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Hyperbolic manifolds in high dimensions

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Abstract

Hyperbolic manifolds play a central role in three-dimensional topology, constituting the richest of the eight Thurston geometries, and arising as knot complements. Fibrations of hyperbolic 3-manifolds over the circle are also well understood, with tools such as the Thurston norm and Agol's virtual fibering theorem. On the other hand, the theory of hyperbolic manifolds in higher dimensions, as well as their role in the wider context of differential geometry, is not as clear, due to increased theoretical and computational complexities. The present thesis aims to shed some light on this vast field by means of three specific topics.

Firstly, to study fibrations of odd-dimensional hyperbolic manifolds, it is natural to seek a generalization of the Thurston norm. A promising candidate for such an invariant is Friedl and Lück's twisted L^2-Euler characteristic, which also detects the Euler characteristic of a fiber. Our contribution is an algorithm that computes this invariant, given a CW complex and a class in its first cohomology group.

Then, we consider the technique of constructing hyperbolic manifolds by gluing Coxeter polytopes, which is especially useful as it gives some control over the structure of the resulting manifolds. As a result, we construct a family of closed hyperbolic 5-manifolds with $b_1 = 0$ and volume $< 250\,000$. In a joint work with Edoardo Rizzi, we also find cusp-transitive hyperbolic 4-manifolds with every possible cusp section.

Lastly, another line of research is based on a result of Kolpakov, Reid and Slavich, which gives codimension-1 geodesic embeddings of hyperbolic manifolds. By iterating a slight generalization of this result, and relating it to the Stiefel–Whitney characteristic classes, we prove the existence of closed orientable hyperbolic manifolds without spin^c structures in all dimensions ≥ 5; more generally, for all $k \geq 1$, we find such manifolds with $w_{4k-1} \neq 0$ in dimensions $\geq 4k+1$. We also find closed non-cobordant hyperbolic manifolds in all dimensions ≥ 4 not of the form $4k+3$.

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Finally, I would like to thank an anonymous referee of the paper [25] for many helpful improvements to the exposition, which I carried over to Chapter 3.

1. Introduction

Hyperbolic manifolds play a central role in three-dimensional topology, as they form the richest and most complex of the eight Thurston geometries. Indeed, according to Perelman's geometrization theorem [104; 105; 106], every closed orientable 3-manifold can be cut along spheres and tori, and each resulting component can be endowed with one of eight geometric structures, determined almost completely by its fundamental group. Hyperbolic geometry arises when this fundamental group is not virtually solvable and does not have a normal infinite cyclic subgroup, which is arguably the most general case.

Hyperbolic 3-manifolds are also ubiquitous in knot theory, as the complements of most prime knots or links can be given hyperbolic metrics with finite volume, many of which admit fibrations over the circle. Furthermore, for general hyperbolic 3-manifolds, such fibrations are abundant, as shown by Agol's *virtual fibering theorem* [2; 3] and by the *fibered faces* of the Thurston norm [123].

In higher dimensions, however, the theory of hyperbolic manifolds is not as approachable, due to increased theoretical and computational complexities, and their role in the wider context of differential geometry is much less clear. In this thesis, we shall explore this field by considering three directions of research: fibrations over the circle, the construction of hyperbolic manifolds from Coxeter polytopes, and the relationship between characteristic classes of hyperbolic manifolds and geodesic embeddings.

1.1 Fibrations and the twisted L^2-Euler characteristic

In dimension 3, it is well known that nontrivial cohomology classes of hyperbolic manifolds are *virtually fibered*, that is, they represent fibrations over the circle in a finite cover. The fibered classes have a remarkable structure, as they constitute a union of cones on some open facets of the unit ball of the Thurston norm, called the *fibered faces*.

By some recent developments of Italiano, Martelli and Migliorini [67], we now know that there exist fibrations of (cusped) hyperbolic manifolds in dimension 5, including the *Ratcliffe–Tschantz manifold* [112], but their structure is much less understood. A far-reaching generalization of the Thurston norm is the *twisted L^2-Euler characteristic* $\chi^{(2)}(\widetilde{M};\phi)$, an L^2-invariant for a CW complex M, introduced by Friedl and Lück [50] as a variation on the definition of the classical L^2-Betti numbers. Under certain hypotheses on M, this invariant associates a real number to each cohomology class $\phi \in H^1(M;\mathbb{R})$; like the Thurston norm, the function $\chi^{(2)}(\widetilde{M};\phi)$ is positively homogeneous, multiplicative with respect to finite coverings, and in the case of a fibered class, it equals the Euler characteristic of the fiber.

In Chapter 3, which is based on the author's paper [25], we outline an algorithm for the computation of $\chi^{(2)}(\widetilde{M}; -)$:

Theorem 1.1. *There exists an algorithm that, given a finite L^2-acyclic CW complex M, such that its fundamental group G is residually finite and satisfies the Atiyah conjecture, and a character $\phi : G \to \mathbb{Z}$, computes the twisted L^2-Euler characteristic $\chi^{(2)}(\widetilde{M}; \phi)$.*

The proof consists of several steps, involving the *universal L^2-torsion* of Friedl and Lück [49] and Oki's *matrix expansion algorithm* [100] to compute valuations of Dieudonné determinants; moreover, in its final step, the algorithm relies on an effective version of *Lück's approximation theorem*, due to Löh and Uschold [88].

Theoretically, the bounds given by the theorem are much too loose for computation; however, we find that a truncated, human-assisted version of the algorithm, implemented in the computer algebra system SageMath [116] and provided as a GitHub repository [22], produces very good experimental results in a wide variety of cases, including:

- **Dunfield's link** L10n14**.** As mentioned at the beginning, in the case of 3-manifolds, our algorithm provides an experimental method for the computation of the Thurston norm; we remark that exact algorithms exist, both for manifolds having 2-generator 1-relator fundamental groups [51] and for general manifolds, via normal surface methods [36]. In order to get a quick estimate, one can also bound the Thurston norm with the Alexander norm, by *McMullen's inequality* [95]; among other link complements, we demonstrate an example, first constructed by Dunfield [43] as the complement of the two-component link L10n14, where our algorithm can correctly approximate the former even when the two norms differ.

- **Closed census 3-manifolds.** We apply our algorithm to all closed manifolds in the census with rank-1 first homology. For these examples, the Thurston norm is naturally reduced to a single integer, greatly simplifying experimentation. Surprisingly, our algorithm produces the exact Thurston norm, even when Lück's theorem is applied with the trivial quotient. We also investigate the unique census manifold with rank-2 homology v1539$(5, 1)$, with good results.

- **Free-by-cyclic groups.** Free-by-cyclic groups can be thought of as extending the class of 3-manifolds with boundary, by analogy with the semidirect product structure of such a manifold when it fibers over the circle. Considering this, we study a couple of randomly generated examples of free-by-cyclic groups.

- **The fiber of the Ratcliffe-Tschantz manifold.** Lastly, of particular interest is the computation of the L^2-Betti numbers of the fiber F of the Ratcliffe-Tschantz 5-manifold studied in [67]. In this 4-dimensional example, we skip the matrix expansion step, computing untwisted L^2-Betti numbers, which equal the twisted L^2-Betti numbers of the whole Ratcliffe-Tschantz manifold.

In this last case, we find strong experimental evidence for the *Singer conjecture* for F about the vanishing of the L^2-Betti numbers outside the middle dimension:

Conjecture 1.2 (Singer)**.** *If M is a closed aspherical n-manifold, then all its L^2-Betti numbers vanish, except possibly for $b^{(2)}_{n/2}(\widetilde{M})$ if n is even. In that case, if M also admits a metric with negative sectional curvature, then $b^{(2)}_{n/2}(\widetilde{M}) > 0$.*

In turn, this inspires an analogous conjecture involving the twisted L^2-Betti numbers:

Conjecture 1.3. *Let M be a closed aspherical $(2n+1)$-manifold. Then, for all $i \neq n$, the twisted L^2-Betti number $b^{(2)}_i(\widetilde{M}; -)$ is identically zero. Moreover, the twisted L^2-Euler characteristic $\chi^{(2)}(\widetilde{M}; -) = (-1)^n \cdot b^{(2)}_n(\widetilde{M}; -)$ is a seminorm up to sign.*

In particular, in the case of a fibered cohomology class $\phi \in H^1(M; \mathbb{Z})$, the fiber F is an aspherical manifold, satisfying $b^{(2)}_i(\widetilde{F}) = b^{(2)}_i(\widetilde{M}; \phi)$; hence, our conjecture reduces to the Singer conjecture for F.

Moreover, this conjecture gives a description of the twisted L^2-Euler characteristic as (plus or minus) a seminorm, which we may expect to exhibit fibered faces in analogy with the Thurston norm. In this case, it is also worth noting that we can determine the entire invariant $\chi^{(2)}(\widetilde{M}; -)$ by running the algorithm a finite number of times. Indeed, any enumeration of integral cohomology classes will eventually subsume all vertices of the unit ball, plus one interior point for each facet. The values of the seminorm at these points determine the unit ball, giving us a stopping condition that can be checked algorithmically. Hence, we also obtain:

Corollary 1.4. *In the same context as Theorem 1.1, there exists an algorithm that computes $\chi^{(2)}(\widetilde{M}; -)$ in finite time whenever it is a seminorm up to sign.*

In order to conduct more experimental investigation, it would also be interesting to generalize the method of Italiano, Martelli and Migliorini and construct explicit closed hyperbolic 5-manifolds that fiber over the circle, as is tentatively explored in Chapter 4.

1.2 Hyperbolic manifolds and Coxeter polytopes

The main technique for the explicit construction of hyperbolic manifolds is the gluing of hyperbolic Coxeter polytopes. Recall that, by Selberg's lemma, any Coxeter group has a finite-index torsion-free subgroup, which corresponds to a finite-volume manifold cover of the associated Coxeter polytope. This method is rather non-constructive, and to obtain manifolds with certain desirable properties, a more refined gluing procedure is needed. In Chapter 4, which is based on [28] and [27], we discuss two problems that can be solved via explicit or semi-explicit constructions of hyperbolic manifolds.

1.2.1 Cusp-transitive manifolds

The paper [28], written jointly with Edoardo Rizzi, is a continuation of his previous work [115] on *cusp-transitive* manifolds. Recall that, as a consequence of Margulis' lemma, every complete hyperbolic manifold admits a *thick-thin decomposition*, and the unbounded thin components are called *cusps*.

Definition 1.5. A complete, finite-volume hyperbolic manifold is *cusp-transitive* if its isometry group acts transitively on its cusps.

Hyperbolic manifolds with only one cusp are trivially cusp-transitive and constitute an infinite class of examples in dimensions 2 and 3, to which we can add some 1-cusped 4-manifolds of Kolpakov and Martelli [75]. By contrast, there are no known 1-cusped manifolds in higher dimensions; cusp-transitive manifolds with more than one cusp can be constructed, in dimension between 3 and 8, by applying the coloring method (see Section 2.2.5) to the right-angled Coxeter polytopes P^n [107, Section 3].

Since all cusps of a cusp-transitive manifold M are isometric, we can define the *cusp type* of M as the diffeomorphism class of any cusp section of M. Cusp types are closed flat manifolds of codimension 1.

Our work focuses on the 4-dimensional case. There are 10 closed flat 3-manifolds [35], which can potentially arise as cusp types; six of them are orientable:

- E_1, the 3-torus;
- E_2, the $\frac{1}{2}$-twist manifold;
- E_3, the $\frac{1}{3}$-twist manifold;
- E_4, the $\frac{1}{4}$-twist manifold;
- E_5, the $\frac{1}{6}$-twist manifold;
- E_6, the Hantzsche–Wendt manifold;

while four are non-orientable: we will call them B_1, B_2, B_3 and B_4 (respectively denoted by $+a1$, $-a1$, $+a2$ and $-a2$ in [35]). These 10 manifolds can be constructed as in Figures 4.2 and 4.3 by gluing facets of polyhedral fundamental domains; the latter can be obtained, along with the associated gluing maps, by inspecting [35, Table 12].

The cusp types E_1 [75], E_2 [79], B_1 [80] and B_2 [113] can be realized by 1-cusped 4-manifolds, while E_3 and E_5 cannot [84]. If we allow more than one cusp, we can additionally realize the types E_6 [48] and E_4 [115].

We improve on the technique of [115] (which already realizes E_1, E_2, E_4 and E_6) by requiring a less explicit construction which can actually handle all the cusp types. Specifically, we show:

Theorem 1.6. *For each closed flat* 3*-manifold* N *there exists a cusp-transitive hyperbolic* 4*-manifold* M *with cusps of type* N*.*

By Remark 4.2, if N is orientable, we can also take M to be orientable.

Using an argument inspired by a paper of Nimershiem [99], this result is strengthened in Section 4.5 as follows:

Theorem 1.7. *For every closed flat* 3*-manifold* N*, the set of flat metrics on* N *which can be realized as cusp sections of a cusp-transitive* 4*-manifold is dense in the space of all flat metrics of* N*.*

Moreover, with the same technique, we show an analogous result in dimension 3 (where the possible cusp types are the torus and the Klein bottle).

Finally, in Section 4.6, a variant of our method, combined with some arguments from [17], and [10], enables the construction of a lot of manifolds with pairwise isometric cusps:

Theorem 1.8. *For every closed flat* 3*-manifold* N*, there exists a positive constant* c *such that, for sufficiently large* $V > 0$*, there exist at least* V^{cV} *complete hyperbolic* 4*-manifolds with pairwise isometric cusps of type* N *and volume* $\leq V$*.*

Note that, by [17], there are $V^{k(n)V}$ complete hyperbolic n-manifolds without boundary of volume $\leq V$ at all, where $k(n)$ depends only on the dimension n.

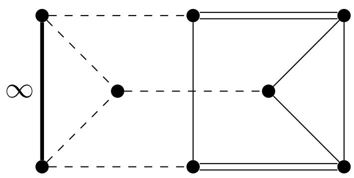
In a similar fashion to Rizzi's paper [115], our construction proceeds by gluing copies of certain Coxeter polytopes to obtain a so-called *developable reflectofold* with one cusp having a given section. Then, by applying Davis' *basic construction* [40], we obtain a cusp-transitive manifold without boundary.

As the main limitation of our technique, any Coxeter polytope we use must satisfy two properties:

(a) it has exactly one ideal vertex;

(b) if a compact facet and a non-compact facet meet, their dihedral angle is an even submultiple of π.

We have found only two hyperbolic Coxeter n-polytopes with $n \geq 4$ satisfying (a) and (b), both in dimension 4: a polytope P_1 associated to Napier cycles, due to Im Hof [66] (see also the web page [47]), and a seemingly novel 4-polytope P_2 with 8 facets, given by the following diagram:



The latter polytope has a prism over an equilateral triangle as the link of its ideal vertex, which is crucial to our construction, as it tessellates the manifolds E_3 and E_5.

Our approach can potentially be extended to higher dimensions upon finding suitable Coxeter polytopes, which is challenging due to increasing combinatorial complexity. For instance, it would be interesting to realize all closed flat 4-manifolds as cusp sections of cusp-transitive 5-manifolds. Hence, we find it natural to ask:

Question 1.9 (compare [115, Question 1.2])**.** Do there exist other finite-volume hyperbolic Coxeter polytopes satisfying (a) and (b), especially in dimension $n \geq 5$?

1.2.2 Closed hyperbolic 5-manifolds

As anticipated above, another problem asks for the explicit description of a closed hyperbolic 5-manifold in a form that allows the computation of topological invariants. Our discussion is based on a preprint by the author [27].

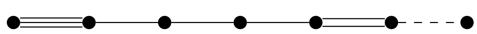
It is well known that hyperbolic manifolds of finite volume, both compact and non-compact, exist in every dimension $n \geq 2$, as a consequence of the Borel–Harish-Chandra theorem [12]. However, these constructions are rather abstract and not very suitable for further investigation. Hence, we again turn our attention to manifolds obtained by gluing Coxeter polytopes.

This strategy is especially fruitful in the cusped case, as mentioned above for cusp-transitive 4-manifolds, and for the polytopes P^n in dimension from 3 to 8. By contrast, the compact case appears to be more complex. Already in dimension 4, the smallest known manifolds [33; 83] are tessellated by 115 200 copies of a Coxeter simplex, and in dimension 5, there are no known explicit descriptions of any closed hyperbolic manifold. In particular, there is no closed hyperbolic 5-manifold M for which $b_1(M)$ is known.

Our main result in this direction is the description of a family of such manifolds with relatively small volume:

Theorem 1.10. *There exists a family* $\{N^{\pm}_{i,I}\}$ *of closed orientable hyperbolic* 5*-manifolds having* $b_1 = 0$ *and volume* $< 250\,000$*.*

These manifolds are tessellated by $117\,964\,800 = 512 \cdot 16 \cdot 14400$ copies of an arithmetic Coxeter polytope P_0, first studied by Bugaenko [16], with diagram


(1.1)

A notable property of P_0, which is combinatorially a prism on a 4-simplex, is that 14400 copies of it can be joined to make a 120-cell prism Q (Figure 4.1), whose two bases have dihedral angles of $2\pi/3$ and $\pi/2$. Moreover, the sides of Q make angles of $\pi/2$ and $\pi/4$ with the bases, respectively.

At this point, we would like to use the aforementioned coloring method, which constructs a manifold cover of a right-angled polytope (or right-angled manifold), given a coloring of its facets with binary vectors. In dimension 5 there are no compact right-angled polytopes, hence we start by gluing many copies of P_0 into a hyperbolic 5-manifold with right-angled corners. A careful choice of coloring, based on the theory of *linear binary codes*, allows us to substantially reduce the size of the final manifolds.

In more detail, our construction is based on some hyperbolic 4-manifolds Z_i, $i = 1, \ldots, 8$, of Euler characteristic 8, found by Long [83]. Their 17-fold Galois coverings X_i can then be used to construct orientable right-angled 5-manifolds Y_i, which can undergo the coloring construction. The definition of the Long manifolds Z_i is algebraic in nature, which simplifies the study of the adjacency graph of the facets of Y_i (which turns out to be the same for all i). With the help of SageMath, we single out and classify some symmetrical 17-colorings of this graph.

This already gives many closed manifolds (which we call $M_{i,I}$) composed of 2^{17} copies of Y_i. These colorings can be converted into more efficient $\mathbb{Z}_2^9$-valued colorings, using the properties of an exceptionally symmetrical 9-dimensional subspace of $\mathbb{Z}_2^{17}$, called a *quadratic residue code*. We call the resulting manifolds $\widehat{M}_{i,I}^{\pm}$.

Finally, these 5-manifolds can be quotiented by a 17-fold symmetry, resulting in relatively small manifolds $N_{i,I}^{\pm}$, tessellated by 117 964 800 copies of P_0, and with hyperbolic volume less than 250 000 (computed exactly in Section 4.10). The final manifolds can be classified by computer and fall into 1 600 432 isometry classes.

We then proceed to study various properties of the 5-manifolds: in particular, we show that they are orientable, and we compute their volume from a formula for $\mathrm{vol}(P)$ [45]. We also compute the first Betti number for $N_{i,I}^{\pm}$ and $\widehat{M}_{i,I}^{\pm}$ as follows. In the case of manifolds obtained by coloring a right-angled polytope, the Betti numbers can be obtained through a combinatorial formula of Choi–Park [30]. While the pieces Y_i have nontrivial topology, they can be made contractible by collapsing their embedded copy of X_i to a point, which is enough to apply the formula; the collapsing operation does not change the first Betti number, since $b_1(X_i) = 0$. Using SageMath and a custom program, we find that all the manifolds $N_{i,I}^{\pm}$ and $\widehat{M}_{i,I}^{\pm}$ have vanishing first Betti number. We note that computing higher Betti numbers is much harder (Remark 4.41) and remains an open question.

We also discuss parallelizability of manifolds obtained with this method through the following general result, which may be of independent interest:

Theorem 1.11. *Let $n \in \{1, 3, 5, 7\} \cup \{4k + 1 \mid k \geq 2\}$. Then every closed hyperbolic n-manifold M is virtually parallelizable.*

Lastly, one may notice that the definition of the manifolds $N_{i,I}^{\pm}$ involves many non-canonical choices, starting from the initial Long manifold Z_i. In Section 4.11, we introduce another 4-manifold X, tessellated by 650 120-cells, as opposed to 136 for the X_i; its definition, based on [46], is arguably much more elegant. We also construct the adjacency graph of the resulting right-angled 5-manifold Y and study a few possible colorings.

It is natural to ask whether a similar procedure can be used to obtain relatively small closed hyperbolic manifolds in higher dimensions, or even in dimension 5 but in different commensurability classes. Recalling the topic of fibrations, we even expect low-degree covers of these manifolds to admit many descriptions as fiber bundles over the circle, possibly simple enough to be studied with existing techniques.

The code used to carry out the computations required for this problem can be found in a GitHub repository [23].

1.3 Geodesic embeddings and characteristic classes

According to results of Kolpakov, Reid and Slavich [77], there are certain conditions under which an orientable arithmetic hyperbolic manifold admits a codimension-1 geodesic embedding into another orientable arithmetic hyperbolic manifold, with trivial normal bundle. By iterating this construction, Martelli, Riolo and Slavich [94] showed that there exist compact hyperbolic manifolds without spin structures in every dimension ≥ 4.

The argument relies on the following observation about the Stiefel–Whitney characteristic classes. If $j\colon M \to N$ is a codimension-1 embedding, then

$$\begin{aligned} j^*(w_k(N)) &= w_k(TN\big|_M) \\ &= w_k(TM \oplus \nu M) \\ &= w_k(M) + w_{k-1}(M) \smile w_1(\nu M). \end{aligned}$$

Since in our case νM is trivial, given a non-spin orientable 4-manifold (that is, having nontrivial w_2), we can iterate the embedding and obtain non-spin orientable n-manifolds for all $n > 4$.

In Chapter 5, we generalize the embedding result as stated in [94, Lemma 5.1] and [78, Theorem 2.2] to the following theorem, which may be of independent interest:

Theorem 1.12. *Let M^n be a k-good hyperbolic n-manifold and let $c \in H^1(M^n; \mathbb{Z}_2)$. Then M^n geodesically embeds in a k-good hyperbolic $(n+1)$-manifold M^{n+1} such that*

$$w_1(\nu_{M^{n+1}}(M^n)) = c. \tag{1.2}$$

If $c = w_1(M^n)$, we can take M^{n+1} to be orientable.

(The definition of *k-good*, for k a totally real number field, is given in Section 5.3.)

We proceed to apply this result to solve two problems concerning the Stiefel–Whitney classes of high-dimensional hyperbolic manifolds, following the preprints [24] and [26].

1.3.1 Manifolds without spin^c structures

Let M be a smooth n-manifold. Various properties of M, such as orientability and existence of spin and spin^c structures, have a relatively simple characterization involving the Stiefel–Whitney classes of the tangent bundle $w_i(M) := w_i(TM)$. Indeed, as we discuss in Section 5.2, we have:

- M is orientable $\iff$ $w_1(M) = 0$;
- M is spin $\iff$ $w_1(M) = w_2(M) = 0$;
- M is spin^c $\iff$ $w_1(M) = 0$ and $w_2(M)$ lifts to $H^2(M;\mathbb{Z})$.

Moreover, the lifting condition for $w_2(M)$ is equivalent to the vanishing of the *third integral Stiefel–Whitney class* $W_3(M) \in H^3(M;\mathbb{Z})$, which reduces modulo 2 to $w_3(M)$ for orientable manifolds M.

A spin manifold is also orientable and spin^c. We also recall that, in dimension up to 3, every closed orientable manifold is spin; moreover, by a theorem of Hirzebruch and Hopf [63], every closed orientable 4-manifold is spin^c (see also [120] for non-closed 4-manifolds).

As mentioned before, some recent research has been focused on finding orientable hyperbolic manifolds that do not admit spin structures. In [86], Long and Reid found cusped non-spin manifolds in all dimensions ≥ 5, while the compact case was studied by Martelli, Riolo and Slavich [94], as mentioned before. As for spin^c structures, even more recently, Reid and Sell [114] showed that there are infinitely many commensurability classes of non-spin^c cusped manifolds in all dimensions ≥ 6.

The main result of [24] covers the compact non-spin^c case:

Theorem 1.13. *For every $n \geq 5$, there exist infinitely many commensurability classes of closed orientable hyperbolic n-manifolds that have no spin^c structure.*

Since having no spin^c structure also implies having no spin structure, this provides an alternate proof to the main result of [94] in dimension > 4.

The main idea is to use Theorem 1.12 to recursively construct an infinite sequence of manifolds

$$(M^2, M^3, M^4, M^5, \dots), \tag{1.3}$$

starting from a k-good surface M^2. By carefully choosing the class c at each step and applying the Whitney sum formula, we can ensure the following chain of implications:

$$\begin{aligned} w_1(M^2) \neq 0 &\implies w_2(M^3) \neq 0 \\ &\implies w_3(M^4) \neq 0 \implies w_3(M^i) \neq 0 \text{ for } i \geq 5. \end{aligned} \tag{1.4}$$

Moreover, we also have that M^i is orientable if $i \geq 5$. We then use the fact that different fields k yield non-commensurable manifolds to obtain infinitely many commensurability classes of non-spin^c manifolds in each dimension, proving Theorem 1.13.

In fact, the same argument can be used to prove a generalization of Theorem 1.13:

Theorem 1.14. *For every $m \geq 1$ and $n \geq 4m+1$, there exist infinitely many commensurability classes of closed orientable hyperbolic n-manifolds M such that $w_{4m-1}(M) \neq 0$.*

A direct consequence is that there exist closed orientable hyperbolic manifolds with nonzero Stiefel–Whitney classes in arbitrarily high degree.

Since the proof of Theorem 1.13 is non-constructive, in Section 5.4 we also give a semi-explicit procedure to obtain a compact 5-manifold without spin^c structures, based on two right-angled hyperbolic polytopes (the dodecahedron and the 120-cell) and on the Coxeter 5-polytope with diagram (1.1).

Some natural questions arise. For instance, regarding the existence of more non-spin^c manifolds, we could try to escape the constraints given by our arithmetic tools:

Question 1.15. Do there exist closed, orientable, non-arithmetic hyperbolic manifolds without spin^c structures?

Question 1.16. Are there infinitely many commensurability classes of such manifolds?

A variation of our method based on the results of [78] might give an affirmative answer to both questions.

Furthermore, as the vanishing of W_3 does not *a priori* imply that of w_3, we could ask:

Question 1.17. Do there exist closed orientable hyperbolic manifolds without spin^c structures and with $w_3 = 0$?

Using characteristic classes, one could also investigate higher spin structures beyond spin and spin^c: for instance, the primary obstruction to a spin^h structure is the fifth integral Stiefel–Whitney class W_5 [4]. While this class is outside the scope of Theorem 1.14, it is still natural to ask:

Question 1.18. What can be said about the existence of orientable hyperbolic manifolds without higher spin structures?

1.3.2 Non-cobordant manifolds

The (unoriented) cobordism relation on smooth, closed n-manifolds gives rise to the *cobordism group* $\mathcal{N}_n$, whose elements are equivalence classes of cobordant manifolds, and whose group operation is induced by the disjoint union: $[M] + [M'] := [M \sqcup M']$. Moreover, the direct sum of all $\mathcal{N}_n$ for $n \geq 0$ can be given a graded ring structure, with multiplication induced by the Cartesian product of manifolds. The structure of the *cobordism ring* $\mathcal{N}_*$ was described by Thom in his seminal paper [121]: it is a free polynomial ring over $\mathbb{Z}_2$

$$\mathcal{N}_* \simeq \mathbb{Z}_2[x_2, x_4, x_5, x_6, \dots], \tag{1.5}$$

with one generator x_i for each $i \neq 2^k - 1$. Of course, this allows one to easily determine the individual groups $\mathcal{N}_n$.

It is also of interest to impose additional structure on manifolds and the cobordisms between them: for instance, we may study the *oriented cobordism* or *spin cobordism* groups. In the context of hyperbolic geometry, Long and Reid [84] introduced the concept of a manifold that *bounds geometrically*, that is, a hyperbolic manifold that is the geodesic boundary of another hyperbolic manifold.

Returning to unoriented cobordism, we may instead impose conditions on the Riemannian metric: it is a classical result of Hamrick and Roysted [60] that all flat closed manifolds are boundaries; on the other hand, for even n, the real projective spaces $\mathbb{RP}^n$ are examples of non-cobordant spherical manifolds, since their Euler characteristic is 1. Moreover, by Ontaneda's Riemannian hyperbolization [102, Corollary 1], for any given $\varepsilon > 0$, we can realize *every* cobordism class with closed manifolds of pinched negative sectional curvature $K \in [-1-\varepsilon, -1]$.

Along this direction, in the latter half of Chapter 5, we will analyze the unoriented cobordism relation between closed hyperbolic manifolds. Specifically, we prove:

Theorem 1.19. *For each $n \geq 4$, $n \not\equiv 3 \pmod 4$, there exists a connected, non-cobordant closed hyperbolic n-manifold.*

Note that the two-dimensional case is realized by any hyperbolic surface diffeomorphic to $\mathbb{T}^2 \# \mathbb{RP}^2$, while the case $n = 4$ is realized by two manifolds of Euler characteristic 17, due to Ratcliffe and Tschantz [111]. We have not been able to find any other examples of non-cobordant closed hyperbolic manifolds in the literature.

By contrast, we may arrange for our examples to have even Euler characteristic, which implies that in dimension 4 they realize a different cobordism class from either Ratcliffe–Tschantz 4-manifold (see Section 5.8.3).

The proof of Theorem 1.19 relies on describing the cobordism class of a manifold M in terms of *projective bundles* on the fixed submanifolds X of a nontrivial involution $\tau\colon M \to M$. By a formula of Thom, we can compute the value of a certain homomorphism $\varphi^n\colon \mathcal{N}_n \twoheadrightarrow \mathbb{Z}_2$ on such a projective bundle, discovering that it depends only on the Stiefel–Whitney classes of the normal bundle νX (and not on the tangent classes $w_i(X)$). The formula, stated in Theorem 5.20, may be of independent interest. It takes the following form:

$$\varphi^n[M] = \sum_{d=0}^{n-1} \sum_{X \in F_d(\tau)} I_{n,d}(w_1(\nu X), \dots, w_d(\nu X)), \tag{1.6}$$

where $F_d(\tau)$ is the set of all fixed submanifolds of τ having dimension d, and $I_{n,d}$ are certain polynomial expressions.

In Section 5.7, we use Theorem 1.12 in the special case $c = 0$, and extend it to the context of arithmetic manifolds with involutions. Moreover, we show that by cutting M' along M and reattaching with a twist given by the involution, we obtain another $(n+1)$-manifold M'' in which M also embeds geodesically.

The right hand side of (1.6) is a well-defined invariant $I(n, M, \tau)$ for a manifold with involution (M, τ) of any dimension, coinciding with $\varphi^n[M]$ when $\dim M = n$: using this fact, we construct a non-cobordant n-manifold M^n by starting from a particular 3-manifold with $I(n, M, \tau) = 1$, and repeatedly applying the Kolpakov–Reid–Slavich embedding to increase the dimension. At each step, we may choose whether to perform the twist or not; it is exactly this freedom that allows us to ensure $I(n, M, \tau) = 1$ at each step, until we reach dimension n.

Furthermore, for 3-manifolds, the expression $I(n, M, \tau)$ is periodic in n of period 4, so we manage to cover all dimensions $n \equiv 0, 1, 2 \pmod 4$ with only three starting examples. The limitations of our method for the remaining case $n \equiv 3 \pmod 4$, which we discuss in Section 5.8.2, arise from a specific pattern in the expressions $I_{n,d}$. By choosing suitable starting manifolds, we describe how one might construct non-cobordant hyperbolic manifolds in all dimensions $n \neq 2^k - 1$. Note that the cobordism groups $\mathcal{N}_{2^k-1}$ are actually nontrivial for $k \geq 3$, but constructing non-cobordant hyperbolic manifolds in these dimensions requires a different approach.

Some questions remain open:

Question 1.20. Do there exist non-cobordant hyperbolic manifolds of dimension $4m+3$?

Question 1.21. Which cobordism classes can be realized by hyperbolic manifolds? What if the manifolds are required to be connected?

Additionally, since our method does not easily allow for the construction of orientable manifolds, we may ask:

Question 1.22. Do there exist non-cobordant orientable hyperbolic manifolds?

In dimensions 2 and 4 the answer is known to be negative. Indeed, an orientable surface bounds a handlebody; on the other hand, a closed orientable hyperbolic 4-manifold has zero signature by the Hirzebruch signature formula [84], thus it is cobordant.

Similar questions may of course be asked for the orientable cobordism group and its higher analogs.

Finally, it is worth pointing out a connection between the two problems studied in this chapter. A non-cobordant 5-manifold is characterized by the non-vanishing of the Stiefel–Whitney number $w_2 w_3$, which in particular implies $w_3 \neq 0$. Indeed, while the manifolds thus constructed are not orientable, a stronger version of Theorem 1.19 giving orientable manifolds would already imply Theorem 1.13 in dimension 5 (and solve Question 1.22 in the affirmative); see also [26, Question 1.5]. More generally, the technique is based on constructing n-manifolds for which certain *sums* of Stiefel–Whitney numbers do not vanish, and could potentially be adapted to obtain the non-vanishing of specific Stiefel–Whitney classes.

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2. Preliminaries

We will now establish some common concepts and notation that will appear throughout this thesis, mostly pertaining to the three broad areas mentioned in the introduction.

2.1 The von Neumann algebra of a group

In this section, we introduce the von Neumann algebra, which we will then use in Chapter 3 to define L^2-invariants. In what follows we take G to be any group, even if in our applications we will always consider countable, torsion-free groups. Rings are unital but not necessarily commutative, and modules are left modules unless explicitly stated. Our main reference is Lück's book [89].

Considering G as a discrete measure space, we can form the complex Hilbert space $\ell^2(G)$ of square-summable sequences indexed by G, where the inner product

$$\left\langle \sum_{g\in G} \alpha_g g, \sum_{g\in G} \beta_g g \right\rangle := \sum_{g\in G} \alpha_g \overline{\beta_g} \tag{2.1}$$

is antilinear in the second argument. There is a natural left action $G \curvearrowright \ell^2(G)$, defined by formal multiplication; this leads to the following definition.

Definition 2.1. The *von Neumann algebra* $\mathcal{N}(G)$ is the algebra of G-equivariant endomorphisms of $\ell^2(G)$, that is, all linear bounded operators $\ell^2(G) \to \ell^2(G)$ that commute with the left action by G.

To get a better grasp on this object, we note that any element $f \in \mathcal{N}(G)$ is determined by the image of the identity element $e \in G \subset \ell^2(G)$:

$$f\left(\sum_{g\in G} \alpha_g g\right) = \sum_{g\in G} \alpha_g f(g) = \sum_{g\in G} \alpha_g g f(e). \tag{2.2}$$

Therefore, we can identify $\mathcal{N}(G)$ with a subset of $\ell^2(G)$.

Moreover, if we write $f(e) = \sum_{h\in G} \varphi_h h$, we have

$$\sum_{g\in G} \alpha_g g f(e) = \sum_{g\in G} \alpha_g \sum_{h\in G} \varphi_h g h, \tag{2.3}$$

which is formally an action by *right* multiplication.

The complex number $\varphi_e = \langle f(e), e\rangle$ is called the *von Neumann trace*, $\mathrm{tr}_{\mathcal{N}(G)}(f)$ and will play a fundamental role in defining the *von Neumann dimension*.

2.1.1 Von Neumann dimension for $\mathcal{N}(G)$-modules

The theory of modules over certain rings, such as fields, skew fields and principal ideal domains (PIDs), is characterized by the existence of a well-behaved *dimension* (or *rank*) function. This concept also applies to modules over $\mathcal{N}(G)$, with the peculiarity that the dimension is not necessarily an integer.

We start with some general definitions.

Definition 2.2. Let R be a ring and let $M \subseteq N$ be two R-modules. The *closure* of M in N is
$$\overline{M} \coloneqq \bigcap \{\ker f \mid f \in \operatorname{Hom}_R(N, R), f\big|_M \equiv 0\}.$$

Definition 2.3. A *predimension* on R is a function[1]

$$\operatorname{pdim}\colon \{\text{finitely generated projective } R\text{-modules}\} \to [0, +\infty)$$

such that:

(1) $P \simeq Q \implies \operatorname{pdim}(P) = \operatorname{pdim}(Q)$;

(2) $\operatorname{pdim}(P \oplus Q) = \operatorname{pdim}(P) + \operatorname{pdim}(Q)$;

(3) if K is a submodule of Q, then its closure $\overline{K}$ is a direct summand of Q and

$$\operatorname{pdim} \overline{K} = \sup \{\operatorname{pdim}(P) \mid P \subseteq K \text{ fin. gen. projective}\}.$$

Such a function can be defined on $\mathcal{N}(G)$ as follows. Let P be a finitely generated projective $\mathcal{N}(G)$-module. Then, P is a quotient of some finitely generated free module $F = \mathcal{N}(G)^n$; it is also a direct summand by projectivity. The projection on P is an endomorphism of $N(G)^n$ and can be seen as a square matrix A with coefficients in $\mathcal{N}(G)$. Finally, we define

$$\operatorname{pdim}_{\mathcal{N}(G)}(P) \coloneqq \operatorname{tr}_{\mathcal{N}(G)}(A) \coloneqq \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(A_{ii}). \tag{2.4}$$

Extending this definition to a genuine dimension function can also be done in a general framework, as follows.

Definition 2.4. Given a predimension pdim on R, we can define a dimension function on all R-modules:

$$\dim(M) \coloneqq \sup \{\operatorname{pdim}(P) \mid P \subseteq M \text{ fin. gen. projective}\} \in [0, +\infty].$$

[1] Strictly speaking, the term "function" should be taken to mean "class function", to avoid foundational issues when the domain is a proper class.

Theorem 2.5 ([89, Theorem 6.7])**.** *Let* pdim *be a predimension on R. Then:*

(1) dim *agrees with* pdim *on the class of finitely generated projective R-modules;*

(2) given a short exact sequence of R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

we have $\dim(B) = \dim(A) + \dim(C)$*, where the sum is extended to* $[0, +\infty]$ *in the natural way;*

(3) if $\{M_i \mid i \in I\}$ *is a family of submodules of M such that any two M_i, M_j are contained in some M_k, and* $\bigcup_{i\in I} M_i = M$*, then*

$$\dim(M) = \sup\{\dim(M_i) \mid i \in I\};$$

(4) if $K \subseteq M$ *is a submodule, then* $\dim(\overline{K}) = \dim K$*;*

(5) properties (1)–(4) uniquely characterize dim*.*

It follows, of course, that all $\mathcal{N}(G)$-modules have a well-defined dimension $\dim_{\mathcal{N}(G)}$.

2.2 Coxeter polytopes

As mentioned in the introduction, a classical method for the construction of manifolds of constant sectional curvature, especially hyperbolic manifolds, is the gluing of copies of *Coxeter polytopes*. Our exposition follows Vinberg's article [130].

Let us denote by $\mathbb{X}^n$ any of the three spaces of constant sectional curvature, that is, the sphere $\mathbb{S}^n$, Euclidean space $\mathbb{E}^n$ and hyperbolic space $\mathbb{H}^n$.

Definition 2.6. A *polytope* P is the intersection of finitely many half-spaces of $\mathbb{X}^n$. Depending on $\mathbb{X}^n$, we may speak of *spherical*, *Euclidean* or *hyperbolic* polytopes.

The *dimension* of P, $\dim P$, is the least dimension of a hyperbolic subspace containing P, or -1 if P is empty. Such a subspace is called a *supporting subspace* for P.

The boundary of P (relative to its supporting subspace) is naturally stratified by dimension and consists of polytopes of dimension $k \in \{0, 1, \ldots, \dim P - 1\}$, called the *$k$-faces* of P. In particular, faces of dimension 0 and codimension 2 and 1 are respectively called *vertices*, *ridges* and *facets* of P. When two facets meet at a ridge, their *dihedral angle* is well defined, and it is less than π by convexity of P. This is an essential concept in the definition of Coxeter polytopes.

Furthermore, in the hyperbolic case, the closure of P may intersect the boundary at infinity $\partial\mathbb{H}^n$; if P has finite volume, this intersection consists of finitely many isolated points, called *ideal vertices* (as opposed to *real vertices*).

Definition 2.7. A polytope P in $\mathbb{H}^n$ (respectively, $\mathbb{E}^n$ or $\mathbb{S}^n$) is called a *hyperbolic* (respectively, *Euclidean* or *spherical*) *Coxeter polytope* if all its dihedral angles are of the form π/k for $k \geq 2$.

A important special case is given by *right-angled* polytopes, that is, Coxeter polytopes whose dihedral angles are $\pi/2$. Any face of a right-angled polytope is itself a right-angled polytope.

2.2.1 Coxeter diagrams and groups

Every Coxeter polytope P can be completely described by means of a labeled graph, called its *Coxeter diagram*.

Definition 2.8. A *Coxeter diagram* is a labeled complete finite graph, where edges can be of two types:

- *dashed*, and optionally labeled with a real number in $[1, +\infty)$;
- *non-dashed*, and labeled with an integer greater than 1, or ∞.

We will adopt the following common convention when drawing Coxeter diagrams: if a non-dashed edge is labeled 2, then the edge is omitted, while if it is labeled 3, 4 or 5, then the label is omitted, and a single, double or triple edge is drawn instead. This greatly simplifies the visual appearance of many Coxeter diagrams. As such, we may speak of *connected diagrams* or *connected components* of a diagram, by considering non-dashed edges with label 2 as absent.

Now, let $\mathrm{Fac}(P) \coloneqq \{F_1, \ldots, F_k\}$ be the facets of P. The Coxeter diagram of P is a labeled graph D with a vertex for each facet, and an edge for every pair of facets F_i, F_j. We have three cases:

- if the two facets intersect with dihedral angle π/m_{ij}, then the corresponding edge is labeled m_{ij};
- if the supporting hyperplanes of F_i and F_j are parallel, the edge is labeled ∞;
- if the supporting hyperplanes of F_i and F_j are ultraparallel with distance d, a dashed edge is drawn, which can optionally be labeled by $\cosh d \geq 1$.

Definition 2.9. We say that a Coxeter diagram is *hyperbolic* (respectively, *affine* or *spherical*) if it is the diagram of a polytope in $\mathbb{H}^n$ (respectively, $\mathbb{E}^n$ or $\mathbb{S}^n$).

As a consequence of [130, §2.3], a diagram is spherical or affine precisely when its connected components are, respectively, spherical or affine. For a complete classification of connected spherical and affine Coxeter diagrams, see [130, Tables 1–2].

