There is only one KAM curve

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Abstract

We consider the standard family of area-preserving twist maps of the annulus and the corresponding KAM curves. Addressing a question raised by Kolmogorov, we show that, instead of viewing these invariant curves as separate objects, each of which having its own Diophantine frequency, one can encode them in a single function of the frequency which is naturally defined in a complex domain containing the real Diophantine frequencies and which is monogenic in the sense of Borel; this implies a remarkable property of quasianalyticity, a form of uniqueness of the monogenic continuation, although real frequencies constitute a natural boundary for the analytic continuation from the Weierstraß point of view because of the density of the resonances.

Introduction

In this paper, we address what is, to our best knowledge, the oldest open problem in KAM theory. Indeed, in 1954, in his ICM conference [Kol54], Kolmogorov asked whether the regularity of the solutions of small divisor problems with respect to the frequency could be investigated using appropriate analytical tools, suggesting a connection with the theory

of 'monogenic functions' in the sense of Émile Borel [Bo17][4]. We provide evidence that Kolmogorov's intuition was correct by establishing a monogenic regularity result upon a complexified rotation number for the KAM curves of a family of analytic twist maps of the annulus; as a consequence of our result, these curves enjoy a property of '$\mathscr{H}^1$-quasianalyticity' with respect to the rotation number.

We recall that small divisor problems are at the heart of the study of quasiperiodic dynamics: resonances are responsible for the possible divergence of the perturbative series expansions of quasi-periodic motions and their accumulation must be controlled in order to prove convergence. The Kolmogorov–Arnold–Moser (KAM) theory deals with perturbations of completely integrable Hamiltonian systems for which, when frequencies verify a suitable Diophantine condition, the small divisor difficulty can be overcome and one can establish the persistence of quasi-periodic solutions of Hamilton's equations; in the analytic case their parametric expressions depend analytically on angular variables as well as on the perturbation parameter, however they are only defined on closed sets in frequency space, corresponding to the Diophantine condition.

Borel's monogenic functions may be considered as a substitute to holomorphic functions when the natural domain of definition is not open but can be written as an increasing union of closed subsets of the complex plane (monogenicity essentially amounts to Whitney differentiability in the complex sense on these larger and larger closed sets). According to what these closed sets are, monogenic functions may enjoy some of the properties of holomorphic functions (e.g. one may be able to use the Cauchy integral). As pointed out by Herman [He85], Borel's motivation was probably to ensure quasianalytic properties (unique monogenic continuation) by an appropriate choice of the sequence of closed sets, which turns out to be difficult in a general framework.

Kolmogorov's question about the link between small divisor problems and Borel's monogenic functions has already been considered in small divisor problems other than KAM theory, particularly in the context of circle maps [Ar61, He85, Ri99] where the role of frequency is played by the so-called rotation number (see also [BMS00, MS03, CM08, MS11]). In his work on the local linearization problem of analytic diffeomorphisms of the circle, Arnold [Ar61] defined a complexified rotation number, with respect to which he showed the monogenicity of the solution of the linearized problem, but his method did not allow him to prove that the solution of the nonlinear conjugacy problem was monogenic, because it would have required to iterate infinitely many times a process in which the analyticity strip was reduced by a finite amount equal to the imaginary part of the rotation number. This point was dealt with by Herman [He85], who used quite a different method and also reformulated Borel's ideas using the modern terminology, and by Risler [Ri99], who used Yoccoz's renormalization method [Yo88, Yo95] to enlarge the set of complex rotation numbers covered by the regularity result, passing from Siegel's Diophantine condition to Bruno's condition.

