Time-polynomial Lieb-Robinson bounds for finite-range spin-network models

Stefano Chessa* and Vittorio Giovannetti
NEST, Scuola Normale Superiore and Istituto Nanoscienze, Consiglio Nazionale delle Ricerche, I-56126 Pisa, Italy

(Received 28 May 2019; revised manuscript received 26 September 2019; published 8 November 2019)

The Lieb-Robinson bound sets a theoretical upper limit on the speed at which information can propagate in nonrelativistic quantum spin networks. In its original version, it results in an exponentially exploding function of the evolution time, which is partially mitigated by an exponentially decreasing term that instead depends upon the distance covered by the signal (the ratio between the two exponents effectively defining an upper bound on the propagation speed). In the present paper, by properly accounting for the free parameters of the model, we show how to turn this construction into a stronger inequality where the upper limit only scales polynomially with respect to the evolution time. Our analysis applies to any chosen topology of the network, as long as the range of the associated interaction is explicitly finite. For the special case of linear spin networks we present also an alternative derivation based on a perturbative expansion approach which improves the previous inequality. In the same context we also establish a lower bound to the speed of the information spread which yields a nontrivial result at least in the limit of small propagation times.

DOI: 10.1103/PhysRevA.100.052309

I. INTRODUCTION

When dealing with communication activities, information transfer speed is one of the most relevant parameters in order to characterize the communication line performances. This statement applies both to quantum communication, obviously, and to quantum computation, where the effective ability to carry information, for instance from a gate to another one, can determine the number of calculations executable per unit of time. It appears therefore to be useful being able to estimate such speed or, whenever not possible, bound it with an upper value. In the context of communication via quantum spin networks [1] a result of this kind can be obtained exploiting the so-called Lieb-Robinson (L-R) bound [2,3]: defining a suitable correlation function involving two local spatially separated operators $\hat{A}$ and $\hat{B}$, a maximum group velocity for correlations and consequently for signals can be extrapolated. In more recent years this bound has been generalized and applied to attain results in a wider set of circumstances, including for the Lieb-Schultz-Mattis theorem in higher dimensions [4], for the exponential clustering theorem [5], to link spectral gap and exponential decay of correlations for short-range interacting systems [6], for the existence of the dynamics for interactions with polynomial decay [7], for the area law in one-dimensional (1D) systems [8], for the stability of topological quantum order [9], for information and entanglement spreading [10–13], and for black holes physics and information scrambling [14,15]. Bounds on correlation spreading, remaining in the framework set by L-R bounds, have been then generalized to different scenarios such as, for instance, long-range interactions [16–20], disordered systems [21,22], and finite temperature [23–25]. After the original work by Lieb and Robinson the typical shape found to describe the bound has been one that is exponentially growing in time t and depressed with the spatial distance between the supports of the two operators $d(A,B)$, namely,

$$\|[\hat{A}(t),\hat{B}]\| \lesssim e^{v|t|}\, f[d(A,B)], \tag{1}$$

with v a positive constant and $f(\cdot)$ a suitable decreasing function, both depending upon the interaction considered, the size of the supports of $\hat{A}$ and $\hat{B}$, and the dimensions of the system [4–7]. More recently instances have been proposed [25,26] in which such behavior can be improved to a polynomial one,

$$\|[\hat{A}(t),\hat{B}]\| \lesssim \left(\frac{t}{d(A,B)}\right)^{d(A,B)}, \tag{2}$$

at least for Hamiltonian couplings which have an explicitly finite range, and for short enough times. The aim of the present paper is to set these results on a firm ground providing an alternative derivation of the polynomial version (2) of the L-R inequality which, as long as the range of the interactions involved is finite, holds true for arbitrary topology of the spin network and which does not suffer from the short-time limitations that affect previous approaches. Our analysis yields a simple way to estimate the maximum speed at which signals can propagate along the network. In the second part of the paper we focus on the special case of single sites located at the extremal points of a 1D linear spin chain model. In this context we give an alternative derivation of the t-polynomial L-R bound and discuss how the same technique can also be used to provide a lower bound on $\|[\hat{A}(t),\hat{B}]\|$, which at least for small t is nontrivial.

The paper is organized as follows. We start in Sec. II by presenting the model and recalling the original version of the L-R bound. The main result of the paper is hence presented in Sec. III, where by using a simple analytical argument we derive our t-polynomial version of the L-R inequality. In Sec. IV we present the perturbative expansion approach for

*stefano.chessa@sns.it

1D linear spin chain models. In Sec. V we test results achieved in previous sections by comparing them to the numerical simulation of a spin chain. Conclusions are presented finally in Sec. VI.

II. THE MODEL AND SOME PRELIMINARY OBSERVATIONS

Adopting the usual framework for the derivation of the L-R bound [5] let us consider a network $\mathcal{N}$ of quantum systems (spins) distributed on a graph $\mathbb{G} := (V, E)$ characterized by a set of vertices V and by a set E of edges. The model is equipped with a metric $d(x, y)$ defined as the shortest path (least number of edges) connecting $x, y \in V$ ($d(x, y)$, being set equal to infinity in the absence of a connecting path), which induces a measure for the diameter $D(X)$ of a given subset $X \subset V$ and a distance $d(X, Y)$ among the elements $X, Y \subset V$:

$$D(X) := \max_{x,y} \min\{d(x, y)|x, y \in X\},$$
$$d(X, Y) := \min\{d(x, y)|x \in X, y \in Y\}. \tag{3}$$

Indicating with $\mathcal{H}_x$ the Hilbert space associated with spin that occupies the vertex x of the graph, the Hamiltonian of $\mathcal{N}$ can be formally written as

$$\hat{H} := \sum_{X \subset V} \hat{H}_X, \tag{4}$$

where the summation runs over the subsets X of V with $\hat{H}_X$ being a self-adjoint operator that is local on the Hilbert space $\mathcal{H}_X := \otimes_{x\in X}\mathcal{H}_x$, i.e., it acts nontrivially on the spins of X while being the identity everywhere else. Consider then two subsets $A, B \subset V$ which are disjoint, $d(A, B) > 0$. Any two operators $\hat{A} := \hat{A}_A$ and $\hat{B} := \hat{B}_B$ that are local on such subsets clearly commute, i.e., $[\hat{A}, \hat{B}] = 0$. Yet as we let the system evolve under the action of the Hamiltonian $\hat{H}$ this condition will not necessarily hold due to the building up of correlations along the graph. More precisely, given $\hat{U}(t) := e^{-i\hat{H}t}$, the unitary evolution induced by (4), and indicating with

$$\hat{A}(t) := \hat{U}^\dagger(t)\hat{A}\hat{U}(t) \tag{5}$$

the evolved counterpart of $\hat{A}$ in the Heisenberg representation, we expect the commutator $[\hat{A}(t), \hat{B}]$ to become explicitly nonzero for large enough t; the faster this happens, the stronger the correlations that are dynamically induced by $\hat{H}$ (hereafter we set $\hbar = 1$ for simplicity). The Lieb-Robinson bound puts a limit on such behavior, that applies for all $\hat{H}$ which are characterized by couplings that have a finite-range character (at least approximately). Specifically, indicating with $|X|$ the total number of sites in the domain $X \subset V$, and with

$$M_X := \max_{x\in X} \dim[\mathcal{H}_x] \tag{6}$$

the maximum value of its spins' Hilbert-space dimension, we say that $\hat{H}$ is well behaved in terms of long-range interactions, if there exists a positive constant λ such that the functional