We can define a group from every Coxeter diagram, even if it does not come from a polytope:

Definition 2.10. Let $S \coloneqq \{r_1, \dots, r_k\}$ be the vertex set of a Coxeter diagram D, let E be the set of non-dashed edges with finite label, and let m_{ij} be the label of the edge $(r_i, r_j) \in E$.

We define the *Coxeter group associated to D*, or simply the *Coxeter group of D*, as a finitely presented group W_D with generating set S:

$$W_D \coloneqq \langle r_1, \dots, r_k \mid r_i^2 = 1 \ \forall i = 1, \dots, k, (r_i r_j)^{m_{ij}} = 1 \ \forall (i,j) \in E \rangle. \tag{2.5}$$

If the diagram D comes from a polytope P, the group W_D will also be called the *Coxeter group of P*. Finally, the pair (W_D, S) is also called a *Coxeter system.*

Remark 2.11. If a Coxeter diagram D has several connected components $D_1 \sqcup \cdots \sqcup D_m$, then generators in different components commute, and hence we have $W_D \simeq W_{D_1} \times \cdots \times W_{D_m}$.

Remark 2.12. By passing to the Coxeter group, we lose all information distinguishing dashed edges from each other and from non-dashed edges labeled ∞.

Proposition 2.13. *Let P be a polytope in $\mathbb{X}^n$. For each facet $F_i \in \mathrm{Fac}(P)$, let $r_i \in \mathrm{Isom}(\mathbb{X}^n)$ be the reflection in the supporting hyperplane of F_i. Then, the group generated by all reflections $r_1, \dots, r_k$ is isomorphic to the Coxeter group of P, with an isomorphism sending each reflection to the corresponding vertex of the Coxeter diagram.*

Because of this result, we will sometimes abuse notation and identify reflections, facets of the polytope, vertices of the Coxeter diagram, and generators of the Coxeter group. Moreover, we will also describe Coxeter groups obtained from hyperbolic, affine or spherical diagrams using the same adjectives.

2.2.2 The Gram matrix and the canonical representation

The information provided in a Coxeter diagram can also be represented in the form of a matrix.

Definition 2.14. The *Gram matrix G* of a Coxeter diagram D with k vertices $\{r_1, \dots, r_k\}$ is a real symmetric $k \times k$ matrix. If e is the edge (r_i, r_j), the entry G_{ij} is given by:

$$G_{ij} \coloneqq \begin{cases} 1 & \text{if } i = j; \\ -\cos(\pi/m_{ij}) & \text{if } e \text{ is non-dashed and has finite label } m_{ij}; \\ -1 & \text{if } e \text{ is non-dashed and has label } \infty; \\ -z_{ij} & \text{if } e \text{ is dashed and has label } z_{ij}. \end{cases} \tag{2.6}$$

The Gram matrix gives a bilinear form on $\mathbb{R}^k$, and it can be used to define reflections along each basis vector e_i:

$$\rho_i(x) \coloneqq x - 2G(x, e_i)e_i. \tag{2.7}$$

We have the following result:

Proposition 2.15 ([15, Chapter 5, §4.4, Corollary 2]). *The map*

$$\lambda\colon W_D \to \mathrm{Isom}(\mathbb{R}^k, G) := \{M \in \mathrm{GL}(k,\mathbb{R}) \mid M^T G M = G\}, \tag{2.8}$$

sending $r_i \mapsto \rho_i$, is a faithful linear representation of the Coxeter group W_D.

This representation is sometimes called the *canonical representation* of W_D and has an important consequence: Coxeter groups are linear groups, and by Selberg's lemma, they are virtually torsion-free. Hence, given a Coxeter polytope P, the associated Coxeter group W has a finite-index torsion-free subgroup Γ, which acts freely on $\mathbb{X}^n$. The quotient $\mathbb{X}^n/\Gamma$ is then a finite-degree manifold cover of the orbifold $\mathbb{X}^n/W \simeq P$. Hence, any hyperbolic Coxeter polytope gives rise to a hyperbolic manifold, albeit not in an immediately explicit manner (but see Section 4.11 and [46]).

Remark 2.16. Let $\operatorname{rad} G$ be the radical of the bilinear form G and let $\overline{G}$ be the induced non-degenerate bilinear form on the quotient $\mathbb{R}^k/\operatorname{rad} G$. Notice that $\operatorname{rad} G$ is an invariant subspace of λ, and moreover it is fixed by every isometry in $\operatorname{im}\lambda$. Hence, by passing to the quotient, we obtain a faithful representation

$$\overline{\lambda}\colon W_D \hookrightarrow \mathrm{Isom}(\mathbb{R}^k/\operatorname{rad} G, \overline{G}). \tag{2.9}$$

With more work, if the signature of $\overline{G}$ is $(n,1)$, one can show that W_D acts through $\overline{\lambda}$ on a hyperboloid $\mathcal{H}^n \simeq \mathbb{H}^n$ contained in the Lorentzian vector space $(\mathbb{R}^k/\operatorname{rad} G, \overline{G})$. The generators fix hyperbolic hyperplanes, which bound a hyperbolic Coxeter polytope, the fundamental domain for the action. More precisely, we have:

Proposition 2.17. *If a Gram matrix of a connected Coxeter diagram D has signature $(n,1,m)$, then D is the Coxeter diagram of a hyperbolic Coxeter n-polytope.*

Proof. This is a direct consequence of Vinberg's theorem on acute-angled polytopes; see [130, Theorem 2.1] and the preceding discussion. □

2.2.3 Faces of a Coxeter polytope

We will now show how to obtain some combinatorial information from a Coxeter diagram.

Any face F of a Coxeter polytope P can be uniquely described as the intersection of a proper, non-empty subset of $\mathrm{Fac}(P)$, and hence corresponds to a subdiagram D_F of the Coxeter diagram D. In fact, we have:

Proposition 2.18. *Coxeter polytopes are* simple*, that is, any (non-ideal) face of codimension c is the intersection of exactly c facets.*

Proposition 2.19. *The inclusion $D_F \subset D$ induces a well-defined injective homomorphism of Coxeter groups $W_{D_F} \hookrightarrow W_D$.*

This fact may also be proven from the canonical representation. Moreover, the group W_{D_F} has a special significance:

Proposition 2.20. *The group W_{D_F} is the stabilizer of the face F under the action of W_D on $\mathbb{X}^n$.*

Indeed, if $\dim F = d$, then W_{D_F} is a spherical Coxeter group, acting on the unit sphere $\mathbb{S}^{n-d-1}$ of an orthogonal space to F. The opposite is also true:

Proposition 2.21. *The faces of a Coxeter polytope given by a diagram D are in bijection with the spherical subdiagrams of D, and the number of vertices of such a subdiagram equals the dimension of the face.*

We can also find ideal vertices in the hyperbolic, finite-volume case:

Proposition 2.22 (see [130, Chapter 1, §3])**.** *The ideal vertices of a hyperbolic Coxeter polytope of finite volume, with diagram D, correspond to maximal affine subdiagrams of D.*

Let us now consider isometries of a Coxeter polytope P with diagram D. To this end, let $\mathrm{Aut}(D)$ be the group of (label-preserving) automorphisms of the diagram. Any isometry $\gamma \in \mathrm{Isom}(P)$ induces an automorphism $\alpha(\gamma)$ of D by permuting the facets of P. In fact, if we assume that P has at least one vertex v, then the homomorphism $\alpha\colon \mathrm{Isom}(P) \to \mathrm{Aut}(D)$ is injective: if an isometry of P fixes every facet, then it also fixes v and its incident edges, which implies that it fixes P pointwise. On the other hand, by [130, Theorem 2.1], diagrams determine polytopes up to isometry, so every automorphism of D is induced by an isometry of P. Hence, we have:

Proposition 2.23. *Let P be a Coxeter polytope with diagram D, and assume that P has at least one vertex. Then every label-preserving automorphism of D is induced by a unique isometry of P.*

2.2.4 Reflectofolds

Many of the definitions we have seen so far can be extended to manifolds with corners, as follows [38; 115; 28]:

Definition 2.24. A *reflectofold*, or a *Coxeter manifold with corners*, is a spherical, Euclidean or hyperbolic manifold with corners, locally modeled on a Coxeter polytope.

The boundary of manifolds with corners is naturally stratified into faces of varying dimension, and this definition adds the condition that dihedral angles between facets of reflectofolds be submultiples of π. We shall adopt the same naming conventions as for polytopes, calling faces of codimension 1, codimension 2 and dimension 0 *facets*, *ridges* and *vertices* respectively. Ridges of reflectofolds are sometimes also called *corners*.

As the name suggests, a reflectofold has a natural orbifold structure, inherited from those of its local models. However, the analogy between reflectofolds and Coxeter polytopes breaks down in some edge cases, which are described by the following properties:

Definition 2.25. A reflectofold is said to be *developable* if it has the following properties:

(AC) *angle consistency*, if, for any pair of facets, any ridge between them has the same dihedral angle;

(EF) *embedded facets*, if no facet meets itself along a ridge.

A complete, finite-volume developable reflectofold R has a well-defined Coxeter diagram, and the associated Coxeter group has a finite-index torsion-free subgroup. This idea is used in Davis' *basic construction* [39, Chapter 11] to construct a finite-volume manifold cover of R.

2.2.5 Right-angled polytopes and the coloring method

In the case where all dihedral angles are $\pi/2$, we speak of *right-angled polytopes* or, in the case of reflectofolds, *right-angled manifolds*. Such objects are especially well behaved and allow for an explicit description of finite-degree manifold covers (that is, without boundary), via the *coloring method* [124; 125; 38; 55]. Note that property (AC) is automatically true for right-angled manifolds, so developability reduces to having embedded facets (EF).

Let P be a right-angled polytope (or a right-angled developable manifold) with facets $\mathrm{Fac}(P) = \{F_1, \ldots, F_k\}$.

Definition 2.26. A *(proper) coloring* of P is a map $\lambda\colon \mathrm{Fac}(P) \to V$ for some finite-dimensional vector space V over $\mathbb{Z}_2$, assigning to each facet a *color* in V, such that for each vertex of $\mathcal{P}$, the colors of its incident facets form a linearly independent set.

A particularly simple class of colorings is obtained by considering only colors in a fixed basis of V. This is equivalent to coloring the facet adjacency graph of $\mathcal{P}$ in such a way that no two vertices with the same color share an edge; note that if P is a polyhedron, then this graph is the 1-skeleton of the dual polyhedron.

Definition 2.27. Given a coloring λ, we can define the *real toric manifold* $M(P, \lambda)$ as a gluing of $|V|$ copies $\{P_v\}$ of P, indexed by vectors $v \in V$: for each facet $F \in \mathrm{Fac}(P)$ and each vector $v \in V$, glue P_v and $P_{v+\lambda(F)}$ along F, using the identity map of F.

Due to the linear independence condition, the resulting space is indeed a closed manifold; moreover, it is connected if and only if $\mathrm{im}(\lambda)$ is a generating set for V. The construction is clearly invariant under vector space isomorphisms, so we may assume $V = \mathbb{Z}_2^m$ for some m, if needed. If we fix such an isomorphism, along with an ordering of $\mathrm{Fac}(P)$, the coloring λ can be summarized by a binary matrix whose columns are the colors of the facets of P; this is called the *characteristic matrix* of the coloring.

We note that $M(P, \lambda)$ is connected if and only if $\mathrm{im}(\lambda)$ is a generating set for V; otherwise each connected component arises from gluing some set $\{P_v : v \in C\}$, where C is a coset of $\operatorname{span} \mathrm{im}(\lambda)$ in V.

Multiplying the characteristic matrix by an invertible matrix on the left is equivalent to a vector space isomorphism, and does not change the isometry class of the real toric manifold. More generally, if $q\colon \mathbb{Z}_2^m \twoheadrightarrow \mathbb{Z}_2^n$ is a quotient of vector spaces such that $q \circ \lambda$ is also a coloring on $\mathcal{P}$, then $M(P, \lambda)$ covers $M(P, q \circ \lambda)$.

We may also consider *partial colorings* that satisfy the linear independence condition, but are only defined on a subset of $\mathrm{Fac}(P)$; we have:

Remark 2.28. A *partial* coloring on a proper subset $A \subset \mathrm{Fac}(P)$, with values in a $\mathbb{Z}_2$-vector space V, can always be extended to a genuine coloring $\mathrm{Fac}(P) \to V \oplus V'$ by coloring the remaining facets with different basis vectors of V'.

It is convenient to identify P with one of its copies inside M, say P_0. Now, any face F of P is again a right-angled polytope or manifold, contained in a totally geodesic, possibly disconnected submanifold M_F of the same dimension, obtained as the preimage of F under the covering map $M \twoheadrightarrow P$. The following result provides a description of M_F as a real toric manifold over F, generalizing [76, Lemma 2.8].

Proposition 2.29. *Let P be a developable right-angled hyperbolic manifold, and let F be a face of P. Let* $\mathrm{adj}(F)$ *be the set of facets of P that intersect the boundary of F but not its interior, and let* $\mathrm{supp}(F)$ *be the set of facets of P containing F in their boundary. Then* $\mathrm{Fac}(F)$ *maps naturally onto* $\mathrm{adj}(F)$*, and $M_F = M(F, \lambda_F)$, where λ_F is the composition*

$$\mathrm{Fac}(F) \twoheadrightarrow \mathrm{adj}(F) \hookrightarrow \mathrm{Fac}(P) \xrightarrow{\lambda} V \twoheadrightarrow V/\langle \lambda(i) \mid i \in \mathrm{supp}(F)\rangle. \tag{2.10}$$

Proof. The space V acts by isometries on the P-tessellation of $M(P, \lambda)$. This action restricts to the tessellation of M_F by copies of F. Let us fix a copy of F, say F_0; then its stabilizer is precisely $\langle \lambda(F_i) \mid F_i \in \mathrm{supp}(F)\rangle$, and the manifold M_F is obtained by gluing copies of F, canonically indexed by the quotient $V/\langle \lambda(F_i) \mid F_i \in \mathrm{supp}(F)\rangle$. It is not hard to check that the gluing is induced by the coloring λ_F. □

2.3 Arithmetic hyperbolic manifolds

Finite-volume hyperbolic Coxeter polytopes cannot exist in dimension greater than 995 in the non-compact case [109; 70], and 29 in the compact case [129]. In this section, we introduce another way to construct finite-volume hyperbolic manifolds, based on algebraic tools, that works in all dimensions $n \geq 2$.

2.3.1 Arithmetic manifolds of simplest type

Let k be a totally real number field, that is, a finite extension of $\mathbb{Q}$ whose complex embeddings have image in $\mathbb{R}$. Moreover, let f be a quadratic form over $\mathbb{R}^{n+1}$ given by a symmetric matrix with coefficients in k.

Definition 2.30. The form f is *admissible* if it has signature $(n, 1)$ and, for every nontrivial embedding $\sigma\colon k \to \mathbb{R}$, the form f^σ is positive definite.

The set of lines (1-dimensional subspaces) of $\mathbb{R}^{n+1}$ on which f is definite negative can be naturally identified with hyperbolic space $\mathbb{H}^n$, and this gives a natural action of the orthogonal group $\mathrm{O}(f)$ on $\mathbb{H}^n$ by isometries. Equivalently, the group $\mathrm{O}(f)$ acts on the standard hyperboloid model after a basis change that diagonalizes f. Note that the action is faithful up to factoring through the central subgroup $\langle -I_{n+1}\rangle$.

If $\mathcal{O}_k$ is the ring of integers of k, consider the group $\mathrm{O}(f, \mathcal{O}_k)$ of orthogonal matrices with entries in R. Then, by the Borel–Harish-Chandra theorem [12], the quotient $\mathbb{H}^n/\mathrm{O}(f, \mathcal{O}_k)$ is a finite-volume orbifold.

Definition 2.31. We say that a hyperbolic manifold (or orbifold) M is *arithmetic of simplest type* with *field of definition* k if there exists an admissible quadratic form f defined over k such that M is commensurable with $\mathbb{H}^n/\mathrm{O}(f, \mathcal{O}_k)$.

In this case, up to a change of basis, the group $\pi_1(M) < \mathrm{Isom}(\mathbb{H}^n)$ can be regarded as a subgroup of $\mathrm{O}(f)$ which is commensurable with $\mathrm{O}(f, \mathcal{O}_k)$, and is said to be *arithmetic of simplest type*. The field of definition k is a commensurability invariant for arithmetic subgroups of simplest type [9; 127].

2.3.2 Other arithmetic manifolds

In more generality, following [126, §5] and [9], we can define *arithmetic subgroups* of $\mathrm{Isom}(\mathbb{H}^n)$ for $n \geq 2$; in fact, these definitions apply to any non-compact semisimple Lie group with finitely many connected components (note that $\mathrm{Isom}(\mathbb{H}^1) \simeq \mathbb{R} \rtimes \mathbb{Z}_2$ is not semisimple).

Recall that an (affine) *algebraic group* over a field k is an affine variety G over k equipped with a group structure, that is, a distinguished identity element, and two algebraic morphisms $i\colon G \to G$ and $m\colon G \times G \to G$ defining inverses and multiplication.

Examples of algebraic groups include matrix groups such as $\mathrm{GL}(n, k)$, $\mathrm{SL}(n, k)$, $\mathrm{O}(n, k)$, and even $\mathrm{O}(f, k)$, where f is a quadratic form defined over k. In fact, every algebraic group embeds as a closed subgroup of some $\mathrm{GL}(m, k)$ [11, Proposition 1.10, p. 54].

Definition 2.32. A discrete subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is *arithmetic* if there exist:

– an algebraic group G over a totally real number field k,

– an embedding of algebraic groups $\rho\colon G \hookrightarrow \mathrm{GL}(m, k)$,

– an epimorphism $\varphi\colon G(k \otimes_{\mathbb{Q}} \mathbb{R}) \to \mathrm{Isom}(\mathbb{H}^n)$ with compact kernel,

such that $\varphi(\rho^{-1}(\mathrm{GL}(m, \mathcal{O}_k)))$ is commensurable with Γ.

Again, by the Borel–Harish-Chandra theorem, arithmetic subgroups have finite covolume in $\mathrm{Isom}(\mathbb{H}^n)$, and correspond to finite-volume hyperbolic orbifolds.

In the case of arithmetic subgroups of simplest type, the group G is simply $\mathrm{O}(f, k)$, and we have an epimorphism

$$\mathrm{O}(f, k \otimes_{\mathbb{Q}} \mathbb{R}) \simeq \bigoplus_{\sigma\colon k \to \mathbb{R}} \mathrm{O}(f^\sigma, \mathbb{R}) \longrightarrow \mathrm{O}(f, \mathbb{R}) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Isom}(\mathbb{H}^n) \tag{2.11}$$

given by restriction of scalars and projection on the component corresponding to $\sigma = \mathrm{id}$.

In general, arithmetic hyperbolic manifolds are divided into three classes [9, Section 2.4]:

(1) arithmetic manifolds of simplest type;

(2) arithmetic manifolds from Hermitian forms over quaternion algebras;

(3) exceptional 3-manifolds from $\mathrm{PSL}(2, \mathbb{C})$ and trialitarian 7-manifolds.

The first class contains all even-dimensional and all non-compact arithmetic manifolds; moreover, non-compact manifolds are all defined over the field $k = \mathbb{Q}$.

It is worth noting that hyperbolic Coxeter n-polytopes, and their corresponding Coxeter groups, which are subgroups of $\mathrm{Isom}(\mathbb{H}^n)$, can also be classified with respect to their arithmeticity, using Vinberg's arithmeticity criterion [126, Theorem 2]. Moreover, all arithmetic hyperbolic Coxeter groups are of simplest type [126, Lemma 7; 8, Lemma 2.1].

Finally, there is a semi-algorithm, also due to Vinberg [128, §3], which takes as input an arithmetic group of simplest type of the form $\mathrm{O}(f, \mathcal{O}_k)$, and if the group is generated by reflections (and hence is a Coxeter group), it recovers the list of generators, given as vectors orthogonal to the facets of the corresponding polytope; otherwise, it runs forever.

2.4 Stiefel–Whitney classes

We will now recall some basic notions pertaining to the Stiefel–Whitney characteristic classes of a vector bundle. We adopt the notational convention of writing the cup product as a dot or simply as juxtaposition.

Let $E \twoheadrightarrow M$ be a real vector bundle over a manifold M. We denote the *Stiefel–Whitney classes* of E by $w_i(E) \in H^i(M; \mathbb{Z}_2)$ for $i \geq 0$, where $w_0(E)$ is defined to be $1 \in H^0(M; \mathbb{Z}_2)$ by convention. These classes can be collected into an element of the cohomology ring

$$w(E) := 1 + w_1(E) + w_2(E) + \cdots \in H^*(M; \mathbb{Z}_2). \tag{2.12}$$

The Stiefel–Whitney classes can be defined via the following result:

Proposition 2.33 ([97, Section 8])**.** *There is a unique sequence of cohomology classes* $w_i(E) \in H^i(M;\mathbb{Z}_2)$*, defined for every vector bundle* $E \twoheadrightarrow M$*, and satisfying the following properties:*

(1) $w_0(E) = 1$ *and* $w_i(E) = 0$ *for* $i > \operatorname{rk} E$*;*

(2) if $f\colon M \to M'$ *is a smooth map, then* $w(f^*E) = f^*(w(E))$*;*

(3) if E*,* E' *are two vector bundles over* M*, then* $w(E \oplus E') = w(E)w(E')$*;*

(4) $w_1(\gamma_1^1) \neq 0$*, where* γ_1^1 *denotes the tautological line bundle over* $\mathbb{RP}^1$*.*

Property (3) is known as the *Whitney sum formula.* Moreover, when E is the tangent bundle TM, we speak of the Stiefel–Whitney classes of M and write $w_i(M), w(M)$.

2.4.1 Steenrod squares

The cohomology of a topological space X with coefficients in $\mathbb{Z}_2$ is equipped with natural homomorphisms, called the *Steenrod squares* $\mathrm{Sq}^n\colon H^k(X;\mathbb{Z}_2) \to H^{k+n}(X;\mathbb{Z}_2)$. They satisfy the following characterization.

Proposition 2.34 ([118])**.** *The Steenrod squares are uniquely determined by the following axioms:*

(1) for every continuous map $f\colon X \to Y$ *and every* $k, n \geq 0$*, we have* $f^* \circ \mathrm{Sq}^n = \mathrm{Sq}^n \circ f^*$ *as maps* $H^k(Y;\mathbb{Z}_2) \to H^{k+n}(X;\mathbb{Z}_2)$*;*

(2) for every $k \geq 0$*,* $\mathrm{Sq}^0\colon H^k(X;\mathbb{Z}_2) \to H^k(X;\mathbb{Z}_2)$ *is the identity map;*

(3) if $x \in H^n(X;\mathbb{Z}_2)$*, then* $\mathrm{Sq}^n(x) = x^2$ *(whence the name* squares*);*

(4) if $n > \deg(x)$*, then* $\mathrm{Sq}^n(x) = 0$*;*

(5) $\mathrm{Sq}^n(xy) = \sum_{i=0}^n \mathrm{Sq}^i(x)\mathrm{Sq}^{n-i}(y)$*.*

The formal sum of all Steenrod squares is an endomorphism of the whole cohomology ring, named the *total Steenrod square* $\mathrm{Sq}\colon H^*(X;\mathbb{Z}_2) \to H^*(X;\mathbb{Z}_2)$. As a consequence of the *Adem relations* [1], for all $i \geq 0$, we also have the identities

$$\mathrm{Sq}^{2i+1} = \mathrm{Sq}^1 \circ \mathrm{Sq}^{2i}, \quad \mathrm{Sq}^1 \circ \mathrm{Sq}^{2i+1} = 0. \tag{2.13}$$

2.4.2 Wu classes and Wu formulas

In a (connected) closed manifold, we can exploit the Poincaré duality pairing to represent the Steenrod squares with cohomology classes.

Definition 2.35. Let M be a closed n-manifold. For $0 \leq i \leq n$, the i-th *Wu class* of M is defined as the unique class $\nu_i(M) \in H^i(M; \mathbb{Z}_2)$ representing

$$\mathrm{Sq}^i \colon H^{n-i}(M; \mathbb{Z}_2) \to H^n(M; \mathbb{Z}_2) \simeq \mathbb{Z}_2, \tag{2.14}$$

that is, such that $\nu_i(M) \cdot x = \mathrm{Sq}^i(x)$ for all $x \in H^{n-i}(M; \mathbb{Z}_2)$.

It is convenient to let $\nu_i(M) = 0$ for $i > n$, and to define the *total Wu class* of a manifold $\nu(M) := \nu_0(M) + \nu_1(M) + \cdots \in H^*(M; \mathbb{Z}_2)$.

The significance of the Wu classes is at least partially due to their relation with the Stiefel–Whitney classes:

Theorem 2.36 ([131])**.** *Let M be a closed manifold. Then the total Stiefel–Whitney class of M is the total Steenrod square of the total Wu class of M: $w(M) = \mathrm{Sq}(\nu(M))$.*

This result allows us to recursively express the Wu classes in terms of Steenrod squares of Stiefel–Whitney classes, which can be computed as follows.

Theorem 2.37 (Wu's formula [132])**.** *Let w_k denote the Stiefel–Whitney classes of a vector bundle over a closed manifold. For $i < j$, we have*

$$\mathrm{Sq}^i(w_j) = \sum_{t \geq 0} \binom{j-i+t-1}{t} w_{i-t} w_{j+t}. \tag{2.15}$$

Indeed, by combining Theorems 2.36 and 2.37, we obtain the following identities:

$$\nu_0 = 1 \tag{2.16}$$
$$\nu_1 = w_1 \tag{2.17}$$
$$\nu_2 = w_2 + w_1^2 \tag{2.18}$$
$$\nu_3 = w_1 w_2 \tag{2.19}$$
$$\nu_4 = w_4 + w_3 w_1 + w_2^2 + w_1^4 \tag{2.20}$$
$$\nu_5 = w_4 w_1 + w_3 w_1^2 + w_2^2 w_1 + w_2 w_1^3 \tag{2.21}$$
$$\cdots$$

Now note that by Property (4) in Proposition 2.34, the Wu classes of a closed manifold must vanish above the middle dimension; this implies, for example, that in every closed 4-manifold, $\nu_3 = w_1 w_2 = 0$. Such identities between Stiefel–Whitney classes are also called *Wu formulas*.

In fact, the case of 4-manifolds can be generalized as follows:

Proposition 2.38. *Let M be a closed $4k$-manifold. Then we have $w_1(M)w_{4k-2}(M) = 0$.*

Proof. If we write simply w_i, ν_i for $w_i(M), \nu_i(M)$, we have:

$$\begin{aligned}
w_1 w_{4k-2} &= \mathrm{Sq}^1(w_{4k-2}) + w_{4k-1} && (2.22)\\
&= \mathrm{Sq}^1(w_{4k-2}) + \sum_{i=0}^{4k-1} \mathrm{Sq}^i(\nu_{4k-1-i}) && (2.23)\\
&= \mathrm{Sq}^1(w_{4k-2}) + \sum_{i=0}^{2k-1} \mathrm{Sq}^{2i+1}(\nu_{4k-2-2i}) + \sum_{i=0}^{2k-1} \mathrm{Sq}^{2i}(\nu_{4k-1-2i}) && (2.24)\\
&= \mathrm{Sq}^1\left(w_{4k-2} + \sum_{i=0}^{2k-1} \mathrm{Sq}^{2i}(\nu_{4k-2-2i})\right) + \sum_{i=0}^{2k-1} \mathrm{Sq}^{2i}(\nu_{4k-1-2i}) && (2.25)\\
&= \mathrm{Sq}^1\left(\sum_{i=0}^{2k-2} \mathrm{Sq}^{2i+1}(\nu_{4k-3-2i})\right) + \sum_{i=0}^{2k-1} \mathrm{Sq}^{2i}(\nu_{4k-1-2i}) && (2.26)\\
&= \sum_{i=0}^{2k-1} \mathrm{Sq}^{2i}(\nu_{4k-1-2i}), && (2.27)
\end{aligned}$$

using the identities (2.13). The sum (2.27) is zero since the Wu classes of M vanish above the middle dimension $2k$, and $\mathrm{Sq}^i(\nu_j) = 0$ for $i > j$. □

3. Computing the twisted L^2-Euler characteristic

In this chapter, we will describe an algorithm for the computation of the twisted L^2-Euler characteristic, ultimately proving Theorem 1.1.

As anticipated in Section 2.1, the definition of this invariant involves the homology of a CW complex with local coefficients in a von Neumann algebra $\mathcal{N}(G)$. In Section 3.1 we show how, by assuming the Atiyah conjecture for G, we can replace $\mathcal{N}(G)$ by the Linnell skew field $\mathcal{D}(G)$. This enables us, in Sections 3.2 and 3.3, to recast the definition as a homology invariant of the corresponding complex over $\mathcal{D}(G)$, involving the *universal L^2-torsion* of Friedl and Lück [49] and the *Dieudonné determinant.*

Due to the difficulty in computing this determinant explicitly, we apply Oki's *matrix expansion algorithm* [100] to approximate a valuation of it, depending on the cohomology class ϕ. Together with an effective version of Lück's approximation theorem [88], this ultimately reduces the problem to computing ranks of rational matrices, leading to our main result in Section 3.4.

Then, in Section 3.5, we describe the implementation of our algorithm in SageMath [116], and in Section 3.6 we proceed to test it on several examples:

- the complement of Dunfield's link L10n14;
- the closed 3-manifold v1539$(5, 1)$, having $b_1 = 2$;
- closed census 3-manifolds with $b_1 = 1$;
- classifying spaces of free-by-cyclic groups;
- the Ratcliffe–Tschantz hyperbolic 5-manifold (leading to Conjecture 1.3);
- an aspherical 5-manifold, obtained as a product of two hyperbolic manifolds.

Finally, in Section 3.7 we prove Proposition 3.49, and we make some final remarks in Section 3.8.

The contents of this chapter are based on the paper [25].

3.1 L^2-Betti numbers

If X is a G-CW complex, i.e. a CW complex with a properly discontinuous, cellular G-action that is free on open cells, then its cellular chain groups $C_n(X)$ have a natural left $\mathbb{Z}G$-module structure, which is respected by the boundary maps. On the other hand, the von Neumann algebra contains $\mathbb{Z}G$ as a subring and is a right module over it. Therefore, we can form the tensor product of $\mathcal{N}(G)$ and $C_*(X)$ over $\mathbb{Z}G$, and use it to define the *L^2-Betti numbers*.

Definition 3.1. The *L^2-Betti numbers* of a G-CW complex X are defined as

$$b_n^{(2)}(X;G) \coloneqq \dim_{\mathcal{N}(G)} H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*(X)).$$

It is also natural to define the *L^2-Euler characteristic*

$$\chi^{(2)}(X;\mathcal{N}(G)) \coloneqq \sum_{n\geq 0}(-1)^n \cdot b_n^{(2)}(X;\mathcal{N}(G))$$

whenever the series is absolutely convergent. We shall simply write $b_n^{(2)}(X)$ and $\chi^{(2)}(X)$ if the choice of G is clear from the context.

The prime example of a G-CW complex is the universal covering $\widetilde{Y}$ of a connected CW complex Y, where of course $G = \pi_1(Y)$. Here the $C_n(\widetilde{Y})$ are even free over $\mathbb{Z}G$, generated by G-orbits of n-cells in $\widetilde{Y}$, that is, inverse images of n-cells of Y.

By inspecting the definition, we note that a G-homotopy of G-complexes induces a $\mathbb{Z}G$-chain homotopy between the cellular chain complexes, which becomes a $\mathcal{N}(G)$-chain homotopy after tensoring. Hence, the L^2-Betti numbers are G-homotopy invariants (see [89, Theorem 6.54 (1b)] for a more general statement), so $b_n^{(2)}(\widetilde{Y})$ and $\chi^{(2)}(\widetilde{Y})$ are homotopy invariants of Y.

Somewhat surprisingly, for a finite CW complex Y we have $\chi^{(2)}(\widetilde{Y}) = \chi(Y)$ [89, Theorem 1.35, (2)].

Remark 3.2. Definition 3.1 actually applies to the extremely general case of a chain complex C_* of $\mathbb{Z}G$-modules: in fact, we can define its L^2-Betti numbers simply as

$$b_n^{(2)}(C_*;\mathcal{N}(G)) \coloneqq \dim_{\mathcal{N}(G)} H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*).$$

Unlike classical Betti numbers, their L^2 counterparts enjoy a multiplicative property when passing to finite coverings or, equivalently, subgroups of G of finite index:

Proposition 3.3 (compare [89, Theorem 6.54 (6)])**.** *Let X be a G-CW complex and let $i\colon H \hookrightarrow G$ be the inclusion of a subgroup of finite index. Then X is also an H-CW complex i^*X and*

$$b_n^{(2)}(i^*X;\mathcal{N}(H)) = [G:H] \cdot b_n^{(2)}(X;\mathcal{N}(G)).$$

As a consequence, if Y is a CW-complex and Z is a finite covering of degree d, then

$$b_n^{(2)}(\widetilde{Z}) = d \cdot b_n^{(2)}(\widetilde{Y}).$$

A special class of CW complexes is characterized by the vanishing of all L^2-Betti numbers. Such complexes are called *L^2-acyclic* and include all hyperbolic odd-dimensional closed manifolds and all mapping tori of finite connected CW-complexes (such as fibrations over the circle). In this case, we can obtain finer invariants by twisting with an infinite-dimensional $\mathbb{Z}G$-module.

Definition 3.4. Let X be a G-CW complex and let $\phi\colon G \to \mathbb{Z}$ be a character. Then G acts on the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$ via $g \cdot p \coloneqq t^{\phi(g)}p$, and we can define the $\mathbb{Z}G$-chain complex

$$\overline{C}_*(X) \coloneqq C_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$$

with the diagonal action $g \cdot (\sigma \otimes p) \coloneqq g \cdot \sigma \otimes g \cdot p$.

The *twisted L^2-Betti numbers* of X are given by

$$b_n^{(2)}(X; \mathcal{N}(G), \phi) \coloneqq \dim_{\mathcal{N}(G)} H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} \overline{C}_*(X)).$$

Definition 3.5. A G-CW complex X is said to be *ϕ-L^2-finite* if

$$\sum_{n \geq 0} b_n^{(2)}(X; \mathcal{N}(G), \phi) < +\infty.$$

If this is the case, we call

$$\chi^{(2)}(X; \mathcal{N}(G), \phi) \coloneqq \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(X; \mathcal{N}(G), \phi)$$

the *ϕ-twisted L^2-Euler characteristic* of X.

These invariants also have desirable properties, as evidenced by the following results of Friedl-Lück:

Proposition 3.6 ([50, Theorem 2.5, Lemma 2.6])**.** *Let X be a G-CW complex and let $\phi\colon G \to \mathbb{Z}$. Then:*

(1) Let $i\colon H \to G$ be the inclusion of a subgroup of finite index; then the H-CW complex i^*X is $(\phi \circ i)$-L^2-finite if and only if X is ϕ-L^2-finite, and if this is the case, then

$$\chi^{(2)}(i^*X; \mathcal{N}(H), \phi \circ i) = [G : H] \cdot \chi^{(2)}(X; \mathcal{N}(G), \phi);$$

(2) For every integer $k \geq 1$, X is ϕ-L^2-finite if and only if it is $(k\phi)$-L^2-finite, and if this is the case, then

$$\chi^{(2)}(X; \mathcal{N}(G), k\phi) = k \cdot \chi^{(2)}(X; \mathcal{N}(G), \phi);$$

(3) Suppose that ϕ is the trivial character; then X is ϕ-L^2-finite if and only if we have $b_n^{(2)}(X; \mathcal{N}(G)) = 0$ for all $n \geq 0$. If this is the case, then

$$\chi^{(2)}(X; \mathcal{N}(G), 0) = 0.$$

(4) If ϕ is primitive, i.e. surjective, and $i\colon K \to G$ is the inclusion of its kernel, then X is ϕ-L^2-finite if and only if $b_n^{(2)}(i^*X;\mathcal{N}(K)) < \infty$ for all $n \in \mathbb{N}$. If this is the case, then

$$b_n^{(2)}(X;\mathcal{N}(G),\phi) = b_n^{(2)}(i^*X;\mathcal{N}(K)).$$

This invariant is very valuable from a geometric perspective. Just like the untwisted case, Proposition 3.6 (1) gives multiplicativity with respect to finite coverings; however, (4) gives us control on *infinite cyclic coverings*. If Y is a finite connected CW complex and X is its universal cover G-CW complex, then the i^*X appearing in the statement is the K-CW complex associated to $\overline{Y}_\infty$, the $\mathbb{Z}$-covering of Y defined by ϕ. Whenever ϕ is *fibered*, that is, induced by a fibration $Y \twoheadrightarrow S^1$ via the π_1 functor, this covering deformation retracts onto the fiber F. Therefore, we have

$$\chi^{(2)}(\widetilde{Y};\phi) = \chi^{(2)}(\widetilde{\overline{Y}}_\infty) = \chi^{(2)}(\widetilde{F}) = \chi(F). \tag{3.1}$$

This is reminiscent of the *Thurston norm*, which equals the negative of the fiber Euler characteristic for fibered classes of a 3-manifold. Indeed, we have:

Proposition 3.7 ([50, Theorem 0.2])**.** *Let $M \neq S^2 \times D^1$ be a compact, connected, orientable, irreducible 3-manifold, whose boundary is a union of zero or more tori and whose fundamental group is infinite. Then, for any $\phi \in H^1(M;\mathbb{Z})$,*

$$-\chi^{(2)}(\widetilde{M};\phi) = x_M(\phi),$$

where x_M is the Thurston seminorm of M.

Hence, the twisted L^2-Euler characteristic is a natural extension of the Thurston norm to spaces that are more general than 3-manifolds.

The condition of ϕ-L^2-finiteness, necessary for its definition, may seem daunting at first, but it can be elegantly dealt with by assuming the so-called *Atiyah conjecture* for the fundamental group G. This will also ensure that the twisted L^2-Euler characteristic is especially well behaved.

3.1.1 The Atiyah conjecture

Let G be a torsion-free group; examples include, notably, all hyperbolic manifold groups.

Definition 3.8 (Atiyah conjecture)**.** We say that G satisfies the *Atiyah conjecture* if, for any matrix $A \in \mathrm{M}(m,n,\mathbb{Q}G)$, acting by right multiplication as a map

$$r_A\colon (\mathcal{N}(G))^m \to (\mathcal{N}(G))^n,$$

the von Neumann dimension $\dim_{\mathcal{N}(G)} \ker(r_A)$ is an integer.

This conjecture holds for a rather large class of torsion-free groups, containing all torsion-free fundamental groups of 3-manifolds [73] and closed under taking subgroups [50, Theorem 3.2 (1)]. At the time of writing this thesis, there are no known counterexamples to the Atiyah conjecture among torsion-free groups; otherwise, see [57].

This assumption opens a road to the computation of L^2-invariants by replacing the von Neumann algebra with the slightly less elusive *Linnell skew field*.

Definition 3.9. Since $\mathcal{N}(G)$ satisfies the Ore condition with respect to its set of non-zero-divisors [89, Theorem 8.22 (1)], we can define its Ore localization $\mathcal{U}(G)$. The *Linnell skew field* $\mathcal{D}(G)$ is defined as the smallest subring of $\mathcal{U}(G)$ that contains $\mathbb{Q}G$ and is division closed, i.e. it contains inverses of all the $\mathcal{U}(G)$-units in it.

The following result by Friedl-Lück expands on the ideas we introduced earlier.

Theorem 3.10 ([50, Theorem 3.8]). *Let G be a torsion-free group that satisfies the Atiyah conjecture and let C_* be a chain complex of projective $\mathbb{Q}G$-modules. Then:*

(1) The ring $\mathcal{D}(G)$ is a skew field.

(2) For all $n \geq 0$,

$$b_n^{(2)}(C_*; \mathcal{N}(G)) = \dim_{\mathcal{D}(G)} H_n(\mathcal{D}(G) \otimes_{\mathbb{Q}G} C_*),$$

which is either a natural number or $+\infty$.

(3) Let $\phi\colon G \to \mathbb{Z}$ be surjective with kernel K. Then G can be identified with the semidirect product $K \rtimes_c \mathbb{Z}$, where the $\mathbb{Z}$-factor is generated by any element u such that $\phi(u) = 1$ acting on K via the conjugation c. The character ϕ becomes the canonical projection $G = K \rtimes_c \mathbb{Z} \to \mathbb{Z}$.

The conjugation induces an automorphism $t \in \operatorname{Aut}(\mathcal{D}(K))$, enabling us to define the skew Laurent polynomial ring $\mathcal{D}(K)_t[u^{\pm 1}]$, *with multiplication defined on monomials as $au^i \cdot bu^j := at^{-i}(b)u^{i+j}$.*

Then, this ring is a (noncommutative) principal ideal domain and it satisfies the Ore condition with respect to its subset T of non-zero elements. Moreover, there is a canonical isomorphism of skew fields

$$T^{-1}\mathcal{D}(K)_t[u^{\pm 1}] \xrightarrow{\sim} \mathcal{D}(G).$$

(4) Let $\phi\colon G \to \mathbb{Z}$ be surjective with kernel K and let $i : K \hookrightarrow G$ be the inclusion. Suppose that C_* is finitely generated and $b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*) = 0$ for all $n \geq 0$.

Consider the $\mathbb{Q}K$-chain complex i^*C_* obtained by restriction. Then, for all $n \geq 0$:

- *the modules $H_n(\mathcal{D}(K) \otimes_{\mathbb{Q}K} i^*C_*)$ and $H_n(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Q}G} C_*)$ are finitely generated free over $\mathcal{D}(K)$;*

- *the L^2-Betti number $b_n^{(2)}(i^*C_*;\mathcal{N}(K))$ is finite and, in particular,*

$$\begin{aligned} b_n^{(2)}(i^*C_*;\mathcal{N}(K)) &= \dim_{\mathcal{D}(K)}(H_n(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Q}G} C_*)) \\ &= \dim_{\mathcal{D}(K)}(H_n(\mathcal{D}(K) \otimes_{\mathbb{Q}K} i^*C_*)). \end{aligned}$$

As anticipated, this theorem simplifies the treatment of the twisted L^2-Euler characteristic. Indeed, if X is an L^2-acyclic finite connected CW complex, $G \coloneqq \pi_1(X)$ satisfies the Atiyah conjecture and ϕ is a primitive character, then all the ϕ-twisted L^2-Betti numbers are integers, being dimensions of vector spaces over skew fields. By homogeneity, this also holds for non-primitive ϕ; therefore, the twisted L^2-Euler characteristic $\chi^{(2)}(\widetilde{X};\mathcal{N}(G),\phi)$ is always an integer.

