SIAM J. Imaging Sciences
Vol. 8, No. 1, pp. 19–66

Banach-Like Distances and Metric Spaces of Compact Sets*

A. C. G. Mennucci† and A. Duci†

Abstract. In the first part we study general properties of the metrics obtained by isometrically identifying a generic metric space with a subset of a Banach space; we obtain a rigidity result. We then discuss the Hausdorff distance, proposing some less-known but important results: a closed-form formula for geodesics; generically two compact sets are connected by a continuum of geodesics. In the second part we present and study a family of distances on the space of compact subsets of $\mathbb{R}^N$ (that we call "shapes"). These distances are "geometric," that is, they are independent of rotation and translation, and the resulting metric spaces enjoy many interesting properties, as, for example, the existence of geodesics. We view our metric space of shapes as a subset of Banach (or Hilbert) spaces, so we can define a "tangent manifold" to shapes and (in a very weak form) talk of a "Riemannian geometry" of shapes. Some of the metrics that we propose are topologically equivalent to the Hausdorff distance.

Key words. shape, Hausdorff distance, Banach space, compact set, isometric embedding, geodesic

AMS subject classifications. 54E35, 58B20, 58D15, 46B85

DOI. 10.1137/140972512

1. Introduction.

In recent years the study of *shape spaces* has garnered wide interest, in particular within the computer vision community.

There are two different (but interconnected) fields of applications for a good shape space in computer vision:

Shape optimization, where we want to find the shape that best satisfies a design goal, a topic of interest in engineering at large.

Shape analysis, where we study a family of shapes for purposes of statistics, (automatic) cataloging, probabilistic modeling, among others—possibly also to create an a priori model for better shape optimization.

To achieve the above, some structure is clearly needed on the shape space so that our goals can be studied and the problem solved.

1.1. Shape spaces.

A common way to model shapes is by *representation/embedding*:

- we *represent* the shape A by a function u_A;
- then we *embed* this representation in a space E so that we can operate on the shapes A by operating on the representations u_A.

Most often, this representation/embedding scheme does not directly provide a shape space satisfying all desired properties. In particular, in many cases it happens that the representation is "redundant," that is, the same shape has many different possible representations. An

*Received by the editors June 11, 2014; accepted for publication (in revised form) November 4, 2014; published electronically January 6, 2015. A preliminary version of this paper appeared as preprint arXiv:0707.1174v1.
http://www.siam.org/journals/siims/8-1/97251.html

†Scuola Normale Superiore, Pisa 56100, Italy (andrea.mennucci@sns.it, alessandro.duci@gmail.com). The second author was supported by the EU 6th Framework Programme through grant MRTN-CT-2005-019481.

appropriate *quotient* is then introduced.

There are many examples of shape spaces in the literature that are studied by means of the *representation/embedding/quotient* scheme. We review two such examples.

1. The space of embedded curves. When studying embedded curves, usually, for the sake of mathematical analysis, the curves are modeled as smooth immersed parametric curves; a quotient with respect to (w.r.t.) the group of possible reparameterizations of the curve c (that coincides with the group of smooth diffeomorphisms $\mathrm{Diff}(S^1)$) is applied afterward to all the mathematical structures that are defined (such as the manifold of curves, the Riemannian metric, the induced distance, etc.). The resulting Riemannian spaces of embedded curves have been studied by Michor, Mumford, and coauthors [17, 18, 30] and Yezzi and Mennucci [29, 28], and more recently by Mennucci, Yezzi, and Sundaramoorthi [16] and Sundaramoorthi et al. [23, 20, 24, 25, 26, 27, 21, 22].
2. The family $\mathcal{M}$ of all nonempty compact subsets of $\mathbb{R}^N$. This is the shape space that we will study in the main part of this paper.
   A standard representation is obtained by associating a closed subset A to the *distance function*

   (1) $$u_A(x) \overset{\text{def}}{=} \inf_{y\in A} |x-y|$$

   or the *signed distance function*

   (2) $$b_A(x) \overset{\text{def}}{=} u_A(x) - u_{\mathbb{R}^N\setminus A}(x) \ .$$

   See section 4 for a list of properties of u_A.
   We may then define a *topology of shapes* by deciding that $A_n \to A$ when $u_{A_n} \to u_A$ uniformly on compact sets. This convergence coincides with the Kuratowski topology of closed sets (see section 5.3).
   We may also operate *"linearly"* on shapes by operating on u_A or b_A. So we may define *shape averages* and *shape principal component analysis*; see [11] and (4) here.
   When this shape space is used for shape analysis, a *registration* of the shapes to a common pose is often performed, or a quotient is enacted, as explained in section 2.1.1.

1.2. Goals. To a certain degree, our theory should be independent of rotation and translation; that is, whatever we do with shapes should not depend on "where in the space" we do it.

In the rest of the paper we will denote by $\mathcal{M}$ the family of the nonempty compact sets in $\mathbb{R}^N$ and build many examples of metrics d on $\mathcal{M}$. We will always require these metrics to be *Euclidean invariant*: if R is a Euclidean transformation of the space (a rigid transformation), then

(3) $$d(R\Omega_1, R\Omega_2) = d(\Omega_1, \Omega_2) \ .$$

What other properties and operations may be interesting for applications?

1.2.1. Means and averages. As mentioned before, a goal of shape analysis is to define *shape metrics*, *shape averages*, *shape principal component analysis*, *shape probabilities*, etc.

For example, if we represent shapes A_j, $j = 1, \dots, n$, by their *signed distance function* b_{A_j}, then we may define *signed distance level set averaging* as

$$\bar{A} = \Big\{ x \,|\, f(x) \le 0 \Big\}, \text{ where } f(x) = \frac{1}{N} \sum_{n=1}^{N} b_{A_n}(x) \ . \tag{4}$$

(Note that in general a linear combination of (signed) distance functions will not be a (signed) distance function.) A benefit of this definition is that it is easily computable; a defect is that if the shapes are far away, then $\bar{A}$ will be empty. Another defect is that this definition is quite ad hoc: it is not coupled with any other structure that we may wish to add to the shape space, such as a metric d. We may then look at the problem from a different point of view.

Considering a generic metric space (M, d), define the *distance-based averaging*[1] of any given collection $a_1 \dots a_n \in M$, as a minimum point $\bar{a}$ of the sum of its squared distances:

$$\bar{a} = \arg\min_{a} \sum_{j=1}^{n} d(a, a_j)^2 \ . \tag{5}$$

Supposing now that the shape space $\mathcal{M}$ is given a metric d, we can use the abstract definition above to define *shape averages*; this definition has many advantages.

- It comes from a minimality criterion, so it is *"optimal"* in a certain sense (contrary to the definition in (4)).
- If the distance is invariant w.r.t. a group action, then the *shape average* is invariant as well (see section 2.1.1). For example, in the case of *geometric curves*, where the distance is independent of parameterization, then the *shape average* will be independent of the parameterization of $a_1 \dots a_n$.
- Suppose that $\mathcal{M}$ is a smooth Riemannian manifold and that d is the distance derived from the metric; then, when $a_1 \dots a_n$ are near enough, $\bar{a}$ exists and is unique [10].
- It coincides with the arithmetic mean in Euclidean spaces; more generally, when $\mathcal{M}$ is a smooth submanifold of a Hilbert space and $a_1 \dots a_n$ are near enough, then $\bar{a}$ is an approximation of the arithmetic mean.

1.2.2. Averages, midpoints, and geodesics. Let (M, d) be a metric space. A *geodesic* is a continuous path connecting x to y that has minimum length in the class of all such paths. The metric space (M, d) is *intrinsic* if the distance $d(x, y)$ between $x, y \in M$ is equal to the infimum of the length of all continuous paths connecting x to y. See section 2.1 for details.

Definition 1.1 (midpoint). *Let $x, y \in M$; a point $z \in M$ such that*

$$d(z, y) = d(z, x) = \frac{1}{2} d(x, y)$$

is called a midpoint.

[1] Also known as the *Karcher mean* due to the seminal work [10], but it is also sometimes attributed to Fréchet, in 1948.

It is easily verified that if (M, d) is *intrinsic* and x, y are connected by a geodesic, a point halfway through the geodesic is a midpoint; see Lemma 2.4.8 in section 2.4.3 in [5]. In contrast, suppose (M, d) is complete and that for every $x, y \in M$ there exists a midpoint z; then d is intrinsic, and every two points in M may be joined by a geodesic; see Theorem 2.4.16 in section 2.4.4 in [5].

Consider now a shape space that is a complete finite-dimensional Riemannian metric; let d be the distance between shapes; then the average shape A of two shapes A_1, A_2 (as defined in (5) above) is also a *midpoint*.

The above shows that averages, midpoints, and geodesics are deeply linked.

For this reason, we will end up studying whether the shape space admits geodesics.

1.2.3. Motions and tangent spaces. Many operations performed in shape optimization may be related to these concepts and operations:

- given the motion of a shape, we would like to define its derivative, that is, the *infinitesimal motion*;
- given a vector field of *infinitesimal motions*, we would like to be able to flow shapes according to the field;
- the family of all such *infinitesimal motions* should define a *tangent space* to the shape space.

One easy way to define all the above is again by representation/embedding: if we embed the shape space in a vector space E, then we can define the *infinitesimal motions* as vectors in E. At the same time, if E is a Banach space with norm $\|\cdot\|$, we can define a *distance of shapes* simply by $d(A, B) \stackrel{\text{def}}{=} \|u_A - u_B\|$ (so that the embedding $A \mapsto u_A$ is isometric). For this reason, in the first part of the paper we will study general properties of isometric embeddings of metric spaces into Banach spaces.

1.3. The proposed framework. In this paper we will study the family $\mathcal{M}$ of all nonempty compact subsets of $\mathbb{R}^N$.

Having fixed a decreasing smooth function $\varphi : [0, \infty) \to (0, \infty)$, we will define in (31) the L^p-like distance of compact sets by

$$d_{p,\varphi}(A, B) \stackrel{\text{def}}{=} \|\varphi \circ u_A - \varphi \circ u_B\|_{L^p} \ .$$

Under appropriate hypotheses on φ, we will prove that this distance satisfies the requirements listed in the previous sections, and then some, as follows.

- The metric space $(\mathcal{M}, d_{p,\varphi})$ is complete (Proposition 6.12).
- The mapping $A \mapsto \varphi \circ u_A$ associates isometrically $\mathcal{M}$ to a closed subset of $L^p(\mathbb{R}^N)$.
- $d_{p,\varphi}$ is Euclidean invariant.
- $d_{p,\varphi}$ induces a well-defined distance on the shape space of compact sets up to Euclidean transformation (Proposition 6.14).
- Compact sets can be connected by minimizing geodesics; the *geodesic distance-based averaging* of shapes exists (Theorem 6.20).
- The metric spaces $(\mathcal{M}, d^g_{p,\varphi})$, $(\mathcal{M}, d_{p,\varphi})$, and $(\mathcal{M}, d_H)$ have the same topology. Here $d^g_{p,\varphi}$ is the distance induced by the length of paths in $(\mathcal{M}, d_{p,\varphi})$ and d_H is the well-known Hausdorff distance of compact sets (Theorems 6.11 and 6.30).

- Certain motions can be infinitesimally represented by vectors in $L^p(\mathbb{R}^N)$. In particular, to any Lipschitz path $\gamma(t) : \mathbb{R} \to \mathcal{M}$ of compact sets in $(\mathcal{M}, d_{p,\varphi})$ we can associate the path $f : \mathbb{R} \to L^p(\mathbb{R}^N)$ by $f(t,x) = \varphi(u_{\gamma(t)}(x))$; then we can represent (for almost all t) the motion of $\gamma(t)$ by the weak partial derivative $\partial_t f$ (see section 6.4).
- In the case $p = 2, N = 2$, for compact sets with smooth boundary, the metric can be explained as a Riemannian metric of deformations of the boundary (see section 6.6).

The last two properties are what distinguishes this framework from the Hausdorff distance of compact sets.

1.4. Plan of the paper. The plan of the paper is as follows.

In section 2 we foremost review the theory of metric spaces, and provide definitions of the length of a path, the metric derivative, the *induced distance* d^g, and geodesics. We define the action of a group on a metric space and the properties of quotient distances. We propose some properties of metric spaces isometrically embedded in Banach spaces.

In section 3 we list definitions and notation.

In section 4 we review properties of the distance function u_A and the *fattening* of sets.

Considering the space $\mathcal{M}$ of nonempty compact subsets of $\mathbb{R}^N$, in section 5 we define on $\mathcal{M}$ the renowned *Hausdorff distance* d_H and review some of its properties; we also provide some original results, such as a closed-form formula for geodesics and a generic condition for nonuniqueness of geodesics.

In section 6 we discuss the main subject of this paper. We define in (31) the L^p-like distance of compact sets. We prove many properties regarding the metric space $(\mathcal{M}, d_{p,\varphi})$: we prove in Proposition 6.11 that this metric space has the same topology as $(\mathcal{M}, d_H)$, and in Proposition 6.12 that it is complete; adding some more hypotheses on φ, we prove in Theorem 6.20 that any two compact sets may be joined by a geodesic in $(\mathcal{M}, d_{p,\varphi})$. In section 6.4 we show a variational description of geodesics, and in section 6.6 (when $p = 2$, $N = 2$) we use it to describe $(\mathcal{M}, d_{p,\varphi})$ as a weak kind of Riemannian manifold that has an explicit description for sets with smooth boundary. In section 6.7, assuming that φ is convex and $\varphi(|x|) \in W^{1,p}(\mathbb{R}^N)$, we show that the metric space $(\mathcal{M}, d^g_{p,\varphi})$ has the same topology as $(\mathcal{M}, d_{p,\varphi})$ and $(\mathcal{M}, d_H)$, where $d^g_{p,\varphi}$ is the distance induced by the length of paths in $(\mathcal{M}, d_{p,\varphi})$. In section 6.8 we present a simple numerical method for computing geodesics and two results.

We conclude in section 7 by showing some possible further expansions of the presented framework.

2. Metric spaces and embeddings in Banach spaces.

2.1. Metric spaces. We recall some root definitions and results in the abstract theory of metric spaces.

Suppose that (M, d) is a metric space. We will denote with

$$\mathbb{B}(x, \rho) \stackrel{\text{def}}{=} \{x \mid d(x, y) < \rho\}, \tag{6}$$

$$\mathbb{D}(x, \rho) \stackrel{\text{def}}{=} \{x \mid d(x, y) \leq \rho\} \tag{7}$$

the open ball and the closed disc in this metric space; note that in general $\mathbb{D}$ contains the closure of $\mathbb{B}$, but it may be strictly larger.

Definition 2.1. *We induce from d the length* $\operatorname{Len}^d \gamma$ *of a continuous path*

$$\gamma : [\alpha, \beta] \to M$$

by using the total variation

$$\operatorname{Len}^d \gamma \stackrel{\text{def}}{=} \sup_T \sum_{i=1}^n d\big(\gamma(t_{i-1}), \gamma(t_i)\big) \ , \tag{8}$$

where the supremum is computed over all finite subsets $T = \{t_0, \ldots, t_n\}$ *of* $[\alpha, \beta]$ *and* $t_0 \leq \cdots \leq t_n$*. When* $\operatorname{Len}^d \gamma < \infty$ *we will say that* γ *is* rectifiable.

Definition 2.2. *We define the* metric derivative [3, 2]

$$|\dot\gamma|(t) \stackrel{\text{def}}{=} \lim_{s \to 0} \frac{d\big(\gamma(t+s), \gamma(t)\big)}{s} \ . \tag{9}$$

(The above notation does not imply that there is an actual object "$\dot\gamma$" and that the metric derivative is the "norm" of this object—the symbol $|\dot\gamma|$ is atomic.)

The metric derivative enjoys the following properties.

Lemma 2.3.

- *If* γ *is absolutely continuous, then the above limit* (9) *exists for almost all* t*.*
- *For any absolutely continuous* γ*, let*

$$\operatorname{len}^d \gamma \stackrel{\text{def}}{=} \int_\alpha^\beta |\dot\gamma|(t) \, \mathrm{d}t \ ; \tag{10}$$

  then

$$\operatorname{Len}^d \gamma = \operatorname{len}^d \gamma \ .$$

- *If* $\operatorname{Len}^d \gamma < \infty$*, then there exists a continuous monotonic* $\theta : [0, \operatorname{Len}^d(\gamma)] \to [\alpha, \beta]$ *such that for* $c = \gamma \circ \theta$ *we have* $|\dot c| = 1$ *for almost all* θ*. Such a path c is called the* reparameterization to arc parameter *of* γ*.*

See Theorem 1.1.2 and Lemma 1.1.4 in [2], and Theorem 4.1.1 in [3].

Definition 2.4. *We define the* induced distance d^g *by*

$$d^g(x, y) \stackrel{\text{def}}{=} \inf_\gamma \operatorname{Len}^d \gamma, \tag{11}$$

where the infimum is taken in the class of all continuous paths γ *connecting* x *to* y*. If the infimum is a minimum, the path providing the minimum is called a* geodesic.

Note that it may be the case that $d^g(x, y) = \infty$ for some choices of x, y. Note also that $d^g \geq d$.

The topology of (M, d) and (M, d^g) may be quite different, as we see in this example.

Example 2.5. Consider

$$M = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, x_2 = 0\} \cup \bigcup_{n \geq 1} \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, \quad x_2 = x_1/n\}$$

and d the Euclidean distance (see Figure 1). Then (M, d) is compact but (M, d^g) is not.

When $d = d^g$, we will say that the metric space is *path-metric*, or that d is *intrinsic*.

The following results hold.

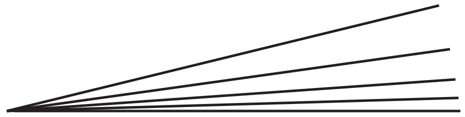


Figure 1. *Example 2.5.*

Proposition 2.6.
- *A path $\gamma : [a,b] \to M$ is continuous and rectifiable in (M,d) iff it is continuous and rectifiable in (M,d^g).*
- *The length Len^{d^g} defined by d^g coincides with Len^d on all such paths.*
- *$d^g = (d^g)^g$, that is, the space (M,d^g) is always intrinsic.*

These results are found in [5] (for the first point, look at Exercises 2.1.4 and 2.1.5 in [5]). We will use the following propositions.

Proposition 2.7. *If for a choice of $\rho > 0$*

$$\mathbb{D}^g(x,\rho) \stackrel{\text{def}}{=} \{x \mid d^g(x,y) \le \rho\} \tag{12}$$

is compact in the (M,d) topology, then x and any $y \in \mathbb{D}^g(x,\rho)$ may be connected by a geodesic.

The proof is simply obtained by the direct method in the calculus of variations (see Theorem 9.2 in [15]).

Proposition 2.8. *Suppose that $a_1 \dots a_n \in M$ are given; a sufficient condition for the existence of the* geodesic distance-based averaging *$\bar a$ of $a_1 \dots a_n$,*

$$\bar a = \mathrm{argmin}_a\, \tau(a)\ ,\ \textit{where}\ \tau(a) \stackrel{\text{def}}{=} \sum_{j=1}^n d^g(a,a_j)^2, \tag{13}$$

is that, defining

$$\rho^* = \min_{i=1,\dots,n} \tau(a_i)\ ,$$

we have that $\rho^* < \infty$ and that $\mathbb{D}^g(a_1, 2\sqrt{\rho^*+\varepsilon})$ is compact in the (M,d) topology for $\varepsilon > 0$ small.

The proof is in section A.2.

A similar proposition can be stated for d.

