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Concentration Phenomena for NLS: Recent Results and New Perspectives ⋆

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Summary. We survey some results on (NLS_ε), discussing also new perspectives and open problems.

1.1 Introduction

Beginning from the pioneering paper by A. Floer and A. Weinstein [14], a great deal of work has been devoted to the study of nonlinear Schrödinger equations like

$$\begin{cases} -\varepsilon^2 \Delta u + V(x) u = K(x) u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases} \tag{NLS_ε}$$

where ε is a small parameter and where

$$1 < p < \frac{n+2}{n-2}.$$

For the applications to Quantum Mechanics, it is particularly relevant to understand the behavior of the solutions of (NLS_ε) as $\varepsilon \to 0$. In this respect, one of the most characteristic features of (NLS_ε) is that its solutions u_ε *concentrate* as $\varepsilon \to 0$, in the sense that, out of a certain concentration set, the function $u_\varepsilon(x)$ decays uniformly to zero as $\varepsilon \to 0$. When this concentration set is a single point, resp. a finite (or infinite) collection of points, these solutions

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are usually called *spikes*, resp. *multi-bump solutions*. We anticipate that in some cases it is also possible to find solutions concentrating at a manifold.

First, in section 1.2, we discuss the well studied case in which V is bounded and such that $\inf_{\mathbb{R}}^{n} V(x) > 0$. An interesting open question is related with the existence of solutions concentrating on manifolds. This actually arises in the radial case, when there are solutions concentrating on a sphere, see section 1.3 (we also refer to the last section for more general results). In sections 1.4 and 1.5 we will outline some more recent results dealing with potentials such that $\inf_{\mathbb{R}^n} V(x) = 0$ and/or $V(x) \to 0$ as $|x| \to \infty$. A last section is devoted to new perspectives and some open problems.

Some of the above topics, especially those contained in sections 1.2 and 1.3, are also discussed in the recent Monograph [3, Chapters 8 and 10], which also contains a broad bibliography, where we address the reader for references and for further details.

1.2 The case of potentials bounded away from zero

In this section we shortly outline the case in which V is smooth and satisfies

$$\exists\, a, b > 0 \quad \text{such that} \quad a \leq V(x) \leq b. \tag{V_1}$$

We also take $p \in \left(1, \frac{n+2}{n-2}\right)$, and $K(x) > 0$ smooth and bounded. In order to prove the existence of solution to (NLS_ε) it is convenient to introduce the auxiliary potential

$$Q(x) = V^{\theta}(x) K^{-2/(p-1)}(x), \qquad \theta = \frac{p+1}{p-1} - \frac{n}{2}.$$

The role of Q is highlighted by the fact that a necessary condition for the concentration of solutions of (NLS_ε) at a point x_0 is that $Q'(x_0) = 0$. Conversely, one can prove:

Theorem 1.2.1 *Let* $1 < p < \frac{n+2}{n-2}$ *and* $K(x) > 0$ *be smooth and bounded, and assume that* V *satisfies* (V_1) *and*

$$\|V(x)\|_{C^2(\mathbb{R}^n)} < +\infty. \tag{V_2}$$

Moreover let x_0 *be a* stable *stationary point of* Q*, in the sense that the local degree of* Q' *at* x_0 *is different from zero. Then, for* ε *sufficiently small,* (NLS_ε) *has a solution concentrating at* x_0 *as* $\varepsilon \to 0$.

Remarks 1.2.2

(i) Maxima, minima and non-degenerate stationary points of Q are stable in the sense specified above. One could also suppose that Q possesses a bounded set $\mathcal{Q} \subset \mathbb{R}^n$ of critical points such that $deg(Q', \mathcal{Q}_\delta, 0) \neq 0$, for $\delta \ll 1$, where $\mathcal{Q}_\delta$ denotes a δ-neighborhood of $\mathcal{Q}$. In this case, however, one merely obtains the existence of a solution concentrating at some point of $\mathcal{Q}$, but not determined a priori.

(ii) One can be more precise about the behavior of u_ε near the concentration point x_0. Actually, if $U_0 \in W^{1,2}(\mathbb{R}^n)$ denotes the radial positive solution of

$$-\Delta u + V(x_0)u = K(x_0)u^p, \quad \nabla u(0) = 0,$$

then $u_\varepsilon(x) \sim U_0(\frac{x-x_0}{\varepsilon})$, as $\varepsilon \to 0$.

(iii) In the preceding theorem, the parameter ε is sufficiently small. Existence results for $\varepsilon = 1$ are different in nature and will not be discussed here.

(iv) For $p \geq \frac{n+2}{n-2}$ and, say, $V \equiv K \equiv 1$, the Pohozaev identity implies that there are no solutions in $W^{1,2}(\mathbb{R}^n)$, for any $\varepsilon > 0$.

