

Anomalies and phases of strongly coupled chiral gauge theories: Recent developments

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After many years of investigations, our understanding of the dynamics of strongly coupled chiral gauge theories is still quite unsatisfactory today. Conventional wisdom about strongly coupled gauge theories, successfully applied to QCD, is not always as useful in chiral gauge theories. Recently, some new ideas and techniques have been developed, which involve concepts of generalized symmetries, of gauging a discrete center symmetry, and of generalizing the 't Hooft anomaly matching constraints to include certain mixed symmetries. This new development has been applied to chiral gauge theories, leading to many interesting, sometimes quite unexpected, results. For instance, in the context of generalized Bars–Yankielowicz and generalized Georgi–Glashow models, these new types of anomalies give a rather clear indication in favor of the dynamical Higgs phase, against confining, flavor symmetric vacua.

Another closely related topic is strong anomaly and the effective low-energy action representing it. It turns out that they have significant implications on the phase of chiral gauge theories, giving indications consistent with the findings based on the generalized anomalies.

Some striking analogies and contrasts between the massless QCD and chiral gauge theories seem to emerge from these discussions. The aim of this work is to review these developments.

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1. Introduction

One of the mysteries of the world we live in is the fact that it has a nontrivial chiral property. The macroscopic structures such as biological bodies have often

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approximately left–right symmetric forms, but not exactly. At the molecular level, $O(10^{-6} \text{ cm})$, the structure of DNA possesses a definite chiral spiral form. At the microscopic scales of the fundamental interactions, $O(10^{-14} \text{ cm})$, the left-handed and right-handed quarks and leptons have distinct couplings to the $SU(3) \times SU(2)_L \times U(1)_Y$ gauge bosons. The parity violation in the “weak interactions” processes, though it was first considered somewhat weird, later found a natural explanation¹: the fundamental entity of our world is a set of Weyl fermions in the $(\frac{1}{2}, 0)$ representation of the Lorentz group. If such fermions are in a generic complex representation of the gauge group, the resulting theory will break parity. In other words, the origin of parity violation is to be traced to the type of building blocks our world is made of, rather than to a peculiar property of the Fermi interactions.

Most grand unification schemes such as those based on $SU(5)$, $SO(10)$ or E_7 groups are also all based on chiral gauge theories.

The Glashow–Weiberg–Salam (GWS) $SU(2)_L \times U(1)_Y$ theory (as well as its GUT generalizations) is a weakly coupled theory and, as such, is well understood within the framework of perturbation theory. But this also means that the theory should be regarded, at best, as a very good low-energy effective theory. In particular, it is unlikely that the gauge symmetry breaking sector described by a potential term for the Higgs scalar, though phenomenologically quite successful, is a self-consistent, fundamental description. Nevertheless, attempts to replace it by new, QCD-like strongly coupled gauge theories (such as Technicolor, Extended technicolor and Walking technicolor) have not been entirely successful so far.

On the other hand, our understanding of *strongly coupled chiral gauge theories* is today uncomfortably limited.^{2–21} This is in a striking contrast to the case of vector-like gauge theories, for which we have an extensive literature. They include some general theorems,^{22,23} lattice simulations,^{24–27} the effective Lagrangians,^{28–36} the conventional ’t Hooft anomaly analysis,² the powerful exact results in $\mathcal{N} = 2$ supersymmetric theories,^{37–43} space compactification with semi-classical approximation,^{44–46} and so on.

These theoretical tools are unfortunately unavailable for the analysis of strongly coupled *chiral* gauge theories, with a few exceptions such as the large N approximation, the ’t Hooft anomaly matching constraints, and some general considerations based on the renormalization group. Taken together, they yield some useful but not very detailed knowledge about the dynamics and phases of the chiral gauge theories. Partial list of the papers on these efforts are found in Refs. 2–21. This kind of situation is certainly limiting rather severely our capability of finding the place for chiral gauge theories in the context of a realistic theory of the fundamental interactions beyond the standard model, e.g. in the context of composite models for the quarks and leptons, the composite Higgs boson models, the composite dynamical models for dark matter. The need for new ideas for making true progress in this research field is badly felt.

In our view, a little hope for a breakthrough comes from the very recent ideas involving the concept of generalized symmetries.^{47–50} Pure Yang–Mills theories and QCD-like theories have been analyzed by using new, stronger, version of

't Hooft anomaly matching constraints, involving 0-form and 1-form symmetries together.^{51–68}

The generalized symmetries are symmetries which do not act on local field operators, as the conventional symmetries, but only on extended objects, such as closed lines and surfaces. As the corresponding gauge functions are now 1-form, 2-form fields (in the standard gauging of global symmetries the gauge transformation parameters which appear in the exponent are just — 0-form — functions of the space–time), these new types of symmetries are sometimes called 1-form-, 2-form-, etc., symmetries.

A familiar example of the 1-form symmetry is the \mathbb{Z}_N center symmetry in Euclidean $SU(N)$ Yang–Mills theory at finite temperature, which acts on the Polyakov loop. The unbroken (or broken) center symmetry by the vacuum expectation value (VEV) of the Polyakov loop, is a valid criterion for confinement (or de-confinement) phase. Similarly, the area law/perimeter law of the VEV of the Wilson loops can be considered as the criterion for confinement/Higgs phase. Note however that the center symmetry \mathbb{Z}_N as used this way is a *global* 1-form symmetry. The main input of the new development^{51–68} is the idea of *gauging* 1-form symmetries.

In fact, these generalized symmetries *are* symmetries of the systems being considered, even if they act in a way different from the conventional symmetry action. We are free to decide to “gauge” these new types of symmetries. Anomalies one encounters in doing so are obstructions of gauging a symmetry, which is by definition a 't Hooft anomaly. As in the usual requirement of the ultraviolet (UV)-infrared (IR) “anomaly matching”, a similar conditions arise in gauging the generalized (higher-form) symmetries together with some conventional (“0-form”) symmetries, which have come to be known in the recent literature as a “mixed 't Hooft anomaly”. As will be seen below, these constraints carry significant information on the dynamics of wide classes of chiral gauge theories as well, which is our main interest.

Very recently, the present authors have realized⁹¹ that the strong anomaly, which plays a prominent role in the solution of the so-called $U(1)_A$ problem in QCD, can also be significant in the study of the phases of chiral gauge theories, such as those discussed in this review. A key observation is that the well-known low-energy strong-anomaly effective action for QCD, which reproduces the effects of the strong anomaly in the low-energy effective action, has a nontrivial implication on the symmetry breaking pattern itself. For some reason these ideas have not been applied much to the study of strongly interacting chiral gauge theories until now.

It is found that, quite remarkably, the considerations based on the strong anomaly yield similar indications on the phase of the chiral gauge theories, as those found by applying the generalized 't Hooft anomalies. Even though there are arguments fully independent of each other, the agreement of the results should not probably be considered entirely accidental. Indeed they both originate from the proper treatment of the strong anomaly on various $U(1)$ symmetries present in the theory.

The rest of the work is organized as follows. In Sec. 2, the procedure of the computing anomalies associated with *the gauging of certain 1-form discrete symmetry* is discussed, as this constitutes one of the main theoretical tools of our analysis. The detail of the discussion is further divided in two parts. The first part, Subsec. 2.1, concerns models with 1-form center symmetry $\mathbb{Z}_k \subset \mathbb{Z}_N \subset SU(N)$ which does not act on the matter fermions. These models have ordinary (0-form) discrete symmetries also, call \mathbb{Z}_ℓ , which are nonanomalous remnants of anomalous $U(1)$ symmetries.

The second class of models contain some matter fermions in the fundamental or anti-fundamental representation of $SU(N)$. Normally, one would conclude that 1-form center symmetry \mathbb{Z}_N is simply absent in such models. However, as explained in Subsec. 2.2, it turns out that it is still possible to define a color-flavor locked 1-form center symmetry \mathbb{Z}_N .

In Secs. 3 and 4, applications to various chiral gauge theories of these new 't Hooft anomaly constraints are explored. In Sec. 3, physics of various chiral gauge theories of the first kind are discussed, by using the general results of Subsec. 2.1. Sec. 4 is dedicated to the applications of the formulas found in Subsec. 2.2 to two large classes of chiral gauge theories, the so-called generalized Bars–Yankielowicz (BY) and Georgi–Glashow (GG) models.

After discussing the new, generalized anomalies and their implications in various kinds of chiral gauge theories, we explore in Sec. 5 the implications of the strong anomaly on the phases of the same classes of chiral gauge theories, studied in Sec. 2 to Sec. 4.

We conclude in Sec. 6 by summarizing the results found, and by discussing interesting analogies and contrasts between the dynamics of massless QCD and chiral gauge theories. A clearer picture of infrared dynamics of many strongly coupled chiral gauge theories seems to emerge.

Appendices A–H are a collection of tables summarizing the massless fermions and their quantum numbers in various possible phases of BY and GG models.

2. Computation of the Mixed Anomalies

Gauging of a discrete (1-form) center symmetry and calculating anomalies induced by it in some otherwise nonanomalous global discrete symmetry — a generalized 't Hooft anomaly — is the central theme of the work reviewed here. Let us go through the basic elements of the analysis and enlist main formulas needed to get to the physics results discussed in the following sections. For more general introduction and theoretical considerations on the generalized symmetries, the reader can consult the original literature.^{49–69}

We need to distinguish two different classes of models: the first concerns the systems where the fermions do not transform under a \mathbb{Z}_k (k is a divisor of N) subgroup of the $SU(N)$ center. These systems possess a \mathbb{Z}_k 1-form symmetry (the “center symmetry”), which acts naturally on fundamental Wilson loops. Their analysis is relatively straightforward. In the second class of systems, the fundamental

fermions transform nontrivially under the center of the gauge group, \mathbb{Z}_N^c , and only the diagonal combination $\mathbb{Z}_N \subset \mathbb{Z}_N^c \times G_f$ (being G_f the flavor symmetry group) leaves them invariant. The study of these models requires a careful determination of the global structure of the symmetry group involved.

2.1. Gauging a 1-form $\mathbb{Z}_k \subset \mathbb{Z}_N$ center symmetry

First consider $SU(N)$ theories with an exact center $\mathbb{Z}_k \subset \mathbb{Z}_N$ symmetry, k being a divisor of N , under which the matter fermions do not transform. Examples are: pure $SU(N)$ YM theory or the adjoint QCD, where $k = N$, or various models with fermions neutral with respect to some \mathbb{Z}_k , see Sec. 3.

The procedure was formulated in Ref. 50 building upon some earlier results,^{47–49} and used in Ref. 51 for $SU(N)$ Yang–Mills theory at $\theta = \pi$. The methods have been further developed and found in other areas of applications.^{52–66}

1-form center symmetry can be simply understood in the formalism of principal bundles. Here the gauge and the fermions fields are defined locally on open patches U_i of our space–time. These local definitions are glued together by $SU(N)$ -valued transitions functions, $g_{ij} : U_i \cap U_j \rightarrow SU(N)$. In particular,

$$\psi_i(x) = R(g_{ij}(x))\psi_j(x) \quad x \in U_i \cap U_j, \quad (2.1)$$

where ψ_i and ψ_j are the local expressions of the field ψ (which transform in the representation R) in the patches U_i and U_j .

To require that the theory is an $SU(N)$ theory (i.e. the fundamental Wilson loops are meaningful) enforces the cocycle condition

$$g_{ij}g_{jk}g_{ki} = \mathbb{1}, \quad (2.2)$$

in the triple intersection, $U_{ijk} = U_i \cap U_j \cap U_k$.

In this language, a *global 1-form symmetry* transformation multiplies the transition functions g_{ij} by \mathbb{Z}_k elements, z_{ij} (one for each simple intersection, $U_{ij} = U_i \cap U_j$), which satisfy their own consistency condition

$$z_{ij}z_{jk}z_{ki} = 1. \quad (2.3)$$

This transformation is a symmetry of the system if it does not spoil Eq. (2.1), i.e. if \mathbb{Z}_k does not act on fermions. However, it can act nontrivially on fundamental Wilson loops.^a

If one relaxes the cocycle consistency condition, allowing

$$z_{ij}z_{jk}z_{ki} = z_{ijk} \in \mathbb{Z}_N, \quad (2.4)$$

one obtains a *gauge 1-form symmetry* transformation. In this case, the condition (2.2) does not make sense, and must be replaced by

$$g_{ij}g_{jk}g_{ki} = B_{ijk} \in \mathbb{Z}_k, \quad (2.5)$$

^aIt acts on noncontractible Wilson loop, therefore global 1-form gauge transformations are indexed by elements of $H^1(\mathcal{M}, \mathbb{Z}_k)$. The z_{ij} implements a Čech version of this cohomology group.

where the new data B_{ijk} , are (a discretized version of) a 2-form connection.^b This construction defines an $\frac{SU(N)}{\mathbb{Z}_k}$ gauge bundle. If one considers the B_{ijk} data dynamical, summing on them in the functional integral, one obtains a $\frac{SU(N)}{\mathbb{Z}_k}$ gauge theory.

In Refs. 48 and 50, a useful construction is presented that reproduces this gauging in terms of continuous fields. We adopt this description, which is reviewed below briefly.

The rough idea is to replace the discrete \mathbb{Z}_k 1-form symmetry with a $U(1)$ 1-form symmetry, at the price of introducing other new degrees of freedom. Gauge fixing these new degrees of freedom, one can gauge-fix most of the continuous 1-form symmetry. What remains is the discrete 1-form symmetry.

As a first step, one must introduce a pair of $U(1)$ 2-form and 1-form^c gauge fields $(B_c^{(2)}, B_c^{(1)})$ such that⁵⁰

$$kB_c^{(2)} = dB_c^{(1)}, \tag{2.6}$$

satisfying

$$B_c^{(2)} \rightarrow B_c^{(2)} + d\lambda_c, \quad B_c^{(1)} \rightarrow B_c^{(1)} + k\lambda_c. \tag{2.7}$$

λ_c is a 1-form gauge function. The $SU(N)$ gauge field a is embedded into a $U(N)$ field

$$\tilde{a} = a + \frac{1}{k}B_c^{(1)}, \tag{2.8}$$

and one requires invariance under $U(N)$ gauge transformation. The gauge field tensor $F(a)$ is replaced by

$$F(a) \rightarrow \tilde{F}(\tilde{a}) - B_c^{(2)}. \tag{2.9}$$

This fixes the manner these \mathbb{Z}_k^c gauge fields are coupled to the standard gauge fields a . The matter fields must also be coupled to the $U(N)$ gauge fields, so that the 1-form gauge invariance (2.7) is respected. For a Weyl fermion ψ in the representation R this can be done having the kinetic term as

$$\bar{\psi}\gamma^\mu \left(\partial + R(\tilde{a}) - \frac{\mathcal{N}(R)}{k}B_c^{(1)} \right)_\mu P_L\psi, \tag{2.10}$$

where $R(\tilde{a})$ is the appropriate matrix form for the representation; $\mathcal{N}(R)$ is the N -ality of R . P_L is the projection operator on the left-handed fermions.

We introduce an external $U(1)_\psi$ gauge field A_ψ to study the anomaly, e.g. of a $U(1)_\psi$ symmetry $\psi \rightarrow e^{i\alpha}\psi$, or of a discrete subgroup of it, and couple it to the fermion as

$$\bar{\psi}\gamma^\mu \left(\partial + R(\tilde{a}) - \frac{\mathcal{N}(R)}{k}B_c^{(1)} + A_\psi \right)_\mu P_L\psi. \tag{2.11}$$

^bSimilarly, the B_{ijk} are representatives of the second Čeck cohomology group, $H^2(M, \mathbb{Z}_k)$. The closeness of B can be seen on quadruple overlaps.

^cIn most part of this review, a compact differential form notation is used. For instance, $a \equiv T^c A_\mu^c(x) dx^\mu$; $F = da + a^2$; $F^2 \equiv F \wedge F = \frac{1}{2}F^{\mu\nu}F^{\rho\sigma}dx_\mu dx_\nu dx_\rho dx_\sigma = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}d^4x = F^{\mu\nu}\tilde{F}_{\mu\nu}d^4x$, and so on.

The standard anomaly calculation for $\psi \rightarrow e^{i\alpha}\psi \simeq \psi + i\alpha\psi$, gives

$$\delta S_{\delta A_\psi^{(0)}} = \frac{2T(R)}{8\pi^2} \int \text{tr} F^2 \delta\alpha = 2T(R)\mathbb{Z}\delta\alpha. \quad (2.12)$$

\mathbb{Z} is the integer instanton number, and it leads to the well-known result that a discrete subgroup

$$\mathbb{Z}_{2T(R)} \subset U(1)_\psi \quad (2.13)$$

remains. $T(R)$ is twice the Dynkin index

$$\text{tr} T^a T^b = \delta^{ab} D(R), \quad D\left(\square\right) = \frac{1}{2}, \quad T(R) \equiv 2D(R). \quad (2.14)$$

With $(B_c^{(2)}, B_c^{(1)})$ fields in Eq. (2.11), $U(1)_\psi$ symmetry can further be broken due to the replacement

$$\text{tr} F^2 \rightarrow \text{tr}(\tilde{F} - B_c^{(2)})^2. \quad (2.15)$$

Indeed

$$\frac{1}{8\pi^2} \int_{\Sigma_4} \text{tr}(\tilde{F} - B_c^{(2)})^2 = \frac{1}{8\pi^2} \int_{\Sigma_4} \{\text{tr}\tilde{F}^2 - N(B_c^{(2)})^2\}; \quad (2.16)$$

we recall that $B_c^{(2)}$ is Abelian, $\propto \mathbb{1}_N$, and that $\text{tr}\tilde{F} = NB_c^{(2)}$. The first term is an integer.

The second is

$$-\frac{N}{8\pi^2} \int_{\Sigma_4} (B_c^{(2)})^2 = -\frac{N}{8\pi^2 k^2} \int_{\Sigma_4} dB_c^{(1)} \wedge dB_c^{(1)} = \frac{N}{k^2} \mathbb{Z}, \quad (2.17)$$

which is generally fractional. This explains the origin of various 0-form-1-form (mixed) anomalies and the consequent stronger anomaly conditions in many models discussed in Sec. 3.

2.2. Color-flavor locked \mathbb{Z}_N center symmetry: Master formula

Subtler situations present themselves, when a gauge theory of our interest contains matter Weyl fermions in the fundamental, or anti-fundamental, representation of the gauge group $SU(N)$. Ordinarily, this means that the center symmetry is lost, leaving no possibilities of gauging the 1-form \mathbb{Z}_N center symmetry. Actually, in order to consider the 't Hooft anomalies one must externally gauge also the flavor symmetry group, G_f . Having done so, in the systems of our interest $SU(N)_c \times G_f$ is found not to act faithfully. In particular, there is a \mathbb{Z}_N subgroup that leaves all the fields invariant.

