PHILOSOPHICAL TRANSACTIONS A

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Research

Article submitted to journal

Subject Areas:
76F25, 60H50

Keywords:
Turbulence, eddy diffusion, vortex patch, covariance matrix, Dirichlet boundary condition, first eigenvalue

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THE ROYAL SOCIETY PUBLISHING

Eddy heat exchange at the boundary under white noise turbulence

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We prove the existence of an eddy heat diffusion coefficient coming from an idealized model of turbulent fluid. A difficulty lies in the presence of a boundary, with also turbulent mixing and the eddy diffusion coefficient going to zero at the boundary. Nevertheless enhanced diffusion takes place.

1. Introduction

Eddy viscosity and eddy diffusion are two recognized phenomena which appear in experiments and real situations under suitable fluid regimes. In this note we focus on a particular case, but of main interest: the fact that the heat exchange through a boundary may be increased by turbulence of the conducting fluid. The problem can be investigated using different models from the one used here, see e.g. Boussinesq problem. Here we model the phenomenon in the following very simplified way: temperature $T = T(t, \mathbf{x})$ is subject to the equation

$$\partial_t T = \kappa \Delta T + \mathbf{u}_Q \circ \nabla T \qquad \text{in } [0, T] \times D \tag{1.1}$$
$$T|_{\partial D} = 0, \qquad T|_{t=0} = T_0 \qquad \text{in } D$$

in a open connected bounded domain $D \subset \mathbb{R}^d$ with piecewise regular boundary; $\kappa > 0$ is the diffusion constant, that we should think to be small; the velocity field, similarly to investigations for passive scalars [1–6], is a given random, divergence free, vector field $\mathbf{u}_Q(t,\mathbf{x})$, Gaussian, white noise in time, with a prescribed covariance matrix function $Q(\mathbf{x},\mathbf{y})$ in space, simulating in a simplified fashion an incompressible turbulent fluid. The property of being white in time is certainly artificial compared to real fluids; we consider this investigation a first step, to be completed in the future with the understanding of more realistic regimes. We aim to recognize in a quantitative way that, due to the random turbulent transport, heat diffusion is enhanced.

Without noise and fluid motion, the temperature would decay to zero due to the Dirichlet boundary conditions (the cold boundary absorbs heat) but the rate of decay would be given by $\kappa\lambda_D$, where $-\lambda_D$ is the first eigenvalue of the Laplacian operator Δ with zero boundary condition. But when the fluid is turbulent, we expect a faster decay.

There are specific technical difficulties due to the boundary that we have to overcome to prove the result. One problem is that the fluid fluctuations are at rest on ∂D (e.g. [7,8]) namely $Q(\mathbf{x},\mathbf{x}) = 0$ for $\mathbf{x}\in\partial D$. Hence the strength of the mixing mechanism is depleted near the boundary, exactly where the fluid comes in interaction with the cold boundary which is responsible for cooling. We therefore have to understand the balance between these phenomena.

One of the main ideas used in this work goes back to [9,10], see also [11,12], but several other aspects are new, first of all the way to overcome the difficulties due to the boundary, but also the more quantitative presentation of the results, which required new proofs.

Let us state the main result of this work. Let J be a finite or countable index set and $\left(\mathbf{u}_j(\mathbf{x})\right)_{j\in J}$ be divergence free vector fields $\mathbf{u}_j : \overline{D}\to\mathbb{R}^d$:

$$\mathbf{u}_j|_{\partial D} = 0, \qquad \operatorname{div}\mathbf{u}_j = 0$$

with smoothness $\sum_{j\in J}\|\mathbf{u}_j\|^2_{W^{1,2}(D)\cap C(\overline{D})} < \infty$, which in particular allows us to define the covariance matrix-valued function $Q : \overline{D}\times\overline{D}\to\mathbb{R}^{d\times d}$

$$Q(\mathbf{x},\mathbf{y}) = \sum_{j\in J}\mathbf{u}_j(\mathbf{x})\otimes\mathbf{u}_j(\mathbf{y}), \qquad \mathbf{x},\mathbf{y}\in\overline{D}.$$

Associated to it define the bounded linear operator

$$\mathbb{Q} : L^2(D;\mathbb{R}^d)\to L^2(D;\mathbb{R}^d), \qquad (\mathbb{Q}\mathbf{v})(\mathbf{x}) = \int_D Q(\mathbf{x},\mathbf{y})\,\mathbf{v}(\mathbf{y})\,d\mathbf{y}$$

and introduce two important quantities:

$$q(\mathbf{x}) := \min_{\xi\neq\mathbf{0}}\frac{\xi^T Q(\mathbf{x},\mathbf{x})\,\xi}{\xi^T\xi},$$

$$\epsilon_Q := \|\mathbb{Q}^{1/2}\|^2_{L^2\to L^2} = \sup_{\mathbf{v}\neq\mathbf{0}}\frac{\int_D\int_D \mathbf{v}(\mathbf{x})^T Q(\mathbf{x},\mathbf{y})\,\mathbf{v}(\mathbf{y})\,d\mathbf{x}d\mathbf{y}}{\int_D \mathbf{v}(\mathbf{x})^T\,\mathbf{v}(\mathbf{x})\,d\mathbf{x}}.$$

Denoted by $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}\right)$ a filtered probability space with expectation $\mathbb{E}$, let $\left(W_t^j\right)_{j\in J}$ be a family of independent Brownian motions; the generalized process

$$\mathbf{u}_Q(t,\mathbf{x}) = \sum_{j\in J}\mathbf{u}_j(\mathbf{x})\,\frac{dW_t^j}{dt}$$

is a white noise in time, divergence free, with space-covariance $Q(\mathbf{x},\mathbf{y})$. We interpret the equation above as a stochastic equation with Stratonovich noise (the precise interpretation is in weak form with smooth test functions)

$$dT = \kappa\Delta T\,dt + \sum_{j\in J}\mathbf{u}_j\cdot\nabla T\circ dW_t^j. \tag{1.2}$$

Call $D(A)$ the space $W^{2,2}(D)\cap W_0^{1,2}(D)$ where $W^{k,2}(D)$ are the classical Sobolev spaces of square integrable k-times weakly differentiable functions and $W_0^{1,2}(D)$ is the set of $W^{1,2}(D)$-functions equal to zero at the boundary. Define two linear operators $A, A_Q : D(A)\to L^2(D)$ by setting

$$Af = \kappa \Delta f, \qquad A_Q f = \left(\kappa \Delta + \mathcal{L}_Q\right) f$$

where

$$\left(\mathcal{L}_Q f\right)(\mathbf{x}) = \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \partial_\beta \left(Q_{\alpha\beta}(\mathbf{x},\mathbf{x}) \partial_\alpha f(\mathbf{x})\right).$$

Both operators A, A_Q generate analytic semigroups [13] which we denote by e^{tA}, e^{tA_Q}, $t \geq 0$. The function $T_Q(t,\mathbf{x}) := \left(e^{tA_Q} T_0\right)(\mathbf{x})$ is the solution of the modified heat equation

$$\partial_t T_Q(t,\mathbf{x}) = \operatorname{div}\left[\left(\kappa I + \frac{1}{2} Q(\mathbf{x},\mathbf{x})\right) \nabla T_Q(t,\mathbf{x})\right]$$

and thus, in view of the following result, we may call $Q(\mathbf{x},\mathbf{x})$ the **eddy diffusion coefficient**.