(v) Theorem 1.2.1 can be improved by considering the case in which V satisfies (V_1) and (V_2), but possesses a smooth manifold $\mathcal{M}$ of critical points which is non-degenerate, in the sense that $\mathrm{Ker}[V''(x)] = T_x\mathcal{M}$, $\forall\, x \in \mathcal{M}$. In such a case one can show that (NLS_ε) has at least cuplong($\mathcal{M}$) (see e.g. [3] for the definition) solutions concentrating on points of $\mathcal{M}$. We believe it is possible to establish the locations of these concentrations considering the second-order derivatives of V or the geometry of $\mathcal{M}$, although such a result has never been explicitly carried out.

(vi) Another improvement is concerned with a nonlinearity more general than a pure power u^p. See, e.g. [1, Section 6] and [11].

(vii) Further existence results are concerned with solutions that have many peeks, namely multi-bump solutions. These solutions can be obtained gluing together two or more spikes whose peeks are sufficiently far away each other (usually at a scale bigger than ε). ■

1.3 Solutions concentrating on manifolds

It has been conjectured for some time (see e.g. [20]) that (NLS_ε) should admit solutions concentrating at non-trivial sets (curves or manifolds) when $\varepsilon \to 0$. The same conjecture has been raised around the end of the 80's by W.M. Ni for the following related Neumann problem (concerning pattern formation in chemistry or biology)

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega \subseteq \mathbb{R}^n; \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega; \\ u > 0 & \text{in } \Omega. \end{cases} \tag{P_ε}$$

The first general results on (P_ε) are very recent.

Theorem 1.3.1 [16, 17] *Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded set, and let $p > 1$. Then there exists* a sequence $\varepsilon_j \to 0$ *such that (P_{ε_j}) possesses solutions u_j concentrating at $\partial\Omega$.*

Theorem 1.3.2 [15] *Let $\Omega \subseteq \mathbb{R}^3$ be a smooth, bounded domain, and let $p > 1$. Let $h : S^1 \to \partial\Omega$ be a simple closed non-degenerate geodesic. Then there exists $\varepsilon_j \to 0$ such that (P_{ε_j}) possesses solutions u_j concentrating at h.*

Also in the case of (NLS_ε), very few results are known. They are concerned with radial potentials: $V(x) = V(r)$, $r = |x|$, and with radial solutions u_ε concentrating at spheres. Here the counterpart of Q is played by a combination of volume and potential energy, and is expressed by the *auxiliary weighted potential*

$$M(r) = r^{n-1}V^{\ell}(r), \qquad \ell = \frac{p+1}{p-1} - \frac{1}{2}.$$

It is possible to show that if a radial solution concentrates at a sphere of radius r_0, then $M'(r_0) = 0$. Conversely, there holds

Theorem 1.3.3 [4] *Let V be radial and satisfy (V_1), (V_2). Moreover, let $p > 1$ and assume that $r_0 > 0$ is a local maximum or a minimum of M. Then, for ε small enough, (NLS_ε) has a radial solution concentrating at the sphere $\{|x| = r_0\}$.*

Remarks 1.3.4

(i) For $n = 1$, the critical points of M coincide with those of V. On the contrary, for $n > 1$ they are always different because

$$M'(r) = 0 \qquad \Leftrightarrow \qquad V'(r) = -\frac{n-1}{\ell}\frac{V(r)}{r} \neq 0.$$

Moreover, since $M(r) \sim r^{n-1}$ at 0 and at infinity, then generically solutions arise in pairs.

(ii) The functions u_ε scale in one variable only, and their profile is asymptotic to the solution of the one-dimensional problem $-v'' + v = v^p$, where p can be arbitrary. For this reason, any exponent $p > 1$ is allowed. In general, one can prove the existence of solutions concentrating near a k-dimensional sphere, $1 \leq k \leq n-1$. In such a case, the corresponding weighted auxiliary potential becomes $M_k = r^k V^{\ell_k}(r)$, where $\ell_k = \frac{p+1}{p-1} - \frac{1}{2}(n-k)$ and the solutions are asymptotic to those of $-\Delta u + u = u^p$, $u \in W^{1,2}(\mathbb{R}^{n-k})$. As a consequence, it turns out that the exponent p has to be taken subcritical w.r.t. $\mathbb{R}^{n-k}$, namely $1 < p < \frac{n-k+2}{n-k-2}$ if $n - k > 2$, and $p > 1$ if $n - k \leq 2$.

