j}^*(n)$ to $\mathcal{W}_{\alpha, j+1}^*(n)$ for $0 \leq j < N$.

The points $\mathcal{W}_{\alpha, j}^*(n)$, for $0 \leq j \leq N$, are called the *vertices* of $\mathcal{W}_\alpha^*(n)$.

The following properties hold.

- For $n \geq 1$, any vertex of $\mathcal{W}_\alpha^*(n)$ is also a vertex of $\mathcal{W}_\alpha^*(n-1)$.
- The first vertex $\mathcal{W}_{\alpha, 0}^*(n)$ is equal to 0.
- The last vertex $\mathcal{W}_{\alpha, N}^*(n)$ is equal to $\zeta_\alpha^{(0)}$.

Let $p_h : H_1(M, \Sigma, \mathbb{R}) \rightarrow \mathbb{R}$ be the linear form determined by integration of $\Re\omega$. The restriction of p_h to $\mathcal{W}_\alpha^*(n)$ is a piecewise-affine homeomorphism onto the interval $[0, \lambda_\alpha^{(0)}]$. The image of the segment from $\mathcal{W}_{\alpha, j}^*(n)$ to $\mathcal{W}_{\alpha, j+1}^*(n)$ is a subinterval of length $\lambda_{\beta_j}^{(n)}$.

6.7. The functions $\Omega_\alpha^*(n)(\mathcal{Y})$. — In this subsection, we assume that M is *positively* KZ-hyperbolic of (*direct*) Roth type. Let $v \in \mathbb{R}_0^\Sigma$.

Using the identification between $H_1(M \setminus \Sigma, \mathbb{R})$ and $H_1(M \setminus \Sigma, \mathbb{R})$, similarly to Proposition 6.8, one can show the existence of classes $\mathcal{Y} \in H_1(M \setminus \Sigma, \mathbb{R})$ be a class satisfying the conclusion of Proposition ?? . Let $\mathcal{Y} \in H_1(M \setminus \Sigma, \mathbb{R})$ be any such class.

Let $p_{\mathcal{Y}} : H_1(M, \Sigma, \mathbb{R}) \rightarrow \mathbb{R}^2$ be the linear map defined by

$$p_{\mathcal{Y}}(\zeta) = (p_h(\zeta), i(\mathcal{Y}, \zeta)).$$

From the above considerations (cf. § 6.6), the image $p_{\mathcal{Y}}(\mathcal{W}_{\alpha}^*(n))$ is the graph of a piecewise-affine function $\Omega_{\alpha}^{*(n)}(\mathcal{Y}) : [0, \lambda_{\alpha}^{(0)}] \rightarrow \mathbb{R}$. With arguments similar to the proof of Proposition 6.13, where the role of dual Birkhoff sums is replaced by special Birkhoff sums and using the bounds for Birkhoff sums of Hölder under the dual Roth type condition proved in this paper, one can prove the following.

Proposition 6.15. — *Assume that (π, λ) is of dual Roth type. For any $\alpha \in \mathcal{A}$, $\Omega_{\alpha}^{*(n)}(\mathcal{Y}) : [0, \lambda_{\alpha}^{(0)}] \rightarrow \mathbb{R}$ converge as $n \rightarrow +\infty$ as a distribution on Hölder functions, i.e., for any function ψ on the interval $[0, \lambda_{\alpha}^{(0)}]$ which is mean zero and Hölder continuous for some exponent $\eta \in (0, 1)$, the sequence*

$$\int_0^{\lambda_{\alpha}} \psi(x) \Omega_{\alpha}^{*(n)}(\mathcal{Y})(x) dx$$

is convergent as $n \rightarrow +\infty$.

Finally, one can also prove a dual version of Proposition 6.14 and thus show also in this case that the limit distribution is uniquely determined, up to the addition of Holder functions, by the projection of $\mathcal{Y} \in H_1(M \setminus \Sigma, \mathbb{R})$ onto the unstable space $H^u \subset H_1(M \setminus \Sigma, \mathbb{R})$ (along the stable space).

Appendix A

Completeness of backward rotation numbers

In this section we prove Proposition 4.3 (see § 4.1), which shows that when M has no horizontal connection, the backward rotation number $\underline{\gamma}$ defined in § 4.1 is ∞ -complete. The proof which we present exploits the combinatorics of Rauzy-Veech classes and is surprisingly involved. It would be interesting to know if it is possible to find a simpler, perhaps more geometric, proof.