$$\|\hat{H}\|_\lambda := \sup_{x\in V} \sum_{X\ni x} |X| M_X^{2|X|} e^{\lambda D(X)}\ \|\hat{H}_X\| \tag{7}$$

is finite. In this expression the symbol

$$\|\hat{\Theta}\| := \max_{|\psi\rangle} \|\hat{\Theta}|\psi\rangle\| \tag{8}$$

represents the standard operator norm, while the summation runs over all the subset $X \subset V$ that contains x as an element. Variant versions [5,6,27] or generalizations [3,28] of Eq. (7) can be found in the literature; however, as they express the same behavior and substantially differ only by constants, in the following we shall gloss over these differences. The L-R bound can now be expressed in the form of the following inequality [5]:

$$\|[\hat{A}(t), \hat{B}]\| \leqslant 2|A||B|\|\hat{A}\|\|\hat{B}\|(e^{2|t|\|\hat{H}\|_\lambda} - 1)e^{-\lambda\, d(A,B)}, \tag{9}$$

which holds nontrivially for well-behaved Hamiltonian $\hat{H}$ admitting finite values of the quantity $\|\hat{H}\|_\lambda$. It is worth stressing that Eq. (9) is valid irrespective of the initial state of the network and that, due to the dependence upon $|t|$ on the right-hand side (rhs) term, exactly the same bound can be derived for $\|[\hat{A}, \hat{B}(t)]\|$, obtained by exchanging the roles of $\hat{A}$ and $\hat{B}$. Finally we also point out that in many cases of physical interest the prefactor $|A||B|$ on the rhs can be simplified: for instance, it can be omitted for one-dimensional models, while for nearest-neighbor interactions one can replace this by the smaller of the boundary sizes that $\hat{A}$ and $\hat{B}$ support [28].

For models characterized by interactions which are explicitly not finite, refinements of Eq. (9) have been obtained under special constraints on the decaying of the long-range Hamiltonian coupling contributions [5,6]. For instance, assuming that there exist (finite) positive quantities s_1 and μ_1 (s_1 being independent from the total number of sites of the graph $\mathbb{G}$), such that

$$\sup_{x\in V} \sum_{X\ni x} |X|\|\hat{H}_X\|[1 + D(X)]^{\mu_1} \leqslant s_1\,, \tag{10}$$

one gets

$$\|[\hat{A}(t), \hat{B}]\| \leqslant C_1|A||B|\|\hat{A}\|\|\hat{B}\| \frac{e^{v_1|t|} - 1}{[1 + d(A, B)]^{\mu_1}}, \tag{11}$$

with C_1 and v_1 positive quantities that only depend upon the metric of the network and on the Hamiltonian. In contrast, if there exist (finite) positive quantities μ_2 and s_2 (the latter being again independent from the total number of sites of $\mathbb{G}$), such that

$$\sup_{x\in V} \sum_{X\ni x} |X|\|\hat{H}_X\|e^{\mu_2 D(X)} \leqslant s_2\,, \tag{12}$$

we get instead

$$\|[\hat{A}(t), \hat{B}]\| \leqslant C_2|A||B|\|\hat{A}\|\|\hat{B}\|(e^{v_2|t|} - 1)e^{-\mu_2 d(A,B)}, \tag{13}$$

where once more C_2 and v_2 are positive quantities that only depend upon the metric of the network and on the Hamiltonian. The common trait of these results is the fact that their associated upper bounds maintain exponential dependence with respect to the transferring t in Eq. (1).

III. CASTING THE LIEB-ROBINSON BOUND INTO A t-POLYNOMIAL FORM FOR (EXPLICITLY) FINITE-RANGE COUPLINGS

The inequality (9) is the starting point of our analysis: it is indicative of the fact that the model admits a finite speed $v \simeq 2\|\hat{H}\|_\lambda/\lambda$ at which correlations can spread out in the spin network. As $|t|$ increases, however, the bound becomes less and less informative due to the exponential dependence of the rhs: in particular, it becomes irrelevant as soon as the multiplicative factor of $\|\hat{A}\|\|\hat{B}\|$ gets larger than 2. In this limit in fact Eq. (9) is trivially reduced by the inequality

$$\|[\hat{A}(t), \hat{B}]\| \leqslant 2\|\hat{A}(t)\|\|\hat{B}\| = 2\|\hat{A}\|\|\hat{B}\|, \tag{14}$$

that follows by simple algebraic considerations. One way to strengthen the conclusions one can draw from (9) is to consider λ as a free parameter and to optimize with respect to all the values it can assume. As the functional dependence of $\|\hat{H}\|_\lambda$ upon λ is strongly influenced by the specific properties of the spin model, we restrict the analysis to the special (yet realistic and interesting) scenario of Hamiltonians $\hat{H}$ (4) which are strictly short ranged. Accordingly we now impose $\hat{H}_X = 0$ to all the subsets $X \subset V$ which have a diameter $D(X)$ that is larger than a fixed finite value $\bar{D}$, i.e.,

$$D(X) > \bar{D} \quad \Longrightarrow \quad \hat{H}_X = 0, \tag{15}$$

which is clearly more stringent than both those presented in Eqs. (10) and (12). Under this condition $\hat{H}$ is well behaved for all $\lambda \geqslant 0$ and one can write

$$\|\hat{H}\|_\lambda \leqslant \zeta\, e^{\lambda \bar{D}}, \quad \forall \lambda \geqslant 0, \tag{16}$$

with ζ being a finite positive constant that for sufficiently regular graphs does not scale with the total number of spins of the system. For instance, for regular arrays of nearest-neighbor coupled spins we get $\zeta = 2CM^4\|\hat{h}\|$, where C is the maximum coordination number of the graph (i.e., the number of edges associated with a given site),

$$\|\hat{h}\| := \sup_{X \subset V} \|\hat{H}_X\| \tag{17}$$

is the maximum strength of the interactions, and $M := \max_{x\in V} \dim[\mathcal{H}_x]$ is the maximum dimension of the local spins' Hilbert space of the model. More generally for graphs $\mathbb{G}$ characterized by finite values of C it is easy to show that ζ cannot be greater than $C^{\bar{D}} M^{C^{\bar{D}}} \|\hat{h}\|$.

Using (16) we can now turn (9) into a more treatable expression,

$$\|[\hat{A}(t), \hat{B}]\| \leqslant 2|A||B|\|\hat{A}\|\|\hat{B}\|(e^{2|t|\zeta e^{\lambda \bar{D}}} - 1)e^{-\lambda\, d(A,B)}, \tag{18}$$

the rhs of which can now be explicitly minimized in terms of λ for any fixed t and $d(A,B)$. As shown in Sec. III the final result is given by

$$\begin{aligned}\|[\hat{A}(t), \hat{B}]\| &\leqslant 2|A||B|\|\hat{A}\|\|\hat{B}\| \left(\frac{2\, e\, \zeta\, \bar{D}\, |t|}{d(A,B)}\right)^{\frac{d(A,B)}{\bar{D}}} \mathcal{F}\left(\frac{d(A,B)}{\bar{D}}\right)\\ &\leqslant 2|A||B|\|\hat{A}\|\|\hat{B}\| \left(\frac{2\, e\, \zeta\, \bar{D}\, |t|}{d(A,B)}\right)^{\frac{d(A,B)}{\bar{D}}},\end{aligned} \tag{19}$$