(iii) There is a plethora of non-radial solutions to (NLS_ε) bifurcating from the radial ones. Roughly, let M possess a critical point $\overline{r} > 0$ such that $M''(\overline{r}) \neq 0$, and let u_ε denote the radial solutions concentrating at $\{|x| = \overline{r}\}$. Then, for $\varepsilon_0 \ll 1$, $\Lambda_{\varepsilon_0} = \{(\varepsilon, u_\varepsilon) \ : \ \varepsilon \in (0, \varepsilon_0)\}$ is a smooth curve and there exists a sequence $\varepsilon_l \to 0$ such that from every $(\varepsilon_l, u_{\varepsilon_l}) \in \Lambda_{\varepsilon_0}$ bifurcates a family of non-radial solutions of (NLS_ε). ■

The proof of Theorem 1.3.3 is based on a finite-dimensional reduction. The proof of the bifurcation result mentioned in Remark 1.3.4-(iii) relies on the fact that along Λ_{ε_0} the radial solutions u_ε have diverging Morse index (as $\varepsilon \to 0$) in $W^{1,2}(\mathbb{R}^n)$. For details, see [4] and [3, Ch. 10].

1.4 The case of vanishing potentials

The first attempt to weaken the assumption (V_1) has been made in [8] where V is assumed to satisfy

$$V \in C(\mathbb{R}^n), \quad \inf_{\mathbb{R}^n} V(x) = 0, \quad \text{and} \quad \liminf_{|x|\to\infty} V(x) > 0. \tag{V_3}$$

In this case is no more possible to use the perturbation method used when (V_1) holds because, roughly, the limit equation $-\Delta u + V(\varepsilon\xi)u = K(\varepsilon\xi)u^p$ can have only (assuming decay at infinity) the trivial solution when $\varepsilon\xi$ belongs to the set

$$\mathcal{V} := \{x \in \mathbb{R}^n : V(x) = 0\}.$$

The new strategy consists in considering the constrained minimization problem

$$\begin{aligned} m_\varepsilon \quad &= \inf\{\|u\|_\varepsilon^2 : \textstyle\int_{\mathbb{R}^n} K(\varepsilon x)|u|^{p+1}dx = 1, \\ &\qquad \textstyle\int_{\mathbb{R}^n\setminus\mathcal{V}_\varepsilon^\delta} K(\varepsilon x)|u|^{p+1}dx \le \varepsilon^{\frac{3(p+1)}{p-1}}\}, \end{aligned}$$

where $\mathcal{V}_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon x \in \mathcal{V}\}$ and by $\mathcal{V}_\varepsilon^\delta$ denotes the $\frac{\delta}{\varepsilon}$-neighborhood of $\mathcal{V}_\varepsilon$. One shows that $m_\varepsilon > 0$ is achieved and hence there exist $v_\varepsilon \in W^{1,2}(\mathbb{R}^n)$ and λ_ε, $\mu_\varepsilon \in \mathbb{R}$ such that

$$-\Delta v_\varepsilon + V(\varepsilon x)v_\varepsilon = \lambda_\varepsilon K(\varepsilon x)v_\varepsilon^p + \mu_\varepsilon \chi_{\mathbb{R}^n\setminus\mathcal{V}_\varepsilon^{4\delta}} K(\varepsilon x)v_\varepsilon^p, \quad u_\varepsilon > 0. \tag{1.1}$$

Using the fact that $\liminf_{|x|\to\infty} V(x) > 0$, it is possible to prove that v_ε decays exponentially to zero as $|x| \to \infty$ uniformly, and thus

$$\int_{\mathbb{R}^n\setminus\mathcal{V}_\varepsilon^\delta} K(\varepsilon x)|v_\varepsilon|^{p+1}dx < \varepsilon^{\frac{3(p+1)}{p-1}}. \tag{1.2}$$

Then $\widetilde{v}_\varepsilon = m_\varepsilon^{\frac{1}{p-1}} v_\varepsilon$ is a solution of the equation

$$\begin{cases} -\Delta u + V(\varepsilon x)u = K(\varepsilon x)u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases} \tag{$\widetilde{NLS}_\varepsilon$}$$

obtained by the change of variable $x \mapsto \varepsilon x$. Scaling back, $u_\varepsilon(x) = \widetilde{v}_\varepsilon(\varepsilon^{-1}x)$ solves (NLS_ε). These arguments lead to prove:

Theorem 1.4.1 [8] *Suppose that (V_3) holds and let $1 < p < \frac{n+2}{n-2}$. Then for ε sufficiently small, (NLS_ε) has a solution u_ε, concentrating at some point $x^* \in \mathcal{V}$, as $\varepsilon \to 0$. Moreover, there holds*

$$\lim_{\epsilon\to 0} \|u_\varepsilon\|_\infty = 0, \qquad \text{and} \qquad \liminf_{\varepsilon\to 0} \varepsilon^{\frac{-2}{p-1}} \|u_\varepsilon\|_\infty > 0. \tag{1.3}$$