Proposition 2.9. *Suppose that $a_1 \dots a_n \in M$ are given; let*

$$\rho^* = \min_i \sum_{j=1}^n d(a_i,a_j)^2 \tag{14}$$

and let i^* be the index that achieves the above minimum. Suppose that $\mathbb{D}(a_{i^*}, \sqrt{\rho^*+\varepsilon})$ is compact for $\varepsilon > 0$ small. Then there exists a point $\bar a$ that is the distance-based averaging *of $a_1 \dots a_n$, as defined in (5).*

The proof is similar, so we omit it.

An intrinsic space such that any disc $\mathbb{D}(x,\rho)$ is compact is called *finitely compact* in [6]. Such a space satisfies the hypotheses of the previous propositions. A classical example is given by finite-dimensional complete Riemannian manifolds.

2.1.1. Distances, quotients, and groups. Let $d_M(x,y)$ be a distance on a space M and G a group acting on M. We suppose that d_M is *invariant w.r.t.* G, i.e.,

$$d_M(gx, gy) = d_M(x,y) \quad \forall g \in G \ .$$

(This generalizes the idea of (3).)

A distance d_B may be defined on $B = M/G$ by

$$d_B([x],[y]) = \inf_{x\in[x],y\in[y]} d_M(x,y) = \inf_{g,h\in G} d_M(gx, hy),$$

which is the lowest distance between two orbits; we write $d_B(x,y)$ for simplicity. Since d_M is invariant w.r.t. the group action, d_B coincides with

$$d_B(x,y) = \inf_{g\in G} d_M(gx,y) \ . \tag{15}$$

It is easy to see that d_B satisfies the triangle inequality; but it may be the case that $d_B(x,y) = 0$ even when $x \neq y$. We state a simple sufficient condition.

Lemma 2.10. *If the orbits are compact, then d_B is a distance.*

When studying metrics d on a shape space the quotient is particularly useful in at least two cases.

- For the purpose of shape analysis, shapes are usually intended "up to rotation, translation, and scaling," while in shape optimization each shape has a distinctive position and orientation. For this reason, when we wish to distinguish between the two different ideas of "shape spaces," we will call a space for shape optimization a "*preshape space*."
  When we want to pass from a *preshape space* to a *shape space*, we will apply the quotient above by choosing G to be the Euclidean group of rotations and translation (and sometimes of scaling).
- The second case is when the representation is redundant. In example 1 of embedded curves we proposed in the introduction, we would set $G = \mathrm{Diff}(S^1)$, the family of reparameterizations of the circle.

2.2. Embeddings in Banach spaces. In most of what follows, we will be able to identify M (using an isometry i) with a subset of a Banach space E with norm $\|\cdot\|$. We note that in the following an "isometry" is a map i such that $d(x,y) = \|i(x) - i(y)\|$ (and this should not be confused with the concept of "isometrical immersions of Riemannian manifolds").

2.2.1. Radon–Nikodým property. The following result from [1] will come in handy.

Theorem 2.11. *Suppose that E is the dual of a separable Banach space F. Let $\gamma : [a,b] \to E$ be a Lipschitz path. By Theorem* 8.1 *in* [1], *for almost all t there exists the derivative $\dot\gamma(t)$ that is defined as*

$$\dot\gamma(t) \stackrel{\mathrm{def}}{=} w\text{-}\lim_{\tau\to 0} \frac{\gamma(t+\tau) - \gamma(t)}{\tau} \quad , \tag{16}$$

where the limit is according to the weak- topology; moreover*

$$\|\dot\gamma(t)\| = \lim_{\tau\to 0} \left\| \frac{\gamma(t+\tau) - \gamma(t)}{\tau} \right\| \quad , \tag{17}$$

so $\|\dot\gamma(t)\|$ *coincides with the* metric derivative (9).[2] *We can then define (following* (10)*) the length of* γ *using the integral*

$$\operatorname{len}\gamma \overset{\text{def}}{=} \int_a^b \|\dot\gamma(t)\|\,\mathrm{d}t \ . \tag{18}$$

It follows easily (by applying duality w.r.t. F *in* (16)*) that*

$$\gamma(b)-\gamma(a) = \int_a^b \dot\gamma(t)\,\mathrm{d}t \ , \tag{19}$$

and, by Lemma 2.3*,*

$$\operatorname{Len}^{d_E}\gamma = \operatorname{len}\gamma \ , \tag{20}$$

where $\operatorname{Len}^{d_E}\gamma$ *is the total variation length* (8) *of paths in* E *computed using the usual distance* $d_E(x,y)=\|x-y\|$.

Here is a simple example where the above theorem does not apply. Consider the map $t\mapsto \mathbf{1}_{[t,t+1]}$ in $L^1(\mathbb{R})$. It is Lipschitz, but its derivative should be $t\mapsto \delta_{t+1}-\delta_t$.[3]

2.2.2. The Radon–Nikodým property. It is common to say that E enjoys the *Radon–Nikodým property*, when the limit in (16) exists in the strong sense and for almost all t.

We now recall this root definition.

Definition 2.12. *A Banach space* E *is* uniformly convex *if for all* $\varepsilon>0$ *there exists* $\delta>0$*,*

$$\forall x,y\in E, \|x\|\le 1, \|y\|\le 1, \|x-y\|\ge\varepsilon \Longrightarrow \|(x+y)/2\| < (1-\delta) \ .$$

Examples of uniformly convex Banach spaces include $L^p(\Omega,\mathcal{A},\mu)$ for $p\in(1,\infty)$. Uniformly convex Banach spaces have many interesting properties: for example, they are reflexive (the Milman–Pettis theorem, Theorem, 3.31 in [4]); moreover, if $x_n\to x$ in the weak sense and $\limsup\|x_n\|\le\|x\|$, then $x_n\to x$ in the strong sense (Proposition III.32 in [4]).

So we obtain a sufficient condition.

Corollary 2.13. *If* E *is uniformly convex and separable, then it enjoys the* Radon–Nikodým property *(indeed* (16) *and* (17) *imply that the limit in* (16) *is valid also in the strong sense).*

2.2.3. Embeddings in uniformly convex Banach spaces. If E is uniformly convex, then in particular the closed ball $\{x\in E : \|x\|\le 1\}$ is strictly convex; this has an interesting implication.

Lemma 2.14. *Suppose the closed balls in* E *are strictly convex. Consider* E *as a metric space, with distance* $d_E(x,y)=\|x-y\|$*. The segment connecting* $x,y\in E$ *is the unique (up to reparameterization) geodesic.*

Proof. We will prove that, for x,y, for any geodesic $\gamma:[0,1]\to M$ connecting x to y, if γ is reparameterized to the arc parameter, then $\gamma(1/2)=(x+y)/2$; iterating this reasoning with finer subdivision, we obtain that $\gamma(t)=(tx+(1-t)y)$.

[2] So in this case the metric derivative is the norm of an actual vector.

[3] $\mathbf{1}_A$ is the characteristic, or indicator, function of the set A.

With no loss of generality, up to translation and scaling, suppose $y = -x$ and $\|x\| = 1$. The segment $t \mapsto tx$ is a geodesic for $t \in [-1, 1]$, by Theorem 2.11, and its length is 2. Suppose now that $\gamma : [-1, 1] \to M$ is another geodesic. Then the length of γ is 2 and, up to reparameterization, $\|\dot\gamma\| = 1$ at almost all points. In particular, setting $z = \gamma(0)$, $\|z - y\| \le 1$ and $\|x - z\| \le 1$; but then, by the triangle inequality,

$$\|z - y\| = \|z + x\| = \|x - z\| = 1 \ .$$

Suppose that $z \neq 0$; then $\|(z + x) - (x - z)\| > 0$. By strict convexity, though, this implies that $\|((z + x) + (x - z))/2\| = \|x\| < 1$, and this is a contradiction. ■

Theorem 2.15. *Suppose that (M, d) is a complete space and that $i : M \to E$ is an isometrical immersion in a uniformly convex Banach space E. If, given $x, y \in M$, $d(x, y) = d^g(x, y)$, then the segment connecting $i(x), i(y)$ is all contained in $i(M)$.*

In particular, if (M, d) is intrinsic, then $i(M)$ is convex and then any two points in M can be joined by a unique geodesic (unique up to reparameterization).

Proof. Note that $i(M)$ is complete and thus it is closed in E. We will prove that, for any $x, y \in i(M)$, $(x + y)/2 \in i(M)$. We can iterate this idea to further subdivide. Since $i(M)$ is closed, this proves that the whole segment connecting x, y is in $i(M)$. By the above lemma the segment is the unique geodesic.

We now fix $x, y \in i(M)$: there must be paths $\gamma_n : [-1, 1] \to i(M)$ connecting x to y with length $\mathrm{Len}^d(\gamma_n) < L_n \stackrel{\text{def}}{=} \|x - y\| + 2/n$.

As in the lemma before, we suppose for simplicity that $y = -x$ and $\|x\| = 1$ (so $L_n = 2 + 2/n$); and we reparameterize so that $\|\dot\gamma_n\| le 1 + 1/n$. Hence setting $z_n = \gamma_n(0)$

$$\|z_n - y\| = \|z_n + x\| \le 1 + 1/n \ , \quad \|x - z_n\| \le 1 + 1/n \ .$$

Then by the triangle inequality $\|z_n + x\| \to 1$, $\|z_n - x\| \to 1$. Setting

$$w_n = (z_n + x)/\|z_n + x\| \ , \quad v_n = (x - z_n)/\|z_n - x\| \ ,$$

we can prove that $\|(w_n + v_n)/2\| \to 1$; hence by the uniform convexity of E we obtain that $w_n - v_n \to 0$ and $z_n \to 0$. Since $z_n \in i(M)$ and $i(M)$ is closed, then $0 \in i(M)$. ■

The above is a "rigidity theorem," in that it restricts the class of metric spaces that can be isometrically embedded in a uniformly convex Banach space E.

Corollary 2.16. *A compact finite-dimensional Riemannian manifold M cannot be isometrically embedded*[4] *in a uniformly convex Banach space E: indeed in this space M there are two points that can be joined by more than one geodesic.*

When E is not uniformly convex, on the other hand, strange behavior arises.

Proposition 2.17. *Let $L^\infty = L^\infty(\Omega, \mathcal{A}, \mu)$ and suppose Ω is not an atom of μ, that is, suppose the dimension of L^∞ is greater than 1. Given generic $f, g \in L^\infty$, there is an uncountable number of geodesics connecting them.*

Proof. We can assume without loss of generality that $g = 0$ and that $\|f\| = 1$. We abbreviate

$$\{|f| = 1\} = \{x \in \Omega : |f(x)| = 1\}$$

[4] In the sense explained at the beginning of section 2.2.

and similarly for similar expressions. Let $A = \{|f| = 1\}$. We will prove that if there is only one geodesic, then $|f| = \mathbf{1}_A$. Indeed if $|f| \neq \mathbf{1}_A$, then $\mu\{|f| < 1\} > 0$. Let $0 < t < 1$ be such that $\mu\{|f| < t\} > 0$; obviously $\mu\{|f| \geq t\} > 0$ since $\|f\| = 1$; let $A' = \{|f| \geq t\}$ and $A'' = \{|f| < t\}$. Given any increasing diffeomorphism $b : [0,1] \to [0,1]$ with $b'(s) \leq 1/t$,

$$\gamma(t) \stackrel{\text{def}}{=} t f \mathbf{1}_{A'} + b(t) f \mathbf{1}_{A''}$$

is a geodesic. Indeed its derivative is

$$\gamma'(t) \stackrel{\text{def}}{=} f \mathbf{1}_{A'} + b'(t) f \mathbf{1}_{A''}$$

and $\|\gamma'(t)\| = 1$ by construction.

The family of f such that $|f| = \lambda \mathbf{1}_A$ is closed and has empty interior. ■

The idea of *isometrical embedding* is quite powerful: indeed any separable metric space may be isometrically embedded in ℓ^∞ (which is the dual of the separable space ℓ^1): so the breadth of application of Theorem 2.11 is general and is at the basis of many results in [1]. But the embedding in ℓ^∞ that is studied in [1] is not suited for our practical applications.

- It would not respect the geometric properties of the space (as we discussed in section 1.2).
- It would be too difficult to find a satisfactory notion of *"shooting of geodesics"* using this embedding: that is, to define a way, given a point p and a direction v, to find a (possibly unique) geodesic starting from p and with first derivative v in p.

For all the above reasons, we will consider *isometrical embeddings* in this paper as well, but we will (for the most interesting applications) use an explicitly chosen embedding in uniformly convex Banach spaces.

3. Definitions. We introduce some definitions that will be used in the rest of the paper.

- We will denote by $e_1, \ldots, e_n$ the canonical basis of $\mathbb{R}^N$.
- We will write $s^+ = \max\{s, 0\}$ for $s \in \mathbb{R}$.
- We will write $B(x,r)$ or $B_r(x)$ for the *open ball*

$$B_r(x) = \{y \in \mathbb{R}^N : |x - y| < r\}$$

  of center x and radius $r > 0$ in $\mathbb{R}^N$; we will write B_r for $B_r(0)$. Similarly $D_r(x)$ will be the *disk* (or *closed ball*)

$$D_r(x) = \{y \in \mathbb{R}^N : |x - y| \leq r\} \tag{21}$$

  of center x and radius $r > 0$ in $\mathbb{R}^N$ and $D_r = D_r(0)$.
- We will say that a family $A_{i \in I}$ of sets in $\mathbb{R}^N$ is *equibounded* if there is an $R > 0$ such that $A_i \subseteq D_R$ for all i.
- We denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure, and $\omega_N \stackrel{\text{def}}{=} \mathcal{L}^N(B_1)$; we write $\int_A f(x)\,\mathrm{d}x$ for the Lebesgue integral.

4. Distance function and fattening. Let $A \subseteq \mathbb{R}^N$ be a closed set. We recall here some useful properties of the *distance function* u_A that was defined in (2).

- u_A is the viscosity solution of the *eikonal equation*

$$|\nabla f(x)| - 1 = 0$$

  in $\mathbb{R}^N \setminus A$, with boundary condition that $f = 0$ on A; the viscosity solution is unique in the class of continuous functions f that are bounded from below.
- u_A is Lipschitz of constant 1; hence it is differentiable almost everywhere.
- Suppose that u_A is differentiable at x; when $x \notin A$ we have $|\nabla u(x)| = 1$; otherwise $\nabla u(x) = 0$.
- Fix $x \in \mathbb{R}^N$, $x \notin A$. A *projection point* is a point $y \in A$ of minimum distance, i.e., a point such that $u_A(x) = |x - y|$. There is always at least one projection point.
  The following two facts are equivalent:
  1. u_A is differentiable at x;
  2. there is a unique *projection point* $y \in A$.

  When both hold,

$$\nabla u(x) = \frac{x - y}{|x - y|} \ . \tag{22}$$

- u_A is convex iff A is convex.
- If $\lambda > 0$,

$$u_{\lambda A}(\lambda z) = \lambda u_A(z) \ , \tag{23}$$

  where $\lambda A \stackrel{\text{def}}{=} \{\lambda z : z \in A\}$ is the rescaled set.

For all of the above, see [13] (where the above properties are discussed for general Riemannian manifolds) and references therein.

For $A \subseteq \mathbb{R}^N$ a closed set and $r \geq 0$, we define the *fattened set* to be

$$A + D_r = \{x + y \mid x \in A, |y| \leq r\} = \bigcup_{x \in A} D_r(x) = \{y \mid u_A(y) \leq r\}.$$

The fattened set is closed. The fattening operation is a semigroup in the sense that $A + D_0 = A$, and for $r, s > 0$

$$(A + D_r) + D_s = A + D_{r+s} \ ;$$

similarly the distance function satisfies

$$u_{A+D_r}(x) = (u_A(x) - r)^+ \ . \tag{24}$$

The next lemma will prove useful in what follows.

Lemma 4.1. *Let $r > 0$. Let $F = A + D_r$ and $E = \{x : u_A(x) = r\}$ for convenience.*

- *The boundary ∂F of F is contained in the set E.*
- *E is Lebesgue negligible.*

Proof.

- If $u_A(z) < r$, then z is in the topological interior of F.

- Let $z \in E$ such that $u_A(z) = r$; let $x \in A$ be a projection point of z; then the ball $D_r(x)$ is contained in F and z is in its boundary. Setting $y = (x+z)/2$, the ball $D_{r/2}(y)$ is contained in F and z is in its boundary; but, moreover, for all points $w \in D_{r/2}(y)$ with $w \neq z$, we have $|w - x| < r$, and hence $u_A(w) < r$. We conclude that $D_{r/2}(y)$ intersects E only in z. This proves that the Lebesgue density of the set E in the point z cannot be one, so E is negligible.

(The above proves also that F satisfies an *interior sphere condition.*) ■

5. Hausdorff distance. Let $\mathcal{M}$ again be the family of the nonempty compact sets in $\mathbb{R}^N$. A fundamental example of metric on $\mathcal{M}$ is the *Hausdorff distance*

$$d_H(A,B) \stackrel{\text{def}}{=} \inf\{\delta > 0 \mid B \subseteq (A + D_\delta), A \subseteq (B + D_\delta)\} .$$

The Hausdorff distance may be defined in many equivalent ways,

$$d_H(A,B) = \max\{\max_A u_B, \max_B u_A\} \tag{25}$$

$$= \sup_{x\in\mathbb{R}^N} |u_A(x) - u_B(x)| , \tag{26}$$

as shown in section C in Chapter 4 in [19] and section 2.2 in Chapter 4 in [7].

This metric enjoys many important properties.

Theorem 5.1. *The metric space $(\mathcal{M}, d_H)$ satisfies the following:*

1. *Given $r > 0$, the family of equibounded compact sets*

   $$\{A \in \mathcal{M} \mid A \subseteq D_r\}$$

   is compact; in particular, given $B \in \mathcal{M}$, the set

   $$\{A \in \mathcal{M} \mid d_H(A,B) \leq \rho\}$$

   is compact.
2. *$(\mathcal{M}, d_H)$ is complete.*
3. *$(\mathcal{M}, d_H)$ is intrinsic (that is, $d_H = (d_H)^g$).*
4. *Any two $A, B \in \mathcal{M}$ may be joined by a geodesic γ.*

The first statement is a well-known property of the Hausdorff distance; see, e.g., Example 4.13 and Theorem 4.18 in [19]. By exploiting the characterization (26), it also follows from a diagonal/compactness argument and the results presented in section 5.3. The second follows from the first. The third and fourth properties in Theorem 5.1 derive from Proposition 5.3 below.

We complement the above with this family of nice properties.

Proposition 5.2.

1. *For any fixed $A \in \mathcal{M}$, the fattening map $\lambda \mapsto A + D_\lambda$ is Lipschitz (of constant one) as a map from $[0,\infty)$ to $(\mathcal{M}, d_H)$.*
2. *For any fixed $\lambda > 0$, the* "fattened area map" *$L_\lambda : \mathcal{M} \to \mathbb{R}$ defined as $L_\lambda(A) \stackrel{\text{def}}{=} \mathcal{L}^N(A + D_\lambda)$ is continuous on $(\mathcal{M}, d_H)$.*
3. *The area map $L(A) \stackrel{\text{def}}{=} \mathcal{L}^N(A)$ is upper semicontinuous.*

4. *Let*
$$\# : \mathcal{M} \to \mathbb{N} \cup \infty$$
be the number $\#\Omega$ *of connected components of a compact set* Ω. *Then* $\#$ *is lower semicontinuous in the metric space* $(\mathcal{M}, d_H)$.[5]
As a corollary, the family of connected compact sets is a closed family in $(\mathcal{M}, d_H)$.
5. *Let* $A \in \mathcal{M}$ *and* x *in the topological interior of* A, *and let* $R > 0$ *be such that* $B_R(x) \subseteq A$. *Then we define the* carving motion $\gamma : [0, R] \to \mathcal{M}$ *as* $\gamma(0) = A$ *and*
$$\gamma(t) = A \setminus B_t(x) \tag{27}$$
for $t \in (0, R]$. *Then this motion is an arc parameterized*[6] *geodesic. See also Example 6.23.*
6. *Let* $\Phi : [-T, T] \times \mathbb{R}^N \to \mathbb{R}^N$ *be a locally Lipschitz map. Given* A *compact, let* $A_t = \Phi(t, A)$ *be the image; then the path* $t \mapsto A_t$ *is Lipschitz in* $(\mathcal{M}, d_H)$.