We have denoted above by $\kappa\lambda_D$ the first eigenvalue of $-A$; denote by $\lambda_{D,\kappa,Q}$ the first eigenvalue of $-A_Q$; a priori we only know that $\lambda_{D,\kappa,Q} \geq \kappa\lambda_D$.

Remark 1.1. *$\mathcal{L}_Q$ is a degenerate elliptic operator: since $\mathbf{u}_j|_{\partial D} = 0$ we have also $Q|_{\partial D} = 0$. Therefore it is not clear a priori that A_Q is more "elliptic" than A. However we shall prove that $\lambda_{D,\kappa,Q}$ can be much larger than $\kappa\lambda_D$.*

Denote by $L^2_{\mathcal{F}_0}\left(\Omega; L^2(D)\right)$ the space of square integrable random variables with values in $L^2(D)$, adapted to $\mathcal{F}_0$.

Theorem 1.1. *Assume $T_0 \in L^2_{\mathcal{F}_0}\left(\Omega; L^2(D)\right)$. Then, for every $\phi \in L^\infty(D)$,*

$$\mathbb{E}\left[\left(\int_D \phi(\mathbf{x}) T(t,\mathbf{x})\, dx - \int_D \phi(\mathbf{x}) T_Q(t,\mathbf{x})\, dx\right)^2\right] \leq \frac{\epsilon_Q}{2\kappa} \mathbb{E}\left[\|T_0\|^2_{L^2}\right] \|\phi\|^2_\infty .$$

In particular, if $T_0 \geq 0$,

$$\mathbb{E}\left[\left(\int_D |T(t,\mathbf{x})|\, dx\right)^2\right] \leq \left(\frac{\epsilon_Q}{\kappa} + 2|D| \exp\left(-2\lambda_{D,\kappa,Q} t\right)\right) \mathbb{E}\left[\|T_0\|^2_{L^2}\right].$$

Here $|D|$ is the Lebesgue measure of D. This theorem shows (in a quantitative way) that decay is improved on *finite time intervals* $[0,\tau]$ if:

(i) ϵ_Q is very small,
(ii) $\lambda_{D,\kappa,Q} \gg \kappa\lambda_D$.

Denote by D_δ the set

$$D_\delta = \{\mathbf{x} \in D : dist(\mathbf{x}, \partial D) > \delta\}$$

and assume

$$q(\mathbf{x}) \geq \sigma^2 \quad \text{ in } D_\delta.$$

For very general domains D, we have:

Theorem 1.2. *Let D be an open, bounded, Lipschitz domain in $\mathbb{R}^d$. Then, for any fixed $\kappa > 0$, it holds*

$$\lim_{(\sigma,\delta)\to(+\infty,0)} \lambda_{D,\kappa,Q} = +\infty.$$

Under more restrictive assumptions on the domain D we may also provide the following quantitative lower bound on $\lambda_{D,\kappa,Q}$:

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Theorem 1.3. *There exists a constant $C_{D,d} > 0$ such that*

$$\lambda_{D,\kappa,Q} \geq C_{D,d} \min\left(\sigma^2, \frac{\kappa}{\delta}\right)$$

for every Q such that

$$q(\mathbf{x}) \geq \sigma^2 \quad \text{in } D_\delta.$$

When D is the unit ball, asymptotically as $\delta \to 0$ one can take $C_{D,d} = d/2$ and one also has $\lambda_{D,\kappa,Q} \geq \frac{\kappa d}{\kappa + \delta\sigma^2}\sigma^2$.

We prove all these claims in Section 3. The consequence of the last two theorems is that $\lambda_{D,\kappa,Q}$ is large if the noise has a large intensity function $q(\mathbf{x})$, up to a small layer around the boundary ∂D. Summarizing, the information given by Theorems 1.1, 1.2 and 1.3 is that decay is improved on *finite time intervals* $[0, \tau]$ if:

(i) ϵ_Q is very small,
(ii) $q(\mathbf{x})$ is large, except for a small layer around ∂D.

The question then is: can we find a noise (namely a covariance function $Q(\mathbf{x}, \mathbf{y})$) with both properties, and possibly a similarity with the statistics observed in turbulent fluids?

Remark 1.2. *Notice that ϵ_Q, by definition, is given by the operator norm $\|\mathbb{Q}^{1/2}\|^2_{L^2 \to L^2}$ and thus, loosely speaking, it is related to the operator norm of $\mathbb{Q}$; and $q(\mathbf{x})$ is, loosely speaking, related to the trace of the operator $\mathbb{Q}$:*

$$Tr(\mathbb{Q}) = \int_D TrQ(\mathbf{x}, \mathbf{x})\, d\mathbf{x}.$$

The requirement that ϵ_Q is small and $q(\mathbf{x})$ is large, heuristically translated into the requirements that the operator norm of $\mathbb{Q}$ is small and the trace is large is not strange: many operators have finite norm and infinite trace.

First, we would like to explain an heuristic idea, which however we think of relevance. We refer to a noise in full space; the translation in bounded domain is a nontrivial issue under investigation. Consider the homogeneous covariance ($Q(\mathbf{x}, \mathbf{y}) = Q(\mathbf{x} - \mathbf{y})$) of Kraichnan type

$$Q(\mathbf{z}) = \sigma^2 k_0^\zeta \int_{k_0 \leq |\mathbf{k}| \leq k_1} \frac{1}{|\mathbf{k}|^{d+\zeta}} e^{i\mathbf{k}\cdot\mathbf{z}} \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2} d\mathbf{k}.$$

There are two cases where conditions 1 and 2 above are satisfied:

- if $\zeta > 0$, $k_1 = +\infty$, σ^2 large, and k_0 is so large that $\sigma^2 k_0^{-d}$ is small, then $q(\mathbf{x})$ is large and ϵ_Q is small; recall [14] that K41 is $\zeta = \frac{4}{3}$;
- if $-d \leq \zeta \leq 0$, $k_0 = 1$, σ^2 small, and k_1 is so large that $\sigma^2 \int_{1 \leq k \leq k_1} \frac{1}{k^{\zeta+1}} dk$ is large, then $q(\mathbf{x})$ is large and ϵ_Q is small; notice that $\zeta = -d$ is the case of white in space; and $\zeta = 0$ is, in dimension 2, the so called enstrophy measure.

