Quantum-capacity bounds in spin-network communication channels

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(Received 10 June 2019; published 9 September 2019)

Using the Lieb-Robinson inequality and the continuity property of the quantum capacities in terms of the diamond norm, we derive an upper bound on the values that these capacities can attain in spin-network communication i.i.d. models of arbitrary topology. Different from previous results we make no assumptions about the encoding mechanisms that the sender of the messages adopts in loading information on the network.

DOI: 10.1103/PhysRevA.100.032311

I. INTRODUCTION

In the flying qubit model of quantum communication messages are conveyed from the sender (Alice) to the intended receiver (Bob) after being encoded into some degree of freedom which actually "moves" from the location of the first party to the location of the second party [1–3]. This scenario is the most widely studied in the literature as it finds application in many realistic scenarios which, for instance, employ electromagnetic pulses as quantum carriers. An intriguing alternative is provided by the spin-network communication (SNC) model where instead Alice and Bob are assumed to have access to different portions of an extended many-body quantum medium formed by interacting particles which occupy fixed locations but which are mutually coupled via an assigned, fixed Hamiltonian that, as in a solid, allows the spread of local perturbations along the medium; see, e.g., Ref. [4] and references therein. While being intrinsically limited to short distance applications, SNC schemes have been suggested as an effective way to avoid interfacing issues in the engineering of connections between clusters of otherwise independent quantum processors [5–12]. The study of these models is also motivated by the need for better understanding how the many-body system reacts to the spreading of local perturbations. The main result in this context is the well known bound by Lieb and Robinson (LR) [13,14] on the maximum group velocity for two-points correlation functions of the network; see also [15–18]. For sufficiently regular models, it basically identifies the presence of an effective light cone with exponentially decaying tails implying that information that leaks out to spacelike separated regions is negligible, so that for large enough distances nonsignaling is preserved. Several applications of the LR inequality in a quantum information theoretical treatment of SNC models have been presented in the literature. For instance, in Ref. [19] the LR bound was used to set a limit on the entanglement that can develop across the boundary of a distinguished region for short times. In Ref. [20] instead the bound was used to show that dynamics of 1D quantum spin systems can be approximated efficiently. In Ref. [21] finally, making use of the Fannes inequality [22], Bravyi *et al.* succeeded in linking the LR inequality to the Holevo information capacity C_1 [23,24] attainable for a special example of SNC model where Alice tries to communicate classical messages to Bob by "overwriting" them into the initial state of the spin network she controls. A generalization of this result was presented in Ref. [25], where the LR bound was employed to set the limits within which high-fidelity quantum state transfer and entanglement generation can be performed in general spin-network systems. The aim of the present manuscript is to go beyond these findings, by generalizing the inequality derived in Ref. [21] to the whole plethora of quantum channel capacities [26] that one can associate to the underlying SNC model and to the arbitrary encoding strategies Alice may adopt to upload her messages into the network. For this purpose we shall make explicit use of the continuity argument of Refs. [27,28], which allows one to connect the capacities values of two channels via their relative distance measured in terms of the diamond norm metric [29,30]. While our derivation in many respects mimics the one presented by Bravyi *et al.*, we stress that in order to account for all possible encoding strategies, we have explicitly to deal with the dimension of the ancillary memory element Q Alice can use in the process. The presence of such element, which does not enter in the definition of the spin network (and hence in the associated LR inequality), introduces a divergent contribution which, if not properly tamed, tends to spoil the connection between the LR bound and the diamond norm distance, compromising the possibility of using the results of Refs. [27,28] to constrain the capacity values of the underlying SNC model (a problem which, due to the intrinsic subadditivity of the Holevo information C_1, needed not to be addressed in Ref. [21]).

The manuscript is organized as follows: we start in Sec. II by introducing the SNC scheme and reviewing some basic facts about the LR bound. The main results of the paper are presented in Sec. III. Here, in Sec. III A, first we exploited the LR inequality to put an upper limit on the induced trace-norm distance [3] between the map associated with the SNC scheme and a (zero-capacity) completely depolarizing channel [31,32]. From this, in Sec. III B we hence derive an analogous bound for the diamond distance [29,30] from which ultimately the bounds on the SNC communication capacities follow. The paper ends with the conclusions in Sec. IV. Technical material is presented in the Appendix.

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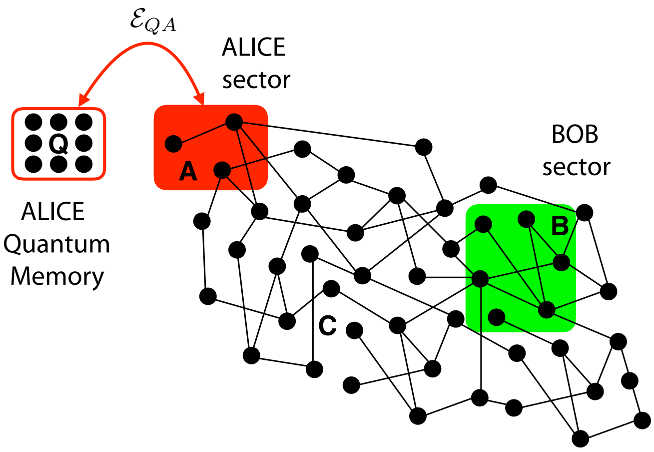

FIG. 1. Schematic representation of a spin-network model for quantum communication. The network $\mathcal{N}$ is divided into three components: the sector A (controlled by the sender of the message Alice), the sector B (controlled by the receiver Bob), and the sector C on which neither Alice nor Bob can operate. The element Q represents an external ancillary memory element Alice uses to store the information she wants to transmit. At time $t = 0$ Alice couples A with Q via an arbitrary encoding mapping $\mathcal{E}_{QA}$ which fully characterizes the adopted communication strategy; Bob, on his side, will try to pick up the message at some later time t from B.