Suppose that G *is a group of diffeomorphisms of* $\mathbb{R}^N$. *This group* G *acts on* $\mathcal{M}$*; for any* $g \in G$, *the action of* g *on* $A \in \mathcal{M}$ *is* $g(A) = \{g(x) : x \in A\}$. *The last result shows that any such action is locally Lipschitz.*

Proof.
1. The proof follows from (24).
2. If $A_n \to A$, then for fixed $\varepsilon > 0$ there exists N such that for all $n \geq N$,
$$A_n \subseteq A + D_\varepsilon \ , \quad A \subseteq A_n + D_\varepsilon$$
and then
$$A_n + D_\lambda \subseteq A + D_{\varepsilon+\lambda} \ , \quad A + D_{\lambda-\varepsilon} \subseteq A_n + D_\lambda \ .$$
Passing to Lebesgue measures,
$$\mathcal{L}^N(A + D_{\lambda-\varepsilon}) \leq \liminf_n \mathcal{L}^N(A_n + D_\lambda) \leq \limsup_n \mathcal{L}^N(A_n + D_\lambda) \leq \mathcal{L}^N(A + D_{\varepsilon+\lambda}) \ .$$
We let $\varepsilon \to 0$: the left-hand side converges to the measure of the set $\{u_A < \lambda\}$, and the right-hand side converges to the measure of the set $\{u_A \leq \lambda\}$; by Lemma 4.1 they are equal.
3. This follows since it is the pointwise limit $L_\lambda(A) \downarrow L(A)$ for $\lambda \to 0$.
4. See Theorem 2.3 in Chapter 4 in [7].
5. For $0 \leq s < t \leq R$ we have $\gamma(t) \subseteq \gamma(s)$ and $\gamma(s) \subseteq \gamma(t) + D_{(t-s)}$.
6. Let $R > 0$ such that $A \subseteq D_R$. Let L be the Lipschitz constant of Φ on $[-T, T] \times A$; then $A_t \subseteq D_{R+LT}$ for all t. Fix $s, t \in [-T, T]$; given $y \in A_t$, let $x \in A$ such that $y = \Phi(t, x)$; then consider $z = \Phi(s, x) \in A_s$, by Lipschitzianity $|y - z| \leq L|t - s|$. Thus $A_t \subseteq A_s + D_{L|t-s|}$. ∎

[5] This result does not hold for closed sets.
[6] In the sense that its metric derivative is 1 for all t; see Lemma 2.3.

5.1. The maximal geodesic. In this section we describe an explicit formula to compute the geodesic connecting two compact sets A to B.

Proposition 5.3 (maximal geodesic). *Let $A, B \in \mathcal{M}$ be two compact sets, and let $\mu = d_H(A, B)$. For all $t \in [0, \mu]$ we define the set*

$$C_t \stackrel{\text{def}}{=} \{z : u_A(z) \le t, u_B(z) \le (\mu - t)\} \ .$$

Then $t \mapsto C_t$ is an arc parameterized geodesic connecting A to B, and in particular its length is μ.

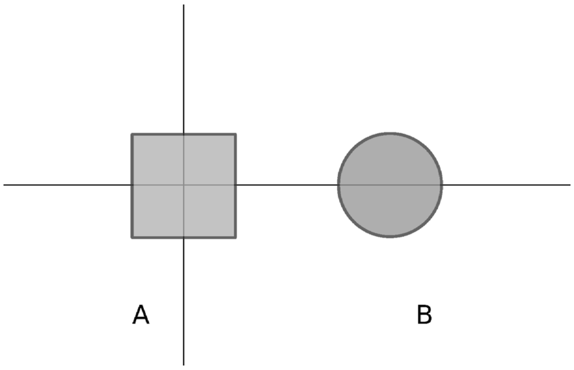
Moreover C is maximal *in the sense that, for any arc parameterized geodesic $\gamma : [0, \mu] \to \mathcal{M}$ connecting A to B, we have $\gamma(t) \subseteq C_t$ for all $t \in [0, \mu]$.*

The proof is in section A.3.

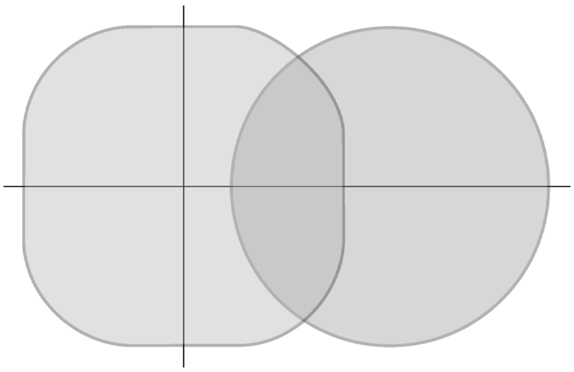
Remark 5.4. The above results still hold for the Hausdorff metric space of compact subsets of a finitely compact intrinsic metric space—in particular they hold for finite-dimensional Riemannian manifolds.

We provide an explicit example.

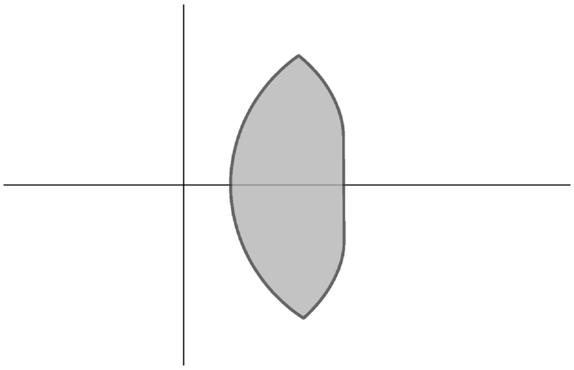
Example 5.5. Let A be a square of side 2 centered at the origin, and B a disc of radius 1 centered at $(4, 0)$. (In the figure, cartesian axes are drawn for easy comparison with the following steps.)



The Hausdorff distance is $\mu = d_H(A, B) = \sqrt{26} - 1$. Let us fix $t = \mu/2$. We now fatten the set A by t and the set B by $\mu - t$ (which in this case is again t) and obtain the shapes below.



Eventually we intersect the two fattenings to obtain C_t.



The above maximal geodesic enjoys some properties.

Corollary 5.6.

- *Let $A, B, \tilde{A}, \tilde{B} \in \mathcal{M}$ with $A \subseteq \tilde{A}$ and $B \subseteq \tilde{B}$ and suppose that*

$$\mu = d_H(A, B) = d_H(\tilde{A}, \tilde{B}) \ .$$

  Let $C_t, \tilde{C}_t$ be maximal geodesics connecting A to B and, respectively, $\tilde{A}$ to $\tilde{B}$; then $C_t \subseteq \tilde{C}_t$ for all $t \in [0, \mu]$.
- *Let E be the convex hull of $A \cup B$, and let $\mu = d_H(A, B)$. For all $t \in [0, \mu]$ we define the set*

$$\tilde{C}_t = C_t \cap E = \{z \in E : u_A(z) \le t, u_B(z) \le (\mu - t)\} \ .$$

  Then $t \mapsto \tilde{C}_t$ is another arc parameterized geodesic connecting A to B.
- *If A, B are convex sets, then C_t and $\tilde{C}_t$ are convex for all $t \in [0, \mu]$.*

Proof. The proof of the first result follows immediately from the definition of C_t. For the second we reread the proof in section A.3, noting that if $z \in E$, then $y \in E$ as well. For the third we recall that the two distance functions are convex. ■

The maximal geodesic defined in Proposition 5.3 may not be suited for applications in computer vision. Consider this example.

Example 5.7. Let $A \subset \mathbb{R}^2$ be a square of unit side, and let $B = A + (4, 0)$ be its translation; then $d_H(A, B) = 4$. The map

$$\gamma : [0, 4] \to \mathcal{M} \quad , \quad \gamma(t) = A + (t, 0)$$

that translates A to B is an arc parameterized geodesic, but is not the maximal geodesic. The maximal geodesic C_t is much larger; the set C_2 (at time $t = 2$) is depicted in Figure 2.

5.2. Multiple geodesics. Unfortunately $(\mathcal{M}, d_H)$ is quite "unsmooth"; we will indeed prove that generically a pair $A, B \in \mathcal{M}$ may be joined by a continuum of geodesics.

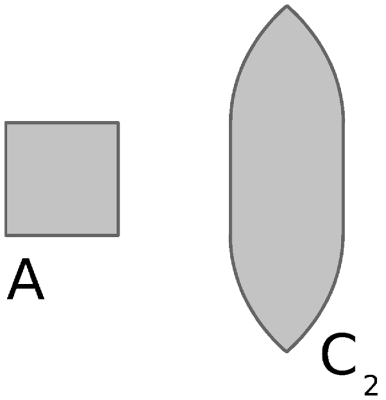
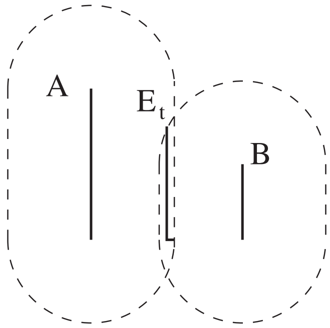


Figure 2. *Example* 5.7.

Example 5.8. We first provide an example.

$$A = \{x = 0, 0 \le y \le 2\} \ ,$$
$$B = \{x = 2, 0 \le y \le 1\} \ ,$$
$$E_t = \left\{x = 1, 0 \le y \le \tfrac{3}{2}\right\} \cup \{y = 0, 1 \le x \le t\}$$
$$\text{with } 1 \le t \le \sqrt{5}/2 \ ;$$



in the picture we represent (dashed) the fattened sets $A + D_{\sqrt{5}/2}$ and $B + D_{\sqrt{5}/2}$. Note that $d_H(A, B) = \sqrt{5}$, while $d_H(A, E_t) = d_H(B, E_t) = \sqrt{5}/2$. So for all $t \in [1, \sqrt{5}/2]$, E_t is a midpoint of a distinct geodesic between A and B.

The above idea is generalized in the following results.

Lemma 5.9. *Let $A, B \in \mathcal{M}$, suppose $A \setminus B$ has nonempty topological interior, and let $\theta = \max_A u_B(x) > 0$. For any nonempty open $G \subseteq A \setminus B$ such that $G \cap \{u_B = \theta\} = \emptyset$, set $E = A \setminus G$ then $d_H(A, B) = d_H(E, B)$.*

The proof is in section A.4. Note that such sets G exist, since the set $\{u_B = \theta\}$ is compact and negligible, due to Lemma 4.1.

Lemma 5.10. *Let $A, B \in \mathcal{M}$, with $A \neq B$, and let $\mu = d_H(A, B)$. Let C_t denote the maximal geodesic. Suppose that for a $t \in (0, \mu)$ the set $C_t \setminus (A \cup B)$ has nonempty interior: then there is a continuum of different*[7] *geodesics connecting A to B.*

The proof is in section A.5.

We then propose a general theorem.

Theorem 5.11. *Let $A, B \in \mathcal{M}$, with $A \neq B$, and let $\mu = d_H(A, B)$. Suppose that there is an x in the boundary of A such that $0 < u_B(x) < \mu$; then there is a continuum of different geodesics connecting A to B.*

[7] That is, not "equal up to reparameterization."

Proof. Let C_t be the maximal geodesic connecting A to B, as defined in Proposition 5.3. Choose $\varepsilon > 0$ small such that $2\varepsilon < u_B(x) < \mu - 2\varepsilon$. There is a point y near x such that $|x-y| < \varepsilon$ but $y \notin A$; such a point satisfies $u_A(y) < \varepsilon$ and $\varepsilon < u_B(y) < \mu - \varepsilon$, and hence it is in the topological interior of C_t when $t = \varepsilon$, but is outside of A and B. So we can apply the previous lemmas. ■

(Note the similarity of the above arguments to the proof of Proposition 2.17—and for a reason!)

Lemma 5.12. *Let $(A_n), A \subseteq \mathcal{M}$. Suppose that $A_n \to A$ in the sense of Hausdorff convergence; then for any $x \in \partial A$ there exists a sequence with $x_n \in \partial A_n$ such that $x_n \to x$.*

Theorem 5.13. *Generically any pair $A, B \in \mathcal{M}$ is connected by a continuum of different geodesics.*

Proof. If a pair $A, B \in \mathcal{M}$ does not satisfy the hypothesis of Theorem 5.11, then

$$\partial A \subseteq \{u_B = \mu\} \cup B \ , \quad \partial B \subseteq \{u_A = \mu\} \cup A \ , \tag{28}$$

where $\mu = d_H(A, B)$. Let $\mathcal{U}$ in $\mathcal{M}^2$ be the set of all such pairs; note that any pair (A, A) is in $\mathcal{U}$.

We will prove that $\mathcal{U}$ is closed and has empty interior (w.r.t. the Hausdorff convergence).

The fact that $\mathcal{U}$ is closed follows from the previous lemma and (26).

Fix $(A, B) \in \mathcal{U}$.

We choose an "exposed boundary point" $x \in A \cup B$; to fix ideas, we let $a = \max\{x_1 : x \in A \cup B\}$ (where x_1 is the first component of x), and let x be a point providing the above maximum; then $A \cup B$ is contained in the half space $H = \{z : z_1 \le a\}$.

Suppose without loss of generality that $x \in A$; let $y = x + \varepsilon e_1$ with $\varepsilon > 0$ small; then add to A the segment xy to create $\tilde{A}$. The pair $(\tilde{A}, B) \notin \mathcal{U}$, since the segment xy is contained in the boundary of $\tilde{A}$, but xy is not contained in B (since by construction xy is outside of H) and u_B is not constant on it.

Since this construction holds for any $\varepsilon > 0$, then $\mathcal{U}$ has empty interior. ■

We close this section by reviewing some examples of pairs A, B that are connected by a unique geodesic. Note how the condition (28) is satisfied in these examples.

Definition 5.14. *Following [8], F is a set of* positive reach *if there exists $r > 0$ such that for any x with $u_F(x) < r$ there exists a unique projection point $y \in F$; this defines a projection map $\pi_F(x) = y$. The* reach *is the largest such r. For any $0 < s < r$ the projection π_F is Lipschitz on $\{u_F \le s\}$.*

Examples of sets of positive reach are

- *convex sets (in this case $r = \infty$);*
- *compact sets with nonempty interior and C^2-regular boundary;*
- *compact submanifolds of $\mathbb{R}^N$ that are C^2-regular.*

Example 5.15. Suppose that F is a set of *positive reach* r, and choose $\mu \in (0, r)$. Suppose that A is a compact subset of ∂F, and B is a compact subset of $\{u_F = \mu\}$ such that $\pi_F(B) = A$. Then $d_H(A, B) = \mu$; there is a unique geodesic C_t connecting A to B, and C_t is given by all interpolated points

$$\frac{tx + (\mu - t)\pi_F(x)}{\mu}$$

for all $x \in B$.

When $A = \partial F$, the above is known as *grassfire evolution.*

The above encompasses many examples.

Example 5.16 (orthogonal translation). Let $A \in \mathcal{M}$, suppose that there is an $(N-1)$-dimensional affine space H containing A, choose a vector v orthogonal to H, and define $B = v + A$ (B is the translation of A by the vector v); then $d_H(A,B) = |v|$ and the translation $C_t = A + tv/|v|$ is the unique geodesic connecting A to B.

Example 5.17 (fattening). Suppose that A is a set of *positive reach* r, let $\mu \in (0,r)$, and let $B = A + D_\mu$ be a fattening; then $d_H(A,B) = \mu$ and the fattening $C_t = A + D_t$ is the unique geodesic connecting A to B.

Note that in general the "fattening" is a geodesic, but it may fail to be unique.

5.3. Hausdorff and Kuratowski convergence. We provide some extra definitions.

Definition 5.18 (Kuratowski convergence). *Let Ω, Ω_n be nonempty closed sets in $\mathbb{R}^N$. We will say that $\Omega_n \to \Omega$ in the Kuratowski sense if these equivalent facts hold:*

- *$u_{\Omega_n} \to u_\Omega$ pointwise;*
- *$u_{\Omega_n} \to u_\Omega$ pointwise on a dense subset of $\mathbb{R}^N$;*
- *$u_{\Omega_n} \to u_\Omega$ uniformly on compact subsets of $\mathbb{R}^N$.*

This definition is not the standard one, but it is equivalent; see section 4.B in [19]. The equivalence of the statements in the above definition is due to the fact that distance functions are 1-Lipschitz functions. The above three equivalent facts express a "rigidity" of distance functions, which is again seen in the following.

Lemma 5.19. *Let Ω_n be nonempty closed sets and suppose that $\lim_n u_{\Omega_n}(x)$ exists and is finite for all x in a dense subset D of $\mathbb{R}^N$; call $f(x) = \lim_n u_{\Omega_n}(x)$. Then there is a nonempty closed set Ω such that $\Omega_n \to \Omega$ in the Kuratowski sense and $u_\Omega(x) = f(x)$ for all $x \in D$.*

Proof. The proof may follow from the theory of viscosity solutions: as remarked in section 4, indeed u_Ω is the unique solution to the *eikonal equation*; moreover viscosity solutions do enjoy the required rigidity property. The proof is anyway easily derived by a direct argument and the Ascoli–Arzelà theorem (similarly to the arguments of Chapter 2 in [7] and of Chapter 4 in [19][8]). We propose a direct proof in section A.6. ■

The Kuratowski convergence and the Hausdorff convergence coincide for equibounded families.

Lemma 5.20. *Suppose that Ω is compact and nonempty and that Ω_n are nonempty closed sets. These facts are equivalent:*

- *Ω_n is equibounded and $\Omega_n \to \Omega$ in the Kuratowski sense;*
- *$d_H(\Omega_n, \Omega) \to 0$;*
- *$u_{\Omega_n} \to u_\Omega$ uniformly.*

Proof. The equivalence of the first two facts is proved in section C in Chapter 4 [19]. Equation (26) shows that the third condition is equivalent to the second. ■

6. L^p-like metrics of shapes. The definition of the Hausdorff distance by (26) leads us back to the paradigm of *representation/embedding*; but in this case it is unfortunately not precise, since the Banach metric that we use, namely,

$$\|f\| = \|f\|_\infty \stackrel{\text{def}}{=} \sup_x |f(x)| \ ,$$

[8]But see an important correction in Remark 2.7 in [14].

is usually associated to the space of continuous bounded functions—whereas the distance function u_A is not bounded! What follows is a simple yet effective workaround.

Hypotheses 6.1. We fix $p \in [1, \infty]$; we fix a function $\varphi : [0, \infty) \to (0, \infty)$ monotonically strictly decreasing and of class C^1, such that

$$\varphi(|x|) \in L^p(\mathbb{R}^N) \ . \tag{29}$$

When $p = \infty$ we also ask that $\lim_{t\to\infty} \varphi(t) = 0$ as an extra hypothesis.

In the rest of the paper φ will always satisfy the above assumption (and possibly others, which will be specified when needed).