In Section (a) below we prove these claims. The previous arguments require an excellent quantitative spectral knowledge which is not so obvious in bounded domains; one could work with the eigenfunctions and eigenvalues of Stokes operator, mimicking the previous claims, but it is difficult to have explicit information to control the quantities. We have preliminary results corresponding to the white noise case ($\zeta = -d$), not reported here. Below, in Section (b), we present a different class of noise which, we believe, is new, suitable for bounded domains and of interest in itself.

2. Vortex patch noise

The purpose of this section is the construction of a noise, in 2D, based on the idea of vortex patches. The reader will recognize that a similar construction can be done also in dimension 3 but the resulting objects look artificial, since coherent vortex structures in 3D are closer to curves and surfaces. But before, in order to identify a key step, we show why Kraichnan noise works.

(a) Preliminaries on Kraichnan noise

Above we have claimed that Kraichnan noise produces large $q(\mathbf{x})$ and small ϵ_Q under certain conditions. Let us prove that claim because it requires a nontrivial argument in one step. Missing that detail would spoil the understanding of the vortex patch noise below. The control, for Kraichnan noise, on $q(\mathbf{x})$ is given by

$$\begin{aligned}\mathbf{v}^T Q(\mathbf{x},\mathbf{x})\,\mathbf{v} = \mathbf{v}^T Q(0)\,\mathbf{v} &= \sigma^2 k_0^\zeta \int_{k_0\le|\mathbf{k}|\le k_1} \frac{1}{|\mathbf{k}|^{d+\zeta}} \frac{(\mathbf{k}\cdot\mathbf{v})^2}{|\mathbf{k}|^2} d\mathbf{k} \\ &\ge \frac{|\mathbf{v}|^2}{4}\sigma^2 k_0^\zeta \int_{\substack{k_0\le|\mathbf{k}|\le k_1\\ \mathbf{k}\cdot\mathbf{v}\ge|\mathbf{k}||\mathbf{v}|/2}} \frac{1}{|\mathbf{k}|^{d+\zeta}} d\mathbf{k} \\ &= \frac{|\mathbf{v}|^2}{4}\sigma^2 k_0^\zeta C \int_{k_0}^{k_1} \frac{1}{r^{d+\zeta}} r^{d-1} dr = \frac{|\mathbf{v}|^2}{4}\sigma^2 C' \left(1-\left(\frac{k_0}{k_1}\right)^\zeta\right)\end{aligned}$$

for suitable constants $C, C' > 0$. The control on ϵ_Q, is given by

$$\begin{aligned}&\int\int \mathbf{v}(\mathbf{x})^T Q(\mathbf{x},\mathbf{y})\,\mathbf{v}(\mathbf{y})\, d\mathbf{x}d\mathbf{y} \\ &= \sigma^2 k_0^\zeta \int_{k_0\le|\mathbf{k}|\le k_1} \frac{1}{|\mathbf{k}|^{d+\zeta}} \frac{|\mathbf{k}\cdot\widehat{\mathbf{v}}(\mathbf{k})|^2}{|\mathbf{k}|^2} d\mathbf{k} \\ &\le \sigma^2 k_0^{-d} \int_{k_0\le|\mathbf{k}|\le k_1} |\widehat{\mathbf{v}}(\mathbf{k})|^2\, d\mathbf{k} \le \sigma^2 k_0^{-d} \|\mathbf{v}\|_{L^2}^2 .\end{aligned}$$

It is here that one step must be performed in the right way. If we just estimate from above as

$$\int\int \mathbf{v}(\mathbf{x})^T Q(\mathbf{x},\mathbf{y})\,\mathbf{v}(\mathbf{y})\, d\mathbf{x}d\mathbf{y} \le \int\int \sigma^2 k_0^\zeta \int_{k_0\le|\mathbf{k}|\le k_1} \frac{1}{|\mathbf{k}|^{d+\zeta}} |\mathbf{v}(\mathbf{x})|\,|\mathbf{v}(\mathbf{y})|\, d\mathbf{k}d\mathbf{x}d\mathbf{y}$$

then, first, we are in trouble since the L^1 norm of $\mathbf{v}$ is difficult to estimate. Second, even if the space domain is a Torus (in this case the integral over wave numbers is a series) we would end-up with an estimate of the form

$$\le \sigma^2 k_0^\zeta \|\mathbf{v}\|_{L^2}^2 \sum_{k_0\le|\mathbf{k}|\le k_1} \frac{1}{|\mathbf{k}|^{d+\zeta}} \le C\sigma^2 \|\mathbf{v}\|_{L^2}^2 \left(1-\left(\frac{k_0}{k_1}\right)^\zeta\right)$$

which is not sufficient. The result would be that there is no difference in estimating the norm or the trace. The key is using the presence of an orthonormal family of functions (here $e^{i\mathbf{k}\cdot\mathbf{z}}$).

(b) The vortex noise in 2D

Thus consider $d=2$ and assume that D is a smooth bounded connected open domain. We are going to describe a noise of the form $\sum_{j\in J}\mathbf{u}_j(\mathbf{x})\, dW_t^j$ with

$$\mathbf{u}_j(\mathbf{x}) = \mathbf{w}_r(\mathbf{x}-\mathbf{x}_j), \qquad \mathbf{w}_r(\mathbf{x}) = r^{-1}\mathbf{w}\left(\frac{\mathbf{x}}{r}\right)$$

for suitable r and $\mathbf{w}$. The ingredients are therefore the points $\mathbf{x}_j$, called the "centers" of the vortex blobs below, and a vector field $\mathbf{w}$.

(i) The centers of the vortex blobs

Given a positive integer N such that $\frac{1}{N} \leq \delta$, consider the set Λ_N of all points of D_δ having coordinates of the form $\left(\frac{k}{N}, \frac{h}{N}\right)$ with $k, h \in \mathbb{Z}$. For the purpose of the example developed here, the centers $\mathbf{x}_j$ of the blobs will be taken equal to the points of Λ_N; with some effort one can generalize to more flexible distributions of points, also random.