Let M be a translation surface constructed from combinatorial data $\pi \in \mathcal{R}$, length data $\lambda \in C$ and suspension data $\tau \in \Theta_{\pi}$. We assume throughout this appendix that M has no horizontal saddle connections. We associate to τ the quantities

$$H_k^t(\pi, \tau) := \sum_{\pi_t \alpha \leq k} \tau_{\alpha}, \quad H_k^b(\pi, \tau) := \sum_{\pi_b \alpha \leq k} \tau_{\alpha}, \quad \forall 1 \leq k < d.$$

We also define $H_d^t(\pi, \tau) = H_d^b(\pi, \tau) = H_d(\pi, \tau) = \sum_{\alpha} \tau_{\alpha}$ and $H_0^t(\pi, \tau) = H_0^b(\pi, \tau) = H_0(\pi, \tau) = 0$.

The domain for the suspension data τ is the open cone Θ_π in $\mathbb{R}^{\mathcal{A}}$ defined by the Veech conditions on suspension data:

$$H_k^t(\pi, \tau) > 0, \quad H_k^b(\pi, \tau) < 0, \quad \forall 1 \leq k < d.$$

As usual, we write $\zeta_\alpha = \lambda_\alpha + i\tau_\alpha$. Since M has no horizontal saddle connections, by iterating the *backward* Rauzy Veech algorithm defined in § 4.1, we obtain a *backward rotation number*, i.e., an infinite path $\underline{\gamma} = \dots \gamma_{-1} \star \gamma_0$ in \mathcal{D} with terminal point $\pi^{(0)}$. We want to show that it is ∞ -complete.

A.1. The quantity $H(n)$. — Given T with no connection, let us denote as usual by $T^{(n)}$ the i.e.m. with data $(\pi^{(n)}, \lambda^{(n)})$ obtained via Rauzy-Veech induction. We define for $n \leq 0$ the quantities

$$\begin{aligned} H_k^t(n) &:= H_k^t(\pi^{(n)}, \lambda^{(n)}), & \text{for } 1 \leq k < d; \\ H_k^b(n) &:= H_k^b(\pi^{(n)}, \lambda^{(n)}), & \text{for } 1 \leq k < d; \\ H(n) &:= \sum_{1 \leq k < d} H_k^t(n) - H_k^b(n). \end{aligned}$$

This is the sum of $(2d - 2)$ positive numbers. We have

$$H(n - 1) = H(n) + H_{k-1}^b(n) - H_{d-1}^t(n),$$

with $k := \pi_b^{(n)}(\alpha_w)$, when $\gamma(n, n + 1)$ is of top type. Similarly

$$H(n - 1) = H(n) - H_{k-1}^t(n) + H_{d-1}^b(n),$$

with $k := \pi_t^{(n)}(\alpha_w)$, when $\gamma(n, n + 1)$ is of bottom type.

Therefore we always have $H(n - 1) < H(n)$; the sequence $H(n)$ thus converges to a limit $H(-\infty) \geq 0$ when n goes to $-\infty$.

Lemma A.1. — For any $\alpha \in \mathcal{A}$, any $n \leq 0$, one has

$$|\tau_\alpha^{(n)}| \leq H(n).$$

Proof. — Indeed, if for instance $\pi_t(n)(\alpha) =: k < d$, one has

$$|\tau_\alpha^{(n)}| \leq |H_k^t(n) - H_{k-1}^t(n)|,$$

with $H_k^t(n), H_{k-1}^t(n) \in [0, H(n)]$. □

A.2. Switching times

Lemma A.2. — The path $\underline{\gamma}$ contains infinitely many arrows of each type.

Proof. — If the conclusion of the lemma does not hold, we may assume, after truncating $\underline{\gamma}$, that all arrows $\gamma(n, n + 1)$ have the same type. Assume for instance that all arrows are of top type. Then they have all the same winner α_w . Let $\pi_b^{(0)}(\alpha_w) = k$. We

must have $k = 1$ because $H_{k-1}^b(n)$ does not depend on n and therefore must vanish (as $H(n)$ is convergent). The $\tau_\alpha^{(n)}$ with $\alpha \neq \alpha_w$ do not depend on n . One also has

$$\tau_{\alpha_w}^{(n-d+1)} = \tau_{\alpha_w}^{(n)} + \sum_{\alpha \neq \alpha_w} \tau_\alpha, \quad \forall n \leq 0.$$

However, we have $\sum_{\alpha \neq \alpha_w} \tau_\alpha = H_{d-1}^t > 0$, hence τ_{α_w} cannot be negative for all $n \leq 0$, a contradiction. \square

Denote by \mathcal{F} (resp. \mathcal{B}) the set of $n \leq 0$ such that $\gamma(n, n+1)$ is of top type (resp. bottom type).