In other words, there is a \mathbb{Z}_N “color-flavor-locked” 1-form symmetry.^d Similarly to the previous case, gauging of this 1-form symmetry allows us to gauge the faithful symmetry group of the system, $\frac{SU(N)_c \times G_f}{\mathbb{Z}_N}$.^e

To introduce this kind of systems, and to discuss the method of analysis developed, we consider the concrete example of an $SU(N)$ gauge theory with matter left-handed fermions in the reducible, complex representation,

$$\square \oplus (N + 4) \bar{\square} \tag{2.18}$$

that is

$$\psi^{\{ij\}}, \eta_i^B, \quad i, j = 1, 2, \dots, N, \quad B = 1, 2, \dots, N + 4, \tag{2.19}$$

(which is the simplest of the so-called BY models). This model will be referred to as the “ $\psi\eta$ ” model below.

The symmetry group of the model is

$$G_f = SU(N + 4) \times U(1)_{\psi\eta}, \tag{2.20}$$

where $U(1)_{\psi\eta}$ indicates the anomaly free combination of $U(1)_\psi$ and $U(1)_\eta$, associated with the two types of matter Weyl fermions of the theory. In this model, the conventional ’t Hooft anomaly matching discussion allows, apparently, a confining phase, with no condensates and with full unbroken global symmetry, and with some simple set of massless composite fermions — “baryons” — saturating the anomaly matching equations, see App. A. Notably, the anomaly constraints are *also* consistent with a dynamical Higgs phase, in which the color and (part of) the flavor symmetry are dynamically broken by certain bi-fermion condensates, see App. B.

The situation is analogous in a large class of chiral gauge theories, to be discussed in Sec. 4. Clearly, the conventional ’t Hooft anomaly matching requirement is not powerful enough to discriminate among possible (confining or dynamical Higgs) vacua.

To go beyond the conventional (perturbative) ’t Hooft anomaly analyses, it is necessary to consider the global properties of the symmetry groups, not only the algebra. For even N , the true symmetry group of the model is found to be⁶⁸

$$SU(N)_{\text{color}} \times G_f, \quad G_f = \frac{SU(N + 4) \times U(1)_{\psi\eta} \times (\mathbb{Z}_2)_F}{\mathbb{Z}_N \times \mathbb{Z}_{N+4}}, \tag{2.21}$$

and not (2.20), where $(\mathbb{Z}_2)_F$ is the fermion parity, $\psi, \eta \rightarrow -\psi, -\eta$.

^dThis hinges upon a quite remarkable property of the generalized symmetries, that they are all Abelian. This reflects the fact it is not possible to define time ordering between two extended operators, hence impossible to define equal-time commutators between them. In the case of particles, how the (equal-time) commutators can arise in the operator formalism, as a limit of time-ordered products taken in different orders, is best explained in Feynman’s book on Path-Integral formulation of quantum mechanics.⁷⁰

^eOne should keep in mind that there are “more” configurations in $\frac{SU(N)}{\mathbb{Z}_N}$ ($\frac{SU(N)_c \times G_f}{\mathbb{Z}_N}$) than in the $SU(N)$ ($SU(N) \times G_f$) gauge bundles, i.e. any of the latter always belong to the former, but not the other way around.

Indeed, as promised, there is a subgroup of $SU(N)_c \times SU(N+4) \times U(1)_{\psi\eta} \times (\mathbb{Z}_2)_F$,

$$\mathbb{Z}_N = SU(N) \cap \{U(1)_{\psi\eta} \times (\mathbb{Z}_2)_F\}, \quad (2.22)$$

which leaves the matter fields invariant.^f The gauge transformation with $e^{\frac{2\pi i}{N}} \in \mathbb{Z}_N \subset SU(N)$

$$\psi \rightarrow e^{\frac{4\pi i}{N}}\psi, \quad \eta \rightarrow e^{-\frac{2\pi i}{N}}\eta, \quad (2.23)$$

can be undone by the following $(\mathbb{Z}_2)_F \times U(1)_{\psi\eta}$ transformation:

$$\psi \rightarrow (-1)e^{i\frac{N+42\pi}{2N}}\psi = e^{-i\frac{N+2\pi}{2N}}e^{i\frac{N+42\pi}{2N}}\psi, \quad \eta \rightarrow (-1)e^{-i\frac{N+22\pi}{2N}}\eta = e^{i\frac{N+2\pi}{2N}}e^{-i\frac{N+22\pi}{2N}}\eta. \quad (2.24)$$

A relevant fact is that the odd elements of \mathbb{Z}_N belong to the disconnected component of $U(1)_{\psi\eta} \times (\mathbb{Z}_2)_F$ whereas the even elements belong to the connected component of the identity.

The presence of a subgroup which acts trivially means that there is a 1-form global symmetry. Again, in the discrete language introduced before, it acts on transition functions. In particular, if g_{ij}^c , u_{ij} and q_{ij} are the transition functions for $SU(N)$, $U(1)_{\psi\eta}$ and $(\mathbb{Z}_2)_F$, one may introduce some \mathbb{Z}_N transitions functions (a \mathbb{Z}_N gauge field), z_{ij} , and transform

$$g_{ij} \rightarrow z_{ij}g_{ij}, \quad u_{ij} \rightarrow (z_{ij})^{-1}u_{ij}, \quad \text{and} \quad q_{ij} \rightarrow (z_{ij})^{-\frac{N}{2}}q_{ij}. \quad (2.25)$$

If one drops the cocycle condition for z_{ij} , one gauges the 1-form symmetry. In this case, one must introduce also the 2-form connection,^g described by the new data $B_{ijk} \in \mathbb{Z}_N$, which are read from the transition functions

$$g_{ij}g_{jk}g_{ki} = B_{ijk}, \quad u_{ij}u_{jk}u_{ki} = (B_{ijk})^{-1}, \quad q_{ij}q_{ji}q_{ki} = (B_{ijk})^{-\frac{N}{2}}. \quad (2.26)$$

This definition assures that all fields (matter, gauge) are well defined in the triple intersections.

Again, let us turn to the continuous language. As a first step, we have to gauge $U(1)_{\psi\eta} \times (\mathbb{Z}_2)_F$, introducing

- (1) A : $U(1)_{\psi\eta}$ 1-form gauge field,
- (2) $A_2^{(1)}$: $(\mathbb{Z}_2)_F$ 1-form gauge field,

in addition to the dynamical color gauge $SU(N)$ field, a .

^fThere is another independent subgroup, \mathbb{Z}_{N+4} , which does not act on matter fields, leading to another \mathbb{Z}_{N+4} 1-form center symmetry. In Ref. 68, the effects of gauging this flavor center symmetry and the resulting mixed anomalies in the $\psi\eta$ model have also been taken into account. None of the main results however were found to depend on it. Here for simplicity, we consider only the gauging of the color-flavor locked center symmetry \mathbb{Z}_N , together with $U(1)_{\psi\eta}$ and $(\mathbb{Z}_2)_F$.

^gAgain, an element of $H^2(M, \mathbb{Z}_N)$, $B_{ijk} \in \mathbb{Z}_N$.

The gauging of 1-form discrete \mathbb{Z}_N symmetry is done by introducing

$$NB_c^{(2)} = dB_c^{(1)}. \tag{2.27}$$

These 2-form gauge fields must be coupled to the $SU(N)$ gauge fields a and $U(1)_{\psi\eta} \times (\mathbb{Z}_2)_F$ gauge fields $(A$ and $A_2^{(1)})$ appropriately. We first embed a in a $U(N)$ gauge field \tilde{a} as

$$\tilde{a} = a + \frac{1}{N} B_c^{(1)} \tag{2.28}$$

and requiring the invariance

$$B_c^{(2)} \rightarrow B_c^{(2)} + d\lambda_c, \quad B_c^{(1)} \rightarrow B_c^{(1)} + N\lambda_c, \quad \tilde{a} \rightarrow \tilde{a} + \lambda_c. \tag{2.29}$$

In these equations, λ_c is a properly normalized $U(1)$ gauge field, which satisfies its own Dirac quantization condition.

To reproduce (2.25) correctly in this continuous language, $U(1)_{\psi\eta}$ and $(\mathbb{Z}_2)_F$ gauge fields must also transform

$$A \rightarrow A - \lambda_c, \quad A_2^{(1)} \rightarrow A_2^{(1)} + \frac{N}{2} \lambda_c. \tag{2.30}$$

The last equation needs a comment, as $A_2^{(1)}$ is a \mathbb{Z}_2 gauge field, while λ_c is a $U(1)$ gauge field. To be precise, it is more correct to proceed as it has been done with the $SU(N)$ gauge field. In particular, one should write a $U(1)$ gauge connection

$$\tilde{A}_2 = A_2 + \frac{1}{2} B_c^{(1)}, \tag{2.31}$$

and impose

$$\tilde{A}_2 \rightarrow \tilde{A}_2 + \frac{N}{2} \lambda_c. \tag{2.32}$$

As before a is not a globally defined $SU(N)$ gauge field while \tilde{a} is a correctly normalized $U(N)$ gauge field, and now \tilde{A}_2 is a correctly normalized $U(1)$ field.

One has now an $\frac{SU(N)}{\mathbb{Z}_N}$ connection rather than $SU(N)$. It implies that

$$\frac{1}{2\pi} \int_{\Sigma_2} B_c^{(2)} = \frac{n_1}{N}, \quad n_1 \in \mathbb{Z}_N, \tag{2.33}$$

in a closed two-dimensional subspace, Σ_2 . On a topologically nontrivial four-dimensional space–time of Euclidean signature which contains such sub-spaces one has

$$\frac{1}{8\pi^2} \int_{\Sigma_4} (B_c^{(2)})^2 = \frac{n}{N^2}, \tag{2.34}$$

where $n \in \mathbb{Z}_N$.

The fermion kinetic term with the background gauge fields is determined by the minimal coupling procedure as

$$\begin{aligned} & \bar{\psi}\gamma^\mu \left(\partial + \mathcal{R}_S(\tilde{a}) + \frac{N+4}{2}A + \tilde{A}_2 \right)_\mu P_L \psi \\ & + \bar{\eta}\gamma^\mu \left(\partial + \mathcal{R}_{F^*}(\tilde{a}) - \frac{N+2}{2}A - \tilde{A}_2 \right)_\mu P_L \eta. \end{aligned} \quad (2.35)$$

(With an obvious notation). Note that each of the kinetic terms is invariant under (2.29) and (2.30).

The 1-form gauge invariance of our system can be made completely manifest, by rewriting the above as

$$\begin{aligned} & \bar{\psi}\gamma^\mu \left(\partial + \left[\mathcal{R}_S(\tilde{a}) - \frac{2}{N}B_c^{(1)} \right] + \frac{N+4}{2} \left[A + \frac{1}{N}B_c^{(1)} \right] + \left[\tilde{A}_2 - \frac{1}{2}B_c^{(1)} \right] \right)_\mu P_L \psi \\ & + \bar{\eta}\gamma^\mu \left(\partial + \left[\mathcal{R}_{F^*}(\tilde{a}) + \frac{1}{N}B_c^{(1)} \right] - \frac{N+2}{2} \left[A + \frac{1}{N}B_c^{(1)} \right] \right. \\ & \left. - \left[\tilde{A}_2 - \frac{1}{2}B_c^{(1)} \right] \right)_\mu P_L \eta. \end{aligned} \quad (2.36)$$

Written this way, the expression inside each square bracket is invariant under (2.29) and (2.30). This leads to the gauge field strength for the ψ and η , in the form used in the analysis à la Stora–Zumino descent procedure,^{71,72} in Refs. 68 and 69. The final answer of course does not depend on the rewriting of the kinetic terms as (2.36); the original form (2.35) is perfectly adequate for the calculation of the anomaly in a more straightforward approach explained below.

Under the fermion parity, such as $\psi \rightarrow -\psi$, $\eta \rightarrow -\eta$ in the $\psi\eta$ model, the contribution to the $(\mathbb{Z}_2)_F - [\mathbb{Z}_N]^2$ anomaly from a fermion in the representation R is given by the phase in the partition function

$$c_2 \frac{1}{8\pi^2} \int_{\Sigma_4} \text{tr}_R[(F(\tilde{a}))^2](\pm\pi), \quad (2.37)$$

where c_2 is the \mathbb{Z}_2 charge of the fermion. In the case of the $\psi\eta$ model, for instance, $c_2(\psi) = 1$, $c_2(\eta) = -1$, see (2.35).

Now

$$\begin{aligned} \text{tr}_R[(F(\tilde{a}))^2] &= \text{tr}_R[(F(\tilde{a}) - B_c^{(2)} + B_c^{(2)})^2] \\ &= \text{tr}[(\mathcal{R}_R(F(\tilde{a}) - B_c^{(2)}) + \mathcal{N}(R)B_c^{(2)}\mathbf{1}_{d(R)})^2] \\ &= \text{tr}[\mathcal{R}_R(F(\tilde{a}) - B_c^{(2)})^2 + \mathcal{N}(R)^2(B_c^{(2)})^2\mathbf{1}_{d(R)}]. \end{aligned} \quad (2.38)$$

\mathcal{R}_R is the matrix form for the representation R and $\mathcal{N}(R)$ its N -ality, and we used the fact that

$$\text{tr}_R(F(\tilde{a}) - B_c^{(2)}) = 0, \quad (2.39)$$

valid for an $SU(N)$ element in a general representation. $\mathbb{1}_{d(R)}$ is the $d(R) \times d(R)$ unit matrix ($d(R)$ is the dimension of the representation R). One finds

$$\begin{aligned} \text{tr}_R[(F(\tilde{a}))^2] &= D(R)\text{tr}_F[(F(\tilde{a}) - B_c^{(2)})^2] + d(R)\mathcal{N}(R)^2(B_c^{(2)})^2 \\ &= D(R)\text{tr}_F[F(\tilde{a})]^2 + [-D(R) \cdot N + d(R)\mathcal{N}(R)^2](B_c^{(2)})^2, \end{aligned} \quad (2.40)$$

where $D(R)$ is twice the Dynkin index T_R , (2.14). Note that

$$\frac{1}{8\pi^2} \int_{\Sigma_4} \text{tr}_F[F(\tilde{a})^2] \in \mathbb{Z} \quad (2.41)$$

the first term in Eq. (2.40) corresponds to the conventional instanton contribution to the $(\mathbb{Z}_2)_F$ anomaly. In all models of interest here, however, the *sum* of the instanton contribution from the fermions is of the form

$$(\text{an even integer}) \times \frac{1}{8\pi^2} \int_{\Sigma_4} \text{tr}_F[F(\tilde{a})^2] \times (\pm\pi) = 2\pi\mathbb{Z}, \quad (2.42)$$

which is trivial.

The fact that $(\mathbb{Z}_2)_F$ anomaly is absent in the standard instanton analysis because of a (nonvanishing) *even* coefficient, and of the quantized instanton flux, but *not* because of an algebraic cancellation from different fermions, *is of utmost importance*. Indeed, the gauging of the 1-form \mathbb{Z}_N by the introduction of the 2-form gauge fields $B_c^{(2)}$ basically amounts to the fractionalization of the instanton flux^{73–75} à la 't Hooft (2.34), and as a consequence, a nonvanishing mixed anomaly involving $(\mathbb{Z}_2)_F$ can appear.

Thus, the non-vanishing mixed $(\mathbb{Z}_2)_F - [\mathbb{Z}_N]^2$ anomaly comes only from the second term of Eq. (2.40), containing the 2-form gauge field

$$\Delta S^{(\text{Mixed anomaly})} = (\pm\pi) \cdot \sum_{\text{fermions}} c_2(d(R)\mathcal{N}(R)^2 - N \cdot D(R)) \frac{1}{8\pi^2} \int_{\Sigma_4} (B_c^{(2)})^2. \quad (2.43)$$

This is the master formula.

The formula (2.43) gives the result for the mixed anomaly in *all* models considered in Refs. 68 and 69 at once. For instance, for the $\psi\eta$ model, one gets

$$\begin{aligned} \Delta S^{(\text{Mixed anomaly})} &= \frac{\pm\pi}{N^2} \left[\left(\frac{N(N+1)}{2} \cdot 4 - N(N+2) \right) - (N+4)(N \cdot 1 - N \cdot 1) \right] \\ &= \pm\pi, \end{aligned} \quad (2.44)$$

which means that there is a $(\mathbb{Z}_2)_F$ anomaly in the presence of the \mathbb{Z}_N gauging. More precisely, there is an obstruction for gauging simultaneously \mathbb{Z}_N , $U(1)_{\psi\eta}$ and $(\mathbb{Z}_2)_F$, by keeping the equivalence

$$\mathbb{Z}_N \subset SU(N) \sim \mathbb{Z}_N \subset \{U(1)_{\psi\eta} \times (\mathbb{Z}_2)_F\}. \quad (2.45)$$

One can repeat the same calculation in the possible confining phase without symmetry breaking, App. A. The result is very simple: there is no such anomaly.

This can be traced back to the fact that the massless baryons are $SU(N)_c$ singlet, and any appearance of $B_c^{(1)}$ in their covariant derivative simply cancel. Clearly this mismatch of anomaly *forbids confinement without symmetry breaking*.

The same cannot be said in the dynamical Higgs scenario, see App. B, as the color group is broken.

Even though, for concreteness, we discussed above the particularly simple model, the $\psi\eta$ model, the master formula found above is actually applicable to any theory, after the correct symmetry is found and after the fermion kinetic terms, invariant under the 1-form \mathbb{Z}_N gauge symmetry are written down. The results for the $(\mathbb{Z}_2)_F - [\mathbb{Z}_N]^2$ anomaly in the $\chi\eta$ model as well as all other generalized BY and GG models^{68,69} discussed in Sec. 4 indeed follow straightforwardly from the master formula (2.43) this way. See Sec. 4.

2.3. Comments on the paper⁷⁶

In a recent paper,⁷⁶ the $\psi\eta$ model and $\chi\eta$ model (in our notation) are studied, and the authors claim that “there is no $(\mathbb{Z}_2)_F$ anomaly”, of the type discussed in the previous section. Such a statement is however intrinsically ambiguous. It is unclear whether the authors’ claim is that there is no $(\mathbb{Z}_2)_F$ symmetry, or that there is one but is nonanomalous.

Indeed, their argument in their Sec. 2.1 seems to indicate the former; but as we have made explicit here in (2.21),⁶⁹ an independent $(\mathbb{Z}_2)_F$ symmetry exists, but only in the $SU(N)/\mathbb{Z}_N$ gauge theory, not in the original $SU(N)$ theory. Their comments in Subsec. 2.5 also seem to be in line with the first. But the fact that the $(\mathbb{Z}_2)_F$ symmetry we are interested in here coincides with the angle 2π space rotation, is well known, and it has been taken into account in our papers.^{68,69} Any $(\mathbb{Z}_2)_F$ anomaly could be canceled by a space rotation, so in that sense, there would never be a $(\mathbb{Z}_2)_F$ anomaly. But this is not the point. As there is no *a priori* guarantee that the Lorentz invariance cannot be dynamically broken, a $(\mathbb{Z}_2)_F$ anomaly arising from the gauge dynamics cannot be allowed, if the Lorentz invariance is to be maintained.

Their discussion in Sec. 3 about the Higgs phase in these models does not contain anything new, as compared to what we have discussed about the Higgs phase, see Ref. 69, and Subsec. 2.5 and in Apps. B, D, F, H, in this work, for the $\chi\eta$, $\psi\eta$ models and for all other BY and GG models.