The index set J will be Λ_N itself and points of Λ_N will be denoted by $\mathbf{z}$. Notations below in this section will adapt to this choice; for instance we write the noise in the form

$$\Gamma \sum_{\mathbf{z} \in \Lambda_N} \mathbf{w}_r\left(\mathbf{x}-\mathbf{z}\right) dW_t^{\mathbf{z}}.$$

We have

$$\min_{\mathbf{z}_1 \neq \mathbf{z}_2 \in \Lambda_N} \left|\mathbf{z}_1 - \mathbf{z}_2\right| = \frac{1}{N}, \qquad \min_{\mathbf{z} \in \Lambda_N} d\left(\mathbf{z}, \partial D\right) \geq \delta.$$

Given a positive integer M (in the sequel M will be finite, while $N \to \infty$), the set Λ_N is decomposed as the disjoint union of the sets

$$\Lambda_N = \bigcup_{(k_0,h_0) \in \{0,1,\ldots,M-1\}^2} \Lambda_N^{(M,k_0,h_0)}$$

defined as follows: the points $\left(\frac{k}{N}, \frac{h}{N}\right)$ of $\Lambda_N^{(M,k_0,h_0)}$ have the property that $k = Mn + k_0$, $h = Mm + h_0$, with $n, m \in \mathbb{Z}$. Therefore

$$\min_{\mathbf{z}_1 \neq \mathbf{z}_2 \in \Lambda_N^{(M,k_0,h_0)}} \left|\mathbf{z}_1 - \mathbf{z}_2\right| = \frac{M}{N}$$

for each $(k_0, h_0) \in \{0, 1, ..., M-1\}^2$.

(ii) The vector field $\mathbf{w}$

The construction of vector field $\mathbf{w}$ requires some care. First, in order to have that $\sum_{\mathbf{z} \in \Lambda_N} \mathbf{w}_r\left(\mathbf{x}-\mathbf{z}\right) dW_t^{\mathbf{z}}$ is an admissible noise for our investigation, we need that each $\mathbf{u}_{\mathbf{z}}\left(\mathbf{x}\right) := \mathbf{w}_r\left(\mathbf{x}-\mathbf{z}\right)$ is divergence free, smooth enough and zero at ∂D. Therefore we need $\operatorname{div} \mathbf{w} = 0$, $\mathbf{w}$ smooth enough; and we look for a vector field with compact support, say in the closed ball $\overline{B\left(0,1\right)}$, so that for $r \in (0, \delta)$ and $\mathbf{z} \in \Lambda_N \subset D_\delta$ the rescaled and shifted vector field $\mathbf{w}_r\left(\mathbf{x}-\mathbf{z}\right)$ is zero on ∂D. Moreover, we need other two properties.

One is that $\mathbf{w}\left(\mathbf{x}\right)$ is close to $\frac{1}{2\pi} \frac{\mathbf{x}^\perp}{|\mathbf{x}|^2}$ near $\mathbf{x} = 0$; this is central to the proof that the function $q\left(\mathbf{x}\right)$ is large. The other is that the vector fields $\mathbf{w}_r\left(\mathbf{x}-\mathbf{z}\right)$ are (up to the constant $\int \left|\mathbf{w}\left(\mathbf{x}\right)\right|^2 d\mathbf{x}$, which is not zero since $\mathbf{w}$ is close to $\frac{1}{2\pi} \frac{\mathbf{x}^\perp}{|\mathbf{x}|^2}$ near $\mathbf{x} = 0$) "almost" orthonormal in L^2, which is guaranteed by the fact that the supports are "almost" disjoint. To be precise, if we take truly disjoint supports, then the action of $\mathbf{w}_r\left(\mathbf{x}-\mathbf{z}\right)$ does not cover the full set $D_{2\delta}$: there are intermediate zones between the supports, where $\mathbf{w}_r\left(\mathbf{x}-\mathbf{z}\right)$ does not move space points and this is in contrast with the requirement that $q\left(\mathbf{x}\right)$ should be large everywhere in $D_{2\delta}$. This is why we have introduced M and the sets $\Lambda_N^{(M,k_0,h_0)}$ above: inside each one of these classes the supports will be disjoint and this is sufficient for our estimates; in order to have the supports disjoint for elements of $\Lambda_N^{(M,k_0,h_0)}$ we ask $r \leq \frac{M}{2N}$.

Therefore, summarizing, we look for a vector field $\mathbf{w}$, defined on $\mathbb{R}^2$, smooth, with compact support in $\overline{B\left(0,1\right)}$, $\operatorname{div} \mathbf{w} = 0$, close to $\frac{1}{2\pi} \frac{\mathbf{x}^\perp}{|\mathbf{x}|^2}$ near $\mathbf{x} = 0$. We construct it as

$$\mathbf{w} = \nabla^\perp \psi$$

so that it is divergence free. Thus we look for a smooth function ψ on $\mathbb{R}^2$, compactly supported in $\overline{B\left(0,1\right)}$, close to $\frac{1}{2\pi} \log|\mathbf{x}|$ near $\mathbf{x} = 0$. Such function exists and can be constructed in several ways.

Let $\psi_0 \in C^\infty\left(\mathbb{R}^2 \setminus \{0\}\right)$ be a radial function such that

$$\psi_0(\mathbf{x}) = \frac{1}{2\pi}\log|\mathbf{x}| \text{ for } |\mathbf{x}| \le \frac{1}{3} \text{ and } \psi_0(\mathbf{x}) = 0 \text{ for } |\mathbf{x}| > \frac{2}{3}.$$

Let $f \in C^\infty\left(\mathbb{R}^2\right)$ be a probability density function with support in $B(0,1)$. Given $\epsilon > 0$ small (at least $\epsilon < \frac{1}{6}$), define

$$f_\epsilon(\mathbf{x}) = \epsilon^{-2} f\left(\frac{\mathbf{x}}{\epsilon}\right), \qquad \psi(\mathbf{x}) = \int_{\mathbb{R}^2} \psi_0(\mathbf{x}-\mathbf{y}) f_\epsilon(\mathbf{y})\, d\mathbf{y}.$$

This function satisfies our requirements: its support is in $\overline{B(0,1)}$, it is smooth everywhere and, if we take ϵ small, it is close to ψ_0 which is equal to $\frac{1}{2\pi}\log|\mathbf{x}|$ near $\mathbf{x}=0$. The corresponding vector field $\mathbf{w} = \nabla^\perp\psi$ has the required properties.