Note that, when $p < \infty$, the above (29) is equivalent to asking that

$$\int_0^\infty t^{N-1} \varphi(t)^p \, \mathrm{d}t < \infty \ , \tag{30}$$

and this implies that $\lim_{t\to\infty} \varphi(t) = 0$.

An example of such a function is $\varphi(t) = \exp(-t)$, or $\varphi = (1 + t)^{-(N+1)/p}$.

We will often write

$$v_A \stackrel{\text{def}}{=} \varphi \circ u_A$$

for simplicity.

Lemma 6.2. *Let $\Omega \subseteq \mathbb{R}^N$ be closed and nonempty, and suppose $p < \infty$; then the following are equivalent:*

(a) $v_\Omega \in L^p(\mathbb{R}^N)$.

(b) Ω *is bounded (and then Ω is compact).*

Proof. We first prove that (a) $\Longrightarrow$ (b) by contradiction. Let us assume that Ω is unbounded. Then there exists a sequence $\{x_k\} \subseteq \Omega$ such that $|x_k| \to \infty$ and $|x_k - x_q| > 2$ for all $k, q \in \mathbb{N}, k \neq q$. The sequence of sets $B_1(x_k)$ is disjoint. It is easy to see that $v_\Omega(x) > \varphi(1)$ for $x \in \bigcup_k B_1(x_k)$ and then $v \notin L^p$.

Then we prove that (b) $\Longrightarrow$ (a). If Ω is bounded, we can find a disk D_R such that $\Omega \subseteq D_R$. Then easily we have $u_\Omega \geq u_{D_R} \Longrightarrow v_\Omega \leq v_{D_R}$, but $v_{D_R} \in L^p$ (as is easily proved by $v_{D_R}(x) = \varphi((|x| - R)^+)$ and by (30)) so that $v_\Omega \in L^p$ as well. ∎

Let $\mathcal{M}$ again be the family of the nonempty compact sets in $\mathbb{R}^N$.

Definition 6.3. *Given $A, B \in \mathcal{M}$, we define*

$$d_{p,\varphi}(A, B) \stackrel{\text{def}}{=} \|\varphi \circ u_A - \varphi \circ u_B\|_{L^p(\mathbb{R}^N)} \ . \tag{31}$$

By the above lemma, this distance is finite. We will often write d for $d_{p,\varphi}$ in the following for simplicity. Similarly we will write d^g for the induced distance $d^g_{p,\varphi}$.

The above distance is obtained by the *representation* of a shape A as v_A, combined with the *embedding* of v_A in $L^p(\mathbb{R}^N)$. For this reason, we may identify our shape space with

$$\mathcal{N} \stackrel{\text{def}}{=} \{v_\Omega \mid \Omega \in \mathcal{M}\} \ , \tag{32}$$

which is a subset of L^p.

Remark 6.4. By the definition of d, the map $\Omega \mapsto v_\Omega$ is an isometrical embedding of $\mathcal{M}$ inside L^p and the image is $\mathcal{N}$; $\mathcal{N}$ is a closed subset of L^p, by the completeness result, Proposition 6.12, that we will prove in the following. We will exploit this embedding in the following, in particular in section 6.6.

It is immediate to verify that $d_{p,\varphi}$ satisfies these properties.

- The embedding $A \mapsto v_A$ is injective: if $v_A = v_B$ almost everywhere, then $u_A = u_B$ almost everywhere (since φ is strictly decreasing, and so it is injective); since distance functions are continuous, this implies that $u_A = u_B$, and then $A = B$. Consequently, for all $A, B \in \mathcal{M}$, $d_{p,\varphi}(A, B) = 0$ iff $A = B$.
- $d_{p,\varphi}$ is Euclidean invariant, as we requested in section 1.2.
- When $p < \infty$, for any A, B compact there holds

$$d_{p,\varphi}(A, B) < \sqrt[p]{\|v_A\|^p_{L^p} + \|v_B\|^p_{L^p}} \ . \tag{33}$$

  Indeed we note that for $a, b > 0$ we have

$$|a - b|^p < \max\{a^p, b^p\} < a^p + b^p \ ;$$

  so

$$d_{p,\varphi}(A, B)^p = \int_{\mathbb{R}^N} |v_A(x) - v_B(x)|^p \,\mathrm{d}x < \int_{\mathbb{R}^N} v_A(x)^p + v_B(x)^p \,\mathrm{d}x \ .$$

- When $p = \infty$ instead

$$d_{\infty,\varphi}(A, B) < \varphi(0) \ .$$

- *(Separation at infinity.)* Given two bounded sets A, B and $\tau \in \mathbb{R}^N$ we have

$$\lim_{|\tau|\to\infty} d_{p,\varphi}(A, B + \tau) = \sqrt[p]{\|v_A\|^p_{L^p} + \|v_B\|^p_{L^p}} \tag{34}$$

  for $p < \infty$, while

$$\lim_{|\tau|\to\infty} d_{\infty,\varphi}(A, B + \tau) = \varphi(0) \ . \tag{35}$$

  Proof. For the case $p = \infty$ it derives from the hypothesis $\lim_{t\to\infty} \varphi(t) = 0$. When $p < \infty$, it comes from a general result for L^p functions; see section A.1. ■

- *(Scaling.)* If $p < \infty$ and $\lambda > 0$ is a rescaling of the space, then the rescaled distance may be expressed as

$$d_{p,\varphi}(\lambda A, \lambda B) = \lambda^{N/p} d_{p,\tilde\varphi}(A, B) \ , \tag{36}$$

  where $\tilde\varphi(r) = \varphi(\lambda r)$; indeed

$$d_{p,\varphi}(\lambda A, \lambda B)^p = \int |v_{\lambda A}(x) - v_{\lambda B}(x)|^p \,\mathrm{d}x \tag{37}$$

$$= \lambda^N \int |v_{\lambda A}(\lambda z) - v_{\lambda B}(\lambda z)|^p \,\mathrm{d}z \tag{38}$$

$$= \lambda^N \int |\varphi(\lambda u_A(z)) - \varphi(\lambda u_B(z))|^p \,\mathrm{d}z \tag{39}$$

$$= \lambda^N d_{p,\tilde\varphi}(A, B)^p \ ,$$

where to go from (37) to (38) we used the change of variable $x = \lambda z$ and the property (23) of the distance function to change (38) to (39).

Remark 6.5. Inequality (33) easily implies that the closed balls of the distance d in general are not compact sets. Indeed it is enough to consider a compact set Ω and the closed ball

$$\mathbb{D} = \{A \mid d(A,\Omega) \le 2r\}$$

with $r = \|v_\Omega\|_{L^p}$. Then the sequence $\{\Omega + n\tau\}_{n\in\mathbb{N}}$ with $\tau \in \mathbb{R}^N \setminus \{0\}$ is contained in $\mathbb{D}$ and it does not have any convergent subsequence.

We will nonetheless prove in the following that the metric space $(\mathcal{M}, d)$ is locally compact.

To continue with our study of d, we prove this fundamental inequality.

Lemma 6.6 (local equiboundedness). *There is a continuous and increasing function $b : \mathbb{R}^+ \to \mathbb{R}^+$ with $b(0) = 0$ and $\lim_{r\to\infty} b(r) = \|\varphi(|x|)\|_{L^p}$ such that, for any $\Omega, \Omega' \in \mathcal{M}$ satisfying*

$$\|v_\Omega - v_{\Omega'}\|_{L^p} < b(r) \ ,$$

we have $\Omega' \subseteq \Omega + D_r$.

Proof. Set $K \stackrel{\text{def}}{=} \Omega + D_r$. By the relation in (24),

$$v_K(x) = \varphi\big((u_\Omega(x) - r)^+\big).$$

To prove the proposition for $p \in [1,\infty)$, suppose that $x_0 \in \Omega'$, but $x_0 \notin K$; for $y \in B(x_0, r/2)$ recall the simple triangle inequality

$$u_\Omega(y) \ge r - |x_0 - y| \ge |x_0 - y| \ge u_{\Omega'}(y) \ .$$

Hence

$$v_\Omega(y) \le \varphi(r - |x_0 - y|) \le \varphi(|x_0 - y|) \le v_{\Omega'}(y) \ ,$$

so

$$\begin{aligned} \|v_\Omega - v_{\Omega'}\|^p_{L^p} &\ge \int_{B(x_0,r/2)} |v_{\Omega'} - v_\Omega|^p \,\mathrm{d}x \\ &\ge \int_{B(x_0,r/2)} |\varphi(|x_0 - y|) - \varphi(r - |x_0 - y|)|^p \,\mathrm{d}x = b(r)^p \ , \end{aligned}$$

where

$$b(r)^p \ \stackrel{\text{def}}{=}\ \omega_N N \int_0^{r/2} t^{N-1} (\varphi(t) - \varphi(r - t))^p \,\mathrm{d}t$$

and where ω_N is the N-volume of the ball B_1 in $\mathbb{R}^N$. It is easy to prove that b is continuous and increasing (by direct derivation), that $b(0) = 0$, and that $\lim_{r\to\infty} b(r) = \|\varphi(|x|)\|_{L^p}$. With some calculus it is also possible to prove that

$$b(r) \sim |\varphi'(0)| r^{1+N/p} \tag{40}$$

for r small. (This estimate is sharp; see Example 6.23.)

The case $p = \infty$ is simpler: in this case we can note that

$$\|v_\Omega - v_{\Omega'}\|_\infty \geq v_{\Omega'}(x_0) - v_\Omega(x_0) \geq \varphi(0) - \varphi(r)$$

and set $b(r) = \varphi(0) - \varphi(r)$. ■

Corollary 6.7. *For $d(\Omega, \Omega')$ small enough*[9]

$$d_H(\Omega, \Omega') \leq b^{-1}\big(d(\Omega, \Omega')\big) \ .$$

Remark 6.8. The above does not hold for arbitrarily large distance $d(\Omega, \Omega')$: indeed, let $\Omega = \{0\}$ and $\Omega_n = \{ne_1\}$: then $d(\Omega, \Omega_n) \to 2^{1/p}\|\varphi(|x|)\|_{L^p}$ (as we mentioned in (34)).

We can also obtain a converse inequality, as follows.

Lemma 6.9. *There is a family of continuous functions $f_R : \mathbb{R}^+ \to \mathbb{R}^+$ with $f_R(0) = 0$ such that for any $\Omega, \Omega' \in \mathcal{M}$ with $\Omega, \Omega' \subseteq B_R$*

$$d(\Omega, \Omega') \leq f_R\big(d_H(\Omega, \Omega')\big) \ .$$

Proof. Note that φ is uniformly continuous; so when $p = \infty$ we choose f_1 to be a (continuous and increasing) modulus of continuity for φ, and then for any $R > 0$ we set $f_R \equiv f_1$ and use (26).

We now provide the proof for $p < \infty$. Since B_R contains both Ω and Ω', then $u_\Omega, u_{\Omega'} \geq u_{B_R}$ and then

$$v_\Omega(x), v_{\Omega'}(x) \leq \varphi(((|x| - R)^+) \ ,$$

so for $r \geq R$

$$\int_{\mathbb{R}^N \setminus B_r} |\max\{v_\Omega(x), v_{\Omega'}(x)\}|^p \, \mathrm{d}x \leq a_R(r) \ ,$$

where

$$a_R(r) \stackrel{\text{def}}{=} \int_{\mathbb{R}^N \setminus B_r} \varphi(|x| - R)^p \, \mathrm{d}x = \omega_N N \int_r^\infty t^{N-1} \varphi(t - R)^p \, \mathrm{d}t \tag{41}$$

(where ω_N is the N-volume of the ball B_1, and $\omega_N N$ is the $(N-1)$-volume of its boundary); note that $a_R(r) \to 0$ for $r \to \infty$. At the same time, let $l(r) = \sup_{[0,r]} |\varphi'|$. Then

$$\forall x \in B_r, \quad |v_\Omega(x) - v_{\Omega'}(x)| \leq l(r + 2R)|u_\Omega(x) - u_{\Omega'}(x)| \ ,$$

so

$$\int_{B_r} |v_\Omega(x) - v_{\Omega'}(x)|^p \, \mathrm{d}x \leq \omega_N r^N l(r + 2R)^p \sup_{x \in B_r} |u_\Omega(x) - u_{\Omega'}(x)|^p$$
$$\leq \omega_N r^N l(r + 2R)^p d_H(\Omega, \Omega')^p \ .$$

Summarizing,

$$d(\Omega, \Omega')^p = \int_{\mathbb{R}^N \setminus B_r} |v_\Omega(x) - v_{\Omega'}(x)|^p \, \mathrm{d}x + \int_{B_r} |v_\Omega(x) - v_{\Omega'}(x)|^p \, \mathrm{d}x$$
$$\leq a_R(r) + \omega_N r^N l(r + R)^p d_H(\Omega, \Omega')^p \ .$$

[9]Precisely, for $d(\Omega, \Omega') < \lim_{r \to \infty} b(r) = \|\varphi(|x|)\|_{L^p}$.

Eventually, for $s \geq 0$, let

$$g_R(s) = \inf_{r \geq R} \left[a_R(r) + \omega_N r^N l(r+2R)^p \; s \right] \tag{42}$$

and note that it is concave and monotonically increasing, and that $\lim_{s \to 0} g_R(s) = 0$; let $f_R(s) = \sqrt[p]{g_R(s^p)}$. ∎

Remark 6.10. Suppose that φ is convex and that $\varphi'(0) = -1$; then $l \equiv 1$ in the above proof. When $s > 0$ the minimum in (42) is obtained by $r = R$ if $s > \varphi(0)^p$, and otherwise by

$$r = R + \varphi^{-1}(\sqrt[p]{s}) \ .$$

A special case is $\varphi(t) = \exp(-t)$, $N = 2$, where we obtain $f_R(s) \sim s|R - \log s|^{2/p}$ for s small.

Another interesting case is $\varphi(t) = (1+t)^{-(N+1)/p}$, in which, for $R > 1$, $N \geq 2$, we obtain $f_R(s) \sim s^{\frac{1}{N+1}}$ for s small.

Combining Lemmas 6.9 and 6.7, we obtain this result.

Theorem 6.11. *The topology induced by $d_{p,\varphi}$ over the space $\mathcal{M}$ coincides with the topology induced by d_H.*

This implies that all topological properties of the Hausdorff distance are valid for the distance d as well.

The two distances are not equivalent, however, since

$$\lim_{|\tau| \to \infty} d_H(A, B + \tau) = \infty \ ,$$

while (34) and (35) show that the limit is finite for $d_{p,\varphi}$. When $p < \infty$, then $d_{p,\varphi}$ and d_H are also not locally equivalent, as seen in Example 6.23 below.

6.1. Completeness. By Theorem 6.11, we know that $(\mathcal{M}, d)$ is locally compact. We now prove that it is complete.

Proposition 6.12. *The space $(\mathcal{M}, d)$ is complete.*

Proof. Let Ω_n be a Cauchy sequence; this means that $\{v_{\Omega_n}\}_n$ is a Cauchy sequence in L^p. Since L^p is complete, $v_{\Omega_n} \to g$ in L^p.

By Lemma 6.6 we know that there exists a compact set K such that the sets $\Omega_n \subseteq K$ for all n. In particular, for any $x \in \mathbb{R}^N$, $u_{\Omega_n}(x) \leq \max_{y \in K} |y - x|$, and then

$$v_{\Omega_n}(x) \geq h(x) \stackrel{\text{def}}{=} \min_{y \in K} \varphi(|y - x|) \ ;$$

note that $h(x) > 0$ at all points.

It is well known (see, e.g., Theorem 4.9 in [4]) that, up to a subsequence which we indicate with $\{v_k\}_k$, there is also a convergence $v_k(x) \to g(x)$ for almost all x. By the above reasoning, $g(x) \geq h(x) > 0$ in all points of convergence.

Let $u_k(x) \stackrel{\text{def}}{=} \varphi^{-1} v_k(x)$ and $u = \varphi^{-1} g$; then $u_k(x) \to u(x)$ on a dense subset, so by Lemmas 5.19 and 5.20, $u = u_\Omega$, where $\Omega \stackrel{\text{def}}{=} \{u = 0\}$. ∎

Summarizing, this proposition together with Theorems 5.1 and 6.11 implies that $\mathcal{N}$ is a complete (that is, closed) and locally compact subset of L^p. ($\mathcal{N}$ was defined in (32) as the family of all functions v_A for A compact sets.)

Remark 6.13. The above implies an interesting property of the subset $\mathcal{N}$ of L^p: it admits a small neighborhood U on L^p such that, for $f \in U$, there is at least a $v \in \mathcal{N}$ providing the minimum of the distance $\inf_{v\in\mathcal{N}} \|f - v\|$. As far as we know, this minimum may fail to be unique.

6.2. Shape analysis. The family of distances is suitable for shape analysis.

Proposition 6.14. *Let $G = \mathcal{O}(N) \ltimes \mathbb{R}^N$ be the Euclidean group (i.e., the group generated by rotations, translations, and reflections); as in (15), we can define the quotient metric by*

$$d_q([A],[B]) = \inf_{g\in G} d(gA, B). \tag{43}$$

Then the above infimum is a minimum; so $d_q([A],[B]) > 0$ when $[A] \neq [B]$.

Proof. Choose a minimizing sequence $\{g_n = (R_n, T_n)\}_{n\in\mathbb{N}}$, that is,

$$\inf_{g\in G} d(gA, B) = \lim_{n\to\infty} d(g_n A, B) = \lim_{n\to\infty} d(R_n A + T_n, B).$$

Then $\{T_n\}_{n\in\mathbb{N}}$ must be bounded; we prove this by contradiction. Let us assume that $|T_n| \to \infty$; then by (33) we would have that

$$d(A,B) < \sqrt[p]{\|v_A\|^p_{L^p} + \|v_B\|^p_{L^p}}$$

and by Lemma A.1 that

$$\lim_{n\to\infty} d(R_n A + T_n, B) = \sqrt[p]{\|v_A\|^p_{L^p} + \|v_B\|^p_{L^p}}\ ,$$

so $\{g_n\}$ is not a minimizing sequence.

So the translation part of every minimizing sequence of (43) must be bounded. By compactness we have that there exists a limit transformation $g = (R,T) \in G$ and subsequence such that $g_{n_k} \to_k g$; by continuity of $d(fA,B)$ w.r.t. $f \in G$, we have that $d(gA,B) = d_q([A],[B])$. ■

6.3. d^g and geodesics. Let $d = d_{p,\varphi}$ in the following.

Proposition 6.15. *Given any two $A, B \in \mathcal{M}$ with $A \neq B$, then for all $\lambda \in (0,1)$, $\lambda v_A + (1-\lambda) v_B \notin \mathcal{N}$.*

Proof. It is easy to show that $f_\lambda = \lambda v_A + (1-\lambda) v_B$ assumes the value $\varphi(0)$ only on the intersection of the two sets $A \cap B$ for any $\lambda \in (0,1)$. Then $f_\lambda \in \mathcal{N}$ implies that $f_\lambda = v_{A\cap B}$. If $A \cap B = \emptyset$, the proof ends. Suppose without loss of generality that $A \setminus B \neq \emptyset$. Let $x \in A \setminus B$; then $\varphi(0) = v_A(x) > v_B(x) \geq v_{A\cap B}(x)$; so $f_\lambda(x) > v_{A\cap B}(x)$, achieving a contradiction. ■

Similarly, the convex combination $\lambda u_A + (1-\lambda) u_B$ of two distance functions u_A, u_B is not a distance function (but for the special cases $\lambda \in \{0,1\}$ or $A = B$).

Corollary 6.16. *Suppose that $p \in (1,\infty)$. Given any two $A, B \in \mathcal{M}$ with $A \neq B$, we have that $d(A,B) < d^g(A,B)$.*

The result follows from the previous proposition and Theorem 2.15.