Lemma A.3. — *One has*

$$\lim_{n \rightarrow -\infty, n \in \mathcal{F}} H_{d-1}^t(n) = \lim_{n \rightarrow -\infty, n \in \mathcal{B}} H_{d-1}^b(n) = 0.$$

Proof. — This follows from the formulas relating $H(n)$ and $H(n-1)$. \square

Lemma A.4. — *If $\gamma(n, n+1)$ is of top type, one has*

$$H_{d-1}^t(n-1) = H_{d-1}^t(n), \quad H_{d-1}^b(n-1) = H_d(n).$$

If $\gamma(n, n+1)$ is of bottom type, one has

$$H_{d-1}^b(n-1) = H_{d-1}^b(n), \quad H_{d-1}^t(n-1) = H_d(n).$$

Proof. — Clear from the formulas of an elementary Rauzy-Veech step. \square

Definition A.5. — A negative integer n is a *switching time* if $\gamma(n, n+1)$ and $\gamma(n+1, n+2)$ are of different types. Let $\mathcal{S}w$ be the set of switching times. It is infinite by Lemma A.2.

Lemma A.6. — *One has*

$$\lim_{n \rightarrow -\infty, n \in \mathcal{S}w} H_{d-1}^t(n) = \lim_{n \rightarrow -\infty, n \in \mathcal{S}w} H_{d-1}^b(n) = 0.$$

Proof. — This follows from Lemmas A.3 and A.4. \square

Lemma A.7. — *One has also*

$$\lim_{n \rightarrow -\infty, n \in \mathcal{S}w} H_d(n) = 0.$$

Proof. — Let $n \in \mathcal{S}w$. Assume for instance that $\gamma_{n+1} = \gamma(n, n+1)$ is of top type and $\gamma_{n+2} = \gamma(n+1, n+2)$ is of bottom type. Let α_w be the winner of γ_{n+2} and define $k := \pi_b(n+1)(\alpha_w) \in [1, d)$. We have

$$H_d(n) = H(n+1) - H(n) + H_{k+1}^b(n+1).$$

When n is large, $H(n)$ and $H(n+1)$ are close, $H_d(n)$ is positive (as $\gamma(n)$ is of bottom type) and $H_{k+1}^b(n+1)$ is negative (even when $k = d-1$ as γ_{n+2} is of top type). Therefore $H_d(n)$ is small. \square

A.3. The subset $\mathcal{A}' \subset \mathcal{A}$. — Let \mathcal{A}' be the set of letters in \mathcal{A} which are the winners of at most finitely many arrows of $\underline{\gamma}$. After cutting a terminal subpath off $\underline{\gamma}$, we may assume that none of the letters in \mathcal{A}' is the winner of an arrow of $\underline{\gamma}$. Therefore the quantities $\tau_\alpha^{(n)}$, for $\alpha \in \mathcal{A}'$, do not depend on n . Observe that, as M has no horizontal saddle-connection, these quantities are *different from 0*.

Lemma A.8. — *For any $\alpha \in \mathcal{A} \setminus \mathcal{A}'$, we have*

$$\lim_{n \rightarrow -\infty, n \in \mathcal{S}^w} \tau_\alpha^{(n)} = 0.$$

Proof. — Let $\alpha \in \mathcal{A} \setminus \mathcal{A}'$. Between consecutive switching times, all arrows have the same winner. Let $0 > n_1 > \dots > n_k > \dots$ be the switching times such that α is the winner of $\gamma(n_k, n_{k+1})$, and let $n'_k < n_k$ be the switching time immediately inferior to n_k . The value of $\tau_\alpha^{(n)}$ stays the same for $n_{k+1} \leq n \leq n'_k$. On the other hand, it follows from Lemmas A.6 and A.7 that

$$\lim_{k \rightarrow +\infty} \tau_\alpha^{(n_k)} = 0. \quad \square$$

Remark A.9. — At this stage of the proof we *do not claim* that $\tau_\alpha^{(n)}$ is small for $n'_k < n < n_k$.

We have to prove that \mathcal{A}' is empty. We assume by contradiction that \mathcal{A}' is not empty.