For this reason, we will assume that the Atiyah conjecture is satisfied for all the groups we consider in the remainder of this chapter.

3.2 The polytope homomorphism

Having translated all our machinery in the language of skew fields, we will see how to unify all the twisted L^2-Euler characteristics for every character ϕ into a single object, by leveraging the *Dieudonné determinant* for skew field square matrices.

Consider a finitely generated torsion-free group G with a free abelian quotient $\nu\colon G \twoheadrightarrow H$.

Definition 3.11. We call *integral polytopes* in $H \otimes \mathbb{R}$ the subsets obtained by taking convex hulls of finitely many points of H. Such polytopes form a commutative, cancellative monoid $\mathfrak{P}(H)$ under the Minkowski sum operation; we define the *integral polytope group* $\mathcal{P}(H)$ to be the Grothendieck group of $\mathfrak{P}(H)$.

Elements of $\mathcal{P}(H)$ can be seen as formal differences of two polytopes, where $P - Q = P' - Q'$ if and only if $P + Q' = P' + Q$ in $\mathfrak{P}(H)$. Moreover, since $\mathfrak{P}(H)$ is cancellative, it is actually embedded in its Grothendieck group $\mathcal{P}(H)$ as the subset consisting of those differences that can be written in the form $P - 0$.

The next step is a multidimensional generalization of Theorem 3.10 (3), obtained with ν in place of ϕ:

Proposition 3.12 ([50, (6.1)])**.** *Let $\nu\colon G \twoheadrightarrow H$ be a free abelian quotient as before, and let K be its kernel. Then the crossed product ring $\mathcal{D}(K) * H$ admits an Ore localization with respect to the set T of its non-zero elements, and there is an isomorphism*

$$T^{-1}(\mathcal{D}(K) * H) \simeq \mathcal{D}(G)$$

of $\mathcal{D}(K)$-modules.

Remark 3.13. For our purposes, the above *crossed product* can be thought of as a *multivariate skew polynomial ring* $\mathcal{D}(K)[u_1^{\pm 1}, \dots, u_r^{\pm 1}]$, where $\{u_1, \dots, u_r\} \subset G$ is a (fixed) lift of a basis for H. Elements are sums of monomials of shape $ku_1^{\alpha_1}\dots u_r^{\alpha_r}$, or ku^α (with $\alpha \in H$) using multi-index notation. Multiplication is naturally defined as

$$ku^\alpha \cdot hu^\beta := k(u^\alpha h u^{-\alpha})\tau(\alpha,\beta)u^{\alpha+\beta}, \tag{3.2}$$

where $\tau(\alpha,\beta) := u^\alpha u^\beta (u^{\alpha+\beta})^{-1}$ can be written as a product of commutators of G, and hence belongs to K (and $\mathcal{D}(K)$). Just like in Theorem 3.10 (3), since u^α is an element of G, it induces an automorphism of $\mathcal{D}(K)$ by conjugation, making the above definition sound. For a formal, general definition of crossed product, see e.g. [89, Section 10.3.2].

Owing to this, we will make extensive use of a polynomial-like notation

$$q(u_1,\dots,u_r)^{-1}p(u_1,\dots,u_r) \tag{3.3}$$

for elements of the Linnell skew field $\mathcal{D}(G)$.

Every non-zero polynomial $p \in \mathcal{D}(K)[u_1^{\pm1},\dots,u_r^{\pm1}]$ has a *noncommutative Newton polytope* $P(p)$, given by the convex hull in $H \otimes \mathbb{R}$ of all exponents α of its non-zero monomials. It can be proved [50, Section 6.1] that P is independent of the choice of lifts $u_1,\dots,u_n$ and sends products to Minkowski sums, making it a monoid homomorphism

$$P\colon (\mathcal{D}(K)[u_1^{\pm1},\dots,u_r^{\pm1}] \setminus \{0\}, \cdot) \to \mathfrak{P}(H). \tag{3.4}$$

By localizing, this can be augmented to a group homomorphism

$$P_\nu\colon \mathcal{D}(G)^\times_{\mathrm{ab}} \to \mathcal{P}(H), \quad P_\nu(q^{-1}p) := P(p) - P(q), \tag{3.5}$$

where we can abelianize the domain since the codomain is abelian.

Consider now a primitive character ϕ containing K in its kernel; clearly, ϕ factors as $\omega \circ \nu$ for some $\omega\colon H \twoheadrightarrow \mathbb{Z}$. Any $p \in \mathcal{D}(K)[u_1^{\pm1},\dots,u_r^{\pm1}]$ can, of course, be seen as a single-variable skew Laurent polynomial in $\mathcal{D}(\ker\phi)[v^{\pm1}]$ whose *degree* (the difference between the greatest and the least exponents of v in p) can be directly read off from $P(p)$. In fact, every multivariate monomial ku^α contributes to the monomial $v^{\omega(\alpha)}$, and there can be no simplifications between terms with the same $\omega(\alpha)$, as they are still linearly independent over $\mathcal{D}(K)$.

Therefore, the degree $\deg_\phi p$ of p, as a polynomial over $\mathcal{D}(\ker\phi)$, is simply the *thickness* of $P(p)$ along the ω direction. Since $P(p)$ is convex and compact, its thickness function is an integer-valued *seminorm* on $\mathrm{Hom}(H,\mathbb{Z})$, which can be extended by homogeneity and continuity to a full-fledged seminorm on $\mathrm{Hom}(H,\mathbb{R})$.

We can also define the degree of a rational element $q^{-1}p \in \mathcal{D}(G)^\times$ as $\deg_\phi p - \deg_\phi q$. With an analogous reasoning, we get that the degree is the *difference* of the two seminorms associated to $P(p)$ and $P(q)$.

Remark 3.14. As the degree is a homomorphism to $\mathbb{Z}$, it is in fact well defined for elements of $\mathcal{D}(G)^\times_{\mathrm{ab}}$.

Remark 3.15. The degree is independent of the quotient ν. In fact, it is easy to see that taking a smaller quotient corresponds to restricting the seminorm to a subspace or to projecting the polytope onto a subspace. Therefore, we will generally consider $\nu = \mathrm{ab}\colon G \twoheadrightarrow \mathrm{ab}(G)$, the largest free abelian quotient.

3.2.1 The Dieudonné determinant

The determinant is a natural invariant of square matrices over commutative rings. Apart from its well-known desirable properties, such as multilinearity and multiplicativity, it can also capture some fine-grained information about endomorphisms over PIDs. As an example:

Proposition 3.16. *We have the following:*

- *if* $A \in \mathrm{M}(n, \mathbb{Z})$ *is invertible over* $\mathbb{Q}$*, then* $\operatorname{coker} A$ *has cardinality* $|\det A|$*;*
- *if* $A \in \mathrm{M}(n, \mathbb{K}[x])$ *is invertible over* $\mathbb{K}(x)$*, then* $\operatorname{coker} A$ *has* $\mathbb{K}$*-dimension* $\deg \det A$*.*

Both facts follow easily from the Smith normal form and can be generalized extensively.

In the context of skew fields, noncommutativity makes it impossible to define a determinant with the sum-of-permutations approach, as multiplicativity would fail. A better definition starts with the *Bruhat decomposition* of a matrix:

Proposition 3.17 (Bruhat decomposition, [32, Theorem 9.2.2])**.** *Let A be an invertible square matrix over a skew field F. Then there is a decomposition $A = LDPU$, where L and U are lower and upper unitriangular, D is diagonal and P is a permutation matrix. Moreover, D and P are uniquely determined.*

Now we can define the *Dieudonné determinant*

$$\det A := \operatorname{sign}(P) \cdot d_1 \dots d_n \mod [F^\times, F^\times] \in F^\times_{\mathrm{ab}}, \tag{3.6}$$

where $\operatorname{sign}(P)$ is the sign of the corresponding permutation, and $D = \operatorname{diag}(d_1, \dots, d_n)$.

The Dieudonné determinant is the unique map $\det\colon \mathrm{GL}(n, F) \to F^\times_{\mathrm{ab}}$ such that:

(1) $\det AB = \det A \det B$ for $A, B \in \mathrm{GL}(n, F)$;

(2) if $E = I_n + cE_{ij}$ with $i \neq j$ is an elementary matrix, then $\det E = 1$;

(3) $\det \operatorname{diag}(d_1, \dots, d_n) = d_1 \dots d_n \mod [F^\times, F^\times]$.

Of course, these properties (see [100, Section 3.1]) are enabled by descending to the coarser group $F^\times_{\mathrm{ab}}$.

Let us now retrace our steps by asking if there is a noncommutative version of Proposition 3.16. Indeed, we can consider the determinant of a suitable matrix A, apply the degree function we defined earlier and relate the result to the cokernel of A.

Proposition 3.18. *Given a free abelian quotient $\nu\colon G \twoheadrightarrow H$ with kernel K, let A be a square matrix with coefficients in $\mathcal{D}(K)[u_1^{\pm 1}, \dots, u_r^{\pm 1}]$, invertible over $\mathcal{D}(G)$. Let $\phi = \omega \circ \nu$ be a primitive character with kernel K_ϕ. Then*

$$\dim_{\mathcal{D}(K_\phi)} \operatorname{coker} A = \deg_\phi \det A.$$

Proof. This is essentially proved in [50, Lemma 6.16]. The main idea is to consider A as a matrix over $\mathcal{D}(K_\phi)[v^{\pm 1}]$ (a noncommutative PID), then bring it into the Smith normal form with elementary moves, just like in the commutative case. □

Remark 3.19. This argument also proves that $\det A$ has a representative in $\mathcal{D}(K_\phi)[v^{\pm 1}]$: the Dieudonné determinant is the product of the diagonal entries in the Smith normal form, up to left and right multiplication by monomials that accumulate during the diagonalization process.

Comparing Proposition 3.18 with Theorem 3.10, we get a tentative first step toward the computation of the twisted L^2-Euler characteristic. Indeed, let $H = \mathrm{ab}(G) \simeq \mathbb{Z}^r$ and $K = \ker \mathrm{ab}$ and let C_* be the following $\mathbb{Z}G$-chain complex:

$$\ldots \longrightarrow 0 \longrightarrow C_1 \xrightarrow{A} C_0 \longrightarrow 0 \tag{3.7}$$

with $A \in \mathrm{M}(n, \mathbb{Z}G)$ invertible over $\mathcal{D}(G)$. If ϕ is a primitive character, we can consider the twisted L^2-Betti numbers

$$\begin{aligned} b_0^{(2)}(C_*; \mathcal{N}(G), \phi) &= \dim_{\mathcal{D}(K_\phi)} H_0(\mathcal{D}(K_\phi)[v^{\pm}] \otimes_{\mathbb{Q}G} C_*) && (3.8)\\ &= \dim_{\mathcal{D}(K_\phi)} \mathrm{coker}(\mathcal{D}(K_\phi)[v^{\pm}] \otimes_{\mathbb{Q}G} A) && (3.9)\\ b_1^{(2)}(C_*; \mathcal{N}(G), \phi) &= \dim_{\mathcal{D}(K_\phi)} H_1(\mathcal{D}(K_\phi)[v^{\pm}] \otimes_{\mathbb{Q}G} C_*) && (3.10)\\ &= \dim_{\mathcal{D}(K_\phi)} \ker(\mathcal{D}(K_\phi)[v^{\pm}] \otimes_{\mathbb{Q}G} A) && (3.11) \end{aligned}$$

Since A is injective over $\mathcal{D}(G)$, it must be so over $\mathcal{D}(K_\phi)[v^{\pm}]$, so $b_1^{(2)}(C_*; \mathcal{N}(G), \phi) = 0$. It follows that

$$\chi^{(2)}(C_*; \mathcal{N}(G), \phi) = b_0^{(2)}(C_*; \mathcal{N}(G), \phi) = \deg_\phi \det A. \tag{3.12}$$

Therefore, in this toy example, we see that the twisted L^2-Euler characteristic is a rather concrete, combinatorial object, being the difference of two polyhedral seminorms; when extended to the real vector space $\mathrm{Hom}(G, \mathbb{R})$, it is also a continuous function. In the following section, we will see that this reasoning can be extended to *all* L^2-acyclic free $\mathbb{Z}G$-chain complexes.

3.3 Universal L^2-torsion

Our aim for this section is to define some sort of matrix-valued invariant for $\mathbb{Z}G$-chain complexes in order to carry out the above plan; again, G will be a finitely generated torsion-free group. We will mostly follow Friedl and Luck's article [49].

Definition 3.20. An endomorphism $A \in \mathrm{M}(n, \mathbb{Z}G)$ is called a *weak isomorphism* when

$$\mathrm{id}_{\mathcal{N}(G)} \otimes_{\mathbb{Z}G} A \colon \ell^2(G)^n \to \ell^2(G)^n$$

is an injective operator with dense image.

Remark 3.21. Since we are taking G to satisfy the Atiyah conjecture, the property of being a weak isomorphism can be replaced by the more intuitive *invertibility over* $\mathcal{D}(G)$. This is ultimately a consequence of [89, Theorem 6.24] (see also [49, Lemma 1.21], where it is applied).

Definition 3.22. The *weak* K_1 *group* $K_1^w(\mathbb{Z}G)$ is the abelian group generated by all weak isomorphisms (with coefficients in $\mathbb{Z}G$), subject to the relations:

(1) $g \circ f = f + g$ for all f, g compatible weak isomorphisms;

(2) $\begin{bmatrix} f & h \\ 0 & g \end{bmatrix} = f + g$ for all f, g weak isomorphisms and h compatible $\mathbb{Z}G$-matrices.

Definition 3.23. The *weak Whitehead group* $\mathrm{Wh}^w(\mathbb{Z}G)$ is the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup generated by all the 1×1 matrices $\begin{bmatrix} \pm g \end{bmatrix}$ for $g \in G$. We also define a finer quotient $\widetilde{K}_1^w(\mathbb{Z}G) \coloneqq K_1^w(\mathbb{Z}G)/\langle -\mathrm{id}_{\mathbb{Z}G} \rangle$.

Definition 3.24. Let C_* be a $\mathbb{Z}G$-chain complex with differentials c_*. A *weak chain contraction* for C_* is a pair (u_*, γ_*), where:

- $u_n \colon C_n \to C_n$ is a weak isomorphism for all n;
- $\gamma_n \colon C_n \to C_{n+1}$ is a null homotopy, that is, $\gamma_n \circ u_n = u_{n+1} \circ \gamma_n$.

In the case of a free $\mathbb{Z}G$-chain complex C_*, the differentials c_n can be thought of as $\mathbb{Z}G$-matrices acting by right multiplication, and have *adjoints* c_n^* induced by transposition and by the $\mathbb{Z}$-linear self-map of $\mathbb{Z}G$ sending g to g^{-1} (called the *antipode* or *coinverse* in the context of Hopf algebras). We can employ this construction to define the *combinatorial Laplacian*, taking inspiration from the Laplacian of the de Rham complex:

$$\Delta_n \coloneqq c_n^* c_n + c_{n+1} c_{n+1}^* \colon C_n \to C_n. \tag{3.13}$$

It is easy to verify that c_{*+1} is a null homotopy for Δ_*. Indeed, since $c_* c_{*+1} = 0$, we have

$$\begin{aligned} \Delta_n c_{n+1} &= (c_n^* c_n + c_{n+1} c_{n+1}^*) c_{n+1} && (3.14) \\ &= c_{n+1} c_{n+1}^* c_{n+1} && (3.15) \\ &= c_{n+1} (c_{n+1}^* c_{n+1} + c_{n+2} c_{n+2}^*) && (3.16) \\ &= c_{n+1} \Delta_{n+1}. && (3.17) \end{aligned}$$

Hence, it is natural to ask if the combinatorial Laplacian is a weak isomorphism, making (Δ_*, c_{*+1}) a weak chain contraction. By [49, Lemma 1.5], this is the case if and only if C_* is L^2-acyclic.

Now, let C_* be a finitely generated free L^2-acyclic $\mathbb{Z}G$-chain complex such that $C_n \neq 0$ for finitely many n, and fix an unordered $\mathbb{Z}G$-basis for each C_n (we will say that C_* is *finite based free L^2-acyclic*). If we define

$$C_{\text{even}} \coloneqq \bigoplus_{2|n} C_n, \quad C_{\text{odd}} \coloneqq \bigoplus_{2\nmid n} C_n, \tag{3.18}$$

the two modules inherit unordered bases, which are of the same cardinality by the L^2-acyclicity of C_*: let $b\colon C_{\text{even}} \to C_{\text{odd}}$ be an isomorphism induced by a bijection of the two bases. If $f\colon C_{\text{odd}} \to C_{\text{even}}$ is a $\mathbb{Z}G$-homomorphism, then $[b \circ f]$ is a well-defined class in $\widetilde{K}_1^w(\mathbb{Z}G)$, independent of the choice of bijection. Of course, it is necessary to quotient by the subgroup $\langle -\mathrm{id}_{\mathbb{Z}G}\rangle$ because the bases are unordered.

Now we can define the *universal L^2-torsion* of C_* as

$$\rho_u^{(2)}(C_*) \coloneqq [(b \circ (\Delta c + c^*))] - [\Delta] \in \widetilde{K}_1^w(\mathbb{Z}G), \tag{3.19}$$

where $\Delta c + c^*$ and Δ are seen respectively as maps $C_{\text{odd}} \to C_{\text{even}}$ and $C_{\text{odd}} \to C_{\text{odd}}$.

On a more practical note, the interplay between even and odd degrees in the first term translates to large, unwieldy $\mathbb{Z}G$-matrices. Luckily, it is possible to express this invariant solely in terms of the Laplacians.

Proposition 3.25 ([49, Lemma 1.17])**.** *Let C_* be a finite based free L^2-acyclic $\mathbb{Z}G$-chain complex. Then its combinatorial Laplacians Δ_n are weak isomorphisms and*

$$\rho_u^{(2)}(C_*) + *(\rho_u^{(2)}(C_*)) = -\sum_{n\geq 0}(-1)^n \cdot n \cdot [\Delta_n],$$

where $*\colon \widetilde{K}_1^w(\mathbb{Z}G) \to \widetilde{K}_1^w(\mathbb{Z}G)$ is the involution induced by adjunction of $\mathbb{Z}G$-matrices.

The importance of this invariant stems from a universal property that makes it, in a sense, the *most general* invariant for the chain complexes of our interest.

Definition 3.26. We say that a short exact sequence $0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$ of finite based free $\mathbb{Z}G$-modules, with bases B_0, B_1, B_2, is *based exact* if $i(B_0) \subseteq B_1$ and p induces a bijection between $B_1 \setminus i(B_0)$ and B_2. Analogously, we define *short based exact sequences* of finite based free $\mathbb{Z}G$-chain complexes.

Definition 3.27. An *additive L^2-torsion invariant* is a map defined on all finite based free L^2-acyclic $\mathbb{Z}G$-chain complexes, with values in an abelian group, such that:

- if $0 \to C_* \to D_* \to E_* \to 0$ is short based exact, then $a(D_*) = a(C_*) + a(E_*)$;
- $a(\ldots \to 0 \to \mathbb{Z}G \xrightarrow{\pm\mathrm{id}} \mathbb{Z}G \to 0) = 0$.

Theorem 3.28 ([49, Remark 1.16]). *The universal L^2-torsion $\rho_u^{(2)}$ is a $\widetilde{K}_1^w(\mathbb{Z}G)$-valued additive L^2-torsion invariant. Moreover, any additive L^2-torsion invariant uniquely factors through $\rho_u^{(2)}$.*

In fact, the twisted L^2-Euler characteristic is also an additive L^2-torsion invariant with values in the group of set-maps $\mathrm{Hom}(G, \mathbb{Z}) \to \mathbb{Z}$. Therefore, we can expect to be able to write it in terms of $\rho_u^{(2)}$, possibly with the help of the Dieudonné determinant.

Lemma 3.29. *The Dieudonné determinant* $\det\colon \mathrm{GL}(n, \mathcal{D}(G)) \to \mathcal{D}(G)^\times_{\mathrm{ab}}$ *extends to the weak K_1 group as a map*

$$\det\colon K_1^w(\mathbb{Z}G) \to \mathcal{D}(G)^\times_{\mathrm{ab}}.$$

Proof. By Remark 3.21, the weak K_1 group is generated by $\mathcal{D}(G)$-invertible matrices, which admit a Dieudonné determinant. We can extend det $\mathbb{Z}$-linearly, provided that the relations in Definition 3.22 are respected. Relation (1) amounts to multiplicativity of the determinant, while (2) requires that

$$\det \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \det A \det B. \tag{3.20}$$

This is a well-known property of the Dieudonné determinant: to prove it, one can nullify C with elementary moves, exploiting invertibility of A or B, and then take the Bruhat normal form. □

The universal L^2-torsion is not quite an element of the weak K_1 group, being defined up to sign, but we still obtain an invariant

$$\det \rho_u^{(2)}(C_*) \in \mathcal{D}(G)^\times_{\mathrm{ab}}/\{\pm 1\}. \tag{3.21}$$

Since the sign is inconsequential in the definition of the noncommutative Newton polytope, we can define the L^2*-torsion polytope*

$$P(C_*; G) := P_{\mathrm{ab}}(\det \rho_u^{(2)}(C_*)) \in \mathcal{P}(\mathrm{ab}(G)). \tag{3.22}$$

3.3.1 What about manifolds?

Up to now, this section has remained relatively abstract, focusing on chain complexes instead of G-CW complexes. Given a finite free L^2-acyclic G-CW complex X, the main hurdle is that the cellular $\mathbb{Z}G$-modules $C_n(X)$ do not have a canonical basis. However, if we choose a representative n-cell for each G-orbit, we always get the same basis, up to reordering and multiplying every basis element b_i by some $\pm g_i$ for $g_i \in G$.

As the ambiguity mirrors the definition of the weak Whitehead group, it is not hard to see that there is a well-defined universal L^2-torsion for G-CW complexes:

$$\rho_u^{(2)}(X) := [\rho_u^{(2)}(C_*(X))] \in \mathrm{Wh}^w(\mathbb{Z}G). \tag{3.23}$$

Its determinant is defined up to multiplication by signed elements of G. Such elements are *monomials* in the skew polynomial ring $\mathcal{D}(K)[u_1^{\pm 1}, \dots, u_r^{\pm 1}]$, so they act on Newton polytopes by translation. Accordingly, we will extend the notation for the polytope homomorphism:

$$P_\nu \colon \mathcal{D}(G)^\times_{\mathrm{ab}} / \left\langle [\pm g] \mid g \in G \right\rangle \to \mathcal{P}_T(H), \tag{3.24}$$

where $\nu \colon G \twoheadrightarrow H$ is a free abelian quotient and $\mathcal{P}_T(H)$ is the Grothendieck group of integral polytopes of H *up to translation*. This leads to the *L^2-torsion polytope of a G-CW complex*

$$P(X; G, \nu) \coloneqq P_\nu(\det \rho_u^{(2)}(X)) \in \mathcal{P}_T(H). \tag{3.25}$$

When $\nu = \mathrm{ab} \colon G \twoheadrightarrow \mathrm{ab}(G)$, we will simply write $P(X; G)$.

We will now formalize the polytope thickness function of Section 3.2 as the *seminorm homomorphism*.

Definition 3.30. An integral polytope $P \in \mathfrak{P}(H)$ induces a seminorm on $\mathrm{Hom}(H, \mathbb{Z})$:

$$\|\phi\|_P \coloneqq \max\{\phi(p) - \phi(p') \mid p, p' \in P\}.$$

Definition 3.31. The above construction extends to the integral polytope group as the *seminorm homomorphism*

$$\mathfrak{N} \colon \mathcal{P}(H) \to \{\mathrm{Hom}(G, \mathbb{Z}) \to \mathbb{Z}\}, \quad \mathfrak{N}(P - Q)(\phi) \coloneqq \|\phi\|_P - \|\phi\|_Q.$$

Since $\mathfrak{N}$ is translation invariant, it is well defined as a homomorphism from $\mathcal{P}_T(H)$, also called $\mathfrak{N}$.

Theorem 3.32 (compare [52, Theorem 3.52])**.** *Let X be a finite free L^2-acyclic G-CW complex and let $\nu \colon G \twoheadrightarrow H$ be a free abelian quotient. If ϕ is a primitive character that factors through ν, then X is ϕ-L^2-finite and*

$$\chi^{(2)}(X; \mathcal{N}(G), \phi) = \mathfrak{N}(P_\nu(X; G))(\phi).$$

Like we just said, the seminorm $\|\phi\|_P$ measures the thickness of the polytope P in the direction ϕ. When P is a noncommutative Newton polytope, this amounts to the ϕ-degree of the underlying polynomial. Therefore, we get

$$\chi^{(2)}(X; \mathcal{N}(G), \phi) = \deg_\phi \det \rho_u^{(2)}(X), \tag{3.26}$$

where the degree is, of course, considered as a function on $\mathcal{D}(G)^\times_{\mathrm{ab}} / \left\langle [\pm g] \mid g \in G \right\rangle$.

Recall now Proposition 3.25, where the adjoint of the universal L^2-torsion appears. By [49, Lemma 3.18], the involution $* \colon \widetilde{K}_1^w(\mathbb{Z}G) \to \widetilde{K}_1^w(\mathbb{Z}G)$ corresponds to reflecting polytopes with respect to the origin, and hence has no effect on the associated seminorm. We can summarize this as:

Proposition 3.33. *Let X be a finite free L^2-acyclic G-CW complex and let Δ_n be the combinatorial Laplacians of its $\mathbb{Z}G$-chain complex. Then*

$$\chi^{(2)}(X;\mathcal{N}(G),\phi) = \deg_\phi \det \rho_u^{(2)}(X) = -\frac{1}{2}\sum_{n\geq 0}(-1)^n \cdot n \cdot \deg_\phi \det \Delta_n.$$

If we have an L^2-acyclic compact manifold M, we can take $X = \widetilde{M}$ as a G-CW complex, with $G = \pi_1(M)$. While there is no unique way to do so, all such G-CW structures give $\mathbb{Z}G$-chain homotopic cellular chain complexes, and the L^2-torsion polytope is invariant under $\mathbb{Z}G$-chain homotopies [62]. Hence, we can use Proposition 3.33 to compute this invariant for manifolds using any G-CW structure on $\widetilde{M}$.

3.4 Computing the twisted L^2-Euler characteristic

It is time to tie up loose ends and outline a procedure for the computation of the twisted $\chi^{(2)}$ for universal coverings. In what follows, M will be a finite L^2-acyclic CW complex or, more commonly, a manifold with the homotopy type of one. Examples include:

- closed hyperbolic manifolds of odd dimension [41];
- manifolds that fiber over the circle, such as complements of fibered links.

We will also assume that $G \coloneqq \pi_1(M)$ is residually finite.

The algorithm starts by computing the $\mathbb{Z}G$-chain complex of the universal cover of M. Of course, we also take a character $\phi\colon G \to \mathbb{Z}$ as input; by homogeneity of the twisted L^2-Euler characteristic, we can assume that ϕ is primitive, with kernel K_ϕ.

Next, we compute the Laplacians of the chain complex: indeed, thanks to Proposition 3.33, we only have to find $\deg_\phi \det \Delta_n$ for $n = 0, \ldots, \dim M$. However, as is rightly noted in [49, Remark 3.20], computing the Dieudonné determinant is very hard, due to our lack of a concrete representation for the elements of $\mathcal{D}(G)$ for general G (see, however, [31] for free groups and [69] for locally indicable groups). Fortunately, recent developments by Oki [100] allow us to probe *valuations* of det without computing it fully, and without so much as leaving the ring $\mathbb{Z}G \subset \mathcal{D}(K)[u_1^{\pm 1}, \ldots, u_r^{\pm 1}]$.

3.4.1 Degrees of determinants

Much like commutative fields, skew fields can be equipped with *valuations*:

Definition 3.34. A *valuation* on a skew field F is a map $v\colon F \to \mathbb{R} \cup \{+\infty\}$ such that:

(1) $v(ab) = v(a) + v(b)$ for all $a, b \in F$;

(2) $v(a+b) \geq \min\{v(a), v(b)\}$ for all $a, b \in F$;

(3) $v(1) = 0$;

(4) $v(0) = +\infty$.

It is easy to see that any valuation on F is also well defined on $F_{\mathrm{ab}}^{\times}$. We will restrict our attention to *discrete valuations*, that is, those with image in $\mathbb{Z} \cup \{+\infty\}$. Examples include the p-adic valuations on $\mathbb{Q}$ and the *order* of a skew Laurent polynomial.

Definition 3.35. Let $F = T^{-1}(D_t[u^{\pm 1}])$ be the Ore quotient skew field of a skew Laurent polynomial ring. The *order* $\operatorname{ord}\colon F \to \mathbb{Z} \cup \{+\infty\}$ is defined on a polynomial as the minimum degree of its monomials, and on a rational element $q(u)^{-1}p(u) \in F$ as $\operatorname{ord} p - \operatorname{ord} q$.

The degree of a skew Laurent polynomial (or polynomial fraction) is not a valuation; however, if $x \in D_t[u^{\pm 1}]$ has *symmetric* support, then clearly

$$\operatorname{ord} x = -2 \deg x.$$

In the case $F = \mathcal{D}(G)$, we can generalize this fact via the following technical lemma:

Lemma 3.36. *Let $B \in \mathrm{M}(n, \mathbb{Z}G)$ be self-adjoint and invertible over $\mathcal{D}(G)$, and let $x \in \mathcal{D}(G)^{\times}$ be a representative of the Dieudonné determinant of B. Considering $\mathcal{D}(G) = T^{-1}(\mathcal{D}(K_\phi)[u^{\pm 1}])$ with the order valuation ord_ϕ, we have $\deg_\phi x = -2 \operatorname{ord}_\phi x$.*

Proof. Recall [49, Lemma 3.18], which we applied to obtain Proposition 3.33: adjunction of matrices corresponds to the central symmetry involution $*\colon \mathcal{P}(\mathrm{ab}(G)) \to \mathcal{P}(\mathrm{ab}(G))$.

Let $P_{\mathrm{ab}}(x) = P - Q \in \mathcal{P}(\mathrm{ab}(G))$. Since B is self-adjoint, we have $P - Q = *P - *Q$ or, equivalently, $P + *Q = *(P + *Q)$. The degree and order of x can be read off the integral polytope of x over $\mathbb{Z}$, that is, $P_\phi(x)$. In turn, this is simply obtained by applying to $P - Q$ the linear projection $\pi_\phi\colon \mathrm{ab}(G) \otimes \mathbb{R} \to \mathbb{R}$ induced by ϕ. Such a projection clearly preserves Minkowski sums and the central symmetry involution.

Let $\pi_\phi(P) = [a, b]$ and $\pi_\phi(Q) = [c, d]$. Then

$$\deg_\phi x = (b - a) - (d - c), \quad \operatorname{ord}_\phi x = a - c. \tag{3.27}$$

However, $\pi_\phi(P + *Q) = [a, b] + *[c, d] = [a - d, b - c]$ is centrally symmetric, so $b - c = d - a$. The initial claim follows from a routine verification. □

We can now introduce Oki's *matrix expansion algorithm*, following [100, Sections 6–7]. Given a square $n \times n$ matrix A, with coefficients in $R := D_t[u^{\pm 1}]$ and invertible over F, the algorithm computes $\operatorname{ord} \det A$ as the D-rank of a certain matrix A' with entries in the coefficient skew field D. Moreover, the entries of A' are obtained from the coefficients of the entries of A using only ring operations in D and the automorphism $t \in \mathrm{Aut}(D)$. In the case $F = \mathcal{D}(G)$, if A has entries in $\mathbb{Z}G$, then we can see them as skew polynomials with coefficients in $\mathbb{Z}[K_\phi]$. Hence, the matrix A' has entries in $\mathbb{Z}[K_\phi]$ and can be constructed with ring operations in $\mathbb{Z}G$. (See Section 3.5.2 for more details on this algorithm.)

As a consequence, we can apply Lemma 3.36 and Oki's matrix expansion algorithm to obtain the degree of the determinant of each Laplacian; this step reduces our problem to computing the rank of a square $\mathbb{Z}[K_\phi]$-matrix A over $\mathcal{D}(K_\phi)$. It is worth noting that K_ϕ is usually not finitely generated, but its subgroup S generated by the elements appearing in A is.

3.4.2 Lück's approximation theorem

At this point, it is necessary to leave the familiar skew field realm and return to von Neumann algebras. Indeed, Theorem 3.10 (2) applied to the chain complex

$$0 \longrightarrow \mathbb{Z}K_\phi^N \xrightarrow{A} \mathbb{Z}K_\phi^N \longrightarrow 0$$

easily implies

$$\mathrm{rk}_{\mathcal{D}(K_\phi)} A = \dim_{\mathcal{N}(K_\phi)} \mathrm{im}(r_A), \tag{3.28}$$

where $r_A\colon \mathcal{N}(K_\phi)^N \to \mathcal{N}(K_\phi)^N$ is right multiplication by A. By [89, Theorem 6.29 (2)], we can replace $\mathcal{N}(K_\phi)$ with $\mathcal{N}(S)$: the induction functor $\mathcal{N}(K_\phi) \otimes_{\mathcal{N}(S)} -$ is right exact and preserves the cokernel of r_A and its dimension, from which the rank can be computed via subtraction from N. Now we can present the titular result of this section.

Theorem 3.37 ([88, Section 2])**.** *Let Γ be a countable residually finite group and let A be any matrix over $\mathbb{Z}\Gamma$. Let $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \Gamma_2 \geq \dots$ be a residual chain of Γ, that is, a sequence of finite index normal subgroups with trivial intersection.*

If, for $k \geq 0$, $A_k \in \mathrm{M}(n, \mathbb{Z}[\Gamma/\Gamma_k])$ is the image of A under the canonical projection, then we have

$$\dim_{\mathcal{N}(\Gamma)} \mathrm{im}(r_A) = \lim_{k\to+\infty} \dim_{\mathcal{N}(\Gamma/\Gamma_k)} \mathrm{im}(r_{A_k}).$$

Since the groups Γ/Γ_k on the right hand side are finite, the von Neumann algebras are just complex group algebras $\mathbb{C}[\Gamma/\Gamma_k]$. Hence, every A_k is but an endomorphism acting on a complex vector space of dimension $n \cdot [\Gamma : \Gamma_k]$, whose von Neumann rank is given by

$$\dim_{\mathcal{N}(\Gamma/\Gamma_k)} \mathrm{im}(r_{A_k}) = \frac{1}{[\Gamma : \Gamma_k]} \cdot \dim_{\mathbb{C}} \mathrm{im}(r_{A_k}), \tag{3.29}$$

as a consequence of [89, Theorem 6.29, (2)] applied to the inclusion $1 \hookrightarrow \Gamma/\Gamma_k$.

In our case, the matrices A_k will have integer coefficients, so they can be represented exactly without rounding errors.

Remark 3.38. There is a simple (but extremely inefficient) algorithm that computes a residual chain: enumerate all homomorphisms from Γ to larger and larger symmetric groups; since Γ is residually finite and every finite quotient appears as a subgroup of a symmetric group, the kernels of these homomorphisms will intersect trivially.

3.4.3 Estimating the error

The statement of Theorem 3.37, as written, does not provide an effective rate of convergence, but there is a way to make Lück's theorem quantitative, as proved by Löh and Uschold:

Definition 3.39 ([82, Definition 6.3])**.** Let Γ be a countable residually finite group and let $A \in \mathrm{M}(n, \mathbb{Z}\Gamma)$ be self-adjoint. We say that a sequence $(\Gamma_k)_{k\in\mathbb{N}}$ of finite index normal subgroups of Γ is *adapted to* A if for all $k \geq 0$, all diagonal entries of $A^0, A^1, \dots, A^{k^2}$ have support in $\Gamma \setminus \Gamma_k \cup \{e\}$.

Proposition 3.40 (compare [82, Proposition 6.6])**.** *Let* $\Gamma, (\Gamma_k)_k, A$ *be as in Definition 3.39 and let* A_k *be the image of* A *in* $\mathrm{M}(n, \mathbb{Z}[\Gamma/\Gamma_k])$. *We have the estimate*

$$|\dim_{\mathcal{N}(\Gamma)} \mathrm{im}(r_A) - \dim_{\mathcal{N}(\Gamma/\Gamma_k)} \mathrm{im}(r_{A_k})| \leq n \cdot \left[\left(1 - \frac{1}{kd}\right)^{k^2} + \frac{\log d}{\log k}\right],$$

where d *is the operator norm of* $r_A \colon \ell^2(\Gamma)^n \to \ell^2(\Gamma)^n$.

Following [89, p. 194], we can bound the operator norm by $n\sqrt{2} \cdot \max_{i,j} \|A_{ij}\|_1$, where $\|\sum_{g\in\Gamma} \lambda_g g\|_1 \coloneqq \sum_{g\in\Gamma} |\lambda_g|$ is the ℓ^1-norm on $\mathbb{Z}\Gamma$. If Γ satisfies the Atiyah conjecture, as our group S does, then the von Neumann rank over Γ is an integer. To compute it, we choose k such that the right hand side in Proposition 3.40 is less than $1/2$, and then round the von Neumann rank over Γ/Γ_k, computed as in (3.29), to the nearest integer.

An adapted sequence can be computed provided that we have a solution for the word problem in Γ (that is, S): see the proof of [82, Lemma 6.4]. Since we represent elements of S as elements of G, which is finitely presented and residually finite, the proof goes through in the same way, using the algorithm in [96, Theorem 5.3]. The latter consists of a parallel enumeration of relations of G and homomorphisms to symmetric groups, in order to eventually certify that two elements are respectively equal or different; clearly, this is of theoretical interest only, owing to the overwhelming computational complexity. Moreover, the bound in Proposition 3.40 is unfortunately much too loose, because of the term $1/\log k$. Again, this is no obstacle to experimentation.

Remark 3.41. The requirement that A be self-adjoint in Proposition 3.40 is not generally met by our expanded combinatorial Laplacians. However, as noted in [82, Remark 2.2], we may replace r_A by r_{AA^*} without changing the kernel as a module over $\mathcal{N}(\Gamma)$, for the same reason as in more elementary contexts: if $f \in \mathcal{N}(\Gamma)$, then $ff^* = 0 \implies f = 0$. Here we must point out that the *adjoint* induced by the antipode of $\mathbb{Z}\Gamma$ is none other than the Hermitian adjoint for operators on $\ell^2(\Gamma)$.

In conclusion, in order to compute the twisted L^2-Euler characteristic, we construct the $\mathbb{Z}G$-chain complex of the universal cover and the combinatorial Laplacians, in order to apply Proposition 3.33. By Lemma 3.36 it suffices to compute $\operatorname{ord}\det \Delta_n$ for each Laplacian Δ_n. This can be done using the matrix expansion algorithm and Lück's theorem, with the error bound of Proposition 3.40. In the end, the problem reduces to computing the ranks of several matrices with integer entries. Hence, we have proved:

Theorem 3.42. *There exists an algorithm that, given a finite L^2-acyclic CW complex M, such that its fundamental group G is residually finite and satisfies the Atiyah conjecture, and a character $\phi\colon G \to \mathbb{Z}$, computes the twisted L^2-Euler characteristic $\chi^{(2)}(\widetilde{M};\phi)$.*

3.5 Implementation details

An implementation of our algorithm is available on GitHub [22]. It relies on the SageMath software system [116] and its interface with GAP [54], especially the HAP computational homotopy package [44]. We also use Regina [18] and SnapPy [37] for testing purposes.

In what follows, we give further explanations of the internal workings of our algorithm.

3.5.1 Chain complex construction

The SageMath system allows for computations in finitely presented groups and their group algebras. Hence, we can represent the differentials of the $\mathbb{Z}G$-chain complex of our space M as matrices over $\mathbb{Z}G$. In practice, we carry out computations over $\mathbb{Z}F$ or $\mathbb{Q}F$, where F is a free group with the same number of generators as G; this is because some ring operations in $\mathbb{Z}G$ cause SageMath to attempt to solve the word problem for G.

We start by representing M as a *regular CW complex*, that is, we require that every *closed* cell is injectively included into M via its attaching map. This can be ensured by taking the barycentric subdivision of a triangulation of M, which is easily obtained using Regina and SnapPy in the case of 3-manifolds. Given adjacency information between $(i+1)$- and i-cells, we can use the HAP method ChainComplexOfUniversalCover to construct the aforementioned differentials.

3.5.2 The matrix expansion algorithm

After constructing the Laplacians, the next major step is applying Oki's algorithm. Recall the problem from the previous section: given a square matrix $A \in \mathrm{M}(n,R) \cap \mathrm{GL}(n,F)$, where $R \coloneqq D_t[u^{\pm 1}]$ and F is the Ore quotient skew field of R, compute $\operatorname{ord}\det A$. The algorithm involves a sequence of matrices over D of increasing size (whence the name), whose ranks converge to $\operatorname{ord} A$.

Remark 3.43. In our case, any element $a = \sum_{g\in G} a_g g \in \mathbb{Z}G$ can be seen as a skew Laurent polynomial in SageMath, by choosing an element $u \in \phi^{-1}(G)$ and collecting monomials with the same exponent of u:

$$a = \sum_{d\in\mathbb{Z}} \left[\sum_{g\in\phi^{-1}(d)} a_g \cdot g u^{-d} \right] u^d. \tag{3.30}$$

Clearly, every coefficient of the resulting polynomial is in $\mathbb{Z}[K_\phi]$.

First, we assume that A does not contain any monomials with negative exponents. In fact, this is ensured by multiplying rows with appropriate powers of u and keeping track of the total multiplier u^N; at the end of the computation, we subtract N to get the correct order. In this way, A can be written as

$$A = \sum_{d=0}^{\ell} A_d u^d, \quad A_d \in \mathrm{M}(n, D). \tag{3.31}$$

Consequently, for all $i \geq 0$,

$$u^i A = \sum_{d=0}^{\ell} u^i A_d u^d \tag{3.32}$$

$$= \sum_{d=0}^{\ell} (u^i A_d u^{-i}) \cdot u^{d+i} \tag{3.33}$$

$$= \sum_{d=i}^{\ell+i} (u^i A_{d-i} u^{-i}) \cdot u^d, \tag{3.34}$$

where $A_d^{(i)} \coloneqq u^i A_{d-i} u^{-i} \in \mathrm{M}(n, D)$, since conjugation by u is an automorphism of D.