In this paper, we consider the Lagrangian formulation of KAM theory for symplectic twist maps of the annulus [SZ89, LM01] and prove that the parametrization of the invariant KAM curves can be extended to complex values of the rotation number, that their dependence on real and complex rotation numbers is an example of Borel's monogenic

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[4] Towards the end of [Kol54], he considers the example of a real analytic vector field on the two-dimensional torus which depends analytically on a real parameter θ: he claims that, if the ratio of mean frequencies is not constant, then there is a full measure set R of parameters for which the vector field is analytically conjugate to a constant normal form, giving rise to a discrete set of eigenfuctions φ_{mn} which are analytic functions on the torus; he then writes: It is possible that the dependence of φ_{mn} on the parameter θ on the set R is related to the class of functions of the type of monogenic Borel functions and, despite its everywhere-discontinuous nature, will admit investigation by appropriate analytical tools.

function and furthermore that it enjoys the property introduced in [MS11] under the name '$\mathscr{H}^1$-quasianalyticity'. This is sufficient to get an interesting uniqueness property for the monogenic continuation of the KAM curve because such functions are determined by their restriction to any subset of their domain of definition which has positive linear measure.

With a view to avoid too many technicalities, we do not try to work in the most general context. We restrict ourselves to the standard family of area-preserving twist maps of the annulus because we find it suggestive enough and this allows us to contrast our monogenicity result, which entails holomorphy with respect to complex non-real frequencies, with another result that we prove, according to which real frequencies do constitute a natural boundary.

We do not try either to reach optimal results for the arithmetical condition; we content ourselves with imposing a Siegel-type Diophantine condition on the rotation number (instead of a Bruno-type condition).

Notice that Whitney smooth dependence on *real* Diophantine frequencies has been established long ago by Lazutkin [Laz73] and Pöschel [Pö82] in this kind of small divisor problem (see also [BT07]), but the question we address in this paper is quite different in its spirit: what is at stake here is the complex extension, its regularity and the uniqueness property this regularity implies (see section 1.4 for a comment). Indeed, we show that from the point of view of classical analytic continuation, the real axis in frequency space appears as a natural boundary, because of the density of the resonances, but our quasianalyticity result is sufficient to prove that some sort of 'generalized analytic continuation'[5] through it is indeed possible: the knowledge of the parametrizations on a set of positive linear measure of rotation numbers (real or complex) is sufficient to determine all the parametrized KAM curves: in this sense there is only one KAM curve, parametrized by one monogenic function of the rotation number.

Plan of the paper. In section 1 we formulate the KAM problem which is investigated in this paper: the existence of analytic invariant curves for the standard family of area-preserving twist maps of the cylinder and the dependence of the parametrization of the invariant curves on the rotation number ω when ω is allowed to take real and complex values. We state our main result, theorem 1, about the $\mathscr{C}^1$-holomorphy of this parametrization, and connect it with Borel's monogenic functions and their quasianalytic properties. We also give a theorem 2 about the impossibility of having an analytic continuation in the Weierstraß sense through the real frequencies.

In section 2, we introduce an algebra norm on the space of $\mathscr{C}^1$-holomorphic functions, which is useful to deal with nonlinear analysis (in particular composition of functions). In section 3, we provide the small divisor estimates for the linearized conjugacy equation: these estimates must be uniform with respect to both real Diophantine and complex rotation numbers.

In section 4, inspired by Levi–Moser's 'Lagrangian' proof of the KAM theorem for twist mappings [LM01], we adapt their algorithm to construct a sequence of approximations which converges to the parametrization of the invariant curve in our Banach algebra of $\mathscr{C}^1$-holomorphic functions, so as to prove our complexified KAM theorem (theorem 1).

Section 5 is devoted to proving that the real line in the complex frequency space is a natural boundary, in the classical sense, for the analytic continuation of the parametrization of the invariant curve (theorem 2).

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[5] We borrow this expression from [RS02].

1. Statement of the results

1.1. Invariant graphs for the standard family

Let f be a 1-periodic real analytic function with zero mean value. We consider the standard family, i.e. the discrete dynamical system defined by

$$T_\varepsilon \colon (x, y) \mapsto (x_1, y_1), \qquad \begin{cases} x_1 = x + y + \varepsilon f(x) \\ y_1 = y + \varepsilon f(x) \end{cases} \tag{1}$$

in the phase space $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and ε is a real parameter (when $f(x) = \cos(2\pi x)$, T_ε is the so-called standard map). For ε close to 0, this is an exact symplectic map that we can view as a perturbation of the integrable twist map $(x, y) \mapsto (x + y, y)$.