Lemma A.10. — *One has*

$$\sum_{\alpha \in \mathcal{A}'} \tau_\alpha = 0.$$

Proof. — Indeed, as the switching time n goes to $-\infty$, the quantity $H_d(n)$ is alternately positive and negative and the $\tau_\alpha^{(n)}$ with $\alpha \in \mathcal{A} \setminus \mathcal{A}'$ converge to zero. \square

A.4. Decompositions of $\mathcal{A} \setminus \mathcal{A}'$. — Let n be a switching time. Define

$$\mathcal{A}_0^t(n) = \left\{ \alpha \in \mathcal{A} \setminus \mathcal{A}' \mid \sum_{\substack{\beta \in \mathcal{A}' \\ \pi_t^{(n)}(\beta) < \pi_t^{(n)}(\alpha)}} \tau_\beta = 0 \right\},$$

$$\mathcal{A}_{>0}^t(n) = \left\{ \alpha \in \mathcal{A} \setminus \mathcal{A}' \mid \sum_{\substack{\beta \in \mathcal{A}' \\ \pi_t^{(n)}(\beta) < \pi_t^{(n)}(\alpha)}} \tau_\beta > 0 \right\},$$

$$\mathcal{A}_0^b(n) = \left\{ \alpha \in \mathcal{A} \setminus \mathcal{A}' \mid \sum_{\substack{\beta \in \mathcal{A}' \\ \pi_b^{(n)}(\beta) < \pi_b^{(n)}(\alpha)}} \tau_\beta = 0 \right\},$$

$$\mathcal{A}_{<0}^b(n) = \left\{ \alpha \in \mathcal{A} \setminus \mathcal{A}' \mid \sum_{\substack{\beta \in \mathcal{A}' \\ \pi_b^{(n)}(\beta) < \pi_b^{(n)}(\alpha)}} \tau_\beta < 0 \right\},$$

Lemma A.11. — For n negative enough, one has $\mathcal{A}_0^t(n) = \mathcal{A}_0^b(n) = \mathcal{A} \setminus \mathcal{A}'$.

Proof. — We first observe that, for n negative enough, we must have

$$\mathcal{A} \setminus \mathcal{A}' = \mathcal{A}_0^t(n) \sqcup \mathcal{A}_{>0}^t(n) = \mathcal{A}_0^b(n) \sqcup \mathcal{A}_{<0}^b(n).$$

Indeed, if some $\alpha \in \mathcal{A} \setminus \mathcal{A}'$ satisfies

$$\sum_{\substack{\beta \in \mathcal{A}' \\ \pi_t(n)(\beta) < \pi_t(n)(\alpha)}} \tau_\beta < 0,$$

with $n \in \mathcal{S}^w$ large and negative, we would contradict the suspension data conditions as $\tau_\beta(n)$ goes to zero for $\beta \in \mathcal{A} \setminus \mathcal{A}'$ (Lemma A.8). Therefore we have $\mathcal{A} \setminus \mathcal{A}' = \mathcal{A}_0^t(n) \sqcup \mathcal{A}_{>0}^t(n)$ for n negative enough. We prove that $\mathcal{A} \setminus \mathcal{A}' = \mathcal{A}_0^b(n) \sqcup \mathcal{A}_{<0}^b(n)$ in the same way.

To prove that $\mathcal{A}_{>0}^t(n)$ is eventually empty, we prove that this set is shrinking as $n \in \mathcal{S}^w$ goes to $-\infty$.

Consider $n_{k+1} < n_k$ two consecutive switching times.

If $\gamma_{n+1} = \gamma(n, n+1)$ is of top type for $n_{k+1} < n \leq n_k$, then $\pi_t^{(n_k)} = \pi_t^{(n_{k+1})}$ hence $\mathcal{A}_{>0}^t(n_k) = \mathcal{A}_{>0}^t(n_{k+1})$. Observe that, by Lemma A.10, the winner of the arrows γ_{n+1} , for $n_{k+1} < n \leq n_k$, belongs to $\mathcal{A}_0^t(n_k)$.

Assume that $\gamma_{n+1} = \gamma(n, n+1)$ is of bottom type for $n_{k+1} < n \leq n_k$. Let α_w be the winner of the arrows γ_{n+1} , $n_{k+1} < n \leq n_k$. Define $m := \pi_t^{(n_k)}(\alpha_w)$. The successive losers of the arrows γ_{n_k-j} , $j = 1, \dots, n_k - n_{k+1}$ are the letters $\alpha(\bar{j})$ with $\pi_t^{(n_k)}(\alpha(\bar{j})) = m + \bar{j}$, $1 \leq \bar{j} \leq d - m$ and $j \equiv \bar{j} \pmod{d - m}$. There are two cases.