II. MODEL

In the scenario we are interested in, two distant parties (Alice the sender and Bob the receiver) try to exchange (classical or quantum) messages by locally manipulating portions of a many-body quantum system $\mathcal{N}$ that, as schematically shown in Fig. 1, acts as the mediator of the information exchange [4–11,21]. An exhaustive characterization of $\mathcal{N}$ is provided by the spin network formalism [16] where the (fixed) locations of the quantum subsystems are specified by a graph $\mathbb{G} := (V, E)$ defined by a set of vertices V and by a set E of edges. The model is equipped with a metric $d(x, y)$ defined as the shortest path (least number of edges) connecting $x, y \in V$ [$d(x, y)$ being set equal to infinity in the absence of a connecting path], which induces a measure for the diameter $D(X)$ of a given subset $X \subset V$, and a distance $d(X, Y)$ between the subsets $X, Y \subset V$,

$$D(X) := \max_{x,y} \min\{d(x, y)|x, y \in X\},$$
$$d(X, Y) := \min\{d(x, y)|x \in X, y \in Y\}. \tag{1}$$

Indicating with $\mathcal{H}_x$ the Hilbert space associated with the spin that occupies the vertex x of the graph, the Hamiltonian of $\mathcal{N}$, which ultimately is responsible for the information propagation in the medium, can be expressed as

$$\hat{H} := \sum_{X\subset V} \hat{H}_X, \tag{2}$$

where the summation runs over the subsets X of V with $\hat{H}_X$ being a self-adjoint operator that is local on the Hilbert space $\mathcal{H}_X := \otimes_{x\in X}\mathcal{H}_x$, i.e., it acts nontrivially on the spins of X while being the identity everywhere else.

Assume then that Alice and Bob control respectively two nonoverlapping sections A and B of the network $\mathcal{N}$, their distance being $d(A, B) > 0$. The model includes also a domain C of $\mathcal{N}$ that represents the spins which are neither under Bob's nor Alice's control. The two parties agree about a protocol according to which Alice signals to Bob by locally perturbing the input state of the chain $\hat{\tau}_{ABC}$ via a set of local operations acting on the spins belonging to her domain A. Such actions will hence propagate according to the natural Hamiltonian (2) of the network for some transferring time t after which Bob will try to recover them via some proper local operations on the domain B. The question we want to address is how much Bob will be able to discern about Alice's encoding action by performing arbitrary (local) operations on the output state (12). In the next section we shall approach this problem by generalizing the work of Ref. [21] where, using the Lieb-Robinson (LR) inequality [13,14], an upper limit was set for the Holevo capacity C_1 [26] attainable using a specific spin-network communication strategy [explicitly the model defined in Eq. (16) below]. We recall that the LR is a universal bound on the correlations that can be established between distant portions of the network due to the dynamics induced by the system Hamiltonian $\hat{H}$ under minimal assumptions about the structure of involved couplings. In particular, given any two operators $\hat{A}$ and $\hat{B}$ that are local on Alice's and Bob's subsets A and B, respectively, the LR inequality imposes the constraint

$$\frac{\|[\hat{A}(t), \hat{B}]\|}{\|\hat{A}\|\,\|\hat{B}\|} \leqslant \epsilon_{AB}(t), \tag{3}$$

where

$$\|\hat{\Theta}\| := \max_{|\psi\rangle} \|\hat{\Theta}|\psi\rangle\| \tag{4}$$

represents the standard operator norm and where, given

$$\hat{U}(t) := \exp[-i\hat{H}t], \tag{5}$$

the unitary operator associated with the network Hamiltonian (2) ($\hbar = 1$),

$$\hat{A}(t) := \hat{U}^\dagger(t)\hat{A}\hat{U}(t), \tag{6}$$

is the evolved counterpart of $\hat{A}$ in the Heisenberg representation. According to the LR analysis, the quantity $\epsilon_{AB}(t)$ appearing on the right-hand side (RHS) of (3) exhibits an explicit dependence upon the coupling strengths but is independent of the actual state of the network $\hat{\tau}_{ABC}$. Most importantly it depends upon t via its absolute value $|t|$, and tends to zero when this parameter is small and/or $d(A, B)$ is large enough, pointing out that modifications on A sites require a certain time to affect the sector B when the two are disjoint. In particular, as shown in Ref. [33], for finite range Hamiltonians admitting $\bar{D}$ such that $\hat{H}_X = 0$ whenever $D(X) > \bar{D}$, we can express the LR quantity $\epsilon_{AB}(t)$ in the following compact form:

$$\epsilon_{AB}(t) = 2|A||B|\left(\frac{2\,e\,\zeta\,\bar{D}\,|t|}{d(A, B)}\right)^{\frac{d(A,B)}{\bar{D}}}, \tag{7}$$

where $|X|$ is the total number of sites in the domain $X \subset V$, and where ζ is a finite, positive constant characterizing the graph topology and the intensity of the couplings (but not on the size of the graph). If instead the Hamiltonian is explicitly of long-range couplings but sufficiently well behaved so that there exist μ, s positive constants such that $\sup_{x\in V}\sum_{X\ni x}|X|\|\hat{H}_X\|e^{\mu_2 D(X)}\leqslant s$ (exponential decay), or $\sup_{x\in V}\sum_{X\ni x}|X|\|\hat{H}_X\|[1+D(X)]^{\mu}\leqslant s$ (power-law decay), then Eq. (7) gets replaced by

$$\epsilon_{AB}(t)=C|A||B|(e^{v|t|}-1)e^{-\mu d(A,B)}, \tag{8}$$

in the first case, and by

$$\epsilon_{AB}(t)=C|A||B|\frac{e^{v|t|}-1}{[1+d(A,B)]^{\mu}}, \tag{9}$$

in the second case, v and C being positive constants that again depend upon the metric of the network and on the Hamiltonian, but do not scale with the size of the model [16,18].

SNC channels

Without loss of generality we can describe the perturbation induced by Alice on the network in an effort to communicate with Bob as a linear, completely positive, trace preserving (LCPT) [1,2,31,34] encoding map $\mathcal{E}_{QA}$ which at time $t=0$ locally couples the portion A of $\mathcal{N}$ with an external memory element Q that stores the information she wants Bob to receive; see Fig. 1. Specifically, indicating with $\hat{\tau}_{ABC}$ the initial state of the network we have

$$\hat{\rho}_Q\rightarrow\mathcal{E}_{QA}[\hat{\rho}_Q\otimes\hat{\tau}_{ABC}]:=(\mathcal{E}_{QA}\otimes\mathcal{I}_{BC})[\hat{\rho}_Q\otimes\hat{\tau}_{ABC}], \tag{10}$$

where $\mathcal{I}_{BC}$ represents the identity superoperator on the BC domains. Once introduced into the system, the perturbation (10) propagates freely for a transferring time t along the spin network, i.e.,