Now, choose an integer *expansion parameter* $\mu \geq 1$ and construct the block matrix

$$\Omega_\mu(A) \coloneqq \begin{bmatrix} A_0^{(0)} & A_1^{(0)} & \cdots & \cdots & A_{\mu-1}^{(0)} \\ 0 & A_1^{(1)} & A_2^{(1)} & \cdots & A_{\mu-1}^{(1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & A_{\mu-1}^{(\mu-2)} \\ 0 & \cdots & \cdots & 0 & A_{\mu-1}^{(\mu-1)} \end{bmatrix} \in \mathrm{M}(\mu n, D). \tag{3.35}$$

Lastly, define

$$\omega_\mu(A) \coloneqq \operatorname{rk} \Omega_\mu(A), \quad \psi_\mu(A) \coloneqq \mu n - \omega_\mu(A). \tag{3.36}$$

Proposition 3.44. *The sequence $(\psi_\mu(A))_\mu$ is non-decreasing, concave (i.e. its first differences are non-increasing) and eventually equal to* $\operatorname{ord}\det A$. *In particular, if M is an upper bound on* $\operatorname{ord}\det A$*, then the limit value is attained by $\mu = M$.*

Proof. Firstly, we have $\psi_{d+1}(A) - \psi_d(A) = n - (\omega_{d+1}(A) - \omega_d(A))$. By [100, (36)+(33)], the difference $\omega_{d+1}(A) - \omega_d(A)$ can be expressed as a quantity $N_d \leq n$ non-decreasing in d. Therefore, $(\psi_\mu(A))_\mu$ is a non-decreasing concave sequence.

The second part follows directly from [100, Lemma 6.5]. □

Importantly, [100, Proposition 7.1] gives an effective upper bound $M = \ell n$, where ℓ is the maximum exponent of u appearing in A. Hence, we have an algorithm to reduce the computation of $\operatorname{ord}\det A$ to a rank computation over D.

The *matrix expansion algorithm* runs in polynomial time in n and ℓ, provided that elementary arithmetic operations in the skew field D cost $O(1)$.

Remark 3.45. Oki's article describes another algorithm, the *combinatorial relaxation algorithm*, which is more efficient if the complexity of matrix multiplication is at least $O(n^{2.25})$. Unfortunately, it requires division in D, which is inconvenient in our case $D = \mathcal{D}(\ker\phi)$. Instead, the above algorithm works entirely inside the group ring $\mathbb{Z}G$ up to the rank computation.

Remark 3.46. About the expansion parameter μ, Proposition 3.44 suggests that we run the algorithm at $\mu = 1, 2, 3\ldots$ until the valuations of the determinants stop increasing. An even better strategy is to take $\mu = 1, 2, 2^2, 2^3, \ldots$, and then optionally perform a binary search to find the smallest μ at which valuations stop increasing. This value may differ for each Laplacian; however, for the sake of simplicity, our implementation applies the same μ to all Laplacians.

3.5.3 Finite quotients

Since the quantitative version of Lück's approximation theorem cannot be used in practice, our strategy is to compute a residual chain and heuristically infer the limit value of the von Neumann rank from the approximating sequence in Theorem 3.37. As the efficiency of the simple algorithm in Remark 3.38 is abysmal, we are urged to find a shortcut.

Looking at the statement of Lück's theorem, it is natural to search for large finite quotients of S, in the hope that the kernels of the associated epimorphisms are part of a residual chain. Given a finitely presented group, a prime number p and an integer $c \geq 0$, the GAP function EpimorphismPGroup returns the largest p-group quotient having nilpotency class c. This can be used to construct a function finite_quotient, taking as input a finitely presented group and a finite list of classes $c = (c_2, c_3, c_5, c_7, \ldots)$, and returning the product of all p-quotients, each of class c_p. By applying finite_quotient to G and then restricting the epimorphism to S, we obtain the desired quotient $S \twoheadrightarrow L$.

Remark 3.47. This method exposes sizable quotients quickly, but it may not produce a residual system of subgroups even if we let c range over all possible classes. Since the quotients we obtain in this way are always nilpotent, some examples are given by groups that are not residually nilpotent, such as the *modular group* $\mathrm{PSL}(2, \mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$ or some free-by-cyclic groups [7]. On the other hand, it seems that the resulting kernels are deep enough to approximate the von Neumann rank in every case we examined.

Remark 3.48. In order to compute the classical L^2-Betti numbers, we can skip the matrix expansion step and use finite_quotient as a practical way to generate large finite quotients for Lück's approximation theorem. This gives the von Neumann ranks of the differentials, from which the L^2-Betti numbers can be obtained via the rank-nullity formula.

It is also worth noting that the process of Remark 3.46 becomes heuristic if this method is used, as the former depends on the quality of the Lück approximation step: roughly speaking, if L is large enough, the computed rational value will be close to an integer, which can be inferred to be "underlying" value of the order valuation. Hence, in general, manual inspection may be required.

This rounding strategy works whenever the error in the approximation is less than 1/2. However, intuition based on commutative rings suggests that the rank decreases when passing to a finite approximation (while the valuation increases); see also Funke and Kielak's *determinant comparison problem* [71, Question 3.6]. Hence, if a stronger heuristic is necessary, we can round up the computed ranks, allowing for an error up to 1. Following [68], if G is a (locally indicable) *Lewin group*, then this intuition is correct: every subgroup $S < G$ is also Lewin and the rank $\mathrm{rk}_{\mathcal{D}(S)}$ is an upper bound for $\mathrm{rk}_{\{1\}}$, that is, the rank arising from the trivial quotient of S. By [68, Corollary 4.3], the inequality holds for every finite quotient of S. It is worth noting that locally indicable free-by-cyclic groups, and conjecturally all locally indicable groups, are Lewin [68, Theorem 1.1, Conjecture 1].

3.5.4 Non-determinism

The internal representation of the homomorphism ϕ depends on the presentation of G, and ultimately on GAP internal workings, in its treatment of CW complex objects. The GAP method edgeToWord allows one to relate the basis of $\mathrm{Hom}(G, \mathbb{Z})$ in which ϕ is expressed to concrete 1-cycles on a given CW complex; however, the GAP code loses track of edges during simplification steps, which must then be skipped. In most cases, we have chosen to dispense with this refinement, since doing so only affects the shape of the resulting polytope by a linear transformation.

An additional source of uncertainty is in the construction of the triangulations. Indeed, we often rely on Regina and, in dimension 3, SnapPy to construct and simplify these structures with non-deterministic algorithms. Hence, simpler G-CW complex structures may arise from different runs of the routines.

3.6 Examples and applications

In this section we summarize experimental data on L^2-invariants while simultaneously showcasing the algorithm's various aspects; all the experiments are also available as SageMath notebooks in the GitHub repository [22]. All examples under consideration have residually finite fundamental group, and most are known to satisfy the Atiyah conjecture.

For the sake of brevity, we will use an alternate product-like notation $2^{c_2} \cdot 3^{c_3} \cdot 5^{c_5} \cdot \ldots$ for the nilpotency class vector $c = (c_2, c_3, c_5, \ldots)$. Moreover, we will often represent characters ϕ by their coordinates in some fixed basis of $\mathrm{Hom}(G, \mathbb{Z})$, denoted by $\{\mathsf{v}_i\}$ in figures (see Section 3.5.4). Reported running times refer to computations carried out on an Intel i9-12900HX CPU.

3.6.1 Convergence rate of finite approximation

To test the convergence given by Lück's approximation theorem, we apply the untwisted algorithm to a *naturally occurring* matrix. More specifically, let M be the census hyperbolic 3-manifold v1539(5, 1). We use HAP to compute the fundamental group

$$G \coloneqq \langle a, b, c \mid (a^2c^{-5}b^{-3})^4a^2b^2, a^2c^{-5}b^{-2}c, c^9a^{-3}b^3c^{-3}\rangle \tag{3.37}$$

and a $\mathbb{Z}G$-chain complex for $\widetilde{M}$, of the form

$$\mathbb{Z}G \longrightarrow \mathbb{Z}G^3 \xrightarrow{A} \mathbb{Z}G^3 \longrightarrow \mathbb{Z}G. \tag{3.38}$$

Recall that by [89, Theorem 1.62] the above chain complex is L^2-acyclic. Therefore, a routine calculation shows that the von Neumann ranks of the three differentials must be $1, 2, 1$. We can test our algorithm on the second differential A with various nilpotency class vectors c, expecting it to approximate $\operatorname{rk} A = 2$ (Table 3.1).

Empirically, the algorithm spends most of its time in the final rank computation over $\mathbb{Z}$ or $\mathbb{Q}$; this suggests a time complexity superquadratic in $|L|$, which is experimentally confirmed. It is therefore essential to keep $|L|$ as low as possible.

Looking at p-group quotients, we see that when $p \neq 5$, the error is exactly $1/|L|$; this suggests a trend that holds for all but a finite number of *special* primes depending on the group G. Indeed, choosing $c = 5$, 5^2 and even $2 \cdot 3 \cdot 5$ severely degrades the accuracy, even incurring a large computational cost in the latter case. This behavior can possibly be traced to the matrix A involving only finitely many group elements; we also speculate that the special primes are related to the set of primes p for which G is not residually a (finite) p-group.

It also seems that combining two or more non-special primes could be detrimental, as in the case $c = 2 \cdot 3$. Hence, we will generally try single primes first, in order to find the special primes, and only then use different primes together.

3.6.2 Convergence rate of the whole algorithm

This time, we apply the full algorithm to the 3-manifold M defined as the complement of the Borromean rings (Thistlethwaite notation L6a4, see Figure 3.1). Since M is hyperbolic and therefore L^2-acyclic, we can compute the twisted L^2-Euler characteristic of a primitive character and study the dependence of the approximation on the expansion parameter μ.

As before, we find a presentation for the fundamental group

$$G \coloneqq \pi_1(M) = \langle a, b, c \mid c^{-1}b^{-1}cac^{-1}a^{-1}baca^{-1}, b^{-1}abc^{-1}b^{-1}ca^{-1}c^{-1}bc\rangle, \tag{3.39}$$

and a rather simple $\mathbb{Z}G$-chain complex of the form

$$\mathbb{Z}G^2 \xrightarrow{d_2} \mathbb{Z}G^3 \xrightarrow{d_1} \mathbb{Z}G. \tag{3.40}$$

c	$\lvert L\rvert$	Output r	$\lvert L\rvert \cdot (2-r)$	Time (ms)
2	4	1.75	1	62
3	9	1.888 89	1	32
5	25	1.64	9	40
7	49	1.979 59	1	55
11	121	1.991 74	1	140
13	169	1.994 08	1	203
17	289	1.996 54	1	551
2^2	32	1.968 75	1	75
3^2	243	1.995 88	1	609
5^2	3125	1.990 72	29	133 111
$2 \cdot 3$	36	1.861 11	5	40
$3 \cdot 7$	441	1.997 73	1	1 150
$2 \cdot 3 \cdot 5$	900	1.985 56	13	4 877

Table 3.1: Rank of A as computed by the algorithm with nilpotency class vector c, rounded to five decimal places. Multiplying the error with $|L|$ (smaller is better) reveals an inverse proportionality phenomenon.

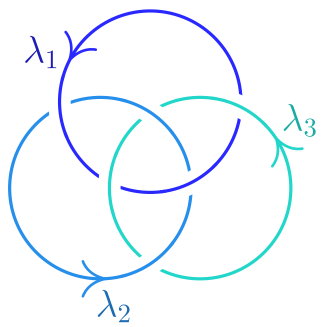


Figure 3.1: The Borromean rings.

The rank of G is 3, so we arbitrarily fix a character ϕ defined by $(a, b, c) \mapsto (0, 0, 1)$. First, we try class-1 p-quotients using $\mu = 6$, for p ranging through the first 10 primes. In every run the finite quotient has size $|L| = p^2$, and is therefore abelian, for all three Laplacians. Furthermore, the computed valuations v_i and determinant degrees δ_i are

$$v_0 = 0, \quad v_1 = \frac{12}{p}, \quad v_2 = \frac{8}{p}, \quad \delta_0 = 2, \quad \delta_1 = 6 - \frac{24}{p}, \quad \delta_2 = 8 - \frac{16}{p}, \tag{3.41}$$

leading to computed values of $\chi^{(2)}(\widetilde{M}; \phi) = -1 + \frac{4}{p}$. This suggests that all the v_i are in fact 0, so we could have chosen $\mu = 1$, greatly reducing computational time.

We take advantage of this observation and run our code again, with larger quotients and $\mu = 1$. Whenever only primes with exponent 1 are involved, the corresponding $|L|$ is always the squared product of the primes, while the error is proportional to $|L|^{-1/2}$. Further tests show that the coefficient of $|L|^{-1/2}$ for $\mu = 1, 2, 3, 4, 5, 6$ is respectively $2, 4, 6, 8, 6, 4$. This suggests that, by choosing μ as low as possible, we minimize the error (and the computational time due to matrix expansion). If class-2 p-quotients are involved, the proportionality holds only approximately, with a coefficient around 10 for $p = 2$ and 20 for $p = 3$.

Hence, the most efficient finite quotients appear to be by far the ones involving only class-1 p-groups: in this case, L is a subgroup of a product of 1-nilpotent p-groups, and therefore abelian. This is somewhat surprising, in that kernels of abelian quotients cannot form a residual chain, as they all contain the derived subgroup: effectively, we are approximating the twisted L^2-Euler characteristic of the quotient chain complex with coefficients in $\mathbb{Z}[\mathrm{ab}(G)]$. In fact, this phenomenon is related to the Alexander norm, which can also be computed from the latter complex, as the following result shows:

Proposition 3.49. *Let M be a compact orientable 3-manifold whose boundary is a union of zero or more tori and whose Alexander polynomial is non-zero. Let $\phi\colon \pi_1(M) \twoheadrightarrow \mathbb{Z}$ be a primitive character and let $\chi^{(2)}(\widetilde{M}; \mathrm{ab}(G), \phi)$ be the twisted L^2-Euler characteristic of the $\mathbb{Z}[\mathrm{ab}(G)]$-chain complex C_*^{ab}, i.e. the cellular chain complex associated to the $\mathrm{ab}(G)$-covering of M. The latter is related to the Alexander norm of ϕ via:*

$$-\chi^{(2)}(\widetilde{M}; \mathrm{ab}(G), \phi) = \|\phi\|_A - \begin{cases} 0 & \text{if } b_1(M) \geq 2, \\ 1 & \text{if } b_1(M) = 1 \text{ and } \partial M \neq \varnothing, \\ 2 & \text{if } b_1(M) = 1 \text{ and } \partial M = \varnothing. \end{cases}$$

Proof. See Section 3.7; compare also [50, Theorem 8.4]. □

Hence, when the Alexander and Thurston norms do not agree (up to the correction term in Proposition 3.49), using only abelian quotients is destined to fail. For many link complements, including the Borromean rings, equality holds: see [95, Section 7] for a more exhaustive discussion.

Remark 3.50. This result can be generalized to quotients coming from the *rational derived series* $(G_r^{(k)})$ of G, provided that the induced $\mathbb{Z}[G/G_r^{(k)}]$-chain complex is L^2-acyclic. In that case, following [50, Section 8], the twisted L^2-Euler characteristic is minus the k-th *higher-order Alexander norm* of Harvey [61]. (More inequalities between these generalized Alexander norms and the Thurston norm are proved in [53, Theorem 2.29] and [74, Section 6.2].) Our finite quotients are instead approximating the *lower central series*, a similar but distinct construction.

Remark 3.51. Compared to the untwisted case, the error decays more slowly in $|L|$, which is the main driver of the running time. We expect this pattern to hold in greater generality.

3.6.3 A few Thurston norm unit balls

We shall now compute the entire unit balls of the Thurston norms for a few manifolds, including the already mentioned v1539(5, 1) and the Borromean rings complement. This can be done iteratively via Proposition 3.7:

- compute $-\chi^{(2)}(\widetilde{M};\phi)$ for some values of ϕ;
- plot the corresponding points on the unit sphere of the Thurston seminorm;
- manually infer a possible shape for the unit ball B;
- compute $-\chi^{(2)}(\widetilde{M};-)$ for all vertices of B and for one point in the interior of each face of B, expecting a result of 1;
- if unsuccessful, retry with more points.

The Borromean rings complement

This is one of the examples in Thurston's original article [123], where he shows that the unit ball is a regular octahedron up to linear isomorphisms. We will continue to use the $\mathbb{Z}G$-chain complex found earlier (3.40).

Remark 3.52. The following non-rigorous argument can be used to choose the finite quotient. We heuristically assume that the error in the computation of $\chi^{(2)}(\widetilde{M};\phi)$ is bounded above by $20 \cdot |L|^{-1/2}$, where $|L|$ is the largest quotient pertaining to any of the three Laplacians. Suppose that the algorithm outputs a value m' within $1/4$ of an integer m; then m is the unique integer within distance $3/4$ of m'. By the heuristic error bound, we want $3/4 > 20 \cdot |L|^{-1/2}$, that is, $|L| \geq 712$; if this is the case, we round the output to m. We use the same argument to round the individual valuations v_i to the nearest integers.

We can accomplish this with $c = 29$; of course, choosing abelian quotients is only acceptable to the extent that the Thurston norm equals the Alexander norm for this manifold. According to the discussion in Section 3.6.2, it is crucial to choose μ as small as possible (as in Remark 3.46), both for speed and to create more leeway in the error bound. We start by applying the algorithm to a few values of ϕ with $\mu = 1, 2$ (Table 3.2), finding that the rounded valuations do not change between $\mu = 1, 2$; hence, we deduce that $\mu = 1$ suffices.

Therefore, the three basis vectors all have Thurston norm 1; by symmetry, we get six lattice points on the unit sphere. Thurston's argument for this manifold shows that the only six unit norm lattice points are the vertices of the (octahedral) unit ball, represented by the three link components and their opposites. Hence, expecting to find eight points of norm 3 at $(\pm 1, \pm 1, \pm 1)$, we run the algorithm once again (see Table 3.3).

Again by symmetry, we get that each point in $\{(\pm 1/3, \pm 1/3, \pm 1/3)\}$ (lying in the interior of a face of the octahedron) has unit norm. This already determines the unit ball

ϕ	$-\chi^{(2)}(\widetilde{M};\phi)$
$(1,0,0)$	1
$(0,1,0)$	1
$(0,0,1)$	1

Table 3.2: A few trials of the algorithm with $c = 29$. Each computed value is rounded to the unique integer within distance $1/4$. The maximum $|L|$ is always $29^2 > 712$. Total computational time was around 1 minute.

ϕ	μ	v_0	v_1	v_2	$-\chi^{(2)}(\widetilde{M};\phi)$
$(1,1,1)$	1	0	1	0	3
$(-1,1,1)$	2	0	2	0	3
$(1,-1,1)$	2	0	2	0	3
$(1,1,-1)$	2	0	2	0	3

Table 3.3: Four lattice points taken from the cones over open faces of the octahedron. Outputs are already integer numbers and $|L|$ is always 29^2. We report the smallest μ at which Oki's algorithm attains its limit value. Total computational time was around 3 minutes.

completely (Figure 3.2); the shape is consistent with further runs on the remaining lattice points with coordinates in $\{-1,0,1\}$.

The L10n14 link complement

Next, we test the algorithm on the complement M of the two-component link L10n14 (Figure 3.3), introduced by Dunfield [43] as a counterexample to the equality of Thurston and Alexander unit ball faces.

As always, we first compute a presentation for the fundamental group

$$G \coloneqq \langle a, b \mid a(ba^{-1}b^{-2}a^{-1})^2ba^3ba^{-1}b^{-2}a^{-1}ba^{-1}b^{-1}(ab^2ab^{-1})^2a^{-3}b^{-1}ab^2ab^{-1}\rangle \tag{3.42}$$

and a $\mathbb{Z}G$-chain complex, of the form

$$\mathbb{Z}G \xrightarrow{d_2} \mathbb{Z}G^2 \xrightarrow{d_1} \mathbb{Z}G. \tag{3.43}$$

The presentation for G is the same as the one given directly by SnapPy, up to a cyclic permutation of the relator. Hence we can access information about the link meridians, obtaining that the cohomology classes of the components λ_1, λ_2 are respectively $(1,-1)$ and $(-1,0)$ in our basis. We can also compute the Thurston norm using an algorithm

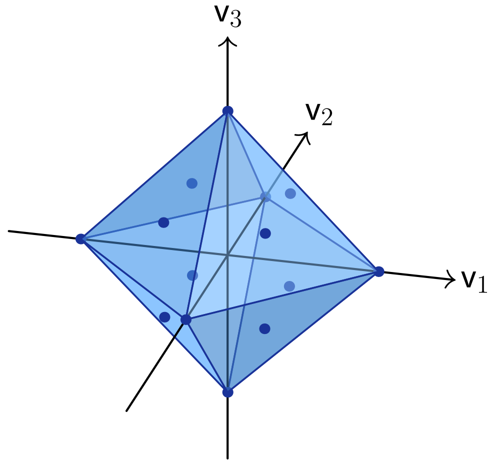


Figure 3.2: The unit ball is determined by the 14 marked points.

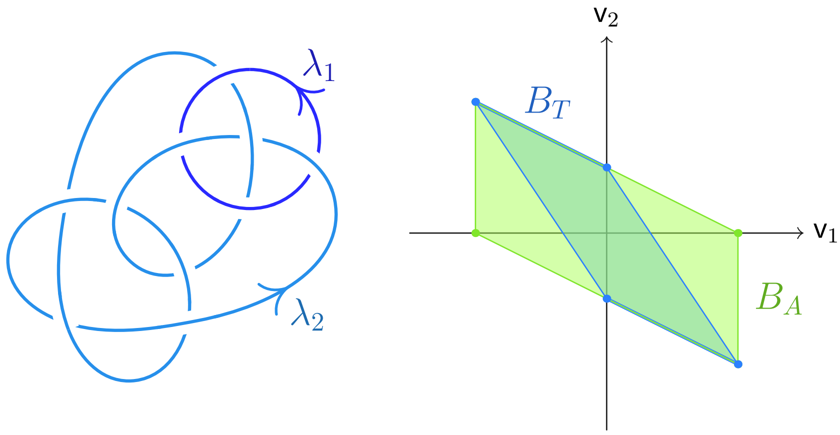


Figure 3.3: The two-component link L10n14 and the unit balls for its Thurston and Alexander norms.

by Friedl, Tillmann and Schreve for two-generator one-relator groups [51], verifying the strict inclusion between the two norm balls (again Figure 3.3).

Specifically, on a few selected points, the Thurston and Alexander norms take on the values in Table 3.4, which completely determine the Thurston norm. The last three values are shared by the Alexander norm; indeed, for $\phi = (0, 1), (-1, 2), (-1, 1)$ our algorithm gives consistent results across several finite quotients, both abelian and non-abelian.

On the other hand, the two norms differ for $\phi = (1, 0)$: accordingly, we expect the Thurston norm to be harder to approximate, requiring highly non-abelian quotients. As even $c = 2^2$ gives an abelian $L \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$, we anticipate a very demanding computation, due to the fast growth of the order of L with respect to the nilpotency class.

In the end, it is hard to get very close to the true value of the Thurston norm, even with

ϕ	$x_M(\phi)$	$\|\phi\|_A$
$(1,0)$	3	1
$(0,1)$	2	2
$(-1,2)$	3	3
$(-1,1)$	1	1

Table 3.4: A few points in $H^1(M;\mathbb{Z})$ and their Thurston and Alexander norms.

c	$\|L\|$	v_0	v_1	v_2	δ_0	δ_1	δ_2	$-\chi^{(2)}(\widetilde{M};\phi)$	Time (s)
2	2	0	4	2	2	6	4	1	0.23
2^2	8	0	4	2	2	6	4	1	0.17
2^3	128	0	3	1	2	8	6	2	0.51
3^3	2187	0	2.667	0.667	2	8.667	6.667	2.333	76
2^4	8192	0	2.5	0.5	2	9	7	2.5	1726
limit	–	0	2	0	2	10	8	3	–

Table 3.5: Results obtained from the point $\phi = (1,0)$ with $\mu = 2$, including the inferred limit values. Outputs are rounded to three decimal places.

30 minutes of computational time; however, the results (collected in Table 3.5) are still consistent with Table 3.4.

The census manifold v1539(5, 1)

This manifold is unique in the orientable closed census for having first Betti number greater than one (in fact, equal to 2). Unlike our first example, we will use a simpler chain complex

$$\mathbb{Z}G \xrightarrow{d_3} \mathbb{Z}G^2 \xrightarrow{d_2} \mathbb{Z}G^2 \xrightarrow{d_1} \mathbb{Z}G, \tag{3.44}$$

with

$$G \coloneqq \langle a,b \mid b^{-4}a^2(b^{-1}a^{-1})^3b^{-1}a^2b^5a^2b^{-1}a^{-3}b^4, (ba^{-2}ba^3)^2ba^{-2}b^{-5}\rangle. \tag{3.45}$$

It turns out that taking $c = 3^2$ is sufficient to obtain exact integer outputs, provided μ is as large as necessary for the valuations to stop changing: the minimal values of μ are listed along with the outputs in Table 3.6. In more detail, we first try the vectors $\phi = (1,0),(0,1),(1,1),(1,-1)$ (in a basis dual to the generators a,b), which give 8 points on the boundary of the unit ball (refer to Figure 3.4, left). There is still some ambiguity, resolved by the input $\phi = (2,-1)$, which gives the top-left and bottom-right corners of the unit ball.

For many choices of c, even involving higher-class quotients, the group L turns out to be abelian. Fortunately, this is not a problem: by the discussion in [19, Example 2.1], *every*

ϕ	μ	δ_0	δ_1	δ_2	δ_3	$-\chi^{(2)}(\widetilde{M};\phi)$
$(1,0)$	4	2	10	10	2	2
$(0,1)$	10	2	12	14	4	2
$(1,1)$	11	2	18	22	6	4
$(1,-1)$	16	2	10	10	2	2
$(2,-1)$	18	4	14	12	2	2

Table 3.6: Computed values of the Thurston norm of various classes with $c = 3^2$. Total running time was under 10 seconds.

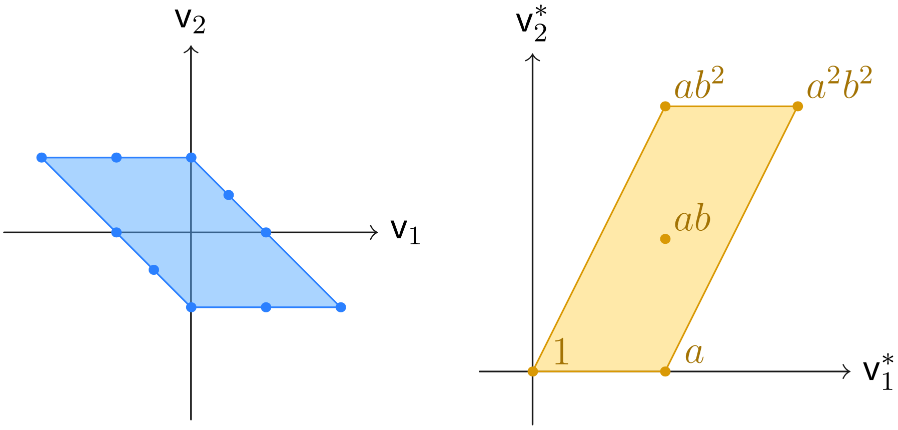


Figure 3.4: Left: the Thurston norm unit ball for v1539(5, 1), determined by the 10 marked points. Right: the Newton polytope for the Alexander polynomial obtained from the group presentation (3.45). The two polytopes are dual to each other.

non-vertex cohomology class of v1539(5, 1) is fibered, so the Thurston norm equals the Alexander norm. Of course, the Alexander polynomial can be easily computed from the group presentation (3.45):

$$\Delta_M = 1 + a + ab + ab^2 + a^2b^2 \in \mathbb{Z}[a^{\pm 1}, b^{\pm 1}]. \tag{3.46}$$

We observe that its Newton polytope (Figure 3.4, right) is dual to the norm ball we computed, providing further confirmation.

Manifolds with rank-1 first homology

Whenever the first Betti number of a manifold is 1, its Thurston norm becomes a simple, integer-valued invariant, requiring in principle no pattern recognition in order to determine unit ball vertices. If we restrict our attention to closed 3-manifolds, McMullen's inequality gives

$$\|\phi\|_A \leq x_M(\phi) + 2, \tag{3.47}$$

where ϕ is a generator of $H^1(M;\mathbb{Z}) \simeq \mathbb{Z}$.

The hyperbolic 3-manifold census lists 127 closed 3-manifolds with first Betti number equal to 1, of which 86 are fibered and 41 non-fibered [19]; surprisingly, for all of them, equality holds in (3.47). Indeed, this is part of McMullen's statement for the fibered manifolds, while the rest can all be shown with SageMath to contain appropriate normal surfaces of Euler characteristic -2 and to have Alexander polynomials of degree 4. Combining this with Proposition 3.49, we obtain

$$\chi^{(2)}(\widetilde{M};\phi) = \chi^{(2)}(\widetilde{M};\mathrm{ab}(G),\phi), \tag{3.48}$$

for M any of the 127 manifolds we are considering.

When we attempt to apply our algorithm to these manifolds, we encounter a limitation of the finite quotient subroutine: nontrivial p-group quotients are found only if $H_1(M;\mathbb{Z})$ has p-torsion. This appears to be strongly related to $\ker\phi$ having finite, or even trivial, abelianization. However, due to (3.48), it is not necessary to consider non-abelian quotients: in practice, it turns out that the trivial quotient always suffices, attaining exactly (!) the Thurston norm, if μ is high enough. The results are listed in Table 3.7.

3.6.4 Free-by-cyclic groups

Another rich class of test cases is given by *free-by-cyclic* groups:

Definition 3.53. A *free-by-cyclic group* is a semidirect product $F_n \rtimes_\varphi \mathbb{Z}$, where F_n is a free group of finite rank $n \geq 0$ and φ is an automorphism of F_n. Equivalently, it is an extension of a finitely generated free group by the infinite cyclic group $\mathbb{Z}$.

Every compact 3-manifold with boundary that fibers over S^1 has a free-by-cyclic fundamental group: the fiber is a compact surface with boundary and so has free fundamental group, while φ is simply the monodromy of the fibration. Thus, it is natural to consider such groups as generalizations of fibered 3-manifolds. Following [89, Definition 6.50] and [49, Section 4.1], we define the twisted L^2-Euler characteristic of a group G as that of the universal cover of its classifying space BG, if the latter is finite and L^2-acyclic.

A free-by-cyclic group $G = F_n \rtimes_\varphi \mathbb{Z}$ has a canonical presentation

$$\langle x_1,\dots,x_n,t \mid tx_it^{-1} = \varphi(x_i)\ \forall i \in \{1,\dots,n\}\rangle, \tag{3.49}$$

which is *combinatorially aspherical* as defined in [29], by the Corollary of Theorem 3.4 in the same article. To obtain a model for BG, we further prove that the presentation complex K is aspherical. By [29, Proposition 1.3], it suffices to check that:

- no relation $r_i \coloneqq tx_it^{-1}\varphi(x_i)^{-1}$ is a proper power;
- no two relations are freely conjugate to each other or their inverses.

M	x_M	M	x_M	M	x_M	M	x_M
m160(3,1)	2	**s644(-4,3)**	**2**	v2345(5,1)	2	v2984(-1,3)	4
m159(4,1)	2	**s643(-5,1)**	**2**	v3209(-3,1)	2	**v3145(3,2)**	**2**
m199(-4,1)	2	s646(5,2)	4	v3077(5,1)	2	**v3181(-3,2)**	**2**
m122(-4,1)	2	s789(-5,1)	2	v2959(-3,1)	2	v3209(5,1)	2
s942(-2,1)	2	s719(7,1)	2	v2671(-2,3)	4	v3019(5,2)	6
m336(-1,3)	2	v1373(-2,3)	2	**v3209(-1,2)**	**2**	**v3036(3,2)**	**2**
m345(1,2)	4	**v2018(-4,1)**	**2**	**v2593(4,1)**	**2**	v3212(1,3)	4
m289(7,1)	2	v3209(3,1)	2	s928(2,3)	6	**v3209(1,3)**	**2**
m280(1,4)	2	v2420(-3,1)	2	v3390(3,1)	2	**v3269(4,1)**	**2**
m304(-5,1)	2	v2099(-4,1)	2	v3209(4,1)	2	**v3209(-3,2)**	**2**
m305(-1,3)	2	v2101(3,1)	2	v2913(-3,2)	2	**v3209(2,3)**	**2**
s385(5,1)	4	s789(5,1)	2	v3505(-3,1)	2	**v3313(3,1)**	**2**
s296(-1,3)	2	v1539(-5,1)	2	v3261(4,1)	2	**v3239(3,2)**	**2**
s297(5,1)	2	**v1436(-5,1)**	**2**	v3262(3,1)	2	**v3209(5,2)**	**2**
s912(0,1)	2	v1721(1,4)	8	v2678(-5,1)	2	**v3209(-1,3)**	**2**
m401(-2,3)	2	**s750(4,3)**	**2**	**v3209(3,2)**	**2**	v3209(-5,1)	2
m371(-1,3)	2	**s749(5,1)**	**2**	**v3027(-3,1)**	**2**	v3425(-3,2)	4
m368(-4,1)	2	**s789(-5,2)**	**2**	**v2896(-6,1)**	**2**	v3209(6,1)	2
s580(-5,1)	2	**v1539(5,2)**	**2**	**v2683(-6,1)**	**2**	**v3209(4,3)**	**2**
s581(-1,3)	2	**v2238(-5,1)**	**2**	**v2796(4,1)**	**2**	v3318(4,1)	6
s869(-1,2)	2	**v3209(1,2)**	**2**	**v2797(-3,4)**	**2**	**v3244(4,3)**	**2**
s861(3,1)	2	**s828(-4,3)**	**2**	v3107(3,2)	4	**v3243(-4,1)**	**2**
v1191(-5,1)	2	**v1695(5,1)**	**2**	v3216(4,1)	2	v3352(1,4)	6
v1076(-5,1)	2	v2771(-4,1)	4	v3217(-1,3)	2	v3398(2,3)	4
s528(-1,3)	**2**	s836(-6,1)	6	v3320(4,1)	4	v3378(-1,4)	6
s527(-5,1)	**2**	v2986(1,2)	4	v3091(-2,3)	4	v3408(1,3)	8
s924(3,1)	2	v2209(2,3)	4	**v2948(-6,1)**	**2**	v3467(-2,3)	8
v1408(4,1)	2	**s862(7,1)**	**2**	**v2794(-6,1)**	**2**	v3445(6,1)	10
s677(1,3)	2	**v2190(4,1)**	**2**	v3214(1,3)	2	v3509(4,3)	4
s676(5,1)	2	v2054(-7,1)	2	v3215(-4,1)	2	v3508(4,1)	4
v2641(-4,1)	2	v3066(-1,2)	4	**v3183(-3,2)**	**2**	v3504(-2,3)	6
s745(3,2)	2	v2563(5,1)	2	v3209(-4,1)	2		

Table 3.7: Thurston norms of closed hyperbolic 3-manifolds in the census with rank-1 first homology. Rows in bold denote non-fibered manifolds [19, Table 4].

The former follows from the relations being cyclically reduced and containing only one occurrence of the *stable letter* t; to prove the latter, note that each relation (and its inverse) has a unique cyclically reduced representative starting with t, and that the second letter is x_i for r_i and x_i^{-1} for r_i^{-1}.

At this point, it remains to show that our model for the classifying space is L^2-acyclic. This can be done by applying either Theorem 1.39 or Theorem 7.2 (5) in [89]. As a bonus, free-by-cyclic groups satisfy the Atiyah Conjecture by [89, Theorem 10.22], and are residually finite by [14].

Constructing test cases becomes a matter of finding free group automorphisms, which can be done by composing elements from a generating set of $\mathrm{Aut}(F_n)$, such as the following involutions inspired by [5]:

– τ_i, inverting the generator x_i;

– $\sigma_{i,j}$, swapping x_i and x_j;

– $\eta_{i,j}$, sending $x_i \mapsto x_j^{-1}x_i$, $x_j \mapsto x_j^{-1}$, and leaving other generators unchanged.

In order to determine the invariant $-\chi^{(2)}(G;-)$ in a finite number of evaluations, we exploit the fact that it is a genuine seminorm, proved in [53, Corollary 3.5], just like the Thurston norm case. This is not strictly necessary, as the Laplacian degrees are themselves seminorms by Kielak's *single polytope theorem* [72, Theorem 3.14].

The first example we consider is the automorphism $\varphi\colon F_3 \to F_3$ given by

$$\varphi \coloneqq \eta_{2,1} \cdot \sigma_{1,3} \cdot \eta_{2,1} \cdot \eta_{3,2} \cdot \eta_{3,1}, \tag{3.50}$$

where the leftmost factor is applied first. We proceed by constructing the necessary GAP objects. Since G has an aspherical group presentation, we can compute the $\mathbb{Z}G$-chain complex directly using the HAP method `ResolutionAsphericalPresentation`. We choose a basis of $\mathrm{Hom}(G,\mathbb{Z}) \simeq \mathbb{Z}^2$ such that $\phi = (0,1)$ is the canonical projection on $\mathbb{Z}$, while $\phi = (1,0)$ has the stable letter t in its kernel. More precisely, the images of the generators of G are:

ϕ	$\phi(x_1)$	$\phi(x_2)$	$\phi(x_3)$	$\phi(t)$
$(1,0)$	1	1	1	0
$(0,1)$	0	0	0	1

(3.51)

Key values of $\chi^{(2)}(G;-)$ are collected in Table 3.8: the two runs are consistent with each other and with the value 2 for each value of ϕ we considered, determining the square unit ball shown in Figure 3.5.

Another (considerably more complex) rank-2 test case arises from the automorphism $\varphi : F_4 \to F_4$, defined by

$$\begin{aligned}\varphi \coloneqq\ & \eta_{2,3} \cdot \sigma_{2,3} \cdot \tau_4 \cdot \eta_{4,3} \cdot \eta_{3,1} \cdot \tau_3 \cdot \sigma_{3,4} \cdot \sigma_{2,4} \cdot \eta_{3,1} \cdot \tau_4 \cdot \eta_{2,1} \cdot \sigma_{1,4} \cdot \eta_{1,4} \\ & \cdot \eta_{1,4} \cdot \eta_{3,2} \cdot \sigma_{1,4} \cdot \eta_{2,3} \cdot \sigma_{1,3} \cdot \eta_{3,4} \cdot \eta_{3,1} \cdot \eta_{1,2} \cdot \sigma_{3,4} \cdot \eta_{2,4} \cdot \eta_{1,4} \cdot \eta_{3,2} \cdot \eta_{3,1} \\ & \cdot \eta_{4,3} \cdot \eta_{2,3} \cdot \eta_{2,3} \cdot \sigma_{1,4} \cdot \sigma_{3,4} \cdot \tau_2 \cdot \eta_{1,2} \cdot \eta_{1,2} \cdot \sigma_{3,4} \cdot \sigma_{2,3} \cdot \eta_{4,3} \cdot \eta_{1,2} \cdot \eta_{2,1} \\ & \cdot \tau_2 \cdot \eta_{4,3} \cdot \sigma_{2,4} \cdot \sigma_{1,2} \cdot \sigma_{1,2} \cdot \eta_{3,1} \cdot \eta_{4,1} \cdot \sigma_{1,2} \cdot \eta_{3,4} \cdot \sigma_{1,3} \cdot \eta_{2,4} \cdot \eta_{2,4} \cdot \eta_{1,4}.\end{aligned} \tag{3.52}$$

We fix a basis given by the following table:

ϕ	$\phi(x_1)$	$\phi(x_2)$	$\phi(x_3)$	$\phi(x_4)$	$\phi(t)$
$(1, 0)$	1	-2	-3	-1	0
$(0, 1)$	0	0	0	0	1

(3.53)

In this case, the algorithm has more trouble converging: when choosing an odd prime p, the quotient routine returns cyclic groups, incurring the risk of not approximating the invariant correctly, like in Proposition 3.49; on the other hand, even the 2-quotient from $c = 2^3$ is much too large (order 1024), producing runtimes on the order of hours.

Therefore, we use the improved heuristic from Section 3.5.3 to infer results from the smaller quotient with $c = 2^2$; that is, we increase the expansion parameter μ until $\lfloor v_1 \rfloor$ and $\lfloor v_2 \rfloor$ stabilize, and use the latter values to compute Laplacian degrees and the twisted L^2-Euler characteristic. We can also check that the quotients L are all non-abelian, of the form $(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$ or $\mathbb{Z}_2 \times (\mathbb{Z}_4 \rtimes \mathbb{Z}_4)$.

The ten lattice points of Table 3.9 outline a decagonal unit ball (Figure 3.5) when appropriately scaled. In some cases the error in the valuations is very close to 1, indicating that it could be even larger, which in turn would lead to incorrect inferences about v_1 and v_2. This is somewhat mitigated by the general consistency of the shape and by a cross-check with $c = 3^2$.

In both examples, the class $\phi = (0, 1)$ dual to the stable letter may be considered analogous to a fibered class, since it describes the group G as an extension of its finitely generated, free kernel by $\mathbb{Z}$, just like in the 3-manifold case. This can be taken as a definition of fibered class; such matters are discussed in the aforementioned article [72], which relates them to the *Bieri-Neumann-Strebel* (or *BNS*) *invariant* $\Sigma(G)$. It is worth noting that, in both cases, the class $(0, 1)$ lies in an open cone, and the twisted L^2-Euler characteristic agrees with the Euler characteristic of the free group: $\chi(F_n) = 1 - n$.

3.6.5 The fiber of the Ratcliffe-Tschantz manifold

In a recent article, Italiano, Martelli and Migliorini [67] constructed fibrations over the circle on a few cusped hyperbolic 5-manifolds of finite volume including, notably, the smallest known hyperbolic 5-manifold of Ratcliffe and Tschantz N (see [112]).

c	μ	ϕ	v_1	v_2	$-\chi^{(2)}(G;\phi)$
2^3	4	$(0,1)$	$0 \to 0$	$0 \to 0$	$2 \to 2$
		$(1,1)$	$2.4375 \to 2$	$1.578\,12 \to 1$	$1.281\,25 \to 2$
		$(1,0)$	$1 \to 1$	$1 \to 1$	$2 \to 2$
		$(-1,1)$	$3.406\,25 \to 3$	$1.4375 \to 1$	$1.531\,25 \to 2$
7^2	5	$(0,1)$	$0 \to 0$	$0 \to 0$	$2 \to 2$
		$(1,1)$	$2.078\,72 \to 2$	$1.1516 \to 1$	$1.775\,51 \to 2$
		$(1,0)$	$1 \to 1$	$1 \to 1$	$2 \to 2$
		$(-1,1)$	$3 \to 3$	$1 \to 1$	$2 \to 2$

Table 3.8: Experimental results for the twisted L^2-Euler characteristic of the first free-by-cyclic example; for simplicity, we omit the values of v_0, as they do not affect the result.