Therefore, if $|\mathbf{x}| \le \frac{1}{6}$ and $\epsilon < \frac{1}{6}$ (so that the support of f_ϵ is in $B\left(0,\frac{1}{6}\right)$) we have

$$\mathbf{w}(\mathbf{x}) = \int_{\mathbb{R}^2} \nabla^\perp\psi_0(\mathbf{x}-\mathbf{y}) f_\epsilon(\mathbf{y})\, d\mathbf{y} = \int_{\mathbb{R}^2} \frac{1}{2\pi}\frac{(\mathbf{x}-\mathbf{y})^\perp}{|\mathbf{x}-\mathbf{y}|^2} f_\epsilon(\mathbf{y})\, d\mathbf{y},$$

$$\mathbf{w}_r(\mathbf{x}) = \frac{1}{2\pi r}\int_{\mathbb{R}^2} \frac{(\mathbf{x}/r-\mathbf{y})^\perp}{|\mathbf{x}/r-\mathbf{y}|^2} f_\epsilon(\mathbf{y})\, d\mathbf{y} = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(\mathbf{x}-\mathbf{y})^\perp}{|\mathbf{x}-\mathbf{y}|^2} (\epsilon r)^{-2} f(\mathbf{y}/(\epsilon r))\, d\mathbf{y}.$$

(iii) Estimates on $q(\mathbf{x})$ and ϵ_Q.

We now check that, with proper choices of the parameters, the noise $\Gamma\sum_{\mathbf{z}\in\Lambda_N}\mathbf{w}_r(\mathbf{x}-\mathbf{z})\, dW_t^{\mathbf{z}}$ with $\mathbf{w}_r(\mathbf{x}) = r^{-1}\mathbf{w}\left(\frac{\mathbf{x}}{r}\right)$ has large $q(\mathbf{x})$ and small ϵ_Q.

We choose r with more than one constraint. We have already assumed above

$$r \le \frac{M}{2N}, \qquad r \le \delta.$$

The first inequality implies that the supports of $\mathbf{w}_r(\mathbf{x}-\mathbf{z})$ are disjoint for $\mathbf{z}$ in the same subset $\Lambda_N^{(M,k_0,h_0)}$. The second inequality implies that they are zero at the boundary of D.

The covariance of this noise is

$$Q(\mathbf{x},\mathbf{y}) = \Gamma^2 \sum_{\mathbf{z}\in\Lambda_N} \mathbf{w}_r(\mathbf{x}-\mathbf{z}) \otimes \mathbf{w}_r(\mathbf{y}-\mathbf{z}).$$

We therefore have, for the estimate of ϵ_Q,

$$\begin{aligned}
\int\!\!\int \mathbf{v}(\mathbf{x})^T Q(\mathbf{x},\mathbf{y})\mathbf{v}(\mathbf{y})\, d\mathbf{x}d\mathbf{y} &= \Gamma^2 \sum_{\mathbf{z}\in\Lambda_N}\left(\int \mathbf{w}_r(\mathbf{x}-\mathbf{z})\cdot\mathbf{v}(\mathbf{x})\, d\mathbf{x}\right)^2 \\
&= \|\mathbf{w}\|_{L^2}^2\,\Gamma^2 \sum_{(k_0,h_0)\in\{0,1,\dots,M-1\}^2}\ \sum_{\mathbf{z}\in\Lambda_N^{(M,k_0,h_0)}}\left(\int \frac{\mathbf{w}_r(\mathbf{x}-\mathbf{z})}{\|\mathbf{w}\|_{L^2}}\cdot\mathbf{v}(\mathbf{x})\, d\mathbf{x}\right)^2 \\
&\le M^2\,\|\mathbf{w}\|_{L^2}^2\,\Gamma^2\,\|\mathbf{v}\|_{L^2}^2.
\end{aligned}$$

We have used a basic property, similarly to the most important step in the verification done above for Kraichnan noise: the family $\left\{\frac{\mathbf{w}_r(\mathbf{x}-\mathbf{z})}{\|\mathbf{w}\|_{L^2}}\right\}_{\mathbf{z}\in\Lambda_N^{(M,k_0,h_0)}}$ is orthonormal (not complete), because of the disjoint supports and the property $\int|\mathbf{w}_r(\mathbf{x})|^2\, d\mathbf{x} = \|\mathbf{w}\|_{L^2}^2$. One can easily check that

$$\|\mathbf{w}\|_{L^2}^2 \le C\log\frac{1}{\epsilon}$$

and therefore, taking $\epsilon = \frac{1}{N}$ leads to

$$\epsilon_Q \le M^2\Gamma^2 C\log N$$

which is small if, given N, Γ is small enough.

Concerning $q(\mathbf{x})$, we have, for every $\mathbf{x} \in D$ and every unitary vector $\mathbf{v} \in \mathbb{R}^2$,

$$\mathbf{v}^T Q(\mathbf{x}, \mathbf{x}) \mathbf{v} = \Gamma^2 \sum_{\mathbf{z} \in \Lambda_N} (\mathbf{w}_r(\mathbf{x} - \mathbf{z}) \cdot \mathbf{v})^2 .$$

Now, consider a point $\mathbf{x} \in D_{2\delta}$. If N is large enough with respect to the curvature of ∂D near $\mathbf{x}$, we may find $\mathbf{z} \in \Lambda_N$ close to $\mathbf{x}$, precisely with $\frac{1}{2N} \le |\mathbf{x} - \mathbf{z}| < \frac{2}{N}$, such that

$$\left| \mathbf{v} \cdot \frac{(\mathbf{x} - \mathbf{z})^{\perp}}{|\mathbf{x} - \mathbf{z}|} \right| \ge \frac{1}{4}.$$

Then, if $\left|\frac{\mathbf{x}-\mathbf{z}}{r}\right| \le \frac{1}{6}$, which is true if $\frac{2}{rN} \le \frac{1}{6}$, namely $r \ge \frac{12}{N}$,

$$|\mathbf{w}_r(\mathbf{x} - \mathbf{z}) \cdot \mathbf{v}| = \left| \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathbf{v} \cdot \frac{(\mathbf{x} - \mathbf{z} - \mathbf{y})^{\perp}}{|\mathbf{x} - \mathbf{z} - \mathbf{y}|^2} (\epsilon r)^{-2} f(\mathbf{y}/(\epsilon r)) \, d\mathbf{y} \right| .$$

The constraints $r \le \frac{M}{2N}$, $r \le \delta$, $r \ge \frac{12}{N}$ are all satisfiable if we take $M > 24$ and N large enough; of course we may reduce quantitatively the constraint $M > 24$ by different choices of some parameters above. Recalling that $\epsilon = \frac{1}{N}$; in the above integral, we have $|\mathbf{y}| \le \epsilon r \sim \frac{1}{N^2}$ which means that $\mathbf{y}$ is an infinitesimal perturbation of $\mathbf{x} - \mathbf{z}$ due to $|\mathbf{x} - \mathbf{z}| \sim \frac{1}{N}$. Thus, for N big enough the last integral is bounded below by

$$\ge \frac{1}{2\pi} \frac{1}{8} N = \frac{N}{16\pi}.$$

It follows

$$q(\mathbf{x}) \ge \frac{\Gamma^2 N}{16\pi}.$$

Therefore we may choose N and Γ so that ϵ_Q is small as we want and $q(\mathbf{x})$, on $D_{2\delta}$, is large as we want.