- Assume that the letter α_w belongs to $\mathcal{A}_0^t(n_k)$. Then the loser of $\gamma_{n_{k+1}+2}$, which is also the winner of $\gamma_{n_{k+1}+1}$, also belongs to $\mathcal{A}_0^t(n_k)$ (from Lemmas A.6 and A.7). One has $\mathcal{A}_{>0}^t(n_k) = \mathcal{A}_{>0}^t(n_{k+1})$.
- Assume that the letter α_w belongs to $\mathcal{A}_{>0}^t(n_k)$. Then we must have (from Lemmas A.6 and A.7) that the loser α_ℓ of $\gamma_{n_{k+1}+2}$, which is also the winner of $\gamma_{n_{k+1}+1}$, satisfies $\alpha_\ell \in \mathcal{A} \setminus \mathcal{A}'$, $\pi_t^{(n_k)}(\alpha_w) < \pi_t^{(n_k)}(\alpha_\ell) \leq d - 2$ and

$$\sum_{\substack{\beta \in \mathcal{A}' \\ \pi_t^{(n_k)}(\alpha_w) < \pi_t^{(n_k)}(\beta) < \pi_t^{(n_k)}(\alpha_\ell)}} \tau_\beta = 0$$

Therefore α_ℓ belongs to $\mathcal{A}_{>0}^t(n_k)$ and to $\mathcal{A}_0^t(n_{k+1})$. On the other hand, no element α of $\mathcal{A}_0^t(n_k)$ satisfies $\pi_t^{(n_k)}(\alpha_w) < \pi_t^{(n_k)}(\alpha) < \pi_t^{(n_k)}(\alpha_\ell)$, therefore $\mathcal{A}_0^t(n_k) \subset \mathcal{A}_0^t(n_{k+1})$.

We now can conclude that $\mathcal{A}_{>0}^t(n)$, $n \in \mathcal{S}^w$, is eventually empty. Indeed, a letter which would belong to $\mathcal{A}_{>0}^t(n)$ for all $n \in \mathcal{S}^w$ is the winner of γ_n for infinitely many γ_n , $n \in \mathcal{S}^w$. But we have seen that it cannot be the winner of infinitely many arrows of top type, and that each time that it is the winner of an arrow of bottom type, the cardinality of $\mathcal{A}_{>0}^t(n)$ is decreasing.

We prove similarly that $\mathcal{A}_{<0}^b(n)$, $n \in \mathcal{S}^w$, is eventually empty. □

A.5. Proof of Proposition 4.3. — Take $n \in \mathcal{S}^w$ large enough such that $\mathcal{A}_0^t(n) = \mathcal{A} \setminus \mathcal{A}'$. Let m be the smallest value of $\pi_t^{(n)}(\alpha)$, for $\alpha \in \mathcal{A}'$. From Lemma A.11, there exists $m' > m$ such that $\beta \in \mathcal{A}'$ for $m \leq \pi_t^{(n)}(\beta) \leq m'$, and

$$\sum_{m \leq \pi_t^{(n)}(\beta) \leq m'} \tau_\beta = 0, \quad \sum_{m \leq \pi_t^{(n)}(\beta) \leq m_1} \tau_\beta > 0, \quad \forall m \leq m_1 < m'.$$

Then we have an horizontal saddle connection!

Appendix B

Adaptation of the proofs of results on cohomological equation for absolute Roth type i.e.m.

Our goal in this appendix is to check that the conclusion of Theorem 3.10 of [38], namely Theorem 3.14, as well as the conclusions of Theorem 3.11 and Theorem 3.22 of [38], are still valid under the assumption that the i.e.m. is of restricted *absolute* Roth type (instead that under the stronger requirement of being of restricted Roth type). The generalization of Theorem 3.10 of [38] was already stated as Theorem 3.14 in § 3.5. We begin the appendix by stating the generalizations of Theorems 3.11 and 3.22 of [38] (stated as Theorem B.1 and B.4 below), then review their proofs indicating which (small) modifications are needed.

B.1. Estimates for special Birkhoff sums with C^1 data. — The following result extends Theorem 3.11 of [38], which was proved for i.e.m. of restricted Roth type, to i.e.m. of restricted *absolute* Roth type.