ϕ	μ	v_1	v_2	$-\chi^{(2)}(G;\phi)$
$(1,0)$	10	30	9	10
$(1,1)$	13	32.375	10.375	9
$(2,3)$	23	64.406 25	19.406 25	19
$(1,2)$	11	32	9	10
$(1,3)$	13	31.5	10.25	12
$(1,6)$	15	24.9375	13.9375	18
$(0,1)$	1	0	0	3
$(-1,7)$	10	7.406 25	8.406 25	21
$(-1,4)$	9	14	17	15
$(-1,3)$	13	18.875	20.625	13

Table 3.9: Results for the second free-by-cyclic example, run with $c = 2^2$. The reported value of μ is the smallest such that $\lfloor v_1 \rfloor, \lfloor v_2 \rfloor$ coincide for $\mu, \mu+1$; values of $-\chi^{(2)}(G;\phi)$ are computed from $\lfloor v_1 \rfloor, \lfloor v_2 \rfloor$.

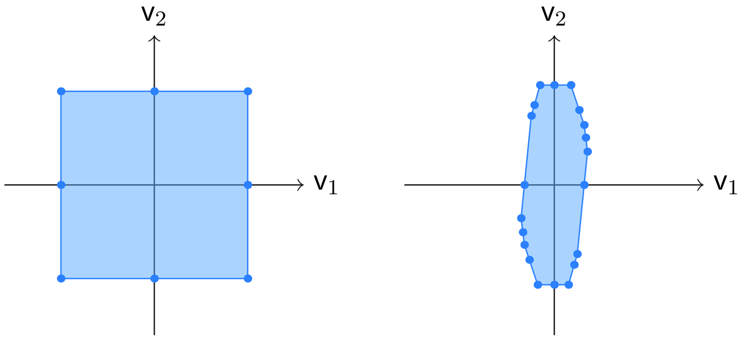


Figure 3.5: The unit balls of the seminorms $-\chi^{(2)}(G;-)$ for both free-by-cyclic examples, including the boundary points that determine them.

c	$\lvert L\rvert$	$\mathrm{rk}(d_1)$	$\mathrm{rk}(d_2)$	$\mathrm{rk}(d_3)$
2	16	0.9375	4.125	2.8125
2^2	4096	0.999 76	4.986 82	3.843 99
limit	–	1	5	4

Table 3.10: Ranks of the differentials of $C_*(\widetilde{F})$ with inferred limit values. The computations for $c = 2^2$ took respectively around 500, 4500 and 900 seconds.

Despite the manifold having a relatively simple and concrete combinatorial description, GAP appears to be unable to produce its cellular chain complex, hanging for an extremely long time. Thus, we study its four-dimensional fiber F instead, which is an aspherical manifold of Euler characteristic 1 admitting no hyperbolic structure.

As always, we construct the $\mathbb{Z}G$-chain complex of $\widetilde{F}$, this time starting from the Regina isomorphism signature given in the paper [67, Section 2.3], obtaining

$$\mathbb{Z}G^4 \xrightarrow{d_3} \mathbb{Z}G^{10} \xrightarrow{d_2} \mathbb{Z}G^6 \xrightarrow{d_1} \mathbb{Z}G. \tag{3.54}$$

Since $G^{\mathrm{ab}} \simeq \mathbb{Z}_4^4$, we expect no nontrivial p-quotients for $p \neq 2$: this is supported by a few quick tests. Hence, we try to approximate the von Neumann rank of the three differentials using only 2-quotients of G. In practice, setting $c = 2^2$ in the algorithm provides good results (Table 3.10).

Note that the ranks are *as high as possible*: indeed, d_1 and d_3 are full rank, while $\mathrm{rk}(d_2)$ is bounded above by $6 - \mathrm{rk}(d_1)$. We easily recover the L^2-Betti numbers of the fiber via the rank-nullity-type formula

$$b_i^{(2)}(\widetilde{F}) = \dim C_i(\widetilde{F}) - \mathrm{rk}(d_i) - \mathrm{rk}(d_{i+1}), \tag{3.55}$$

obtaining

$$b_i^{(2)}(\widetilde{F}) = \begin{cases} 1 & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.56}$$

This confirms $\chi(F) = 1$ and is reminiscent of the *Singer conjecture* for closed aspherical manifolds (which we restate here from the introduction):

Conjecture 1.2 (Singer)**.** *If M is a closed aspherical n-manifold, then all its L^2-Betti numbers vanish, except possibly for $b_{n/2}^{(2)}(\widetilde{M})$ if n is even. In that case, if M also admits a metric with negative sectional curvature, then $b_{n/2}^{(2)}(\widetilde{M}) > 0$.*

This conjecture is known to hold for closed hyperbolic manifolds [41].

Remark 3.54. The computation we carried out suggests a broadening of its scope to include F and possibly some other non-compact manifolds. Let H be an aspherical n-manifold which is the interior of a compact manifold $\overline{H}$ with L^2-acyclic boundary, as is the case for F (see [67, Section 3] and note that the boundary components are flat 3-manifolds, hence virtually fibered and L^2-acyclic). Following [101, p. 2565], we use the *Davis reflection group trick* to construct a compact aspherical n-manifold H'. Moreover, if we assume the Singer conjecture for H' and the *cadim conjecture* in dimension $n-1$ (which can be ensured for $n \leq 4$ [101, Corollary 4.15]), we obtain the cadim conjecture for H, i.e. $b_i(\widetilde{H}) = 0$ for $i > n/2$. We can strengthen the condition to $i \neq n/2$ as a consequence of Poincaré duality, the L^2-homology long exact sequence of $(\overline{H}, \partial\overline{H})$, and L^2-acyclicity of $\partial\overline{H}$; in other words, H satisfies the Singer conjecture.

Some musings and conjectures

A direct consequence of our computation is that the twisted L^2-Betti numbers $b_i^{(2)}(\widetilde{N};\phi)$ of the Ratcliffe-Tschantz manifold all vanish, except for $b_2^{(2)}(\widetilde{N};\phi) = 1$. Of course, a similar argument can be made for any fibered manifold with fiber satisfying the Singer conjecture:

Lemma 3.55. *Assume the Singer conjecture for all even-dimensional closed aspherical manifolds. Let M be a closed aspherical $(2n+1)$-manifold and let $\phi\colon \pi_1(M) \twoheadrightarrow \mathbb{Z}$ be a virtually fibered cohomology class (i.e. its image in a finite cover represents a fibration over the circle). Then $b_i^{(2)}(\widetilde{M};\phi) = 0$ for all $i \neq n$.*

Proof. Since twisted L^2-Betti numbers are multiplicative with respect to finite covers, we can assume that ϕ is a fibered class. The infinite cyclic cover associated to ϕ is homotopically equivalent to the fiber, which must then be aspherical. By the Singer conjecture, its (untwisted) L^2-Betti numbers vanish in dimension not equal to n, and so do the twisted L^2-Betti numbers of M. $\square$

Recall that fibering is an open condition in cohomology, so that by assuming the existence of a single virtually fibered class, we obtain a whole open cone on which $b_i^{(2)}(\widetilde{M};-) = 0$ for $i \neq n$. If we add a tameness condition, such as requiring that the twisted L^2-Betti numbers be seminorms in the variable ϕ, we are led to formulate Conjecture 1.3:

Conjecture 1.3. *Let M be a closed aspherical $(2n+1)$-manifold. Then, for all $i \neq n$, the twisted L^2-Betti number $b_i^{(2)}(\widetilde{M};-)$ is identically zero. Moreover, the twisted L^2-Euler characteristic $\chi^{(2)}(\widetilde{M};-) = (-1)^n \cdot b_n^{(2)}(\widetilde{M};-)$ is a seminorm up to sign.*

Note that the above holds for $n = 0, 1$: in the case of the circle, the twisted L^2-Betti numbers are just the L^2-Betti numbers of the real line (as a G-CW complex with trivial G), which vanish in dimension other than 0; as for 3-manifolds, this follows from [50, Theorem 5.5] by taking $\mu = \mathrm{id}_{\pi_1(M)}$. Indeed, the condition $\mathrm{im}(\mu) \cap \ker(\phi) \neq 1$, or equivalently $\pi_1(M) \neq \mathbb{Z}$, holds because M is a closed 3-manifold and a $K(\pi_1(M), 1)$; hence, $H^3(\pi_1(M); \mathbb{Z}_2) = \mathbb{Z}_2$, while $H^3(\mathbb{Z}; \mathbb{Z}_2) = H^3(S^1; \mathbb{Z}_2) = 0$. Conjecture 1.3 would strengthen the link between the Thurston norm and the twisted L^2-Euler characteristic. As an example, if the latter invariant has a polytopal unit ball B, it becomes natural to ask whether the set of fibered classes is a union of cones over some open faces of B, as in the case of 3-manifolds.

3.6.6 A closed aspherical 5-manifold

At the time of writing this thesis, there is no explicit description of a closed hyperbolic 5-manifold as a CW complex small enough to run the algorithm on it (but see Theorem 1.10). Hence, we will settle for an aspherical manifold M defined as the product of two closed hyperbolic manifolds: $M_3 := \mathrm{m160}(3,1)$ and the genus-2 surface S_2. The first factor is the first entry in Table 3.7 and has S_2 as a fiber.

By the Künneth formula, the first cohomology of M is the direct sum

$$H^1(M; \mathbb{R}) \simeq H^1(M_3; \mathbb{R}) \oplus H^1(S_2; \mathbb{R}) \simeq \mathbb{R} \oplus \mathbb{R}^4. \tag{3.57}$$

After fixing a basis for the second factor, we observe that $(1,0,0,0,0)$ is a fibered class, corresponding to the fiber $S_2 \times S_2$ of Euler characteristic 4. Hence,

$$\chi^{(2)}(\widetilde{M}; (t,0,0,0,0)) = 4|t|. \tag{3.58}$$

In fact, the whole function $\chi^{(2)}(\widetilde{M};-)$ can be determined:

Lemma 3.56. *For all $(t,x,y,z,w) \in \mathbb{Z}^5$, we have $\chi^{(2)}(\widetilde{M}; (t,x,y,z,w)) = 4|t|$.*

Proof. First, we prove that every class (t,x,y,z,w) with $t \neq 0$ is fibered; this is the same as exhibiting a non-vanishing closed 1-form representing the class. Let $\pi_3 \colon M \twoheadrightarrow M_3, \pi_2 \colon M \twoheadrightarrow S_2$ be the natural projections. The M_3 factor has a closed non-vanishing 1-form ω given by the differential of the fibration, multiplied by t. This can be extended to all of M by pulling back, defining a closed 1-form $\alpha \coloneqq \pi_3^*(\omega) \in \Omega^1(M)$, which represents the class $(t, 0, 0, 0, 0)$.

Now choose $\psi \in \Omega^1(S_2)$ representing (x, y, z, w) and extend it similarly to $\beta \coloneqq \pi_2^*(\psi)$. Clearly, $\alpha + \beta$ represents (t, x, y, z, w) and is non-vanishing: given a point $(p, q) \in M$, we may choose $v \notin \ker \omega_p$ as $\omega_p \neq 0$, and then $(v, 0) \notin \ker (\alpha + \beta)_{(p,q)}$.

In a neighborhood of a fibering class, the fiber Euler characteristic is a linear function of the class: this is part of [123, Theorem 3] and applies to any compact oriented manifold. Therefore, the function $\chi^{(2)}(\widetilde{M}; -)$ is locally linear at each point with $t \neq 0$. Since $\{t > 0\}$ is connected, there is a single linear functional $f\colon \mathbb{R}^5 \to \mathbb{R}$ equal to $\chi^{(2)}(\widetilde{M}; -)$ on the upper half-space.

As the extension of $\chi^{(2)}(\widetilde{M}; -)$ to $\mathbb{R}^5$ is continuous, being a difference of two seminorms (Theorem 3.32), it must equal f on $\{t = 0\}$; being an even function, this gives $f(0, x, y, z, w) = 0$. Finally, by (3.58), we infer that f is four times the projection on the first coordinate. □

Following Section 3.5.4, the product structure of M makes it easier to keep track of the basis in which ϕ is expressed. Exploiting this, we ran our algorithm using a $\mathbb{Z}G$-chain complex of dimensions $(1, 7, 16, 16, 7, 1)$, on the class $(1, 0, 0, 0, 0)$, with parameters $\mu = 28$, $c = 2$, obtaining exactly the correct result (that is, 4) in under 20 seconds.

3.7 The Alexander norm and the $\mathbb{Z}[\mathrm{ab}(G)]$-chain complex

In this section we give an overview of the relationship between the Alexander norm and the twisted L^2-Euler characteristic of the $\mathbb{Z}[\mathrm{ab}(G)]$-chain complex C_*^{ab} of a compact orientable 3-manifold, ultimately proving Proposition 3.49. We assume that the boundary of M is a union of zero or more tori. We refer to McMullen's article [95] for an overview of the Alexander norm, recalling from it the definition of the quantity

$$p \coloneqq \begin{cases} 0 & \text{if } b_1(M) = 1, \\ 1 & \text{if } b_1(M) \geq 2 \text{ and } \partial M \neq \varnothing, \\ 2 & \text{if } b_1(M) \geq 2 \text{ and } \partial M = \varnothing. \end{cases} \tag{3.59}$$

By the proof of [95, Theorem 5.1], the Alexander module of G has a presentation as the cokernel of the second differential d_2 of C_*^{ab}. Let r be the rank of G. There are a few cases, mirroring the proof we just mentioned.

If M is closed, we can arrange for C_*^{ab} to be of the form

$$\mathbb{Z}[\mathrm{ab}(G)] \xrightarrow{d_3} \mathbb{Z}[\mathrm{ab}(G)]^n \xrightarrow{d_2} \mathbb{Z}[\mathrm{ab}(G)]^n \xrightarrow{d_1} \mathbb{Z}[\mathrm{ab}(G)], \tag{3.60}$$

where $d_1^T = d_3 = \begin{bmatrix} 1 - g_1 & 1 - g_2 & \dots & 1 - g_n \end{bmatrix}$, $\{g_1, \dots, g_r\}$ is any basis of $\mathrm{ab}(G)$ and $g_{r+1} = \dots = g_n = 1$. Since $\phi\colon \mathrm{ab}(G) \twoheadrightarrow \mathbb{Z}$ has a section, we can choose a basis where $\phi(g_1) = 1$ and $g_2, \dots, g_n \in \ker \phi$; we shall write $t = g_1$.

Moreover, let d_2' be d_2 without its first column and d_2'' be d_2 without its first column and row. Following McMullen's proof, if the rank of G is at least 2, we have that the multivariate Alexander polynomial Δ satisfies $\det d_2'' = \pm(1-t)^2\Delta \neq 0$, so that

$$\deg_\phi \det d_2'' = 2 + \deg_\phi \det \Delta = 2 + \|\phi\|_A. \tag{3.61}$$

If instead $r \leq 1$, then only the $(1,1)$ minor is nonzero and $\det d_2'' = \pm\Delta$; that is to say, $\deg_\phi \det d_2'' = \|\phi\|_A$. Combining both subcases, we have

$$\deg_\phi \det d_2'' = \|\phi\|_A + p. \tag{3.62}$$

As for the twisted L^2-Euler characteristic of C_*^{ab}, we refer to the proof of [50, Theorem 5.1], where a chain complex such as (3.60) is decomposed using two short exact sequences. First, we quotient C_*^{ab} by the image of the inclusion of

$$A_* : \qquad \mathbb{Z}[\mathrm{ab}(G)] \xrightarrow{1-g_1} \mathbb{Z}[\mathrm{ab}(G)] \longrightarrow 0 \longrightarrow 0, \tag{3.63}$$

obtaining a complex

$$0 \longrightarrow \mathbb{Z}[\mathrm{ab}(G)]^{n-1} \xrightarrow{d_2'} \mathbb{Z}[\mathrm{ab}(G)]^n \xrightarrow{d_1} \mathbb{Z}[\mathrm{ab}(G)], \tag{3.64}$$

where d_2' is d_2 without its first column. This complex has a natural projection onto

$$B_* : \qquad 0 \longrightarrow 0 \longrightarrow \mathbb{Z}[\mathrm{ab}(G)] \xrightarrow{1-g_1} \mathbb{Z}[\mathrm{ab}(G)], \tag{3.65}$$

with kernel

$$D_* : \qquad 0 \longrightarrow \mathbb{Z}[\mathrm{ab}(G)]^{n-1} \xrightarrow{d_2''} \mathbb{Z}[\mathrm{ab}(G)]^{n-1} \longrightarrow 0, \tag{3.66}$$

where d_2'' is d_2 without its first column and row. Note that A_*, B_*, D_* are all L^2-acyclic, as their non-trivial differentials have non-zero determinant.

Since the universal L^2-torsion is additive along based short exact sequences (Theorem 3.28), we have

$$\chi^{(2)}(\widetilde{M}; \mathrm{ab}(G), \phi) = \chi^{(2)}(A_*) + \chi^{(2)}(B_*) + \chi^{(2)}(D_*) \tag{3.67}$$
$$= \deg_\phi \det[1-g_1] - \deg_\phi \det d_2'' + \deg_\phi \det[1-g_1]. \tag{3.68}$$

The second equality follows from the discussion around (3.12), with signs added in order to account for the complexes being shifted to the left.

Clearly, the first and last terms in (3.68) are both 1, while the middle term was computed above; summing up the closed case, we get

$$\chi^{(2)}(\widetilde{M}; \mathrm{ab}(G), \phi) = -\|\phi\|_A + 2 - p. \tag{3.69}$$

A similar argument (with only one short exact sequence) shows that if $\partial M \neq \emptyset$, then $\chi^{(2)}(\widetilde{M}; \mathrm{ab}(G), \phi) = -\|\phi\|_A + 1 - p$. Both cases can be condensed into

$$\chi^{(2)}(\widetilde{M}; \mathrm{ab}(G), \phi) = -\|\phi\|_A + \begin{cases} 0 & \text{if } b_1(M) \geq 2, \\ 1 & \text{if } b_1(M) = 1 \text{ and } \partial M \neq \varnothing, \\ 2 & \text{if } b_1(M) = 1 \text{ and } \partial M = \varnothing. \end{cases} \tag{3.70}$$

□

3.8 Final remarks

As we have seen, even if the effective bounds for our algorithm are unusable in practice, we are still able to produce a sequence of approximations that appear to converge at an acceptable rate. In this sense, the algorithm may prove useful as an empirical tool for further research into the twisted L^2-Euler characteristic. In fact, the latter is arguably the most natural candidate for the problem of extending the Thurston norm construction to higher-dimensional spaces, due to its behavior with respect to finite coverings and fibrations over the circle. However, there seems to be a glaring contrast between the concrete and geometric *minimal complexity* definition of the Thurston norm and the abstract objects involved in L^2-invariant theory.

While computational experiments may prove difficult in high dimension due to the sheer combinatorial complexity of the spaces involved, dimension 5 is probably still accessible, even more so with an optimized version of the algorithm. In particular, if M is a closed aspherical 5-manifold, we expect results consistent with $\chi^{(2)}(\widetilde{M}; -)$ being a seminorm, as in Conjecture 1.3, bridging the gap with the Thurston norm.

Of course, as noted above, more research is also needed on the theoretical front: a geometric interpretation of the twisted L^2-Euler characteristic, possibly in a framework that generalizes the Thurston norm, would be incredibly illuminating on these matters.

The Singer conjecture, from which we took inspiration for Conjecture 1.3, might play an important role, as it determines the sign of even-dimensional closed aspherical manifolds, and this sign is inherited by the twisted L^2-Euler characteristic of virtually fibered classes in odd-dimensional closed aspherical manifolds. This "sign constancy" is exploited in the definition of the Thurston norm by considering only surfaces of non-positive characteristic, which are exactly the aspherical ones.

While our "algebraic" approach allows for relative ease of computation, relying on well-established computational homology methods, such a geometric description could produce a more efficient, direct algorithm for the computation of this invariant, of a combinatorial nature akin to normal surface methods in 3- and 4-manifold theory. This would, in turn, enable more experimentation and a better understanding of the twisted L^2-Euler characteristic.

4. Hyperbolic manifolds and Coxeter polytopes

In this chapter, we study two problems related to the construction of hyperbolic manifolds, which can be solved by gluing Coxeter polytopes in a more or less explicit pattern.

Specifically, in Sections 4.1 to 4.6, which are based on a joint work with Edoardo Rizzi [28], we construct cusp-transitive hyperbolic 4-manifolds of finite volume, which realize every possible closed flat 3-manifold as the cusp section (Theorem 1.6). We also study the problem of approximating all flat metrics on such cusp sections (Theorem 1.7), and discuss the weaker condition of having pairwise isometric cusps (Theorem 1.8).

Then, in Sections 4.7 to 4.11, we construct a family of closed hyperbolic 5-manifolds with relatively small volume and vanishing first Betti number, based on a hyperbolic Coxeter prism P_0, proving Theorem 1.10. We classify the manifolds and analyze various properties of them, and we incidentally prove Theorem 1.11. Finally, we outline a variant construction based on the same polytope. This part is based on a preprint by the author [27].

4.1 From reflectofolds to cusp-transitive manifolds

Let N be a closed flat 3-manifold. In this section we will see that, in order to show the existence of a cusp-transitive manifold with cusp type N, it suffices to prove that there exists a 1-cusped developable reflectofold with cusp type N (see Section 2.2.4).

Proposition 4.1. *Let R be a 1-cusped developable reflectofold with complete cusp section, and built by gluing finitely many finite-volume polytopes. Then, there exists a cusp-transitive manifold M having cusps isometric to that of R.*

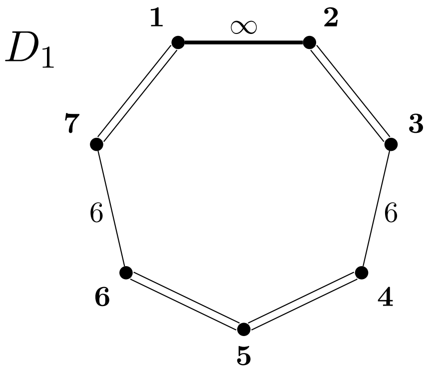
Proof. The desired manifold M can be constructed as in [115, Section 2] by gluing finitely many copies of R following Davis' basic construction [40, Chapter 11]. Since R has finite volume, M will also have finite volume. Note that, unlike [115], we do not require reflectofolds to be complete; however, M will be complete since it is obtained by gluing polytopes, and has complete cusp sections (see [110, Theorem 11.1.6]). □

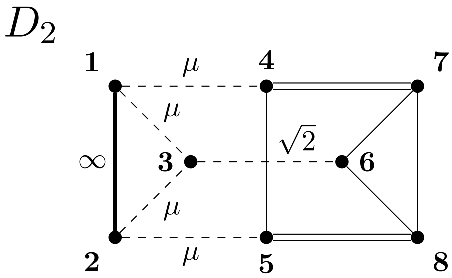
Remark 4.2. If the cusp section of the cusp-transitive manifold M thus constructed is orientable, we can also construct an orientable cusp-transitive manifold with the same cusp section, namely the orientable double cover of M, as in [115, Corollary 2.4].

4.2 Some polytopes

In this section we will introduce and construct the polytopes P_1 and P_2, which we will use to construct developable reflectofolds as required by Proposition 4.1.

Consider the following Coxeter diagrams D_1 and D_2, where $\mu := \sqrt{7/3}$:





We shall refer to the vertices of D_1 and D_2 (and to the associated facets) using boldface numbers.

The diagram D_1 corresponds to a finite-volume arithmetic hyperbolic Coxeter 4-polytope P_1, introduced in [66], which satisfies (a) and (b). Indeed, by Proposition 2.22, the ideal vertices correspond to the maximal affine subdiagrams of the Coxeter diagram, and in D_1 we have exactly one of this kind (see [130, Table 2]), spanned by the vertices $\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{6}$. The polytope L_1 corresponding to this subdiagram is the horospherical link of the ideal vertex of P_1, and it is a Euclidean right prism, whose base is a triangle with interior angles $(\pi/2,\ \pi/4,\ \pi/4)$. Arithmeticity can be verified through Vinberg's criterion [126, Theorem 2], which is implemented in the program CoxIter [58; 59].

The diagram D_2 can also be shown to define a hyperbolic Coxeter 4-polytope using Vinberg's theorem (Proposition 2.17): it is obviously connected, and we can check that its Gram matrix has signature $(4, 1, 3)$. Using CoxIter, we also find that P_2 has finite volume and is non-arithmetic. Moreover, like the polytope P_1, it satisfies (a) and (b): there is exactly one ideal vertex, corresponding to the unique maximal affine subdiagram of D_2, which is spanned by $\mathbf{1}, \mathbf{2}, \mathbf{6}, \mathbf{7}, \mathbf{8}$. Its link L_2 is a Euclidean right prism over an equilateral triangle.

4.2.1 Visualizing polytopes with one ideal vertex

In order to better understand the geometry of a 4-polytope with one ideal vertex, such as P_1 or P_2, it may prove useful to *project* the compact boundary facets onto the horospherical link of the ideal vertex as follows.

Place such a polytope P in the half-space model of $\mathbb{H}^4$ with the ideal vertex at infinity, so that the non-compact facets of P become vertical, and let $\pi\colon \mathbb{H}^4 \to \mathbb{R}^3$ be the orthogonal

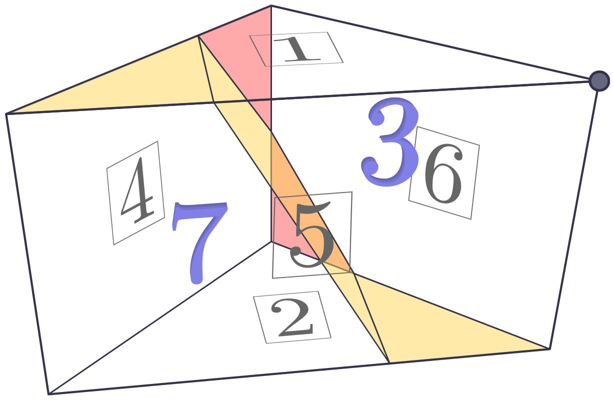


Figure 4.1: The projection of P_1 onto the link L of its ideal vertex. The five faces of L and the two interior regions are labeled with the corresponding facets of P_1: five non-compact $(\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{6})$ and two compact $(\mathbf{3}, \mathbf{7})$.

projection onto a horizontal hyperplane, such as a horosphere or the ideal boundary. The image of P under π is the three-dimensional link L of the ideal vertex, and the non-compact facets are mapped to its boundary.

We can compute the face lattice of P by enumerating all spherical subdiagrams of D, and therefore determine the shape of the regions associated to the compact facets, by applying the following result.

Proposition 4.3. *Let Z be a hyperbolic n-polytope in $\mathbb{H}^n$ with one ideal vertex v. Let Λ be the link of v. The images of the compact facets of Z under the projection onto a horosphere centered at v partition Λ into a polyhedral complex, whose face lattice is combinatorially isomorphic to the lattice of compact faces of Z.*

Proof. We consider the half-space model where v is at infinity. We say that a k-subspace of the half-space model is *vertical* if its ideal boundary contains v.

Let F be a compact k-face of Z. Then, in the half-space model, F is supported on a k-hemisphere H. Let Σ be the unique vertical $(k+1)$-space containing H. The face F is bounded by the supporting $(k-1)$-spaces of its facets $F_1, \ldots, F_m$. Each F_i lies on a unique vertical k-space S_i. As such, F is the intersection of H and some hyperbolic half-$(k+1)$-spaces bounded by the S_i and contained in Σ. Let P_F be the Euclidean k-polytope obtained by intersecting the projections of these half-spaces on the horosphere; then we have $F = (P_F \times (0, +\infty)) \cap H$. Hence, the projection is a homeomorphism between F and P_F, which preserves facets. Since Z has only one ideal vertex, every compact face of Z is contained in a compact facet. Hence, by induction on the codimension of faces, the projection preserves the face lattice of the compact boundary of Z. □

The resulting projection in the case of the polytope P_1 is shown in Figure 4.1. Note that dihedral angles between facets of P are defined along ridges (codimension-2 faces),

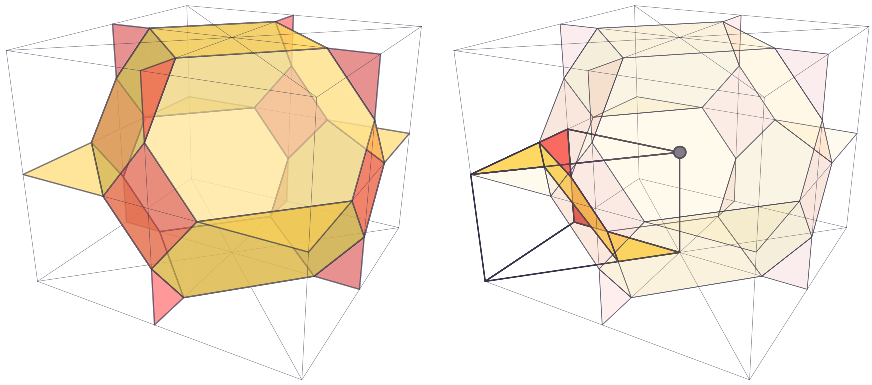


Figure 4.2: On the left, the projection of Q_1 onto the link C. The truncated octahedron corresponds to a facet made of 16 copies of the facet **3** of P_1. The other 8 regions appear when copies of the facet **7** merge together two at a time. On the right, we emphasize one copy of L inside C.

which appear in Figure 4.1 either as 2-faces (if the angle involves a compact facet) or as edges of the link L (if the angle is between two non-compact facets). We mark the former by coloring certain 2-faces of the projection, with the following rule. Every 2-face corresponds to a ridge between either two compact facets or a compact facet and a non-compact one. In the first case, we assign the dihedral angle to the 2-face, while in the second case, we assign twice the dihedral angle. We use yellow for $\pi/2$, red for $\pi/3$, and we leave the 2-face transparent for π.

This rule keeps track of the fact that, if we glue two copies of P along a non-compact facet F, the angle along any ridge it shares with a compact facet gets doubled. For instance, in the case of P_1, the dihedral angle between facets **1** and **7** is $\pi/4$; when doubling P_1 along **1**, the angle becomes $\pi/2$, so we color in yellow the corresponding triangle in Figure 4.1.

Moreover, after doubling, some ridges end up with an angle of π, meaning that the compact facet is coplanar with its mirrored copy. Leaving the 2-face transparent has the advantage of visually representing when compact facets merge together in an arbitrary gluing of copies of P.

4.2.2 The construction of the polytope Q_1

In this section we will glue 16 copies of P_1 in order to form a bigger polytope Q_1.

Consider the subdiagram of D spanned by the vertices $\mathbf{1}, \mathbf{5}, \mathbf{6}$. This induces a subgroup $G \simeq \mathbb{Z}_2 \times D_4$, of cardinality 16, which is the stabilizer of a non-compact edge of P_1. This edge projects to the marked vertex in Figure 4.1.

We can define a new polytope Q_1 by taking the orbit of P_1 under G. Equivalently, we glue 16 copies of P_1 along their non-compact facets $\mathbf{1}, \mathbf{5}, \mathbf{6}$ using the identity map. By construction, Q_1 has exactly one ideal vertex, whose link is a right prism C with a square base, obtained by placing 16 copies of L around the marked vertex (Figure 4.2).

The facets **3** merge into a single facet of Q_1 (which we will also call **3**), which projects to an irregular truncated octahedron in C, with 8 hexagonal and 6 quadrilateral facets. The latter correspond to ridges of Q_1 where the facet **3** meets the non-compact facets: the dihedral angles are $\pi/6$ for the vertical quadrilaterals and $\pi/4$ for the horizontal ones.

The height, width and depth of C are in a ratio of $\cos(\pi/4) : \cos(\pi/6) : \cos(\pi/6) = \sqrt{2} : \sqrt{3} : \sqrt{3}$. Indeed, suppose that C is the cuboid $[-x_1, x_1] \times [-x_2, x_2] \times [-x_3, x_3]$. Then, in the conformal half-space model, each non-compact facet of Q_1 is contained in the product of a facet of C and $(0, +\infty)$. The facet **3** is supported on a hemisphere, which is centered at $(0, 0, 0, 0)$ by symmetry. It is not hard to see that, if this hemisphere has radius r, then the acute angles with the supporting hyperplanes of the non-compact facets of Q_1 are $\theta_i := \arccos(x_i/r)$. Hence, the cosines of the θ_i are in the same ratios as the x_i.

Because of this, all symmetries of C must preserve or exchange the two horizontal faces. There are 16 such symmetries, and they are generated by reflections in the facets $\mathbf{1}, \mathbf{5}, \mathbf{6}$ of L, so they extend to symmetries of Q_1 and they form a group isomorphic to G. This group can also be defined *a priori* as the group of symmetries of a *combinatorial* cube preserving or exchanging a certain pair of opposite facets.

Remark 4.4. The polytope C naturally tessellates $\mathbb{R}^3$ by translations. This gives a tessellation into truncated octahedra in the following way: some are centered and contained in the copies of C, as in Figure 4.2, while the others are centered at the vertices of the tessellation; one-eighth of a truncated octahedron can be seen near each vertex of C in Figure 4.2. The two types of truncated octahedra are actually congruent, because of the symmetry of D_1 that exchanges **3** and **7**. Indeed, passing to the dual tessellation by copies of C exchanges the two types of truncated octahedra.

4.3 Layered tessellations of 3-manifolds

In this section we will prove that if a flat 3-manifold N admits a tessellation in cubes with some properties, then there is a cusp-transitive hyperbolic 4-manifold with cusp type N.

Definition 4.5 (Layered tessellation)**.** Let T be a tessellation of a flat 3-manifold N into Euclidean cubes, with some chosen *special* 2-cells. We say that T is *layered* if each cube has two opposite special facets, and whenever two cubes C_1, C_2 share a facet F, the reflection through F sends the special facets of C_1 to those of C_2, and vice versa.

The above condition causes the special facets to fall into several embedded geodesic surfaces tessellated by squares, in a discrete analog of a foliation.

Remark 4.6. We will only make use of the combinatorial properties of layered tessellations; hence, we may define generalizations, such as layered tessellations by rectangular cuboids, in the same way.

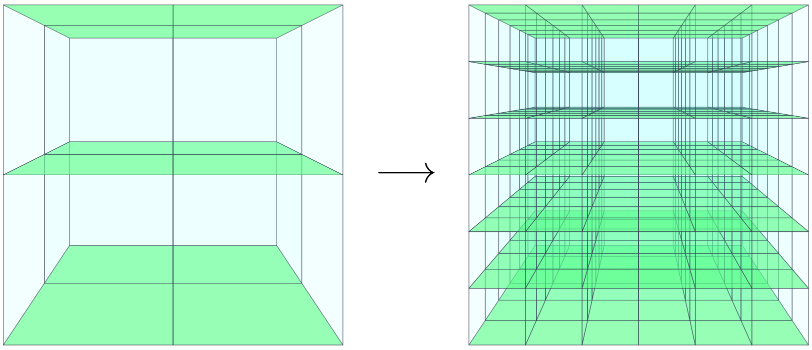


Figure 4.1: Subdividing a layered tessellation. Special faces are colored green.

Remark 4.7. A layered tessellation can be *subdivided* by replacing every cube with a block of $n \times n \times n$ cubes, $n \geq 1$, in which we mark as special all facets parallel to the two original special facets (see Figure 4.1).

Definition 4.8. We say that a layered tessellation is *proper* if the following dual conditions hold:

- every cube is distinct from its six neighbors, which are pairwise distinct;
- every vertex is distinct from its six neighbors, which are pairwise distinct.

Lemma 4.9. *Every layered tessellation can be subdivided into a proper one.*

Proof. First, we realize each cube of the tessellation as a cube with side length 1. This gives a flat Riemannian metric on a compact 3-manifold, which has an injectivity radius $r > 0$. We then subdivide the tessellation as in Remark 4.7, in such a way that the side length of the little cubes is less than r. The resulting tessellation is proper because, for every cube, the centers of it and its neighbors are contained in an embedded ball, and a similar argument applies to the vertices. □

Theorem 4.10. *Let N be a closed 3-manifold that admits a layered tessellation. Then there exists an arithmetic cusp-transitive hyperbolic 4-manifold with cusp type N.*

Proof. By Proposition 4.1, it suffices to prove that there exists a 1-cusped developable reflectofold with cusp type N, obtained by gluing copies of Q_1.

By Lemma 4.9, we may assume that the tessellation is proper.

We have that N is obtained by gluing some copies of C along their facets, where the gluing maps are restrictions of isometries of G: this gluing is induced by the layered tessellation, where the special facets correspond to the horizontal facets of C. Since there is a natural correspondence between facets of C and non-compact facets of Q_1, and every isometry in G extends to Q_1 accordingly, we can glue copies of Q_1 in the same pattern. This gives a 1-cusped reflectofold with cusp type N.

It remains to prove that N is developable. The facets of N are hyperbolic truncated octahedra, which arise from the merging of 16 facets **3** or **7**; we will call them *of types* **3** and **7** respectively. The former are centered at the centers of the cubes, while the latter are centered at the vertices of the tessellation.

If a facet of type **3** were adjacent to itself, it would be so along a quadrilateral corner, and so it would also occupy a neighboring cube. However, the two cubes must be distinct by properness of the tessellation. A similar argument involving vertices works for facets of type **7**.

As for angle consistency, if two facets of different type intersect, they do so in a hexagonal corner with a dihedral angle of $\pi/2$. If a facet intersects another of the same type, say **3**, it must do so at a single quadrilateral corner, since the neighbors of a given cube are pairwise distinct by properness; angle consistency follows trivially. A dual argument deals with the case of two facets of type **7**.

The final 4-manifold we obtain is arithmetic, since it covers the arithmetic orbifold P_1. □

Corollary 4.11. *Let N be one of the manifolds $E_1, E_2, E_4, E_6, B_1, B_2, B_3, B_4$. Then there exists a cusp-transitive arithmetic hyperbolic 4-manifold with cusp type N.*

Proof. As shown in Figure 4.2, the manifold N admits a layered tessellation, so the result follows from Theorem 4.10. □

4.4 Marked tessellations of 3-manifolds

In this section we will construct cusp-transitive manifolds having the remaining two cusp sections: the $\frac{1}{3}$-twist and $\frac{1}{6}$-twist manifolds.

Recall the polytope P_2 introduced in Section 4.2. Using Proposition 4.3, we can draw the compact facets in the projection onto the link L_2 (Figure 4.1), and color the 2-faces of the resulting polyhedral complex with the rule of Section 4.2.1.

From the diagram D_2, we can see that the polytope P_2 has a symmetry σ that exchanges the facets **1** and **2**, **4** and **5**, **7** and **8** (see Proposition 2.23). This induces a half-turn rotation of L_2 swapping each vertex with the other one of the same color in Figure 4.1. This rotation is the only nontrivial color-preserving combinatorial automorphism of L_2.

Mirroring the previous sections, we will prove that if a flat 3-manifold N admits a tessellation in prisms over equilateral triangles with some properties, then there is a cusp-transitive hyperbolic 4-manifold with cusp type N.

Definition 4.12 (Marked tessellation)**.** Let T be a tessellation of a flat 3-manifold N into right prisms over equilateral triangles, such that the vertices are colored red, yellow

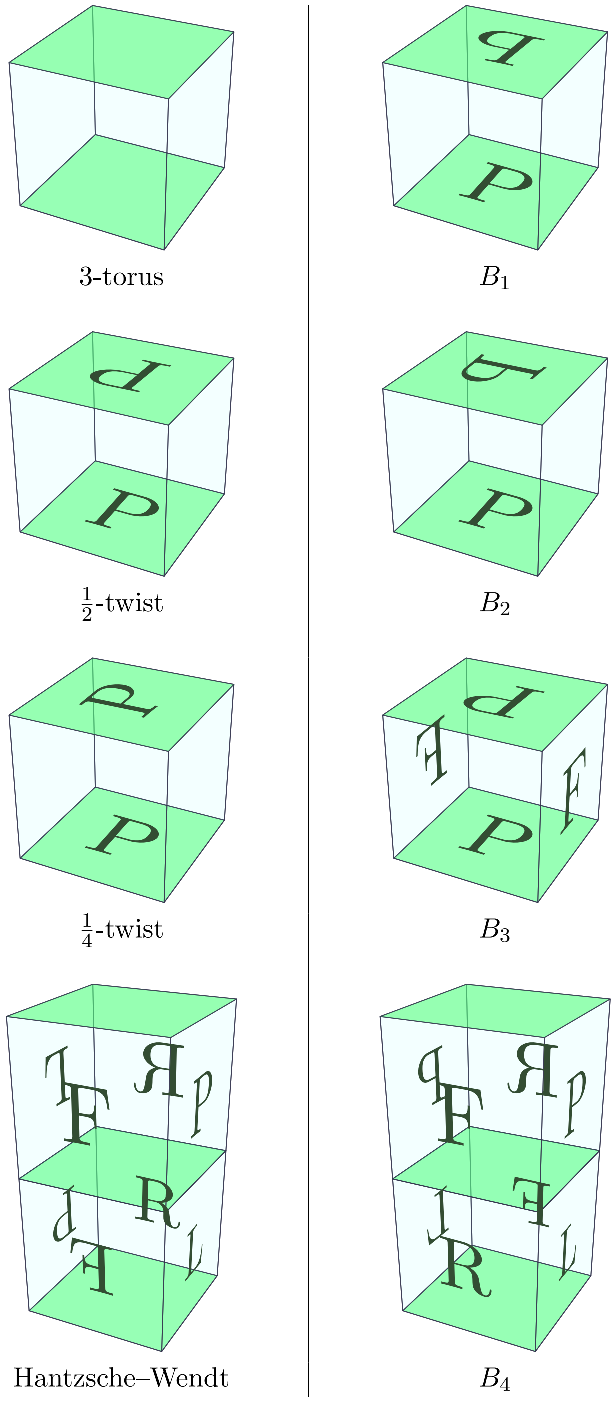


Figure 4.2: Layered tessellations for eight closed flat 3-manifolds. Labeled facets are glued according to their labels, while unlabeled facets are glued to the opposite facets by translation. Fundamental domains and gluing maps can be deduced from the algebraic descriptions of [35, Table 12].

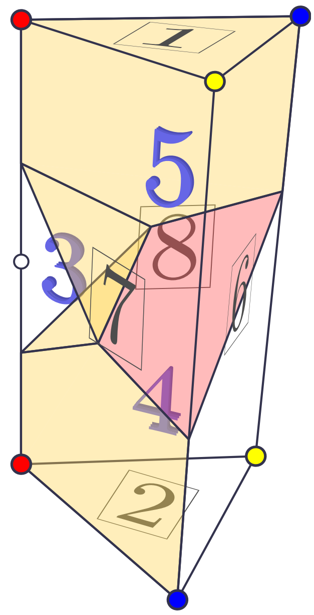


Figure 4.1: The projection of P_2 onto the link L_2 of its ideal vertex, with labels indicating the five non-compact facets ($\mathbf{1}, \mathbf{2}, \mathbf{6}, \mathbf{7}, \mathbf{8}$) and three compact facets ($\mathbf{3}, \mathbf{4}, \mathbf{5}$).

or blue. We say that T is *marked* if the vertices of each prism are colored as in Figure 4.1 (up to combinatorial isomorphism), and whenever two prisms share a facet F, they are symmetrical with respect to F, including their vertex colorings.