The main point of Ref. 76 seems to be in Subsec. 2.2, which apparently leads to the second conclusion, that there is a $(\mathbb{Z}_2)_F$ symmetry but is nonanomalous. They argue that by choosing the normalization of the \mathbb{Z}_2 gauge field as

$$\int_{\Sigma} dA_2^{(1)} = 2\pi\mathbb{Z}, \tag{2.46}$$

(a formula in the line below Eq. (2.13) of Ref. 76), the relation

$$2A_2^{(1)} - B_c^{(1)} - B_f^{(1)} = dA_2^{(0)}, \tag{2.47}$$

leads to

$$2dA_2^{(1)} - NB_c^{(2)} - (N + 4)B_f^{(2)} = 0: \tag{2.48}$$

by taking the derivatives of both sides, hence to the constraints on the fluxes of \mathbb{Z}_2 and \mathbb{Z}_{N+4} gauge fields

$$\int_{\Sigma_2} NB_c^{(2)} + \int_{\Sigma_2} (N + 4)B_f^{(2)} = 4\pi k, \quad k \in \mathbb{Z}. \tag{2.49}$$

If one chooses not to introduce the 1-form gauging $B_f^{(2)}$ (as we did in Ref. 69) one would simply get

$$\int_{\Sigma_2} NB_c^{(2)} = 4\pi k, \quad k \in \mathbb{Z}, \tag{2.50}$$

and our anomaly (2.43) would indeed disappear. The rest of Sec. 2 in Ref. 76 all follows from the normalization (2.46).

However, (2.46) means that their background \mathbb{Z}_2 gauge field corresponds to

$$\psi \rightarrow \psi, \quad \eta \rightarrow \eta, \tag{2.51}$$

i.e. no transformation (the trivial element of \mathbb{Z}_2). The fact that one finds no anomaly in such a background is certainly correct, but it is not what one is interested in.

The correct normalization for a \mathbb{Z}_2 gauge field is the one we have adopted

$$\oint A_2^{(1)} = \frac{2\pi m}{2}, \quad m \in \mathbb{Z}, \tag{2.52}$$

that (for the nontrivial element) corresponds to the holonomy

$$\psi \rightarrow -\psi, \quad \eta \rightarrow -\eta. \tag{2.53}$$

This leads to (we ignore the \mathbb{Z}_{N+4} gauge field)

$$\oint dx^\mu (2A_2^{(1)} - B_c^{(1)})_\mu = \oint dA_2^{(0)} = 2\pi n, \quad n \in \mathbb{Z}, \tag{2.54}$$

and

$$\int_{\Sigma_2} NB_c^{(2)} = 2\pi k, \quad k \in \mathbb{Z}, \tag{2.55}$$

and this leads to the anomaly, (2.43).

In other words, our assumption is that it is possible to choose the smallest cycle of $B^{(2)}$ compatible with the Dirac flux quantization for $B^{(1)}$, i.e.

$$\int_{\Sigma} B^{(2)} = \frac{1}{N} \int_{\Sigma} dB^{(1)} = \frac{2\pi}{N}, \tag{2.56}$$

without any topological obstruction. In the discrete language, the analogous assumption is to be able to choose B_{ijk} to be any element in $H^2(\mathcal{M}, \mathbb{Z}_N)$.

Actually, if one insists on working with the theory on a smooth manifold, without any topological defect for the \mathbb{Z}_2 gauge field A_2 , the assumption made above cannot be maintained, as pointed out by ourselves.⁶⁸ This seems to be the point on which the authors of Ref. 76 are trying to make a clean mathematical statement.

However, A_2 is not a proper $(\mathbb{Z}_2)_F$ gauge field, as a is not an $SU(N)$ gauge field. In particular, its cocycle condition in triple overlap might fail, leading to curvature-like insertions at discrete points. Moreover, the 1-form gauging of \mathbb{Z}_N invalidates the naive Dirac quantization condition for A_2 , as it can be checked directly: if one separates a 2D cycle Σ through the curve γ in $\Sigma_1 \cup \Sigma_2$, one obtains

$$\begin{aligned} \int_{\Sigma} dA_2 &= \int_{\Sigma_1} dA_2 + \int_{\Sigma_2} dA_2 = \int_{\gamma} (A_2)_{\Sigma_1} - \int_{\gamma} (A_2)_{\Sigma_2} \\ &= \frac{N}{2} \int_{\gamma} \lambda_{12} = k\pi, \quad k \in \mathbb{Z} \end{aligned} \tag{2.57}$$

(see (2.30)) as λ_{12} (1-form gauge transition functions)^h is a \mathbb{Z}_N 1-form gauge field, satisfying

$$\oint_{\gamma} \lambda = \frac{2\pi}{N}. \tag{2.58}$$

This leads to more general flux quantization (2.56), and allows to insert an odd number of flux insertion in the surface.

To recapitulate, in half of Ref. 76 the authors argue that there is no $(\mathbb{Z}_2)_F$ symmetry; in the other half, they discuss the background $(\mathbb{Z}_2)_F - \mathbb{Z}_N$ 1-form gauge fields, corresponding however to the trivial element of the 1-form $(\mathbb{Z}_2)_F$ transformation, finding no anomaly.

2.4. Comments on the papers^{77,78}

Two interesting papers appeared recently, which discuss the $\psi\eta$ and $\chi\eta$ models. The authors of Refs. 77 and 78 start from the $\mathcal{N} = 1$ supersymmetric version of the models, and introduce a particular (“anomaly-mediated”) supersymmetry breaking perturbation. In the second paper (on the $\psi\eta$ model) this is done by making use of the known (Seiberg-) duality for these systems, at the origin of the moduli space of this model. In the $\psi\eta$ model, with $SU(N)$ gauge group and with a global symmetry group

$$G_{\text{f}} = SU(N + 4) \times U(1)_{\psi\eta}, \tag{2.59}$$

the authors claim⁷⁸ that for $N \geq 21$ the global symmetry is broken to $SO(N)$, with no massless composite fermions, whereas for $N < 21$ the system flows into a conformal fixed point in the IR.

^hThe simple fact that the \mathbb{Z}_2 gauge field transforms nontrivially and changes from a patch to another, means that gauging of 1-form symmetry has been appropriately implemented. Indeed, thanks to (2.57), a Wilson loop of the form $e^{\oint A_2}$ is not 1-form gauge invariant, i.e. it is not a proper line operator. This is where we differ from part of the analysis of Ref. 76.

For the $\chi\eta$ model, with odd N , they argue⁷⁷ that the global symmetry $SU(N - 4)$ is spontaneously broken to $Sp(N - 5)$. For N even the unbroken symmetry is claimed to be $Sp(N - 4)$.

We shall not go into the details and merits of their analyses, but will make only a few general comments on their use of the supersymmetric models as the starting point of the analysis. First of all, in the supersymmetric version of these models, there are often nontrivial quantum moduli space of vacua (vacuum degeneracies, or flat directions), whereas in the nonsupersymmetric chiral models we are studying here the vacuum is always unique and strongly coupled. It is a nontrivial question which point in the moduli space of the supersymmetric theory (apart from which perturbation to use) is the correct one to choose, to start the analysis.ⁱ

Second, all bifermion condensates such as $\langle\psi\eta\rangle$ (in the $\psi\eta$ model) and $\langle\chi\eta\rangle$ and $\langle\chi\chi\rangle$ (in the $\chi\eta$ model), which are analogue of the quark condensate in QCD, and play the central roles in the (candidate) Higgs vacua of these models, are forbidden in supersymmetric version of the models, as can be easily proven by use of supersymmetric Ward–Takahashi identities.⁸⁰ In other words, these condensates are absent, unless supersymmetry is dynamically broken, which does not occur in general, supersymmetric chiral gauge theories.⁸⁰ Also, in supersymmetric models, the global symmetry breaking occurs due to the condensation of scalar fields, which do not exist in nonsupersymmetric theories. Because of all this, the infrared dynamics of supersymmetric and nonsupersymmetric theories are usually very different, even though the gauge group and the global symmetry group are the same. A strong bifermion condensates such as $\langle\psi\eta\rangle \sim \Lambda^3$ or $\langle\chi\eta\rangle \sim \Lambda^3$ are intrinsically nonperturbative effects. They cannot be found via small perturbations in a theory in which they vanish by symmetries.

The crucial question whether or not a phase transition occurs when the supersymmetry breaking mass parameters introduced reach some critical values, seems to be unanswered in Refs. 77 and 78.

2.5. Higgs phase and anomaly matching

As said above, it is possible to satisfy the standard 't Hooft anomaly matching also in the Higgs phase, see App. B. Even though these results are known from the earlier work,^{5,18} and in Ref. 69 the remarkable way it works, as compared to the matching equations in the “confining vacua”, is perhaps not generally known. The Higgs phase of these chiral theories is, in general, described by massless NG bosons together with some massless fermions. These fermions saturate the conventional 't Hooft anomaly triangles with respect to the unbroken flavor symmetries. The way they do is, however, quite remarkable, and in our view, truly significant. As can be seen from

ⁱThis subtle problem is discussed in Ref. 79 in a slightly different but basically similar context, of perturbing a $\mathcal{N} = 2$ supersymmetric model to $\mathcal{N} = 0$ (nonsupersymmetric) model, in the attempt of finding out the correct infrared dynamics of the nonsupersymmetric $SU(2)$ theories with different number of (adjoint) flavors.

Tables B.1 and D.1, and in similar Tables F.1, F.2, H.1 and H.2 for the generalized BY and GG models (in Apps. B, D, F, H), the set of fermions remaining massless in UV and those in the IR are *identical* in their quantum numbers, charges, and multiplicities. Therefore, the matching of anomalies (in the unbroken global symmetries) is completely automatic, and natural. No arithmetic equations need to be solved. We may further argue that this way the system “solves” ’t Hooft’s anomaly-matching conditions in the true sense. Note that this solution (Higgs phase vacua, with given sets of condensates) is stable, in the sense that any extra (1-form) gauging or possible new mixed anomalies would not introduce any new constraints: the matching continues to be automatic.

3. Physics of Models with $\mathbb{Z}_k \subset \mathbb{Z}_N$ Center Symmetry

In this section, we review the study of symmetry breaking, implied by the various mixed anomalies of the type, $\mathbb{Z}_\ell^{(0)} - [\mathbb{Z}_k^{(1)}]^2$, where $\mathbb{Z}_\ell^{(0)}$ is some 0-form (ordinary) discrete symmetry, and $\mathbb{Z}_k^{(1)}$ is a 1-form symmetry, based on a subgroup, $\mathbb{Z}_k \subset \mathbb{Z}_N$ of the color $SU(N)$ center. The method of analysis has already been explained in Subsec. 2.1. \mathbb{Z}_ℓ and \mathbb{Z}_k depend on the particular model considered, but as we will see, many interesting models can be analyzed this way.⁶⁷

3.1. Self-adjoint anti-symmetric tensor matter

We start with a class of $SU(N)$ gauge theories (N even) where the matter fermions are in the $\frac{N}{2}$ rank fully anti-symmetric representation. There is a 1-form $\mathbb{Z}_{\frac{N}{2}}^c$ center symmetry present, and we wish to know if some mixed ’t Hooft anomaly with the 0-form (ordinary) symmetries might arise.

3.1.1. $SU(6)$ gauge group

Our first example is an $N = 6$ theory with N_f flavors of Weyl fermions in the representation

$$\underline{20} = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}. \tag{3.1}$$

This ($SU(6)$) is the simplest nontrivial case of interest, as we will see. Moreover, if $N_f \leq 10$ the theory is asymptotic free, and discussions are basically similar to those below can be worked out.⁶⁷

There is a $U(1)_\psi$ global symmetry, in all these cases, broken by the instantons to a global discrete $\mathbb{Z}_{6N_f}^\psi$ symmetry, which is further broken by the 1-form gauging to $\mathbb{Z}_{2N_f}^\psi$. This last step is due to a mixed ’t Hooft anomaly, that is, there is an obstruction to gauge such a $\mathbb{Z}_{\frac{N}{2}}^c$ discrete center symmetry, together with the global $\mathbb{Z}_{6N_f}^\psi$ symmetry.

Below, we will take $N_f = 1$. The model was studied first in Ref. 58 and then in Ref. 67.

This model has a nonanomalous \mathbb{Z}_6^ψ symmetry,

$$\mathbb{Z}_6^\psi : \psi \rightarrow e^{\frac{2\pi i}{6}j} \psi, \quad j = 1, 2, \dots, 6. \tag{3.2}$$

Which is a subset of $U(1)_\psi$. The system has also an exact center symmetry acting Wilson loops as

$$\mathbb{Z}_3^c : e^i \oint^A \rightarrow e^{\frac{2\pi i}{3}k} e^i \oint^A, \quad k = 2, 4, 6, \tag{3.3}$$

which does not act on ψ .

By introducing the \mathbb{Z}_3^c gauge fields

$$3B_c^{(2)} = dB_c^{(1)}, \tag{3.4}$$

use of (2.15)–(2.17) gives for the anomalous \mathbb{Z}_6^ψ transformation

$$\left(\frac{6}{8\pi^2} \int \text{tr} \tilde{F}^2 - \frac{6N}{8\pi^2} \int (B_c^{(2)})^2 \right) \delta A_\psi^{(0)}, \quad \delta A_\psi^{(0)} = \frac{2\pi \mathbb{Z}_6^\psi}{6}. \tag{3.5}$$

The first term in (3.5) is trivial, as

$$\frac{1}{8\pi^2} \int \text{tr} \tilde{F}^2 \in \mathbb{Z}, \quad A_\psi = dA_\psi^{(0)}, \tag{3.6}$$

(which is the standard gauge anomaly, reducing $U(1)_\psi \rightarrow \mathbb{Z}_6^\psi$).

Due to the second term in (3.5), $\delta A_\psi^{(0)}$ gets now multiplied by ($N = 6$ here)

$$-\frac{6N}{8\pi^2} \int (B_c^{(2)})^2 = -6N \left(\frac{1}{3} \right)^2 \mathbb{Z} = -6 \frac{2}{3} \mathbb{Z}. \tag{3.7}$$

We see the reduction of the global chiral \mathbb{Z}_6^ψ symmetry

$$\delta A_\psi^{(0)} = \frac{2\pi \ell}{6}, \quad \ell = 1, 2, \dots, 6 \tag{3.8}$$

to its subgroup ($\ell = 3, 6$),

$$\mathbb{Z}_6^\psi \rightarrow \mathbb{Z}_2^\psi. \tag{3.9}$$

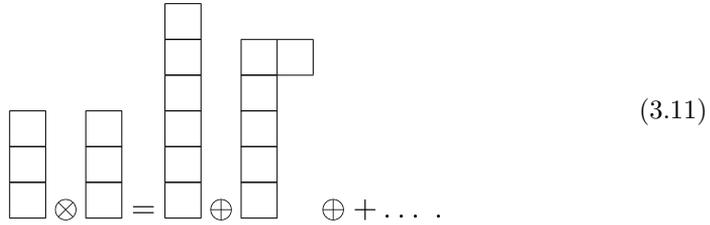
Thus, it is not possible that the vacuum of this system is confining, has a mass gap, and with no condensates breaking the \mathbb{Z}_6^ψ symmetry.

What are the implications of (3.9) on the phase of the theory? First of all, it implies a three-fold vacuum degeneracy, under the assumption that the system confines (with mass gap) and that no massless fermions are present in the infrared, on which $\frac{\mathbb{Z}_6^\psi}{\mathbb{Z}_2^\psi}$ can act. A natural assumption is that there are some condensates which “explain” such a reduction of the symmetry in the infrared. However, physics

depends on which condensates form. The simplest assumption is that a bi-fermion condensate

$$\langle \psi\psi \rangle \sim \Lambda^3 \neq 0 \tag{3.10}$$

forms. As $\psi \in \underline{20}$ a scalar bi-fermion composite may be in one of the irreducible representations of $SU(6)$, appearing on the right-hand side of the composition-decomposition



$$\tag{3.11}$$

The most attractive channel is the first, $\underline{1}$, but it vanishes by the Fermi statistics. This leaves us with the second best possibility that $\psi\psi$ in the adjoint representation acquires a VEV (dynamical Higgs mechanism).^{3,17,18}

Note that even though such a condensate should necessarily be understood as a gauge-dependent form of some gauge invariant VEV, it *unambiguously* determines the breaking of global discrete chiral symmetry as (3.9), where the broken symmetry $\frac{\mathbb{Z}_6^\psi}{\mathbb{Z}_2^\psi}$ acts on the degenerate vacua, permuting them. The reason for this is that as the global symmetry group \mathbb{Z}_6^ψ commutes with the color $SU(6)$ a gauge transformation cannot undo the nontrivial transformation of the condensate under \mathbb{Z}_6^ψ .

Of course, four-fermion, gauge-invariant condensates such as

$$\langle \psi\psi\psi\psi \rangle \neq 0, \quad \text{or} \quad \langle \bar{\psi}\bar{\psi}\psi\psi \rangle \neq 0, \tag{3.12}$$

could also form, first of which also breaks \mathbb{Z}_6^ψ as in (3.9).^j

The bi-fermion $\psi\psi$ condensate being in the adjoint representation, it is possible that physics in the infrared is described by full dynamical Abelianization.^{17,18} The low-energy theory could be an Abelian $U(1)^5$ theory. In such a case, although the infrared theory may look trivial, there is a remnant of the \mathbb{Z}_6 symmetry of the UV theory. Domain walls which connect the three vacua would exist, and nontrivial infrared 3D physics can appear there.

$SU(6)$ models with $N_f \geq 2$ have been studied also. Physics implications from the mixed anomaly turn out to depend quite nontrivially on the value of N_f .⁶⁷

3.1.2. $SU(N)$ models

We next consider $SU(N)$ (N general, even) theory, with left-handed fermions ψ in the self-adjoint, totally anti-symmetric representation. It exhibits some interesting features of the generalized anomalies.

^jWe however do not share the view expressed in Ref. 58 that the gauge noninvariance of the bi-fermion composite $\psi\psi$ means $\langle \psi\psi \rangle = 0; \langle \psi\psi\psi\psi \rangle \neq 0$.

The first coefficient of the beta function is

$$b_0 = \frac{11N - 2N_f T_R}{3}. \tag{3.13}$$

The twice Dynkin index is given by

$$2T_R = \left(\frac{N - 2}{2} \right). \tag{3.14}$$

See Table in (3.15) for $2T_R$ and $d(R)$ for some even values of N .

N	4	6	8	10	12
$2T_R$	2	6	20	70	252
$d(R)$	6	20	70	252	924

(3.15)

We limit ourselves to asymptotically free theories ($N \leq 10$).