Theorem B.1 (Generalization of Theorem 3.11 of [38]). — *Let T be an i.e.m. which satisfies conditions (a'), (c), (d) of Sections 3.1 and 3.3. Let $\Gamma_u(T)$ be a subspace of $\Gamma_\partial(T)$ which is supplementing $\Gamma_s(T)$ in $\Gamma_\partial(T)$. The operator $L_1 : \varphi \mapsto \chi$ of Theorem 3.14 extends to a bounded operator from $C_0^1(\bigsqcup I_\alpha(T))$ to $\Gamma_u(T)$. For $\varphi \in C_0^1(\bigsqcup I_\alpha(T))$, the function $\chi = L_1(\varphi) \in \Gamma_u(T)$ is characterized by the property that the special Birkhoff sums of $\varphi - \chi$ satisfy, for any $\tau > 0$*

$$\|S(0, n)(\varphi - \chi)\|_{C^0} \leq C(\tau) \|\varphi\|_{C^1} \|B(0, n)\|^\tau.$$

The inequality of the theorem for special Birkhoff sums implies a similar inequality for general Birkhoff sums:

Corollary B.2. — *For any T and φ as in Theorem B.1, for all $\tau > 0$, we have*

$$\left\| \sum_{j=0}^{N-1} (\varphi - \chi) \circ T^j \right\|_{C^0} \leq C(\tau) \|\varphi\|_{C^1} N^\tau.$$

B.2. Higher differentiability. — To formulate a result allowing for smoother solutions of the cohomological equation which extend Theorem 3.22 in [38], we introduce the same finite-dimensional spaces than in [37] (although the notations are slightly different). In the following definitions, T is an i.e.m., κ is a nonnegative integer, and r is a real number ≥ 0 .

Definition B.3. — We denote by

- $C_{\kappa}^{\kappa+r}(\bigsqcup I_{\alpha}(T))$ the subspace of $C^{\kappa+r}(\bigsqcup I_{\alpha}(T))$ consisting of functions φ which satisfy $\partial\kappa^i\varphi = 0$ for $0 \leq i \leq \kappa$.
- $\Gamma(\kappa, T)$ the subspace of $C_{\kappa}^{\infty}(\bigsqcup I_{\alpha}(T))$ consisting of functions χ whose restriction to each $[u_{i-1}(T), u_i(T)]$ is a polynomial of degree $\leq \kappa$.
- $\Gamma_s(\kappa, T)$ the subspace of $\Gamma(\kappa, T)$ consisting of functions χ which can be written as $\chi = \psi \circ T - \psi$, for some function ψ of class C^{κ} on the closure of $[u_0(T), u_d(T)]$.

The proof given in [37] (p.1602) that if T is of restricted Roth type then the dimension of $\Gamma(\kappa, T)$ is equal to $2g + \kappa(2g - 1)$, and that the dimension of $\Gamma_s(\kappa, T)$ is equal to $g + \kappa$, also holds, essentially unmodified, under the assumption that T is of absolute restricted Roth type.

Theorem B.4 (Generalization of Theorem 3.22 of [38]). — *Let T be an i.e.m. of absolute restricted Roth type. Let κ be a nonnegative integer, and let r be a real number > 1 . Let $\Gamma_u(\kappa, T)$ be a subspace of $\Gamma(\kappa, T)$ supplementing $\Gamma_s(\kappa, T)$. There exist a real number $\bar{\delta} > 0$, and bounded linear operators $L_0 : \varphi \mapsto \psi$ from $C_{\kappa}^{\kappa+r}(\bigsqcup I_{\alpha}(T))$ to $C^{\kappa+\bar{\delta}}([u_0(T), u_d(T)])$ and $L_1 : \varphi \mapsto \chi$ from $C_{\kappa}^{\kappa+r}(\bigsqcup I_{\alpha}(T))$ to $\Gamma_u(\kappa, T)$ such that any $\varphi \in C_{\kappa}^{\kappa+r}(\bigsqcup I_{\alpha}(T))$ satisfies*

$$\varphi = \chi + \psi \circ T - \psi.$$

The number $\bar{\delta}$ is the same than in Theorem 3.14.

The derivation of Theorem 3.22 from Theorem 3.10 (the case $\kappa = 0$) is easy and done in Section 3.2 of [37] (see p.1602-1603). Following the same arguments, Theorem B.4 follows from Theorem 3.14. We hence will now focus on the adaptation of the proofs of Theorems 3.10 and 3.11, needed to prove Theorems 3.14 and B.1.

B.3. Growth of $B(0, n_{\ell})$. — At many places in the proof of Theorems 3.10 and 3.11, one uses the following estimate, which we now prove under condition (a') of absolute Roth type instead than condition (a) of Roth type.