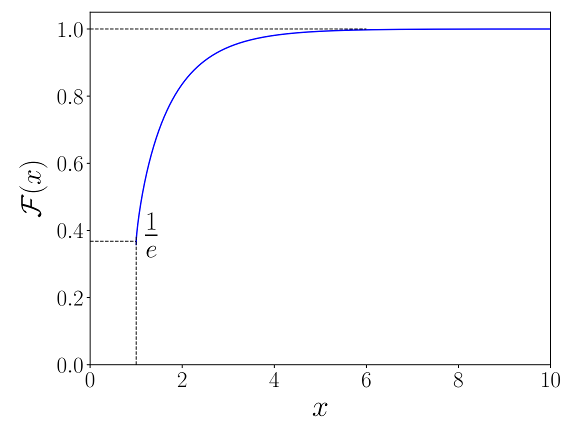

FIG. 1. Plot of the function $\mathcal{F}(x)$ entering into the derivation of Eq. (19): for $x = \frac{d(A,B)}{\bar{D}} \geqslant 1$ it is monotonically increasing, reaching the value $1/e \simeq 0.37$ for $x = 1$ and quickly approaching the asymptotic value 1 for large enough x.

where in the second inequality we used the fact that the function $\mathcal{F}(x)$ defined in Eq. (31) below and plotted in Fig. 1 is monotonically increasing and bounded from above by its asymptotic value 1. At variance with Eq. (9), the inequality (19) contains only terms which are explicit functions of the spin-network parameters. Furthermore the new bound is polynomial in t with a scaling that is definitely better than the linear behavior one could infer from the Taylor expansion of the rhs of Eq. (9). Looking at the spatial component of (19) we notice that correlations still decrease with distance as well as in bounds (9), (11), and (13) but with a scaling $(1/x)^x = e^{-x\log x}$ that is more than exponentially depressed. Also, fixing a (positive) target threshold value $R_* < 1$ for the ratio

$$R(t) := \|[\hat{A}(t), \hat{B}]\|/(2|A||B|\|\hat{A}\|\|\hat{B}\|), \tag{20}$$

Eq. (19) predicts that it will be reached not before a time interval

$$t_* = \frac{d(A,B) R_*^{\bar{D}/d(A,B)}}{2e\zeta\bar{D}} \tag{21}$$

has elapsed from the beginning of the dynamical evolution. Exploiting the fact that $\lim_{z\to\infty} R_*^{1/z} = 1$, in the asymptotic limit of very distant sites [i.e., $d(A,B) \gg \bar{D}$], this can be simplified to

$$t_* \simeq \frac{d(A,B)}{2e\zeta\bar{D}}, \tag{22}$$

that is independent from the actual value of the target $R_* \neq 0$, leading us to identify the quantity

$$v_{\max} := 2e\zeta\bar{D} \tag{23}$$

as an upper bound for the maximum speed allowed for the propagation of signals in the system.

Explicit derivation of Eq. (19)

We start by noticing that by neglecting the negative contribution $-e^{-\lambda\, d(A,B)}$ we can bound the rhs of Eq. (18) by a form which is much easier to handle, i.e.,

$$\|[\hat{A}(t),\hat{B}]\| \leqslant 2|A||B|\|\hat{A}\|\|\hat{B}\|e^{2|t|\zeta e^{\lambda\bar{D}}-\lambda\, d(A,B)}. \tag{24}$$

One can observe that for $t > d(A,B)/(2\zeta\bar{D})$ the approach yields an inequality that is always less stringent than (14). In contrast, for $|t| \leqslant d(A,B)/(2\zeta\bar{D})$, imposing the stationary condition on the exponent term, i.e., $\partial_\lambda(e^{2\,\zeta\, e^{\lambda\bar{D}(X)}|t|-\lambda\, d(A,B)}) = 0$, we found that the optimal value for λ is provided by

$$\lambda_{\rm opt} := \frac{1}{\bar{D}} \ln\left(\frac{d(A,B)}{2\,|t|\,\zeta\,\bar{D}}\right), \tag{25}$$

which plugged into Eq. (24) yields directly (19). More generally, we can avoid passing through Eq. (24) by looking for minima of the rhs of Eq. (9), obtaining the first inequality given in Eq. (19), i.e.,

$$\|[\hat{A}(t),\hat{B}]\| \leqslant 2|A||B|\|\hat{A}\|\|\hat{B}\|\left(\frac{2\,e\,\zeta\,\bar{D}\,|t|}{d(A,B)}\right)^{\frac{d(A,B)}{\bar{D}}} \times\mathcal{F}\left(\frac{d(A,B)}{\bar{D}}\right). \tag{26}$$

For this purpose we consider a parametrization of the coefficient λ in terms of the positive variable z as indicated here:

$$\lambda := \frac{1}{\bar{D}} \ln\left(\frac{z d(A,B)}{2\,|t|\,\zeta\,\bar{D}}\right). \tag{27}$$

With this choice the quantity we are interested in becomes

$$\begin{aligned} &2|A||B|\|\hat{A}\|\|\hat{B}\|(e^{2|t|\zeta e^{\lambda\bar{D}}}-1)e^{-\lambda\, d(A,B)} \\ &= 2|A||B|\|\hat{A}\|\|\hat{B}\|\left(\frac{2et\zeta}{x}\right)^x f_x(z), \end{aligned} \tag{28}$$

where in the rhs term for ease of notation we introduced $x = d(A,B)/\bar{D}$ and the function

$$f_x(z) := \frac{e^{xz}-1}{z^x e^x}. \tag{29}$$

For a fixed value of $x \geqslant 1$ the minimum of Eq. (29) is attained for $z = z_{\rm opt}$, fulfilling the constraint

$$x = -\frac{\ln(1-z_{\rm opt})}{z_{\rm opt}}. \tag{30}$$

By formally inverting this expression and by inserting it into Eq. (28) we hence get (26) with

$$\mathcal{F}(x) := \frac{z_{\rm opt}(x)}{1-z_{\rm opt}(x)}\left(\frac{1}{e z_{\rm opt}(x)}\right)^x \tag{31}$$

being the monotonically increasing function reported in Fig. 1.

IV. PERTURBATIVE EXPANSION APPROACH

An alternative derivation of a t-polynomial bound similar to the one reported in Eq. (19) can be obtained by adopting a perturbative expansion of the unitary evolution of the operator $\hat{A}(t)$ that allows one to express the commutator $[\hat{A}(t),\hat{B}]$ as a sum over a collection of "paths" connecting the locations A and B [see, e.g., Eq. (41) below]. This derivation is somehow analogous to the one used in [25,26]. Yet in these papers the number of relevant terms entering in the calculation of the norm of $[\hat{A}(t),\hat{B}]$ could be underestimated by just considering those paths which are obtained by concatenating adjacent contributions and resulting in corrections that are negligible only for small times t. In what follows we shall overcome these limitations by focusing on the special case of linear spin chains, which allows for a proper account of the relevant paths. Finally we shall see how it is possible to exploit the perturbative expansion approach to also derive a lower bound for $\|[\hat{A}(t),\hat{B}]\|$.