3. Proofs

For reasons of space, we omit some secondary details in the following proofs; for instance we do not write explicitly the definition of solution, the proof that energy and maximum principle estimates are satisfied, the proof that we may pass from the weak to the mild formulation.

(a) Proof of Theorem 1.1

The first key ingredient is the reformulation of the Stratonovich equation in Itô form

$$d_t T = \left(\kappa \Delta T + \widetilde{\mathcal{L}}_Q T\right) dt + \sum_{j \in J} \mathbf{u}_j \cdot \nabla T dW_t^j ,$$

where

$$\left(\widetilde{\mathcal{L}}_Q T\right)(\mathbf{x}) := \frac{1}{2} \sum_{j \in J} \mathbf{u}_j(\mathbf{x}) \cdot \nabla \left(\mathbf{u}_j(\mathbf{x}) \cdot \nabla T(\mathbf{x})\right) .$$

One has

$$\widetilde{\mathcal{L}}_Q = \mathcal{L}_Q .$$

This is a well known fact, see for instance [15]; indeed

$$\widetilde{\mathcal{L}}_Q T = \frac{1}{2}\sum_{j\in J}\sum_{\alpha,\beta=1}^{d} u_j^{\alpha}\partial_\alpha u_j^{\beta}\partial_\beta T + \frac{1}{2}\sum_{j\in J}\sum_{\alpha,\beta=1}^{d} u_j^{\alpha}u_j^{\beta}\partial_\alpha\partial_\beta T.$$

The second sum is equal to $\frac{1}{2}\sum_{\alpha,\beta=1}^{d} Q_{\alpha\beta}(\mathbf{x},\mathbf{x})\,\partial_\alpha\partial_\beta T$. The first one, due to the property $\operatorname{div}\mathbf{u}_j = 0$, is equal to

$$\frac{1}{2}\sum_{j\in J}\sum_{\alpha,\beta=1}^{d}\partial_\alpha\left(u_j^{\alpha}u_j^{\beta}\right)\partial_\beta T = \frac{1}{2}\sum_{\alpha,\beta=1}^{d}\partial_\alpha Q_{\alpha\beta}(\mathbf{x},\mathbf{x})\,\partial_\beta T$$

where we have also used the assumptions of uniform convergence of the series of the derivatives.

From the previous facts we have

$$d_t\left(T - T_Q\right) = \left(\kappa\Delta + \mathcal{L}_Q\right)\left(T - T_Q\right)dt + \sum_{j\in J}\mathbf{u}_j\cdot\nabla T\, dW_t^j.$$

The mild formulation of this identity, furthermore applied in a weak sense to a smooth test function ϕ with compact support in D, is:

$$\langle\phi, T(t) - T_Q(t)\rangle = \sum_{j\in J}\int_0^t\left\langle e^{(t-s)A_Q}\phi, \mathbf{u}_j\cdot\nabla T(s)\right\rangle dW_s^j$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(D)$ and we have also used the fact that the semigroup $e^{(t-s)A_Q}$ is self adjoint. By the isometry formula for Itô integrals,

$$\mathbb{E}\left[\langle\phi, T(t) - T_Q(t)\rangle^2\right] = \sum_{j\in J}\int_0^t\mathbb{E}\left[\left\langle e^{(t-s)A_Q}\phi, \mathbf{u}_j\cdot\nabla T(s)\right\rangle^2\right]ds.$$

We have (we write $T_s(\mathbf{x})$ for $T(s,\mathbf{x})$ and $\phi_{t,s}(\mathbf{x})$ for $\left(e^{(t-s)A_Q}\phi\right)(\mathbf{x})$ to shorten notations)

$$\begin{aligned}&\sum_{j\in J}\left\langle e^{(t-s)A_Q}\phi, \mathbf{u}_j\cdot\nabla T_s\right\rangle^2\\ &= \sum_{\alpha,\beta=1}^{d}\int_D\int_D\phi_{t,s}(\mathbf{x})\,\phi_{t,s}(\mathbf{y})\,Q_{\alpha\beta}(\mathbf{x},\mathbf{y})\,\partial_\alpha T_s(\mathbf{x})\,\partial_\beta T_s(\mathbf{y})\,d\mathbf{x}d\mathbf{y}.\end{aligned}$$

The semigroup e^{tA_Q} satisfies the Maximum Principle, namely $\|e^{tA_Q}\phi\|_\infty \le \|\phi\|_\infty$. Hence, recalling the definition of $\mathbb{Q}$ and ϵ_Q,

$$\begin{aligned}\sum_{j\in J}\left\langle e^{(t-s)A_Q}\phi, \mathbf{u}_j\cdot\nabla T_s\right\rangle^2 &\le \epsilon_Q\int_D\left|\phi_{t,s}(\mathbf{x})\,\nabla T_s(\mathbf{x})\right|^2 d\mathbf{x}\\ &\le \|\phi\|_\infty^2\,\epsilon_Q\int_D\left|\nabla T(s,\mathbf{x})\right|^2 d\mathbf{x}.\end{aligned}$$

Moreover, for the original stochastic equation (1.2) we have the inequality

$$\int_0^\infty\int_D\left|\nabla T(t,\mathbf{x})\right|^2 d\mathbf{x}dt \le \frac{1}{2\kappa}\int_D T_0^2(\mathbf{x})\,d\mathbf{x}.$$

Together they imply

$$\mathbb{E}\left[\langle\phi, T(t) - T_Q(t)\rangle^2\right] \le \frac{\epsilon_Q}{2\kappa}\mathbb{E}\left[\|T_0\|_{L^2}^2\right]\|\phi\|_\infty^2.$$

If $T_0 \ge 0$, then both $T(t)$ and $T_Q(t)$ are nonnegetive. Choose a sequence ϕ_n converging to 1 in D. We deduce

$$\mathbb{E}\left[\left(\int_D T(t,\mathbf{x})\,d\mathbf{x} - \left\langle 1, e^{tA_Q}T_0\right\rangle\right)^2\right] \le \frac{\epsilon_Q}{2\kappa}\mathbb{E}\left[\|T_0\|_{L^2}^2\right].$$