$$\mathcal{E}_{QA}[\hat{\rho}_Q\otimes\hat{\tau}_{ABC}]\longrightarrow\hat{U}(t)\mathcal{E}_{QA}[\hat{\rho}_Q\otimes\hat{\tau}_{ABC}]\hat{U}^{\dagger}(t), \tag{11}$$

with $\hat{U}(t)$ being the unitary transformation (5) defining the dynamics of $\mathcal{N}$. Bob on his sites will have hence the possibility of perceiving it as a modification of the reduced density matrix of the portion of spin network he controls, i.e.,

$$\begin{aligned}\hat{\rho}_B(t)=\Phi[\hat{\rho}_Q]&:=\mathrm{Tr}_{QAC}(\hat{U}(t)\mathcal{E}_{QA}[\hat{\rho}_Q\otimes\hat{\tau}_{ABC}]\hat{U}^{\dagger}(t))\\&=\mathrm{Tr}_{AC}(\hat{U}(t)\mathcal{E}_{A}[\hat{\tau}_{ABC}]\hat{U}^{\dagger}(t)),\end{aligned} \tag{12}$$

where in the second line we used the fact that $\hat{U}(t)$ does not operate on Q to introduce the LCPT mapping locally acting on A

$$\hat{\tau}_{ABC}\rightarrow\mathcal{E}_A[\hat{\tau}_{ABC}]:=\mathrm{Tr}_Q(\mathcal{E}_{QA}[\hat{\rho}_Q\otimes\hat{\tau}_{ABC}]) \tag{13}$$

that depends on the selected message $\hat{\rho}_Q$ and encoding operation $\mathcal{E}_{QA}$.

Equation (12) defines the SNC channel Φ connecting Alice's quantum memory Q to Bob's location. By construction it is explicitly LCPT and besides the properties of the network (namely its Hamiltonian $\hat{H}$ and its input state $\hat{\tau}_{ABC}$) and the propagation time t, it explicitly depends upon Alice's choice of the encoding transformation $\mathcal{E}_{QA}$. A trivial option is represented for instance by the case where $\mathcal{E}_{QA}$ is the identity mapping $\mathcal{I}_{QA}$: under this assumption no information is transferred from Q either to the A or to the B portion of the network, leading (12) to coincide with the depolarizing map [31,32] $\Phi^{(0)}_{DP}$ defined by the identity

$$\Phi^{(0)}_{DP}[\hat{\rho}_Q]:=\hat{\rho}^{(0)}_B(t)\mathrm{Tr}[\hat{\rho}_Q], \tag{14}$$

where

$$\hat{\rho}^{(0)}_B(t):=\mathrm{Tr}_{AC}[\hat{U}(t)\hat{\tau}_{ABC}\hat{U}^{\dagger}(t)] \tag{15}$$

is the state Bob would have received if Alice decided not to perturb her spins at time $t=0$. Identifying instead $\mathcal{E}_{QA}$ with a control gate activated by different choices of $\hat{\rho}_Q$, we can force $\mathcal{E}_A$ to belong to a generic list $\{\mathcal{E}^{(\alpha)}_A\}_\alpha$ of possible operations, each associated with a classical symbol labeled by the index α. With this choice the scheme (12) induces the mapping

$$\alpha\longrightarrow\hat{\rho}^{(\alpha)}_B(t):=\mathrm{Tr}_{AC}\big(\hat{U}(t)\mathcal{E}^{(\alpha)}_A[\hat{\tau}_{ABC}]\hat{U}^{\dagger}(t)\big) \tag{16}$$

that corresponds to the the signaling strategy analyzed in Ref. [21] to allow the transfer of classical messages from A to B. On the contrary, by identifying Q with a memory element Q_A that is isomorphic with A and taking $\mathcal{E}_{QA}$ to be a unitary swap gate, Eq. (13) reduces to

$$\hat{\tau}_{ABC}\rightarrow\hat{\rho}_A\otimes\hat{\tau}_{BC}, \tag{17}$$

with $\hat{\rho}_A$ being the isomorphic copy of $\hat{\rho}_{Q_A}$ on A and $\hat{\tau}_{BC}:=\mathrm{Tr}_A[\hat{\tau}_{ABC}]$ being the reduced state of the BC domains obtained by tracing away A from the input $\hat{\tau}_{ABC}$. Accordingly, under this construction the SNC channel (12) becomes

$$\Phi_{SW}[\hat{\rho}_{Q_A}]=\mathrm{Tr}_{AC}(\hat{U}(t)[\hat{\rho}_A\otimes\hat{\tau}_{BC}]\hat{U}^{\dagger}(t)), \tag{18}$$

which represents the swap-in–swap-out spin-network communication strategy extensively studied in the literature (see, e.g., Refs. [4–12]) that, at least in principle, is capable of conveying both classical and quantum messages.

Of course, Eqs. (14), (16), and (18) are just three examples out of a large (possibly infinite) set of possible maps (12) that we can realize for fixed $\hat{\tau}$, $\hat{H}$, and t, by using different choices of the mapping $\mathcal{E}_{QA}$. Determining what is the optimal option in terms of communication efficiency is a rather complex problem which arguably depends upon the property of the network, the value of transferring time t, and the relative distance of the locations A and B, as well as upon the kind of messages (classical, private classical, quantum, etc.) one wishes to transfer. Our aim is to show that, however, irrespective of the freedom to select the encoding $\mathcal{E}_{QA}$, the LR inequality (3) poses a fundamental limitation on the resulting communication efficiency.

III. DISTANCE OF THE RECEIVED MESSAGE FROM THE NONSIGNALING STATE

To determine the amount of information that can be effectively retrieved by Bob at the end of the transmission (12) associated with an arbitrary coding strategy $\mathcal{E}_{QA}$, we have to compute the distance between the SNC channel Φ and the depolarizing channel $\Phi^{(0)}_{DP}$ of Eq. (14) associated with the nonsignaling protocol. Specifically in Sec. III A we first analyze the induced trace-norm distance [3] between Φ and $\Phi^{(0)}_{DP}$ showing that irrespective of the choice of $\mathcal{E}_{QA}$ we get the inequality