While in principle the perturbative expansion approach can be adopted to discuss arbitrary topologies of the network, in order to get a closed formula for the final expression we shall restrict the analysis to the case of two single sites (i.e., $|A| = |B| = 1$) located at the end of a N-long, 1D spin chain with nearest-neighbor interactions (i.e., $d = N-1$). Accordingly we shall write the Hamiltonian (4) as

$$\hat{H} := \sum_{i=1}^{N-1}\hat{h}_i, \tag{32}$$

with $\hat{h}_i$ operators acting nontrivially only on the ith and $(i+1)$th spins, hence fulfilling the condition

$$[\hat{h}_i,\hat{h}_j] = 0, \quad \forall |i-j| > 1. \tag{33}$$

A. Upper bound

Adopting the Baker-Campbell-Hausdorff formula we write

$$[\hat{A}(t),\hat{B}] = [\hat{A},\hat{B}] + \sum_{k=1}^{\infty}\frac{(it)^k}{k!}[[\hat{H},\hat{A}]_k,\hat{B}], \tag{34}$$

where for $k \geqslant 1$

$$[\hat{H},\hat{A}]_k := \overbrace{[\hat{H},[\hat{H},[\cdots,[\hat{H},[\hat{H},\hat{A}]]}^{k\text{ times}}\cdots]] \tag{35}$$

indicates the kth-order nested commutator between $\hat{H}$ and $\hat{A}$. Exploiting the structural properties of Eqs. (32) and (33) it is easy to check that the only terms which may give us a nonzero contribution to the rhs of Eq. (34) are those with $k \geqslant d$. Accordingly we get

$$[\hat{A}(t),\hat{B}] = \sum_{k=d}^{\infty}\frac{(it)^k}{k!}[[\hat{H},\hat{A}]_k,\hat{B}], \tag{36}$$

which leads to

$$\|[\hat{A}(t),\hat{B}]\| \leqslant \sum_{k=d}^{\infty}\frac{|t|^k}{k!}\|[[\hat{H},\hat{A}]_k,\hat{B}]\|, \tag{37}$$

via subadditivity of the norm. To proceed further we observe that

$$\|[[\hat{H},\hat{A}]_k,\hat{B}]\| \leqslant 2\|\hat{A}\|\|\hat{B}\|\|2\hat{H}\|^k, \tag{38}$$

which for sufficiently small times t yields

$$\begin{aligned} \|[\hat{A}(t),\hat{B}]\| &\simeq \frac{|t|^d}{d!}\|[[\hat{H},\hat{A}]_d,\hat{B}]\| \leqslant 2\|\hat{A}\|\|\hat{B}\|\frac{(2\|\hat{H}\||t|)^d}{d!} \\ &\leqslant \frac{2\|\hat{A}\|\|\hat{B}\|}{\sqrt{2\pi d}}\left(\frac{2\,e\|\hat{H}\|\,|t|}{d}\right)^d, \end{aligned} \tag{39}$$

where in the last passage we adopted the lower bound on $d!$ that follows from Stirling's inequalities:

$$(d/e)^d\sqrt{e^2 d} \geqslant d! \geqslant (d/e)^d\sqrt{2\pi d}. \tag{40}$$

Equation (39) exhibits a polynomial behavior similar to the one observed in Eq. (19) (notice that if instead of nearest-neighbor interaction we had next-$\bar{D}$-neighbor interaction the first not null order will be the $\lceil \frac{d}{\bar{D}} \rceil$th one and accordingly, assuming $d/\bar{D}$ to be an integer, the above derivation will still hold with d replaced by $d/\bar{D}$). Yet the derivation reported above suffers from two main limitations: first of all it only holds for sufficiently small t due to the fact that we have neglected all the terms of (37) but the first one; second the rhs of Eq. (39) has a direct dependence on the total size N of the system carried by $\|\hat{H}\|$, i.e., on the distance d connecting the two sites. Both these problems can be avoided by carefully considering each "nested" commutator $[[\hat{H},\hat{A}]_k,\hat{B}]$ entering (37). Indeed, given the structure of the Hamiltonian and the linearity of commutators, it follows that we can write

$$[[\hat{H},\hat{A}]_k,\hat{B}] = \sum_{i_1,i_2,\cdots,i_k=1}^{N-1} \left[\hat{C}^{(k)}_{i_1,i_2,\cdots,i_k}(\hat{A}),\hat{B}\right], \tag{41}$$

where for $i_1, i_2, \cdots, i_k \in \{1, 2, \cdots, N-1\}$ we have

$$\hat{C}^{(k)}_{i_1,i_2,\cdots,i_k}(\hat{A}) := \left[\hat{h}_{i_k}, \left[\hat{h}_{i_{k-1}}, \cdots, \left[\hat{h}_{i_2}, \left[\hat{h}_{i_1}, \hat{A}\right]\right]\cdots\right]\right]. \tag{42}$$

Now taking into account the commutation rule (33) and the fact that $\hat{A}$ and $\hat{B}$ are located at the two opposite ends of the chain, it turns out that only a limited number

$$n_k \leqslant \binom{k}{d} d^{k-d} = \frac{k!\, d^{k-d}}{d!(k-d)!} \tag{43}$$

of the N^k terms entering (41) will have a chance of being nonzero. For the sake of readability we postpone the explicit derivation of this inequality (as well as the comment on alternative approaches presented in [25,26]) to Sec. IV C: here instead we observe that using

$$\left\|\left[\hat{C}^{(k)}_{i_1,i_2,\cdots,i_k}(\hat{A}),\hat{B}\right]\right\| \leqslant 2\|\hat{A}\|\|\hat{B}\|(2\|\hat{h}\|)^k, \tag{44}$$

where now $\|\hat{h}\| := \max_i \|\hat{h}_i\|$, allows us to transform Eq. (37) into

$$\begin{aligned}\|[\hat{A}(t),\hat{B}]\| &\leqslant 2\|\hat{A}\|\|\hat{B}\|\sum_{k=d}^{\infty} n_k \frac{(2|t|\|\hat{h}\|)^k}{k!} \\ &\leqslant 2\|\hat{A}\|\|\hat{B}\|\frac{(2|t|\|\hat{h}\|)^d}{d!}\sum_{k=0}^{\infty}\frac{(2|t|\|\hat{h}\|d)^k}{k!} \\ &= 2\|\hat{A}\|\|\hat{B}\|\frac{(2|t|\|\hat{h}\|)^d}{d!}e^{2|t|\|\hat{h}\|d},\end{aligned}$$

which presents a scaling that closely resembles the one obtained in [29] for finite-range quadratic Hamiltonians for harmonic systems on a lattice. Invoking hence the lower bound for $d!$ that follows from (40) we finally get

$$\|[\hat{A}(t),\hat{B}]\| \leqslant \frac{2\|\hat{A}\|\|\hat{B}\|}{\sqrt{2\pi d}}\left(\frac{2\,e\|\hat{h}\|\,|t|}{d}\right)^d e^{2|t|\|\hat{h}\|d}, \tag{45}$$

which explicitly shows that the dependence from the system size present in (39) is lost in favor of a dependence on the interaction strength $\|\hat{h}\|$ similar to what we observed in Sec. III. In particular for small times the new inequality mimics the polynomial behavior of (19): as a matter of fact, in this regime, due to the presence of the multiplicative term $1/\sqrt{d}$, Eq. (45) tends to be more strict than our previous bound (a result which is not surprising as the derivation of the present section takes full advantage of the linear topology of the network, while the analysis of Sec. III holds true for a larger, less regular, class of possible scenarios). At large times, in contrast, the new inequality is dominated by the exponential trend $e^{2|t|\|\hat{h}\|d}$ which, however, tends to be overruled by the trivial bound (14).