We are interested in the KAM curves associated with Diophantine frequencies. For $\omega \in \mathbb{R} - \mathbb{Q}$, we call *invariant graph of frequency ω for T_ε* the graph $G = \{(x, \varphi(x))\} \subset \mathbb{T} \times \mathbb{R}$ of a continuous map $\varphi \colon \mathbb{T} \to \mathbb{R}$ such that T_ε leaves G invariant and the map $\Phi \colon x \mapsto x + \varphi(x) + \varepsilon f(x)$ on $\mathbb{R}$ is conjugate to the translation $x \mapsto x + \omega$ by a homeomorphism of $\mathbb{R}$ of the form $\mathrm{id} + u$, where u is a 1-periodic function (observe that Φ is a lift of the circle map induced by the restriction of T_ε to G).

There is a natural way of viewing an invariant graph of frequency ω as a parametrized curve $G = \gamma(\mathbb{T})$: finding G is equivalent to finding continuous functions $u, \varphi \colon \mathbb{T} \to \mathbb{R}$ such that $\theta \mapsto U(\theta) := \theta + u(\theta)$ defines a homeomorphism of $\mathbb{R}$ and the curve $\gamma \colon \mathbb{T} \to \mathbb{T} \times \mathbb{R}$ defined by $\gamma(\theta) = \big(U(\theta), \varphi(U(\theta))\big)$ satisfies

$$\gamma(\theta + \omega) = T_\varepsilon\big(\gamma(\theta)\big), \qquad \theta \in \mathbb{T}. \tag{2}$$

Setting $v = -\omega + \varphi \circ U$, i.e. writing the curve γ as

$$\gamma(\theta) = \big(\theta + u(\theta), \omega + v(\theta)\big) \tag{3}$$

we see that equation (2) is equivalent to the system

$$v(\theta) = u(\theta) - u(\theta - \omega), \tag{4}$$

$$u(\theta + \omega) - 2u(\theta) + u(\theta - \omega) = \varepsilon f\big(\theta + u(\theta)\big) \tag{5}$$

(using the fact that $x_1 = x + y_1$ in (1) and writing (3) at the points θ and $\theta + \omega$). It is in fact sufficient to know the function u: any 1-periodic solution u of the second-order difference equation (5) such that $\mathrm{id} + u$ is injective parametrizes an invariant graph of frequency ω through formulas (3)–(4).

The continuity of φ and the irrationality of ω are enough to ensure uniqueness: if it exists, the invariant graph of frequency ω is unique[6] and the corresponding parametrization γ is unique up to a shift in the variable θ. We can then normalize the parametrization by adding the requirement that u have zero mean value: equation (5) cannot have more than one continuous solution u of zero mean value such that $\mathrm{id} + u$ is injective and finding such a solution is equivalent to finding an invariant graph of frequency ω.

The classical KAM theorem for twist maps [Mo62] guarantees the existence of an invariant graph of frequency ω for every Diophantine ω provided $|\varepsilon|$ is small enough. More precisely,

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[6] The argument relies on the positive twist map condition verified by T_ε: Suppose that G and G^* are invariant graphs of frequency ω and define φ, Φ, γ and φ^*, Φ^*, γ^* accordingly; if G and G^* did not intersect, we would have $\varphi < \varphi^*$ or $\varphi > \varphi^*$ on $\mathbb{T}$, which would imply $\Phi < \Phi^*$ or $\Phi > \Phi^*$ on $\mathbb{R}$, and hence contradict $\omega \in \mathbb{R} - \mathbb{Q}$ (as it is known for rotation numbers [HK97] that $\Phi_1 < \Phi_2$ implies $\rho(\Phi_1) < \rho(\Phi_2)$ or $\rho(\Phi_1) = \rho(\Phi_2) \in \mathbb{Q}$); we thus can find $\theta_0, c \in \mathbb{T}$ such that $\gamma(c + \theta_0) = \gamma^*(\theta_0)$, but the shifted curve $\theta \mapsto \gamma(c + \theta)$ is a solution of (2) as well as γ and iterating this equation we find $\gamma(c + \theta_0 + k\omega) = T_\varepsilon^k\big(\gamma(c + \theta_0)\big) = T_\varepsilon^k\big(\gamma^*(\theta_0)\big) = \gamma^*(\theta_0 + k\omega)$ for all integers k; the irrationality of ω thus implies that γ^* coincides with the shifted curve on a dense subset of $\mathbb{T}$, hence everywhere by continuity, whence $G = G^*$.