The last formula (1.3) relies on the fact that $m_\varepsilon \to 0$ as $\varepsilon \to 0$ and shows that the case of a vanishing potential has features different from those seen when $\inf V > 0$. This is confirmed by a sharp analysis carried out in [8] which permits to find the asymptotic profile of the solutions u_ε. Roughly, if e.g. $\mathcal{V} = \{0\}$ and $V(x) = |x|^m + o(|x|^m)$ as $|x| \to 0$, then one proves that for any $\varepsilon_j \to 0$ there is a subsequence (denoted still by ε_j) such that $\varepsilon_j^{-\frac{2}{p-1}\frac{m}{m+2}} u_{\varepsilon_j}(\varepsilon_j^{\frac{2}{m+2}} x)$ converges uniformly to a ground state (i.e. a Mountain-Pass) solution of

$$-\Delta w + |x|^m w = K(0) w^p. \tag{1.4}$$

Other flatness condition on V are studied in [8], where we refer for more details.

1.5 The case of potentials decaying to zero

In this section we are concerned with the case in which $V > 0$ is such that $\lim_{|x|\to\infty} V(x) = 0$. Precisely we will assume

$$\exists\, \alpha, a_1, a_2 > 0 \qquad \text{such that} \qquad \frac{a_1}{1+|x|^\alpha} \le V(x) \le a_2. \tag{V_4}$$

In addition, in Theorem 1.5.1 below, we will require that

$$\exists\, \beta, a_3 > 0 \qquad \text{such that} \qquad 0 < K(x) \le \frac{a_3}{1+|x|^\beta}\ . \tag{K_1}$$

As before, one studies the scaled equation $(\widetilde{NLS}_\varepsilon)$ whose solutions are, formally, the critical points of

$$I_\varepsilon(u) = \tfrac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + V(\varepsilon x) u^2\right) - \tfrac{1}{p+1} \int_{\mathbb{R}^n} K(\varepsilon x) |u|^{p+1}. \tag{1.1}$$

Unfortunately, I_ε is not well defined on $W^{1,2}(\mathbb{R}^n)$ and, to circumvent this difficulty, one can work on the weighted Sobolev space

$$H_\varepsilon = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left[|\nabla u(x)|^2 + V(\varepsilon x) u^2(x)\right] dx < \infty\},$$

endowed with scalar product and norm

$$(u|v)_\varepsilon = \int_{\mathbb{R}^n} [\nabla u(x) \cdot \nabla v(x) + V(\varepsilon x) u(x) v(x)]\, dx, \qquad \|u\|_\varepsilon^2 = (u|u)_\varepsilon. \tag{1.2}$$

Let

$$\sigma = \sigma_{n,\alpha,\beta} = \begin{cases} \frac{n+2}{n-2} - \frac{4\beta}{\alpha(n-2)}, & \text{if } 0 < \beta < \alpha, \\ 1 & \text{otherwise.} \end{cases}$$

If $\sigma \leq p \leq \frac{n+2}{n-2}$, then H_ε is embedded into the weighted Lebesgue space $L^q_K(\mathbb{R}^n) = \{u : \int_{\mathbb{R}^n} K(\varepsilon x)|u|^q dx < \infty\}$. Moreover the embedding is compact provided $\sigma < p < \frac{n+2}{n-2}$. Thus I_ε is well defined (and smooth) on H_ε and the Mountain-Pass theorem yields a critical point $v_\varepsilon \in H_\varepsilon$ of I_ε. The energy of v_ε is smaller than any other critical point of I_ε and, for this reason, will be called a *ground state*. Moreover, if $0 < \alpha < 2$, there exists $\gamma > 0$ such that $\|v_\varepsilon\|^2_{H_\varepsilon} \leq \gamma\,\varepsilon^n$. From this fact and using some integral estimates, one proves that $v_\varepsilon \in L^2(\mathbb{R}^n)$, so that $v_\varepsilon \in W^{1,2}(\mathbb{R}^n)$. Finally, one shows that v_ε has a pointwise exponential decay at infinity, depending on $0 < \alpha < 2$: $\exists\, C,\, d,\, R > 0$, such that

$$|v_\varepsilon(x)| \leq C|x|^d \exp\left\{ -\tfrac{1}{4}\left|\log \tfrac{3}{4}\right| \frac{\left(|x|^{\frac{2-\alpha}{2}} - (\frac{R}{\varepsilon})^{\frac{2-\alpha}{2}}\right)}{\varepsilon^{\alpha/2}} \right\}, \quad \forall\, |x| \gg 1. \tag{1.3}$$

This allows us to prove that the rescaled solution u_ε concentrates at a global minimum of Q.