In the above cases d is not intrinsic. So, to prove that the metric d admits geodesics, we have to study d^g as well.

We will need some extra hypotheses in many of the following results.

Hypotheses 6.17. Let φ be as defined in 6.1. We moreover suppose that there is a constant $T > 0$ such that $\varphi(t)$ is convex for $t \geq T$. When $p < \infty$, we suppose that

$$\varphi'(|x|) \in L^p(\mathbb{R}^N). \tag{44}$$

The above implies that $\lim_{t\to\infty} \varphi'(t) = 0$. Note also that (44) is equivalent to asking that

$$\int_0^\infty t^{N-1} |\varphi'(t)|^p \, \mathrm{d}t < \infty \ . \tag{45}$$

Proposition 6.18. *If Hypotheses 6.17 hold, then the space $(\mathcal{M}, d_{p,\varphi})$ is Lipschitz-arc connected.*

The proof is in section A.7.

When $\mathcal{M}$ is Lipschitz-arcwise connected, the induced metric $d^g = (d_{p,\varphi})^g$ is a finite metric, that is, $d^g(A, B) < \infty$ for all A, B compact.

We can prove an equiboundedness result for d^g (which is stronger than the one in Proposition 6.6).

Proposition 6.19. *Suppose that Hypotheses 6.17 hold. Fix a compact nonempty set Ω and an $r > 0$; then there is a K compact such that for any closed set Ω' satisfying $d^g(\Omega, \Omega') < r$ we have $\Omega' \subseteq K$.*

Proof. Let $b(r)$ be defined by Proposition 6.6. Let $d^g(\Omega, \Omega') < r$ and $\gamma : [0, 1] \to \mathcal{N}$ be a Lipschitz path (of constant L) connecting $\gamma(0) = \Omega$ to $\gamma(1) = \Omega'$ such that

$$\mathrm{Len}^d \gamma \leq d^g(\Omega, \Omega') + 1 \ .$$

Up to reparameterization, we also assume that $L \leq r + 2$. Let n be large so that $(r + 2)/n \leq b(r)$, and let $K = \Omega + D_{rn}$ (note that n depends only on r). Let $A_i = \gamma(i/n)$ for $i = 0, \ldots, n$; we know that

$$d(A_i, A_{i+1}) \leq d^g(A_i, A_{i+1}) \leq L/n < (r + 2)/n \leq b(r)$$

since γ is L-Lipschitz; so we apply recursively Lemma 6.6 on each A_i. We obtain that

$$A_{i+1} \subseteq A_i + D_r \ ,$$

and hence $\Omega' \subseteq \Omega + D_{rn} = K$. ■

The above results have many interesting consequences.

Theorem 6.20. *Suppose that Hypotheses 6.17 hold. For any $\rho > 0$,*

$$\mathbb{D}^g(A, \rho) \stackrel{\text{def}}{=} \{A \mid d^g(A, B) \leq \rho\}$$

is compact in the $(\mathcal{M}, d)$ topology; so

- *we obtain by Propositions 2.7 and 6.18 that geodesics do exist,*

- *and by Proposition* 2.8 *that the* geodesic distance-based averaging

$$\bar{A} = \operatorname{argmin}_A \sum_{j=1}^{n} d^g(A, A_j)^2 \tag{46}$$

of any given collection $A_1, \dots, A_n$ *exists.*

Two examples of geodesics are in Figures 3 and 4 in section 6.8.

6.4. Variational description of geodesics. In this section we restrict $p \in (1, \infty)$.

We first state these general results, based on the well-known L^p theory.

Proposition 6.21. *Suppose that* $t \mapsto f(t, \cdot)$ *is a Lipschitz path from* $t \in [0, 1]$ *to* $L^p(\mathbb{R}^N)$*; then, for almost all* t*,* f *admits a strong derivative* $\frac{df}{dt}$ *that is the limit*

$$\frac{df}{dt}(t, \cdot) \stackrel{\text{def}}{=} \lim_{\tau \to 0} \frac{f(t + \tau, \cdot) - f(t, \cdot)}{\tau} \tag{47}$$

in $L^p(\mathbb{R}^N)$*. This follows from Corollary* 2.13*. Moreover*

- f *admits a weak partial derivative* $\partial_t f$*, and* $\partial_t f = \frac{df}{dt}$ *for almost all* t*;*
- *if* f *admits a pointwise partial derivative in* t *for almost all* t, x*, which we will call* h*, then* $\partial_t f = h$*.*

The proof is in section A.8.

If $\gamma(t)$ is a Lipschitz path in $(\mathcal{M}, d)$, then it is associated to a function $f(t, x) = \varphi(u_{\gamma(t)}(x))$. If γ is Lipschitz, then $f(t, \cdot)$ satisfies the hypotheses of the above proposition. The first point means that we can represent the "abstract" derivative $\frac{d\gamma}{dt}$ by means of the weak derivative $\partial_t f(t, \cdot) \in L^p(\mathbb{R}^N)$. The second point is used to compute the derivative in practical cases, such as the following examples.

The above proposition can also be used to provide a variational description of geodesics. Let $\gamma : [0, 1] \to \mathcal{M}$ be a path; by Lemma 2.3 the variational length $\operatorname{Len}^d \gamma$ can be computed using the integral length $\operatorname{len}^d \gamma$ of the metric derivative $|\dot{\gamma}(t)|$; by Theorem 2.11 the metric derivative $|\dot{\gamma}(t)|$ coincides with the norm $\|\dot{v}_{\gamma(t)}\|_{L^p}$ of the derivative $\dot{v}_{\gamma(t)}$ in the Banach space L^p; by the above result, $\|\dot{v}_{\gamma(t)}\|_{L^p} = \|\partial_t f(t, x)\|_{L^p}$. Summarizing,

$$\operatorname{Len}^d \gamma = \int_0^1 \|\partial_t f(t, \cdot)\|_{L^p} \, \mathrm{d}t \ . \tag{48}$$

So to find the geodesic between two compact sets A, B, we need to minimize the above with the following constraints:

- $f(0, \cdot) = v_A$, $f(1, \cdot) = v_B$;
- for any fixed t, $\varphi^{-1} \circ f(t, \cdot)$ is a distance function.

It is possible to prove (using a reparameterization lemma and Hölder inequality) that the geodesic is also the minimum of the *action*

$$J(\gamma) = \int_0^1 \|\partial_t f(t, x)\|_{L^p}^p \, \mathrm{d}t = \int_0^1 \int_{\mathbb{R}^N} |\partial_t f(t, x)|^p \, \mathrm{d}x \, \mathrm{d}t \ . \tag{49}$$

Equivalently, setting $g(t, x) = u_{\gamma(t)}(x)$, to find geodesics we can minimize

$$J(\gamma) = \int_a^b \int_{\mathbb{R}^N} |\varphi'(g) \partial_t g(t, x)|^p \, \mathrm{d}x dt$$

with the constraint that $g(0,\cdot) = u_A$, $g(1,\cdot) = u_B$, and, for any fixed t, $g(t,\cdot)$ is a distance function.

6.4.1. Examples.

Example 6.22. Let $N = 2$, $p = 2$, $\varphi(t)$ smooth, $r \ge 0$. Consider the path $\gamma(t)$ of disks of center $(t,0) \in \mathbb{R}^2$ and radius r in $\mathbb{R}^2$ for $t \in [0,1]$. We want to compare the length of this path as computed using the Hausdorff distance and using the distance d.

- We have $d_H(\gamma(0),\gamma(1)) = 1$, which is also the length $\mathrm{Len}^{d_H}\gamma$ of the path γ.
- We use the expression (48). We have

$$v_{\gamma_t}(x_1,x_2) = \begin{cases} \varphi(0) & \text{if } x_2^2 + (x_1-t)^2 \le r^2 \ , \\ \varphi\Big(\sqrt{x_2^2+(x_1-t)^2} - r\Big) & \text{otherwise .} \end{cases}$$

Upon derivation with respect to t,

$$\partial_t v_{\gamma_t}(x_1,x_2) = \begin{cases} 0 & \text{if } x_2^2 + (x_1-t)^2 \le r^2 \ , \\ \frac{t-x_1}{\sqrt{x_2^2+(x_1-t)^2}}\varphi'(\sqrt{x_2^2+(x_1-t)^2} - r) & \text{otherwise ,} \end{cases}$$

so (using polar coordinates around $(t,0)$)

$$\begin{aligned} \|\partial_t v_{\gamma_t}\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |\partial_t v_{\gamma_t}(x_1,x_2)|^2 \,\mathrm{d}x \\ &= \pi \int_r^\infty \Big(\varphi'(\rho - r)\Big)^2 \rho \,\mathrm{d}\rho = \pi b + r a \pi \ , \end{aligned}$$

where

$$a = \int_0^\infty \big(\varphi'(s)\big)^2 \,\mathrm{d}s \ , \quad b = \int_0^\infty \big(\varphi'(s)\big)^2 s \,\mathrm{d}s \ .$$

For example, when $\varphi(t) = e^{-t}$ then the length $\mathrm{Len}^d\gamma$ is

$$\frac{\sqrt{\pi(1+2r)}}{2} \ .$$

A more general computation of the length of motions of convex bodies will be performed in section 6.6.1.

Example 6.23. Let A be compact. We consider again the *carving* motion that we saw in Proposition 5.2(5); to simplify the matter, suppose that the origin is in the topological interior of A; let $R > 0$ such that $B_R \subseteq A$; for $t \in [0,R]$ let $\gamma(t) = A_t = A \setminus B_t$ be the carving of a small ball from A.

We suppose $p \in (1,\infty)$, and we also suppose that $\varphi'(0) \ne 0$, for simplicity.

We can explicitly compute (for $r > 0, s > 0$ with $r + s \le R$)

$$f(r,x) = v_{\gamma(r)}(x) = \begin{cases} \varphi(r - |x|) & \text{if } |x| \le r \ , \\ v_A(x) & \text{if } |x| > r \ ; \end{cases}$$

hence

$$\begin{aligned}\|v_{A_s} - v_{A_{r+s}}\|^p_{L^p} &= \omega_N N \int_s^{r+s} t^{N-1}\Big(\varphi(0) - \varphi(s+r-t)\Big)^p dt \\ &\quad + \omega_N N \int_0^s t^{N-1}\Big(\varphi(s-t) - \varphi(s+r-t)\Big)^p dt \\ &\le \omega_N r^p L^p (r+s)^N \ ,\end{aligned}$$

where L is the Lipschitz constant of $\varphi(t)$ for small t. We have thus proved that the *carving motion* is Lipschitz for the distance $d_{p,\varphi}$.

Note that $d_H(A, A_t) = t$, but

$$d_{p,\varphi}(A, A_t) \sim t^{1+N/p} \quad ;$$

so the two distances are not locally equivalent when $p \in [1,\infty)$; moreover the estimate (40) is sharp.

Suppose now, moreover, that $p \in (1,\infty)$. Using Proposition 6.21, we can compute the metric derivative of γ,

$$|\dot\gamma|(r) = \|\partial_r f(r,\cdot)\|_{L^p} = \sqrt[p]{\omega_N N \int_0^r s^{N-1}\Big|\varphi'(r-s)\Big|^p \,\mathrm{d}s} \ ,$$

and using Theorem 2.11 we obtain that the length of γ_t for $t \in [a,b]$ is

$$\mathrm{Len}^d \gamma|_{[a,b]} = \int_a^b \sqrt[p]{\omega_N N \int_0^r s^{N-1}\Big(\varphi'(r-s)\Big)^p \,\mathrm{d}s} \,\mathrm{d}r \ ;$$

in particular for r small we obtain

$$\mathrm{Len}^d \gamma|_{[0,r]} \sim r^{1+N/p} \ . \tag{50}$$

So $d^g(A, A_r) \le O(r^{1+N/p})$, and hence the two distances d^g and d_H are not locally equivalent when $p \in (1,\infty)$.

6.5. Tangent bundle. Let $p \in (1,\infty)$. We identify $\mathcal{M}$ with $\mathcal{N} \subseteq L^p$, as by Remark 6.4. Given a $v \in \mathcal{N}$, let $T_v\mathcal{N} \subseteq L^p$ be the *contingent cone*

$$\begin{aligned} T_v\mathcal{N} &\overset{\mathrm{def}}{=} \Big\{\lim_n t_n(v_n - v) \mid t_n > 0, v_n \in \mathcal{N}, v_n \to v\Big\} \\ &= \left\{\lambda \lim_n \frac{v_n - v}{\|v_n - v\|_{L^p}} \mid \lambda \ge 0, v_n \to v\right\} \ ,\end{aligned}$$

where we consider all sequences t_n, v_n such that the limit exists; it is intended that the above limits are in the sense of strong convergence in L^p.

According to Corollary 2.13 if $\gamma : [a,b] \to \mathcal{N}$ is a Lipschitz path, then $\dot\gamma(t)$ exists (in the strong sense) in $L^p(\mathbb{R}^N)$ for almost all t; so $\dot\gamma(t) \in T_\gamma\mathcal{N}$ for almost all t.

In the following example we write explicitly the element of the contingent cone relative to a particular path.

Example 6.24. We fix $\Omega \in \mathcal{M}$. We define the fattening $\Omega_t = \Omega + D_t$ for $t \geq 0$. We are interested in evaluating the derivative $\dot{\gamma}(t)$. As previously done, we use the relationship (24), namely,

$$u_{\Omega_t}(x) = (u_\Omega(x) - t)^+ \ , \tag{51}$$

and note that this map is jointly Lipschitz in (t, x); hence both $u_{\Omega_t}(x)$ and $v_{\Omega_t}(x)$ are almost everywhere differentiable. The pointwise derivative is given by

$$w = \lim_{\tau \to 0} \frac{1}{\tau} \left[v_{\Omega_{t+\tau}} - v_{\Omega_t} \right] = \begin{cases} -\varphi'(u_\Omega(x) - t) & \text{for } \ x \notin \Omega_t \ , \\ 0 & \text{for } \ x \in \mathring{\Omega}_t \ , \end{cases} \tag{52}$$

where $\mathring{\Omega}_t$ is the topological interior. Note that the derivative may not exist for $x \in \partial\Omega_t$. If $\varphi'(|x|) \in L^p$, then $w \in L^p$, and it can be shown that

$$w = \lim_{\tau \to 0} \frac{1}{\tau} \left[v_{\Omega_{t+\tau}} - v_{\Omega_t} \right]$$

in the L^p sense; then w is in the contingent cone. In particular, by Remark 1.1.3 in [2], we obtain that the path γ is Lipschitz for $t \in [0, T]$.

Unfortunately the contingent cone is not capable of expressing some shape motions.

Example 6.25. We consider again the *carving* motion that we saw in Proposition 5.2(5) and in Example 6.23. We define for convenience the functions

$$w_t \stackrel{\text{def}}{=} \frac{v_{A_t} - v_A}{\|v_{A_t} - v_A\|_{L^p}} \ .$$

These do not admit a limit in $L^p(\mathbb{R}^N)$ when $t \to 0+$.

Suppose by contradiction that $\lim_{t \to 0+} w_t = w$ in $L^p(\mathbb{R}^N)$. Let us fix $r > t > 0$; then for any x outside of B_r we have that $v_{A_t}(x) = v_A(x)$ and consequently $w_t(x) = 0$. Hence $w(x) = 0$ for almost any x outside of B_r. By arbitrariness of r this would imply that $w = 0$ for almost every x. At the same time since $\|w_t\|_{L^p} = 1$ for all $t > 0$, then $\|w\|_{L^p} = 1$; so w cannot exist.

We conclude that the "velocity" of the carving motion does not admit a representation in $T_v\mathcal{N}$ at the time $t = 0$.

6.6. Riemannian metric. Now let $p = 2$. The set $\mathcal{N}$ may fail to be a smooth submanifold of L^2; yet we will, as much as possible, pretend that it is, in order to induce a sort of "Riemannian metric" on $\mathcal{N}$ from the standard L^2 metric.

We define the "Riemannian metric" on $\mathcal{N}$ simply by

$$\langle h, k \rangle \stackrel{\text{def}}{=} \langle h, k \rangle_{L^2}$$

for $h, k \in T_v\mathcal{N}$ and correspondingly a norm by

$$|h| \stackrel{\text{def}}{=} \sqrt{\langle h, h \rangle} \ .$$

Remark 6.26. We also argue that the distance induced by this "Riemannian metric" coincides with the geodesically induced distance d^g. Indeed let $\gamma : [a,b] \to M$ be a Lipschitz path in $\mathcal{N}$; by Corollary 2.13 the derivative $\dot\gamma(t)$ exists in $L^2(\mathbb{R}^N)$ for almost all t; so we may define the "Riemannian length" of the path as in (18), namely,

$$\operatorname{len}\gamma \stackrel{\text{def}}{=} \int_a^b \|\dot\gamma(\theta)\|_{L^2}\,\mathrm{d}\theta\ .$$

Then we define the "Riemannian distance" $d^R(x,y)$ as the infimum of $\operatorname{len}\gamma$ for all γ connecting x to y. But by Theorem 2.11 $\operatorname{len}\gamma = \operatorname{Len}^d\gamma$ and $d^R = d^g$.

6.6.1. Riemannian metric for smooth convex sets. We propose an explicit computation of the Riemannian metric. We fix $p=2$, $N=2$. Let $\Omega \subseteq \mathbb{R}^2$ be a convex set with smooth boundary of length L. Let $y(\theta) : [0,L] \to \partial\Omega$ be a parameterization of the boundary, and $\nu(\theta)$ the unit vector normal to $\partial\Omega$ and pointing external to Ω; then the following *"polar"* change of coordinates holds:

$$\psi : \mathbb{R}^+ \times [0,L] \to \mathbb{R}\setminus\Omega\ ,\quad \psi(\rho,\theta) = y(\theta) + \rho\nu(\theta)\ .$$

We suppose that $y(\theta)$ moves on $\partial\Omega$ in a counterclockwise direction; so

$$\nu = J\partial_s y\ ,\quad \partial_{ss}y = -\kappa\nu\ ,\quad \partial_s\nu = \kappa\partial_s y\ ,$$

where J is the rotation matrix (of angle $-\pi/2$), κ is the curvature, and $\partial_s y$ is the tangent vector (obtained by deriving y w.r.t. the arc parameter).

We can then express a generic integral through this change of coordinates as

$$\int_{\mathbb{R}^2\setminus\Omega} f(x)\ \mathrm{d}x = \int_{\mathbb{R}^+}\int_{\partial\Omega} f(\psi(\rho,s))|1+\rho\kappa(s)|\ \mathrm{d}\rho\,\mathrm{d}s\ ,$$

where s is the arc parameter and $\mathrm{d}s$ is the integration in arc parameter.