given $\omega \in \mathbb{R} - \mathbb{Q}$ for which there exist $M > 0$ and $\tau \geqslant 0$ such that $|\omega - (n/m)| \geqslant (1/Mm^{2+\tau})$ for all $(n, m) \in \mathbb{Z} \times \mathbb{N}^*$, the map T_ε admits an invariant graph of frequency ω as soon as $|\varepsilon|$ is smaller than a constant ρ which depends only on f, M, τ. Moreover, the corresponding curve γ_ω is known to be analytic in the angle θ and to depend analytically on ε.

The aim of this paper is to investigate the regularity of the map $\omega \mapsto \gamma_\omega$, which for the moment is defined on the set of real Diophantine numbers. More specifically, we are interested in the quasianalytic properties of this map; this will lead us to extend it to certain complex values of the frequency ω.

1.2. $\mathscr{C}^1$-holomorphy and $\mathscr{H}^1$-quasianalyticity of a complex extension

Throughout the paper, we use the notation

$$S_R = \{x \in \mathbb{C}/\mathbb{Z} \mid |\Im m x| < R\}, \qquad \mathbb{D}_\rho = \{\varepsilon \in \mathbb{C} \mid |\varepsilon| < \rho\} \tag{6}$$

for any $R, \rho > 0$. Let $R_0 > 0$ be such that f extends holomorphically to a neighbourhood of the closed strip $\overline{S}_{R_0}$. Let $R \in (0, R_0)$. We shall be interested in invariant graphs whose parametrizations extend holomorphically for $\theta \in S_R$ and $\varepsilon \in \mathbb{D}_\rho$ for some $\rho > 0$. Let

$$\mathbb{B}_{R,\rho} = H^\infty(S_R \times \mathbb{D}_\rho) \tag{7}$$

denote the complex Banach space of all bounded holomorphic functions of $S_R \times \mathbb{D}_\rho$.

We fix $\tau > 0$ and consider

$$A_M^{\mathbb{R}} = \left\{ \omega \in \mathbb{R} \mid \forall (n, m) \in \mathbb{Z} \times \mathbb{N}^*, \ \left| \omega - \frac{n}{m} \right| \geqslant \frac{1}{Mm^{2+\tau}} \right\} \tag{8}$$

for $M > M(\tau) = 2\zeta(1 + \tau)$ (Riemann's zeta function): this is a closed subset of the real line, of positive measure, which has empty interior and is invariant by the integer translations. The KAM theorem gives us a $\rho = \rho(M) > 0$ (which depends also on f, R_0, R and τ) and a function

$$\omega \in A_M^{\mathbb{R}} \mapsto u \in \mathbb{B}_{R,\rho}$$

such that, for ε real, the restriction of $u(., \varepsilon)$ to $\mathbb{T}$ has zero mean value and parametrizes through (3)–(4) an invariant graph of frequency ω.

Clearly, u depends 1-periodically on ω, we shall thus rather view it as a function of $q = E(\omega)$, where

$$E: \omega \in \mathbb{C} \mapsto q = e^{2\pi i \omega} \in \mathbb{C}^*,$$

and consider that we have a function $u_M: E(A_M^{\mathbb{R}}) \to \mathbb{B}_{R,\rho}$. Observe that $E(A_M^{\mathbb{R}})$ is a subset of the unit circle $\mathbb{S}$ which avoids all roots of unity and has Haar measure $\geqslant 1 - (M(\tau)/M)$. If we consider $M_1 > M$, then we get a larger set $E(A_{M_1}^{\mathbb{R}})$ and, by the aforementioned uniqueness property, the corresponding function u_{M_1} is an extension of u_M, provided we take $\rho_1 \leqslant \rho$ in the KAM result and we regard the new target space $\mathbb{B}_{R,\rho_1}$ as containing $\mathbb{B}_{R,\rho}$.