Remark 4.13. Given a marked tessellation T and two natural numbers a, b, we can *subdivide* each prism as follows. First, note that an equilateral triangle can be subdivided into a^2 triangles (whose sides are a times smaller), by cutting it with three sets of equally spaced lines parallel to the sides; similarly, a prism can be subdivided into a^2 thin prisms. Each of those can then be cut, parallel to the bases, into b equal prisms. The new tessellation T' can be marked provided that b is odd and that $a-1$ is a multiple of 3 (see Figure 4.2 for the case $a = 4$, $b = 3$). Indeed, for each prism t of T, we now have $b+1$ layers of vertices of T'. It is not hard to see that, whenever $a \equiv 1 \pmod 3$, there is a unique way to 3-color the bottom layer consistently with the lower three vertices of t. Each subsequent layer will be the same as the one below it, but with yellow and blue swapped. If b is odd, then the top and bottom layers are the same with yellow and blue swapped, ensuring consistency with the coloring of t.

Remark 4.14. When gluing copies of P_2 along non-compact facets, the compact facets merge into two kinds of new facets, which we can visualize with the help of the projection in Figure 4.1. The first kind arises from 6 copies of the facet $\mathbf{3}$ around the white dot, while the second comes from 12 copies of the facet $\mathbf{4}$ or $\mathbf{5}$ around the corresponding yellow dot. Both kinds of facets are bounded.

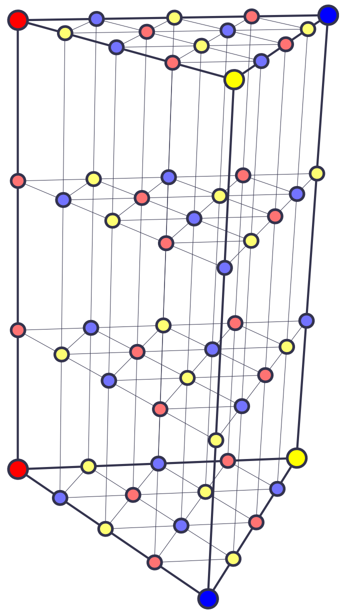


Figure 4.2: Subdividing a marked tessellation ($a = 4$, $b = 3$). The pattern on the bottom face of the prism works whenever $a - 1$ is a multiple of 3.

Theorem 4.15. *Let N be a closed 3-manifold that admits a marked tessellation. Then there exists a non-arithmetic cusp-transitive hyperbolic 4-manifold with cusp type N.*

Proof. The proof is similar in many aspects to that of Theorem 4.10, so we will only go over the main points. Again, by Proposition 4.1, it suffices to prove that there exists a 1-cusped developable reflectofold with cusp type N, obtained by gluing copies of P_2.

A marked tessellation of N induces a way to glue copies of P_2 along their non-compact facets; a key fact is that every color-preserving combinatorial automorphism of L_2 is induced by a symmetry of P_2. This gives a 1-cusped reflectofold with cusp type N.

In order to ensure (EF) and (AC), it suffices that each facet of the reflectofold, together with its adjacent facets, is inside an embedded ball. Indeed, this ensures that every facet is embedded and that the intersection of two adjacent facets is a single corner. Since the facets are bounded by Remark 4.14, this can be done by an argument involving the injectivity radius and a sufficiently fine subdivision of the marked tessellation; the reflectofold constructed from the subdivided tessellation is developable.

The 4-manifold thus constructed is non-arithmetic since it covers the non-arithmetic orbifold $P_2/\langle\sigma\rangle$, where σ is the order-2 isometry of P_2 defined at the beginning of this section. □

Corollary 4.16. *Let N be one of the manifolds E_3, E_5. Then there exists a cusp-transitive hyperbolic 4-manifold with cusp type N.*

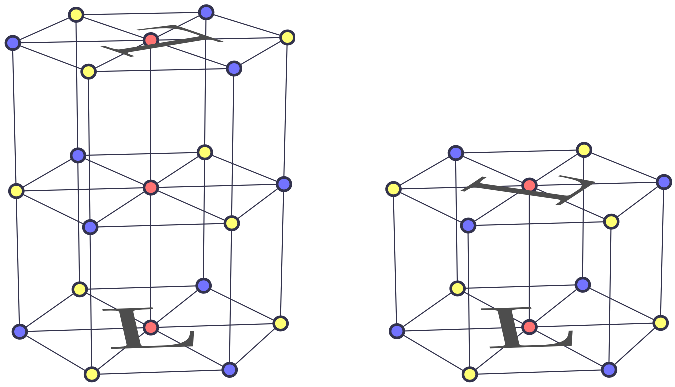

$\frac{1}{3}$-twist $\frac{1}{6}$-twist

Figure 4.3: Marked tessellations for E_3 and E_5. Labeled facets are glued according to their labels, while unlabeled facets are glued to the opposite facets by translation. Fundamental domains and gluing maps can be deduced from the algebraic descriptions of [35, Table 12].

Proof. The manifolds E_3 and E_5 have marked tessellations, as shown in Figure 4.3; the result follows from Theorem 4.15. □

4.5 Density

In this section we will strengthen the previous results by investigating the possible Euclidean structures that can be realized by the cusp sections of a cusp-transitive 4-manifold, and we show how the same techniques can be used in the 3-dimensional case. We start by proving Theorem 1.7:

Theorem 1.7. *For every closed flat* 3*-manifold* N*, the set of flat metrics on* N *which can be realized as cusp sections of a cusp-transitive* 4*-manifold is dense in the space of all flat metrics of* N*.*

Proof. The argument is inspired by the proof of [99, Theorem 2]. We divide the proof into two cases: the manifolds $E_1, E_2, E_4, E_6, B_1, B_2, B_3, B_4$ (tessellated by cubes) and the manifolds E_3, E_5 (tessellated by triangular prisms).

We start with the first case. Any flat metric on N is induced by a subgroup $\Gamma < \text{Isom}(\mathbb{R}^3)$. Generators for Γ are found in [99, Table 1] in the form (A, t_u), where $A \in \text{O}(3)$ and t_u is a translation by the vector u, denoting the map $v \mapsto u + Av$.

The group Γ can be conjugated as in [99, pp. 128–129] so that all the matrices A_i are certain fixed signed permutation matrices, which preserve the x axis, and preserve or exchange the y and z axes. In this *normalized form*, the translation vectors have some zero entries, while the k-uple of the other *free* entries can take any value in an open set of

$\mathbb{R}^k$, containing $(\mathbb{R} \setminus \{0\})^k$. A dense subset of these forms can be obtained by considering only vectors of the form $v_i \coloneqq (\sqrt{2}a_i, \sqrt{3}b_i, \sqrt{3}c_i)$, where a_i, b_i, c_i are in $\mathbb{Q} \setminus \{0\}$ or $\{0\}$, depending on whether the corresponding entry of v_i is free or not. Let Γ' be a group with parameters in this dense subset; we shall prove that the metric resulting from Γ' can be realized by a layered tessellation by copies of C (see Remark 4.6).

The group Γ' preserves a lattice of the form

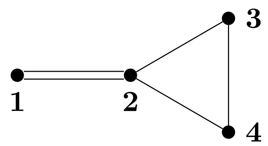
$$\left\{ \frac{1}{d}(\sqrt{2}m, \sqrt{3}n, \sqrt{3}p) \mid m, n, p \in \mathbb{Z} \right\},$$

where d is a common denominator of all the a_i, b_i, c_i. Furthermore, it preserves a layered tessellation of $\mathbb{R}^3$ by copies of C (having side lengths $\sqrt{2}/d$ and $\sqrt{3}/d$) with the special facets orthogonal to the x axis; this is because the x axis is preserved by the A_i. This tessellation descends to the quotient $\mathbb{R}^3/\Gamma'$. The result follows as in Theorem 4.10.

As for the second case, we refer to [35, Section 4]. The space of flat metrics on N has two parameters: in the hexagonal prism fundamental domains of Figure 4.3, they can be taken as the side length and height of the prisms. When subdividing a marked tessellation in the proof of Theorem 4.15, we can choose two parameters a and b provided that b is odd, $a-1$ is a multiple of 3, and they are sufficiently large. The effect of these parameters is to multiply the side length and height of the fundamental domains by a and b respectively. Since we can realize the tessellation with arbitrarily small copies of L_2, by choosing a and b in an appropriate way, we can approximate any values of the two parameters of the metric of N. The result follows as in Theorem 4.15. □

4.5.1 Dimension 3

We consider the following diagram D_3, found in [21, Appendice, Rang 4]:



The corresponding Coxeter polytope P_3 is a finite-volume hyperbolic tetrahedron with one ideal vertex, corresponding to the subdiagram spanned by **2**, **3**, **4**; its link is an equilateral triangle. The polytope has the properties (a) and (b).

Theorem 4.17. *For every closed flat 2-manifold S (i.e. the torus and the Klein bottle), the set of flat metrics on S which can be realized as cusp sections of a cusp-transitive 3-manifold is dense in the space of all flat metrics of S.*

Proof. We begin by gluing six copies of P_3 around the edge between the facets **3** and **4**. The resulting polytope Q_3 has one ideal vertex, whose link is a regular hexagon, and one compact facet, which meets the other six facets at an angle of $\pi/4$.

If S has a tessellation by regular hexagons, such that every hexagon is embedded, then we can glue copies of Q_3 along their non-compact facets in the same pattern. The resulting manifold with corners R is a developable reflectofold: its dihedral angles are all equal to $2 \cdot \pi/4 = \pi/2$, also implying (AC), and its facets are embedded hyperbolic hexagons (EF). Moreover, the section of its unique cusp is isometric to S up to global rescaling. As usual, by Proposition 4.1, we can construct a cusp-transitive manifold which covers R, with cusps isometric to S.

It remains to show that, up to arbitrarily small perturbations, every flat metric on the torus or the Klein bottle admits a tessellation by regular hexagons.

First, consider a parallelogram fundamental domain D_T for the torus, and overlay it onto a tessellation of small regular hexagons. We can perturb D_T by moving three vertices to the nearest hexagon center, and the fourth in such a way as to make a parallelogram; the fourth vertex will also fall into the center of a hexagon. This gives our desired tessellation.

As for the Klein bottle, the generic fundamental domain D_K is a rectangle, with two opposite sides s_1,s_2 identified with a twist. We overlay D_K onto a tessellation of small regular hexagons with some sides parallel to s_1,s_2. We can perturb D_K by translation or scaling along either axis, obtaining a rectangle with all vertices at the centers of hexagons. Note that the tessellation is symmetrical with respect to the line joining the midpoints of the perturbed s_1 and s_2. Hence, we have a tessellation of the Klein bottle. □

4.6 Lots of manifolds

In this section we prove Theorem 1.8:

Theorem 1.8. *For every closed flat 3-manifold N, there exists a positive constant c such that, for sufficiently large $V > 0$, there exist at least V^{cV} complete hyperbolic 4-manifolds with pairwise isometric cusps of type N and volume $\leq V$.*

Proof. Let R be a reflectofold constructed as in the proof of Theorem 4.10 or 4.15 (according to N). Let D be the Coxeter diagram with one vertex for each facet of R, where if two facets meet with dihedral angle of π/k (respectively, do not intersect), the corresponding vertices are joined by an edge labeled k (respectively, a dashed edge).

If R is constructed from a sufficiently fine subdivision of a tessellation, then the diagram D is connected. Indeed, two non-adjacent facets are connected by a dashed edge, while for any two adjacent facets F, F' there exists a third one not adjacent to them (in the projection onto the link, it can be found in a large embedded ball centered on, say, F). As a consequence, we can also assume that D has a dashed edge.

Let G be the Coxeter group associated to D. It is neither affine nor spherical, since D has a dashed edge (compare [130, Tables 1–2]). By [93, Corollary 2], G has a finite-index subgroup H with a quotient isomorphic to the free group F_2. Since H is a subgroup of a Coxeter group, it has a torsion-free subgroup H' such that $d := [G : H'] < +\infty$.

The image of H' in F_2 is free of rank at least 2, so H' also has a quotient isomorphic to F_2. By [98, Theorem 2], there exists a constant $\alpha > 0$ such that F_2 has at least $\alpha r \cdot r!$ subgroups of index $\leq r$. Hence, by pulling back to H', we obtain at least $\alpha r \cdot r!$ torsion-free subgroups of G of index $\leq dr$. These correspond to manifold covers of R with degree $\leq dr$, which have pairwise isometric cusps of type N by construction. As in the proof of Proposition 4.1, we can show that these manifolds are complete.

Let v be the volume of R and let $V := vdr$. Then our manifolds have volume $\leq V$. Using Stirling's approximation, we have the following estimate (for some $k, k', c > 0$ and for r large):

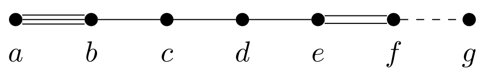
$$\begin{aligned}
\log(\alpha r \cdot r!) &= \log \alpha + \log r + \log r! \\
&= \log \alpha + \log r + r \log r - r + O(\log r) \\
&\geq kr \log r \\
&= k' \frac{V}{vd} \log \frac{V}{vd} \\
&\geq cV \log V \\
&= \log(V^{cV}).
\end{aligned}$$

Hence, we have at least V^{cV} torsion-free subgroups of G. The associated manifolds (of volume $\leq V$) are not necessarily distinct; however, the same estimate holds on the number of isometry classes, with a smaller constant c. Indeed, if G is non-arithmetic, we conclude with the same argument of [17, "The lower bound"] using Margulis' theorem on the commensurator [92, Theorem 1, page 2], while if G is arithmetic, we conclude as in [10, Section 5.2] using the Kazhdan–Margulis theorem. □

Remark 4.18. Note that, since we take subgroups of G which are not necessarily normal, the group G does not act on the associated covers. Hence, we do not necessarily get cusp-transitive manifolds.

4.7 A compact hyperbolic Coxeter prism

Let us now turn to the problem of constructing closed hyperbolic 5-manifolds. To this end, we introduce a hyperbolic Coxeter 5-polytope P_0, corresponding to the Coxeter group

$\Gamma :=$ 

(4.1)

This polytope was first studied by Bugaenko [16], and it is remarkably simple: its number of facets exceeds its dimension by 2 (as opposed to 1 for simplices). In fact, P_0 is a prism with bases f and g, which are combinatorially isomorphic to the fundamental domain of the subgroup $\langle a, b, c, d, e \rangle$ acting on $\mathbb{H}^4$; what is more, both bases are Coxeter 4-simplices.

The facet g makes right angles with all the other facets it intersects; hence, it is isometric to the Coxeter 4-simplex $[5, 3, 3, 3]$, which is the fundamental orthoscheme of a hyperbolic 120-cell with dihedral angles of $2\pi/3$ (of order 3), and is described by the subdiagram


(4.2)

induced by $\{a, b, c, d, e\}$. The situation for f is slightly more complex: to compute the dihedral angle between $d \cap f$ and $e \cap f$ as facets of f, we note that the link of $d \cap e \cap f$ is a spherical triangle with angles of $\pi/3, \pi/2, \pi/4$. By spherical trigonometry, the angle between $d \cap f$ and $e \cap f$ is $\pi/4$. Hence, the facet f is the simplex $[5, 3, 3, 4]$, i.e., the fundamental orthoscheme of a right-angled (order-4) 120-cell:


(4.3)

The orbit of P_0 with respect to the subgroup $\langle a, b, c, d \rangle$ is a 120-cell prism Q_0, which realizes a *cobordism* between two hyperbolic 120-cells with different dihedral angles (Figure 4.1). The sides of Q_0 make angles of $\pi/2$ with the order-3 base and $\pi/4$ with the order-4 base.

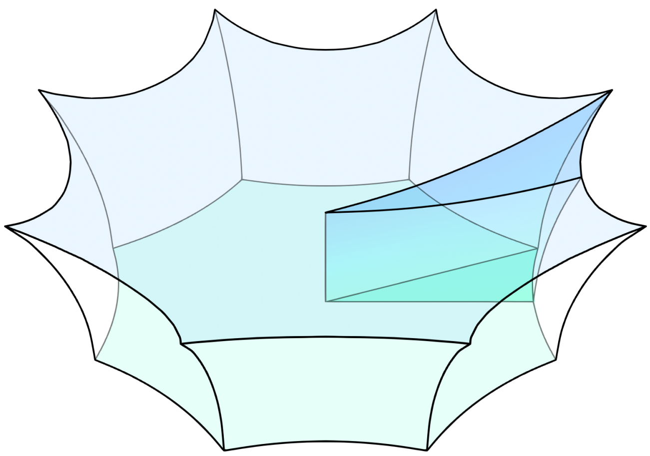


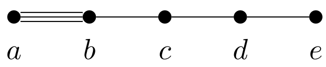
Figure 4.1: A schematic three-dimensional representation of the simplicial prism P_0 within the 120-cell prism Q_0.

The order-3 base sits naturally inside a 120-cell honeycomb which tiles a copy of $\mathbb{H}^4$. By placing copies of Q_0 onto $\mathbb{H}^4$ in the same pattern, we obtain a combinatorially infinite polytope, where one facet is $\mathbb{H}^4$, and the rest are infinitely many right-angled 120-cells meeting at right angles with each other. The vertices thus formed have the geometry of a 5-cube vertex.

While this object is infinite, it is still possible to find compact quotients of the order-3 120-cell honeycomb. We can glue copies of Q_0 onto such a manifold, obtaining an orientable 5-manifold with corners, with right angles between its codimension-1 boundary components (Figure 4.2). We shall then apply the *coloring method* (see Section 2.2.5) to obtain closed orientable manifolds, as we will show in the following sections.

4.8 Some closed 4-manifolds

The goal of this section is to find closed 4-manifolds tessellated by copies of a 120-cell D with dihedral angle $2\pi/3$, which is equivalent to finding finite-index torsion-free subgroups of the Coxeter group

$G :=$  (4.4)

Some research has already been done on this problem, starting with the smallest known closed hyperbolic 4-manifold, of Euler characteristic 8, constructed by Conder–Maclachlan [33], which was later accompanied by more examples of Long [83] with the same volume. Our construction will take inspiration from the latter manifolds, which have a more canonical, algebraic definition.

4.8.1 Long's manifolds

Long's construction starts from noticing that G has a subgroup

$$S := \langle a, c, e, decd, bacbab \rangle \tag{4.5}$$

of index 85. The permutation action ρ of G on its right cosets defines a quotient of G with image isomorphic to the simple group $\mathrm{O}(5,4)$, an orthogonal group over $\mathbb{F}_4$ of cardinality $979\,200 = 2^8 \cdot 3^2 \cdot 5^2 \cdot 17$; these claims and subsequent ones may be verified with a computer algebra system such as GAP [54].

In order to check that a subgroup has no torsion, it suffices to ensure that it does not intersect the conjugacy classes of prime-order elements of vertex stabilizers of G. Representatives of such classes were computed in [33].

The normal subgroup $K := \ker \rho$ contains the prime torsion conjugacy class of $(abcd)^{15}$ and no others. Consider now a 17-Sylow subgroup $C_{17} < \mathrm{O}(5,4)$: it turns out that the preimage $C_{17}K$ has the same property (and has index $57\,600$ in G). Finally, 8 of the 15 nontrivial maps in $\mathrm{Hom}(C_{17}K, \mathbb{Z}_2)$ do not annihilate $(abcd)^{15}$: equivalently, this torsion element is sent to a non-zero vector in $(C_{17}K)^{\mathrm{ab}} \otimes \mathbb{Z}_2$. Such maps define eight torsion-free subgroups H_i ($i = 1, \ldots, 8$) of index $115\,200$ in G, which correspond to manifolds Z_i tessellated by $115\,200$ simplices and $115\,200/14\,400 = 8$ copies of the 120-cell D.

As can be expected from the small size of these manifolds, their tessellations all involve self-adjacencies in copies of D, which makes them unsuitable for constructing developable right-angled manifolds (recall Definition 2.25). Thus, we will look for larger manifolds: to this end, we consider the subgroups $K_i := K \cap H_i$, which have index 2 in K and index 17 in H_i. They are normalized by the whole $C_{17}K$, so they correspond to 17-fold coverings of the Long manifolds with deck transformations of order 17. Consequently, they are tessellated by $136 = 17 \cdot 8$ copies of D. We will call these manifolds X_i.

The *orientation subgroup* $G^+ := \langle ab, bc, cd, de \rangle$ of index 2, consisting of all words of even length, does not contain any K_i. Hence, the X_i are non-orientable, and the subgroup

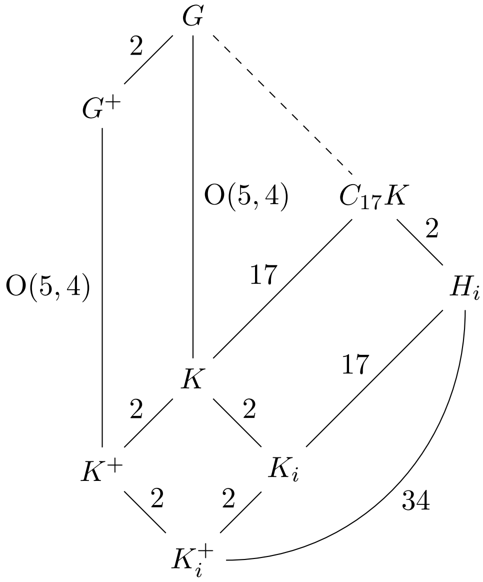


Figure 4.1: Subgroup lattice of G. Solid lines denote normal subgroups, labeled by the quotient group or by an integer n, standing for $\mathbb{Z}_n$.

$K_i^+ := K_i \cap G^+$ corresponds to the orientable double cover of X_i, a manifold tessellated by 272 copies of D.

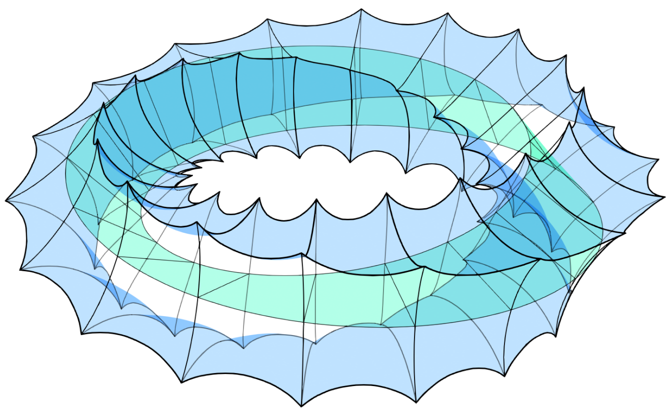


Figure 4.2: A three-dimensional representation of an orientable manifold with right-angled corners, obtained by arranging copies of Q_0 (here, a square prism) on a non-orientable hypersurface (here, a Möbius strip). Note that we will consider only 4-manifolds without boundary, unlike the Möbius strip; hence, only the blue faces are relevant to our case.

We can now construct an orientable hyperbolic 5-manifold with right-angled corners Y_i, whose interior is diffeomorphic to the determinant line bundle on X_i, by gluing copies of the prism Q_0 on both sides of each copy of D in X_i. The *facets* of Y_i, that is, its codimension-1 boundary components, are right-angled 120-cells. In summary, we have:

Theorem 4.19. *For each $i = 1, \dots, 8$, there exists an orientable right-angled hyperbolic 5-manifold Y_i, whose 4-dimensional facets are 272 right-angled 120-cells.*

4.8.2 The adjacency graph

In order to apply the coloring method to Y_i, we first need to check that none of its facets is glued to itself via some dodecahedron. Surprisingly, the facet adjacency graph is the same for all the Y_i and can be described in a relatively simple way. First, note that the graph also naturally describes adjacencies in the orientable double cover X_i^+ of X_i, tessellated by 272 120-cells. Each $[5, 3, 3, 3]$ simplex in X_i^+ corresponds to a coset $K_i^+\gamma$ for some $\gamma \in G$, while each 120-cell corresponds to a double coset $K_i^+\gamma\Sigma$, where $\Sigma := \langle a, b, c, d\rangle$ is the symmetry group of the 120-cell.

This subset is actually a left coset: since K_i^+ does not intersect the conjugacy class of $(abcd)^{15}$, we have $K^+ := K \cap G^+ = K_i^+ \sqcup K_i^+ \cdot \gamma(abcd)^{15}\gamma^{-1}$ and

$$K_i^+\gamma\Sigma = K_i^+\gamma[\Sigma \cup (abcd)^{15} \cdot \Sigma] = [K_i^+ \cup K_i^+ \cdot \gamma(abcd)^{15}\gamma^{-1}]\gamma\Sigma = K^+\gamma\Sigma, \tag{4.6}$$

which can be written as $\gamma\Sigma K^+$ by normality of K^+ in G. (Notice that this expression is already independent of i.) The vertices of our adjacency graph are left cosets of ΣK^+ in G. Passing to the quotient $G/K^+ \simeq \mathrm{O}(5, 4) \times \mathbb{Z}_2$, these are in bijection with left cosets of $\overline{\Sigma}$, the image of Σ in G/K^+. The map $\Sigma \to G/K^+$ has kernel of order 2, so $|\overline{\Sigma}| = 7200$.

Regarding the edges, if we represent cells in the 120-cell honeycomb as left cosets of Σ in G, all pairs of adjacent cells are of the form $(\gamma\Sigma, \gamma e\Sigma)$, corresponding to the adjacency between the simplices γ and $\gamma e = (\gamma e\gamma^{-1})\gamma$.

Summing up, the adjacency graph $\mathcal{G}$ we seek is given by:

$$V(\mathcal{G}) = \{\gamma\overline{\Sigma} \mid \gamma \in G/K^+\}, \tag{4.7}$$

$$E(\mathcal{G}) = \{(\gamma\overline{\Sigma}, \gamma e\overline{\Sigma}) \mid \gamma \in G/K^+\}, \tag{4.8}$$

which is the same for all the X_i^+ (and the Y_i).

The graph $\mathcal{G}$ has no loops: indeed, it suffices that $\gamma\overline{\Sigma} \neq \gamma e\overline{\Sigma}$ for all $\gamma \in G/K^+$, which is equivalent to (the image of) e not being an element of $\overline{\Sigma}$; we can easily check this using GAP. As a consequence, for all i, no facet of Y_i meets itself along a ridge.

Remark 4.20. The space $\mathbb{H}^5/K^+$ is an orbifold cover of the $[5, 3, 3, 3]$ simplex, and is itself doubly covered by all the X_i^+. It is tessellated by 272 copies of a “half-120-cell”, obtained as a quotient of the standard 120-cell by the action of $(abcd)^{15}$, which corresponds to multiplication by -1 in $\mathbb{R}^4$. The graph $\mathcal{G}$ may also be described as the adjacency graph of this tessellation.

Remark 4.21. There is a more convenient way to represent the set $V(\mathcal{G})$. The right action of G on the cosets of $S \cap G^+ = \langle ac, ae, decd, bacbab\rangle$, of index 170, has kernel K^+ and leads to a faithful permutation representation $G/K^+ \hookrightarrow S_{170}$. The orbits of $\overline{\Sigma}$ have sizes 150, 10, 10; choose a 10-element orbit ω. It can be checked that $\overline{\Sigma}$, acting on the subsets of $\{1, 2, \dots, 170\}$, is the stabilizer of ω. Therefore, $V(\mathcal{G})$ can be constructed as the orbit of ω under the action of G/K^+ on subsets of $\{1, 2, \dots, 170\}$.

4.9 Closed 5-manifolds

In this section, we apply the coloring method to our right-angled manifolds Y_i, obtaining some large closed hyperbolic 5-manifolds with many symmetries, which we then use to obtain smaller manifolds as quotients.

4.9.1 Coloring the adjacency graph

We start by defining some colorings of the right-angled manifold Y_i by taking advantage of its symmetries. If we inspect the diagram of Figure 4.1, we see that a group of isometries H_i/K_i^+ isomorphic to $\mathbb{Z}_{17} \times \mathbb{Z}_2$ acts on both X_i and its double cover, inducing isometries of Y_i. We will fix two generators that work for all i: $\psi = (abcde)^2 \cdot K_i^+$ of order 17, and σ, the involution of Y_i that fixes X_i pointwise and acts by negation on its normal bundle. Note that no cell of Y_i, of any dimension, can be fixed by ψ^j, $j = 1, \dots, 16$, because 17 does not divide the order of any finite subgroup of Γ. Hence, ψ acts freely on the graph $\mathcal{G}$, with 16 orbits of size 17.

Now, recall that an *independent set* of a graph is a subset of its vertices inducing a subgraph with no edges. We call an independent set I of $\mathcal{G}$ *good* if it contains exactly one vertex from each orbit of ψ. Such a set with ψ induces a partition of $\mathcal{G}$ into 17 independent sets $(I, \psi(I), \dots, \psi^{16}(I))$; in turn, this defines a coloring $\lambda_{i,I} \colon \mathrm{Fac}(Y_i) \to \mathbb{Z}_2^{17}$, sending each facet in $\psi^j(I)$ to the basis vector e_j. We shall call the resulting manifold $M_{i,I} \coloneqq M(Y_i, \lambda_{i,I})$.

4.9.2 Counting good independent sets

All good independent sets of $\mathcal{G}$ can be obtained in SageMath by constructing all 16-cliques of an auxiliary graph $\mathcal{G}^*$, obtained from $\mathcal{G}$ by adding a 17-clique for each orbit of ψ and then taking the complement. In this way, we find that there are exactly 13 548 660 good independent sets. However, this number does not take into account the symmetries of the manifolds Y_i. Let Π_i be the group of isometries of Y_i that preserve the natural tessellation by copies of P_0; we have $\Pi_i \simeq N_G(K_i)/K_i^+$. The order $|\Pi_i|$ divides $[G : K_i^+] = 2^{10} \cdot 3^2 \cdot 5^2 \cdot 17$; hence, $\langle\psi\rangle < \Pi_i$ is a 17-Sylow subgroup and is unique up to conjugacy.

Let Λ_i be the normalizer of $\langle\psi\rangle$ in Π_i; any two good independent sets in the same orbit of Λ_i produce identically colored Y_i, up to isometries of Y_i and reordering of colors (which simply corresponds to a permutation of the canonical basis of $\mathbb{Z}_2^{17}$). Using SageMath and GAP, we can count orbits of independent sets under the action of Λ_i (Table 4.1).

Groups	Size	Orbit sizes	Number of orbits
$\Lambda_1, \Lambda_3, \Lambda_6, \Lambda_8$	136	$1618 \cdot [34] + 198436 \cdot [68]$	200 054
$\Lambda_2, \Lambda_4, \Lambda_5, \Lambda_7$	272	$809 \cdot [68] + 99218 \cdot [136]$	100 027

Table 4.1: Sizes of orbits of the groups Λ_i, expressed as sums of terms of the form $n \cdot [m]$, meaning n orbits of size m.

Moreover, we also find that there are two well-defined *types* of nontrivial elements of $\langle\psi\rangle$, preserved by the groups Λ_i:

Definition 4.22. The *type* of ψ^k, where $k \in \{\pm1, \pm2, \dots, \pm8\}$, is $+$ if $k \in \{\pm1, \pm2, \pm4, \pm8\}$ (the *squares* modulo 17) and $-$ if $k \in \{\pm3, \pm5, \pm6, \pm7\}$ (the *non-squares* modulo 17).

Lemma 4.23. *For every i, the conjugation action $\Lambda_i \curvearrowright \langle\psi\rangle \setminus \{\mathrm{id}\}$ preserves the type of an isometry.*

Proof. By Table 4.1, the order of Λ_i divides $272 = 2 \cdot 136$. Now, notice that Λ_i contains the order-34 subgroup $\langle\psi, \sigma\rangle$, which centralizes $\langle\psi\rangle$. Hence, the conjugation action $\Lambda_i \to \mathrm{Aut}(\langle\psi\rangle) \simeq \mathbb{Z}_{17}^\times$ factors through the quotient $\Lambda_i/\langle\psi, \sigma\rangle$, whose order divides $272/34 = 8$. It follows that its image is contained in the unique index-2 subgroup of $\mathrm{Aut}(\langle\psi\rangle) \simeq \mathbb{Z}_{16}$, whose orbits on $\langle\psi\rangle \setminus \{\mathrm{id}\}$ are exactly the two types. ☐

In conclusion, we obtain:

Theorem 4.24. *There exists a family of closed hyperbolic 5-manifolds $M_{i,I}$, tessellated by 2^{17} copies of Y_i and by $2^{17} \cdot 272 \cdot 14400 = 513\,382\,809\,600$ copies of the simplicial prism P_0.*

By abuse of notation, the subscript I should be taken to mean the orbit of I under Λ_i.

Using more general, vector-valued colorings, the number of prisms can be reduced by quite a lot; the main ingredient for this optimization is the theory of *linear binary codes.*

4.9.3 Linear binary codes

A *linear code* is a vector subspace C of $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is any finite field. We will only consider the case $q = 2$, whence the adjective "binary". The ambient dimension n is called the *length* of the code. The main use case of codes is in the correction of errors in communication channels. By interpreting vectors as messages, two parties may exchange only elements of C, or *codewords*, carrying $\dim C$ bits of information per codeword. It is desirable for any two distinct elements of C to differ in many bits, say at least d. In this way, as many as $\lfloor \frac{d-1}{2} \rfloor$ bit-flip errors per codeword may be corrected by reverting to the nearest codeword. The maximum such d is called the *minimum distance* of C.

Consider the two binary matrices

$$A \coloneqq \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \tag{4.9}$$

$$B \coloneqq \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}. \tag{4.10}$$

Their row spaces define linear binary codes $C(A), C(B)$ of length 17 and dimensions 9 and 8, respectively, which are *dual* to each other, meaning that AB^t and BA^t are zero matrices. Note that since $9 + 8 = 17$, we have $\mathrm{im}(B^t) = C(B) = \ker(A)$. These codes arise naturally as *quadratic residue codes*, in a construction based on the fact that 2, the order of the field, is a quadratic residue modulo 17 [91, p. 481].

The minimum distance of $C(B)$ is known to be 6, implying that no row vector with less than 6 ones is in $C(B)$. Any relation of linear dependence between k columns of A gives a vector with k ones in $\ker A = C(B)$. Hence, no 5 columns of A are linearly dependent.

The vertex figure of the order-3 120-cell honeycomb is a 4-simplex, so cells of X_i (and facets of Y_i) meet at most 5 at a time. Therefore, if we take one of the 17-colorings of $\mathcal{G}$ defined previously from an independent set I, and replace the basis vectors with the 17 columns $A_0, A_1, \ldots, A_{16}$ of A, we obtain a coloring over $\mathbb{Z}_2^9$.

We can generalize this by introducing a permutation of the columns; specifically, given a good independent set I and a generator $\psi^k \in \langle\psi\rangle$, $k \in \mathbb{Z}_{17}^\times$, define a coloring $\lambda_{i,I}^{(k)}$, respectively assigning colors $(A_0, A_1, \ldots, A_{16})$ to $(I, \psi^k(I), \ldots, \psi^{16k}(I))$. This results in a manifold $\widehat{M}_{i,I}^{(k)}$. Later on, the particular symmetry of these colorings will allow a further reduction in size.

Proposition 4.25. *The isometry class of $\widehat{M}_{i,I}^{(k)}$ depends only on the type of ψ^k and on the Λ_i-orbit of I.*

Proof. Define a binary 9×17 matrix $A^{(k)}$, whose j-th column $A_j^{(k)}$ is the color assigned to $\psi^j(I)$, that is, $A_{k^{-1}j}$ (where column indices are seen as elements of $\mathbb{Z}_{17}$). For example,

$A^{(1)} = A$. We say that $A^{(k)}$ and $A^{(k')}$ are *left equivalent* if there exists a matrix $M \in \mathrm{GL}(9,2)$ such that $A^{(k)} = MA^{(k')}$; in that case, the characteristic matrices are also left equivalent and there is an isometry $\widehat{M}_{i,I}^{(k)} \simeq \widehat{M}_{i,I}^{(k')}$. This is a consequence of the fact that the coloring $\lambda_{i,I}^{(k)}$ assigns the color $A_j^{(k)}$ to each facet in $\psi^j(I)$, for all j.

Using SageMath, we can check that the $A^{(k)}$ are divided into two left equivalence classes, corresponding exactly to the two types. Moreover, by Lemma 4.23, acting on I with Λ_i has the same effect as replacing ψ^k with another isometry ψ^h of the same type; that is, it does not change the manifold. □

As a consequence, we may call these manifolds $\widehat{M}_{i,I}^{\pm}$. They are covered by the corresponding $M_{i,I}$, and are tessellated by 2^9 copies of Y_i and by $2^9 \cdot 272 \cdot 14400 = 2\,005\,401\,600$ copies of the simplicial prism P_0.

4.9.4 Even smaller manifolds

We can still reduce this number: let us construct an extension of the order-17 isometry group $\langle\psi\rangle$ to the whole of $\widehat{M}_{i,I}^{\pm}$. For concreteness, we work with a specific $\widehat{M}_{i,I}^{(k)}$ and seek to extend ψ^k. The cyclic symmetry of the code $C(A)$ entails the existence of a matrix

$$R \coloneqq \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.11}$$

such that $R \cdot A_i = A_{i+1}$ for all $i \in \mathbb{Z}_{17}$. Therefore, if $v_j \in \mathbb{Z}_2^9$ is the color of facet F_j, then Rv_j is the color of $\psi^k(F_j)$.

We define our isometry $\Psi : \widehat{M}_{i,I}^{(k)} \to \widehat{M}_{i,I}^{(k)}$ as follows. If $x \in Y_i$, let x_v be its corresponding point in a copy $Y_i^{(v)}, v \in \mathbb{Z}_2^9$, and let

$$\Psi(x_v) \coloneqq \psi^k(x)_{Rv}. \tag{4.12}$$

It is not hard to check that Ψ is well-defined and gives indeed an isometry. For instance, if x_v is on a boundary facet F_j with color v_j, then it belongs to both $Y_i^{(v)}$ and $Y_i^{(v+v_j)}$. Since $\psi^k(x)$ is on a facet with color Rv_j, the points $\Psi(x_v) = \psi^k(x)_{Rv}$ and $\Psi(x_{v+v_j}) = \psi^k(x)_{Rv+Rv_j}$ are glued together, as required.

This isometry has order 17 and has no fixed points; again, this is because no finite subgroup of Γ has order a multiple of 17. Moreover, it uniquely extends ψ^k to $\widehat{M}_{i,I}^{(k)}$, so the action of $\langle\Psi\rangle$ depends only on the type of $\widehat{M}^{(k)}_{i,I}$. Thus, we can define quotients

$$N^{\pm}_{i,I} \coloneqq \widehat{M}^{\pm}_{i,I}/\langle\Psi\rangle, \tag{4.13}$$

which are tessellated by $2^9 \cdot 272/17 = 8192$ copies of Q_0 and $2^9 \cdot 272 \cdot 14400/17 = 117\,964\,800$ copies of P_0. The volume of these manifolds, as we will see in the next section, is less than 250 000.

4.9.5 Classification of the 5-manifolds

We now discuss the classification of the manifolds $N^{\pm}_{i,I}$ up to isometry.

Theorem 4.26. *There are exactly* 1 600 432 *pairwise non-isometric manifolds among the* $N^{\pm}_{i,I}$*, completely classified by an index* $i \in \{1,4,5,2,3,7\}$*, a* Λ_i*-orbit* I *of good independent sets in* $\mathcal{G}$*, and a type in* $\{+,-\}$*.*

The number 1 600 432 can be obtained from the statement of the theorem and Table 4.1. The proof is rather technical and heavily computer-assisted, and can be divided into three steps. First, we find that there are two pairs of Long manifolds related by tessellation-preserving isometries; then, we extend this result to the manifolds $N^{\pm}_{i,I}$, classifying them up to tessellation-preserving isometries; finally, we show that the Coxeter tessellation of $N^{\pm}_{i,I}$ is uniquely determined. The precise statements are as follows:

Proposition 4.27. *Consider the Long manifolds* Z_i*,* $i = 1,\dots,8$ *with their natural tessellations into* $[5,3,3,3]$ *simplices. Then there is a tessellation-preserving isometry* $Z_i \simeq Z_j$*,* $i < j$*, if and only if* $(i,j) \in \{(1,8),(3,6)\}$*.*

Proposition 4.28. *Given a manifold* $N^{\pm}_{i,I}$ *with its natural tessellation by copies of* P_0*, we can uniquely recover:*

- *the Long manifold* Z_i*, up to tessellation-preserving isometry;*
- *the* Λ_i*-orbit of* I*;*
- *the type* $\pm$*.*

Proposition 4.29. *Every* $N^{\pm}_{i,I}$ *is uniquely tessellated by copies of* P_0*.*

The proofs of these three results can be found in the GitHub repository [23] as SageMath notebooks with accompanying explanations; even if they are mostly computational, it might still be interesting to summarize the main ideas.

In the case of Proposition 4.27, we start by noticing that a tessellation-preserving isometry corresponds to conjugacy of the fundamental groups $\pi_1(Z_i)$ in the Coxeter group. Hence, proving the two isometries in the statement is a matter of finding an appropriate conjugating element in G; on the other hand, we find that the other Long manifolds Z_i can be distinguished by their first homology groups (see [83, Table 1]) or by counting quotients of $\pi_1(Z_i)$ isomorphic to the symmetric group S_3.

As for Proposition 4.28, the idea is to consider the embedded submanifolds of $N^{\pm}_{i,I}$ built from facets corresponding to the generator g, of which there are 2 isometric to Z_i and 30 isometric to X_i; the latter are found inside 30 copies of Y_i. At this point, it remains to determine the type and the orbit of I: indeed, the adjacency relations between the copies of Y_i allow us to define an auxiliary graph, and ultimately to extract the remaining information.

Finally, the proof of Proposition 4.29 is by far the most technical of the three. The main idea is to classify hyperbolic elements of Γ having a certain translation length, up to conjugacy. This is accomplished by searching the dual graph of the tessellation and finding conjugacy representatives via an algorithm based on [81]. The associated geodesics can only occur in certain positions with respect to the tessellation. Eventually, by exploiting patterns in the arrangement of these geodesics, we recover a flag of hyperbolic subspaces of $\mathbb{H}^5$, which determines the tessellation uniquely.

4.10 Properties

In this section, we study various properties of the manifolds $N^{\pm}_{i,I}$ and related ones.