We want to study a smooth deformation of Ω, which we call Ω_t; then the boundary parameterization $y(\theta,t)$ depends on a time parameter t. Suppose also that $\kappa(\theta) > 0$, that is, that the set is strictly convex; then for small smooth deformations, the set Ω_t will still be strictly convex ("small" is intended in the C^2 norm). By deriving

$$\partial_t\partial_s y = \partial_s(\partial_t y) - \partial_s y\langle\partial_s y, \partial_s(\partial_t y)\rangle = \pi_\nu(\partial_s(\partial_t y))\ ,$$

where

$$\pi_\nu(w) \stackrel{\text{def}}{=} \nu\langle\nu,w\rangle = w - \partial_s y\langle\partial_s y, w\rangle$$

is the projection of w on the line generated by ν. Supposing now that $\rho = \rho(t)$ as well, we can express the point $\psi(\rho,\theta)$ in a first order approximation w.r.t. changes in t,θ as

$$d\psi = \Big(\partial_t y + \rho'\nu + \rho J\pi_\nu(\partial_s(\partial_t y))\Big)dt + \Big(\partial_\theta y + \rho\partial_\theta\nu\Big)d\theta\ ,$$

where moreover

$$\Big(\partial_\theta y + \rho\partial_\theta\nu\Big)d\theta = \Big(\partial_s y + \rho\partial_s\nu\Big)ds = \Big(1+\rho\kappa\Big)\partial_s y ds\ .$$

If $y(\theta,t),\rho(t)$ are expressing a constant point $x=\psi(\rho,\theta)$, then $d\psi=0$; we apply scalar products w.r.t. ν and $\partial_s y$ to the above relations, obtaining

$$\langle \nu,(\partial_t y)\rangle+\rho'=0\ ,\quad \langle \partial_s y,\partial_t y\rangle dt-\rho\langle \nu,\partial_s(\partial_t y)\rangle dt+(1+\rho\kappa)ds=0\ .$$

We assume now that each point of $\partial\Omega_t$ moves orthogonally to it; this means that $\partial_t y\perp\partial_s y$, so we can express the motion using a scalar field $\alpha=\alpha(\theta,t)\in\mathbb{R}$ by setting $\partial_t y=\alpha\nu$. So we simplify the above to obtain the relationships

$$\rho'=-\alpha\ ,\quad \frac{ds}{dt}=\frac{\rho\langle\nu,\partial_s(\alpha\nu)\rangle}{(1+\rho\kappa)}=\frac{\rho\ \partial_s\alpha}{(1+\rho\kappa)}\ .$$

Now let

$$h_\alpha(x)\stackrel{\text{def}}{=}\partial_t v_{\Omega_t}(x)\ ,$$

so h_α is the vector in $T_v\mathcal{N}$ that is associated to the velocity field α that is moving the border of Ω.

Now, for $x\notin\Omega_t$ we can write

$$x=\psi(\rho,\theta)=y(\theta,t)+\rho(t)\nu(\theta,t)\ ,$$

so by following the above relations we know that $u_{\Omega_t}(x)=\rho(t)$ and hence

$$h_\alpha(x)=-\varphi'(\rho(t))\alpha(\theta,t)\ ,$$

whereas $h_\alpha(x)=0$ for $x\in\mathring{\Omega}_t$ (the topological interior of Ω_t).

We now wish to use the above computation to pull back the "Riemannian metric" that we presented in the beginning of section 6.6 to the family of orthogonal deformations of $\partial\Omega$. So let us fix two smooth vector fields $\alpha(s)\nu(s)$ and $\beta(s)\nu(s)$, each orthogonal to $\partial\Omega$; these represent two possible infinitesimal deformations of $\partial\Omega$; those correspond to two vectors $h_\alpha,h_\beta\in T_v\mathcal{N}$. By our initial definition,

$$\langle h,k\rangle\stackrel{\text{def}}{=}\langle h,k\rangle_{L^2}=\int_{\mathbb{R}^2}h_\alpha(x)h_\beta(x)\,\mathrm{d}x\ ,$$

so by pullback we impose that

$$\langle\alpha,\beta\rangle\stackrel{\text{def}}{=}\int_{\mathbb{R}^2}h_\alpha(x)h_\beta(x)\,\mathrm{d}x\ .$$

Using the previous computation, we can then expand and obtain that

$$\begin{aligned}\langle\alpha,\beta\rangle&=\int_{\mathbb{R}^2\setminus\Omega}h_\alpha(x)h_\beta(x)\,\mathrm{d}x\\&=\int_{\partial\Omega}\left[\int_{\mathbb{R}^+}(\varphi'(\rho))^2(1+\rho\kappa(s))\ d\rho\right]\alpha(s)\beta(s)\,\mathrm{d}s\ ,\end{aligned}$$

that is,

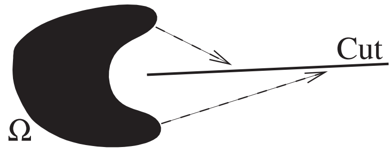
$$\langle\alpha,\beta\rangle=\int_{\partial\Omega}(a+b\kappa(s))\alpha(s)\beta(s)\,\mathrm{d}s \tag{53}$$

with

$$a=\int_{\mathbb{R}^+}(\varphi'(\rho))^2\ d\rho\ ,\quad b=\int_{\mathbb{R}^+}(\varphi'(\rho))^2\rho\ d\rho\ .$$

6.6.2. Riemannian metric for smooth sets. If Ω is smooth but not convex, then the above formula holds up to the cutlocus. We define a function $R(s) : [0, L] \to \mathbb{R}^+$ that spans the cutlocus, that is,

$$\text{Cut} = \{\psi(R(s), s), s \in [0, L]\} \ .$$



ψ is a diffeomorphism between the sets

$$\{(\rho, s) \ s \in [0, L], 0 < \rho < R(s)\} \leftrightarrow \mathbb{R}^2 \setminus (\Omega \cup \text{Cut}) \ ;$$

moreover $R(s)$ is Lipschitz (by results in [9, 12]).

In this case the metric has the form

$$\langle \alpha, \beta \rangle = \int_{\partial\Omega} \left[\int_0^{R(s)} (\varphi'(\rho))^2 (1 + \rho\kappa(s)) \ d\rho \right] \alpha(s)\beta(s) \, \mathrm{d}s \ . \tag{54}$$

Remark 6.27. The above metric (53) resembles the metric presented in [17] for the motion of planar curve. This latter, though, is

$$\int_{\partial\Omega} (a + b\kappa^2(s)) \ k(s) \cdot h(s) \, \mathrm{d}s \ , \tag{55}$$

where $h(s), k(s)$ are vectors that represent infinitesimal displacements of the curve (not necessarily orthogonal to the curve).

In section 3.6 in [17] it is proved that the completion of the space of smooth curves according to the distance derived from the metric (55) is contained in the space of Lipschitz curves.

Now let Ξ be the family of all connected compact sets in $\mathbb{R}^2$, and let Ξ^∞ be the subfamily of all connected compact sets whose boundary is a smooth curve. It is known (see Proposition 5.2) that Ξ is a closed subset of $\mathcal{M}$, according to the Hausdorff distance, and it is reasonably easy to show that Ξ^∞ is dense in Ξ. Since these are topological results, they hold also for the metrics presented in this paper by Theorems 6.11 and 6.30.

So there is a fundamental difference between the metric in (53) and (55).

Let us discuss intuitively what happens when a family of sets $(A_n)_n \subset \Xi^\infty$ approximates a generic connected compact set.

- On one hand, when the set A_n is not convex the metric (53) is substituted by the metric (54), where the cutlocus plays an important part in reducing the cost of moving the boundary.
- But, even more importantly, the term κ in (54) allows for the formation of kinks in the boundaries of A_n so that A_n can approximate A, whereas the term κ^2 in (55) is stronger and the boundaries are not allowed to form singularities.

6.7. Bounds on d^g. We can propose a converse of Proposition 6.19 when additional assumptions hold. In this section we will assume that Hypotheses 6.17 hold, and also that φ is convex (for simplicity).

Lemma 6.28. *We note that*

$$|\nabla v_A(x)| = |\varphi'(u_A(x))| \tag{56}$$

holds for almost all $x \notin A$; indeed we remarked in section 4 that u_A is Lipschitz (hence almost everywhere differentiable) and $|\nabla u_A(x)| = 1$ almost everywhere.

Lemma 6.29. *Let $p \in [1, \infty)$. Assume that Hypotheses 6.17 hold and that φ is convex. Let $r > 0$ and*

$$c^1_{p,\varphi}(r) \stackrel{\text{def}}{=} \sup_{A \subset D_r} \|\nabla v_A\|_{L^p} \ . \tag{57}$$

Then

$$c^1_{p,\varphi}(r)^p = \omega_N r^N |\varphi'(0)|^p + \omega_N N \int_r^\infty t^{N-1} |\varphi'(t-r)|^p \, \mathrm{d}t \ . \tag{58}$$

Proof. By convexity φ' is monotonically nondecreasing and $\varphi'(0) \le \varphi'(t) < 0$. Let $A \subset D_r$ be compact; then for $x \in D_r$, if u_A is differentiable at x, we have

$$|\nabla v_A(x)| = |\varphi'(u_A(x))| |\nabla u_A(x)| \le |\varphi'(0)| \ ,$$

whereas for $x \notin D_r$

$$u_A(x) \ge u_{D_r}(x) = |x| - r \Longrightarrow \varphi'(u_A(x)) \ge \varphi'(|x| - r) \Longrightarrow |\varphi'(u_A(x))| \le |\varphi'(|x| - r)| \ .$$

Equality is obtained by choosing $A = A_\varepsilon$ to be finite collections of points that are ε-nets in D_r (i.e., $A_\varepsilon + D_\varepsilon \supseteq D_r$) and letting $\varepsilon \to 0$. ∎

Theorem 6.30. *Let $p \in (1, \infty)$. Assume again that Hypotheses 6.17 hold and that φ is convex. Then for any continuous path γ and any $r > 0$ such that, for all $t, \gamma(t) \subset D_r$, we have*

$$\mathrm{Len}^d(\gamma) \le c^1_{p,\varphi}(r) \ \mathrm{Len}^{d_H}(\gamma) \ , \tag{59}$$

and then

$$\forall A, B \subset D_r, \quad d^g(A, B) \le c^1_{p,\varphi}(2r) \ d_H(A, B) \ .$$

As a corollary, the topology induced by d^g on $\mathcal{M}$ coincides with the topology induced by d and by d_H. Note, however, that the intrinsic distances d_H and d^g are not equivalent; see Remark 6.31 below.

Proof. Let γ be a path as above. Up to reparameterization Lemma 2.3, we assume that $\gamma : [0, l] \to \mathcal{M}$ with $l = \mathrm{Len}^{d_H} \gamma$ and that the metric derivative is $|\dot\gamma| \equiv 1$; hence

$$d_H(\gamma(t), \gamma(s)) \le |t - s| \ ,$$

which means that

$$\forall x, \quad |u_{\gamma(t)}(x) - u_{\gamma(s)}(x)| \le |t - s| \ ,$$

so $u_{\gamma(t)}(x)$ is jointly Lipschitz continuous, and whenever it is differentiable we have that

$$\left|\frac{\partial}{\partial t}u_{\gamma(t)}(x)\right| \le 1$$

and then

$$\left|\frac{\partial}{\partial t}v_{\gamma(t)}(x)\right| \le |\varphi'(u_{\gamma(t)}(x))| \left|\frac{\partial}{\partial t}u_{\gamma(t)}(x)\right| .$$

By Theorem 2.11, Proposition 6.21, and (48)

$$\mathrm{Len}^d \gamma = \int_0^l \|\frac{\partial}{\partial t}v_{\gamma(t)}\|_{L^p} \, \mathrm{d}t$$

and we use the previous lemma and (56).

For the second inequality, letting $A, B \in \mathcal{M}$ and D_r as above, we use Theorem 5.1 to obtain a geodesic for the Hausdorff metric connecting A to B; then, by the triangle inequality, $\gamma(t) \subset D_{2r}$ for all t, and again we apply the same reasoning. ■

Remark 6.31. Example 6.23 shows that there is no constant c such that

$$\mathrm{Len}^d(\gamma) \ge c \ \mathrm{Len}^{d_H}(\gamma)$$

(i.e., the reverse of (59) does not hold). Indeed in that example we noted in (50) that $\mathrm{Len}^d \gamma|_{[0,r]} \sim r^{1+N/p}$ but $\mathrm{Len}^{d_H} \gamma|_{[0,r]} = r$ for r small.

By the same equation (50) we also obtain that $d^g(A, A_r) \le c r^{1+N/p}$ for $r > 0$ small and $c > 0$ a constant depending on φ. Indeed we recall the definition equation (11) and remark that the path γ is one of the possible paths that connect A to A_r, so $d^g(A, A_r) \le \mathrm{Len}^d \gamma|_{[0,r]}$. At the same time $d^g(A, A_r) = r$. This shows that the intrinsic distances d_H and d^g are not equivalent.

6.8. Numerical approximation. In this section we explain a simple method to numerically approximate the geodesic between two given compact sets A, B. We assume that $p \in (1, \infty)$. Two examples of geodesics computed with this method (choosing $p = 2, \varphi(t) = \exp(-t)$) are in Figures 3 and 4.

Definition 6.32 (cube). *We let $Q = [-1, 1]^N$ be the closed cube of center in the origin and side equal to 2. Let $Q(x, r) \stackrel{\text{def}}{=} x + rQ$ be the cube of center x and side $2r > 0$.*

Definition 6.33 (discretization grids). *Let us fix n_t, n_s large and define $\delta_s > 0, \delta_t = 1/n_t$ small (the "thinness" parameters); consider the following equispaced partitions:*

$$R_{\delta_s, n_s} \stackrel{\text{def}}{=} \{i\delta_s : i = -n_s, \dots, n_s\}^N \subseteq \mathbb{R}^N , \tag{60}$$

$$T_{n_t} \stackrel{\text{def}}{=} \{i\delta_t : i = 0, \dots, n_t\} = \{0, \delta_t, 2\delta_t, \dots 1\} \subseteq [0,1] ; \tag{61}$$

for simplicity in the following we call $T = T_{n_t}$ the time grid *and $R = R_{\delta_s, n_s}$ the* space grid*.*

Note that $R \subset Q(0, \delta_s n_s)$.

Definition 6.34 (pixelization). *Given A closed, the* pixelization *of A to R is*

$$\Pi_R(A) \stackrel{\text{def}}{=} \{x \in R : A \cap Q(x, \delta_s/2) \ne \emptyset\} .$$

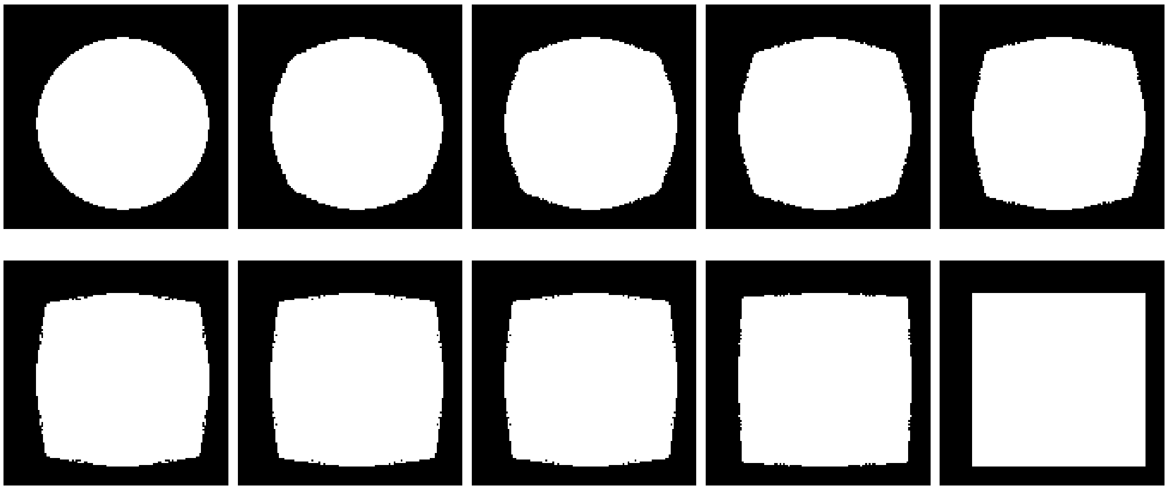


Figure 3. *Example of geodesics connecting a disk and a square.*

Note that $\Pi_R \circ \Pi_R = \Pi_R$. Note also that Π_R is not continuous.[10]

Definition 6.35. *We also define the* discretized (pseudo)distance

$$d_R(A,B) \stackrel{\text{def}}{=} \left[\delta_s^N \sum_{x \in R} |v_A(x) - v_B(x)|^p \right]^{1/p} \tag{62}$$

and the time-discretized length *of a path,*

$$\operatorname{len}_T^d(C) \stackrel{\text{def}}{=} \sum_{t \in T, t<1} d\big(C(t), C(t+\delta_t)\big) \ . \tag{63}$$

Combining the two, we obtain a time-and-space–discretized length $\operatorname{len}_T^{d_R}(C)$.

Remark 6.36. We called d_R a *pseudodistance* since it is not guaranteed that $d_R(A,B) = 0 \implies A = B$. This can be seen by setting $\delta_s < 1/2$, $A = Q(0,1)$, and $B = A \setminus E$, where $E = B(e_1\delta_s/4, \delta_s/8)$ is a small open ball contained in A and that does not intersect R. (e_1 is the first vector of the canonical basis.) At the same time, though, d_R is symmetric and satisfies the triangle equation.

6.8.1. Finding numerical geodesics. We fix $A, B \subseteq \mathbb{R}^N$ compact. We assume that $p \in (1,\infty)$. We assume that n_s is large so that

$$A \subseteq Q(0, n_s\delta_s) \ , \quad B \subseteq Q(0, n_s\delta_s) \ .$$

We let $T = T_{n_t}$ and $R = R_{\delta_s,n_s}$. We define $\mathcal{C}(T,R)$ the space of all the *discretized paths* $C : T \to \mathcal{P}(R)$ such that

$$C(0) = \Pi_R(A) \ , \quad C(1) = \Pi_R(B) \ .$$

[10] As a map from $(\mathcal{M}, d_H)$ into itself, where again $\mathcal{M}$ is the family of the nonempty compact sets in $\mathbb{R}^N$ and d_H is the Hausdorff distance.

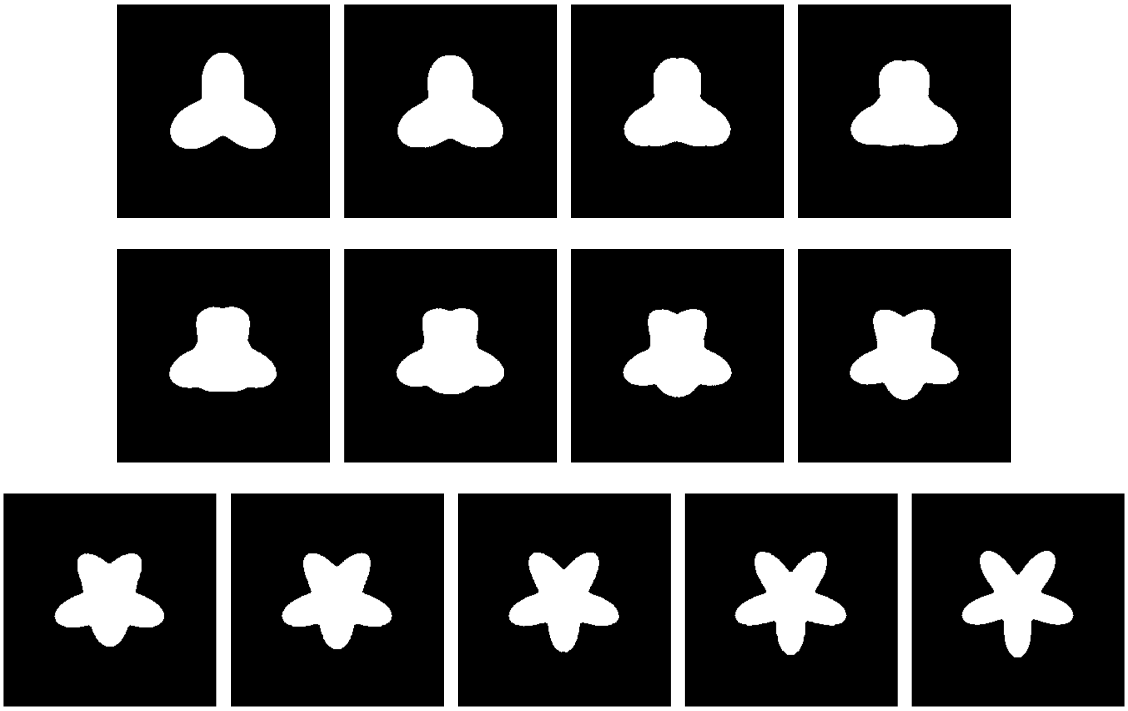


Figure 4. *Example of a geodesic connecting nonconvex sets.*

To find a numerical approximation to the geodesic connecting A to B, we solve the problem

$$d^g_{T,R}(A,B) \stackrel{\text{def}}{=} \min_{C\in\mathcal{C}(T,R)} \text{len}^{d_R}_T(C) \ . \tag{64}$$

The complexity of the minimization problem is exponential,[11] and thus we reduce it using an iterative method. To this end we define

$$P^+_{C(t)} = \{x \in R : x \notin C(t), \mathcal{B}(x) \cap C(t) \neq \emptyset\} \ , \tag{65}$$

$$P^-_{C(t)} = \{x \in R : x \in C(t), \mathcal{B}(x) \setminus C(t) \neq \emptyset\} \ , \tag{66}$$

where $\mathcal{B}(x)$ are the $(2N)$ points (at most) that are nearest neighbors to x in the grid R. We notice that for any $t \in T$ both $P^+_{C(t)}$ and $P^-_{C(t)}$ are discretized versions of the boundary of $C(t)$, the first one from the outside and second from the inside.