The system has an exact 1-form symmetry

$$\mathbb{Z}_{\frac{N}{2}}^c : e^i \mathcal{F}^A \rightarrow e^{\frac{2\pi i k}{N}} e^i \mathcal{F}^A, \quad k = 2, 4, \dots, N, \tag{3.16}$$

as well as a global discrete symmetry

$$\mathbb{Z}_{2T_R} : \psi \rightarrow e^{\frac{2\pi i j}{2T_R}} \psi, \quad j = 1, 2, \dots, 2T_R. \tag{3.17}$$

By introducing a 1-form gauge fields $(B_c^{(2)}, B_c^{(1)})$

$$\frac{N}{2} B_c^{(2)} = dB_c^{(1)} \tag{3.18}$$

(see Sec. 3), one arrives at the conclusion that the phase of the partition function is transformed by

$$-\frac{2\pi k}{2T_R} 2T_R \frac{4}{N} \mathbb{Z} = -2\pi k \frac{4}{N} \mathbb{Z}, \quad k = 1, 2, \dots, 2T_R, \tag{3.19}$$

under \mathbb{Z}_{2T_R} . In other words, \mathbb{Z}_{2T_R} became in general anomalous. The consequence of this mixed anomaly however depends on N nontrivially:

- (i) For $N = 4$, the mixed anomaly vanishes

$$\frac{4}{N} = 1. \tag{3.20}$$

- (ii) For $N = 4\ell$, $\ell \geq 2$, instead,

$$\frac{4}{N} = \frac{1}{\ell}, \tag{3.21}$$

and the discrete symmetry breaking takes the form

$$\mathbb{Z}_{2T_R}^\psi \rightarrow \mathbb{Z}_{\frac{2T_R}{\ell}}^\psi. \tag{3.22}$$

For $N = 4\ell$, we note that $2T_R$ is an integer multiple of ℓ .

(iii) Finally, for $N = 4\ell + 2$,

$$2T_R \cdot \frac{4}{N} = 2T_R \cdot \frac{2}{2\ell + 1}, \tag{3.23}$$

thus the breaking of the discrete symmetry is

$$\mathbb{Z}_{2T_R}^\psi \rightarrow \mathbb{Z}_{\frac{2T_R}{2\ell+1}}^\psi. \tag{3.24}$$

Just for a check, for $N = 4\ell + 2$, $2\ell + 1$ is a divisor of $2T_R$ (see App. A of Ref. 67).

The systematics of different cases, (i) ~ (iii), above, can be nicely understood in terms of the properties of the fractional instantons (torons) of this model. See Ref. 67.

3.2. Adjoint QCD

$SU(N)$ theories with N_f Weyl fermions λ in the adjoint representation, are sometimes called “the adjoint QCD”. These systems have been extensively studied by using different methods: by semi-classical analysis,⁴⁶ by direct lattice simulations,²⁴ and more recently, by studying the mixed anomalies.^{51,52,55} See also Ref. 81, and more recent work.^{59,60}

Here the color \mathbb{Z}_N^c center 1-form symmetry is exact, which can be fully gauged. The system has also a nonanomalous ordinary (0-form) discrete chiral symmetry,

$$\mathbb{Z}_{2N_f N}^\lambda : \lambda \rightarrow e^{\frac{2\pi i}{2N_f N} k} \lambda, \quad k = 1, 2, \dots, 2N_f N, \tag{3.25}$$

as is well known. A set of gauge fields are introduced:

- $A_\lambda : \mathbb{Z}_{2N_f N}^\lambda$ 1-form gauge field, for (3.25);
- $B_c^{(2)} : \mathbb{Z}_N^c$ 2-form gauge field.

$\mathbb{Z}_{2N_f N}^\lambda$ is found to induce a phase change in the partition function

$$\left(\frac{2NN_f}{8\pi^2} \int \text{tr} \tilde{F}^2 - \frac{2N^2 N_f}{8\pi^2} \int (B_c^{(2)})^2 \right) \delta A_\lambda^{(0)}, \tag{3.26}$$

$$\delta A_\lambda^{(0)} \in \frac{2\pi i}{2NN_f} \mathbb{Z}. \tag{3.27}$$

The first term conserves $\mathbb{Z}_{2NN_f}^\lambda$; the second term

$$\Delta S(\delta A_\lambda^{(0)}) \in \frac{2\pi i}{N} \mathbb{Z}, \tag{3.28}$$

breaks the chiral discrete symmetry further as

$$\mathbb{Z}_{2NN_f}^\lambda \rightarrow \mathbb{Z}_{2N_f}^\lambda \tag{3.29}$$

as found in Refs. 51, 52 and 55.

The case of $SU(2)$, $N_f = 2$, is of particular interest. In this case, the discrete chiral symmetry \mathbb{Z}_8^λ is broken by the 1-form \mathbb{Z}_2^c gauging as

$$\mathbb{Z}_8^\lambda \rightarrow \mathbb{Z}_4^\lambda. \tag{3.30}$$

The invariance of the standard $SU(2)$ theory

$$\lambda \rightarrow e^{\pm \frac{2\pi i}{8}} \lambda, \tag{3.31}$$

becomes anomalous.

What is the implication of these results on the physics in the infrared? A familiar lore about the infrared dynamics of this system is (for instance, see Ref. 81) that a condensate

$$\langle \lambda^{\{I\}} \lambda^{\{J\}} \rangle \neq 0, \quad SU(2)_f \rightarrow SO(2)_f \tag{3.32}$$

($I, J = 1, 2$ being the flavor $SU(2)_f$ indices) forms. That would lead to four-fold degenerate vacua, and in each of them, two NG bosons.

In an interesting work,⁵⁵ Anber and Poppitz proposed that the system instead may develop a four-fermion condensate

$$\langle \lambda \lambda \lambda \lambda \rangle \neq 0, \quad \text{with } \langle \lambda \lambda \rangle = 0. \tag{3.33}$$

Such a condensate breaks \mathbb{Z}_8^λ spontaneously to \mathbb{Z}_4^λ , leaving only doubly degenerate $SU(2)_f$ symmetric vacua (and no NG bosons). Massless baryons of spin $\frac{1}{2}$

$$B \sim \lambda \lambda \lambda \tag{3.34}$$

(which is necessarily a doublet of the unbroken $SU(2)_f$) should appear in the infrared spectrum to saturate all the conventional 't Hooft and Witten anomaly matching conditions. The action of the broken $\mathbb{Z}_8^\lambda/\mathbb{Z}_4^\lambda$ is a permutation between the two degenerate vacua

$$\langle \lambda \lambda \lambda \lambda \rangle \rightarrow -\langle \lambda \lambda \lambda \lambda \rangle. \tag{3.35}$$

As usual in an anomaly-matching discussion one can tell some dynamical scenario is consistent, but not that such a vacuum is necessarily realized. It remains to establish which between the familiar $SO_f(2)$ symmetric vacuum and the proposed $SU(2)_f$ symmetric one, is actually realized.

The adjoint QCD with general N reduces to $\mathcal{N} = 1$ supersymmetric Yang–Mills theory, for the special case of $N_f = 1$. A great number of results on nonperturbative aspects are known^{36,80,82,83} there. Note that in this case (3.29) leads to an N -fold vacuum degeneracy, in agreement with the well-known Witten index of pure $\mathcal{N} = 1SU(N)$ Yang–Mills theory.

One may also start from the $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang–Mills theory, where many exact results for the infrared physics are known.^{37,38,43} It can be deformed to $\mathcal{N} = 1$ theory by an adjoint-scalar mass perturbation, which yields a confining, chiral symmetry breaking vacua. Exact calculation of gauge fermion condensates $\langle \lambda \lambda \rangle$ from this viewpoint can be found in Refs. 84 and 85. The pure

$\mathcal{N} = 2$ theory could also be perturbed directly to $\mathcal{N} = 0$.⁷⁹ In principle, such an approach can give hints about $N_f = 2$ adjoint QCD, even though it is not a simple task to identify correctly the vacuum which can be reached by such a deformation.

3.3. QCD with “tensor quarks”

We now move to theories with matter fermions in N_f pairs of

$$\psi, \tilde{\psi} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \bar{\square} & \bar{\square} \\ \hline \end{array} \quad (3.36)$$

or

$$\psi, \tilde{\psi} = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \bar{\square} \\ \hline \bar{\square} \\ \hline \end{array}. \quad (3.37)$$

For reference, the standard QCD quarks are in $\square \oplus \bar{\square}$.

The first beta-function coefficient is

$$b_0 = \frac{11N - 2N_f(N \pm 2)}{3}. \quad (3.38)$$

As the $k = \frac{N}{2}$ element of the center \mathbb{Z}_N does not act, there is a

$$\mathbb{Z}_2^c \subset \mathbb{Z}_N^c \quad (3.39)$$

center symmetry.^k Also there is a discrete axial symmetry subgroup

$$\mathbb{Z}_{2N_f(N \pm 2)}^\psi : \psi \rightarrow e^{\frac{2\pi i}{2N_f(N \pm 2)}} \psi, \quad \tilde{\psi} \rightarrow e^{\frac{2\pi i}{2N_f(N \pm 2)}} \tilde{\psi}, \quad (3.40)$$

respected by instantons. The \pm refer, respectively, to two types of models, Eqs. (3.36) and (3.37).

Let us study for simplicity the $N_f = 1$ theory: the analysis is similar to the cases discussed in the preceding sections. The anomaly is given by

$$\left(-\frac{2(N \pm 2)}{8\pi^2} \text{tr} \tilde{F}^2 + \frac{2N(N \pm 2)}{8\pi^2} (B_c^{(2)})^2 \right) \delta A_\psi^{(0)}, \quad (3.41)$$

where

$$\delta A_\psi^{(0)} \in \frac{2\pi}{2(N \pm 2)} \mathbb{Z}_{2(N \pm 2)}. \quad (3.42)$$

Now

$$\frac{2(N \pm 2)}{8\pi^2} \int \text{tr} \tilde{F}^2 \in 2(N \pm 2)\mathbb{Z}, \quad (3.43)$$

^kThis model has been considered by Cohen,⁸⁶ in particular in relation with such a center symmetry, and concerning the possible existence of an order parameter for confinement.

means that the first term of (3.41) is trivial. By using

$$\frac{1}{8\pi^2} \int (B_c^{(2)})^2 = \frac{1}{4} \mathbb{Z}, \tag{3.44}$$

the second term gives an anomaly

$$A = 2\pi \frac{N}{4} \mathbb{Z}. \tag{3.45}$$

We find therefore no anomaly for $N = 4\ell$; for $N = 4\ell + 2$, the (1-form) gauging breaks the discrete symmetry as

$$\mathbb{Z}_{2(N\pm 2)}^\psi \rightarrow \mathbb{Z}_{N\pm 2}^\psi. \tag{3.46}$$

The assumption of the ‘‘quark condensate’’

$$\langle \psi \tilde{\psi} \rangle \neq 0, \tag{3.47}$$

is consistent with (3.46). The bi-fermion condensate (3.47) however breaks the discrete symmetry as

$$\mathbb{Z}_{2(N\pm 2)}^\psi \rightarrow \mathbb{Z}_2^\psi, \tag{3.48}$$

i.e. stronger than suggested by (3.46).

3.4. Chiral theories with $\frac{N-4}{k}\psi^{\{ij\}}$, s and $\frac{N+4}{k}\bar{\chi}_{[ij]}$, s

Our next theoretical laboratory is chiral $SU(N)$ gauge theories with matter fermions in a complex representation, $\frac{N-4}{k}\psi^{\{ij\}}$, s and $\frac{N+4}{k}\bar{\chi}_{[ij]}$, or

$$\frac{N-4}{k} \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \frac{N+4}{k} \begin{array}{|c|} \hline \bar{\square} \\ \hline \end{array}. \tag{3.49}$$

k is a common divisor of $(N-4, N+4)$ and $N \geq 5$. Asymptotic freedom requirement

$$11N - \frac{2}{k}(N^2 - 8) > 0, \tag{3.50}$$

is compatible with various possible choices for (N, k) . We studied two simple models in Ref. 67:

- (i) $(N, k) = (6, 2)$: $SU(6)$ with

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus 5 \begin{array}{|c|} \hline \bar{\square} \\ \hline \end{array}; \tag{3.51}$$

(ii) $(N, k) = (8, 4)$: $SU(8)$ with

$$\square \square \oplus 3 \begin{array}{c} \square \\ \square \end{array}. \quad (3.52)$$

Below we review the results of the analysis of the first model, $SU(6)$ theory with matter fields in $\underline{21} \oplus 5 \otimes \underline{15}^*$. The implications of the mixed anomalies turn out to be quite subtle even for this simple model, as will be seen.

Classically, the symmetry group is

$$SU(5) \times U(1)_\psi \times U(1)_\chi. \quad (3.53)$$

The anomalies

$$\begin{aligned} U(1)_\psi - [SU(6)]^2 &= \frac{T_{\square\square}}{T_{\square}} = N + 2 = 8, \\ U(1)_\chi - [SU(6)]^2 &= \frac{5T_{\begin{array}{c} \square \\ \square \end{array}}}{T_{\square}} = 5(N - 2) = 20, \end{aligned} \quad (3.54)$$

fix the charges of the nonanomalous $U(1)_{\psi\chi} \subset U(1)_\psi \times U(1)_\chi$ symmetry

$$(Q_\psi, Q_\chi) = (5, -2). \quad (3.55)$$

There are also unbroken discrete groups

$$U(1)_\psi \rightarrow \mathbb{Z}_8^\psi, \quad U(1)_\chi \rightarrow \mathbb{Z}_{20}^\chi. \quad (3.56)$$

By studying the overlap of $\mathbb{Z}_8^\psi \times \mathbb{Z}_{20}^\chi$ and $U(1)_{\psi\chi}$ one arrives at

$$\frac{U(1)_{\psi\chi} \times \mathbb{Z}_8^\psi \times \mathbb{Z}_{20}^\chi}{\mathbb{Z}_{40}} \sim U(1)_{\psi\chi} \times \mathbb{Z}_4 \quad (3.57)$$

(see (3.63) and (3.64)). Taking furthermore the color center and $SU_f(5)$ center into account, the true anomaly free symmetry group is

$$\frac{SU(5) \times U(1)_\psi \times U(1)_\chi}{\mathbb{Z}_6^c \times \mathbb{Z}_5^f} \rightarrow \frac{SU(5) \times U(1)_{\psi\chi} \times \mathbb{Z}_4}{\mathbb{Z}_6^c \times \mathbb{Z}_5^f}. \quad (3.58)$$

From the consideration of the standard 't Hooft anomaly analysis and by the impossibility of finding an appropriate set of massless baryons,⁶⁷ one concludes that if the system confines the global symmetry (3.57) must be broken spontaneously, at least partially. Therefore, the question is whether the 1-form gauging of a center symmetry can tell us anything useful.

First of all, one finds that both \mathbb{Z}_8^ψ and \mathbb{Z}_{20}^χ are broken by the 1-form \mathbb{Z}_2^c gauging:

$$\mathbb{Z}_8^\psi \rightarrow \mathbb{Z}_4, \quad \mathbb{Z}_{20}^\chi \rightarrow \mathbb{Z}_{10}^\chi. \quad (3.59)$$

$$\delta A_\psi^{(0)} = \frac{2\pi k}{8}, \quad k = 2, 4, \dots, 8, \quad \delta A_\chi^{(0)} = -\frac{2\pi \ell}{20}, \quad \ell = 2, 4, \dots, 20. \quad (3.60)$$

In order for the system to “match” the reduction of the symmetry (3.59) in the infrared, some condensates are expected to form. A more careful analysis is, however, needed to find out which bifermion condensates actually occur in the infrared, in order to be consistent with the systematics of the mixed-anomalies.

The division by \mathbb{Z}_{40}

$$\psi \rightarrow e^{5i\alpha}\psi, \quad \chi \rightarrow e^{-2i\alpha}\chi, \quad (3.61)$$

$$\alpha = \frac{2\pi k}{40}, \quad k = 1, 2, \dots, 40, \quad (3.62)$$

in the global symmetry group (3.57), is relevant. The quotient

$$\mathbb{Z}_4 \sim \frac{\mathbb{Z}_{20} \times \mathbb{Z}_8}{\mathbb{Z}_{40}} \quad (3.63)$$

also forms a subgroup acting as

$$\psi \rightarrow e^{2\pi i \frac{2k}{8}}\psi = e^{2\pi i \frac{k}{4}}\psi, \quad \chi \rightarrow e^{-2\pi i \frac{5k}{20}}\chi = e^{-2\pi i \frac{k}{4}}\chi, \quad (3.64)$$

or

$$\delta A_\psi^{(0)} = \frac{2\pi k}{4}, \quad \delta A_\chi^{(0)} = -\frac{2\pi k}{4}, \quad k = 1, 2, 3, 4. \quad (3.65)$$

The subtlety is that \mathbb{Z}_{40} remains nonanomalous, even after 1-form gauging of \mathbb{Z}_2^c

$$-\left(8 \cdot \frac{2\pi k}{8} - 20 \cdot \frac{2\pi k}{20}\right) \frac{1}{8\pi^2} \left[\int \text{tr} \tilde{F}^2 - 6(B_c^{(2)})^2 \right] = 0. \quad (3.66)$$

At the same time, \mathbb{Z}_4 itself is affected by the gauging of the center \mathbb{Z}_2^c symmetry. We find⁶⁷ the generalized anomaly

$$-3 \cdot 2\pi k \frac{1}{8\pi^2} \left[\int \text{tr} \tilde{F}^2 - 6(B_c^{(2)})^2 \right] = 2\pi k \cdot \left(\mathbb{Z} + 3 \cdot 6 \cdot \frac{\mathbb{Z}}{4} \right). \quad (3.67)$$

Thus, \mathbb{Z}_4 is reduced to \mathbb{Z}_2 ($k = 2, 4$).

These are the fates of the discrete symmetries

$$\mathbb{Z}_{20} \times \mathbb{Z}_8 \sim \mathbb{Z}_{40} \times \mathbb{Z}_4 \quad (3.68)$$

under the gauged 1-form center symmetry \mathbb{Z}_2^c . What do they imply on possible condensates such as

$$\psi\chi, \quad \psi\psi, \quad \chi\chi. \quad (3.69)$$

The MAC criterion might suggest formation of condensates in one or more of the channels

$$\begin{aligned}
 A : \psi \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right) \psi \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right) & \text{ forming } \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} ; \\
 B : \chi \left(\begin{array}{|c|} \hline \bar{\square} \\ \hline \square \\ \hline \end{array} \right) \chi \left(\begin{array}{|c|} \hline \bar{\square} \\ \hline \square \\ \hline \end{array} \right) & \text{ forming } \begin{array}{|c|} \hline \bar{\square} \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} ; \\
 C : \psi \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right) \chi \left(\begin{array}{|c|} \hline \bar{\square} \\ \hline \square \\ \hline \end{array} \right) & \text{ forming adjoint representation.}
 \end{aligned} \tag{3.70}$$

The one-gluon exchange strengths corresponding to these scalar composites are proportional to $\frac{16}{6}, \frac{28}{6}, \frac{32}{6}$, respectively. Of these, the last is the most attractive, and thus one might be tempted to assume that the only condensate in the system is

$$\langle (\psi\chi)_{\text{adj}} \rangle \neq 0. \tag{3.71}$$

However, the mixed anomaly analysis as sketched here shows that at least two different types of condensates must form in the infrared. For instance, the $\mathbb{Z}_4 \rightarrow \mathbb{Z}_2$ breaking would not be accounted for if (3.71) were the only condensate. See Ref. 67 for more details. One is led to conclude that two or all of the condensates (3.70), are generated by the system.

4. Generalized Anomalies and Phases of the Generalized BY and GG Models

In this section, we review the generalized anomaly in a large class of chiral gauge theories, BY and GG models. The procedure for computing the anomaly against gauging color-flavor locked 1-form \mathbb{Z}_N symmetry has been exhibited in Subsec. 2.2, by using the simplest of this class of models, $\psi\eta$ model. The master formula found there, however, can be applied to any of the BY and GG models.