Theorem 1.5.1 [2] *If $0 < \alpha < 2$, $\beta > 0$, and $\sigma < p < \frac{n+2}{n-2}$ then (NLS_ε) has, for every $\varepsilon > 0$. a ground-state u_ε. Furthermore, u_ε concentrates at a global minimum of Q, as $\varepsilon \to 0$.*

Remark 1. The existence result is true for all $\varepsilon > 0$. Moreover, if $\sigma < p < \frac{n+2}{n-2}$, Q is bounded below and tends to infinity as $|x| \to \infty$ and hence has a global minimum. ■

Theorem 1.5.1 holds for $\sigma < p < \frac{n+2}{n-2}$. Actually, if $1 < p < \sigma$, one shows that there are no ground state at all. However, one could ask whether there are solutions concentrating at a stable stationary point of Q for p in all the range $1 < p < \frac{n+2}{n-2}$. An answer to this question is given by the following result:

Theorem 1.5.2 [5] *Suppose that (V_4) holds with $0 < \alpha \leq 2$, that K satisfies*

$$K \in C^1(\mathbb{R}^n), \quad \textit{and} \quad \exists\, \kappa 0 \textit{ such that } 0 < K(x) \leq \kappa \quad \forall\, x \in \mathbb{R}^n, \tag{K_2}$$

and that $1 < p < \frac{n+2}{n-2}$. Then for any stationary point x_0 of Q, (NLS_ε) has, for every $\varepsilon > 0$. Moreover u_ε concentrates at x_0, as $\varepsilon \to 0$.

It is worth pointing out that we can handle the case $\alpha = 2$, and we do not need to require that $K \to 0$ as $|x| \to \infty$, namely we can take $\beta = 0$ in (K_1). On the other hand, in Theorem 1.5.2 the existence of a solution is proved for ε sufficiently small, only.

The proof is based on an appropriate perturbation procedure, which we are going to outline. Below we will proceed formally, in the sense that we look for critical points of I_ε, although I_ε might be not well defined on H_ε. The argument can be made precise by means of a truncation. However, we will find critical points such that $Ku^{p+1} \in L^1(\mathbb{R}^n)$. Let $U_{\varepsilon\xi}$ denote the radial positive solution (decaying to zero at infinity) of

$$-\Delta u + V(\varepsilon\xi)u = K(\varepsilon\xi)u^p,$$

and consider the manifold

$$Z_\varepsilon = \{z_{\varepsilon,\xi} = U_{\varepsilon\xi}(x-\xi) : \xi \in \mathbb{R}^n\}.$$

Since for ε small, $U_{\varepsilon\xi}$ are approximate solutions of $(\widetilde{NLS_\varepsilon})$, one can make the *Ansatz* that there exists a critical point of I_ε of the form $u = z_{\varepsilon,\xi} + w$, with a suitable w (small) and an appropriate choice of ξ. To carry out this procedure, one starts by using the Lyapunov-Schmidt reduction method to split the equation $I'_\varepsilon(z_{\varepsilon,\xi} + w) = 0$ into the equivalent system

$$\begin{cases} P_\varepsilon I'_\varepsilon(z_{\varepsilon,\xi} + w) &= 0, \qquad \text{(Auxiliary equation)} \\ \widehat{P}_\varepsilon I'_\varepsilon(z_{\varepsilon,\xi} + w) &= 0, \qquad \text{(Bifurcation equation)} \end{cases}$$

where $\widehat{P}_\varepsilon$ and P_ε denote the projections onto the tangent space $T_{z_{\varepsilon,\xi}} Z_\varepsilon$ and to its orthogonal $(T_{z_{\varepsilon,\xi}} Z_\varepsilon)^\perp$, respectively. The next step consists in solving the Auxiliary equation. Expanding I'_ε, we get $I'_\varepsilon(z_{\varepsilon,\xi}+w) = I'_\varepsilon(z_{\varepsilon,\xi}) + I''_\varepsilon(z_{\varepsilon,\xi})[w] + R(\varepsilon,\xi,w)$. Since for ε small the operator $L_{\varepsilon.\xi} = P_\varepsilon \circ I''_\varepsilon(z_{\varepsilon,\xi})$ is shown to be invertible for all $\xi \in \mathbb{R}^n$ such that $|\varepsilon\xi| < 1$, then the auxiliary equation takes the form

$$w = S_\varepsilon(w) := -L^{-1}_{\varepsilon,\xi}\left[P_\varepsilon I'_\varepsilon(z_{\varepsilon,\xi}) + P_\varepsilon R(\varepsilon,\xi,w)\right]. \tag{1.4}$$

In order to show that S_ε possesses a fixed point, a new device has been used. Given $R, \gamma > 0$ and a function $\phi : \mathbb{R} \to \mathbb{R}$, let W_ε denote the set of the functions $w \in H_\varepsilon$ such that

$$|w(x+\xi)| \le \begin{cases} \gamma_R \sqrt{\varepsilon}\, \phi(|x|), & \text{if } |x| \ge R, \\ \sqrt{\varepsilon}, & \text{if } |x| \le R, \end{cases} \tag{1.5}$$

and set, for $c > 0$,

$$\Gamma_\varepsilon = \{w \in H_\varepsilon : \|w\|_\varepsilon \le c\,\varepsilon, \quad w \in W_\varepsilon \cap (T_{z_{\varepsilon,\xi}})^\perp, \qquad (|\varepsilon\xi| \le 1)\}.$$