For any $t \in T$, $t \neq 0, 1$, and $x \in R$ we define the one-point variation

$$V_{x,t} : \mathcal{C}(T,R) \to \mathcal{C}(T,R)$$

by

$$(V_{x,t}C)(t') = \begin{cases} \{x\}\Delta C(t') & \text{if } t = t' \ , \\ C(t') & \text{otherwise} \ , \end{cases} \tag{67}$$

[11] There are indeed $2^{(2n_s+1)^N(n_t-1)}$ elements in $\mathcal{C}(T,R)$.

where Δ is the set symmetric difference.

Let $C_0 \in \mathcal{C}(T,R)$ be a starting path; in our experiment we defined it by the level set of the linear interpolation of signed distance functions (2), that is,

$$C_0(t) = \{x \in R : t b_B(x) + (1-t) b_A(x) \le 0\} \ .$$

We then evolve it in such a way that at any step $n \in \mathbb{N}$ we decrease the quantity $\mathrm{len}_T^{d_R}(C_n)$. Let

$$(\hat{x},\hat{t}) = \operatorname*{argmin}_{(x,t)\in P^+_{C_s}\cup P^-_{C_s}} \mathrm{len}_T^{d_R}(V_{x,t}C_n) \ ; \tag{68}$$

then we define $C_{n+1} = V_{\hat{x},\hat{t}}C_n$.

We tested this algorithm on some simple shapes; two examples of geodesics (choosing $p=2$, $\varphi(t) = \exp(-t)$) are in Figures 3 and 4. To produce the examples below, we iterated the above algorithm until the energy seemed to stabilize.

Remark 6.37. This numerical method is presented only for the sake of exemplification. No study of the actual convergence of this algorithm has been performed (yet). The approximation step may be ameliorated; in its current form it does not explore carving motions (although it allows for changes in topology).

7. Other Banach-like metrics of shapes. The paradigm that we presented in the previous section may be exploited in other similar ways; to conclude the paper, we shortly present some different embeddings (leaving to a possible future paper the detailed study of their properties).

7.1. Signed distance–based representation. We may use the signed distance function b_A, which was defined in (2), to define a metric of shapes:

$$d'(A,B) \stackrel{\text{def}}{=} \|\varphi(b_A) - \varphi(b_B)\|_{L^p(\mathbb{R}^N)} \ .$$

In this case, we require that the function $\varphi : \mathbb{R} \to (0,\infty)$ be monotonically decreasing and of class C^1 and such that

$$\varphi(|x| - t) \in L^p(\mathbb{R}^N) \quad \forall t. \tag{69}$$

The resulting metric is slightly stronger than the one we studied in the preceding sections; in particular, see the following remark.

Remark 7.1. Let $\mathcal{F}$ be the class of all finite subsets of $\mathbb{R}^N$; this class is dense in $\mathcal{M}$ when we use the metric $d_{p,\varphi}$, or the Hausdorff metric; but it is not dense when we use the metric d'.

7.2. $W^{1,p}$ metrics. Another interesting choice of metric is obtained by embedding the representation in $W^{1,p}$ for $p \in (1,\infty)$.

We require that all of the hypotheses in Hypotheses 6.1 and 6.17 hold. Namely, $\varphi : [0,\infty) \to (0,\infty)$ is Lipschitz, C^1, and monotonically decreasing, and $\varphi(|x|) \in W^{1,p}(\mathbb{R}^N)$; for the case $p < \infty$ we are equivalently asking that

$$\int_0^\infty t^{N-1}(\varphi(t)^p + |\varphi'(t)|^p)\, dt < \infty \ ,$$

and this implies that $\lim_{t\to\infty}\varphi(t) = 0 = \lim_{t\to\infty}\varphi'(t)$. We also assume that there is a $T > 0$ such that $\varphi(t)$ is convex for $t \in [T, \infty)$.

Proposition 7.2. *For any A compact we have $v_A \in W^{1,p}(\mathbb{R}^N)$.*

Proof. We already know by Lemma 6.2 that $v_A \in L^p(\mathbb{R}^N)$.

By hypotheses above, v_A is Lipschitz; and then, for almost all x, $\nabla v_A = \varphi'(u_A)\nabla u_A$, where $|\nabla u_A| = 1$ for almost all $x \notin A$, while $\nabla u_A = 0$ for almost all $x \in A$. We also know that when $t > T$, $\varphi'(t) < 0$, φ' is increasing and $\varphi'(t) \uparrow 0$.

Let $R > 0$ be large so that $A \subseteq B_R$. Then

$$u_A(x) \geq |x| - R\ ,$$

and then when $|x| \geq R + T$ we obtain that

$$\varphi'(u_A(x)) \geq \varphi'(|x| - R)\ ,$$

that is,

$$\int_{\mathbb{R}^N\setminus B_{R+T}} |\varphi'(u_A(x))|^p \ \mathrm{d}x \leq \int_{\mathbb{R}^N\setminus B_{R+T}} |\varphi'(|x| - R)|^p \ \mathrm{d}x < \infty\ .$$

At the same time, since v_A is Lipschitz, then $\int_{B_{R+T}} |\nabla v_A|\,\mathrm{d}x$ is finite. ■

Definition 7.3. *Given $A, B \in \mathcal{M}$, we define*

$$d_{1,p,\varphi}(A, B) \stackrel{\text{def}}{=} \|\varphi(u_A) - \varphi(u_B)\|_{W^{1,p}(\mathbb{R}^N)}\ .$$

We just state a simple property of this metric.

Proposition 7.4. *Let $\mathcal{F}$ again be the class of all finite subsets of $\mathbb{R}^N$: this class is dense in $\mathcal{M}$ iff $\varphi'(0) = 0$.*

The proof is in section A.9.

We just conclude with one last remark.

Remark 7.5. The embedding of $\varphi \circ u_A$ in $W^{2,p}$ is not feasible: if A is smooth but is not convex, the second derivative of u_A along the cutlocus is expressed by a measure (see Corollary 4.13 in [13]) and then $\varphi \circ u_A \notin W^{2,p}$.

8. On the choice of hypotheses. Using arguments in geometric measure theory, it is possible to prove this result.

Proposition 8.1. *Let $f : [0, \infty) \to [0, \infty)$ be a measurable function such that $\int_0^\infty f(t)^p t^{N-1}\,\mathrm{d}t$. Suppose that $A \subset \mathbb{R}^N$ is compact; let u_A be its distance function. Then $f \circ u_A \in L^p(\mathbb{R}^N)$.*

Using this result, it is possible to prove Propositions 6.18 and 7.2 without using the *"convexity"* hypothesis listed in Hypotheses 6.17; hence this hypothesis may be dropped in many other results. Similarly it is possible to prove Lemma 6.2 without using the fact that φ is "strictly decreasing" (as listed in Hypotheses 6.1). The above Proposition 8.1, though, requires a long proof, and relaxing requirements in Hypotheses 6.1 and 6.17 would require longer and more complex proofs in many other propositions. At the same time these generalizations would not improve the usefulness of this theory in applications. So we decided to omit them.

9. Conclusions. We have studied a metric space of shapes $(\mathcal{M}, d_{p,\varphi})$; this space has a "weak distance" in that it has many compact sets, and geodesics do exist; but it can be associated in some cases to a smooth Riemannian metric, as we saw in (53). Moreover, by the properties that we saw in section 2.2 (and, in particular, by the properties of L^p spaces for $p \in (1, \infty)$ that we proved in Theorem 2.15) we can also hope that geodesics can be studied in the ODE sense (although possibly in a very weak sense).

Appendix A. Proofs.

A.1. Proof of (34).

Lemma A.1. *Let $p \in [1, \infty)$. Suppose that $f, g \in L^p(\mathbb{R}^N)$; let $\tau \in \mathbb{R}^N$ and define the translates $g_\tau(x) = g(x - \tau)$; moreover, let $\sigma \in O(N)$ be an orthogonal transformation and $f_\sigma(x) = f(\sigma(x))$ be a rotation of f. Then*

$$\lim_{|\tau| \to \infty} \|f_\sigma - g_\tau\|_{L^p} = \sqrt[p]{\|f\|_{L^p}^p + \|g\|_{L^p}^p} \ , \tag{70}$$

where the limit is uniform in σ.

Proof. The result is obviously true if $f, g \in C_c(\mathbb{R}^N)$. We will prove that the set of f, g such that (70) holds is closed; since C_c is dense in L^p, this will prove the lemma. Choose sequences $(f_n)_n, (g_n)_n \subset L^p$ such that $f_n \to f, g_n \to g$ in $L^p(\mathbb{R}^N)$; define the translates $g_{n,\tau}(x) = g_n(x - \tau)$ and the rotated versions $\hat{f}_n(x) = f_n(\sigma(x))$, where $\sigma \in O(N)$; suppose, moreover, that f_n, g_n satisfy (70), that is,

$$\lim_{|\tau| \to \infty} \|\hat{f}_n - g_{n,\tau}\|_{L^p} = \sqrt[p]{\|f_n\|_{L^p}^p + \|g_n\|_{L^p}^p} \ , \tag{71}$$

where the limit, for each fixed n, is uniform w.r.t. the choice of $\sigma \in O(N)$. We estimate

$$\begin{aligned} \Big| \|f_\sigma - g_\tau\|_{L^p} - \|\hat{f}_n - g_{n,\tau}\|_{L^p} \Big| &\le \|f_\sigma - g_\tau - \hat{f}_n + g_{n,\tau}\|_{L^p} \\ \le \|f_\sigma - \hat{f}_n\|_{L^p} + \|g_\tau - g_{n,\tau}\|_{L^p} &= \|f - f_n\|_{L^p} + \|g - g_n\|_{L^p} \end{aligned}$$

(the last equality derives from Euclidean invariance of the Lebesgue measure). This proves that the term $\|\hat{f}_n - g_{n,\tau}\|_{L^p}$ converges to $\|f_\sigma - g_\tau\|_{L^p}$ as $n \to \infty$ and uniformly w.r.t. τ and σ. Passing to limits in (71) on the left-hand side, we can write

$$\lim_{n \to \infty} \lim_{|\tau| \to \infty} \|\hat{f}_n - g_{n,\tau}\|_{L^p} = \lim_{|\tau| \to \infty} \lim_{n \to \infty} \|\hat{f}_n - g_{n,\tau}\|_{L^p} = \lim_{|\tau| \to \infty} \|f_\sigma - g_\tau\|_{L^p} \ ,$$

whereas clearly the right-hand side of (71) converges to the right-hand side of (70). ■

A.2. Proof of Proposition 2.8.

Proof. Note first that the infimum of $\tau(a)$ is finite, since it does not exceed ρ^*. Recall that

$$d^g(a, a_j) = \inf_{\gamma_j} l_j \ ,$$

where l_j is the length of a Lipschitz path γ_j connecting a, a_j. So we can rewrite the problem (13) as

$$\inf_{\gamma_1 \cdots \gamma_n} \theta(\gamma_1 \ldots \gamma_n) \ , \quad \text{where } \theta(\gamma_1 \ldots \gamma_n) \stackrel{\text{def}}{=} \sum_{j=1}^n (l_j)^2 \ ,$$

where the infimum is computed on all choices of Lipschitz paths $\gamma_1 \dots \gamma_n$ of length $l_1 \dots l_n$ connecting a_i to a common point $x \in M$; for simplicity we represent them as $\gamma_i : [0, l_i] \to M$ parameterized by an arc parameter. By the triangle inequality

$$d^g(a_i, \gamma_j(t)) \le d^g(a_i, x) + d^g(x, \gamma_j(t)) \le l_i + l_j \ .$$

Then let $\gamma_{i,k}$ be a sequence of choices that converges to the infimum:

$$\theta(\gamma_{1,k} \dots \gamma_{n,k}) \to_k \inf_{\gamma_1 \cdots \gamma_n} \theta(\gamma_1 \dots \gamma_n) \ ,$$

so for large k,

$$\theta(\gamma_{1,k} \dots \gamma_{n,k}) \le \rho^* + \varepsilon \quad ,$$

but then in particular $l_{i,k} \le \sqrt{\rho^* + \varepsilon}$. Hence

$$d^g(a_1, \gamma_{j,k}(t)) \le 2\sqrt{\rho^* + \varepsilon}$$

for all $j = 1, \dots, n$ and $t \in [0, l_{j,k}]$. So all the paths are contained in a compact set. By the Ascoli–Arzelà theorem, we can then extract a uniformly convergent subsequence and use the fact that the length is lower semicontinuous. ■

A.3. Proof of Proposition 5.3.

Proof.

- Obviously C_t is compact and $C_0 = A$, $C_\mu = B$.
- We prove that C_t is not empty. Let $z \in B$; if $u_A(z) = 0$, then $z \in A$, so $z \in C_t$. If $u_A(z) > 0$, let $x \in A$ be a projection point of z so that $u_A(z) = |x - z| = l > 0$, and let
$$y = x + \min\{t, l\} \frac{(z - x)}{|z - x|} \ .$$
Obviously $u_A(y) \le |x - y| \le t$. If $t \ge l$, then $y = z$, so $u_B(y) = 0$ and $y \in C_t$. If $t < l$, then $|z - y| = l - t \le \mu - t$, so $u_B(y) \le \mu - t$ and again $y \in C_t$.
- We prove that for all $0 \le s < t \le \mu$,
$$d_H(C_s, C_t) \le t - s \ .$$
Figure 5 may help in reading the following step.[12]
Letting $z \in C_t$, we will prove that $u_{C_s}(z) \le t - s$. Since $u_A(z) \le t$, there is an $x \in A$ such that $|x - z| \le t$; let y be the interpolated point
$$y = x + \frac{s\,(z - x)}{|z - x|}$$
so that $|y - x| = s$ and $|z - y| = |x - z| - s \le t - s$ and then $u_A(y) \le s$. Since $z \in C_t$, then $u_B(z) \le \mu - t$, and also $|z - y| \le t - s$, so by the triangle inequality $u_B(y) \le \mu - s$. We already noted $u_A(y) \le s$, so we proved that $y \in C_s$; eventually $u_{C_s}(z) \le |z - y| \le t - s$.
Working symmetrically, we can prove that for any $z \in C_s$ we have that $u_{C_t}(z) \le t - s$.

[12]Note that the drawings of C_s and C_t in the figure are not faithful to the actual C_s and C_t—the maximal geodesic is much larger than that, and corners are rounded out. Exact examples of maximal geodesics are in section 5.1.

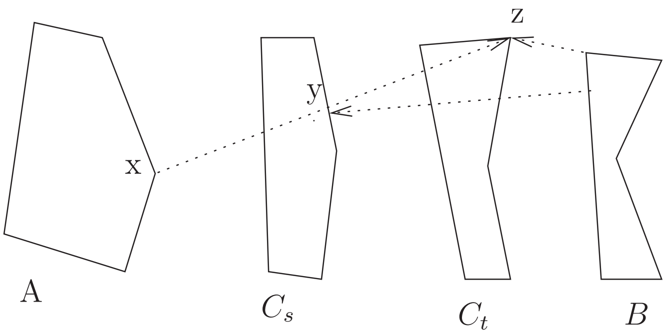


Figure 5. *Helpful illustration for the proof of Proposition 5.3.*

- We prove that for all $0 \le s < t \le \mu$

$$d_H(C_s, C_t) = t - s \quad ;$$

indeed

$$\mu = d_H(A,B) \le d_H(A,C_s) + d_H(C_s,C_t) + d_H(C_t,B) \le \mu \quad ,$$

but then the inequalities must be equalities.

- $d_H(A, \gamma(s)) \le s$ implies that for all $x \in \gamma(s)$, $u_A(x) \le s$; and similarly for $d_H(\gamma(s), B) \le \mu - s$; so $x \in C_s$. ■

A.4. Proof of Lemma 5.9.

Proof. We provide a detailed proof for convenience of the reader.

- We prove that $\max_A u_B(x) = \max_E u_B(x)$. We foremost note that $\max_A u_B(x) \ge \max_E u_B(x)$ since $E \subseteq A$. From $G \cap \{u_B = \theta\} = \emptyset$ we obtain $A \cap \{u_B = \theta\} = E \cap \{u_B = \theta\}$, so we conclude that $\max_A u_B(x) = \max_E u_B(x)$.
- We prove that $\max_B u_A(x) = \max_B u_E(x)$ by proving that $u_A(z) = u_E(z)$ for all $z \in B$. We foremost note that $u_A \le u_E$. Let $z \in B$. We have that $z \in A$ iff $z \in E$, in which case $u_A(z) = u_E(z) = 0$. Otherwise let $x \in A$ be a projection point; necessarily x is a boundary point of A, so $x \in E$ and $u_A(z) \ge u_E(z)$; hence they are equal. ■

A.5. Proof of Lemma 5.10.

Proof. Let $\theta_B = \max_{C_t} u_B(x)$, $\theta_A = \max_{C_t} u_A(x)$ for any nonempty open G contained in $C_t \setminus (A \cup B)$, and such that $G \cap \{u_B = \theta_B\} = \emptyset$ and $G \cap \{u_A = \theta_A\} = \emptyset$ we set $E = C_t \setminus G$, by Lemma 5.9,

$$d_H(A, E) = t \ , \quad d_H(E, B) = \mu - t \ ;$$

so we can build a geodesic from A to B that passes through E. ■

A.6. Proof of Lemma 5.19.

Proof. We set $u_n \stackrel{\text{def}}{=} u_{\Omega_n}$; since u_n is 1-Lipschitz, that is,

$$\forall x, y \in \mathbb{R}^N \ , \quad |u_n(x) - u_n(y)| \le |x - y| \ , \tag{72}$$

passing to the limit in the above (72), we obtain

$$\forall x, y \in D \ , \quad |f(x) - f(y)| \le |x - y| \ . \tag{73}$$

So there is a unique extension of f to a positive function $g : \mathbb{R}^N \to \mathbb{R}$ that is again 1-Lipschitz, that is,

$$\forall x, y \in \mathbb{R}^N \ , \quad |g(x) - g(y)| \le |x - y| \ . \tag{74}$$

It is easy to prove that $u_n(x) \to g(x)$ for all x; then (by imitating the proof of the Ascoli–Arzelà theorem) we prove that $u_n \to g$ uniformly on compact sets.