$$\big\|\Phi-\Phi^{(0)}_{DP}\big\|_1\leqslant M_A^2\epsilon_{AB}(t), \tag{19}$$

where M_A is the dimension of the Hilbert space associated with the spins of the domain A under Alice's control and where $\epsilon_{AB}(t)$ is the LR quantity appearing on the RHS of Eq. (3). Equation (19) is a clear indication that for small enough values of t and/or large enough values of $d(A,B)$, the spin-network channel performances are close to the nonsignaling regime, irrespective of the initial state $\hat{\tau}_{ABC}$ of the network and from the encoding procedure $\mathcal{E}_{QA}$ selected by Alice. In particular, from Eq. (75) of Ref. [28] it is possible to use Eq. (19) to bound the value of the Holevo capacity [23,24] associated with Φ as

$$C_1(\Phi) \leqslant \frac{M_A^2\epsilon_{AB}(t)}{2}\log_2 M_B + g\left(\frac{M_A^2\epsilon_{AB}(t)}{2}\right), \tag{20}$$

where we exploited the fact that $C_1(\Phi_{DP}^{(0)})$ is trivially null (no information being transferred via the depolarizing map) and where $g(x)$ is a function that tends to zero as $x \to 0$, defined by the identities

$$g(x) := (1+x)H_2[x/(1+x)], \tag{21}$$

$$H_2(y) := -y\,\log_2 y - (1-y)\log_2(1-y). \tag{22}$$

Equation (20) generalizes an analogous result obtained in Ref. [21] in the special case of the classical-to-quantum encoding strategy (16). Extending this to all possible encodings and to the full set of communication capacities [1,2,26] [i.e., the classical capacity $C(\Phi)$ [23,24], the private capacity $C_P(\Phi)$ [35], the quantum capacity $Q(\Phi)$ [35–37], and the entanglement assisted capacity $C_E(\Phi)$ [38,39] of the map Φ] requires, however, a little more effort. For this purpose in Sec. III B we focus on the diamond distance [29,30] between Φ and a slightly different version of the depolarizing channel $\Phi_{DP}^{(0)}$, namely the channel

$$\Phi_{DP}^{(1)}[\hat{\rho}_Q] := \hat{\rho}_B^{(1)}(t)\mathrm{Tr}[\hat{\rho}_Q], \tag{23}$$

obtained by replacing in Eq. (14) the state $\hat{\rho}_B^{(0)}(t)$ of (15) with the density matrix

$$\hat{\rho}_B^{(1)}(t) := \mathrm{Tr}_{AC}[\hat{U}(t)(\hat{\tau}_A\otimes\hat{\tau}_{BC})\hat{U}^\dagger(t)], \tag{24}$$

with $\hat{\tau}_A := \mathrm{Tr}_{BC}[\hat{\tau}_{ABC}]$ and $\hat{\tau}_{BC} := \mathrm{Tr}_A[\hat{\tau}_{ABC}]$ the reduced density matrices of the sectors (A and BC, respectively) of the input state of the network $\hat{\tau}_{ABC}$. According to our analysis we shall see that the following inequality holds:

$$\left\|\Phi - \Phi_{DP}^{(1)}\right\|_\diamond \leqslant M\epsilon_{AB}(t), \tag{25}$$

where again $\epsilon_{AB}(t)$ is the LR quantity and where M is upper bounded by $2M_A^4$, specifically

$$M := 2\min\{M_A^4, M_A^3 M_B M_C\}. \tag{26}$$

Notice that, as for Eq. (19), the RHS of this inequality involves only quantities that ultimately just depend upon properties of the spin network: specifically the distance of the sectors A and B, the number of spins they contain, the transferring time t, and the dimension of the Hilbert space of A. From the results of Leung and Smith [27] and the subsequent improvement by Shirokov [28] we can now turn Eq. (25) into a bound for the communication capacities [1,2,26] of the map Φ in terms of the corresponding ones associated with the depolarizing map $\Phi_{DP}^{(1)}$. Explicitly, observing that by definition we have

$$\begin{aligned} C_1\big(\Phi_{DP}^{(1)}\big) &= C\big(\Phi_{DP}^{(1)}\big) = 0,\\ C_P\big(\Phi_{DP}^{(1)}\big) &= Q\big(\Phi_{DP}^{(1)}\big) = 0,\\ C_E\big(\Phi_{DP}^{(1)}\big) &= 0, \end{aligned} \tag{27}$$

Eqs. (81) and (82) of Ref. [28] lead us to

$$Q(\Phi), C(\Phi) \leqslant M\epsilon_{AB}(t)\log_2 M_B + g\left(\frac{M\epsilon_{AB}(t)}{2}\right), \tag{28}$$

while Eq. (76) of Ref. [28] to

$$C_E(\Phi) \leqslant M\epsilon_{AB}(t)\log_2 M' + g\left(\frac{M\epsilon_{AB}(t)}{2}\right), \tag{29}$$

where M' is the minimum between the dimensions of A and B, i.e.,

$$M' := \min\{M_A, M_B\}. \tag{30}$$

As a matter of fact the last of the inequalities presented above happens to be the strongest of all: indeed due to the natural ordering among the capacities [40]

$$C_P(\Phi) \leqslant C(\Phi) \leqslant C_E(\Phi), \quad Q(\Phi) \leqslant C_E(\Phi)/2, \tag{31}$$

our final bounds read

$$C_P(\Phi), C(\Phi), C_E(\Phi) \leqslant M\epsilon_{AB}(t)\log_2 M' + g\left(\frac{M\epsilon_{AB}(t)}{2}\right), \tag{32}$$

$$Q(\Phi) \leqslant \frac{M\epsilon_{AB}(t)}{2}\log_2 M' + \frac{1}{2}g\left(\frac{M\epsilon_{AB}(t)}{2}\right). \tag{33}$$

A. Induced trace-norm distance

The induced trace distance between Φ of Eq. (12) and the depolarizing channel $\Phi_{DP}^{(0)}$ of Eq. (14) related to the nonsignaling protocol is defined as

$$\left\|\Phi - \Phi_{DP}^{(0)}\right\|_1 := 2\max_{\hat{\rho}_Q} D\big(\Phi(\hat{\rho}_Q), \Phi_{DP}^{(0)}(\hat{\rho}_Q)\big), \tag{34}$$

where the maximum is taken over the whole set of possible input states $\hat{\rho}_Q$ of the memory Q and $D(\Phi(\hat{\rho}_Q), \Phi_{DP}^{(0)}(\hat{\rho}_Q))$ is the trace distance [31] between the corresponding output configurations $\hat{\rho}_B(t)$ and $\hat{\rho}_B^{(0)}(t)$ of Φ and $\Phi_{DP}^{(0)}$. According to the Helstrom theorem [1,2], $D(\Phi(\hat{\rho}_Q), \Phi_{DP}^{(0)}(\hat{\rho}_Q))$ gauges the minimum error probability that one can get trying to discriminate $\Phi(\hat{\rho}_Q)$ from $\Phi_{DP}^{(0)}(\hat{\rho}_Q)$; in particular, it is written