B. Lower bound

By properly handling the identity (36) it is also possible to derive a lower bound for $\|[\hat{A}(t),\hat{B}]\|$. Indeed, using the inequality $\|\hat{O}_1+\hat{O}_2\| \geqslant \|\hat{O}_1\| - \|\hat{O}_2\|$ we can write

$$\begin{aligned}\|[\hat{A}(t),\hat{B}]\| &= \left\|\sum_{k=d}^{\infty}\frac{(it)^k}{k!}[[\hat{H},\hat{A}]_k,\hat{B}]\right\| \\ &\geqslant \frac{|t|^d}{d!}\|[[\hat{H},\hat{A}]_d,\hat{B}]\| - \left\|\sum_{k=d+1}^{\infty}\frac{(it)^k}{k!}[[\hat{H},\hat{A}]_k,\hat{B}]\right\|\end{aligned} \tag{46}$$

(notice that the above bound is clearly trivial if $[[\hat{H},\hat{A}]_d,\hat{B}]$ is the null operator: when this happens, however, we can replace it by substituting d on it with the smallest $k > d$ for which $[[\hat{H},\hat{A}]_k,\hat{B}] \neq 0$). Now we observe that the last term appearing on the rhs of the above expression can be bounded by following the same derivation of the previous paragraphs, i.e.,

$$\begin{aligned}&\left\|\sum_{k=d+1}^{\infty}\frac{(it)^k}{k!}[[\hat{H},\hat{A}]_k,\hat{B}]\right\| \\ &\leqslant 2\|\hat{A}\|\|\hat{B}\|\sum_{k=d+1}^{\infty} n_k\frac{(2|t|\|\hat{h}\|)^k}{k!} \\ &\leqslant 2\|\hat{A}\|\|\hat{B}\|\frac{(2|t|\|\hat{h}\|)^d}{d!}\sum_{k=1}^{\infty}\frac{(2|t|\|\hat{h}\|d)^k}{k!} \\ &= 2\|\hat{A}\|\|\hat{B}\|\frac{(2|t|\|\hat{h}\|)^d}{d!}(e^{2|t|\|\hat{h}\|d}-1) \\ &\leqslant 2\|\hat{A}\|\|\hat{B}\|\left(\frac{2e|t|\|\hat{h}\|}{d}\right)^d\frac{e^{2|t|\|\hat{h}\|d}-1}{\sqrt{2\pi d}}.\end{aligned}$$

Hence by replacing this into Eq. (46) we obtain

$$\begin{aligned}\|[\hat{A}(t),\hat{B}]\| &\geqslant \frac{|t|^d}{d!}\|[[\hat{H},\hat{A}]_d,\hat{B}]\| \\ &\quad -2\|\hat{A}\|\|\hat{B}\|\left(\frac{2e|t|\|\hat{h}\|}{d}\right)^d\frac{e^{2|t|\|\hat{h}\|d}-1}{\sqrt{2\pi d}} \\ &\geqslant \frac{2\|\hat{A}\|\|\hat{B}\|}{\sqrt{2\pi d}}\left(\frac{2e|t|\|\hat{h}\|}{d}\right)^d[\Gamma_d - (e^{2|t|\|\hat{h}\|d}-1)],\end{aligned} \tag{47}$$

where in the last passage we used the upper bound for $d!$ that comes from Eq. (40) and introduced the dimensionless quantity

$$\Gamma_d := \sqrt{\frac{\pi}{2e^2}}\,\frac{\|[[\hat{H},\hat{A}]_d,\hat{B}]\|}{\|\hat{A}\|\|\hat{B}\|(2\|\hat{h}\|)^d}, \tag{48}$$

which can be shown to be strictly smaller than 1 (see Sec. IV C).

It is easy to verify that as long as Γ_d is nonzero (i.e., as long as $[[\hat{H},\hat{A}]_d,\hat{B}]\neq 0$) there exists always a sufficiently small time $\bar{t}$ such that $\forall\, 0<t<\bar{t}$ in the rhs of Eq. (47) is explicitly positive, implying that we could have a finite amount of correlation at a time shorter than that required for a light pulse to travel from A to B at speed c. This apparent violation of causality is clearly a consequence of the approximations that lead to the effective spin Hamiltonian we are working on (the predictive power of the model being always restricted to time scales t which are larger than $\frac{d(A,B)}{c}$). More precisely, for a sufficiently small value of t (i.e., for $2|t|\|\hat{h}\|d \ll 1$) the negative contribution on the rhs of Eq. (47) can be neglected and the bound predicts the norm of $[\hat{A}(t),\hat{B}]$ to grow polynomially as t^d, i.e.,

$$\|[\hat{A}(t),\hat{B}]\| \gtrsim \frac{2\|\hat{A}\|\|\hat{B}\|}{\sqrt{2\pi d}}\left(\frac{2e|t|\|\hat{h}\|}{d}\right)^d\Gamma_d, \tag{49}$$

which should be compared with

$$\|[\hat{A}(t),\hat{B}]\| \lesssim \frac{2\|\hat{A}\|\|\hat{B}\|}{\sqrt{2\pi d}}\left(\frac{2e|t|\|\hat{h}\|}{d}\right)^d, \tag{50}$$

that for the same temporal regimes is instead predicted from the upper bound (45).

C. Counting commutators

Here we report the explicit derivation of the inequality (43). The starting point of the analysis is the recursive identity

$$\hat{C}^{(k)}_{i_1,i_2,\cdots,i_k}(\hat{A}) = \left[\hat{h}_{i_k},\hat{C}^{(k-1)}_{i_1,i_2,\cdots,i_{k-1}}(\hat{A})\right], \tag{51}$$

which links the expression for nested commutators (42) of order k to those of order $k-1$. Remember now that the operator $\hat{A}$ is located on the first site of the chain. Accordingly, from Eq. (33) it follows that

$$\hat{C}^{(1)}_i(\hat{A}) = [\hat{h}_i,\hat{A}] = 0, \quad \forall i \geqslant 2, \tag{52}$$

i.e., the only possibly nonzero nested commutator of order 1 will be the operator $\hat{C}^{(1)}_1(\hat{A}) = [\hat{h}_1,\hat{A}]$ which acts nontrivially on the first and second spin. From this and the recursive identity (51) we can then derive the following identity for the nested commutator of order $k=2$, i.e.,

$$\hat{C}^{(2)}_{1,i_2}(\hat{A}) = 0, \quad \forall i_2 \geqslant 3, \tag{53}$$

$$\hat{C}^{(2)}_{i_1,i_2}(\hat{A}) = 0, \quad \forall i_1 \geqslant 2 \text{ and } \forall i_2 \geqslant 1, \tag{54}$$

the only terms which can be possibly nonzero being now $\hat{C}^{(2)}_{1,1}(\hat{A})$ and $\hat{C}^{(2)}_{1,2}(\hat{A}) = [\hat{h}_2,[\hat{h}_1,\hat{A}]]$, the first having support on the first and second spin of the chain and the second instead being supported on the first, second, and third spin. Iterating the procedure it turns out that for a generic value of k the operators $\hat{C}^{(k)}_{i_1,i_2,\cdots,i_k}(\hat{A})$ which may be explicitly not null are those for which we have

$$\begin{aligned} &i_1 = 1,\\ &i_j \leqslant \max\{i_1,i_2,\cdots,i_{j-1}\}+1, \quad \forall j\in\{2,\cdots,k\}, \end{aligned} \tag{55}$$