4.1. Bars–Yankielowicz models

The BY models are $SU(N)$ gauge theories with Weyl fermions

$$\psi^{ij}, \quad \eta_i^A, \quad \xi^{i,a} \tag{4.1}$$

in

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus (N + 4 + p) \begin{array}{|c|} \hline \bar{\square} \\ \hline \end{array} \oplus p \begin{array}{|c|} \hline \square \\ \hline \end{array}. \tag{4.2}$$

The indices run as

$$i, j = 1, \dots, N, \quad A = 1, \dots, N + 4 + p, \quad a = 1, \dots, p. \quad (4.3)$$

The $\psi\eta$ model corresponds to $p = 0$. The number of the extra fundamental pairs p is limited by $\frac{9}{2}N - 3$ before asymptotic freedom (AF) is lost. The classical symmetry group is

$$SU(N)_c \times U(1)_\psi \times U(N + 4 + p)_\eta \times U(p)_\xi. \quad (4.4)$$

Strong anomaly breaks the symmetry group (4.4) to

$$\begin{aligned} p = 0 : & SU(N)_c \times SU(N + 4)_\eta \times U(1)_{\psi\eta}, \\ p = 1 : & SU(N)_c \times SU(N + 5)_\eta \times U(1)_{\psi\eta} \times U(1)_{\psi\xi}, \\ p > 1 : & SU(N)_c \times SU(N + 4 + p)_\eta \times SU(p)_\xi \times U(1)_{\psi\eta} \times U(1)_{\psi\xi}, \end{aligned} \quad (4.5)$$

where the anomaly free combinations are

$$U(1)_{\psi\eta} : \psi \rightarrow e^{i(N+4+p)\alpha}\psi, \quad \eta \rightarrow e^{-i(N+2)\alpha}\eta, \quad (4.6)$$

with $\alpha \in \mathbb{R}$, and

$$U(1)_{\psi\xi} : \psi \rightarrow e^{ip\beta}\psi, \quad \xi \rightarrow e^{-i(N+2)\beta}\xi, \quad (4.7)$$

with $\beta \in \mathbb{R}$. The choice of these two unbroken $U(1)$'s is arbitrary. For example, another $U(1)_{\eta\xi}$

$$U(1)_{\eta\xi} : \eta \rightarrow e^{i\gamma}\eta, \quad \xi \rightarrow e^{-i(N+4+p)\gamma}\xi, \quad (4.8)$$

with $\gamma \in \mathbb{R}$ may be chosen.

Table 1 summarizes the charges.

The standard 't Hooft anomaly matching study, based on the (perturbative) symmetry group (4.5), led to the observation that the anomaly triangles associated with (4.5) can be all matched in the infrared, assuming confinement, no condensate formation, and assuming a simple set of massless composite fermions–baryons–saturating all the anomaly triangles. This is highly nontrivial, as seen in the summary given in App. E.

Table 1. The multiplicity, charges and the representations. (\cdot) is a singlet representation.

	$SU(N)_c$	$SU(N + 4 + p)$	$SU(p)$	$U(1)_{\psi\eta}$	$U(1)_{\psi\xi}$
ψ	$\square \square$	$\frac{N(N+1)}{2} \cdot (\cdot)$	$\frac{N(N+1)}{2} \cdot (\cdot)$	$N + 4 + p$	p
η	$(N + 4 + p) \cdot \overline{\square}$	$N \cdot \square$	$N(N + 4 + p) \cdot (\cdot)$	$-(N + 2)$	0
ξ	$p \cdot \square$	$Np \cdot (\cdot)$	$N \cdot \square$	0	$-(N + 2)$

At the same time, the anomaly matching equations are consistent with dynamics Higgs phase also, where certain bi-fermion condensates form, which break the color and part of the flavor symmetries dynamically, leaving still some nontrivial unbroken chiral symmetry in the infrared. See the review in App. F.

It is thus important to find out whether or not new, mixed type of anomalies are present in the theories, and whether more stringent anomaly constraints arise, capable of discriminating these two dynamical possibilities.

As already emphasized, the study of the mixed anomalies requires clarifying the global structure of the symmetry group, *beyond* their local properties (4.5). The result of a detailed analysis done in Ref. 69 is that the true symmetry group of the BY model is

$$SU(N)_c \times \frac{SU(N+4+p) \times SU(p) \times \mathcal{H}}{\mathbb{Z}_N \times \mathbb{Z}_{N+4+p} \times \mathbb{Z}_p}, \tag{4.9}$$

where

$$\mathcal{H} = U(1)_1 \times U(1)_2 \times (\mathbb{Z}_2)_F, \tag{4.10}$$

when p and N are both even. That is, it has two disconnected components. $U(1)_1$ and $U(1)_2$ being any two out of $U(1)_{\psi\eta}$, $U(1)_{\psi\xi}$, and $U(1)_{\eta\xi}$.

When p and/or N is odd, instead

$$\mathcal{H} = U(1)_1 \times U(1)_2 \tag{4.11}$$

it has only one connected component. In these cases, symmetry group is connected, and perturbative (algebra) aspects of the 't Hooft anomaly triangles exhaust the UV-IR anomaly matching conditions. See Ref. 68. Thus, the most interesting BY models are those with p and N both even, to which we turn now.

The fact that

$$\mathbb{Z}_N \subset U(1)_{\psi\eta} \times U(1)_{\psi\xi} \times (\mathbb{Z}_2)_F \tag{4.12}$$

for N, p both even, can be shown explicitly, by choosing

$$\alpha = \frac{2\pi}{N}, \quad \beta = -\frac{2\pi}{N}. \tag{4.13}$$

We couple the system to the appropriate background gauge fields

- $A_{\psi\eta}$: $U(1)_{\psi\eta}$ 1-form gauge field,
- $A_{\psi\xi}$: $U(1)_{\psi\xi}$ 1-form gauge field,
- A_2 : $(\mathbb{Z}_2)_F$ 1-form gauge field,
- \tilde{a} : $U(N)_c$ 1-form gauge field,
- $B_c^{(2)}$: \mathbb{Z}_N 2-form gauge field.

Under the 1-form gauge transformation, the fields transform as

$$B_c^{(2)} \rightarrow B_c^{(2)} + d\lambda_c, \quad B_c^{(1)} \rightarrow B_c^{(1)} + N\lambda_c, \quad \tilde{a} \rightarrow \tilde{a} + \lambda_c, \quad \tilde{F}(\tilde{a}) \rightarrow \tilde{F}(\tilde{a}) + d\lambda_c, \\ A_{\psi\eta} \rightarrow A_{\psi\eta} - \lambda_c, \quad A_{\psi\xi} \rightarrow A_{\psi\xi} + \lambda_c, \quad A_2 \rightarrow A_2 + \frac{N}{2}\lambda_c. \tag{4.14}$$

The fermion kinetic terms are

$$\begin{aligned} & \bar{\psi}\gamma^\mu \left(\partial + \mathcal{R}_S(\tilde{a}) + \frac{N+4+p}{2}A_{\psi\eta} + \frac{p}{2}A_{\psi\xi} + A_2 \right)_\mu P_L\psi \\ & + \bar{\eta}\gamma^\mu \left(\partial + \mathcal{R}_{F^*}(\tilde{a}) - \frac{N+2}{2}A_{\psi\eta} - A_2 \right)_\mu P_L\eta \\ & + \bar{\xi}\gamma^\mu \left(\partial + \mathcal{R}_F(\tilde{a}) - \frac{N+2}{2}A_{\psi\xi} + A_2 \right)_\mu P_L\xi. \end{aligned} \tag{4.15}$$

Knowing the $(\mathbb{Z}_2)_F$ charges $+1, -1, +1$ for the fermions $\psi\eta$ and ξ , respectively, and their representations under $SU(N)$, the master formula (2.43) gives straightforwardly the result for the mixed anomaly: the result is

$$N^2 \frac{1}{8\pi^2} \int_{\Sigma^4} (B_c^{(2)})^2 \frac{1}{2} \delta A_2^{(0)} = N^2 \times \frac{\mathbb{Z}}{N^2} (\pm\pi) = \pm\pi \times \mathbb{Z}: \tag{4.16}$$

a $(\mathbb{Z}_2)_F - [\mathbb{Z}_N]^2$ mixed anomaly.

One finds no $(\mathbb{Z}_2)_F$ anomaly in the IR, if the system is in the symmetric vacuum of App. E. Such an inconsistency would be avoided, if one assumed instead that the system is in the dynamical Higgs phase (App. F), as the color-flavor locked 1-form symmetry would be spontaneously broken.

4.2. Georgi–Glashow models

The GG models have matter fermions in

$$\chi^{ij}, \quad \eta_i^A, \quad \xi^{i,a}, \tag{4.17}$$

i.e. in

$$\square\square \oplus (N+4+p)\bar{\square} \oplus p\square. \tag{4.18}$$

where

$$i, j = 1, \dots, N, \quad A = 1, \dots, N-4+p, \quad a = 1, \dots, p. \tag{4.19}$$

The simplest of the GG models (with $p=0$)—call it the $\chi\eta$ model—can be analyzed following the same steps taken in the case of the $\psi\eta$ model, in Sec. 2.2. The (true) symmetry group of the $\chi\eta$ model is

$$SU(N) \times G_f, \quad G_f = \frac{SU(N-4) \times U(1)_{\chi\eta} \times (\mathbb{Z}_2)_F}{\mathbb{Z}_N \times \mathbb{Z}_{N-4}}. \tag{4.20}$$

For even N . $U(1)_{\chi\eta}$ and $(\mathbb{Z}_2)_F$ act as

$$U(1)_{\chi\eta} : \psi \rightarrow e^{i\frac{N-4}{2}\beta}\psi, \quad \eta \rightarrow e^{-i\frac{N-2}{2}\beta}\eta; \tag{4.21}$$

$$(\mathbb{Z}_2)_F : \chi, \eta \rightarrow -\chi, -\eta. \tag{4.22}$$

The division by \mathbb{Z}_N in (4.20) is due to the equivalence relation

$$(e^{i\beta}, (-1)^n) \sim (e^{i(\beta - \frac{2\pi}{N})}, (-1)^n e^{i\frac{2\pi N}{N^2}}), \quad (4.23)$$

meaning that $U(1)_{\chi\eta}$ gauge field A and $(\mathbb{Z}_2)_F$ gauge field $A_2^{(1)}$ have charge -1 and $\frac{N}{2}$, respectively.

By introducing the gauge fields

- A_2 : $(\mathbb{Z}_2)_F$ 1-form gauge field,
- A : $U(1) = U(1)_{\chi\eta}$ 1-form gauge field,
- $B_c^{(2)}$: \mathbb{Z}_N 2-form gauge field,

the analysis follows step by step what is done in the $\psi\eta$ model. The result of the calculation, by use of the master formula (Subsec. 2.2), is that there is a mixed $(\mathbb{Z}_2)_F - [\mathbb{Z}_N]^2$ anomaly in the UV

$$\Delta S_{UV}^{(4)} = \pm i\pi\mathbb{Z}: \quad (4.24)$$

the partition function changes sign under (4.22).

Of course, the “massless baryons” lead to no anomalies in the infrared. The conclusion is that in the confining phase (see App. G) the mixed $(\mathbb{Z}_2)_F - [\mathbb{Z}_N]^2$ anomaly does not UV-IR match. In other words such a symmetric confining phase cannot be the correct vacuum of the system. There is no difficulty for the dynamical Higgs phase. The fact that in this particular (and only) case of the simplest of the GG model ($p = 0$), the $\chi\eta$ model, the confining, no-condensate phase (App. C) and the dynamical Higgs phase (App. D) happen to have the same global symmetry of the massless sector does not necessarily imply that these phases are the same phase (see a more detailed discussion on this issue in Ref. 91).

More general GG models with $p \neq 0$ have also been analyzed in detail, in Ref. 68. The analysis is similar to that done for the $\chi\eta$ model and for the general BY models reviewed above. The conclusion is that the confining, symmetric vacuum (App. G) is not consistent with the implications of the generalized anomalies. The Higgs phase (App. H) seems to be perfectly consistent with these new constraints.

5. Strong Anomaly and Phases of Chiral Gauge Theories

Very recently the present authors have realized⁹¹ that strong anomaly, which plays a prominent role in the solution of the so-called $U(1)_A$ problem in QCD, can also be significant in the study of the phases of chiral gauge theories, such as those discussed so far in this review. More concretely, a key observation is that the well-known low-energy strong-anomaly effective action for QCD, which reproduces the effects of the strong anomaly in the low-energy effective action, has a nontrivial implication on the symmetry breaking pattern itself. Note that there is a subtlety in this argument, as the well-known QCD effective sigma model action already assumes the standard chiral symmetry breaking into vectorlike residual symmetries (see (5.2)).

The criterion we adopt to study chiral gauge theories of unknown low-energy symmetries and phases, is that it should be possible to write a low-energy strong-anomaly local effective action by using the low-energy degrees of freedom (NG bosons and/or massless composite fermions) present in the assumed phase.

The simple form of such a low-energy action we will find in the dynamical Higgs phase, in contrast to the impossibility of writing analogous terms with massless baryons only (in the confining phase), provides another, independent, indication that the first type of phase (dynamical Higgs phase, characterized by certain bi-fermion condensates) represents a more plausible IR phase of this class of models.

Before considering the chiral gauge theories we discussed in the previous sections from this new angle, we first review quickly what is known about the $U(1)_A$ problem in the standard QCD.^{87,88}

5.1. $U(1)_A$ problem and the θ dependence in QCD

In QCD, the $U(1)_A$ symmetry is broken by the strong anomaly, and also spontaneously broken by the quark condensate

$$\langle \bar{\psi}_L \psi_R \rangle \sim -\Lambda^3 \neq 0, \tag{5.1}$$

which breaks the nonanomalous chiral symmetry to its vectorlike subgroup

$$SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V. \tag{5.2}$$

At this point one expects, besides the NG bosons of the $SU(N_f)_A$ symmetry (the pions), another NG boson relative to $U(1)_A$, (η or η'),¹ which would get mass due to the strong anomaly. The chiral Lagrangian allows us to understand how this works qualitatively, and quantitatively in the large N limit.

To reproduce the effects of the strong anomaly, the authors of Refs. 29–33 add to the standard chiral Lagrangian

$$L_0 = \frac{F_\pi^2}{2} \text{Tr} \partial_\mu U \partial^\mu U^\dagger + \text{Tr} M U + \text{h.c.} + \dots, \quad U \equiv \bar{\psi}_R \psi_L \tag{5.3}$$

(F_π is the usual pion-decay constant), a new term

$$\hat{L} = \frac{i}{2} q(x) \log \det U / U^\dagger + \frac{N}{a_0 F_\pi^2} q^2(x) - \theta q(x), \tag{5.4}$$

where $q(x)$ is the topological density

$$q(x) = \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{a,\mu\nu}, \tag{5.5}$$

¹Here η, η' are the $SU(2)$ or $SU(3)$ singlet pseudoscalar mesons of the real world, as in the Particle Data Booklet. The reader will not confuse them with the Weyl fermion in the chiral $\psi\eta$ or $\chi\eta$ models being studied here.

and a_0 is some constant of the order of unity. The variation of \hat{L} under the action of $U(1)_A$

$$\psi_L \rightarrow e^{i\alpha}\psi_L, \quad \psi_R \rightarrow e^{-i\alpha}\psi_R, \quad U \rightarrow e^{2i\alpha}U \quad (5.6)$$

reproduces the variation of the phase of the partition function

$$\Delta S = 2N_f\alpha \int d^4x \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{a,\mu\nu}, \quad (5.7)$$

due to the strong anomaly.

The expression (5.4) can be manipulated as shown in Refs. 29–33 integrating away the topological charge $q(x)$ one obtains an equivalent expression

$$\hat{L} = -\frac{F_\pi^2 a_0}{4N} \left(\theta - \frac{i}{2} \log \det U/U^\dagger \right)^2, \quad (5.8)$$

which is well defined as

$$\langle U \rangle \propto \mathbf{1} \neq 0. \quad (5.9)$$

Expanding (5.8) around this VEV

$$U \propto e^{\frac{i\pi^a t^a}{F_\pi} + i\frac{\eta t^0}{F_\pi^{(0)}}} = \mathbf{1} + i\frac{\pi^a t^a}{F_\pi} + i\frac{\eta t^0}{F_\pi^{(0)}} + \dots, \quad (5.10)$$

one finds the mass term for the would-be NG boson, η .

The idea here is to reverse the logic: one can actually argue that the presence of such an effective action needed for reproducing the strong anomaly *implies* a non-vanishing condensate $\langle U \rangle = \langle \bar{\psi}_R \psi_L \rangle \neq 0$, and hence, indirectly, also the spontaneous breaking of *nonanomalous* chiral symmetry, (5.2), affecting the low-energy physics.

Even if in QCD there is a powerful direct argument²² for such a vectorlike symmetry-breaking pattern, it is interesting to note that the requirement of faithfully representing the $U(1)_A$ anomaly in the infrared seems to imply the same conclusion.

It is this kind of consideration that has led recently the present authors to apply an analogous argument to chiral gauge theories,⁹¹ by requiring the effective low-energy theory to be able to express the strong anomaly appropriately. The following (Subsecs. 5.2–5.5) is the review of some of the results found.

5.1.1. $\mathcal{N} = 1$ supersymmetric theories

Before proceeding to the discussion of chiral gauge theories, let us make a brief comment on $\mathcal{N} = 1$ supersymmetric models. In the context of $\mathcal{N} = 1$ supersymmetric gauge theories, the strong-anomaly effective action is derived by using the so-called Veneziano–Yankielowicz (VY) and Affleck–Dine–Seiberg (ADS) superpotentials.^{34–36} They correctly reproduce in the infrared effective theory, the effects of instantons and supersymmetric Ward–Takahashi identities, and embody the anomaly of Refs. 89 and 90. This last one, known as Konishi anomaly, has direct implications on the vacuum properties of the theory under consideration. It is a

straightforward consequence of the strong anomaly, via supersymmetry.^m The VY and ADS superpotentials are indeed crucial in determining the infrared dynamics and phases of the $\mathcal{N} = 1$ supersymmetric gauge theories. For a review, see, for instance, Ref. 80.

5.2. $\psi\eta$ model and strong anomaly

Let us apply a similar idea, i.e. of writing an effective action which reproduces the strong anomaly of the UV theory in the low-energy theory, to one of the simplest chiral gauge theories, the $\psi\eta$ model (see Subsec. 2.2). Let us remind ourselves briefly of the symmetries of the model. At the infinitesimal level, the quantum symmetry group of $\psi\eta$ is

$$SU(N + 4)_\eta \times U(1)_{\psi\eta}, \tag{5.11}$$

while any combinations of $U(1)_\psi \times U(1)_\eta$ different from $U(1)_{\psi\eta}$ is broken by a strong anomaly. The low-energy effective action must capture this strong anomaly correctly.