Proposition B.5. — *Assume that condition (a') of the definition of absolute Roth type is satisfied. Then, for any $\delta > 0$, there exists a constant C_{δ} such that, for all $k \geq 0$, we have*

$$\sum_{\ell > k} \|B(0, n_{\ell})\|^{-\delta} \leq C_{\delta} \|B(0, n_k)\|^{-\delta}.$$

Proof. — The desired estimate will be a consequence of the exponential growth of $\|B(0, n_\ell)\|$, implied by the absolute Roth type condition, which will guarantee that the whole sum is controlled by its first term. Let v be a fixed vector in $\mathcal{C}(\pi(n_0))$. The sequence $|B(0, n_\ell)v|_1$ (where $|\cdot|_1$ is the ℓ_1 -norm on $\mathbb{R}^{\mathcal{S}}$) is non decreasing and we will show that it grows fast enough to obtain the conclusion of the proposition. Let $v(\ell) := B(0, n_\ell)v$. By Lemma 3.12, one has

$$\max_{\alpha} v(\ell)_{\alpha} \leq C \|B(n_{\ell-1}, n_{\ell})\| \min_{\alpha} v(\ell)_{\alpha}.$$

We have therefore

$$|v(\ell+1)|_1 \geq |B(n_{\ell}, n_{\ell+1})v(\ell)|_1 \geq (1 + c \|B(n_{\ell-1}, n_{\ell})\|^{-1}) |v(\ell)|_1.$$

It follows that the smallest ℓ' such that $|v(\ell')|_1 \geq 2|v(\ell)|_1$ satisfies, for any $\tau > 0$ (and an appropriate constant C_{τ})

$$\ell' \leq \ell + C_{\tau} \|B(0, n_{\ell})\|^{\tau}.$$

Since for any fixed vector $v \in \mathcal{C}(\pi(n_0))$ the sequence $|v(\ell)|_1$ doubles in at most $C_{\tau} \|B(0, n_{\ell})\|^{\tau}$ steps, the same holds for the norm $\|B(0, n_{\ell})\|$. Let $\ell_0 = n_k$ and define recursively ℓ_{j+1} equal to the smallest ℓ' such that $\|B(0, n_{\ell'})\| \geq 2\|B(0, n_{\ell_j})\|$. The bound on $\ell' - \ell$ obtained above gives

$$\sum_{\ell \geq k} \|B(0, n_{\ell})\|^{-\delta} \leq \|B(0, n_k)\|^{-\delta} \sum_{j \geq 0} C_{\tau} \|B(0, n_{\ell_j})\|^{\tau} 2^{-\delta} \|B(0, n_{\ell_j})\|^{-\delta}.$$

Taking $\tau = \frac{1}{2}\delta$ gives the required inequality. \square

Along the same lines, one also have (with a very similar proof):

Proposition B.6. — *Assume that condition (a') is satisfied. Then, for any $\delta > 0$, there exists a constant C_{δ} such that, for all $k \geq 0$, we have*

$$\sum_{0 \leq \ell \leq k} \|B(0, n_{\ell})\|^{\delta} \leq C_{\delta} \|B(0, n_k)\|^{\delta}.$$

B.4. Adaptation of the proof of Theorems 3.10 and 3.11 in [38]. — We here follow Section 3.5 of [38]. As mentioned above (Corollary 3.11), the bound from above for $|I_{\alpha}^{(n)}|$ is still valid under condition (a'). The bound from below in Lemma 3.14 of [38] is not used in the proof of Theorem 3.11.

The sequence (n_k) which appears in the proof of Theorem 3.11 (and was defined at the end of Section 2.5 of [38]) is the sequence that we have called (\tilde{n}_k) in Section 3.3 of the present text. We have to replace it by the sequence (n_k) of condition (a'). The only property satisfied by (\tilde{n}_k) under condition (a) which is not satisfied by (n_k) under condition (a') is the lower bound for $|I_{\alpha}^{(n)}|$ in Lemma 3.14.

B.4.1. *Adaptation of the proof of Lemma 3.15 in [38].* — In [38], we refer for the proof to an annex in [37]. The proof in this paper depends on a claim (Proposition B.7 below) which is still true, but the proof has to be modified.

Proposition B.7. — *Assume that the rotation number $\underline{\gamma}$ satisfies condition (a'). For every $\tau > 0$, there exists $C(\tau) > 0$ such that, for any $0 \leq \ell \leq k$, one has*

$$\|B(n_0, n_k)\| \leq \|B(n_0, n_\ell)\| \|B(n_\ell, n_k)\| \leq C(\tau) \|B(n_0, n_k)\|^{1+\tau}$$