the rule being that passing from $\hat{C}^{(k-1)}_{i_1,i_2,\cdots,i_{k-1}}(\hat{A})$ to $\hat{C}^{(k)}_{i_1,i_2,\cdots,i_k}(\hat{A})$ the new Hamiltonian element $\hat{h}_{i_k}$ entering (51) has to be one of those already touched (except the first one $[\hat{h}_1,A]$) or one at a distance at most 1 to the maximum position reached until there. We also observe that among the elements $\hat{C}^{(k)}_{i_1,i_2,\cdots,i_k}(\hat{A})$ which are not null the ones which have the largest support are those that have the largest value of the indices: indeed, from (51) it follows that the extra commutator with $\hat{h}_{i_k}$ will create an operator the support of which either coincides with the one of $\hat{C}^{(k-1)}_{i_1,i_2,\cdots,i_{k-1}}(\hat{A})$ (this happens whenever i_k belongs to $\{i_1,i_2,\cdots,i_{k-1}\}$), or is larger than the latter by 1 (this happens instead for $i_k = \max\{i_1,i_2,\cdots,i_{k-1}\}+1$). Accordingly among the nested commutators of order k the one with the largest support is

$$\hat{C}^{(k)}_{1,2,\cdots,k}(\hat{A}) = [\hat{h}_k,[\hat{h}_{k-1},\cdots,[\hat{h}_2,[\hat{h}_1,\hat{A}]]\cdots]], \tag{56}$$

that in principle operates nontrivially on all the first $k+1$ elements of the chain. Observe then that in order to get a nonzero contribution in (41) we also need the succession $\hat{h}_i$ entering $\hat{C}^{(k)}_{i_1,i_2,\cdots,i_k}(\hat{A})$ to touch at least once the support of $\hat{B}$. This, together with the prescription just discussed, implies that at least once every element $\hat{h}_i$ between A and B has to appear, and the first appearance of each $\hat{h}_i$ has to happen after the first appearance of $\hat{h}_{i-1}$. In summary we can think of each nested commutator of order k as a numbered set of k boxes fillable with elements $\hat{h}_i$ [see Fig. 2(a)] and, keeping in mind the rules just discussed, we want to count how many fillings give us

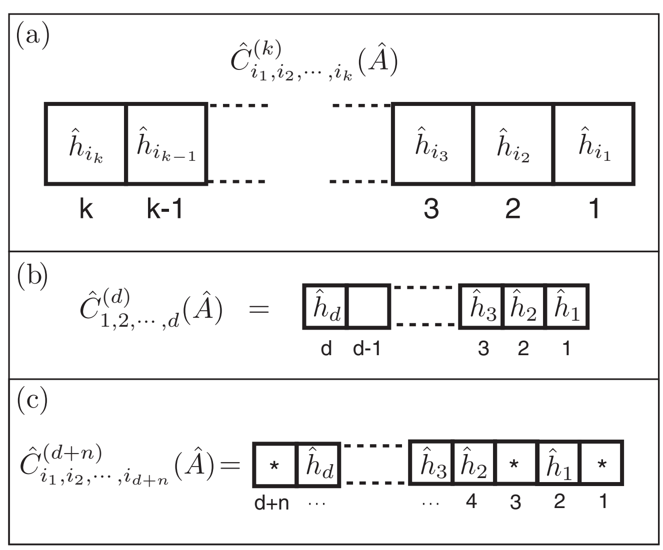

FIG. 2. (a) Pictorial representation of the nested commutator $\hat{C}^{(k)}_{1,2,\cdots,k}(\hat{A})$ as a set of boxes, each one fillable with a $\hat{h}_i$. (b) Representation of the only nested commutator which for the case $k=d$ admits a nonzero value for the commutation with $\hat{B}$. (c) Case $k=d+n$ with $n\geqslant 1$. Here the boxes indicated with the asterisk can be filled depending on their position; for instance, here the box before $\hat{h}_1$ could contain only $\hat{h}_1$ while the one after $\hat{h}_d$ could contain any.

nonzero commutators. Starting from $k = d$, we have only one possibility, i.e., the element $\hat{C}^{(d)}_{1,2,\cdots,d}(\hat{A})$ [see Fig. 2(b)]. This implies

$$[[\hat{H},\hat{A}]_d,\hat{B}] = \left[\hat{C}^{(d)}_{1,2,\cdots,d}(\hat{A}),\hat{B}\right] = [[\hat{h}_d,[\hat{h}_{d-1},\cdots,[\hat{h}_2,[\hat{h}_1,\hat{A}]]\cdots]],\hat{B}], \quad (57)$$

and hence by subadditivity of the norm to

$$\|[[\hat{H},\hat{A}]_d,\hat{B}]\| \leqslant 2\|\hat{A}\|\|\hat{B}\|(2\|\hat{h}\|)^d, \quad (58)$$

which leads to $\Gamma_d \leqslant \sqrt{2\pi/e^2} \simeq 0.923$ as anticipated in the paragraph below Eq. (48). Consider next the case $k = d + n$ with $n \geqslant 1$. In this event we must have at least d boxes filled with each $\hat{h}_i$ between $\hat{A}$ and $\hat{B}$. Once we fix them, the content of the remaining $k = n - d$ boxes [indicated by an asterisk in Fig. 2(c)] depends on their position in the sequence: if one of those is before the first $\hat{h}_1$ it will be forced to be $\hat{h}_1$, if it is before the first $\hat{h}_2$ it will be $\hat{h}_1$ or $\hat{h}_2$, and so on until the one before the first $\hat{h}_d$, which will be any one among the $\hat{h}_i$. So in order to compute the number n_k of nonzero terms entering (41) we need to know in how many ways we can dispose of the empty boxes in the sequence: since empty boxes (as well as the ones necessarily filled) are indistinguishable there are $\binom{k}{n} = \binom{k}{d}$ ways. For each way we would have to count possible fillings, but there is not a straightforward method to do it so we settle for an upper bound. The worst case is the one in which all empty boxes come after the first $\hat{h}_d$, so that we have d^n fillings, and accordingly we can bound n_k with $\binom{k}{n}d^n = \binom{k}{d}d^{k-d}$, leading to Eq. (43).

As mentioned at the beginning of the section a technique similar to the one reported here has been presented in the recent literature expressed in [25,26]. These works also result in a polynomial upper bound for the commutator, yet it appears that the number of contributions entering in the parameter n_k could be underestimated, and this underestimation is negligible only at orders $k \simeq d$ or, equivalently, at small times. Specifically, in [26], which exploits intermediate results from [7,30], the bound is obtained from the iteration of the inequality

$$C_B(t,X) \leqslant C_B(0,X) + 2\sum_{Z\in\partial X}\int_0^{|t|} \mathrm{d}s\, C_B(s,Z)\|\hat{H}_Z\|, \quad (59)$$

where $C_B(t,X) = \|[A(t),\hat{B}]\|$, X is the support of A, and ∂X is the surface of the set X. The iteration adopted in [26] produces an object that involves a summation of the form $\sum_{Z\in\partial X}\sum_{Z_1\in\partial Z}\sum_{Z_2\in\partial Z_1}\cdots$. This selection, however, underestimates the actual number of contributing terms. Indeed, in the first order of iteration $Z \in \partial X$ takes account of all Hamiltonian elements noncommutating with $\hat{A}$, but the next iteration needs to count all noncommuting elements, given by $Z_1 \in \partial Z$ *and* $Z \in \partial X$. So the generally correct statement, as in [7], would be $\sum_{Z\cap X\neq\emptyset}\sum_{Z_1\cap Z\neq\emptyset}\sum_{Z_2\cap Z_1\neq\emptyset}\cdots$. The above discrepancy is particularly evident when focusing on the linear spin chain case we consider here. Taking account only of surface terms in the nested commutators in Eq. (37), among all the contributions which can be nonzero according to Eq. (55), we would have included only those with $i_{j+1} = i_j + 1$. These corrections are irrelevant at the first order in time in Eq. (37) but lead to underestimations in successive orders. In [26] the