In Subsec. 2.2, it was shown that the $\psi\eta$ model cannot confine maintaining the full global symmetry unbroken. The system instead can break the gauge symmetry dynamically (as well as part of the global symmetry), and a color-flavor locking condensate forms:

$$\langle \psi^{ij} \eta_j^A \rangle = \begin{cases} c_{\psi\eta} \delta^{iA}, & A = 1, \dots, N, \\ 0, & A = N + 1, \dots, N + 4. \end{cases} \tag{5.12}$$

Unlike $\bar{\psi}_R \psi_L$ in QCD, $\tilde{\phi} = \sum_{k,j} \psi^{kj} \eta_j^k$, is not gauge invariant. It is convenient to re-express this condensate in a gauge invariant form, i.e.

$$\det U, \quad U_{kl} \equiv \psi^{kj} \eta_j^l. \tag{5.13}$$

Such a gauge invariant condensate is fully equivalent to (5.12). It causes the breaking

$$SU(N + 4) \times U(1) \rightarrow SU(N)_{cf} \times SU(4) \times U(1)'. \tag{5.14}$$

(App. B). $U(1)'$ is the unbroken combination of $U(1)_{\psi\eta}$ and $U(1)_D$, where $U(1)_D$ is $U(1) \subset SU(N + 4)$ generated by $T_D = \text{diag}(4 \cdot \mathbf{1}_{N \times N}, -N \cdot \mathbf{1}_{4 \times 4})$.

At this point, it is useful to look into the NG boson sector of the theory, which leads to an apparent puzzle. From the symmetry breaking, one expects to find $8N$ nonabelian NG bosons relative to $\frac{SU(N+4)}{SU(N) \times SU(4) \times U(1)_D}$, interpolated in a gauge invariant fashion by

$$\phi^A = (\psi^{ij} \eta_j^a)^* (T^A)_b^a (\psi^{ik} \eta_k^b). \tag{5.15}$$

Here T_A are the $8N$ broken generators that connect the N -dimensional subspace (where $SU(N)$ acts) and the 4D one (where $SU(4)$ acts).

^mThe Konishi anomaly can also be viewed as representing an anomalous supersymmetry transformation law for some composite fields.^{89,90}

The problem emerges when one considers $U(1)$ NG boson(s). There is certainly a physical massless NG boson, living in

$$\frac{U(1)_D \times U(1)_{\psi\eta}}{U(1)'} \tag{5.16}$$

The gauge-invariant field that interpolates it can be taken as the (imaginary part of the) condensate $\det U$ itself. However, with the condensate (5.12) alone, there is no space for another possible NG boson, associated with the symmetry breaking of an anomalous $U(1)$ symmetry (any generic combination of $U(1)_\psi$ and $U(1)_\eta$ other than $U(1)_{\psi\eta}$ is in fact spontaneously broken by $\langle\psi\eta\rangle$). This would-be NG boson would get a mass from the strong anomaly, but in any case it needs to be described by an interpolating field (which?). Another related fact is that there is actually a particular anomalous symmetry (a special combination of $U(1)_\psi$ and $U(1)_\eta$)

$$U(1)_A : \begin{cases} \psi \rightarrow e^{i\alpha}\psi, \\ \eta \rightarrow e^{-i\alpha}\eta, \end{cases} \tag{5.17}$$

which is not spontaneously broken by the $\psi\eta$ condensate. How would such a symmetry manifest itself in the infrared? These are the first hints that the description in terms of the condensate $\psi\eta$ (or $\det U$) is not a complete one.

Another reason to look for other condensates is that it is not possible to write an effective Lagrangian which realizes all the (nonanomalous) global symmetries, with the composite field $\det U$ alone. For further details, see Ref. 91.

With these considerations in mind, let us construct the correct form of the strong anomaly effective action systematically. We start from the very beginning,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}^{\text{fermions}}, \tag{5.18}$$

$$\mathcal{L}^{\text{fermions}} = -i\bar{\psi}\bar{\sigma}^\mu(\partial + \mathcal{R}_S(a))_\mu\psi - i\bar{\eta}\bar{\sigma}^\mu(\partial + \mathcal{R}_{F^*}(a))_\mu\eta, \tag{5.19}$$

where a is the $SU(N)$ gauge field, and the matrix representations appropriate for ψ and η fields are indicated with \mathcal{R}_S and \mathcal{R}_{F^*} . We change the variables by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}^{\text{fermions}} + \text{Tr}[(\psi\eta)^*U] + \text{h.c.} + B(\psi\eta\eta)^* + \text{h.c.}, \tag{5.20}$$

where U is the composite scalar of $N \times (N + 4)$ color-flavor mixed matrix form

$$\text{Tr}[(\psi\eta)^*U] \equiv (\psi^{ij}\eta_j^m)^*U^{im}, \tag{5.21}$$

and B are the baryons $B \sim \psi\eta\eta$,

$$B^{mn} = \psi^{ij}\eta_i^m\eta_j^n, \quad m, n = 1, 2, \dots, N + 4, \tag{5.22}$$

anti-symmetric in $m \leftrightarrow n$. In writing down the Lagrangian (5.20), we have anticipated the fact that these baryon-like composite fields, present in the Higgs phase together with the composite scalars $\psi\eta$ (see App. B), are also needed to write down

the strong-anomaly effective action. This allows us to dodge the problem about the NG bosons (and to break $U(1)_A$), as we are going to explain.

Integrating ψ and η out, one gets

$$\mathcal{L}^{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{Tr}(\mathcal{D}U)^\dagger\mathcal{D}U - i\bar{B}\bar{\sigma}^\mu\partial_\mu B - V. \tag{5.23}$$

The potential V is assumed to be such that its minimum is of the following form:

$$\langle U^{im} \rangle = c_{\psi\eta}\Lambda^3\delta^{im}, \quad i, m = 1, 2, \dots, N, \tag{5.24}$$

and contains the strong-anomaly term

$$V = V^{(0)} + \hat{L}_{\text{an}}. \tag{5.25}$$

\hat{L}_{an} of the form

$$\hat{L}_{\text{an}} = \text{const}[\log(\epsilon BB \det U) - \log(\epsilon BB \det U)^\dagger]^2 \tag{5.26}$$

which is analogue of (5.8) in QCD. The argument of the logarithm

$$\epsilon BB \det U \equiv \epsilon^{m_1, m_2, \dots, m_{N+4}} \epsilon^{i_1, i_2, \dots, i_N} B_{m_{N+1}, m_{N+2}} B_{m_{N+3}, m_{N+4}} U_{i_1 m_1} U_{i_2 m_2} \dots U_{i_N m_N}, \tag{5.27}$$

is invariant under the full (nonanomalous) symmetry

$$SU(N)_c \times SU(N+4) \times U(1)_{\psi\eta} \tag{5.28}$$

as it should be. Moreover it contains $N + 2\psi$'s and $N + 4\eta$'s, the correct numbers of the fermion zero modes in the instanton background: it corresponds to a 't Hooft's instanton n -point function, e.g.

$$\langle \psi\eta\eta(x_1)\psi\eta\eta(x_2)\psi\eta(x_3) \dots \psi\eta(x_{N+2}) \rangle. \tag{5.29}$$

This effective Lagrangian is well defined only if the argument of the logarithm takes a VEV. In particular, it is natural to assume

$$\langle \epsilon^{(4)}BB \rangle \neq 0, \quad \langle \det U \rangle \neq 0, \tag{5.30}$$

where

$$\epsilon^{(4)}BB = \epsilon_{\ell_1\ell_2\ell_3\ell_4} B^{\ell_1\ell_2} B^{\ell_3\ell_4}, \quad \ell_i = N + 1, \dots, N + 4. \tag{5.31}$$

As

$$\langle \det U \rangle \propto \mathbf{1}_{N \times N} \tag{5.32}$$

takes up all flavors up to N (the flavor $SU(N+4)$ symmetry can be used to orient the symmetry breaking this way), BB must be made of the four remaining flavors, as in (5.31). These baryons were not among those considered in the earlier studies,^{11,69} but assumed to be massless here, and indicated as $B^{[A_2 B_2]}$ in Table B.1. This is possible because these fermions do not have any perturbative anomaly with respect to the unbroken symmetry group, $SU(N) \times SU(4) \times U(1)$: 't Hooft anomaly

matching considerations cannot tell if they are massive or not, either the two options are possible.

Now we see how the apparent puzzle about the NG bosons hinted at above is solved. We can define the interpolating fields of the two NG bosons by expanding the condensates

$$\begin{aligned} \det U &= \langle \det U \rangle + \dots \propto \mathbf{1} + \frac{i}{F_\pi^{(0)}} \phi_0 + \dots; \\ \epsilon^{(4)} BB &= \langle \epsilon^{(4)} BB \rangle + \dots \propto \mathbf{1} + \frac{i}{F_\pi^{(1)}} \phi_1 + \dots, \end{aligned} \tag{5.33}$$

(here $F_\pi^{(0)}$ and $F_\pi^{(1)}$ are some constants with dimension of mass). Clearly in general, the physical NG boson and the anomalous would be NG boson will be interpolated by *two* linear combinations of ϕ_0 and ϕ_1 . The effective Lagrangian allows us to fix these linear combinations.

Indeed, as the effective Lagrangian (5.26) is invariant under the nonanomalous symmetry group, and in particular $U(1)_{\psi\eta}$ and $U(1)_D$ do not act on $\epsilon BB \det U$, the NG boson which appears in the strong-anomaly effective action as the fluctuation of $\epsilon BB \det U$,

$$\tilde{\phi} \equiv N_\pi \left[\frac{1}{F_\pi^{(0)}} \phi_0 + \frac{1}{F_\pi^{(1)}} \phi_1 \right], \quad N_\pi = \frac{F_\pi^{(0)} F_\pi^{(1)}}{\sqrt{(F_\pi^{(0)})^2 + (F_\pi^{(1)})^2}}, \tag{5.34}$$

cannot be the massless physical one: it is would be NG boson relative to the anomalous symmetry. Indeed the effective action provides a mass term for this NG boson.

The orthogonal combination

$$\phi \equiv N_\pi \left[\frac{1}{F_\pi^{(1)}} \phi_0 - \frac{1}{F_\pi^{(0)}} \phi_1 \right], \tag{5.35}$$

i.e. the interpolating field of the physical NG boson living in the coset (5.16), remains massless.

Before we have included in the low-energy description some massless baryons which are neither required, nor excluded by the 't Hooft anomaly matching. Now one can see their ultimate fate using the strong-anomaly effective Lagrangian. In particular (5.26) contains a 4-fermion coupling between these baryons, which, plugging the VEVs (5.30), provides a mass term for them.

The last remark is that the strong-anomaly effective action does not depend on the absolute value of the condensates, BB and $\det U$, separately. This simply means that these symmetry considerations alone cannot determine the mechanism of condensation. In particular, even if $\langle \det U \rangle \neq 0$ is somehow expected, a VEV for BB is more surprising, and probably due to residual dipole interactions between the baryons. However, a more in-depth study on how these two flat directions are lifted by quantum effects is needed to precisely understand how these two condensates form.

5.3. Strong anomaly: The generalized BY models

As the solution given above on the $\psi\eta$ model is notably subtle, one might wonder whether a similar mechanism is at work in the generalized BY models, an $SU(N)$ gauge theory with Weyl fermions

$$\psi^{ij}, \quad \eta_i^A, \quad \xi^{i,a} \tag{5.36}$$

in the direct-sum representation

$$\square\square \oplus (N + 4 + p) \overline{\square} \oplus p \square. \tag{5.37}$$

Also in this case (Ref. 69 and App. C) the conventional 't Hooft anomaly matching equations allow a chirally symmetric confining vacuum, with massless baryons

$$(B_1)^{[AB]} = \psi^{ij}\eta_i^A\eta_j^B, \quad (B_2)_A^a = \bar{\psi}_{ij}\bar{\eta}_A^i\xi^{j,a} \quad (B_3)_{\{ab\}} = \psi^{ij}\bar{\xi}_{i,a}\bar{\xi}_{j,b}, \tag{5.38}$$

(the first is anti-symmetric in $A \leftrightarrow B$ and the third is symmetric in $a \leftrightarrow b$), saturating all conventional 't Hooft anomaly triangles. The study of the generalized anomaly in Sec. 4.1 has however shown that such a vacuum is not consistent.

A dynamical Higgs phase with condensates

$$\begin{aligned} \langle U^{iB} \rangle &= \langle \psi^{ij}\eta_i^B \rangle = c_{\psi\eta}\Lambda^3\delta^{jB} \neq 0, \quad j, B = 1, \dots, N, \\ \langle V^{aA} \rangle &= \langle \xi^{i,a}\eta_i^A \rangle = c_{\eta\xi}\Lambda^3\delta^{N+4+a,A} \neq 0, \\ a &= 1, \dots, p, \quad A = N + 5, \dots, N + 4 + p, \end{aligned} \tag{5.39}$$

a and with symmetry breaking

$$\begin{aligned} &SU(N)_c \times SU(N + 4 + p)_\eta \times SU(p)_\xi \times U(1)_{\psi\eta} \times U(1)_{\psi\xi} \\ &\xrightarrow{\langle \xi \eta \rangle, \langle \psi \eta \rangle} SU(N)_{cf_\eta} \times SU(4)_\eta \times SU(p)_{\eta\xi} \times U(1)'_{\psi\eta} \times U(1)'_{\psi\xi}, \end{aligned} \tag{5.40}$$

is fully consistent with the gauging of the center symmetry (Ref. 69 and Sec. 4.1).

A strong-anomaly effective action for the BY theories can be constructed in a way similar to the $\psi\eta$ model. Instead of (5.27), one has now

$$\begin{aligned} \epsilon(B_1 B_1 \det U \det V) &\equiv \epsilon^{m_1, m_2, \dots, m_{N+4+p}} \epsilon^{i_1, i_2, \dots, i_N} \epsilon^{k_1, k_2, \dots, k_p} \\ &\times B_1^{[m_{N+1}, m_{N+2}]} B_1^{[m_{N+3}, m_{N+4}]} U^{i_1 m_1} U^{i_2 m_2} \dots U^{i_N m_N} V^{m_{N+5} k_1} \dots V^{m_{N+4+p} k_p}. \end{aligned} \tag{5.41}$$

The rest of the analysis can be completed by closely following that of the $\psi\eta$ model discussed in Subsec. 5.2. We skip the details of the analysis. Let us note only that the strong anomaly effective action with such a logarithm, is perfectly consistent with, and perhaps implies, the condensates, (5.39): i.e. that the system is in dynamical Higgs phase (5.40). It is, instead, not possible to rewrite a strong-anomaly effective action with logarithmic argument (5.41), in terms of massless composite fermions (5.38) alone.

5.4. Strong anomaly and the $\chi\eta$ model

It is an interesting exercise to apply the same reasoning about the strong anomaly to the $\chi\eta$ model. We will find that there are good analogies with the $\psi\eta$ case studied above, but also quite significant differences.

In this model any combination of $U(1)_\chi$ and $U(1)_\eta$, except $U(1)_{\chi\eta}$ (see Table D.1), is anomalous, therefore some term similar to (5.4) (in QCD) should appear.

In the dynamical Higgs scenario for the $\chi\eta$ model, there are two bi-fermion condensates

$$\langle \chi^{ij} \eta_j^m \rangle = c_{\chi\eta} \delta^{im} \Lambda^3, \quad i, m = 1, 2, \dots, N - 4, \quad (5.42)$$

and

$$\langle \chi\chi \rangle \neq 0. \quad (5.43)$$

This implements a two-step breaking

$$\begin{aligned} SU(N) \times SU(N - 4) \times U(1)_{\chi\eta} &\xrightarrow{\langle \chi\eta \rangle} SU(N - 4)_{\text{cf}} \times SU(4)_c \times U(1)' \\ &\xrightarrow{\langle \chi\chi \rangle} SU(N - 4)_{\text{cf}} \times U(1)'. \end{aligned} \quad (5.44)$$

As before, in order to construct a fully consistent effective action, one should keep the full invariance of the original theory, either spontaneously broken or not. To do so, it is convenient to re-express the condensates (5.42) in a gauge invariant way. The answer is to write a single gauge-invariant condensate

$$\begin{aligned} U &= \epsilon_{i_1 i_2 \dots i_N} \epsilon_{m_1 m_2 \dots m_{N-4}} (\chi\eta)^{i_1 m_1} (\chi\eta)^{i_2 m_2} \dots (\chi\eta)^{i_{N-4} m_{N-4}} \chi^{i_{N-3} i_{N-2}} \chi^{i_{N-1} i_N} \\ &\sim \epsilon (\chi\eta)^{N-4} (\chi\chi), \end{aligned} \quad (5.45)$$

which encodes both of the two (gauge depending) ones.

This choice suggests that the correct strong-anomaly effective action for the $\chi\eta$ model is

$$\frac{i}{2} q(x) \log \epsilon (\chi\eta)^{N-4} (\chi\chi) + \text{h.c.}, \quad (5.46)$$

where again, $q(x)$ is the topological density defined in Eq. (5.5). Clearly it is by construction invariant under the whole (nonanomalous) symmetry group

$$SU(N)_c \times SU(N - 4) \times U(1)_{\chi\eta}. \quad (5.47)$$

Let us comment briefly some features that suggest that this is indeed the correct result.

- The argument of the logarithmic function here matches the correct number of the fermion zero modes in the instanton background ($N_\chi = N - 2$ and $N_\eta = N - 4$).
- In contrast, there is no way of writing the strong-anomaly effective action (5.46) in terms of the “baryons”, $B \sim \chi\eta\eta$, of the assumed confining, chirally symmetric

phase (App. C). No combination of the baryons can saturate the correct number of the fermion zero modes, *cfr* (5.45).

This anomaly effective action (5.46) agrees with the one proposed by Veneziano⁶ for the case of $SU(5)$, and generalizes it to all $SU(N)\chi\eta$ models. A key observation we share with⁶ and generalizes to models with any N , is that this strong-anomaly effective action, which should be there in the low-energy theory to reproduce correctly the (anomalous and nonanomalous) symmetries of the UV theory, *implies* nonvanishing condensates

$$\langle\chi\eta\rangle \neq 0, \quad \langle\chi\chi\rangle \neq 0, \tag{5.48}$$

i.e. that the system is in dynamical Higgs phase, App. D.

Upto now the story has been very similar to the one about the $\psi\eta$ model. However there are some differences. Differently from the $\psi\eta$ model, where the baryon condensate must enter in the strong-anomaly effective action, here the structure of the effective action simplifies, and no baryon is needed. Moreover, contrary to the $\psi\eta$ model, the $\chi\eta$ system has no physical $U(1)$ NG boson: it is eaten by a color $SU(N)$ gauge boson. However the counting of the broken and unbroken $U(1)$ symmetries is basically similar in the two models. Of the two nonanomalous symmetries ($U(1)_c$ and $U(1)_{\chi\eta}$), a combination remains a manifest physical symmetry, and the other becomes the longitudinal part of a color gauge boson. Still another, anomalous, $U(1)$ symmetry exists, which is any combination of $U(1)_\chi$ and $U(1)_\eta$ other than $U(1)_{\chi\eta}$. This symmetry is also spontaneously broken, hence it must be associated with a NG boson, even though it will get mass by the strong anomaly.