Proof. — The first inequality is trivial. To prove the second, choose $v \in \mathcal{C}(\pi^{(n_0)})$, $v' \in \mathcal{C}(\pi^{(n_\ell)})$, and define $w := B(n_0, n_\ell) \cdot v$. One has

$$\begin{aligned} \|B(n_0, n_k)\| &\geq \|v\|^{-1} \|B(n_0, n_k)v\| = \|v\|^{-1} \|B(n_\ell, n_k)w\|, \\ \|B(n_0, n_\ell)\| &\leq C(\pi(n_0)) \|v\|^{-1} \|B(n_0, n_\ell)v\| = C(\pi(n_0)) \|v\|^{-1} \|w\|, \end{aligned}$$

and therefore

$$\frac{\|B(n_0, n_\ell)\| \|B(n_\ell, n_k)\|}{\|B(n_0, n_k)\|} \leq C(\pi(n_0)) \frac{\|B(n_\ell, n_k)\| \|w\|}{\|B(n_\ell, n_k)w\|}.$$

From Lemma 3.12, one has

$$\max_{\alpha} w_{\alpha} \leq C \|B(n_{\ell-1}, n_{\ell})\| \min_{\alpha} w_{\alpha}.$$

and therefore

$$\frac{\|B(n_\ell, n_k)\| \|w\|}{\|B(n_\ell, n_k)w\|} \leq C' \|B(n_{\ell-1}, n_{\ell})\| \leq C_{\tau} \|B(n_0, n_k)\|^{\tau},$$

giving the required estimate. \square

The proof of Lemma 3.16 in [38] is completely similar to the proof of Theorem 3.11 of [38] and the same remarks for its generalization apply.

B.4.2. *Space and Time decompositions.* — Following Sections 3.7 and 3.8 in [38], one has just to remember that the sequence n_k in the present setting is not the same than in [38]. Otherwise, the same formalism works and yields the same type of space and time decompositions.

Section 3.9 also does not need any modification (except, as always, for the change of definition of the (n_k)).

B.4.3. *General Hölder estimates.* — We now follow Section 3.10 of [38]. The bound from below in Lemma 3.14 is used (for the first and last time) at the end of this section. However, Proposition 3.13 above is an adequate substitute.

As in Section 3.10 in [38], we write any interval (x_-, x_+) as a countable union of intervals J which belong to some $\mathcal{P}(\ell)$, $\ell \geq \ell_{\min}$ (We use ℓ_{\min} instead of k because we are going to use Proposition 3.13).

Let $J \in \mathcal{P}(\ell)$ be one of these intervals. It is a translated copy of some $I_{\alpha}^{(n_\ell)}$. We apply Proposition 3.13. It gives an integer k such that

- J is the union of at most $(s-1)$ intervals of $\mathcal{P}(k)$;
- $|J| \geq C_{\tau}^{-1} |I^{(0)}| \|B(0, n_k)\|^{-\tau-1}$.

From Lemma 3.20 of [38] and the first item above, we have that

$$(\star) \quad |\Delta(J)| \leq C \|B(0, n_k)\|^{-\delta_2} \|\varphi\|_{C^r}.$$

(where $\Delta(J)$ is the difference between the values of ψ at the endpoints of J).

Let k_{\min} be the minimal value of k when J runs among the intervals of the space decomposition of (x_-, x_+) . We have then

$$|x_+ - x_-| \geq C_\tau^{-1} |I^{(0)}| \|B(0, n_{k_{\min}})\|^{-\tau-1}.$$

For $\ell_{\min} \leq m$, there are at most $\|B(n_m, n_{m+1})\|$ intervals in $\mathcal{P}(m)$ in the space decomposition of (x_-, x_+) . For intervals J with $m \leq k_{\min}$, we use (\star) . For the other intervals with $m > k_{\min}$, we just use Lemma 3.20. We get

$$\begin{aligned} |\psi(x_+) - \psi(x_-)| &\leq C \|\varphi\|_{C^r} \left\{ \|B(0, n_{k_{\min}})\|^{-\delta_2} \left(\sum_{\ell_{\min} \leq m < k_{\min}} \|B(n_m, n_{m+1})\| \right) \right. \\ &\quad \left. + \sum_{m \geq k_{\min}} \|B(n_m, n_{m+1})\| \|B(0, n_m)\|^{-\delta_2} \right\} \\ &\leq C' \|\varphi\|_{C^r} \sum_{m \geq k_{\min}} \|B(0, n_m)\|^{-\delta_2/2} \\ &\leq C'' \|\varphi\|_{C^r} \|B(0, n_{k_{\min}})\|^{-\delta_2/2}. \end{aligned}$$

We have used Propositions B.5 and B.6 (with $\delta = \delta_2/2$). The proof of the required inequality is complete.

This concludes the modifications required to prove Theorem B.1 and Theorem 3.14, and hence also Theorem B.4.

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