Our aim is to find R, γ, c and ϕ in such a way that S_ε maps Γ_ε into itself and is a contraction, for $\varepsilon \ll 1$. To understand which is a good choice of ϕ, one can make an heuristic argument as follows. Setting $\widetilde{w} = S_\varepsilon(w)$, we need to prove that $\widetilde{w}$ satisfies estimates as in the definition of Γ_ε. Let us focus on the first of (1.5). From (1.4) it follows that $\widetilde{w}$ solves an equation like (to simplify notation, we take below $K \equiv 1$)

$$-\Delta\widetilde{w} + V(\varepsilon x + \varepsilon\xi)\widetilde{w} - p z^{p-1}_{\varepsilon,\xi}(x+\xi)\widetilde{w} = h,$$

where h depends upon w in an explicit way. More precisely, since $w \in W_\varepsilon$ it follows that

$$0 < h \le \sqrt{\varepsilon}\, \phi^{2\wedge p}(r), \qquad |x| \ge R \gg 1. \tag{1.6}$$

Since, for any $m > 0$,

$$V(\varepsilon x + \varepsilon\xi) - p z_{\varepsilon,\xi}^{p-1}(x+\xi) \geq \frac{m}{|x|^{\alpha}},$$

provided $|x| \gg 1$, it is convenient to consider, as comparison, the equation:

$$\begin{cases} -\Delta u + \frac{m}{|x|^{\alpha}}\, u = f(|x|), & |x| > R, \\ u(x) = 1 & |x| = R, \\ u(x) \to 0 & |x| \to \infty. \end{cases} \tag{1.7}$$

The corresponding homogeneous equation

$$-\Delta u + \frac{m}{|x|^{\alpha}}\, u = 0$$

has two fundamental radial solutions given by

$$\phi(r) = r^{-\frac{n-1}{2}}\phi_1(r); \qquad \psi(r) = r^{-\frac{n-1}{2}}\psi_1(r),$$

where the functions ϕ_1, ψ_1 depend on α. If $\alpha < 2$ they are expressed by means of Bessel functions. Let us remark that, in such a case, ϕ has an exponential decay of the form

$$\phi \sim r^{\frac{\alpha}{4}-\frac{n-1}{2}} e^{-\frac{2\sqrt{m}}{2-\alpha} r^{\frac{2-\alpha}{2}}}.$$

If $\alpha = 2$, ϕ_1, ψ_1 are given by

$$\phi_1(r) = r^{\frac{1-\sqrt{(n-2)^2+4m}}{2}}, \qquad \psi_1(r) = r^{\frac{1+\sqrt{(n-2)^2+4m}}{2}}.$$

We will choose the preceding ϕ to be the function used in the definition of W_ε and Γ_ε. To prove that $\widetilde{w}$ satisfies the first of (1.5), provided that $w \in \Gamma_\varepsilon$, the following comparison lemma plays a key role.

Lemma 1.5.3 *Let ϕ, ψ be defined as above, and let u be a solution of problem (1.7), where $f : (R, +\infty) \to \mathbb{R}$ is a positive continuous function such that*

$$\int_R^{+\infty} r^{n-1} f(r)\psi(r)\, dr < +\infty. \tag{1.8}$$

Then there exists $\gamma(R) > 0$ such that $u(r) \leq \gamma(R)\phi(r)$ for all $r \in (R, +\infty)$.

In view of (1.6), Lemma 1.5.3 is applied with $f = \phi^{\max\{2,p\}}$. Let us point out that if $\alpha = 2$ the integrability condition (1.8) is satisfied provided we choose m sufficiently large, while any m can be taken in the case $\alpha < 2$. In this way we prove that $\widetilde{w}$ satisfies the first condition in (1.5). Without entering in more details, let us say that it is also possible to choose $R, c > 0$ in such a way that $S_\varepsilon(\Gamma_\varepsilon) \subset \Gamma_\varepsilon$ and is a contraction, provided $\varepsilon \ll 1$. Hence S_ε has a fixed point $w = w_{\varepsilon,\xi}$. Substituting $w_{\varepsilon,\xi}$ into the bifurcation equation, we get $\widehat{P}_\varepsilon I'_\varepsilon(z_{\varepsilon,\xi} + w_{\varepsilon,\xi}) = 0$. If we consider the *reduced functional* $\Phi_\varepsilon(\xi) = I_\varepsilon(z_{\varepsilon,\xi} + w_{\varepsilon,\xi})$, one finds that for any stationary point ξ_ε of Φ_ε, the function $u_\varepsilon = z_{\varepsilon,\xi_\varepsilon} + w_{\varepsilon,\xi_\varepsilon}$ is a solution of the bifurcation equation and hence satisfies $(\widetilde{NLS_\varepsilon})$. Finally, expanding $\Phi_\varepsilon(\xi)$ we get $\Phi_\varepsilon(\xi) = c_0 Q(\varepsilon\xi) + \rho(\varepsilon,\xi)$ with $|\rho(\varepsilon,\xi)| \leq c_1(|\varepsilon|Q'(\varepsilon\xi)| + \varepsilon^2)$. This completes the proof of Theorem 1.5.2. ∎