As in the $\psi\eta$ model, one can describe this situation explicitly, by expanding the composite $\chi\eta$ and $\chi\chi$ fields around their VEV's,

$$\begin{aligned} (\det U)' &= \langle(\det U)'\rangle + \dots \propto \mathbf{1} + i \frac{1}{F_\pi^{(0)}} \phi'_0 + \dots; \\ \chi\chi &= \langle\chi\chi\rangle + \dots \propto \mathbf{1} + i \frac{1}{F_\pi^{(1)}} \phi'_1 + \dots, \end{aligned} \tag{5.49}$$

where $(\det U)'$ is defined in the $N - 4$ dimensional color-flavor mixed space, and

$$\chi\chi \equiv \epsilon_{i_1, i_2, i_3, i_4} \chi^{i_1 i_2} \chi^{i_3 i_4}, \quad N - 3 \leq i_j \leq N. \tag{5.50}$$

Now one can see that the strong-anomaly effective action (5.46) gives mass to

$$\tilde{\phi}' \equiv N_\pi \left[\frac{1}{F_\pi^{(0)}} \phi'_0 + \frac{1}{F_\pi^{(1)}} \phi'_1 \right], \quad N_\pi = \frac{F_\pi^{(0)} F_\pi^{(1)}}{\sqrt{(F_\pi^{(0)})^2 + (F_\pi^{(1)})^2}}, \tag{5.51}$$

whereas an orthogonal combination

$$\phi' \equiv N_\pi \left[\frac{1}{F_\pi^{(1)}} \phi'_0 - \frac{1}{F_\pi^{(0)}} \phi'_1 \right] \tag{5.52}$$

remains massless. The latter corresponds to the potential NG boson which is absorbed by the color T_c gauge boson.

5.5. Generalized GG models and strong anomaly

Let us now turn to the generalized GG models,⁶⁹ i.e. $SU(N)$ gauge theories with Weyl fermions

$$\chi^{[ij]}, \quad \eta_i^A, \quad \xi^{i,a} \tag{5.53}$$

in the direct-sum representation

$$\begin{array}{c} \square \\ \square \end{array} \oplus (N - 4 + p) \overline{\square} \oplus p \square, \tag{5.54}$$

It turns out that the simple structure of the strong-anomaly effective action (5.46), which does not need bi-baryon condensate, works out also in this case

$$\frac{i}{2} q(x) \log \epsilon \chi \chi \det(\chi \eta) \det(\xi \eta) + \text{h.c.}, \tag{5.55}$$

where a short-hand notation

$$\begin{aligned} \epsilon \chi \chi \det(\chi \eta) \det(\xi \eta) &= \epsilon_{i_1 i_2 \dots i_N} \epsilon_{k_1 k_2 \dots k_p} \epsilon_{m_1 m_2 \dots m_{N-4+p}} \\ &\times (\chi \eta)^{i_1 m_1} (\chi \eta)^{i_2 m_2} \dots (\chi \eta)^{i_{N-4} m_{N-4}} (\chi^{i_{N-3} i_{N-2}} \chi^{i_{N-1} i_N}) \\ &\times (\xi \eta)^{m_{N-3} k_1} \dots (\xi \eta)^{m_{N-4+p} k_p} \end{aligned} \tag{5.56}$$

has been used. The strong-anomaly action (5.55) requires the condensates

$$\begin{aligned} \langle \chi^{ij} \eta_i^A \rangle &= \text{const.} \Lambda^3 \delta^{jA} \neq 0, \quad j = 1, \dots, N - 4, \quad A = 1, \dots, N - 4, \\ \langle \xi^{i,a} \eta_i^B \rangle &= \text{const.} \Lambda^3 \delta^{N-4+a,B} \neq 0, \quad a = 1, \dots, p, \quad B = N - 3, \dots, N - 4 + p, \end{aligned} \tag{5.57}$$

and

$$\langle \chi^{j_1 j_2} \chi^{j_3 j_4} \rangle = \text{const.} e^{j_1 j_2 \dots j_4} \Lambda^3 \neq 0, \quad j_1, \dots, j_4 = N - 3, \quad N - 2, \dots, N. \tag{5.58}$$

Hearteningly, these are exactly the set of condensates expected to occur in the dynamical Higgs phase of the GG models (Ref. 69 and App. H).

5.6. Strong anomaly in the chiral gauge theories considered in Sec. 3

Upto now we have analyzed the implications of the strong anomaly in models discussed in Subsecs. 2.2 and in 4. In these models, being able to discriminate between different types of phases (confinement with unbroken global symmetry versus dynamical Higgs phase) was clearly very important, as the conventional 't Hooft anomaly-matching algorithm could not tell us which type of the vacua were the correct ones. We found indeed that both the recent generalized anomaly study

(Subsecs. 2.2 and 4) and strong-anomaly consideration (Subsecs. 5.2–5.5) seem to favor the dynamical Higgs phase, in all these models.

Of course, consideration of the strong-anomaly effective action is relevant also in other models. For illustration, we discuss below a few models studied in Sec. 3.

5.6.1. *SU(6) model with a single fermion in a self-adjoint anti-symmetric representation*

Consider an $SU(6)$ model with a *single* left-handed fermion in the representation,

$$\underline{20} = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \tag{5.59}$$

which was studied in Refs. 58 and 67 and reviewed in Sec. 3.1.1. The lessons learned from the the gauging of the 1-form \mathbb{Z}_3^c symmetry have been that the nonanomalous \mathbb{Z}_6^ψ symmetry must break spontaneously as

$$\mathbb{Z}_6^\psi \rightarrow \mathbb{Z}_2^\psi, \tag{5.60}$$

implying a three-fold vacuum degeneracy.^{58,67} This could either be because of a four-fermion condensate⁵⁸

$$\langle \psi\psi\psi\psi \rangle \sim \Lambda^6 \neq 0, \quad \langle \psi\psi \rangle = 0, \tag{5.61}$$

or due to a gauge-symmetry breaking bi-fermion condensate⁶⁷

$$\langle \psi\psi \rangle \sim \Lambda^3 \neq 0, \tag{5.62}$$

with $\psi\psi$ in the adjoint representation of $SU(6)$. Both scenarios are consistent.

Let us see if considerations on the strong-anomaly can clarify which scenario is actually realized. A particularly simple representation of the strong-anomaly is

$$\frac{i}{2} q(x) \log \psi\psi\psi\psi\psi\psi + \text{h.c.} \tag{5.63}$$

Based on our viewpoint that the argument of the logarithmic function acquires a nonvanishing VEV, the assumption of four-fermion condensate (5.61) appears to leads to a difficulty. In contrast, the assumption of bi-fermion condensates (5.62) looks perfectly consistent, with

$$\langle \psi\psi\psi\psi\psi\psi \rangle \sim \langle \psi\psi \rangle_j^i \langle \psi\psi \rangle_k^j \langle \psi\psi \rangle_i^k \neq 0. \tag{5.64}$$

5.6.2. *Adjoint QCD with $N_c = N_f = 2$*

It is interesting to apply the same logics also to the adjoint QCD, previously analyzed from the point of view of generalized symmetries and their anomalies. In particular, let us focus on the $N_c = 2, N_f = 2$ case, because of the renewed interest in this particular model, raised by the work of Anber and Poppitz.⁵⁵

The conventional thinking holds that a gauge invariant bi-fermion condensate

$$\langle \lambda\lambda \rangle \neq 0 \tag{5.65}$$

forms, breaking the flavor symmetry as $SU(2)_f \rightarrow SO(2)_f$, leading to 2 NG bosons, and reducing the discrete \mathbb{Z}_8 symmetry to \mathbb{Z}_2 resulting four degenerate vacua. The Anber and Poppitz’s proposal⁵⁵ is that, instead, the system might develop a four-fermion condensate but not bifermion condensate:

$$\langle \lambda\lambda\lambda\lambda \rangle \neq 0, \quad \langle \lambda\lambda \rangle = 0. \tag{5.66}$$

The discrete \mathbb{Z}_8 symmetry is now broken to its \mathbb{Z}_4 subgroup (therefore there are only two degenerate vacua). Massless baryons

$$\sim \lambda\lambda\lambda \tag{5.67}$$

(necessarily a doublet of $SU(2)_f$) matches the UV-IR Witten anomaly of $SU(2)_f$.

As said above, the two possibilities are both consistent with the generalized ’t Hooft anomaly matching, therefore an indication from the strong-anomaly effective action would be very welcome.

The analogue of (5.4), (5.46) and (5.63), is in this case

$$\frac{i}{2} q(x) \log \lambda\lambda \dots \lambda + \text{h.c.} \tag{5.68}$$

With eight λ ’s inside the argument of the logarithmic function. Therefore, in contrast to what we saw in the preceding model Subsec. 5.6.1, our strong-anomaly algorithm does not seem to be able to discriminate the two dynamical possibilities (5.65), or (5.66).

Before concluding this section, we note that in the case with $N_f = 1$, arbitrary N_c , the adjoint QCD becomes $\mathcal{N} = 1$ supersymmetric YM theory. The strong-anomaly effective action (5.68) with $2N_f N_c = 2N_c \lambda$ ’s, reduces precisely to the Veneziano–Yankielowicz effective potential,³⁴ implying $\langle \lambda\lambda \rangle \neq 0$. In this case, the assumption of the bi-fermion condensate, $\langle \lambda\lambda \rangle \neq 0$, the breaking of the discrete symmetry $\mathbb{Z}_{2N_c} \rightarrow \mathbb{Z}_2$, and the resulting N_c fold degeneracy of the vacua (Witten’s index), are generally accepted as a well-established fact.

6. Summary and Discussion

We have reviewed in this paper the first applications of generalized anomalies and discussed what the consequent stronger anomaly matching conditions tell us about various chiral gauge theories based on $SU(N)$ gauge group. Our discussion was divided in two classes of models. In the first, the system has a 1-form \mathbb{Z}_k symmetry (k being a divisor of N) under which the matter fermions do not transform. The treatment in this case is relatively straightforward: certain discrete symmetries, respected by instantons, are often found to be further made anomalous, due to the fractional ’t Hooft fluxes accompanying the gauging of the discrete 1-form symmetries. The discussion of Sec. 3 has illustrated that their consequences depend non-trivially on the types of matter fermions present, and an interesting and rich variety

of predictions on the possible condensates, symmetry breaking patterns and phases, have been found.

A second group of models (BY and GG models) have a color-flavor locked \mathbb{Z}_N 1-form symmetry, in which matter fermions transform together with the $SU(N)$ gauge field. A careful analysis of the global properties of the symmetry group is needed before actually introducing the gauging of this discrete 1-form symmetry. This has been worked out in detail in all of the generalized BY and GG models, and the results of the analysis reviewed in Sec. 4. A surprising implication is that, at least for even N theories, color-flavor locked $\mathbb{Z}_2 - U(1) - \mathbb{Z}_N$ 1-form symmetry does not allow its gauging which is a new kind of 't Hooft anomaly, and that this is consistent with the low-energy system being in dynamical Higgs phase characterized by certain bifermion condensates.

After the examination of the results from the generalized anomalies (involving the 1-form symmetries and gauging of some discrete center symmetries), we changed the topics, and turned to the very recent observation⁹¹ about the strong-anomaly associated effects. This is the idea discussed in the context of the so-called $U(1)_A$ problem in QCD many years ago, but for some reason it was almost never applied to the discussion of the physics of strongly coupled chiral gauge theories. It is found that the requirement that the massless degrees of freedom of the hypothesized infrared phase should be able to describe appropriately the strong anomaly giving a rather clear indication on the physics of BY and GG models: the structure of the strong-anomaly effective action favors the dynamical Higgs vacua, against the confining, fully flavor symmetric vacua, in agreement with the implications from the generalized anomaly matching algorithm, explored in the first part of this review.

The fact that both mixed anomalies^{68,69} and the strong-anomaly effective action⁹¹ imply dynamical Higgs phase in chiral BY and GG models is certainly not accidental. Both arise by taking properly the strong chiral $U(1)$ anomalies into account.

These discussions, rather unexpectedly, brought us to note certain analogies and contrasts between the strong-interaction dynamics of vector-like and chiral gauge theories.⁹¹ Let us now compare the standard QCD with N_f light flavors of quarks and anti-quarks, and the $\psi\eta$, $\chi\eta$ models as well as more general BY and GG models.

In many senses, the bifermion condensates such as $U = \psi\eta$ in the $\psi\eta$ model (and $\chi\eta$, $\chi\chi$ condensates in the $\chi\eta$ model), can be regarded as a perfect analogue of the quark condensate $U = \bar{\psi}_R\psi_L$ in QCD. All of these composite scalars enter the strong-anomaly effective action in a similar way, as

$$\hat{L} = \frac{i}{2}q(x)\log \det U/U^\dagger, \quad q(x) = \frac{g^2}{32\pi^2}F_{\mu\nu}^a\tilde{F}^{a,\mu\nu}. \quad (6.1)$$

(See Sec. 5 for more careful discussions.) And in all cases this implies condensation of $\langle U \rangle \propto \mathbf{1}$, i.e. the color-flavor-locked Higgs phase in the $\psi\eta$ or $\chi\eta$ models on the one hand, and the chiral-symmetry broken vacuum in QCD., on the other.

Another fact pointing to a similarity between massless QCD and BY and GG models is the following. In general, BY models (with p pairs of additional matter fermions), we saw that there are two natural bifermion condensate channels:

$$\begin{aligned} & \psi \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right) \eta \left(\begin{array}{|c|} \hline \bar{\square} \\ \hline \end{array} \right) \text{ forming } \square, \\ \text{and} \quad & \xi \left(\begin{array}{|c|} \hline \square \\ \hline \end{array} \right) \eta \left(\begin{array}{|c|} \hline \bar{\square} \\ \hline \end{array} \right) \text{ forming } (\cdot) : \end{aligned} \tag{6.2}$$

the gluon-exchange strengths in the two channels are, respectively, proportional to $-\frac{(N+2)(N-1)}{N}$ and $-\frac{N^2-1}{N}$. The $\psi\eta$ channel is slightly more attractive; the strength is however identical in the large N limit. Note also that $\xi\eta$ has the same quantum numbers as $\bar{\psi}_R\psi_L$ in QCD. Similarly for the comparison between the condensates, $\langle\chi\eta\rangle$ and $\langle\xi\eta\rangle$ in the GG models. These considerations, based on rather naïve MAC³ idea and thus are not rigorous, nevertheless give support to the idea that the quark condensates in QCD and the bifermions condensates in the chiral gauge theories under study in this review, are really on a very similar footing.

Of course, there are significant differences, or contrast, in the vectorlike and chiral gauge theories. The quark condensate $\langle\bar{\psi}_R\psi_L\rangle$ is a color singlet, $SU(N_f)_L \times SU(N_f)_R$ flavor matrix. $\langle\psi\eta\rangle$ is instead in a color-flavor bifundamental form, which means that it breaks the color completely, and reduces (partially or totally) the unbroken flavor symmetry. The most important difference however is the existence of colored NG bosons in the $\psi\eta$ (or in the $\chi\eta$) models. It means that these are coupled linearly to the gauge boson fields, making them massive. These processes are absent in QCD, as all NG bosons are color singlets. It is in this sense that one talks about confinement phase in QCD, in spite of the fact that the inter-quark confining strings can be broken by the spontaneous quark-pair production from the vacuum.

The mass spectra are also qualitatively different in QCD and in the chiral gauge theories discussed here. One is the presence of certain degenerate massive vector bosons (the color-flavor locked $SU(N)_{\text{cf}}$ symmetry) found in the chiral gauge theories in the Higgs phase. But especially, the *massless spectrum* exhibits striking differences. In all chiral gauge theories studied here, it contains in general *both* a number of composite fermions (baryons) as well as some composite scalars (pions), a feature certainly not shared by massless QCD. In other words, the way the chiral symmetries of the theory are realized in the IR, is notably different, in vector-like and chiral gauge theories.

It is possible to see a closer analogy—from a formal point of view—between the vector-like theories and chiral theories, if one considers color superconductivity phase in the high-density region of QCD.^{95,96} The dynamics of QCD in that phase is believed to be such that some colored di-quark condensates form.

$$\langle\psi_L\psi_L\rangle \neq 0, \quad \langle\psi_R\psi_R\rangle \neq 0. \tag{6.3}$$

In particular, in the case with $N_f = 3$ flavors these are condensates of color-flavor diagonal form, showing some similarity to $\langle\psi\eta\rangle$ or $\langle\chi\eta\rangle$ in the chiral theories discussed here. Of course, the details of the dynamics will be quite different.

Summarizing, the implications of new, mixed anomalies and the associated stricter anomaly matching constraints reviewed in this paper, and the consideration of the strong-anomaly effective actions, together, seem to allow us to get a clearer picture of the infrared dynamics of many strongly coupled chiral gauge theories than before. It is to be seen whether some of these developments will turn out to be useful in a future effort to construct a realistic theory of Nature beyond the standard Glashow–Weinberg–Salam-QCD model of the fundamental interactions.

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Appendix A. Chirally Symmetric Confining $\psi\eta$ Model

An interesting possibility pointed out for this model is that no condensates form, the system confines and the flavor symmetry remains unbroken.⁴ The candidate massless degrees of freedom in the IR are

$$B^{[AB]} = \psi^{ij} \eta_i^A \eta_j^B, \quad A, B = 1, 2, \dots, N + 4, \tag{A.1}$$

(baryons), anti-symmetric in $A \leftrightarrow B$. All of the $SU(N + 4)_f \times U(1)$ anomaly triangles are saturated by $B^{[AB]}$, see Table A.1.

Appendix B. Higgs Phase of the $\psi\eta$ Model

Another possibility for the $\psi\eta$ model is that a color-flavor locked phase appears,^{11,17} with

$$\langle \psi^{\{ij\}} \eta_i^B \rangle = c_{\psi\eta} \Lambda^3 \delta^{jB}, \quad j, B = 1, 2, \dots, N, \tag{B.1}$$

where the symmetry is reduced to

$$SU(N)_{\text{cf}} \times SU(4)_f \times U(1)'. \tag{B.2}$$

Table A.1. Chirally symmetric “confining” phase of the $\psi\eta$ model. As in other tables of the text, the multiplicity, charges and the representation are shown for each set of fermions. (·) stands for a singlet representation.

	Fields	$SU(N)_c$	$SU(N + 4)$	$U_{\psi\eta}(1)$
UV	ψ	$\square \square$	$\frac{N(N+1)}{2} \cdot (\cdot)$	$N + 4$
	η^A	$(N + 4) \cdot \overline{\square}$	$N \cdot \square$	$-(N + 2)$
IR	$B^{[AB]}$	$\frac{(N+4)(N+3)}{2} \cdot (\cdot)$	\square	$-N$

Table B.1. Color-flavor locked phase in the $\psi\eta$ model. A_1 or B_1 stand for the first N flavors ($A_1, B_1 = 1, 2, \dots, N$), whereas A_2 or B_2 run over the rest of the flavor indices, $N + 1, \dots, N + 4$. Another set of potentially massless baryons $B^{[A_2 B_2]}$ do not contribute to $SU(N)_{\text{cf}} \times SU(4)_f \times U'(1)$ anomalies.