Let $\Omega = \{g = 0\}$; to prove that Ω is nonempty, let x_n be such that $u_n(x_n) = 0$; letting $y \in D$, then $u_n(y)$ is a bounded sequence; hence x_n is bounded, since $|y - x_n| \le u_n(y)$. Thus, up to a subsequence, x_n converges to a point x such that $g(x) = 0$.

To conclude the proof, we need to prove that $g = u_\Omega$. To this end, we first prove that $g \ge u_\Omega$. Indeed, fixing x, $u_n(x) = |x - y_n|$ for at least one point $y_n \in \Omega_n$; since $u_n(x) \to g(x)$, then the sequence $\{y_n\}$ is bounded, so (up to a subsequence n_k) it converges to a point y; since the family u_n is 1-Lipschitz and $u_n(y_n) = 0$, then $g(y) = 0$, that is, $y \in \Omega$. Hence

$$g(x) = \lim_k u_{n_k}(x) = \lim_k |y_{n_k} - x| = |y - x| \ge u_\Omega(x) \ .$$

Conversely, let $y \in \Omega$ be such that $u_\Omega(x) = |x - y|$; then by (74), $g(x) \le g(y) + |x - y| = |x - y| = u_\Omega(x)$. ■

A.7. Proof of Proposition 6.18.

Proof. For $t \in [0, 1]$, let $\gamma(t) = t\Omega$ be the path that connects the singleton $\{0\}$ to Ω by rescaling; we prove that γ is Lipschitz.

Let $R > 0$ be such that $\Omega \subseteq D_R$, where D_R is the disk centered at zero (see (21)).

Since $\varphi(x)$ is convex for x large and $\lim_{x\to\infty} \varphi'(x) = 0$, then φ is Lipschitz. Let V be the Lipschitz constant of φ.

For convenience, let $f(t, x) = u_{t\Omega}(x)$.

It is not difficult to prove that the map $f(t, x)$ is jointly Lipschitz. Let F be the Lipschitz constant.

This proves the result when $p = \infty$; indeed

$$d_{\infty,\varphi}(s\Omega, t\Omega) \le FV|t - s| \ .$$

When $p < \infty$, we proceed as follows.

As a first step we study the pointwise time derivative of $u_{t\Omega}(x)$. By Rademacher's theorem $u_{t\Omega}(x)$ is differentiable at almost all t, x. Fix such a t, x; note that

$$u_{t\Omega}(x) = t u_\Omega\left(\frac{x}{t}\right) \tag{75}$$

(as in (23)); hence, taking derivatives w.r.t. x, we obtain

$$\nabla u_{t\Omega}(x) = \nabla u_\Omega\left(\frac{x}{t}\right) \ ,$$

while taking derivatives w.r.t. t gives us

$$\partial_t u_{t\Omega}(x) = u_\Omega\left(\frac{x}{t}\right) - \frac{1}{t}\left\langle \nabla u_\Omega\left(\frac{x}{t}\right) \cdot x \right\rangle = \frac{1}{t}\left(u_{t\Omega}(x) - \langle \nabla u_{t\Omega}(x) \cdot x\rangle\right) \ .$$

Suppose moreover that $x \notin t\Omega$ and let $y \in t\Omega$ be the minimum distance point from x; then (as remarked in section 4)

$$u_{t\Omega}(x) = |x-y| \ , \quad \nabla u_{t\Omega}(x) = \frac{x-y}{|x-y|} \ ,$$

so

$$\partial_t u_{t\Omega}(x) = \frac{1}{t}\left(|x-y| - \left\langle \frac{x-y}{|x-y|} \cdot x \right\rangle\right)$$

$$= -\frac{1}{t|x-y|}\langle x-y \cdot y\rangle = -\left\langle \frac{x-y}{|x-y|} \cdot \frac{y}{t} \right\rangle \ . \tag{76}$$

So if $\Omega \subseteq D_R$, we obtain that $|\partial_t u_{t\Omega}(x)| \le R$.

If instead $x \in t\Omega$ and $u_{t\Omega}(x)$ is differentiable at (t,x), then $\nabla u_{t\Omega}(x) = 0$ and $\partial_t u_{t\Omega}(x) = 0$; indeed $u_{t\Omega}(x) = 0$ and $u \ge 0$ everywhere.

As a second step we remark that the time derivative of $v_{t\Omega}(x)$ exists as a strong limit in L^p. For the case $p > 1$ this may follow from Proposition 6.21. Since this would not cover the case $p = 1$, we provide a direct proof that is based on the above computation and on the Lebesgue dominated convergence theorem. Indeed

$$\left|\frac{v_{t\Omega}(x) - v_{s\Omega}(x)}{t-s}\right| = |\varphi'(\xi)| \left|\frac{f(t,x) - f(s,x)}{t-s}\right| \le |\varphi'(\xi)| F \ ,$$

where $\xi = \xi(t,x)$ is a value intermediate between $f(t,x)$ and $f(s,x)$. Clearly $\xi \ge (|x|-R)^+$, and hence

$$|\varphi'(\xi)| \le \left|\varphi'\Big((|x|-R)^+\Big)\right|$$

for $|x|$ large; so by (45) we obtain that $|\varphi'(\xi)| \in L^p(\mathbb{R}^N)$. (See the similar proof of Proposition 7.2 for more details.) To conclude, we compute

$$\|\partial_t v_{t\Omega}(x)\|^p_{L^p} = \int_{\mathbb{R}^N} |\varphi'(u_{t\Omega}(x))|^p |\partial_t u_{t\Omega}(x)|^p \ \mathrm{d}x \le R^p \int_{\mathbb{R}^N} |\varphi'(u_{t\Omega}(x))|^p \, \mathrm{d}x \ , \tag{77}$$

and we argue as above to state that this quantity is finite and bounded uniformly in t. By Remark 1.1.3 in [2], we conclude that γ is Lipschitz. ■

Remark A.2. Asking that φ satisfy both (29) and (44) is equivalent to asking that $\varphi(|x|) \in W^{1,p}$. By using the equality in (77) and in (76), it is possible to show that, for most compact sets, the rescaling is a Lipschitz path iff $\varphi(|x|) \in W^{1,p}$.[13]

A.8. Proof of Proposition 6.21.

Proof.

- We extend $f(t,x) = f(1,x)$ for $t > 1$ and $f(t,x) = f(0,x)$ for $t < 0$. Note that the extended map is still a Lipschitz map $t \mapsto f(t,\cdot)$ with values in $L^p(\mathbb{R}^N)$; let c be its Lipschitz constant. We define

$$g_\tau(t,x) \stackrel{\text{def}}{=} \frac{f(t+\tau,x) - f(t,x)}{\tau} \ ,$$

[13]It is moreover plausible that if (44) does not hold, then in general two compact sets may not be connected by a rectifiable continuous path.

so

$$\|g_\tau(t,x)\|_{L^p(\mathbb{R}^N)} \le c\ .$$

Hence

$$\int_0^1 \int_{\mathbb{R}^N} |g_\tau(t,x)|^p \ \mathrm{d}x\,\mathrm{d}t \le c^p\ .$$

This means that the family g_τ is bounded in $L^p([0,1]\times\mathbb{R}^N)$, so we can find a sequence $\tau_n \to 0$ such that $g_{\tau_n} \to w$ weakly, i.e.,

$$\lim_n \int_0^1 \int_{\mathbb{R}^N} g_{\tau_n}(t,x)\psi(t,x)\ \mathrm{d}x\,\mathrm{d}t = \int_0^1 \int_{\mathbb{R}^N} w(t,x)\psi(t,x)\ \mathrm{d}x\,\mathrm{d}t \tag{78}$$

for all $\psi \in C_c^\infty([0,1]\times\mathbb{R}^N)$. For all such ψ (extending $\psi(t,x)=0$ when $t\notin[0,1]$),

$$\int_0^1 \int_{\mathbb{R}^N} g_\tau(t,x)\psi(t,x)\ \mathrm{d}x\,\mathrm{d}t = \int_0^1 \int_{\mathbb{R}^N} f(t,x)\frac{\psi(t-\tau,x)-\psi(t,x)}{\tau}\ \mathrm{d}x\,\mathrm{d}t\ ;$$

hence

$$\lim_n \int_0^1 \int_{\mathbb{R}^N} g_{\tau_n}(t,x)\psi(t,x)\ \mathrm{d}x\,\mathrm{d}t = -\int_0^1 \int_{\mathbb{R}^N} f(t,x)\partial_t\psi(t,x)\ \mathrm{d}x\,\mathrm{d}t$$

by dominated convergence, so we conclude that f admits a weak derivative and the derivative is w. The relationship (19) in $L^p(\mathbb{R}^N)$, that is,

$$f(b,\cdot)-f(a,\cdot) = \int_a^b \frac{df}{\mathrm{d}t}\,\mathrm{d}t\ ,$$

implies that

$$\int_a^b \xi\frac{df}{dt}\ \mathrm{d}t = -\int_a^b \frac{d\xi}{dt}f\ \ \mathrm{d}t$$

for all $\xi \in C_c^\infty([0,1])$; but then, setting $\psi(t,x)=\xi(t)$, we obtain that $\frac{df}{dt}=\partial_t f$.

- Suppose that the pointwise limit

$$\lim_{\tau\to 0} g_\tau(t,x) = \lim_{\tau\to 0}\frac{f(t+\tau,x)-f(t,x)}{\tau} \tag{79}$$

exists for almost all t,x; we call this limit $h(t,x)$. Reasoning as in (78) above. by dominated convergence we obtain that

$$\lim_{\tau\to 0}\int_0^1 \int_{\mathbb{R}^N} g_\tau(t,x)\psi(t,x)\ \mathrm{d}x\,\mathrm{d}t = \int_0^1 \int_{\mathbb{R}^N} h(t,x)\psi(t,x)\ \mathrm{d}x\,\mathrm{d}t\ , \tag{80}$$

so h is a representative of the weak partial derivative. ■

A.9. Proof of Proposition 7.4. First of all, we remark that $\mathcal{F}$ is dense in $\mathcal{M}$ according to the Hausdorff distance. In particular it is easy to build approximating sequences. One method is as follows. Let A be compact, let $\{x_n\}_n$ be a dense subset of A, and let $A_n \stackrel{\text{def}}{=} \{x_k \mid k \le n\}$; then $d_H(A, A_n) \to_n 0$ (d_H being the Hausdorff distance).

Proof. When $\varphi'(0) < 0$ it is easy to find examples to show that $\mathcal{F}$ is not dense in $\mathcal{M}$. Let $N = 1$, $A = [0,1]$, and suppose by contradiction that there exists $(A_n)_n \subset \mathcal{F}$ such that $A_n \to_n A$ according to $d_{1,p,\varphi}$; then $A_n \to_n A$ according to $d_{p,\varphi}$ and hence $d_H(A, A_n) \to_n 0$ by Theorem 6.11. By direct inspection we observe that u_{A_k} is a piecewise linear function and, for all $x \in (0,1)$ but for finitely many choices, u_{A_k} is differentiable and $|u'_{A_k}(x)| = 1$; we also know that $u_{A_n} \to_n u_A$ uniformly; so we obtain that

$$|v'_{A_k}(x)| = |\varphi'(u_{A_k}(x))\ u'_{A_k}(x)| \to |\varphi'(0)|$$

for almost all $x \in [0,1]$—whereas $v'_A(x) = 0$ for all $x \in (0,1)$.

Conversely, suppose now that $\varphi'(0) = 0$. Fix A compact, and let $\{x_n\}_n$ be a dense subset of A and $A_n \stackrel{\text{def}}{=} \{x_k \mid k \le n\}$; we will prove that $A_n \to_n A$ according to $d_{1,p,\varphi}$.

We remarked above that $d_H(A, A_n) \to_n 0$ and (by Theorem 6.11) that $v_{A_n} \to_n v_A$ in L^p. We need to prove that $\nabla v_{A_n} \to_n \nabla v_A$ in L^p.

For any $x \in A$, $\nabla v_A(x)$ exists and is zero (this is obviously true in the topological interior, whereas when x is in the boundary it derives from $\varphi'(0) = 0$). At the same time, for almost every $x \in A$ and for all n we know (see Lemma 6.28) that ∇v_{A_n} exists and

$$|\nabla v_{A_n}(x)| = |\varphi'(u_{A_n}(x))| \ ,$$

so we can write

$$\int_A |\nabla v_{A_n}(x) - \nabla v_A(x)|^p \, \mathrm{d}x = \int_A |\varphi'(u_{A_n}(x))|^p \, \mathrm{d}x \to_n 0 \ ,$$

exploiting the fact that $u_{A_n} \to_n u_A$ uniformly.

We now consider the complement A^c of A. We will argue that $\nabla v_{A_n} \to_n \nabla v_A$ in $L^p(A^c)$.

Let L be the Lipschitz constant of φ; then L is the Lipschitz constant of any function v_B for B compact. Working as in Proposition 7.2, we select $R > 0$ large so that $A, A_n \subseteq B_R$. For all n and almost every x, when $|x| \le R + T$ we know that $|\nabla v_{A_n}(x)| \le L$, whereas when $|x| \ge R + T$ we know that

$$|\nabla v_{A_n}(x)| = |\varphi'(u_{A_n}(x))| \le |\varphi'(|x| - R)| \ ,$$

where the last function is in L^p. So, by dominated convergence, it suffices to show that $\nabla v_{A_n} \to_n \nabla v_A$ pointwise. We sketch the argument. Suppose now that $x \notin A$ and that, for all n, u_{A_n} and u_A are differentiable at x: we will argue that $\nabla u_{A_n}(x) \to_n \nabla u_A(x)$. By the results presented in section 4 we will prove convergence of the projection points. Let y_n be the projection of x_n to A_n and y the projection of x to A; then $y_n \to y$. If this is not the case, then there is a subsequence and a $z \ne y$ such that $y_{n_k} \to_k z$. But then $z \in A$ is another projection point of x, contradicting the fact that u_A is differentiable at x. ■

Acknowledgments. The authors wish to thank two anonymous reviewers who provided many corrections and useful suggestions. They also wish to thank Prof. Luigi Ambrosio for help and corrections, and for checking Proposition 8.1.

REFERENCES

[1] L. Ambrosio and B. Kirchheim, *Rectifiable sets in metric and Banach spaces*, Math. Ann., 318 (2000), pp. 527–555.

[2] L. Ambrosio, N. Gigli, and G. Savarè, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Birkhäuser, 2005.

[3] L. Ambrosio and P. Tilli, *Selected Topics in "Analysis in Metric Spaces,"* Collana degli appunti, Edizioni Scuola Normale Superiore, Pisa, 2000.

[4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.

[5] D. Burago, Y. Burago, and S. Ivanov, *A Course in Metric Geometry*, Grad. Stud. Math. 33, AMS, Providence, RI, 2001.

[6] H. Busemann, *Local metric geometry*, Trans. Amer. Math. Soc., 56 (1944), pp. 200–274.

[7] M. C. Delfour and J. P. Zolésio, *Shape and Geometries*, Adv. Des. Control 4, SIAM, Philadelphia, 2001.

[8] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc., 93 (1959), pp. 418–491.

[9] J. Itoh and M. Tanaka, *The Lipschitz continuity of the distance function to the cut locus*, Trans. Amer. Math. Soc., 353 (2000), pp. 21–40.

[10] H. Karcher, *Riemannian center of mass and mollifier smoothing*, Comm. Pure Appl. Math., 30 (1977), pp. 509–541.

[11] M. Leventon, E. Grimson, and O. Faugeras, *Statistical shape influence in geodesic active contours*, in IEEE Conference on Computer Vision and Pattern Recognition, Vol. 1, 2000, pp. 316–323.

[12] Y. Li and L. Nirenberg, *The distance function to the boundary, Finsler geometry and the singular set of viscosity solutions of some Hamilton-Jacobi equations*, Comm. Pure Appl. Math., 58 (2005), pp. 85–146.

[13] C. Mantegazza and A. C. Mennucci, *Hamilton–Jacobi equations and distance functions on Riemannian manifolds*, Appl. Math. Optim., 47 (2003), pp. 1–25.

[14] A. C. G. Mennucci, *Regularity and variationality of solutions to Hamilton-Jacobi equations. Part* II*: Variationality, existence, uniqueness*, Appl. Math. Optim., 63 (2011), pp. 191–216.

[15] A. C. G. Mennucci, *Geodesics in asymmetric metric spaces*, Anal. Geom. Metric Spaces, 2 (2014), pp. 115–153.

[16] A. Mennucci, A. Yezzi, and G. Sundaramoorthi, *Properties of Sobolev active contours*, Interfaces Free Bound., 10 (2008), pp. 423–445.

[17] P. W. Michor and D. Mumford, *Riemannian geometries on spaces of plane curves*, J. Eur. Math. Soc. (JEMS), 8 (2006), pp. 1–48.

[18] P. W. Michor and D. Mumford, *An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach*, Appl. Comput. Harmon. Anal., 23 (2007), pp. 76–113.

[19] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Comprehensive Stud. Math. 317, Springer-Verlag, 1998.

[20] G. Sundaramoorthi, J. D. Jackson, A. Yezzi, and A. C. Mennucci, *Tracking with Sobolev active contours*, in Conference on Computer Vision and Pattern Recognition (CVPR06), IEEE Computer Society, 2006, pp. 674–680.

[21] G. Sundaramoorthi, A. Mennucci, S. Soatto, and A. Yezzi, *Tracking deforming objects by filtering and prediction in the space of curves*, in Conference on Decision and Control, 2009, pp. 2395–2401.

[22] G. Sundaramoorthi, A. Mennucci, S. Soatto, and A. Yezzi, *A new geometric metric in the space of curves, and applications to tracking deforming objects by prediction and filtering*, SIAM J. Imaging Sci., 4 (2011), pp. 109–145.

[23] G. Sundaramoorthi, A. Yezzi, and A. Mennucci, *Sobolev active contours*, in Variational, Geometric,

and Level Set Methods in Computer Vision, Third International Workshop (VLSM 2005), Lecture Notes in Comput. Sci. 3752, N. Paragios, O. D. Faugeras, T. Chan, and C. Schnörr, eds., Springer, 2005, pp. 109–120.

[24] G. Sundaramoorthi, A. Yezzi, and A. Mennucci, *Sobolev active contours*, Internat. J. Comput. Vision, 73 (2007), pp. 413–417.

[25] G. Sundaramoorthi, A. Yezzi, and A. Mennucci, *Coarse-to-fine segmentation and tracking using Sobolev active contours*, IEEE Trans. Pattern Anal. Mach. Intell., 30 (2008), pp. 851–864.

[26] G. Sundaramoorthi, A. Yezzi, A. Mennucci, and G. Sapiro, *New possibilities with Sobolev active contours*, in Scale Space Variational Methods 07, Lecture Notes in Comput. Sci. 4485, F. Sgallari, A. Murli, and N. Paragios, eds., Springer, 2007, pp. 153–164.

[27] G. Sundaramoorthi, A. Yezzi, A. Mennucci, and G. Sapiro, *New possibilities with Sobolev active contours*, Internat. J. Comput. Vision, 84 (2009), pp. 113–129.

[28] A. Yezzi and A. Mennucci, *Metrics in the Space of Curves*, preprint, arXiv:math.DG/0412454, 2004.

[29] A. Yezzi and A. Mennucci, *Conformal metrics and true "gradient flows" for curves*, in International Conference on Computer Vision (ICCV05), 2005, pp. 913–919.

[30] L. Younes, P. W. Michor, J. Shah, and D. Mumford, *A metric on shape space with explicit geodesics*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 19 (2008), pp. 25–57.