The preceding result can be improved in several directions. The first one is concerned with the concentration on a sphere in the case in which V is radial but decays to zero at infinity and can possibly vanish.

Theorem 1.5.4 [6] *Let $p > 1$ and suppose that $V \in C^1(\mathbb{R}^n)$ satisfies*

$$V(r) \geq 0, \quad \text{and} \quad \exists\, r_0, a_1, a_2 > 0 \,:\, \frac{a_1}{r^2} \leq V(r) \leq a_2,\ \forall\, r > r_0; \tag{V_5}$$

$$\exists\, V_1' > 0, \quad : \quad |V'(x)| \leq V_1', \quad \forall\, r \in \mathbb{R}_+. \tag{V_6}$$

Moreover let us assume that there exists $r^* > r_0$ such that M has an isolated local maximum or minimum at $r = r^*$. Then for $\varepsilon \ll 1$, equation (NLS_ε) has a solution that concentrates at the sphere $|x| = r^*$.

A second improvement deals with the case in which V also vanish and decays to zero, but we look for ground states concentrating on points where $V = 0$.

Theorem 1.5.5 [7] *Suppose that V satisfies*

$$V \in C(\mathbb{R}^n), \quad \inf_{\mathbb{R}^n} V(x) = 0, \quad \text{and} \quad \exists a_1 > 0,\ \alpha \in]0,2[,\ :\ V(x) \geq \frac{a_1}{1+|x|^\alpha},\ \forall |x| \gg 1. \tag{V_7}$$

Moreover, let (K_1) hold with $\beta > 0$ and let $\sigma < p < \frac{n+2}{n-2}$. Then the same conclusion of Theorem 1.4.1 holds true.

Remark 2. The two preceding theorems are different in nature. In the former, the solutions concentrate at a sphere of radius r^* such that $V(r^*) > 0$ and its profile is like the one found in Theorem 1.3.3. In the latter, the concentration occurs at a point x_0 with $V(x_0) = 0$, the profile being that of the solutions found in Theorem 1.5.5. Some open problem related to these results are listed in the next section. ∎

1.6 New Perspectives

We collect in this section some possible further developments of the results discussed above.

1. In the radial case $V(x) = V(|x|)$, criticality of the function

$$M(r) = r^{n-1}(V(r))^\ell; \qquad \ell = \frac{p+1}{p-1} - \frac{1}{2}$$

gives necessary and sufficient conditions for concentration at spheres.
Heuristically, the energy of a solution concentrating at a k-dimensional manifold Σ is approximately

$$E(V,\Sigma) \simeq \varepsilon^{n-k} \int_{\Sigma} V^{\ell_k} d\sigma; \qquad \ell_k = \frac{p+1}{p-1} - \frac{1}{2}(n-k).$$

Extremizing in Σ one finds

$$\ell_k \nabla^{\perp} V = V\mathbf{H},$$

where $\nabla^{\perp} V$ denotes the component of ∇V perpendicular to Σ and where $\mathbf{H}$ stands for the mean curvature vector of Σ.
We wonder if, under suitable non-degeneracy assumptions, this condition is sufficient for concentration of solutions at Σ. This conjecture has been verified for $n = 2$ (when Σ is a curve) by Del Pino, Kowalczyk and Wei in [12].