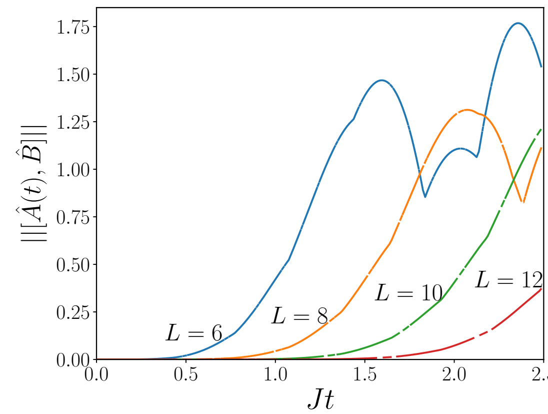

FIG. 3. Simulation of $\|[\hat{A}(t),\hat{B}]\|$ for different chain lengths L for the Heisenberg XY linear spin chain.

discrepancy is mitigated at first orders by the fact that the number of paths of length L considered is upper bounded by $N_1(L) := [2(2\delta - 1)]^L$ with δ dimensions of the graph. But again at higher orders this quantity is overcome by the actual numbers of potentially not null commutators [interestingly, in the case of a two-dimensional (2D) square lattice, $N_1(L)$ could be found exactly, shrinking at the minimum the bound; see [31]]. Similarly, in [25], in the specific case of a 2D square lattice, to estimate the number of paths of length L a coordination number C is used, which gives an upper bound $N_2(L) := (2C - 1)^L$ that for higher orders is again an underestimation. To better visualize why this is the case, let us consider once more the chain configuration. Following rules of Eq. (55) we understood nested commutators $\hat{C}^{(k)}_{i_1,i_2,\cdots,i_k}(\hat{A})$ with repetitions of indices. So with growing k the number of possibilities for successive terms in the commutator grows itself: this is equivalent to a growing dimension $\delta^{(k)}$ or coordination number $C^{(k)}$. For instance we can study the multiplicity of the extensions of the first not null order $\hat{C}^{(d)}_{1,2,\cdots,d}(\hat{A})$. Since the support of this commutator has covered all links between

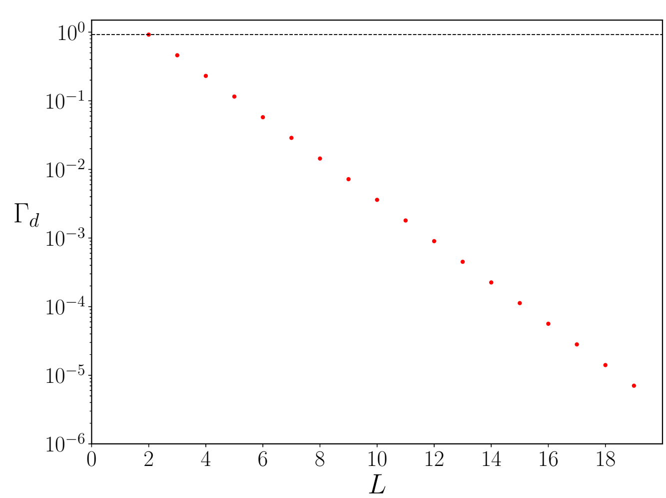

FIG. 4. Plot of the value of Γ_d defined in Eq. (48) for different values of the chain length L, d being fixed equal to $L - 1$. Notice that all values are below $\sqrt{2\pi/e^2}$ (dashed line), which is provably the largest value this parameter can achieve.

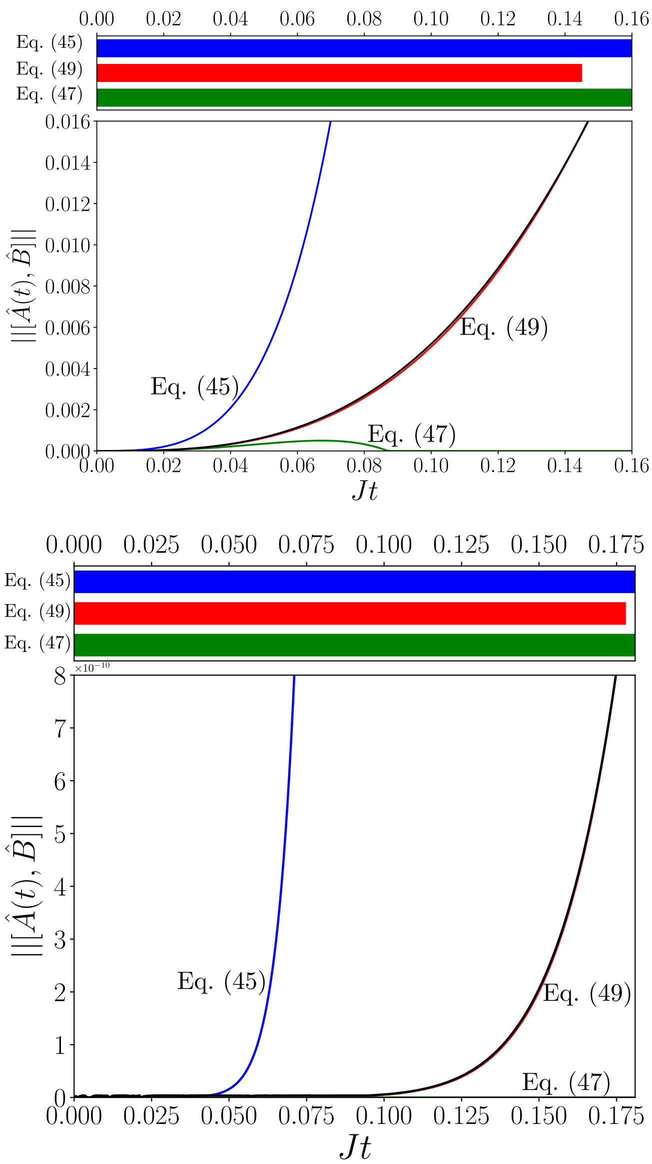

FIG. 5. Simulation and bounds of the function $\|[\hat{A}(t), \hat{B}]\|$ for a $L = 4$ spin chain (upper panel) and for a $L = 10$ spin chain (lower panel). The plot shows upper bound (45) (blue curve), lower bound (47) (green curve), simplified lower bound (49) (red), and numerical simulation (black). The colored bars above the plots outline the time domain in which each bound (identified by the same colors) is valid. As expected the simplified bound stands only for sufficiently small times. In all cases the simulation and the simplified lower bound are comparable in magnitude so that their curves are hardly distinguishable. In the case of $L = 10$ the complete lower bound (47) (green) is considerably small and hence not visible.

A and B that we can choose among d possibilities (not taking into account possible sites beyond B and before A, depending on the geometry of the chain we choose), we will have then d^{L-d} possibilities at the Lth order: for suitable d and n we shall have $d^{L-d} > N_1(L), N_2(L)$. This multiplicity is relative to a single initial path, so we do not even need to count also the different possible initial paths one can construct with $d + l$ steps such that $d + l < L$.