$$\begin{aligned} D\big(\Phi(\hat{\rho}_Q), \Phi_{DP}^{(0)}(\hat{\rho}_Q)\big) &= D\big(\hat{\rho}_B(t), \hat{\rho}_B^{(0)}(t)\big)\\ &:= \tfrac{1}{2}\left\|\hat{\rho}_B(t) - \hat{\rho}_B^{(0)}(t)\right\|_1, \end{aligned} \tag{35}$$

with $\|\hat{X}\|_1 := \mathrm{Tr}[\sqrt{\hat{X}^\dagger\hat{X}}]$ being the trace norm of the operator $\hat{X}$, not to be confused with the operator norm introduced in Eq. (4). A useful way to express (35) is

$$D\big(\hat{\rho}_B(t), \hat{\rho}_B^{(0)}(t)\big) = \max_{\hat{\Theta}_B}\left|\mathrm{Tr}_B\big[\hat{\Theta}_B(\hat{\rho}_B(t) - \hat{\rho}_B^{(0)}(t))\big]\right|, \tag{36}$$

where the maximum can be taken either over the set of positive operators $\hat{\mathbb{1}}_B \geqslant \hat{\Theta}_B \geqslant 0$ or, equivalently, on the set of operators $\hat{\Theta}_B = \hat{V}_B/2$ with $\hat{V}_B$ being a unitary operator acting locally on the spins of the domain B (in what follows we'll find more convenient the latter option). Introducing the operator $\hat{\Theta}_B(t) := \hat{U}^\dagger(t)\hat{\Theta}_B\hat{U}(t)$ and using Eqs. (11), (12), and (15) we can then write

$$D\big(\hat{\rho}_B(t), \hat{\rho}_B^{(0)}(t)\big) = \max_{\hat{\Theta}_B} |\mathrm{Tr}[\hat{\Theta}_B(t)(\mathcal{E}_A[\hat{\tau}_{ABC}] - \hat{\tau}_{ABC})]| = \max_{\hat{\Theta}_B} \left|\sum_{k=1}^{K} \mathrm{Tr}[\hat{M}_k^\dagger \hat{\Theta}_B(t)\hat{M}_k\hat{\tau}_{ABC} - \hat{\Theta}_B(t)\hat{M}_k^\dagger\hat{M}_k\hat{\tau}_{ABC}]\right| = \max_{\hat{\Theta}_B} \left|\sum_{k=1}^{K} \mathrm{Tr}[[\hat{M}_k^\dagger, \hat{\Theta}_B(t)]\hat{M}_k\hat{\tau}_{ABC}]\right|,$$

where $\{\hat{M}_k; k = 1, \dots, K\}$ are a Kraus set of local operators on A which represents the action of the LCPT map $\mathcal{E}_A$, i.e.,

$$\mathcal{E}_A[\cdots] = \sum_{k=1}^{K} \hat{M}_k[\cdots]\hat{M}_k^\dagger, \qquad \sum_{k=1}^{K} \hat{M}_k^\dagger \hat{M}_k = \hat{\mathbb{1}}. \tag{37}$$

Now bounding the expectation value of $[\hat{M}_k^\dagger, \hat{\Theta}_B(t)]\hat{M}_k$ over $\hat{\tau}_{ABC}$ with the associated operator norm (4), exploiting the triangular inequality we obtain

$$D\big(\hat{\rho}_B(t), \hat{\rho}_B^{(0)}(t)\big) \leqslant \max_{\hat{\Theta}_B} \sum_{k}^{K} \|[\hat{M}_k^\dagger, \hat{\Theta}_B(t)]\|\|\hat{M}_k\|. \tag{38}$$

Observe that by unitary equivalence of the norm we have $\|[\hat{M}_k^\dagger, \hat{\Theta}_B(t)]\| = \|[\hat{M}_k^\dagger(-t), \hat{\Theta}_B]\|$ where now $\hat{M}_k^\dagger(t) = \hat{U}^\dagger(t)\hat{M}_k^\dagger\hat{U}(t)$ is the time evolved version of the local operator $\hat{M}_k^\dagger$ of A under the action of the network Hamiltonian. Accordingly we can use (3) and (7) to write

$$\|[\hat{M}_k^\dagger, \hat{\Theta}_B(t)]\| \leqslant \|\hat{M}_k^\dagger\|\|\hat{\Theta}_B\|\epsilon_{AB}(t) \leqslant \epsilon_{AB}(t)/2, \tag{39}$$

where we used the fact that

$$\|\hat{M}_k^\dagger\| = \|\hat{M}_k\| = \sqrt{\|\hat{M}_k^\dagger \hat{M}_k\|} \leqslant 1, \tag{40}$$

due to the normalization condition of the Kraus elements, and $\|\hat{\Theta}_B\| = \|\hat{V}_B/2\| \leqslant 1/2$. Replacing this into the bound on $D(\hat{\rho}_B(t), \hat{\rho}_B^{(0)}(t))$ we hence can write

$$D\big(\hat{\rho}_B(t), \hat{\rho}_B^{(0)}(t)\big) \leqslant (K/2)\epsilon_{AB}(t), \tag{41}$$

with the RHS that depends upon the specific choice of the encoding channel $\mathcal{E}_A$ only via the total number K of Kraus elements that enter the decomposition (37). In case we restrict Alice to adopt only unitary encodings, this yields $K = 1$. Alternatively, if we allow for arbitrary LCPT operations $\mathcal{E}_A$ on A, i.e., arbitrary LCPT operations $\mathcal{E}_{QA}$ on Q and A, a universal bound can be established by recalling that, irrespective of the choice of $\mathcal{E}_A$ it is always possible to have a Kraus set with at most $K = M_A^2$ [34]. This leads to

$$D\big(\hat{\rho}_B(t), \hat{\rho}_B^{(0)}(t)\big) \leqslant \big(M_A^2/2\big)\epsilon_{AB}(t), \tag{42}$$

and hence to Eq. (19) via Eq. (34) exploiting the fact that the RHS of Eq. (42) holds true for all possible choices of the input $\hat{\rho}_Q$.