	Fields	$SU(N)_{\text{cf}}$	$SU(4)_f$	$U'(1)$
UV	ψ		$\frac{N(N+1)}{2} \cdot (\cdot)$	1
	η^{A_1}		$N^2 \cdot (\cdot)$	-1
	η^{A_2}	$4 \cdot$	$N \cdot$	$-\frac{1}{2}$
IR	$B^{[A_1 B_1]}$		$\frac{N(N-1)}{2} \cdot (\cdot)$	-1
	$B^{[A_1 B_2]}$	$4 \cdot$	$N \cdot$	$-\frac{1}{2}$
	$B^{[A_2 B_2]}$	$6 \cdot (\cdot)$		0

A subset of the same baryons ($B^{[A_1 B_1]}$ and $B^{[A_1 B_2]}$ in the notation of Table B.1) saturate all of the triangles for (B.2), see Table B.1. The massless degrees of freedom are $\frac{N^2+7N}{2}$ massless baryons $B^{[A_1 B]}$ and $8N + 1$ NG bosons.

To reproduce correctly the strong anomaly in the IR, however, another condensate $\langle BB \rangle \sim \langle \psi\eta\eta\psi\eta \rangle$ and another set of massless baryon $B^{[A_2 B_2]}$ (see Table B.1) are needed. This does not alter neither the symmetry breaking pattern (B.2), nor the anomaly matching. See Sec. 5.2 for more details.

Appendix C. Symmetric Confining Phase for the $\chi\eta$ Model

Let us first examine the possible confining vacua, with full unbroken global symmetry.⁴ The massless baryons needed for 't Hooft's anomaly matching are

$$B^{\{CD\}} = \chi_{[ij]} \eta^{iC} \eta^{jD}, \quad C, D = 1, 2, \dots, (N - 4), \tag{C.1}$$

symmetric in $C \leftrightarrow D$.

Table C.1. Confinement and unbroken symmetry in the $\chi\eta$ model.

	Fields	$SU(N)_c$	$SU(N - 4)$	$U(1)_{\chi\eta}$
UV	χ		$\frac{N(N-1)}{2} \cdot (\cdot)$	$N - 4$
	η^A	$(N - 4) \cdot$	$N \cdot$	$-(N - 2)$
IR	$B^{\{AB\}}$	$\frac{(N-4)(N-3)}{2} \cdot (\cdot)$		$-N$

Appendix D. Higgs Phase of the $\chi\eta$ Model

It was pointed out ^{11,17} that this system may instead develop a color-flavor locked condensate

$$\langle \chi_{[ij]} \eta^{Bj} \rangle = \text{const} \Lambda^3 \delta_i^B, \quad i, B = 1, 2, \dots, N - 4. \tag{D.1}$$

The symmetry is broken to

$$SU(N - 4)_{\text{cf}} \times U(1)' \times SU(4)_c. \tag{D.2}$$

The massless baryons (C.1) saturate all the anomalies associated with $SU(N - 4)_{\text{cf}} \times U(1)'$. There are $\chi_{i_2 j_2}$ fermions which remain massless and strongly coupled to the $SU(4)_c$. We assume that $SU(4)_c$ confines, and the condensate

$$\langle \chi\chi \rangle \neq 0, \tag{D.3}$$

forms and $\chi_{i_2 j_2}$ acquire mass dynamically. Assume that the massless baryons are

$$B^{\{AB\}} = \chi_{[ij]} \eta^{iA} \eta^{jB}, \quad A, B = 1, 2, \dots, (N - 4), \tag{D.4}$$

the saturation of all of the triangles associated can be seen in Table D.1.

Complementarity⁹² appears to be working here, in the sense that the massless sector of the dynamical Higgs phase has the same $SU(N - 4) \times U(1)$ symmetry as in the confining phase discussed in App. C. See however a discussion in Ref. 91 on this point, which seems to point to the conclusion that the apparent complementarity (which occurs *only* in the $\chi\eta$ model, but in none of other BY and GG models) is just a coincidence.

Table D.1. Color-flavor locking in the $\chi\eta$ model. The color index i_1 or j_1 runs up to $N - 4$ and the rest is indicated by i_2 or j_2 .

	Fields	$SU(N - 4)_{\text{cf}}$	$U'(1)$	$SU(4)_c$
UV	$\chi_{i_1 j_1}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$	N	$\frac{(N-4)(N-5)}{2} \cdot (\cdot)$
	$\chi_{i_1 j_2}$	$4 \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$\frac{N}{2}$	$(N - 4) \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$
	$\chi_{i_2 j_2}$	$\frac{4 \cdot 3}{2} \cdot (\cdot)$	0	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$
	$\eta^{i_1, A}$	$\begin{array}{ c c } \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{ c } \hline \square \\ \hline \end{array}$	$-N$	$(N - 4)^2 \cdot (\cdot)$
	$\eta^{i_2, A}$	$4 \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$-\frac{N}{2}$	$(N - 4) \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$
IR	$B^{\{AB\}}$	$\begin{array}{ c c } \hline \square & \square \\ \hline \end{array}$	$-N$	$\frac{(N-4)(N-3)}{2} \cdot (\cdot)$

Appendix E. Confining Symmetric Phase of the BY Models

The candidate massless composite fermions for the BY models are

$$(B_1)^{[AB]} = \psi^{ij} \eta_i^A \eta_j^B, \quad (B_2)_A^a = \bar{\psi}_{ij} \bar{\eta}^i_A \xi^{j,a}, \quad (B_3)_{\{ab\}} = \psi^{ij} \bar{\xi}_{i,a} \bar{\xi}_{j,b}, \quad (E.1)$$

the first is anti-symmetric in $A \leftrightarrow B$ and the third is symmetric in $a \leftrightarrow b$, see Table E.1.

Explicit anomaly matching checks can be found in e.g. Ref. 69.

Table E.1. Chirally symmetric phase of the BY model.

	$SU(N)_c$	$SU(N+4+p)$	$SU(p)$	$U(1)_{\psi\eta}$	$U(1)_{\psi\xi}$
B_1	$\frac{(N+4+p)(N+3+p)}{2} \cdot (\cdot)$		$\frac{(N+4+p)(N+3+p)}{2} \cdot (\cdot)$	$-N+p$	p
B_2	$(N+4+p)p \cdot (\cdot)$	$p \cdot$	$(N+4+p) \cdot$	$-(p+2)$	$-(N+p+2)$
B_3	$\frac{p(p+1)}{2} \cdot (\cdot)$	$\frac{p(p+1)}{2} \cdot (\cdot)$		$N+4+p$	$2N+4+p$

Appendix F. Higgs Phase in the BY Models

Something nontrivial happens in the dynamical Higgs phase for general BY models. i.e. with $p > 0$. There are now two symmetry breaking channels, $\psi\eta$ and $\xi\eta$. We assume that both condensates occur as

$$\begin{aligned} \langle \psi^{ij} \eta_i^B \rangle &= c_{\psi\eta} \Lambda^3 \delta^{jB} \neq 0, \quad j, B = 1, \dots, N, \\ \langle \xi^{i,a} \eta_i^A \rangle &= c_{\eta\xi} \Lambda^3 \delta^{aA} \neq 0, \quad a = 1, \dots, N, \quad A = N+1, \dots, N+p, \end{aligned} \quad (F.1)$$

where Λ is the renormalization-invariant scale dynamically generated by the gauge interactions and $c_{\eta\xi}, c_{\psi\eta}$ are coefficients both of order one. The resulting symmetry

Table F.1. UV fields in the BY model, decomposed as a direct sum of the representations of the unbroken group of Eq. (F.2).

	$SU(N)_{c\eta}$	$SU(4)_\eta$	$SU(p)_{\eta\xi}$	$U(1)'_{\psi\eta}$	$U(1)'_{\psi\xi}$
ψ		$\frac{N(N+1)}{2} \cdot (\cdot)$	$\frac{N(N+1)}{2} \cdot (\cdot)$	$N+4+p$	p
η_1		$N^2 \cdot (\cdot)$	$N^2 \cdot (\cdot)$	$-(N+4+p)$	$-p$
η_2	$4 \cdot$	$N \cdot$	$4N \cdot (\cdot)$	$-\frac{N+p+4}{2}$	$-\frac{p}{2}$
η_3	$p \cdot$	$Np \cdot (\cdot)$	$N \cdot$	0	$N+2$
ξ	$p \cdot$	$Np \cdot (\cdot)$	$N \cdot$	0	$-(N+2)$

Table F.2. IR fields in the BY model, the massless subset of the baryons in Table E.1 in the Higgs phase.

	$SU(N)_{\text{cf}\eta}$	$SU(4)_\eta$	$SU(p)_{\eta\xi}$	$U(1)'_{\psi\eta}$	$U(1)'_{\psi\xi}$
B_1	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$	$\frac{N(N-1)}{2} \cdot (\cdot)$	$\frac{N(N-1)}{2} \cdot (\cdot)$	$-(N+4+p)$	$-p$
B_2	$4 \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$N \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$4N \cdot (\cdot)$	$-\frac{N+p+4}{2}$	$-\frac{p}{2}$

breaking pattern s

$$SU(N)_c \times SU(N+4+p)_\eta \times SU(p)_\xi \times U(1)_{\psi\eta} \times U(1)_{\psi\xi} \xrightarrow{(\xi\eta), (\psi\eta)} SU(N)_{\text{cf}\eta} \times SU(4)_\eta \times SU(p)_{\eta\xi} \times U(1)'_{\psi\eta} \times U(1)'_{\psi\xi}. \quad (\text{F.2})$$

The color gauge symmetry is completely (dynamically) broken, leaving color-flavor diagonal $SU(N)_{\text{cf}\eta}$ symmetry. $U(1)'_{\psi\eta}$ and $U(1)'_{\psi\xi}$ are combinations, respectively, of $U(1)_{\psi\eta}$ (4.6) and $U(1)_{\xi\eta}$ (4.7) with the element of $SU(N+4+p)_\eta$ generated by

$$t_{SU(N+4+p)_\eta} = \left(\begin{array}{c|c|c} (-\alpha(p+2) - p\beta)\mathbf{1}_{N \times N} & & \\ \hline & \frac{\alpha(N-p) - \beta p}{2} \mathbf{1}_{4 \times 4} & \\ \hline & & (\alpha + \beta)(N+2)\mathbf{1}_{p \times p} \end{array} \right). \quad (\text{F.3})$$

Making the decomposition of the fields one gets Table F.1.

The composite massless baryons (see Table F.2) are subset of those in (E.1)

$$B_1^{[AB]} = \psi^{ij} \eta_i^A \eta_j^B, \quad B_2^{[AC]} = \psi^{ij} \eta_i^A \eta_j^C, \quad A, B = 1, \dots, N, \quad C = N+1, \dots, N+4. \quad (\text{F.4})$$

Appendix G. Confining Symmetric Phase of the GG Models

The candidate massless composite fermions for the generalized Georgi–Glashow models are

$$(B_1)^{\{AB\}} = \chi^{ij} \eta_i^A \eta_j^B, \quad (B_2)_A^a = \bar{\chi}_{ij} \bar{\eta}_i^j \xi^{j,a}, \quad (B_3)_{[ab]} = \chi^{ij} \bar{\xi}_{i,a} \bar{\xi}_{j,b}, \quad (\text{G.1})$$

the first symmetric in $A \leftrightarrow B$ and the third anti-symmetric in $a \leftrightarrow b$.

All anomaly triangles are saturated by these candidate massless composite fermions (Table G.1). For explicit matching equations, see e.g. Ref. 69.

Table G.1. IR massless fermions in the chirally symmetric phase of the GG model.

	$SU(N)_c$	$SU(N-4+p)$	$SU(p)$	$U(1)_{\chi\eta}$	$U(1)_{\chi\xi}$
B_1	$\frac{(N-4+p)(N-3+p)}{2} \cdot (\cdot)$	$\begin{array}{ c c } \hline & \\ \hline \end{array}$	$\frac{(N-4+p)(N-3+p)}{2} \cdot (\cdot)$	$-N+p$	p
B_2	$(N-4+p)p \cdot (\cdot)$	$p \cdot \begin{array}{ c } \hline \bar{} \\ \hline \end{array}$	$(N-4+p) \cdot \begin{array}{ c } \hline \\ \hline \end{array}$	$-(p-2)$	$-(N+p-2)$
B_3	$\frac{p(p-1)}{2} \cdot (\cdot)$	$\frac{p(p-1)}{2} \cdot (\cdot)$	$\begin{array}{ c } \hline \bar{} \\ \hline \\ \hline \end{array}$	$N-4+p$	$2N-4+p$

Appendix H. Higgs Phase in the GG Models

The Higgs phase of the GG model may be described by either of the two possible bifermion channels $\chi\eta$ and $\xi\eta$. We assume that both occur

$$\begin{aligned} \langle \chi^{ij} \eta_i^A \rangle &= c_{\chi\eta} \Lambda^3 \delta^{jA} \neq 0, \quad j = 1, \dots, N-4, \quad A = 1, \dots, N-4, \\ \langle \xi^{i,a} \eta_i^B \rangle &= c_{\eta\xi} \Lambda^3 \delta^{aB} \neq 0, \quad a = 1, \dots, p, \quad B = N-4+1, \dots, N-4+p. \end{aligned} \quad (\text{H.1})$$

The symmetry-breaking pattern is

$$\begin{aligned} &SU(N)_c \times SU(N-4+p)_\eta \times SU(p)_\xi \times U(1)_{\chi\eta} \times U(1)_{\chi\xi} \\ &\xrightarrow{(\xi\eta), (\chi\eta)} SU(4)_c \times SU(N-4)_{\text{cf}_\eta} \times SU(p)_{\eta\xi} \times U(1)'_{\chi\eta} \times U(1)'_{\chi\xi}. \end{aligned} \quad (\text{H.2})$$

The color gauge symmetry is partially (dynamically) broken, leaving color-flavor diagonal global $SU(N-4)_{\text{cf}_\eta}$ symmetry and an $SU(4)_c$ gauge symmetry. $U(1)'_{\chi\eta}$ and $U(1)'_{\chi\xi}$ are a combinations, respectively, of $U(1)_{\chi\eta}$ ((2.16) of Ref. 69) and $U(1)_{\chi\xi}$ ((2.17) of Ref. 69) with the elements of $SU(N)_c$ and $SU(N-4+p)_\eta$ generated by

$$\begin{aligned} t_{SU(N)_c} &= \left(\begin{array}{c|c} 2 \frac{\alpha(N-4+p)+\beta p}{N-4} \mathbf{1}_{(N-4) \times (N-4)} & \\ \hline & -\frac{\alpha(N-4+p)+\beta p}{2} \mathbf{1}_{4 \times 4} \end{array} \right), \\ t_{SU(N-4+p)_\eta} &= \left(\begin{array}{c|c} -\frac{p(\alpha+\beta)(N-2)}{N-4} \mathbf{1}_{(N-4) \times (N-4)} & \\ \hline & (\alpha+\beta)(N-2) \mathbf{1}_{p \times p} \end{array} \right). \end{aligned} \quad (\text{H.3})$$

Making the decomposition of the fields in the direct-sum representations in the subgroup one arrives at Table H.1.

The composite massless baryons (Table H.2) are subset of those in (G.1):

$$B^{\{AB\}} = \chi^{ij} \eta_i^A \eta_j^B, \quad A, B = 1, \dots, N-4. \quad (\text{H.4})$$

In the IR, these fermions saturate all the anomalies of the unbroken chiral symmetry.

Table H.1. UV fields in the GG model, decomposed as a direct sum of the representations of the unbroken group of Eq. (H.2).

	$SU(N-4)_{\text{cf}\eta}$	$SU(4)_c$	$SU(p)_{\eta\xi}$	$U(1)'_{\chi\eta}$	$U(1)'_{\chi\xi}$
χ_1	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	$\frac{(N-4)(N-5)}{2} \cdot (\cdot)$	$\frac{(N-4)(N-5)}{2} \cdot (\cdot)$	$\frac{(N-4+p)N}{(N-4)}$	$p \frac{N}{N-4}$
χ_2	$4 \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$(N-4) \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$4(N-4) \cdot (\cdot)$	$\frac{(N-4+p)N}{2(N-4)}$	$\frac{pN}{2(N-4)}$
χ_3	$6 \cdot (\cdot)$	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	$6 \cdot (\cdot)$	0	0
η_1	$\begin{array}{ c } \hline \square \\ \hline \end{array} \oplus \begin{array}{ c } \hline \square \\ \hline \end{array}$	$(N-4)^2 \cdot (\cdot)$	$(N-4)^2 \cdot (\cdot)$	$-\frac{(N-4+p)N}{(N-4)}$	$-\frac{pN}{N-4}$
η_2	$p \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$p(N-4) \cdot (\cdot)$	$(N-4) \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$-2 - 2 \frac{p}{N-4}$	$N - 2 - \frac{2p}{N-4}$
η_3	$4 \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$(N-4) \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$4(N-4) \cdot (\cdot)$	$-\frac{(N-4+p)N}{2(N-4)}$	$-\frac{pN}{2(N-4)}$
η_4	$4p \cdot (\cdot)$	$p \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$4 \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$\frac{N-4+p}{2}$	$N - 2 + \frac{p}{2}$
ξ_1	$p \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$p(N-4) \cdot (\cdot)$	$(N-4) \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$2 + 2 \frac{p}{N-4}$	$-(N-2) + \frac{2p}{N-4}$
ξ_2	$4p \cdot (\cdot)$	$p \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$4 \cdot \begin{array}{ c } \hline \square \\ \hline \end{array}$	$-\frac{N-4+p}{2}$	$-(N-2) - \frac{p}{2}$

Table H.2. IR field in the GG model in the dynamical Higgs phase.

	$SU(N-4)_{\text{cf}\eta}$	$SU(p)_{\eta\xi}$	$U(1)'_{\chi\eta}$	$U(1)'_{\chi\xi}$
B	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	$\frac{(N-4)(N-3)}{2} \cdot (\cdot)$	$-\frac{(N-4+p)N}{(N-4)}$	$-\frac{pN}{N-4}$

There is a novel feature in the GG models, which is not shared by the BY models. There is an unbroken strong gauge symmetry $SU(4)_c$, with a set of fermions

$$\chi_3, \chi_2, \eta_3, \eta_4, \xi_2, \tag{H.5}$$

charged with respect to it. The pairs $\{\chi_2, \eta_3\}$ and $\{\eta_4, \xi_2\}$ can form massive Dirac fermions and decouple. These are vectorlike with respect to the surviving infrared symmetry, (H.2), thus irrelevant to the anomalies.

The fermion χ_3 can condense

$$\langle \chi_3 \chi_3 \rangle \tag{H.6}$$

forming massive composite mesons, $\sim \chi_3 \chi_3$, which also decouple. It is again neutral with respect to the unbroken symmetry. To summarize, $SU(4)_c$ is invisible—confines—in the IR, and only the unpaired part of the η_1 fermion ($\begin{array}{|c|} \hline \square \\ \hline \end{array}$) remains massless. Its anomalies are reproduced by the composite fermions (H.4).

The massive mesons $\chi_2\eta_3$, $\eta_4\xi_2$, $\chi_3\chi_3$ are not charged with the flavor symmetries surviving in the infrared. It is tempting to consider them as a sort of “dark matter”, as contrasted to the fermions B^{AB} which constitute the ordinary, “visible” sector, in a toy-model interpretation.

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