It implies

$$\mathbb{E}\left[\left(\int_D |T(t,\mathbf{x})|\, d\mathbf{x}\right)^2\right] \leq 2\frac{\epsilon_Q}{2\kappa}\mathbb{E}\left[\|T_0\|_{L^2}^2\right] + 2\mathbb{E}\left[\left\langle 1, e^{tA_Q}T_0\right\rangle_{L^2}^2\right]$$
$$\leq \frac{\epsilon_Q}{\kappa}\mathbb{E}\left[\|T_0\|_{L^2}^2\right] + 2\,|D|\,\mathbb{E}\left[\left\|e^{tA_Q}T_0\right\|_{L^2}^2\right]$$
$$\leq \left(\frac{\epsilon_Q}{\kappa} + 2\,|D|\exp\left(-2\lambda_{D,\kappa,Q}t\right)\right)\mathbb{E}\left[\|T_0\|_{L^2}^2\right].$$

(b) Proof of Theorems 1.2 and 1.3

(i) Proof of Theorem 1.2

We use the variational characterization of $\lambda_{D,\kappa,Q}$ given by

$$\lambda_{D,\kappa,Q} = \inf_{T\in W_0^{1,2}(D):\int_D T^2 d\mathbf{x}=1} \int_D \sum_{\alpha,\beta=1}^{d} \left(\kappa\delta_{\alpha\beta} + Q_{\alpha\beta}(\mathbf{x},\mathbf{x})\right)\partial_\alpha T(\mathbf{x})\,\partial_\beta T(\mathbf{x})\, d\mathbf{x}.$$

We have $\lambda_{D,\kappa,Q} \geq \lambda_{\kappa,\sigma,\delta}$ where

$$\lambda_{\kappa,\sigma,\delta} := \inf_{T\in W_0^{1,2}(D):\int_D T^2 d\mathbf{x}=1} \int_D \left(\kappa + \sigma^2\cdot 1_{D_\delta}(\mathbf{x})\right)|\nabla T(\mathbf{x})|^2 d\mathbf{x}.$$

We want to prove that

$$\lim_{(\sigma,\delta)\to(+\infty,0)} \lambda_{\kappa,\sigma,\delta} = +\infty.$$

Suppose this is not true, then we can find $C>0$ and a sequence $(\sigma_n,\delta_n)\to(+\infty,0)$ such that $\lambda_n := \lambda_{\kappa,\sigma_n,\delta_n} \leq C$; this implies that we can find a sequence $T_n \in W_0^{1,2}(D)$ such that $\|T_n\|_{L^2}=1$ and

$$\int_D (\kappa + \sigma_n^2\cdot 1_{D_{\delta_n}}(\mathbf{x}))|\nabla T_n(\mathbf{x})|^2 d\mathbf{x} = \lambda_n \leq C \quad \forall n\in\mathbb{N}.$$

We deduce as a consequence that $\int_D |\nabla T_n|^2 d\mathbf{x} \leq \kappa^{-1}C$ and the sequence $\{T_n\}_n$ is bounded in $W_0^{1,2}(D)$; by Rellich-Kondrakhov compactness theorem for $W_0^{1,2}(D)$, we can extract a (not relabelled) subsequence such that $T_n \to T$ strongly in $L^2(D)$ and $\nabla T_n \to \nabla T$ weakly in $L^2(D)$ for a suitable $T\in W_0^{1,2}(D)$. On the other hand,

$$\int_{D_{\delta_n}} |\nabla T_n(\mathbf{x})|^2 d\mathbf{x} \leq \frac{C}{\sigma_n^2} \to 0$$

which together with $D_\varepsilon \subset D_{\delta_n}$ for n large enough implies that $\nabla T(\mathbf{x})=0$ for a.e. $x\in D_\varepsilon$ and for any $\varepsilon>0$. Overall this implies that $\|T\|_{L^2}=1$ and $\nabla T=0$, thus T is a constant function which is 0 at the boundary ∂D, giving a contradiction.

(ii) Preparation to the proof of Theorem 1.3

We give the proof of Theorem 1.3 only in the case of the ball $D=B(0,1)$. The case of a star-shaped domain with smooth boundary can be reduced to the ball by relatively easy arguments. We think that the result is true for much more general domains but the details are outside the scope of this work.

Therefore now we have (with the notations of the previous section)

$$\lambda_{\kappa,\sigma,\delta} = \inf_{T\in W_0^{1,2}(B(0,1)):\int_{B(0,1)} T^2 d\mathbf{x}=1} \int_{B(0,1)} \left(\kappa + \sigma^2\cdot 1_{B(0,1-\delta)}(\mathbf{x})\right)|\nabla T(\mathbf{x})|^2 d\mathbf{x}.$$

Classical facts guarantee that there is a unique minimizer for the variational problem which defines $\lambda_{\kappa,\sigma,\delta}$, and it is non-negative. Denote it by $T_{\kappa,\sigma,\delta}$. Since the functional is invariant by rotation, uniqueness implies that also the minimizer is invariant by rotation. Then

$$T_{\kappa,\sigma,\delta}\left(\mathbf{x}\right)=f_{\kappa,\sigma,\delta}\left(\left|\mathbf{x}\right|\right)$$

for some function $f_{\kappa,\sigma,\delta}\in W^{1,2}\left(0,1\right)$. Called ω_d the surface of the unit sphere in $\mathbb{R}^d$, we have $\lambda_{\kappa,\sigma,\delta}=\omega_d\inf J\left(f\right)$,

$$J\left(f\right)=\kappa\int_0^1 f'\left(r\right)^2 r^{d-1}dr+\sigma^2\int_0^{1-\delta} f'\left(r\right)^2 r^{d-1}dr \tag{3.1}$$