B. Diamond norm distance

The diamond distance [29,30] between two channels Φ and Φ' connecting Q to B is defined as

$$\|\Phi - \Phi'\|_\diamond = \max_{|\psi\rangle_{QQ'}} \|(\Phi \otimes \mathcal{I} - \Phi' \otimes \mathcal{I})(|\psi\rangle_{QQ'}\langle\psi|)\|_1, \tag{43}$$

where the maximization now is performed for extensions $\Phi \otimes \mathcal{I}$ and $\Phi' \otimes \mathcal{I}$ of the original channels involving purifications $|\psi\rangle_{QQ'}$ of the possible inputs of Q constructed on an ancillary system Q' that is isomorphic to Q. A naive way to bound this quantity would be given by using the natural ordering with the induced trace-norm distance (see Appendix), according to which one has

$$\|\Phi - \Phi'\|_1 \leqslant \|\Phi - \Phi'\|_\diamond \leqslant 2M_Q\|\Phi - \Phi'\|_1, \tag{44}$$

with M_Q being the dimension of Alice's memory Q. Applying this to the maps Φ, associated with a generic encoding $\mathcal{E}_{QA}$, and to the depolarizing channel $\Phi_{DP}^{(0)}$ of Eq. (14) yields

$$\big\|\Phi - \Phi_{DP}^{(0)}\big\|_\diamond \leqslant 2M_Q\big\|\Phi - \Phi_{DP}^{(0)}\big\|_1 \leqslant 2M_Q M_A^2\epsilon_{AB}(t), \tag{45}$$

where in writing the last term we invoked the bound (19). In many cases of physical interest where M_Q is directly linked to the dimensionality of A, Eq. (45) is sufficiently strong for our purposes. For instance, this happens for the swap-in–swap-out coding map Φ_{SW} of Eq. (18), where by construction the memory element is isomorphic to A, i.e., $M_{Q_A} = M_A$. Accordingly, in this case Eq. (45) leads to

$$\big\|\Phi_{SW} - \Phi_{DP}^{(0)}\big\|_\diamond \leqslant 2M_A^3\epsilon_{AB}(t), \tag{46}$$

which can be used to replace (25) in our study of the channel capacities reported at the beginning of Sec. III. For a generic choice of $\mathcal{E}_{QA}$, however, the presence of M_Q on the RHS of Eq. (45) poses a severe limitation to this inequality as the dimension of Q is not a property of the spin-network model and can in principle assume unbounded values. To deal with this problem we now consider the diamond norm

$$\big\|\Phi - \Phi_{DP}^{(1)}\big\|_\diamond = \max_{|\psi\rangle_{QQ'}} \big\|\big(\Phi \otimes \mathcal{I} - \Phi_{DP}^{(1)} \otimes \mathcal{I}\big)(|\psi\rangle_{QQ'}\langle\psi|)\big\|_1, \tag{47}$$

between the map Φ associated with the encoding operation $\mathcal{E}_{QA}$ and the depolarizing map $\Phi_{DP}^{(1)}$ defined in Eq. (23). Notice that the actions of Φ and $\Phi_{DP}^{(1)}$ can be expressed as a concatenation of two processes, i.e.,

$$\Phi[\cdots] = \Psi \circ \mathcal{E}[\cdots], \tag{48}$$

$$\Phi_{DP}^{(1)}[\cdots] = \Psi_{DP}^{(1)} \circ \mathcal{E}[\cdots], \tag{49}$$

where

$$\mathcal{E}[\cdots] := \mathrm{Tr}_Q[\mathcal{E}_{QA}[\cdots \otimes \hat{\tau}_{ABC}]] \tag{50}$$

is a LCPT channel from Q to ABC and where

$$\Psi[\cdots] := \mathrm{Tr}_{AC}[\hat{U}(t)[\cdots]\hat{U}^\dagger(t)], \tag{51}$$

$$\Psi_{DP}^{(1)}[\cdots] := \mathrm{Tr}_{AC}[\hat{U}(t)(\hat{\tau}_A \otimes \mathrm{Tr}_A[\cdots])\hat{U}^\dagger(t)] \tag{52}$$

are instead LCPT transformations operating from ABC to B which do not depend upon the special choice of $\mathcal{E}_{QA}$.

Consider first the case where the input state $\hat{\tau}_{ABC}$ of the network $\mathcal{N}$ is a pure vector $|\tau\rangle_{ABC}$. For a generic choice of the pure states $|\psi\rangle_{QQ'}$ of QQ' entering the maximization (47), we have that globally the $QQ'ABC$ system is described by the product vector $|\psi,\tau\rangle_{QQ'ABC} := |\psi\rangle_{QQ'}|\tau\rangle_{ABC}$, which, after a Schmidt decomposition of $|\psi\rangle_{QQ'}$ and $|\tau\rangle_{ABC}$ along the partitions Q, Q' and A, BC, respectively, can be written as

$$|\psi,\tau\rangle_{QQ'ABC} = \sum_{i=1}^{r}\sum_{j=1}^{s}\sqrt{\alpha_i\beta_j}|\psi_i,\psi_i,\tau_j,\tau_j\rangle_{Q'QABC},$$
$$|\psi_i,\psi_i,\tau_j,\tau_j\rangle_{Q'QABC} := |\psi_i\rangle_{Q'}|\psi_i\rangle_Q|\tau_j\rangle_A|\tau_j\rangle_{BC}, \tag{53}$$

with $r \leqslant M_Q$ and $s \leqslant \min\{M_A, M_BM_C\}$ with $|\psi_i\rangle_{Q/Q'}$ and $|\tau_j\rangle_{A/BC}$ forming an orthogonal set of pure states of their respective systems. Completing hence $|\psi_i\rangle_Q$ to a basis of Q, we then define the vectors

$$|\tilde{\lambda}_{\ell,q}\rangle_{Q'ABC} := \sum_{i=1}^{r}\sum_{j=1}^{s}\sqrt{\alpha_i\beta_j}|\psi_i\rangle_{Q'}|\chi_{\ell,q,i,j}\rangle_A|\tau_j\rangle_{BC}, \tag{54}$$

where

$$|\chi_{\ell,q,i,j}\rangle_A := {}_Q\langle\psi_q|\hat{N}_\ell|\psi_i,\tau_j\rangle_{QA}, \tag{55}$$

and where $\hat{N}_\ell$ are the Kraus operators associated with the channel $\mathcal{E}_{QA}$