In summary, the polynomial behavior found previously in the literature is solid at the first order but could not be at higher orders.

V. SIMULATION FOR A HEISENBERG *XY* CHAIN

Here we test the validity of our results presented in the previous section for a reasonably simple system such as a uniformly coupled nearest-neighbor Heisenberg XY chain composed by L spin 1/2, described by the following Hamiltonian:

$$\hat{H} = J \sum_{i=0}^{L-1} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y. \tag{60}$$

For local operators $\hat{A}$ and $\hat{B}$ we adopt two $\hat{\sigma}^z$ operators, acting, respectively, on the first and last spin of the chain, so that $\|\hat{A}\| = \|\hat{B}\| = 1$. Employing QUTIP [32,33] we perform the numerical evaluation for $\|[\hat{A}(t), \hat{B}]\|$ varying the length of the chain L (Fig. 3).

We are interested in the comparison between these results with the expressions obtained for the upper bound (45), the lower bound (47), and the simplified lower bound at short times (49). The time domain in which the simplified lower bound stands depends also on the value of the parameter Γ_d specified in Eq. (48), which we understood to be $\leqslant \sqrt{2\pi/e^2}$ but which we need to be reasonably large in order to produce a detectable bound in the numerical evaluation. In Fig. 4 values of Γ_d for different chain lengths L (such that $d = L - 1$) are reported. The magnitude of Γ_d exhibits an exponential decrease with the size of the chain L. The results of our simulations are presented in Fig. 5 for the cases $L = 4$ and 10. The upper bound (45), as well as the lower bound (47), should be universal, i.e., to hold for every t, although the latter being trivial at large times. This condition is satisfied for every L at every t analyzed (we performed the simulation for $2 \leqslant L \leqslant 12$). For what concerns the simplified lower bound (49), we would expect its validity to be guaranteed only for sufficiently small t, and as a matter of fact we find the time domain of validity to be limited at relatively small times (see, e.g., the histograms in Fig. 5).

VI. CONCLUSIONS

The study of the L-R inequality we have presented here shows that for a large class of spin-network models characterized by couplings that are of finite range the correlation function $\|[\hat{A}(t), \hat{B}]\|$ can be more tightly bounded by a constraining function that exhibits a polynomial dependence with respect to time and which, for sufficiently large distances, allows for a precise definition of a maximum speed of the signal propagation [see Eq. (23)]. Our approach does not rely on often complicated graph-counting arguments, but instead is based on an analytical optimization of the original inequality [2] with respect to all free parameters of the model [specifically the λ parameter defining via Eq. (7) the convergence of the Hamiltonian couplings at large distances]. Yet, in the special case of a linear spin chain, we do adopt a graph-counting argument to present an alternative derivation of our result and to show that a similar reasoning can be used to also construct nontrivial lower bounds for $\|[\hat{A}(t), \hat{B}]\|$ when the two sites are located at the opposite ends of the chain. Possible generalizations of the present approach can be foreseen by including a refined evaluation of the dependence upon λ of Eq. (7), that goes beyond the one we adopted in Eq. (16).

Note added. Recently, the same result presented in Eq. (50) for a chain appeared in [34].

ACKNOWLEDGMENTS

The authors would like to thank R. Fazio and B. Nachtergaele for their comments and suggestions.

[1] S. Bose, Contemp. Phys. **48**, 13 (2007).
[2] E. Lieb and D. Robinson, Commun. Math. Phys. **28**, 251 (1972).
[3] B. Nachtergaele, R. Sims, and A. Young, J. Math. Phys. **60**, 061101 (2019).
[4] M. B. Hastings, Phys. Rev. B **69**, 104431 (2004).
[5] B. Nachtergaele and R. Sims, Commun. Math. Phys. **265**, 119 (2006).
[6] M. B. Hastings and T. Koma, Commun. Math. Phys. **265**, 781 (2006).
[7] B. Nachtergaele, Y. Ogata, and R. Sims, J. Stat. Phys. **124**, 1 (2006).
[8] M. B. Hastings, J. Stat. Mech. (2007) P08024.
[9] S. Bravyi, M. B. Hastings, and S. Michalakis, J. Math. Phys. **51**, 9 (2010).
[10] S. Bravyi, M. B. Hastings, and F. Verstraete, Phys. Rev. Lett. **97**, 050401 (2006).
[11] J. Eisert and D. Gross, Phys. Rev. Lett. **102**, 240501 (2009).
[12] S. Hoshino and Y. Kuramoto, Phys. Rev. Lett. **111**, 26401 (2013).
[13] J. M. Epstein and K. B. Whaley, Phys. Rev. A **95**, 042314 (2017).
[14] N. Lashkari, D. Stanford, M. Hastings, T. Osborne, and P. Hayden, J. High Energy Phys. 04 (2013) 022.
[15] D. A. Roberts and B. Swingle, Phys. Rev. Lett. **117**, 091602 (2016).
[16] P. Hauke and L. Tagliacozzo, Phys. Rev. Lett. **111**, 207202 (2013).
[17] J. Eisert, M. van den Worm, S. R. Manmana, and M. Kastner, Phys. Rev. Lett. **111**, 260401 (2013).
[18] Z.-X. Gong, M. Foss-Feig, S. Michalakis, and A. V. Gorshkov, Phys. Rev. Lett. **113**, 030602 (2014).
[19] M. Foss-Feig, Z.-X. Gong, C. W. Clark, and A. V. Gorshkov, Phys. Rev. Lett. **114**, 157201 (2015).
[20] T. Matsuta, T. Koma, and S. Nakamura, Ann. Henri Poincaré **18**, 519 (2017).
[21] C. K. Burrell and T. J. Osborne, Phys. Rev. Lett. **99**, 167201 (2007).
[22] C. K. Burrell, J. Eisert, and T. J. Osborne, Phys. Rev. A **80**, 052319 (2009).
[23] M. B. Hastings, Phys. Rev. Lett. **93**, 126402 (2004).
[24] S. Hernandez-Santana, C. Gogolin, J. I. Cirac, and A. Acìn, Phys. Rev. Lett. **119**, 110601 (2017).
[25] Z. Huang and X.-K. Guo, Phys. Rev. E **97**, 062131 (2018).
[26] K. Them, Phys. Rev. A **89**, 022126 (2014).
[27] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics* (Springer-Verlag, Berlin, 1997).
[28] B. Nachtergaele and R. Sims, in *New Trends in Mathematical Physics: Selected Contributions of the Fifteenth International Congress on Mathematical Physics*, edited by V. Sidoravicius (Springer-Verlag, Berlin, 2009), pp. 591–614.
[29] M. Cramer, A. Serafini, and J. Eisert, in *Quantum Information and Many Body Quantum Systems*, CRM Series Vol. 8, edited by M. Ericsson and S. Montangero (Scuola Normale Superiore, Pisa, Italy, 2008), pp. 51–72.
[30] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims, Commun. Math. Phys. **286**, 1073 (2009).
[31] R. K. Guy, C. Krattenthaler, and B. E. Sagan, Ars. Combin. **34**, 3 (1992).
[32] J. R. Johansson, P. D. Nation, and F. Nori, Comp. Phys. Comm. **184**, 1234 (2013).
[33] J. R. Johansson, P. D. Nation, and F. Nori, Comp. Phys. Comm. **183**, 1760 (2012).
[34] Chi-Fang Chen and A. Lucas, arXiv:1905.03682 (2019).