4.10.1 Orientability

Proposition 4.30. *The manifolds $N^{\pm}_{i,I}$ are orientable.*

Proof. It is enough to prove orientability for $\widehat{M}^{\pm}_{i,I}$, since $\langle\Psi\rangle$ has odd order and thus preserves orientation. Recall that $\widehat{M}^{\pm}_{i,I}$ can be decomposed into 2^9 copies of Y_i, which is orientable; hence, we just need to orient each copy consistently. Following the proof of [76, Lemma 2.4], it suffices to find an isomorphism of $\mathbb{Z}_2^9$ that sends all facet colors (columns of A) to vectors with an odd number of ones.

If u_n is the length-n row vector of ones, this is equivalent to the existence of $M \in \mathrm{GL}(9, 2)$ such that $u_9 MA = u_{17}$. We can easily verify that, by taking

$$w := (1, 0, 0, 1, 1, 1, 0, 0, 1), \tag{4.14}$$

we have $wA = u_{17}$. Hence, any M such that $u_9 M = w$ works. □

Remark 4.31. The relative orientation of the copy of Y_i indexed by $v \in \mathbb{Z}_2^9$ is determined by the value of $w \cdot v \in \mathbb{Z}_2$.

4.10.2 Volume

In general, computing the volume of an odd-dimensional hyperbolic manifold is harder than in the even-dimensional case, as the Chern–Gauss–Bonnet formula is not available.

However, there are number-theoretical techniques that apply to arithmetic orbifolds in all dimensions, based on *Prasad's formula* [108]. In fact, an example of this is an expression for the volume of the simplicial prism P_0, essentially found in [45, Proposition 4]:

$$\mathrm{vol}(P_0) = \frac{9\sqrt{5}^{15}}{32\pi^{15}} \zeta_{k_0}(2)\zeta_{k_0}(4)\zeta_{\ell_2}(3)/\zeta_{k_0}(3), \tag{4.15}$$

where $k_0 = \mathbb{Q}(\sqrt{5})$, $\ell_2 = \mathbb{Q}(\sqrt{\varphi})$, $\varphi = \frac{\sqrt{5}+1}{2}$, and ζ_k is the Dirichlet zeta function for the field k. We can numerically evaluate this expression using PARI/GP [103], obtaining:

$$\begin{aligned} \mathrm{vol}(P_0) &= 0.001\,984\,696\,430\,311\,649\ldots \\ \mathrm{vol}(N^{\pm}_{i,I}) &= 234\,124.317\,462\,427\,649\,199\,813\ldots \end{aligned} \tag{4.16}$$

4.10.3 First homology

As expected in high dimension, the manifolds we constructed are quite combinatorially complex, so that a direct computation of their homology, e.g. from a CW complex structure, seems elusive. However, some simple topological operations can reduce them to well-studied objects known as *real toric spaces*, on which we can apply a formula of Suciu and Trevisan, generalized by Choi and Park [30].

Definition 4.32. Given a simplicial complex K on m vertices, the *real moment-angle complex* $\mathbb{R}\mathcal{Z}_K$ of K is defined as

$$\mathbb{R}\mathcal{Z}_K \coloneqq \bigcup_{\sigma \in K} \{(x_1, \ldots, x_m) \in [-1,1]^m \mid x_i \in \{\pm 1\} \text{ if } i \notin \sigma\}. \tag{4.17}$$

These spaces are cubical subcomplexes of $[-1,1]^m$ and have a natural $\mathbb{Z}_2^m$-action by reflections along the coordinate axes.

Definition 4.33. If $n < m$ and Λ is a binary $n \times m$ matrix, then $\ker \Lambda \subseteq \mathbb{Z}_2^m$ acts on $\mathbb{R}\mathcal{Z}_K$, and we define the *real toric space* $M^{\mathbb{R}}(K, \Lambda)$ as the quotient $\mathbb{R}\mathcal{Z}_K / \ker \Lambda$.

Now consider a manifold $M \coloneqq \widehat{M}^{\pm}_{i,I}$. It contains 2^9 embedded copies of X_i, whose first homology $H_1(X_i) \simeq K_i^{\mathrm{ab}}$ is finite and has only 2-torsion (see Table 4.1).

Manifolds	First homology
$X_1 \simeq X_8, X_4, X_5$	$\mathbb{Z}_2^{32} \oplus \mathbb{Z}_4^{11}$
$X_2, X_3 \simeq X_6$	$\mathbb{Z}_2^{27} \oplus \mathbb{Z}_4^{18}$
X_7	$\mathbb{Z}_2^{27} \oplus \mathbb{Z}_4^{17} \oplus \mathbb{Z}_8$

Table 4.1: First homology groups of the manifolds $X_i, i = 1, \ldots, 8$.

If F is a field with $\operatorname{char} F \neq 2$, then the homology exact sequence

$$0 = H_1(X_i;F) \longrightarrow H_1(M;F) \xrightarrow{\sim} H_1(M,X_i;F) \longrightarrow \widetilde{H}_0(X_i;F) = 0, \tag{4.18}$$

implies $H_1(M;F) \simeq H_1(M,X_i;F) \simeq H_1(M/X_i;F)$. By continuing in this manner, we can collapse all copies of X_i to points, while preserving H_1. The space $\overline{M}$ thus obtained can also be described as a gluing of 2^9 copies of a cone over ∂Y_i.

Consider $\mathcal{G}$ as a 1-dimensional simplicial complex; let $\lambda := \lambda_{i,I}^{(k)}$ be the coloring used in the construction of M and let Λ be a 9×272 characteristic matrix for λ. This enables us to define a real toric space $M_{\mathcal{G}} := M^{\mathbb{R}}(\mathcal{G}, \Lambda)$.

As one could expect, this has a connection with the real toric *manifold* M, underlined by the following result.

Proposition 4.34. *We have $H_1(\overline{M}) \simeq H_1(M_{\mathcal{G}})$ in homology with F-coefficients.*

Proof. Note that M is naturally tessellated by truncated 5-cubes centered at the vertices of each Y_i, whose 4-simplex facets lie on the X_i. Collapsing the latter creates a tessellation of $\overline{M}$ by 5-cubes whose vertices are among the collapsed points.

We are only interested in the 2-skeleton of this tessellation, which is composed of points, edges and squares. The points are in bijection with the copies of Y_i in M, that is, with $\mathbb{Z}_2^9$. Edges correspond to facets of the Y_i in M (i.e. vertices of $\mathcal{G}$), while squares correspond to pairs of adjacent facets (edges of $\mathcal{G}$).

The construction perfectly mirrors that of $M_{\mathcal{G}}$, save for the fact that any two adjacent facets share two opposite dodecahedra (due to the symmetry $(abcd)^{15}$) and contribute to the 2-skeleton with two squares having the same boundary. However, by excision, we may remove one of the two squares without changing the first homology. □

Now we shall apply the aforementioned formula, which we report here:

Theorem 4.35 ([30, Theorem 1.1])**.** *Let $M = M^{\mathbb{R}}(K, \Lambda)$ be a real toric space and R a commutative ring in which 2 is a unit. Then there is an R-linear isomorphism*

$$H^p(M;R) \simeq \bigoplus_{\omega \in \operatorname{row} \Lambda} \widetilde{H}^{p-1}(K_\omega;R), \tag{4.19}$$

where $\operatorname{row} \Lambda$ *is the row space of* Λ*, and* $K_\omega \subseteq K$ *is the subcomplex of* K *induced by the support of* ω *as a subset of* $\{1, \ldots, m\}$*.*

A direct consequence is

$$b_1(M_{\mathcal{G}};F) = \sum_{\omega \in (\operatorname{row} \Lambda) \setminus \{0\}} (b_0(\mathcal{G}_\omega;F) - 1). \tag{4.20}$$

Heuristically, we expect the graphs $\mathcal{G}_\omega$ to be connected: since 5 is the minimum distance of the code $C(A)$, every nonzero $\omega \in \operatorname{row}\Lambda$ has at least $80 = 5 \cdot 16$ ones. Using a custom program (written in C++ for performance reasons) we verify that this is the case for all choices of i, I and a type $\pm$. It follows that the sum (4.20) is always zero:

Theorem 4.36. *Let F be a field with $\operatorname{char} F \neq 2$. Then all manifolds $\widehat{M}^{\pm}_{i,I}$ have vanishing first homology with coefficients in F.*

Corollary 4.37. *The integral homology group $H_1(\widehat{M}^{\pm}_{i,I})$ is a finite abelian 2-group.*

Recall that the manifold $\widehat{M}^{\pm}_{i,I}$ is a 17-fold regular covering space of $N^{\pm}_{i,I}$. Using this fact, we can prove:

Theorem 4.38. *The integral homology group $H_1(N^{\pm}_{i,I})$ is of the form $T_2 \oplus \mathbb{Z}_{17}$, where T_2 is a finite abelian 2-group obtained as a quotient of $H_1(\widehat{M}^{\pm}_{i,I})$.*

Proof. Let $\widehat{\pi} := \pi_1(\widehat{M}^{\pm}_{i,I}), \pi := \pi_1(N^{\pm}_{i,I})$; then $\widehat{\pi}$ is a normal subgroup of π of index 17. Moreover, by the Hurewicz theorem, we have $H_1(\widehat{M}^{\pm}_{i,I}) \simeq \widehat{\pi}/[\widehat{\pi}, \widehat{\pi}]$ and $H_1(N^{\pm}_{i,I}) \simeq \pi/[\pi, \pi]$.

Since the quotient $\pi/\widehat{\pi} \simeq \mathbb{Z}_{17}$ is abelian, the commutator subgroup $[\pi, \pi]$ is contained (and is normal) in $\widehat{\pi}$. If we define $T_2 := \widehat{\pi}/[\pi, \pi]$, which is a quotient of $H_1(\widehat{M}^{\pm}_{i,I})$ (and hence a 2-group by Corollary 4.37), we have $T_2 \triangleleft \pi/[\pi, \pi] = H_1(N^{\pm}_{i,I})$. Finally, since T_2 and $H_1(N^{\pm}_{i,I})/T_2 \simeq \mathbb{Z}_{17}$ have coprime orders, we have $H_1(N^{\pm}_{i,I}) \simeq T_2 \oplus \mathbb{Z}_{17}$. □

Corollary 4.39. *The manifold $\widehat{M}^{\pm}_{i,I}$ is the unique regular 17-fold covering of $N^{\pm}_{i,I}$.*

Remark 4.40. More can be said about the structure of $H_1(\widehat{M}^{\pm}_{i,I})$ and T_2. Since $\widehat{M}^{\pm}_{i,I}$ orbifold covers Y_i, which is injectively included in $\widehat{M}^{\pm}_{i,I}$, we have a retraction $\widehat{M}^{\pm}_{i,I} \twoheadrightarrow Y_i$. Composing with the natural retraction $Y_i \twoheadrightarrow X_i$ and taking homology, we get an injection from $H_1(X_i)$ (see Table 4.1) to (the 2-torsion of) $H_1(\widehat{M}^{\pm}_{i,I})$. Similarly, by considering a 17-fold quotiented Y_i inside $N^{\pm}_{i,I}$, we get an injection $H_i(Z_i) \hookrightarrow T_2$.

Remark 4.41. The computation of higher homology groups is much harder for several reasons, the first of which is the failure of the collapsing trick (4.18), since $H_k(X_i; F)$ is not necessarily 0. Of course, even overlooking this issue, we would have to consider more complicated real toric spaces, arising from higher-dimensional simplicial complexes; computing b_1 and higher for their subcomplexes would be more involved than a simple graph search, as in the case of b_0.

4.10.4 Parallelizability

In this part, we discuss the problem of existence of a parallelizable hyperbolic 5-manifold, and how it relates to the objects studied up to this point. To be more specific, we prove:

Proposition 4.42. *There exists a parallelizable closed hyperbolic 5-manifold tessellated by copies of P_0.*

This is a direct consequence of a more general result, stated in the introduction as Theorem 1.11, and which may be of independent interest:

Theorem 1.11. *Let $n \in \{1, 3, 5, 7\} \cup \{4k + 1 \mid k \geq 2\}$. Then every closed hyperbolic n-manifold M is virtually parallelizable.*

This allows further generalizations of Proposition 4.42, for example involving manifolds tessellated by copies of Y_i.

The proof of Theorem 1.11 starts from a deep result of Deligne–Sullivan [119, p. 553], which states that every closed hyperbolic manifold is virtually stably parallelizable. Thus, let M' be a stably parallelizable finite cover of M. Assume first that $n \in \{1, 3, 7\}$. Then, by [122, p. 652], M' is also parallelizable and we are done.

In the remaining cases $n = 4k + 1, k \geq 1$, again by [122, p. 652], the only obstruction to parallelizability of M' is the Kervaire semi-characteristic:

Definition 4.43. The *Kervaire semi-characteristic* of a manifold X is a $\mathbb{Z}_2$-valued invariant, defined by the formula

$$\kappa(X) \coloneqq \sum_{i \geq 0} b_{2i}(X; \mathbb{Z}_2) \mod 2. \tag{4.21}$$

Suppose that X is a $(4k + 1)$-manifold with $w_{4k}(X) = 0$; then we can find two sections of TM that are linearly independent at all but a finite number of points [122], and there is an index formula for κ [6, Theorem 5.1]. A small $4k$-sphere centered on a singular point naturally maps into the space of 2-frames in $\mathbb{R}^{4k+1}$, i.e. the Stiefel manifold $V_{4k+1,2}$. This defines an element in $\pi_{4k}(V_{4k+1,2}) \simeq \mathbb{Z}_2$, the *index* of the singular point. Finally, the sum of all indices gives $\kappa(X)$.

A direct consequence of the index formula is:

Proposition 4.44. *If X is a $(4k+1)$-manifold with $w_{4k}(X) = 0$, and $\overline{X} \twoheadrightarrow X$ is a cover of even degree d, then $\kappa(\overline{X}) = 0$.*

Proof. Take a singular 2-frame field on X and lift it to $\overline{X}$. Every singular point of X lifts to d singular points with the same index, which cancel out. Hence, $\kappa(\overline{X}) = 0$. □

Since M' is stably parallelizable, its Stiefel–Whitney classes vanish in each positive degree. Hence, by Proposition 4.44, any even degree cover M'' of M' is parallelizable, having $\kappa(M'') = 0$. Finally, by [87, Theorem A], every finitely generated linear group defined over $\mathbb{R}$, such as $\pi_1(M')$, has a subgroup of even index, corresponding to a parallelizable cover $M'' \twoheadrightarrow M' \twoheadrightarrow M$.

This concludes the proof of Theorem 1.11. Note that our proof is highly non-constructive, and hence we have no concrete representation of M'', nor a bound on its volume.

Finally, recall that even-dimensional closed hyperbolic manifolds cannot be parallelizable because of their Euler characteristic; thus it is natural to ask:

Question 4.45. What can be said about virtual parallelizability of closed hyperbolic $(4k+3)$-manifolds, for $k \geq 2$?

4.11 Another class of 5-manifolds

The construction of small 5-manifolds in Section 4.9 involves a certain degree of asymmetry, due to the successive quotients and the choice of an independent set I; moreover, even the simplified definition of the graph $\mathcal{G}$ given in Remark 4.21 falls short of providing a satisfactory description of the vertex set.

There is, however, a different right-angled 5-manifold with a more elegant definition. The construction starts with a 4-manifold tessellated by 650 order-3 120-cells, obtained with Everitt and Maclachlan's method [46]. The general idea is to start with the canonical representation of the group $G = \langle a, b, c, d, e \rangle$ (see Section 2.2.2) over some integral domain, in our case $\mathbb{Z}[\varphi]$, and then pass to a finite field quotient; in most cases (including ours), the kernel of the quotient map gives a torsion-free finite-index subgroup and therefore a manifold.

4.11.1 A finite field quotient

Recall that, by Proposition 2.15, the group G admits a *canonical representation* in $\mathrm{GL}(5, \mathbb{R})$; in fact, since the entries of the Gram matrix only involve cosines of $\pi/2$, $\pi/3$ and $\pi/5$, the images of the generators end up in $\mathrm{GL}(5, \mathbb{Z}[2\cos(\pi/5)]) = \mathrm{GL}(5, \mathbb{Z}[\varphi])$; since the representation is faithful, we will identify group elements and matrices.

By [46, Corollary 1], the natural quotient map $q\colon \mathrm{GL}(5, \mathbb{Z}[\varphi]) \twoheadrightarrow \mathrm{GL}(5, \mathbb{F}_5)$, induced by the ideal $(\varphi - 3)$, has torsion-free kernel. Let $\alpha, \beta, \gamma, \delta, \varepsilon, K$ be the images of a, b, c, d, e, B under q. After a change of basis that diagonalizes K, computed with SageMath, we have:

$$\alpha = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \beta = \begin{bmatrix} 4 & 3 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.22}$$

$$\delta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 3 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 3 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}, K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}. \tag{4.23}$$

The group $Q := \operatorname{im} q = \langle \alpha, \beta, \gamma, \delta, \varepsilon \rangle$ has cardinality 9 360 000, so it has index 2 in the whole orthogonal group $\mathrm{O}(5,5)$ (which does not depend on the choice of a bilinear form). It can be characterized as the kernel of the *spinor norm* $\mathrm{O}(5,5) \to \mathbb{F}_5^\times/(\mathbb{F}_5^\times)^2$ (as defined in [20, p. 178]); its group structure may also be described as the unique nontrivial semidirect product of the simple group $\mathrm{S}_4(5)$ with $\mathbb{Z}_2$, or equivalently as $\mathrm{Aut}(\mathrm{S}_4(5))$.

4.11.2 The 4-manifold

Now we are ready to describe the manifold X associated to $\ker q$. The 120-cell stabilizer $\Sigma = \langle a, b, c, d \rangle$ injects into Q, and the largest subspace of $\mathbb{F}_5^5$ that it preserves is the span of $v_0 := (0,0,0,0,1)$. Since the stabilizer of v_0 is exactly $q(\Sigma)$, there is a bijection between the orbit Qv and the set of 120-cells that comprise X. The adjacency structure between the 120-cells can be obtained just like in Section 4.8.2, by taking the orbit of the pair of adjacent cells $(v_0, \varepsilon v_0)$. The resulting graph $\mathcal{G}'$ has 650 vertices, each of degree 120, and 39 000 edges, and it admits a beautiful description:

$$V(\mathcal{G}') = \{v \in \mathbb{F}_5^5 \mid K(v,v) = 3\}, \tag{4.24}$$

$$E(\mathcal{G}') = \{(v,w) \in \mathbb{F}_5^5 \times \mathbb{F}_5^5 \mid K(v,w) = 1\}. \tag{4.25}$$

Remark 4.46. Interestingly, this alternate definition is reminiscent of hyperbolic 4-space, but as a smooth variety over a finite field: indeed, K is a bilinear form with a single non-square diagonal entry, like the Lorentzian form on $\mathbb{R}^{4,1}$, and by interpreting it as a distance, the edges of $\mathcal{G}'$ appear to describe points that are close together, in some sense.

We could ask if $\mathcal{G}'$ contains enough information to recover the topology of X, since it is naturally embedded in the 1-skeleton of the dual tessellation. However, the latter has 78 000 4-simplices, while the induced flag simplicial complex of $\mathcal{G}'$ has exactly twice as many 5-cliques, which we call *real* if they correspond to a 4-simplex and *virtual* otherwise.

We can check that $\mathrm{Aut}(\mathcal{G}')$ acts transitively on the set of 5-cliques, while Q partitions it into two orbits of size 78 000. It is not hard to see that the action of Q must send real cliques to real cliques, so we can arbitrarily declare one Q-orbit, such as that of $\{v_0, \varepsilon v_0, \delta\varepsilon v_0, \gamma\delta\varepsilon v_0, \beta\gamma\delta\varepsilon v_0\}$, as the set of real cliques, and construct X as a simplicial complex.

Now we can use the GAP package HAP [44] to compute the integral homology of X:

$$H_0(X) = H_4(X) = \mathbb{Z}, \;\; H_1(X) = H_3(X) = \mathbb{Z}^{144}, \;\; H_2(X) = \mathbb{Z}^{936}. \tag{4.26}$$

The Euler characteristic is 650: this is to be expected, since each 120-cell has $\chi = 1$.

As before, in order to construct a 5-manifold, we start with a manifold with corners Y, obtained by arranging 650 copies of the usual 120-cell prism onto one side of X. Indeed, X being orientable, this produces an orientable Y with X as a totally geodesic boundary component. In the coloring method, this facet can be arbitrarily colored, as it does not meet any other facet. The other 650 facets are isometric to the right-angled 120-cell, and we will call them *small*. Their adjacency graph is isomorphic to $\mathcal{G}'$.

Remark 4.47. If F is a small facet of Y corresponding to $v \in V(\mathcal{G}')$, then there is a natural map $\mathrm{Stab}_Q(v) \to \mathrm{Isom}(F)$, which is injective by rigidity of isometries. It can be checked that both domain and codomain have cardinality 14400 (in particular, $\mathrm{Isom}(F)$ is the Coxeter group $[5, 3, 3]$). Hence, the map is also surjective: every isometry of a small facet extends to an isometry of Y.

4.11.3 Colorings

One could expect the number 13 to play a crucial role in coloring $\mathcal{G}'$, if only because of the prime factorization $650 = 2 \cdot 5^2 \cdot 13$. In fact, if λ_1, λ_n are the largest and smallest eigenvalues of $\mathcal{G}'$, at least $1 - \lambda_1/\lambda_n = 1 - 120/(-10) = 13$ colors are needed by the *Hoffman bound* [64]. As we will see in a moment, this bound is realized.

Unlike the case of the manifolds $N^{\pm}_{i,I}$, there is no "good" independent set with one element in each orbit of a 13-Sylow subgroup. However, there is a different, reasonably symmetrical construction. Up to conjugacy, there is exactly one subgroup $L < Q$ whose action on $\mathcal{G}'$ has 26 orbits of size 25, which are all independent sets. It has order 125, and it is generated by the two matrices

$$\begin{bmatrix} 0 & 3 & 2 & 0 & 2 \\ 4 & 0 & 1 & 4 & 2 \\ 1 & 1 & 0 & 1 & 3 \\ 4 & 2 & 3 & 2 & 2 \\ 4 & 3 & 2 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 3 & 2 & 2 & 1 & 3 \\ 0 & 3 & 1 & 3 & 4 \\ 3 & 1 & 2 & 2 & 3 \\ 3 & 1 & 2 & 1 & 4 \end{bmatrix}. \tag{4.27}$$

If a smaller coloring is desired, these orbits can be paired together into 13 independent sets of size 50, in 64 different ways. At the cost of introducing this choice, if we color each independent set with a basis vector of $\mathbb{Z}_2^{13}$, we obtain manifolds tessellated into 2^{13} copies of Y.

Remark 4.48. An even more efficient yet less symmetrical coloring, using binary vectors, arises from the dual of a linear binary code of length 13, dimension 4 and minimum distance 6 [56]. Its columns are elements of $\mathbb{Z}_2^9$, corresponding to 512 copies of Y and $650 \cdot 14\,400 \cdot 512 = 4\,792\,320\,000$ copies of P_0, for a volume of $9\,511\,300.396\,911\ldots$.

5. Geodesic embeddings and characteristic classes

In this chapter, we solve two problems about the existence of closed hyperbolic manifolds in arbitrarily high dimensions with certain properties: in the first part, we find such manifolds without spin^c structures in dimension ≥ 5 (Theorem 1.13), while in the second part, we find such manifolds in nontrivial cobordism classes in dimension $n \geq 4, n \neq 4k+3$ (Theorem 1.19).

Both problems rely on ideas of Kolpakov, Reid and Slavich [77], which we generalize in Section 5.1 with Theorem 1.12, giving a way to embed certain hyperbolic manifolds as codimension-1 totally geodesic submanifolds of another hyperbolic manifold, with a specified normal bundle. Iterating this construction allows us to obtain manifolds with certain properties in arbitrarily high dimensions.

In Section 5.2, we recall how the Stiefel–Whitney classes relate to the existence of spin^c structures. Then, in Section 5.3, we prove Theorems 1.13 and 1.14 using arithmetic methods. Finally, we give an alternate construction of a non-spin^c hyperbolic 5-manifold in Section 5.4.

As for the second problem, in Section 5.5, we define the unoriented cobordism ring $\mathcal{N}_*$, its decomposable classes and the homomorphisms φ^n. Then, in Section 5.6, we relate the cobordism class of a manifold to the fixed point set of an involution defined on it, and we compute its image through φ^n. In Section 5.7, we extend the Kolpakov–Reid–Slavich embedding to the context of manifolds with involutions, and also define an alternate, twisted embedding. Finally, in Section 5.8, we prove Theorem 1.19 and discuss the remaining case $n \equiv 3 \pmod 4$.

This chapter is based on [24] (in Sections 5.1 to 5.4) and [26] (in Sections 5.5 to 5.8).

5.1 Geodesic embeddings

In this section, we introduce and generalize the Kolpakov–Reid–Slavich embedding of arithmetic hyperbolic manifolds (see Section 2.3 for the relevant definitions).

Definition 5.1. We say that a hyperbolic manifold (or orbifold) M is *k-good* if it is connected, arithmetic of simplest type, with field of definition $k \neq \mathbb{Q}$ and admissible quadratic form f, and such that its fundamental group $\pi_1(M) < \mathrm{O}(f)$ is contained in the group of k-points $\mathrm{O}(f, k)$. (The condition $k \neq \mathbb{Q}$ ensures that M is closed.)

Now, one form of the aforementioned embedding is as follows:

Lemma 5.2 (compare [77], [94, Lemma 5.1], [78, Theorem 2.2])**.** *Let M^n be a k-good hyperbolic n-manifold. Then M^n geodesically embeds in an orientable k-good hyperbolic $(n+1)$-manifold.*

Sketch of proof. Let f be the admissible quadratic form associated to M^n. Define a new form $g = f + y^2$ with an additional variable y; this form is also admissible. As outlined in [77], we can embed the group $\mathrm{O}(f,k)$ into $\mathrm{O}(g,k)$. By [77, Corollary 5.2], this can be improved to an embedding $\mathrm{O}^+(f,k) < \mathrm{SO}^+(g,k)$. A separability argument gives a torsion-free, cocompact subgroup $\Lambda < \mathrm{SO}^+(g,k) < \mathrm{Isom}^+(\mathbb{H}^{n+1})$ containing $\pi_1(M^n)$, and such that M^n embeds geodesically in the orientable manifold $M^{n+1} := \mathbb{H}^{n+1}/\Lambda$. □

Note that, by orientability of M^{n+1}, we have $w_1(\nu_{M^{n+1}}(M^n)) = w_1(M^n)$. In fact, Theorem 1.12 (restated below) generalizes this by realizing any choice of a normal bundle for M^n:

Theorem 1.12. *Let M^n be a k-good hyperbolic n-manifold and let $c \in H^1(M;\mathbb{Z}_2)$. Then M^n geodesically embeds in a k-good hyperbolic $(n+1)$-manifold M^{n+1} such that*

$$w_1(\nu_{M^{n+1}}(M^n)) = c. \tag{5.1}$$

If $c = w_1(M^n)$, we can take M^{n+1} to be orientable.

Proof. The key part in the proof of Lemma 5.2 is the embedding of orthogonal groups $\psi\colon \mathrm{O}^+(f,k) \hookrightarrow \mathrm{SO}^+(g,k)$. This is accomplished in [77, Corollary 5.2], on the level of matrices, by sending

$$P \mapsto \left[\begin{array}{c|c} \det(P) & 0 \\ \hline 0 & P \end{array}\right]. \tag{5.2}$$

Let us instead define an embedding $\psi_c\colon \pi_1(M) \hookrightarrow \mathrm{O}^+(g,k)$ by

$$P \mapsto \left[\begin{array}{c|c} c(P) & 0 \\ \hline 0 & P \end{array}\right], \tag{5.3}$$

where c is seen as a map $\pi_1(M) \to \mathbb{Z}_2 \simeq \{\pm 1\}$. After carrying out the rest of the construction in the same way[1], we obtain a manifold M^{n+1} in which M^n embeds geodesically. Moreover, the normal bundle $E := \nu_{M^{n+1}}(M^n)$ is the quotient of $\nu_{\mathbb{H}^{n+1}}(\mathbb{H}^n)$ by the action of $\mathrm{im}(\psi_c)$ on its base space. It is now clear that $w_1(E)$, that is, the monodromy of the associated double cover, is simply c.

If $c = w_1(M^n)$, then $\psi_c = \psi$ on $\pi_1(M)$, and the original proof of Lemma 5.2 ensures global orientability. □

[1] Here an essential fact is that $\mathrm{im}(\psi_c) < \mathrm{Isom}(\mathbb{H}^{n+1})$ is geometrically finite: see the proof of Theorem 1.1 in [77, Section 7].

We conclude this section by showing that the first Stiefel–Whitney class of the normal bundle in a codimension-1 embedding can be tied to the Stiefel–Whitney classes of the two manifolds, as anticipated in the introduction:

Lemma 5.3. *Let $j\colon N \hookrightarrow M$ be a codimension-1 embedding of manifolds. Then, for all $i \geq 1$, we have*

$$j^*(w_i(M)) = w_i(N) + w_{i-1}(N)w_1(\nu_M(N)). \tag{5.4}$$

Proof. We have $j^*(TM) = TM\big|_N \simeq TN \oplus \nu_M(N)$. The statement follows from naturality of w_i and the Whitney sum formula. □

5.2 Stiefel–Whitney classes and spin^c structures

Let us now consider the problem of finding non-spin^c manifolds, and reframe it in terms of characteristic classes.

Stiefel–Whitney classes provide obstructions to various structures on a manifold M. In particular:

- M is orientable $\iff$ $w_1(M) = 0$;
- M is spin $\iff$ $w_1(M) = w_2(M) = 0$;
- M is pin^c $\iff$ $w_2(M)$ lifts to the integral cohomology group $H^2(M;\mathbb{Z})$;
- M is spin^c $\iff$ $w_1(M) = 0$ and M is pin^c.

The conditions for having pin^c or spin^c structures are arguably in a less usable form than the first two, especially for finding counterexamples; in order to resolve this issue, we introduce the *integral Stiefel–Whitney classes.*

The short exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \tag{5.5}$$

induces a long exact sequence on cohomology

$$\cdots \longrightarrow H^i(M;\mathbb{Z}) \longrightarrow H^i(M;\mathbb{Z}) \xrightarrow{\rho} H^i(M;\mathbb{Z}_2) \xrightarrow{\beta} H^{i+1}(M;\mathbb{Z}) \longrightarrow \dots, \tag{5.6}$$

where ρ is the reduction modulo 2 and β is the *Bockstein homomorphism.* We can now define the $(i+1)$-th *integral Stiefel–Whitney class*:

$$W_{i+1}(M) \coloneqq \beta(w_i(M)) \in H^{i+1}(M;\mathbb{Z}). \tag{5.7}$$

By exactness, we immediately see that:

- M is pin^c $\iff$ $W_3(M) = 0$;
- M is spin^c $\iff$ $W_3(M) = 0$ and $w_1(M) = 0$.

Remark 5.4. It is well known that the reduction of W_3 modulo 2 is the first Steenrod square of the second Stiefel–Whitney class $\mathrm{Sq}^1(w_2) = w_2w_1 + w_3$, which is much easier to compute than its integral counterpart. Indeed, we have $\mathrm{Sq}^1(w_2) = w_3$ for closed orientable manifolds (or closed 4-manifolds, by the Wu formulas: see Section 2.4.2).

As such, our general strategy will be to construct orientable manifolds with non-vanishing third Stiefel–Whitney class.

5.3 Manifolds without spin^c structures

The aim of this section is to prove Theorem 1.13. The proof relies on a technique similar to that of [94, Section 5]: we recursively construct a sequence of manifolds $(M^i)_{i\geq 1}$ by embedding each one as a totally geodesic submanifold of the next. To do so, we will use Theorem 1.12, which can be seen as an extension of [94, Lemma 5.1].

Proposition 5.5. *Let M^n be a k-good hyperbolic n-manifold such that $w_{n-1}(M^n) \neq 0$. Then M^n geodesically embeds in a k-good hyperbolic $(n+1)$-manifold M^{n+1} with $w_n(M^{n+1}) \neq 0$.*

Proof. By Poincaré duality, we can find $c \in H^1(M^n; \mathbb{Z}_2)$ such that

$$w_n(M^n) + w_{n-1}(M^n)c \neq 0. \tag{5.8}$$

We then apply Theorem 1.12 to this class, obtaining an embedding $j\colon M^n \hookrightarrow M^{n+1}$. By Lemma 5.3, we have $j^*(w_n(M^{n+1})) = w_n(M^n) + w_{n-1}(M^n)c \neq 0$, so $w_n(M^{n+1}) \neq 0$. □

Using this result, we can attack the problem of finding non-spin^c manifolds by starting from dimension 2.

Theorem 5.6. *Let M^2 be a non-orientable k-good hyperbolic 2-manifold. Then, for all $n \geq 5$, M^2 geodesically embeds in a k-good, orientable, non-spin^c hyperbolic n-manifold M^n.*

Proof. As anticipated, using Theorem 1.12, we recursively construct a sequence of k-good manifolds $(M^2, M^3, M^4, M^5, \dots)$, each one embedding geodesically in the next, and such that M^n is orientable and non-spin^c for $n \geq 5$. By Remark 5.4, the latter condition holds when $w_3(M^n) \neq 0$.

First, since $w_1(M^2) \neq 0$, we can apply Proposition 5.5 twice on M^2, obtaining k-good manifolds (M^2, M^3, M^4) such that $w_{i-1}(M^i) \neq 0$ (in particular $w_3(M^4) \neq 0$).

Next, we repeatedly apply Theorem 1.12 to obtain orientable manifolds $(M^i)_{i\geq 5}$, by choosing the class $c = w_1(M^i)$ at each step. By Lemma 5.3, we have:

$$
\begin{aligned}
j_i^*(w_3(M^{i+1})) &= w_3(M^i) + w_2(M^i)w_1(M^i) && (5.9)\\
&= w_3(M^i) \qquad (*) && (5.10)\\
&\neq 0. \qquad \text{(by induction)} && (5.11)
\end{aligned}
$$

The equality labeled with $(*)$ can be shown to hold by cases: if $i = 4$, then M^4 satisfies $w_2(M^4)w_1(M^4) = 0$ by the Wu formulas (see Section 2.4.2); if $i \geq 5$, then $w_1(M^i) = 0$ by orientability of M^i. It then follows that $w_3(M^{i+1}) \neq 0$, completing the induction. □

We now show the existence of many such 2-manifolds (i.e., surfaces) M^2.

Lemma 5.7. *For every totally real number field $k \neq \mathbb{Q}$, there exists a non-orientable k-good hyperbolic surface Σ_k.*

Proof. Write $k = \mathbb{Q}(\alpha)$, with $\alpha > \beta$ the two largest roots of the minimal polynomial of α, and choose a rational number $q \in (\beta, \alpha)$. Then $\delta := \alpha - q$ is positive, and all Galois conjugates of δ are negative. Thus, the quadratic form $f := \operatorname{diag}(-\delta, 1, 1)$ is admissible.

Let Γ be the arithmetic group of simplest type $\mathrm{O}(f, \mathcal{O}_k)$, up to the appropriate conjugation into $\mathrm{O}(2, 1)$. Then, Γ has finite covolume by the Borel–Harish-Chandra theorem, and it contains the orientation-reversing isometry $\operatorname{diag}(1, -1, 1)$. Hence, $\mathbb{H}^2/\Gamma$ is a compact non-orientable orbifold, and by [85, Theorem 1.2], it is finitely covered by a non-orientable surface Σ_k, which is k-good. □

Since the field of definition k is a commensurability invariant among arithmetic manifolds of simplest type, after applying Theorem 5.6 to Σ_k for infinitely many k, we have:

Corollary (Theorem 1.13)**.** *For every $n \geq 5$, there exist infinitely many commensurability classes of closed orientable hyperbolic n-manifolds that have no spin^c structure.*

Incidentally, from Remark 5.4 and the proof of Theorem 5.6 we also obtain:

Corollary 5.8. *There exist infinitely many commensurability classes of closed hyperbolic 4-manifolds that have no pin^c structure.*

Such manifolds must be non-orientable by the Hirzebruch–Hopf theorem [63].

Finally, using Proposition 2.38, we can easily prove Theorem 1.14, as a generalization of Theorem 5.6:

Theorem 1.14. *For every $m \geq 1$ and $n \geq 4m + 1$, there exist infinitely many commensurability classes of closed orientable hyperbolic n-manifolds M such that $w_{4m-1}(M) \neq 0$.*

Proof. The proof is completely analogous to that of Theorem 5.6: first, starting from a k-good surface M^2, we apply Proposition 5.5 to construct a sequence of k-good manifolds $(M^2, \ldots, M^{4m})$ such that $w_{i-1}(M^i) \neq 0$. Then, we repeatedly apply Theorem 1.12 with $c = w_1(M^i)$, obtaining orientable manifolds $(M^i)_{i \geq 4m+1}$. Crucially, we need the Wu formula $w_{4m-2}(M^{4m})w_1(M^{4m}) = 0$ from Proposition 2.38 in order to have

$$j_{4m}^*(w_{4m-1}(M^{4m+1})) = w_{4m-1}(M^{4m+1}) + w_{4m-2}(M^{4m+1})w_1(M^{4m+1}) \tag{5.12}$$
$$= w_{4m-1}(M^{4m+1}) \neq 0. \tag{5.13}$$

The induction proceeds as in Theorem 5.6, showing that all subsequent manifolds also have $w_{4m-1}(M^i) \neq 0$. □

Corollary 5.9. *There exist closed orientable hyperbolic manifolds with nonzero Stiefel–Whitney classes in arbitrarily high degree.*

5.4 Explicit examples

We will now outline a semi-explicit procedure to construct closed orientable hyperbolic 5-manifolds with no spin^c structure, based on the coloring method (Section 2.2.5). This construction is compatible with that of Theorem 5.6 and yields $\mathbb{Q}(\sqrt{5})$-good manifolds.

5.4.1 Examples of right-angled polytopes

It can be shown that compact right-angled polytopes do not exist in dimension $n \geq 5$ [107], but examples are abundant in lower dimensions. Among these, two are of particular interest to us: the right-angled dodecahedron in dimension 3, and the right-angled 120-cell in dimension 4. These highly symmetric polytopes are hyperbolic analogs of their Euclidean regular counterparts and, as hyperbolic orbifolds, they are both $\mathbb{Q}(\sqrt{5})$-good.

A simple way to construct either polytope starts by placing its Euclidean counterpart inside the Beltrami-Klein model of $\mathbb{H}^3$ or $\mathbb{H}^4$, centered at the origin. This gives a (regular) hyperbolic polytope, since the Beltrami-Klein model preserves collinearity. If we apply a Euclidean scaling to the polytope, taking care that it remains contained within the interior of the disk, we note that the dihedral angle is a continuous function of the scaling factor, and can take on an interval of values. Calculations show that this interval contains $\pi/2$ for both the dodecahedron and the 120-cell, resulting in the corresponding right-angled polytopes.

5.4.2 Small covers

It is not hard to see that, in a compact right-angled polytope P, exactly $\dim P$ facets meet at a vertex. This fact, combined with the linear independence condition on a coloring λ, implies that the colors belong to a vector space of rank $m \geq \dim P$. The equality case corresponds to *small covers*:

Definition 5.10. A *small cover* of a compact right-angled n-polytope P is a connected real toric manifold $M(P, \lambda)$ with $\lambda\colon \operatorname{Fac}(P) \to \mathbb{Z}_2^n$.

These manifolds are especially well studied because they are easier to classify by computer, due to their simplicity, and they have good homological behavior.

Indeed, the small covers of the dodecahedron have been classified by Garrison and Scott [55], and they form 25 isomorphism classes. More recently, Ma and Zheng [90] performed a partial classification of small covers of the 120-cell, by considering the case $|\operatorname{im}(\lambda)| \leq 8$.

As for their algebraic properties, there is an explicit formula for the $\mathbb{Z}_2$-cohomology ring of a small cover [38, Theorem 4.14]. If $\operatorname{Fac}(P) = \{F_1, \ldots, F_k\}$, then we have:

$$H^*(M(P, \lambda); \mathbb{Z}_2) \simeq \mathbb{Z}_2[a_1, \ldots, a_k]/(I + J), \tag{5.14}$$

where the ideal I captures the combinatorics of P:

$$I \coloneqq (a_{i_1} \ldots a_{i_s} \mid F_{i_1} \cap \cdots \cap F_{i_s} = \varnothing), \tag{5.15}$$

and the ideal J captures information about the coloring:

$$J \coloneqq \left(\sum_{i=1}^{k} \lambda(i)_j \cdot a_i \mid j = 1, \ldots, n \right). \tag{5.16}$$

The ring is naturally graded by degree, and the a_i are generators for the first cohomology group. Each a_i can be interpreted geometrically by taking the Poincaré dual; the resulting codimension-1 homology class is represented by the hypersurface M_{F_i} containing F_i.

Definition 5.11. We say that a cohomology class in $H^1(M(P, \lambda); \mathbb{Z}_2)$ is a *sum of hypersurfaces* if it is dual to a sum of codimension-1 cycles of the form M_F, where $F \in \operatorname{Fac}(P)$.

As a consequence of (5.14), we have:

Proposition 5.12. *If $M \coloneqq M(P, \lambda)$ is a small cover, then every class $c \in H^1(M, \mathbb{Z}_2)$ is a sum of hypersurfaces.*

Higher-degree classes involving products of distinct generators can also be interpreted dually, using the fact that cup products are dual to intersections. In this view, the ideal I shows that non-intersecting sets of facets give vanishing cup products; as for J, its j-th generator is dual to the boundary separating the copies P_v with $v_j = 0$ from those with $v_j = 1$.

Finally, the total Stiefel–Whitney class of a small cover has a simple expression in terms of the ring generators [38, Corollary 6.8]:

$$w(M(P, \lambda)) = \prod_{i=1}^{k} (1 + a_i), \tag{5.17}$$

which can be expanded, in each degree, to obtain

$$w_d(M(P,\lambda)) = \sum_{\substack{S\subseteq\{1,\dots,k\}\\ |S|=d}} \prod_{i\in S} a_i. \tag{5.18}$$

5.4.3 The construction

Recall the notion of right-angled hyperbolic manifold from Section 2.2.5. The following result serves as a constructive substitute of Theorem 1.12:

Proposition 5.13. *Let X be a real toric manifold over a right-angled hyperbolic n-polytope P, with characteristic function $\lambda\colon \{F_1,\dots,F_k\} \to \mathbb{Z}_2^m$, and let $c \in H^1(X;\mathbb{Z}_2)$ be a sum of hypersurfaces. Let Q be a right-angled hyperbolic $(n+1)$-manifold with embedded facets. Assume that Q has a facet isometric to P, and that the natural map $\alpha\colon \mathrm{Fac}(P) \twoheadrightarrow \mathrm{adj}(P)$ is a bijection (these conditions hold if Q is a polytope). Then X embeds as a totally geodesic submanifold of a real toric manifold Y over Q, such that $w_1(\nu_Y(X)) = c$. Moreover, if $c = w_1(X)$, then we can take Y to be orientable.*

Proof. For the case $c = w_1(X)$, we can prove that X embeds geodesically in an orientable real toric manifold Y over Q, as in [76, Proposition 2.9]. Hence, $w_1(\nu_Y(X)) = w_1(X) = c$.