$$\mathcal{E}_{QA}[\cdots] = \sum_{\ell=1}^{L}\hat{N}_\ell[\cdots]\hat{N}_\ell^\dagger, \quad \sum_{\ell=1}^{L}\hat{N}_\ell^\dagger\hat{N}_\ell = \hat{\mathbb{1}}, \tag{56}$$

with L, which can be always chosen to be smaller than $M_Q^2M_A^2$. Upon normalization Eq. (54) gives the pure states

$$|\lambda_{\ell,q}\rangle_{Q'ABC} := |\tilde{\lambda}_{\ell,q}\rangle_{Q'ABC}/g_{\ell,q}, \tag{57}$$

with the norms $g_{\ell,q} := \||\tilde{\lambda}_{\ell,q}\rangle_{Q'ABC}\|$ satisfying the constraint

$$\sum_{\ell=1}^{L}\sum_{q=1}^{M_Q} g_{\ell,q}^2 = 1. \tag{58}$$

Notice that since terms (55) are elements of the Hilbert space of $\mathcal{H}_A$, it follows that, for each given q and ℓ, when varying indexes i, j, vectors $|\chi_{\ell,q,i,j}\rangle_A|\tau_j\rangle_{BC}$ span a space of dimension not larger than

$$M_* := M_A \times \min\{M_A, M_BM_C\}$$
$$= \min\{M_A^2, M_AM_BM_C\}. \tag{59}$$

Accordingly this number also bounds the maximum number of nonzero terms entering the Schmidt decomposition of $|\lambda_{\ell,q}\rangle_{Q'ABC}$ along the partition Q', ABC, i.e.,

$$|\lambda_{\ell,q}\rangle = \sum_{m=1}^{M_*}\sqrt{\gamma_m}|m\rangle_{Q'}|m\rangle_{ABC}, \tag{60}$$

for a proper choice of orthogonal sets of vectors $|m\rangle_{Q'}$ and $|m\rangle_{ABC}$. Exploiting the above identities the state of $Q'BC$ after the encoding stage through the mapping Eq. (50) can be cast in the following form:

$$\mathcal{E}\otimes\mathcal{I}[|\psi,\tau\rangle\langle\psi,\tau|] = \sum_{\ell=1}^{L}\sum_{q=1}^{M_Q} g_{\ell,q}^2|\lambda_{\ell,q}\rangle\langle\lambda_{\ell,q}|, \tag{61}$$

where for ease of notation we set $|\psi,\tau\rangle := |\psi,\tau\rangle_{QQ'ABC}$ and $|\lambda_{\ell,q}\rangle := |\lambda_{\ell,q}\rangle_{Q'ABC}$. From (48) and (49) we hence get

$$\big\|\big(\Phi\otimes\mathcal{I} - \Phi_{DP}^{(1)}\otimes\mathcal{I}\big)[|\psi\rangle_{QQ'}\langle\psi|]\big\|_1$$
$$= \left\|\sum_{\ell=1}^{L}\sum_{q=1}^{M_Q} g_{\ell,q}^2\big(\Psi\otimes\mathcal{I} - \Psi_{DP}^{(1)}\otimes\mathcal{I}\big)[|\lambda_{\ell,q}\rangle\langle\lambda_{\ell,q}|]\right\|_1$$
$$\leqslant \sum_{\ell=1}^{L}\sum_{q=1}^{M_Q} g_{\ell,q}^2\big\|\big(\Psi\otimes\mathcal{I} - \Psi_{DP}^{(1)}\otimes\mathcal{I}\big)[|\lambda_{\ell,q}\rangle\langle\lambda_{\ell,q}|]\big\|_1, \tag{62}$$

with the last inequality deriving from Eq. (58) by convexity of the trace norm. Remember now that each one of the vectors $|\lambda_{\ell,q}\rangle$ has Schmidt rank smaller than M_* as indicated in Eq. (60). Therefore, the following steps being identical to those in the Appendix, we get

$$\big\|\big(\Psi\otimes\mathcal{I} - \Psi_{DP}^{(1)}\otimes\mathcal{I}\big)[|\lambda_{\ell,q}\rangle\langle\lambda_{\ell,q}|]\big\|_1 \tag{63}$$
$$\leqslant \sum_{m,m'=1}^{M_*}\sqrt{\gamma_m\gamma_{m'}}\big\|\big(\Psi - \Psi_{DP}^{(1)}\big)[|m\rangle\langle m'|]\otimes|m\rangle\langle m'|\big\|_1$$
$$= \sum_{m,m'=1}^{M_*}\sqrt{\gamma_m\gamma_{m'}}\big\|\big(\Psi - \Psi_{DP}^{(1)}\big)[|m\rangle\langle m'|]\big\|_1$$
$$\leqslant 2M_*\big\|\Psi - \Psi_{DP}^{(1)}\big\|_1, \tag{64}$$

with $\|\Psi - \Psi_{DP}^{(1)}\|_1$ being the induced trace distance between Ψ and $\Psi_{DP}^{(1)}$, i.e., the quantity

$$\big\|\Psi - \Psi_{DP}^{(1)}\big\|_1 := \max_{\hat{\tau}'_{ABC}}\big\|\big(\Psi - \Psi_{DP}^{(1)}\big)[\hat{\tau}'_{ABC}]\big\|_1. \tag{65}$$

A crucial observation now is that, indicating with Q_A Alice's memory which is isometric to A, for all $\hat{\tau}'_{ABC}$ we can write

$$\Psi[\hat{\tau}'_{ABC}] = \mathrm{Tr}_{AC}[\hat{U}(t)\hat{\tau}'_{ABC}\hat{U}^\dagger(t)] = \Phi'_{DP}[\hat{\tau}_{Q_A}],$$
$$\Psi_{DP}^{(1)}[\hat{\tau}'_{ABC}] = \Phi'_{SW}[\hat{\tau}_{Q_A}], \tag{66}$$

where $\hat{\tau}_{Q_A}$ represents the copy of $\hat{\tau}_A$ on Q_A, while Φ'_{DP} and Φ'_{SW} are respectively the nonsignaling and the swap-in–swap-out channels associated with the input state $\hat{\tau}'_{ABC}$ of the network. Hence invoking (19) we can write

$$\big\|\big(\Psi - \Psi_{DP}^{(1)}\big)[\hat{\tau}'_{ABC}]\big\|_1 = \|(\Phi'_{DP} - \Phi')[\hat{\tau}_{Q_A}]\|_1$$
$$\leqslant \|\Phi'_{DP} - \Phi'\|_1 \leqslant M_A^2\,\epsilon_{AB}(t), \tag{67}$$

which, by recalling that $\epsilon_{AB}(t)$ does not depend upon the initial state of the spin network, gives

$$\big\|\Psi - \Psi_{DP}^{(1)}\big\|_1 \leqslant M_A^2\,\epsilon_{AB}(t). \tag{68}$$