2. As mentioned in Remark 1.3.4-(iii), from the solutions concentrating on a sphere branch off solutions with different profile. We suspect that these solutions oscillate along the sphere and that, following the bifurcation branches, they concentrate on points or on lower dimensional manifolds. This bifurcation analysis should be pursued also for solutions considered in the previous item.
3. In [13] the authors studied, for $n = 1$ and $1 < p < 5$, the case in which V is negative in some interval (or intervals) of $\mathbb{R}$. They are able to produce, when $\varepsilon \to 0$, solutions of (NLS_ε) (for $K \equiv 1$) which are highly oscillatory in $\{V \leq 0\}$, and which decay exponentially away from this region. They also characterize the limit profile of the amplitude of the solutions in terms of the potential V.
   Their result relies on ODE techniques, but it is expectable that similar result could be proven for the higher-dimensional case as well.
4. Among the equations that have not been studied intensively, we also mention those in which the potential V has a singularity. For example, if $V \sim |x|^{-a}$ as $x \to 0$ and satisfies at infinity either (V_1) or (V_4). The so called Hardy potential, $V = |x|^{-2}$, has been studied (with $\varepsilon = 1$ and p the critical exponent), but we do not know any result dealing with potentials that do not coincide exactly with that one. The concentration as $\varepsilon \to 0$ has not been investigated, too.
5. In all the results discussed in Section 1.5 we have considered potentials which decay to zero at infinity at most like $|x|^{-2}$. Is it possible to handle potentials with faster decay, or compactly supported? Moreover, what happens if V is a potential well, e.g. V is defined in a bounded domain Ω and $V(x) \to +\infty$ as x tends to the boundary $\partial\Omega$? Clearly, the approach used so far cannot be repeated. However, any result, positive or negative, would be interesting.

6. In the case in which V can vanish and tend to zero at infinity, there are many open questions. The most natural one concerns an improvement of Theorem 1.5.5, by taking $0 < \alpha \leq 2$, any $\beta \geq 0$ and any $1 < p < \frac{n+2}{n-2}$. As we have seen in the results discussed in Section 1.5, we could expect, roughly, that there exist solutions concentrating on a stable stationary point of Q, whose profiles depend on the behavior of V near its zeros. The proof should rely on an approximation procedure, with an appropriate choice of the set Γ_ε, which depends on the asymptotic behavior of V at infinity.
7. Another natural question is to see if, in the case of Theorem 1.5.4, there are solutions concentrating at some $\widetilde{r}$ such that $V(\widetilde{r}) = 0$. It would also be useful to check if the ground state of (1.4) is non-degenerate. This might lead to find solutions (not ground states) on any stable stationary point of Q in a more direct fashion.
8. The case in which (NLS_ε) is replaced by a system of equations is of a great importance. Among the very few results obtained so far on this topic, we mention [18], see also [9, 10], where the following system has been studied:

$$\begin{cases} -\varepsilon^2 \Delta u + u + \Psi(x)u = \nu u^p, & x \in \mathbb{R}^3 \\ -\Delta \Psi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

After an appropriate choice of the parameter ν (depending on ε) and a rescaling, (1.1) is transformed into

$$\begin{cases} -\Delta u + u + \Psi(x)u = u^p, & x \in \mathbb{R}^3 \\ -\Delta \Psi = \varepsilon u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

It is proved that, if $1 < p < 11/7$, (1.2) has a solution concentrating at the sphere of radius

$$\overline{r} = \frac{1}{m_0} \frac{\overline{a}}{(\overline{a}+1)^{\frac{5-p}{2(p-1)}}}, \quad m_0 = \int_{\mathbb{R}} U_0^2, \quad \overline{a} = \frac{8(p-1)}{11-7p},$$

where U_0 is the even, positive solution of $-U'' + U = U^p$, which decays to zero at infinity. It is worth pointing out that the value of $\overline{r}$ satisfies $\widetilde{M}'(r) = 0$, where

$$\widetilde{M}(r) = r f(r) \left[\frac{3p-7}{4} f(r) + p - 1 \right],$$

f being the inverse of $g(s) = m_0^{-1} s(1+s)^{-(5-p)/2(p-1)}$. Such a $\widetilde{M}$ plays the same role of the auxiliary weighted potential M introduced in Section 1.3.

On these class of equations, a lot of problems are still to be addressed. For example, one can study the concentration of solutions of a Schrödinger equation coupled with a nonlinear Poisson equation on, say $\mathbb{R}^3$,

$$\begin{cases} -\varepsilon^2 \Delta u + (V(x) + \Psi(x))u = u^p, \; x \in \mathbb{R}^3 \\ -\Delta\Psi = u^2, \qquad\qquad\qquad\qquad x \in \mathbb{R}^3, \quad \lim_{|x|\to\infty} \Psi(x) = 0, \end{cases} \tag{1.3}$$

with $1 < p < 5$. Similarly, one can consider an equation with a non-local term, such as

$$-\varepsilon^2 \Delta u + V(x)u = (\Phi * u^2)u, \qquad x \in \mathbb{R}^3,$$

where Φ is an appropriate non-negative function. The aim would be the extension of the results discussed in the preceding sections to these classes of systems. For example, one should find the counterpart of the auxiliary potential Q, etc. For some recent interesting results in this direction, we mention [19], where (1.3) is studied, with $V \equiv 1$. This paper improves previous results by other authors, see the bibliography of [19] showing, among other things, that (1.3) has a radial solution iff $2 < p < 5$. A list of open questions is also presented.

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