In the general case, as in the previous section, let $M_i \coloneqq M_{F_i}$ be the geodesic surface of X consisting of all copies of the facet F_i of P; then we can write $c = c_1[M_1] + \cdots + c_k[M_k]$, for some possibly non-unique coefficients $c_i \in \mathbb{Z}_2$.

We will construct Y by coloring Q as follows. Let G_{k+1} be the facet isometric to P, and label its adjacent facets $G_i \coloneqq \alpha(F_i)$ for $i = 1,\dots,k$. Then, define a partial coloring $\lambda^*\colon \{G_1,\dots,G_{k+1}\} \to \mathbb{Z}_2^m \times \mathbb{Z}_2$:

$$\lambda^*(G_i) \coloneqq \begin{cases} (\lambda(F_i), c_i) & \text{if } i \le k, \\ (\quad \mathbf{0},\ 1) & \text{if } i = k+1. \end{cases} \tag{5.19}$$

As in Remark 2.28, we can extend λ^* to a coloring $\widetilde{\lambda}$ of Q. We claim that the resulting 4-manifold $Y \coloneqq M(Q,\widetilde{\lambda})$ satisfies the condition.

First, note that X is embedded in Y as a connected component of the preimage of facet G_{k+1}. Indeed, such a component is a real toric manifold over P; by Proposition 2.29, the coloring is given by the images of $\widetilde{\lambda}(G_i), 1 \le i \le k$ in the quotient by $\mathrm{span}(\widetilde{\lambda}(G_{k+1}))$, and clearly such a coloring is isomorphic to the original λ.

The class $w_1(\nu_Y(X)) \in H^1(X;\mathbb{Z}_2) \simeq \mathrm{Hom}(\pi_1(X);\mathbb{Z}_2)$ is the monodromy of the double cover of X (S^0-bundle) contained in $\nu_Y(X) \setminus X$, and can be computed as follows: given a closed loop $\gamma \in \pi_1(X)$ in general position, intersecting facets $F_{i_1},\dots,F_{i_s}$ in order, we have

$$w_1(\nu_Y(X))(\gamma) = [\widetilde{\lambda}(G_{i_1}) + \cdots + \widetilde{\lambda}(G_{i_s})] \cdot \widetilde{\lambda}(G_{k+1}) = c_{i_1} + \cdots + c_{i_s}. \tag{5.20}$$

On the other hand, we have $c(\gamma) = \sum_{i=1}^{k} c_i(\gamma \cdot M_i)$. Each intersection of γ with facet i_s contributes c_s to the sum, which then equals $w_1(\nu_Y(X))(\gamma)$. Since γ was arbitrary, the proof is complete. □

We will now use this proposition to reproduce the proof of Theorem 5.6.

In the steps up to dimension 4, the class c of Proposition 5.5 is not guaranteed to be a sum of hypersurfaces, since Proposition 5.13 does not necessarily give a small cover. Hence, we will skip to dimension 3 and start from a small cover of the right-angled dodecahedron. Using the computer algebra system SageMath [116], we apply formula (5.18) to the 25 small covers in question, discovering that 22 of these have non-vanishing second Stiefel–Whitney class.[2]

Let X be one of these 22 small covers and let $c_{(3)} \in H^1(X;\mathbb{Z}_2)$ be any class such that $w_2(X) \cdot c_{(3)} \neq 0$. Using Proposition 5.13 with the 120-cell as Q and $c = c_{(3)}$, we obtain a 4-manifold Y with $w_3(Y) \neq 0$.

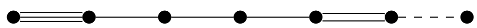
Even if Y is not a small cover, at this point we choose $c = c_{(4)} = w_1(Y)$, which is always a sum of hypersurfaces: for any real toric manifold over a polytope, we have

$$w_1(M(P,\lambda)) = \sum_{F \in E(P,\lambda)} [M_F] \tag{5.21}$$

where

$$E(P,\lambda) := \{F_i \in \mathrm{Fac}(P) \mid \lambda(F_i) \text{ has an even number of ones}\}. \tag{5.22}$$

Hence, to finally obtain an orientable, non-spin^c 5-manifold, we just need to choose a suitable Q having the 120-cell as a facet. However, since compact right-angled hyperbolic 5-polytopes do not exist, we will construct a right-angled 5-manifold based on the hyperbolic Coxeter prism P_0 introduced in Section 4.7, with the following diagram:


(5.23)

This polytope is, perhaps unsurprisingly, also arithmetic of simplest type and defined over $\mathbb{Q}(\sqrt{5})$ [16]. Recall that P_0 is a compact hyperbolic prism over a 4-simplex, with the two bases isometric to the fundamental simplices Δ_3, Δ_4 of the order-3 120-cell and right-angled 120-cell respectively.

We will now construct a right-angled 5-manifold W with embedded facets in a very similar way to Section 4.8.1. Let Z be a closed hyperbolic 4-manifold tessellated by the order-3 hyperbolic 120-cell, such as those in [33; 83]. Up to passing to a finite-index cover, by an injectivity radius argument, we may assume that each 120-cell is embedded in Z and has pairwise distinct neighboring cells. We then glue copies of P_0 along both sides of Z, in such a way as to obtain a right-angled 5-manifold W, homeomorphic to the

[2] Referring to [55, Table 1], the other three are at rows 10 and 13 in the left column, and row 3 in the right column. They are exactly the small covers of the dodecahedron having an isometry of order 3.

determinant line bundle on Z and hence orientable. The boundary of the manifold W consists of many embedded right-angled 120-cells, each having pairwise distinct adjacent facets. Hence, it can be used for our construction, ultimately yielding an orientable, non-spin^c hyperbolic 5-manifold, tessellated by the polytope P_0.

We note that W is likely to have high combinatorial complexity, and so is *a fortiori* the final non-spin^c 5-manifold.

5.5 The unoriented cobordism ring

We now turn our attention to the problem of constructing non-cobordant hyperbolic manifold. To begin, we recall some properties of the unoriented cobordism (henceforth simply *cobordism*) relation on manifolds.

Definition 5.14. Let M, M' be two closed n-manifolds. A *cobordism* between M and M' is a compact $(n+1)$-manifold with boundary W such that $\partial W \simeq M \sqcup M'$. If such a W exists, we say that M and M' are *cobordant* and have the same *cobordism class* $[M] = [M']$. Finally, we say that M is *cobordant* if $[M] = [\varnothing]$.

Cobordism classes of n-manifolds form a group $\mathcal{N}_n$ under disjoint union, with $[\varnothing]$ as the identity; moreover, every class is its own inverse, because $\partial(M \times [0,1]) \simeq M \sqcup M$, making $\mathcal{N}_n$ into an abelian 2-group.

The Cartesian product of manifolds induces a well-defined multiplication of cobordism classes, and hence a graded ring structure on the direct sum of all cobordism groups

$$\mathcal{N}_* \coloneqq \sum_{i \geq 0} \mathcal{N}_i. \tag{5.24}$$

The structure of the cobordism ring is that of a free polynomial algebra on $\mathbb{Z}_2$, as proved by Thom:

Theorem 5.15 ([121, Théorème IV.12])**.** *We have*

$$\mathcal{N}_* \simeq \mathbb{Z}_2[x_2, x_4, x_5, x_6, \dots], \tag{5.25}$$

with a generator x_i of degree i for each $i \neq 2^k - 1$.

Thom also showed [121, p. 80] that the real projective spaces $\mathbb{RP}^i$ can be taken as representatives for x_i, i even, while Dold [42] introduced the manifolds $P(m,n)$ as representatives for x_i, i odd:

Theorem 5.16 ([42, Satz 3])**.** *Representatives for each generator x_i, $i \neq 2^k - 1$, can be chosen as follows. If i is even, then $x_i = [\mathbb{RP}^i]$. If i is odd, write $i = 2^r(2s+1) - 1$; then $x_i = [P(2^r - 1, s2^r)]$, where*

$$P(m,n) \coloneqq (S^m \times \mathbb{CP}^n)/\{(x,[y]) \sim (-x,[\overline{y}])\}. \tag{5.26}$$

5.5.1 Decomposable classes

In order to simplify the problem of constructing non-cobordant hyperbolic n-manifolds, we define nontrivial homomorphisms $\varphi^n \colon \mathcal{N}_n \twoheadrightarrow \mathbb{Z}_2$, and then proceed to find a hyperbolic M^n such that $\varphi^n[M^n] = 1$.

Definition 5.17. A class $y \in \mathcal{N}_*$ is *decomposable* if it is represented by a product of two manifolds of positive dimension.

Decomposable classes are sums of products of two or more generators. In every dimension $n \neq 2^k - 1$, the group $\mathcal{N}_n$ is generated by decomposable classes together with x_n: hence, it is natural to define a map φ^n that sends x_n to 1 and vanishes on decomposable classes.

The value $\varphi^n[M^n]$ can be elegantly computed, following [121] (see also the expository paper [65]). Indeed, given the total Stiefel–Whitney class

$$w(M^n) \coloneqq 1 + w_1 + \cdots + w_n, \tag{5.27}$$

let $\lambda_1, \ldots, \lambda_n$ be the formal roots of the polynomial $p(t) \coloneqq t^n + w_1 t^{n-1} + \cdots + w_n$, obtained by homogenizing $w(M^n)$ with a new degree-1 variable t. Then we have

$$\varphi^n[M^n] = \sum_i \lambda_i^n. \tag{5.28}$$

More rigorously, we observe that the sum of powers in (5.28) can be written in terms of elementary symmetric polynomials in the roots λ_i, which are simply the coefficients $w_1, \ldots, w_n$. As expected, $\varphi^n[M^n]$ is a polynomial expression of degree n in the Stiefel–Whitney classes or, in other words, a sum of Stiefel–Whitney numbers.

5.6 Projective bundles and involutions

Given a rank-k vector bundle ξ over a closed n-manifold M, we can define the *projective bundle* $\mathbb{P}(\xi)$ as a fiberwise quotient of the unit sphere bundle $S(\xi)$ by the antipodal map. This construction results in a bundle over M with fiber $\mathbb{RP}^{k-1}$, whose total space, which we shall also denote by $\mathbb{P}(\xi)$, is a closed $(n + k - 1)$-manifold.

The total Stiefel–Whitney class of $\mathbb{P}(\xi)$ can be described in terms of the Stiefel–Whitney classes of M and ξ, as follows.

Theorem 5.18 ([34, Theorem 23.3], [13, p. 517])**.** *Let ξ be a rank-k vector bundle over a closed n-manifold M and let $p \colon \mathbb{P}(\xi) \twoheadrightarrow M$ be the bundle map. Define the classes $w_i \coloneqq p^*(w_i(M))$, $v_i \coloneqq p^*(w_i(\xi))$. Then we have*

$$H^*(\mathbb{P}(\xi)) \simeq H^*(M)[c]/(c^k + c^{k-1}v_1 + \cdots + v_k), \tag{5.29}$$

where $c \in H^1(\mathbb{P}(\xi))$ is the characteristic class of the double cover $S(\xi) \twoheadrightarrow \mathbb{P}(\xi)$. In particular, $H^*(\mathbb{P}(\xi))$ is a free $H^*(M)$-module with basis $\{1, c, \ldots, c^{k-1}\}$.

Moreover, the total Stiefel–Whitney class of the manifold $\mathbb{P}(\xi)$ *is*

$$w(\mathbb{P}(\xi)) = (1 + w_1 + \cdots + w_n) \cdot ((1+c)^k + (1+c)^{k-1} v_1 + \cdots + v_k). \tag{5.30}$$

As we will see in the following, the cobordism class of a manifold equipped with an involution can be written in terms of projective bundles over the fixed submanifolds of the involution.

5.6.1 Fixed points of involutions

Let M be a connected closed n-manifold equipped with a nontrivial involution τ. Following [34, Section 24], the fixed point set $\mathrm{Fix}(\tau)$ is a disjoint union of closed submanifolds of M. Hence, we may define $F(\tau)$ as the set of all fixed submanifolds, and $F_i(\tau)$ as the set of all such submanifolds of dimension i, for $i \geq 0$; note that $F_i(\tau)$ is trivially empty for $i > n$, and $F_n(\tau)$ is also empty since τ is nontrivial and M is connected.

The following result is of fundamental importance to our discussion.

Theorem 5.19 ([34, Theorem 24.2])**.** *Let M be a connected closed n-manifold and let τ be a nontrivial involution on M. Then we have*

$$[M] = \sum_{X \in F(\tau)} [\mathbb{P}(\nu X \oplus \varepsilon^1)], \tag{5.31}$$

where ε^1 denotes the trivial line bundle.

Note that the Stiefel–Whitney classes of $\nu X \oplus \varepsilon^1$ are the same as those of νX. Hence, by Theorems 5.18 and 5.19, the cobordism class of M is determined by the Stiefel–Whitney classes of X and νX for all fixed submanifolds X.

5.6.2 Applying φ^n

Let us compute the value of φ^n applied to such projective bundles, using Theorem 5.18 and the method of Section 5.5.1.

Let $X \in F_d(\tau)$ be a fixed submanifold of dimension d and let $w_i := w_i(X)$, $v_i := w_i(\nu X)$. Then, by Theorem 5.18, the total Stiefel–Whitney class of $[\mathbb{P}(\nu X \oplus \varepsilon^1)]$ is

$$(1 + w_1 + \cdots + w_d) \cdot ((1+c)^{n-d+1} + (1+c)^{n-d} v_1 + \cdots + (1+c)^{n-2d+1} v_d). \tag{5.32}$$

After homogenizing, we obtain the polynomial

$$p(t) := (t^d + t^{d-1} w_1 + \cdots + w_d)((t+c)^{n-d+1} + (t+c)^{n-d} v_1 + \cdots + (t+c)^{n-2d+1} v_d), \tag{5.33}$$

with formal roots $\lambda_1, \ldots, \lambda_{n+1}$. The number $\varphi^n[\mathbb{P}(\nu X \oplus \varepsilon^1)]$ can then be computed as the formal sum $\lambda_1^n + \cdots + \lambda_{n+1}^n$.

Let $\{\alpha_1, \ldots, \alpha_d\}$ and $\{\beta_1, \ldots, \beta_{n-d+1}\}$ be, respectively, the formal roots of the two polynomials

$$\begin{aligned} p_1(t) &:= t^d + t^{d-1}w_1 + \cdots + w_d, \\ p_2(t) &:= t^{n-d+1} + t^{n-d}v_1 + \cdots + t^{n-2d+1}v_d. \end{aligned} \tag{5.34}$$

Since $p(t) = p_1(t) \cdot p_2(t + c)$, we have

$$(\lambda_1, \ldots, \lambda_{n+1}) = (\alpha_1, \ldots, \alpha_d, \beta_1 + c, \ldots, \beta_{n-d+1} + c). \tag{5.35}$$

Now, note that $\alpha_1^n + \cdots + \alpha_d^n$ is an expression of degree n in the classes w_i, which vanishes since $d < n$. Therefore

$$\varphi^n[\mathbb{P}(\nu X \oplus \varepsilon^1)] = \sum_{i=1}^{n-d+1} (\beta_i + c)^n \tag{5.36}$$

$$= \sum_{j=0}^{d} \binom{n}{j} c^{n-j} \sum_{i=1}^{n-d+1} \beta_i^j, \tag{5.37}$$

where we discard powers of β_i with exponent larger than d. For $j \geq 1$, the sum of j-th powers in (5.37) is a fixed polynomial expression, independent of the number of roots, involving the elementary symmetric polynomials in the β_i (that is, the classes $v_1, \ldots, v_d$): say $p_j(v_1, \ldots, v_d)$. For $j = 0$, it takes the value $n - d + 1$. Hence, we have

$$\varphi^n[\mathbb{P}(\nu X \oplus \varepsilon^1)] = (n - d + 1)c^n + \sum_{j=1}^{d} \binom{n}{j} c^{n-j} p_j(v_1, \ldots, v_d). \tag{5.38}$$

We can simplify the formula further by reducing the powers of c modulo the relation $c^{n-d+1} + c^{n-d}v_1 + \cdots + c^{n-2d+1}v_d = 0$. Define the *dual Stiefel–Whitney classes* $\overline{v}_i$, satisfying the formal relation

$$\left(\sum_{j\geq 0} v_j t^j\right) \cdot \left(\sum_{j\geq 0} \overline{v}_j t^j\right) = 1. \tag{5.39}$$

(Note that we can recursively express each $\overline{v}_i$ as a polynomial over $v_1, \ldots, v_d$.) We claim that c^{n-j} reduces to $c^{n-d}\overline{v}_{d-j}$, plus terms of lower degree in c (and higher degree in the v_i), which can be discarded. This is obvious for $j = d$.

For $j < d$, by using little-o notation $o(c^k)$ for terms of c-degree less than k, we have:

$$0 = \left(\sum_{k=0}^{d-j-1} c^{d-j-1-k}\overline{v}_k\right)\left(\sum_{k=0}^{d} c^{n-d+1-k}v_k\right) \tag{5.40}$$

$$= \left(\sum_{k=0}^{d-j} c^{n-j-k} \sum_{m=0}^{k} \overline{v}_m v_{k-m}\right) - c^{n-d}\overline{v}_{d-j} + o(c^{n-d}) \tag{5.41}$$

$$= c^{n-j} + c^{n-d}\overline{v}_{d-j} + o(c^{n-d}), \tag{5.42}$$

as desired. Hence, by reducing the powers of c in (5.38), we obtain

$$\varphi^n[\mathbb{P}(\nu X \oplus \varepsilon^1)] = c^{n-d}\left[(n-d+1)\overline{v}_d + \sum_{j=1}^{d} \binom{n}{j} \overline{v}_{d-j} p_j(v_1, \dots, v_d)\right]. \tag{5.43}$$

As an element of $H^n(\mathbb{P}(\nu X \oplus \varepsilon^1)) \simeq \mathbb{Z}_2$, this equals the expression in square brackets as an element of $H^d(X) \simeq \mathbb{Z}_2$, since $\{1, c, \dots, c^{n-d}\}$ is a basis. Thus, we have proved:

Theorem 5.20. *Let M be a connected closed n-manifold, $n \neq 2^k - 1$, and let τ be a nontrivial involution on M. Then we have*

$$\varphi^n[M] = \sum_{d=0}^{n-1} \sum_{X \in F_d(\tau)} I_{n,d}(w_1(\nu X), \dots, w_d(\nu X)), \tag{5.44}$$

where $I_{n,d}$ is a polynomial expression in the free variables $v_1, \dots, v_d$, defined for any $n, d \geq 0$ and given by

$$I_{n,d}(v_1, \dots, v_d) \coloneqq (n-d+1)\overline{v}_d + \sum_{j=1}^{d} \binom{n}{j} \overline{v}_{d-j} p_j(v_1, \dots v_d). \tag{5.45}$$

Remark 5.21. By applying the map φ^n, we have eliminated the dependence on the tangent Stiefel–Whitney classes $w_i(X)$. Moreover, the expression $I_{n,d}$ depends on n only through the coefficients $n - d + 1$ and $\binom{n}{j}$, $0 < j \leq d$, whose parity is periodic in n; a common period is the smallest power of two 2^q such that $q > 0$ and $2^q > d$.

Using the computer algebra system SageMath [116], we can easily determine a few values of $I_{n,d}$ (Table 5.1).

A pattern emerges for n of the form $2^k - 1$, where exactly the first 2^k terms of the sequence $(I_{n,d})_{d\geq 0}$ are zero. Indeed, we can prove:

Proposition 5.22. *Let $k, m > 0$ with m odd. We have*

- $I_{m2^k-1,d} = 0$ *for all* $0 \leq d < 2^k$*;*
- $I_{m2^k-1,2^k} = v_1^{2^k}$.

Proof. A common period of $I_{n,0}, \dots, I_{n,2^k}$, as functions of n, is 2^{k+1}. Hence, without loss of generality, we can assume $m = 1$.

For $0 \leq d \leq 2^k$, we have, since $n - d + 1 \equiv 2^k + d \equiv d \pmod 2$,

$$I_{2^k-1,d} = d\overline{v}_d + \sum_{j=1}^{d} \binom{2^k - 1}{j} \overline{v}_{d-j} p_j(v_1, \dots, v_d). \tag{5.46}$$

$n \backslash d$	0	1	2	3	4
0	1	0	$v_1^2 + v_2$	0	$v_1^4 + v_1^2 v_2 + v_2^2 + v_4$
1	0	0	v_1^2	$v_1 v_2 + v_3$	$v_1^4 + v_1 v_3$
2	1	0	v_2	v_1^3	$v_2^2 + v_4$
3	0	0	0	0	v_1^4
4	1	0	$v_1^2 + v_2$	0	$v_1^2 v_2 + v_2^2 + v_4$
5	0	0	v_1^2	$v_1 v_2 + v_3$	$v_1 v_3$
6	1	0	v_2	v_1^3	$v_1^4 + v_2^2 + v_4$
7	0	0	0	0	0
8	1	0	$v_1^2 + v_2$	0	$v_1^4 + v_1^2 v_2 + v_2^2 + v_4$
9	0	0	v_1^2	$v_1 v_2 + v_3$	$v_1^4 + v_1 v_3$
10	1	0	v_2	v_1^3	$v_2^2 + v_4$
11	0	0	0	0	v_1^4
12	1	0	$v_1^2 + v_2$	0	$v_1^2 v_2 + v_2^2 + v_4$
13	0	0	v_1^2	$v_1 v_2 + v_3$	$v_1 v_3$
14	1	0	v_2	v_1^3	$v_1^4 + v_2^2 + v_4$
15	0	0	0	0	0
16	1	0	$v_1^2 + v_2$	0	$v_1^4 + v_1^2 v_2 + v_2^2 + v_4$

Table 5.1: Values of $I_{n,d}$ for $n \leq 16$, $d \leq 4$. Cells in gray correspond to invalid combinations ($n \leq d$ or $n = 2^k - 1$) for which $I_{n,d}$ can still be defined. Note the periodicity of the columns.

The graded polynomial ring $\mathbb{Z}_2[v_1, v_2, \dots]$ is isomorphic to $\Lambda^{\mathbb{Z}_2}$, the *ring of symmetric functions* over $\mathbb{Z}_2$, with the isomorphism sending v_i to the i-th elementary symmetric function. By identifying the two rings, we can interpret $\overline{v}_j$ and p_j as symmetric functions and exploit some known identities. We refer the reader to [117].

The expressions $\overline{v}_i = \overline{v}_i(v_1, \dots, v_i)$ satisfy the same recurrence

$$\overline{v}_i = \sum_{j=0}^{i-1} v_{i-j}\overline{v}_j \tag{5.47}$$

as the complete homogeneous symmetric functions h_i, so we have $\overline{v}_i(v_1, \dots, v_i) = h_i(v_1, \dots, v_i)$ as expressions in $v_1, \dots, v_i$. Moreover, by a well-known identity involving h_j and the power sum symmetric functions p_j we have, for each $d \geq 0$:

$$0 = dh_d + \sum_{j=1}^{d} h_{d-j}p_j = d\overline{v}_d + \sum_{j=1}^{d} \overline{v}_{d-j}p_j. \tag{5.48}$$

The right hand sides of (5.46) and (5.48) are almost the same: the binomial coefficient is always 1, except for the case $j = d = 2^k$. Hence, by subtracting, we have

$$I_{2^k-1,d} = \begin{cases} 0 & \text{if } d < 2^k, \\ \overline{v}_0 p_{2^k} & \text{if } d = 2^k. \end{cases} \tag{5.49}$$

Since $\overline{v}_0 = 1$, it remains to show that $p_{2^k} = v_1^{2^k}$. This is a consequence of the following identity in $\Lambda^{\mathbb{Z}_2}$ as a subring of $\mathbb{Z}_2[\![x_1, x_2, \dots]\!]$:

$$p_{2^k} = \sum_{i\geq 1} x_i^{2^k} = \left(\sum_{i\geq 1} x_i\right)^{2^k} = v_1^{2^k}. \tag{5.50}$$

□

5.7 Involutions and geodesic embeddings

In this section, we adapt the Kolpakov–Reid–Slavich embedding of arithmetic manifolds to the context of manifolds equipped with involutions, and proceed to construct non-cobordant manifolds starting from low dimensions.

In our construction, we will consider k-good manifolds with $k = \mathbb{Q}(\sqrt{5})$; we will denote the ring of integers of k by $\mathcal{O}_k = \mathbb{Z}\left[\frac{\sqrt{5}+1}{2}\right]$. For the sake of brevity, since k will be fixed, we will simply speak of *good manifolds*.

We start by extending Theorem 1.12 to hyperbolic manifolds with involutions.

Definition 5.23. Let M be a hyperbolic manifold equipped with an involution τ. We say that the *manifold with involution* (M, τ) is *good* if M is good and τ is a nontrivial isometry *defined over* k, that is, represented by an element of $\mathrm{O}(f, k)$.

Proposition 5.24. *Let (M, τ) be a good hyperbolic n-manifold with involution. Then M embeds geodesically into a good hyperbolic $(n+1)$-manifold with involution (M', τ'), such that:*

(1) $M \in F(\tau')$;

(2) $\nu_{M'}(M)$ is trivial;

(3) τ' acts on $\nu_{M'}(M)$ by negation;

(4) τ extends to an involution σ of M' defined over k.

The proof requires a technical lemma:

Lemma 5.25. *Let Γ be a subgroup of $\mathrm{O}(f, k)$ which is arithmetic of simplest type, i.e., commensurable with $\mathrm{O}(f, \mathcal{O}_k)$, and let $P \in \mathrm{O}(f, k)$. Then the subgroups Γ and $P^{-1}\Gamma P$ are commensurable.*

Proof. It suffices to show that $\mathrm{O}(f, \mathcal{O}_k)$ is commensurable with $P^{-1}\mathrm{O}(f, \mathcal{O}_k)P$. Let $P = A/a$, $P^{-1} = B/b$, with $A, B \in \mathrm{M}(n, \mathcal{O}_k)$ and $a, b \in \mathbb{Z}$. Consider the congruence subgroup $\Gamma(ab) < \mathrm{O}(f, \mathcal{O}_k)$ consisting of f-orthogonal matrices of the form $I + abM$, which is of finite index in $\mathrm{O}(f, \mathcal{O}_k)$. We shall show that P conjugates it into a subgroup of $\mathrm{O}(f, \mathcal{O}_k)$.

Indeed, given $Q = I + abM \in \Gamma(ab)$, we have

$$P^{-1}QP = I + P^{-1}abMP = I + AMB \in \mathrm{M}(n, \mathcal{O}_k). \tag{5.51}$$

Since $P^{-1}QP \in \mathrm{O}(f, k)$, the claim follows. □

Proof of Proposition 5.24. We start by embedding M into a good manifold M' with trivial normal bundle, using Lemma 5.2 or Theorem 1.12. Let $T \in \mathrm{O}(f, k)$ represent the involution τ and let $S \coloneqq T \oplus [1] \in \mathrm{O}(f + y^2, k)$. Finally, define $T' \coloneqq \mathrm{diag}(1, \ldots, 1, -1) \in \mathrm{O}(f + y^2, k)$.

Note that T' and S commute, and that they both normalize $\pi_1(M)$. Without loss of generality, we can assume that $\pi_1(M')$ is normalized by $H \coloneqq \langle T', S\rangle$, by replacing it with the intersection of all its H-conjugates. The latter contains $\pi_1(M)$, and is of finite index in $\pi_1(M')$ by Lemma 5.25. Hence, T' and S define involutions τ' and σ on M': the former fixes M and reverses its normal bundle, the latter extends τ. □

5.7.1 Twisting the embedding

Let $(M, \tau) \hookrightarrow (M', \tau')$ be an embedding of good manifolds with involutions, as constructed in Proposition 5.24. There is a natural way to modify (M', τ') by a cut-and-paste operation along M.

Definition 5.26. Given an embedding $(M, \tau) \hookrightarrow (M', \tau')$ as in Proposition 5.24, we define the *twist* $T(M', \tau', M, \tau) \coloneqq (M'', \tau'')$, where M'' is obtained by cutting M' along M and re-gluing the boundary components with the isometry τ. The involution τ'' is defined to agree with τ' on $M'' \setminus M \simeq M' \setminus M$, and with τ on M.

It is not hard to check that

$$F(\tau'') = F(\tau) \sqcup F(\tau') \setminus \{M\}. \tag{5.52}$$

As for the normal bundles, we have

$$\nu_{M''}(X) \simeq \begin{cases} \nu_M(X) \oplus \varepsilon^1 & \text{if } X \in F(\tau), \\ \nu_{M'}(X) & \text{if } X \in F(\tau') \setminus \{M\}. \end{cases} \tag{5.53}$$

This construction lets us control the value of the formula of Theorem 5.20 while recursively applying Proposition 5.24. In order to do so, we show that the twist is arithmetically well behaved.

Lemma 5.27. *The manifold* $T(M', \tau', M, \tau) \coloneqq (M'', \tau'')$ *is a good hyperbolic manifold with involution.*

Proof. We distinguish two cases. If M separates M', then M'' is isometric to M'. Up to this isometry, $\tau'' = \sigma\tau'$, where σ extends τ. Hence, we only need to show that τ'' is defined over k, which is true as σ and τ' are.

If M does not separate M', then it defines a map $\lambda \colon \pi_1(M') \twoheadrightarrow \mathbb{Z}_2$ by counting intersections modulo 2. There is a double cover $N \twoheadrightarrow M'$ corresponding to $\ker \lambda$, with deck automorphism s. Then σ lifts to an involution σ' of N defined over k, such that $N/\langle s\sigma' \rangle \simeq M''$ (see Figure 5.1). Therefore, M'' and M' are commensurable. Moreover, σ' is a lift of the involution τ'' of M'', which is hence defined over k. It follows that (M'', τ'') is good. ☐

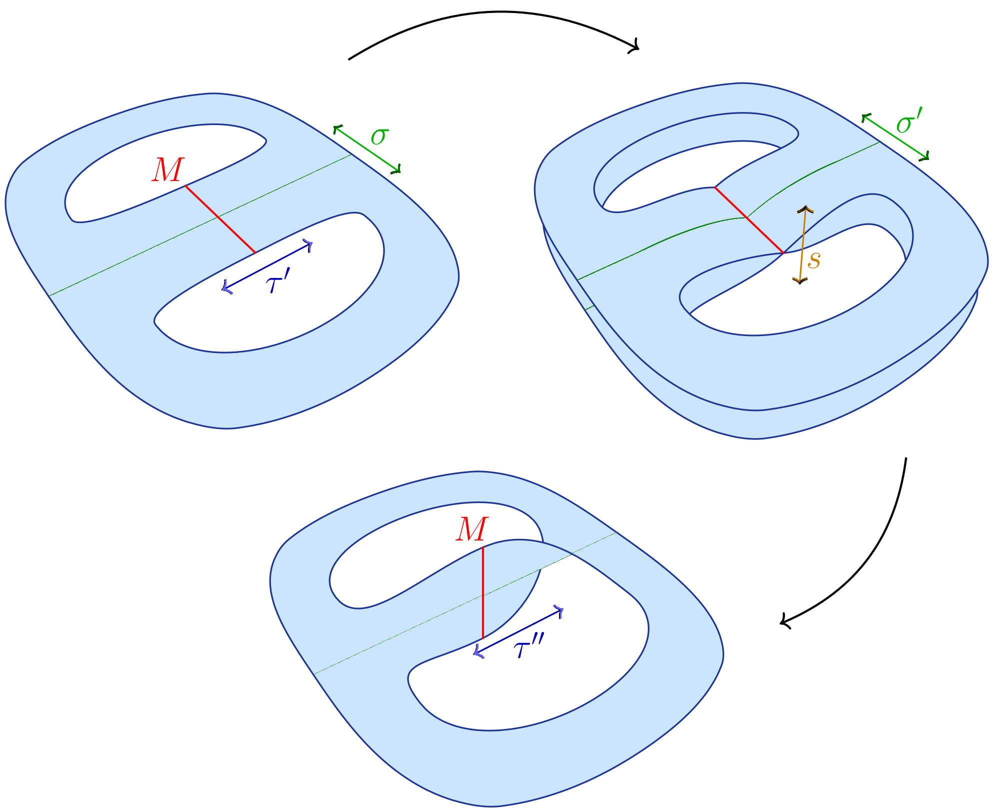


Figure 5.1: Construction of the twist $T(M', \tau', M, \tau)$ when M does not separate M'. Clockwise from top left: the manifold M embedded in M'; the double cover N with deck automorphism s; the twisted quotient M''.

5.8 Non-cobordant hyperbolic manifolds

In this section we show how to construct a hyperbolic n-manifold M^n with $\varphi^n[M^n] = 1$, which is therefore non-cobordant, by starting from a simpler manifold and recursively passing to a (possibly twisted) embedding. We then proceed to apply the method for many values of n.

Theorem 5.28. *Let (M, τ) be a good hyperbolic k-manifold with involution, where $k \leq n$. Suppose that*

$$I(n, M, \tau) := \sum_{d=0}^{n-1} \sum_{X \in F_d(\tau)} I_{n,d}(w_i(\nu_M X)) = 1. \tag{5.54}$$

Then there exists a good hyperbolic n-manifold with involution $(\bar{M}, \bar{\tau})$ such that $\varphi^n[\bar{M}] = 1$. In particular, $\bar{M}$ is non-cobordant.

Proof. We prove by induction that, for $k \leq m \leq n$, there exists a good manifold with involution (M^m, τ^m) such that

$$I(n, M^m, \tau^m) = \sum_{d=0}^{n-1} \sum_{X \in F_d(\tau^m)} I_{n,d}(w_i(\nu_{M^m} X)) = 1. \tag{5.55}$$

(Note that all terms with $d \geq m$ are zero, as $F_d(\tau^m)$ is then empty.) The base case $m = k$ is true by hypothesis. Now, suppose that (5.55) holds for (M^m, τ^m). By Proposition 5.24, we can embed M^m into a manifold with involution (M', τ'). If $I(n, M', \tau') = 1$, we are done by choosing $(M^{m+1}, \tau^{m+1}) := (M', \tau')$.

Otherwise, let $(M^{m+1}, \tau^{m+1}) := T(M', \tau', M^m, \tau^m)$. Then, by (5.52) and (5.53), we have:

$$\begin{aligned} I(n, M^{m+1}, \tau^{m+1}) &= I(n, M', \tau') + I(n, M^m, \tau^m) + I_{n,m}(w_i(\nu_{M'} M^m)) && (5.56) \\ &= I(n, M^m, \tau^m) = 1, && (5.57) \end{aligned}$$

where $I_{n,m}(w_i(\nu_{M'} M^m)) = 0$, since $\nu_{M'} M^m$ is trivial and $I_{n,m}$ is homogeneous of degree $m > 0$. (The case $m = k = 0$ can be excluded, as the sum in $I(n, M, \tau)$ would vanish.)

When $m = n$, we finally obtain a manifold with involution $(\bar{M}, \bar{\tau}) := (M^n, \tau^n)$, such that $\varphi^n[\bar{M}] = I(n, \bar{M}, \bar{\tau}) = 1$. □

5.8.1 A starting manifold

Let D be the first small cover of the hyperbolic right-angled dodecahedron in the left column of [55, Table 1]. Let us number the facets of the dodecahedron 1 to 12, as in [55, Figure 3]. The reflection in facet i induces an isometry τ_i of D, with $F(\tau_i)$ consisting of isolated points, circles, and surfaces of characteristic -1 diffeomorphic to $\mathbb{T}^2 \# \mathbb{RP}^2$; using SageMath, we classify these components in Table 5.1.

		Number of fixed submanifolds of τ_i			
Facet i	Color	Points	Circles	Surfaces X	...with $w_1^2(\nu X) = 1$
$1, 7, 10$	$(0, 0, 1)$	3	1	3	1
$2, 5$	$(0, 1, 0)$	2	4	2	1
11	$(0, 1, 1)$	5	5	1	0
$3, 4$	$(1, 0, 0)$	2	4	2	2
12	$(1, 0, 1)$	1	7	1	1
6	$(1, 1, 0)$	1	7	1	0
$8, 9$	$(1, 1, 1)$	6	2	2	1

Table 5.1: Fixed submanifolds for each isometry τ_i of D.

It is also well known that D covers the arithmetic Coxeter simplex $[5, 3, 4]$, so it is good for $k = \mathbb{Q}(\sqrt{5})$, and that the isometries τ_i are conjugate to a generator of the Coxeter group, so they are defined over k.

By comparing Tables 5.1 and 5.1 we can compute $I(n, D, \tau_i)$ for arbitrary $i = 1, \ldots, 12$ and $n \geq 4$. As an example, let $n = 4m \geq 4$. By periodicity (Remark 5.21), we have

$$I_{n,0} = 1, \quad I_{n,1} = 0, \quad I_{n,2} = v_1^2 + v_2. \tag{5.58}$$

The involution τ_{11} has 5 isolated fixed points and 0 fixed surfaces with nontrivial v_1^2. Hence, $I(4m, D, \tau_{11}) = 1$ for all $m \geq 1$. Similarly, we can check that $I(4m + 1, D, \tau_1) = I(4m + 2, D, \tau_1) = 1$ for all $m \geq 1$. Hence, we have proved Theorem 1.19, which we restate here:

Theorem 1.19. *For each $n \geq 4$, $n \not\equiv 3 \pmod 4$, there exists a connected, non-cobordant closed hyperbolic n-manifold.*

5.8.2 The remaining dimensions

Note that the remaining case $n \equiv 3 \pmod 4$ cannot be handled by starting with a 3-manifold, since by Proposition 5.22, we have $I_{n,0} = I_{n,1} = I_{n,2} = 0$ in that case.

Let us outline a possible general approach to the construction of non-cobordant hyperbolic manifolds in every dimension $n \neq 2^i - 1$, using Proposition 5.22. Since even dimensions are included in Theorem 1.19, assume that $n \neq 2^i - 1$ is odd; then we can write $n = 2^k(2m+1) - 1$ for $k, m \geq 1$. By Proposition 5.22, the first nonzero term in $I(n, M, \tau)$ is $I_{n,2^k} = v_1^{2^k}$. Because of this, we would have to start at the very least from a manifold with involution $(M_{(k)}, \tau_{(k)})$ of dimension $2^k + 1$, such that

$$\sum_{X \in F_{2^k}(\tau_{(k)})} w_1^{2^k}(\nu X) = 1. \tag{5.59}$$

Given such a manifold, by Proposition 5.22 and Theorem 5.28, we would have a connected, non-cobordant closed hyperbolic n-manifold.

Note that the case $k = 1$ corresponds to $n \equiv 1 \pmod 4$ and to the manifold with involution (D, τ_1) described above. In general, the cases $k = 1, 2, 3, 4, \ldots$ correspond to the sets $\{4m+1\}, \{8m+3\}, \{16m+7\}, \{32m+15\}, \ldots$ $(m \geq 1)$, which partition the odd numbers not of the form $2^i - 1$.

As a consequence, finding manifolds with involutions $(M_{(1)}, \tau_{(1)}), \ldots, (M_{(k)}, \tau_{(k)})$ as above implies the existence of connected, non-cobordant closed hyperbolic n-manifolds for all $n \not\equiv -1 \pmod{2^{k+1}}$, $n \notin \{1, 3, 7, \ldots, 2^k - 1\}$: a set of dimensions of asymptotic density $1 - 2^{-k-1}$.

5.8.3 Even dimensions

Recall that, for a manifold of dimension n, the Stiefel–Whitney number w_n is the Euler characteristic modulo 2. Hence, in any even dimension $n = 2k$ there is an alternate method for the construction of non-cobordant hyperbolic manifolds: searching for manifolds of odd characteristic. In the literature, the only such closed manifolds we could find are connected sums $\Sigma_g \# \mathbb{RP}^2$ for $g \geq 1$ (where Σ_g denotes an orientable genus g surface), of characteristic $1 - 2g$, and two 4-manifolds of Euler characteristic 17, constructed by Ratcliffe and Tschantz [111, p. 9] by gluing two copies of the right-angled hyperbolic 120-cell.

On the other hand, we can always ensure that the non-cobordant manifolds from our method have even characteristic, by manipulating the Kolpakov–Reid–Slavich embedding. Indeed, since the twist operation preserves the Euler characteristic, it suffices to arrange for the embedding $(M, \tau) \hookrightarrow (M', \tau')$ of Proposition 5.24 to satisfy $\chi(M') \equiv 0 \pmod 2$.

If M separates M', then the two halves of M' are isometric via τ'. It follows that M' is a double, so $\chi(M')$ is even. If instead M does not separate M', then it defines a double cover of M', associated to the map $\pi_1(M) \twoheadrightarrow \mathbb{Z}_2$ sending $\gamma \mapsto \gamma \cdot D$. In the proof of Proposition 5.24, we can then replace M' by this double cover after applying Theorem 1.12, ultimately obtaining a manifold with even Euler characteristic.

If $|\mathcal{N}_n| = 4$, the two conditions $\varphi^n[M] = 1$ and $w_n(M) = 0$ determine the cobordism class of M. More precisely, we can state:

Proposition 5.29. *There exists a connected, closed hyperbolic 4-manifold M of even Euler characteristic, such that $[M] = [\mathbb{RP}^4 \sqcup (\mathbb{RP}^2 \times \mathbb{RP}^2)] \neq 0$.*

Proof. The cobordism group $\mathcal{N}_4$ is generated by x_4 and x_2^2. By Theorem 1.19, there exists a connected, closed hyperbolic 4-manifold M with $\varphi^4[M] = 1$. Hence, $[M]$ is either $[\mathbb{RP}^4]$ or $[\mathbb{RP}^4 \sqcup (\mathbb{RP}^2 \times \mathbb{RP}^2)]$. From the above discussion, we may also assume that $\chi(M)$ is even, which implies $[M] = [\mathbb{RP}^4 \sqcup (\mathbb{RP}^2 \times \mathbb{RP}^2)]$. ☐

Remark 5.30. In higher even dimensions n, there are at least three linearly independent generators of $\mathcal{N}_n$, that is, x_n, $x_{n-2}x_2$ and $x_2^{n/2}$. Thus, φ^n and w_n do not suffice to determine the cobordism class.

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