Accordingly from Eq. (65) and (62) we have

$$\big\|\big(\Phi\otimes\mathcal{I} - \Phi_{DP}^{(1)}\otimes\mathcal{I}\big)[|\psi\rangle_{QQ'}\langle\psi|]\big\|_1 \leqslant 2M_*M_A^2\,\epsilon_{AB}(t), \tag{69}$$

for all $|\psi\rangle_{QQ'}$, which replaced into Eq. (47) leads to

$$\big\|\Phi - \Phi_{DP}^{(1)}\big\|_\diamond \leqslant 2M_*M_A^2\,\epsilon_{AB}(t), \tag{70}$$

hence proving Eq. (25).

The above argument can be also used to deal with the case where the initial state of the network $\hat{\tau}_{ABC}$ is not pure. Indeed, by writing it as a convex sum over a set of pure states

$$\hat{\tau}_{ABC} = \sum_i p_i |\tau_i\rangle_{ABC}\langle\tau_i|, \tag{71}$$

Eq. (61) gets replaced by

$$\mathcal{E}\otimes\mathcal{I}[|\psi\rangle\langle\psi|\otimes\hat{\tau}_{ABC}] = \sum_i \sum_{\ell=1}^{L}\sum_{q=1}^{M_Q} p_i \big(g_{\ell,q}^{(i)}\big)^2 \big|\lambda_{\ell,q}^{(i)}\big\rangle\big\langle\lambda_{\ell,q}^{(i)}\big|, \tag{72}$$

with $g_{\ell,q}^{(i)}$ and $|\lambda_{\ell,q}^{(i)}\rangle$ being associated with the ith pure vector $|\tau_i\rangle_{ABC}$ entering Eq. (71) via the construction detailed in Eqs. (53)–(57). Consequently, we can still invoke convexity to arrive at

$$\begin{aligned}
&\big\|\big(\Phi\otimes\mathcal{I}-\Phi_{DP}^{(1)}\otimes\mathcal{I}\big)[|\psi\rangle_{QQ'}\langle\psi|]\big\|_1\\
&\leqslant \sum_i\sum_{\ell=1}^{L}\sum_{q=1}^{M_Q} p_i\big(g_{\ell,q}^{(i)}\big)^2\big\|\big(\Psi\otimes\mathcal{I}\\
&\quad -\Psi_{DP}^{(1)}\otimes\mathcal{I}\big)\big[\big|\lambda_{\ell,q}^{(i)}\big\rangle\big\langle\lambda_{\ell,q}^{(i)}\big|\big]\big\|_1,
\end{aligned} \tag{73}$$

which formally replaces (62). From here we can exploit the same steps reported in Eqs. (63)–(70).

IV. CONCLUSIONS

We propose a study of a broad set of information capacities associated with spin networks employed as means of communication. In our analysis we considered as a quantum channel Φ a generic spin network in a generic initial state equipped with an encoding represented by a local LCTP map, which results to be more general with respect to specific solutions adopted previously in the literature. Here we made use of the tools offered by the diamond norm and we exploited established results such as the Lieb-Robinson bound [13], which describes how correlations spread in spin systems, and Fannes inequality [22], which states continuity properties of the von Neumann entropy. We were able in such a way to upper bound the whole set of quantum capacities of the map Φ. Possible extensions of our work should include non i.i.d. models and the presence of memory effects [41] in the information transferring, which may arise when allowing Alice to perform sequences of encoding operations during the time it takes for one of them to reach Bob's location.

ACKNOWLEDGMENTS

The authors would like to thank M. Shirokov, F. Verstraete, L. Lami, and A. Winter for useful comments and suggestions.

APPENDIX: BOUNDS ON THE DIAMOND NORM

The lower bound in Eq. (44) is a direct consequence of the definition of the diamond norm [3,29,30]. To prove the upper bound of (44) let us observe that introducing the Schmidt decomposition of the state $|\psi\rangle$ of Q and Q', $|\psi\rangle := \sum_{j=1}^{M_Q}\lambda_j|j\rangle\otimes|j\rangle$, we can write

$$\begin{aligned}
&2D\big((\Phi\otimes\mathcal{I})(|\psi\rangle\langle\psi|),\big(\Phi_{DP}^{(0)}\otimes\mathcal{I}\big)(|\psi\rangle\langle\psi|)\big)\\
&=\Bigg\|\sum_{j,j'=1}^{M_A}\lambda_j\lambda_{j'}\big(\Phi-\Phi_{DP}^{(0)}\big)[|j\rangle\langle j'|]\otimes|j\rangle\langle j'|\Bigg\|_1\\
&\leqslant\sum_{j,j'=1}^{M_A}\lambda_j\lambda_{j'}\big\|\big(\Phi-\Phi_{DP}^{(0)}\big)[|j\rangle\langle j'|]\otimes|j\rangle\langle j'|\big\|_1\\
&\leqslant\sum_{j,j'=1}^{M_A}\lambda_j\lambda_{j'}\big\|\big(\Phi-\Phi_{DP}^{(0)}\big)[|j\rangle\langle j'|]\big\|_1\\
&\leqslant 2\big\|\Phi-\Phi_{DP}^{(0)}\big\|_1\Bigg(\sum_{j=1}^{M_A}\lambda_j\Bigg)^2\leqslant 2M_A\big\|\Phi-\Phi_{DP}^{(0)}\big\|_1,
\end{aligned}\tag{A1}$$

where first we used the convexity of the trace distance, then the fact that for all $|j\rangle$, $|j'\rangle$ we have

$$\big\|\big(\Phi-\Phi_{DP}^{(0)}\big)[|j\rangle\langle j'|]\big\|_1\leqslant 2\big\|\Phi-\Phi_{DP}^{(0)}\big\|_1, \tag{A2}$$

and finally the Chauchy-Schwarz inequality and the normalization condition for the Schmidt coefficients. Replacing hence (A1) into (43), Eq. (44) finally follows.

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