the infimum being taken over all $f\in W^{1,2}\left(0,1\right)$ such that $f\left(1\right)=0$ and $\int_0^1 f\left(r\right)^2 r^{d-1}dr=1/\omega_d$. The function $f_{\kappa,\sigma,\delta}$, non-negative, is non-increasing; let us prove this by contradiction. Indeed, if there are $r_1<r_2$ with $f_{\kappa,\sigma,\delta}\left(r_1\right)<f_{\kappa,\sigma,\delta}\left(r_2\right)$, by continuity of $f_{\kappa,\sigma,\delta}$ (it is of class $W^{1,2}\left(0,1\right)$) there exists a point $r_{\min}<r_2$ of minimum in $\left[0,r_2\right]$, with $f_{\kappa,\sigma,\delta}\left(r_{\min}\right)<f_{\kappa,\sigma,\delta}\left(r_2\right)$. Given $l\in\left(f_{\kappa,\sigma,\delta}\left(r_{\min}\right),f_{\kappa,\sigma,\delta}\left(r_2\right)\right)$, let r_l^+ be the minimum of all points $r>r_{\min}$ such that $f_{\kappa,\sigma,\delta}\left(r\right)=l$; it exists again by continuity of $f_{\kappa,\sigma,\delta}$. If $r_{\min}=0$, we complete the contradiction as follows: for any such l we introduce the function $\widetilde{f}_l$ equal to $f_{\kappa,\sigma,\delta}$ on $\left[r_l^+,1\right]$ and constantly equal to $f_{\kappa,\sigma,\delta}(r_l^+)=l$ in $\left[0,r_l^+\right]$. It is of class $W^{1,2}\left(0,1\right)$, $\widetilde{f}_l\left(1\right)=0$, $J(\widetilde{f}_l)<J\left(f_{\kappa,\sigma,\delta}\right)$. In itself this is not a contradiction yet because $a_l^2:=\int_0^1\widetilde{f}_l\left(r\right)^2 r^{d-1}dr$ is not equal to $1/\omega_d$; but $a_l^2>1/\omega_d$, because $\widetilde{f}_l\left(r\right)=f_{\kappa,\sigma,\delta}(r_l^+)>f_{\kappa,\sigma,\delta}\left(r\right)$ in $\left[0,r_l^+\right)$; hence the function $f_l=\widetilde{f}_l/(|a_l|\sqrt{\omega_d})$ satisfies all the constraints and has the property $J\left(f_l\right)<J(\widetilde{f}_l)$, hence $J\left(f_l\right)<J\left(f_{\kappa,\sigma,\delta}\right)$. If $r_{\min}>0$, it is sufficient to introduce the maximum r_l^- of all points $r<r_{\min}$ such that $f_{\kappa,\sigma,\delta}\left(r\right)=l$ and repeat the previous argument on $\left[r_l^-,r_l^+\right]$ instead of $\left[0,r_l^+\right]$.

(iii) Proof of Theorem 1.3

Therefore $\lambda_{\kappa,\sigma,\delta}=\omega_d J\left(f_{\kappa,\sigma,\delta}\right)$, where we know that $f:=f_{\kappa,\sigma,\delta}$ is of class $W^{1,2}\left(0,1\right)$, non-negative and non-increasing, $f\left(1\right)=0$, $\int_0^1 f\left(r\right)^2 r^{d-1}dr=1/\omega_d$. We prove now several inequalities, some of them inspired by the Poincaré inequality. First,

$$\begin{aligned}\delta\int_0^1 f'\left(r\right)^2 r^{d-1}dr &\geq \delta\int_{1-\delta}^1 f'\left(r\right)^2 r^{d-1}dr\geq\left(1-\delta\right)^{d-1}\left(\int_{1-\delta}^1\left|f'\left(r\right)\right|dr\right)^2\\ &=\left(1-\delta\right)^{d-1}\left(-\int_{1-\delta}^1 f'\left(r\right)dr\right)^2=\left(1-\delta\right)^{d-1}f^2\left(1-\delta\right).\end{aligned}\tag{3.2}$$

Second, since $f\left(r\right)=f\left(1-\delta\right)-\int_r^{1-\delta}f'\left(s\right)ds$, we have

$$f\left(r\right)^2\leq\left(1+\gamma\right)f^2\left(1-\delta\right)+\left(1+\gamma^{-1}\right)\int_r^{1-\delta}f'\left(s\right)^2 ds$$

for all $\gamma>0$, hence we get, for $g\left(\delta\right):=\int_{1-\delta}^1 f\left(r\right)^2 r^{d-1}dr$,

$$\begin{aligned}\frac{1}{\omega_d}-g\left(\delta\right)&=\int_0^{1-\delta}f\left(r\right)^2 r^{d-1}dr\\ &\leq\frac{1+\gamma}{d}f^2\left(1-\delta\right)+\left(1+\gamma^{-1}\right)\int_0^{1-\delta}\int_r^{1-\delta}f'\left(s\right)^2 r^{d-1}dsdr.\end{aligned}$$

The double integral can be manipulated and shown to be equal to $\frac{1}{d}\int_0^{1-\delta}s^d f'\left(s\right)^2 ds$, and a factor s in this integral can be bounded above by 1. Notice also that, by monotonicity of f, $g\left(\delta\right)\leq\delta f^2\left(1-\delta\right)$. We deduce

$$\int_0^{1-\delta}s^{d-1}f'\left(s\right)^2 ds\geq\frac{d}{\left(1+\gamma^{-1}\right)\omega_d}-\frac{1+\gamma}{1+\gamma^{-1}}f^2\left(1-\delta\right)-\frac{\delta d}{1+\gamma^{-1}}f^2\left(1-\delta\right).$$

Therefore, combining this inequality with (3.2) and (3.1) yields

$$\lambda_{\kappa,\sigma,\delta}\geq\omega_d\left(\frac{\kappa}{\delta}\left(1-\delta\right)^{d-1}-\sigma^2\frac{1+\gamma+\delta d}{1+\gamma^{-1}}\right)f^2\left(1-\delta\right)+\frac{d}{1+\gamma^{-1}}\sigma^2.$$

We now choose γ such that

$$\frac{1+\gamma+\delta d}{1+\gamma^{-1}} = \frac{\kappa}{\delta\sigma^2}(1-\delta)^{d-1}$$

which is easily seen to be always possible. With this choice we have $\lambda_{\kappa,\sigma,\delta} \geq \frac{d}{1+\gamma^{-1}}\sigma^2$. The algebraic computations to complete the proof of the theorem are now elementary but cumbersome, so let us give them only asymptotically as $\delta \to 0$. We thus have $\frac{1+\gamma}{1+\gamma^{-1}} = \frac{\kappa}{\delta\sigma^2}$ which gives $\gamma = \frac{\kappa}{\delta\sigma^2}$, hence $\lambda_{\kappa,\sigma,\delta} \geq \frac{\kappa d}{\kappa+\delta\sigma^2}\sigma^2$, as stated in the theorem. It can also be rewritten as

$$\lambda_{\kappa,\sigma,\delta} \geq \frac{\frac{\kappa}{\delta}d}{\frac{\kappa}{\delta}+\sigma^2}\sigma^2$$

which easily proves it is larger than $\frac{d}{2}\min\left(\sigma^2, \frac{\kappa}{\delta}\right)$ (if $\sigma^2 \leq \frac{\kappa}{\delta}$, then $\frac{\frac{\kappa}{\delta}d}{\frac{\kappa}{\delta}+\sigma^2} \geq \frac{\frac{\kappa}{\delta}d}{2\frac{\kappa}{\delta}} = \frac{d}{2}$; similarly in the opposite case).

Funding. The last named author would like to thank the financial supports of the National Key R&D Program of China (No. 2020YFA0712700) and the National Natural Science Foundation of China (No. 11688101).

Acknowledgements. Insert acknowledgment text here.

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