Let

$$A_M^{\mathbb{C}} = \left\{ \omega \in \mathbb{C} \mid \exists \omega_* \in A_M^{\mathbb{R}} \text{ such that } |\Im m \omega| \geqslant |\omega_* - \Re e \omega| \right\} \tag{9}$$

and

$$K_M = E(A_M^{\mathbb{C}}) \cup \{0, \infty\} \subset \widehat{\mathbb{C}}, \tag{10}$$

where $\widehat{\mathbb{C}}$ denotes the Riemann sphere—see figures 1 and 2. Observe that $A_M^{\mathbb{C}}$ is a perfect subset of $\mathbb{C}$ and K_M is a perfect subset of $\widehat{\mathbb{C}}$.

Our main result is that the above function u_M extends to a $\mathscr{C}^1$-holomorphic function from K_M to $\mathbb{B}_{R,\rho}$ (possibly with a smaller ρ). The reader is referred to section 2.1 for the
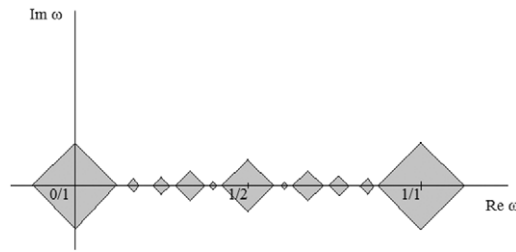
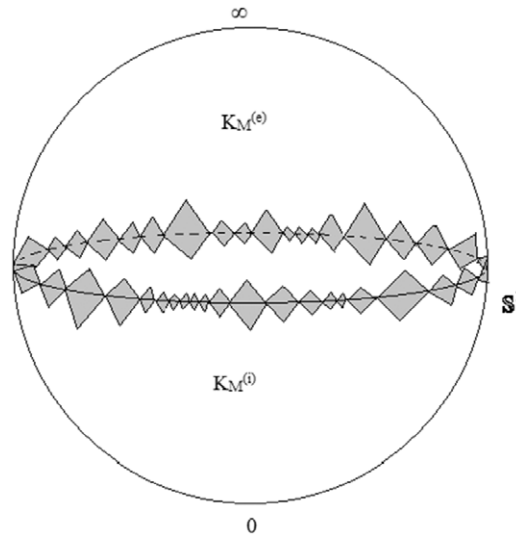**Figure 1.** The perfect subset $A_M^{\mathbb{C}} \subset \mathbb{C}$.

Figure 2. The perfect subset $K_M = K_M^{(i)} \cup K_M^{(e)} \subset \widehat{\mathbb{C}}$.

definition of the Banach space $\mathscr{C}^1_{\text{hol}}(K, B)$ of all $\mathscr{C}^1$-holomorphic functions from a perfect subset K of $\mathbb{C}$ or $\widehat{\mathbb{C}}$ to a Banach space B ($\mathscr{C}^1$-holomorphy essentially means complex differentiability in the sense of Whitney, i.e. real Whitney differentiability on a closed subset with partial derivatives which satisfy the Cauchy–Riemann equations).

Theorem 1. *Suppose* $\tau > 0$, $0 < R < R_0$, *f real analytic and holomorphic in a neighbourhood of* $\overline{S}_{R_0}$, *with zero mean value. Then there exist* $c > 0$ *(depending on* τ, R_0, *f and R) and, for each* $M > 2\zeta(1 + \tau)$, *a function*

$$\tilde{u}_M \in \mathscr{C}^1_{\text{hol}}(K_M, \mathbb{B}_{R,\rho}), \qquad \text{with } \rho = cM^{-8}$$

such that, for each $\omega \in A_M^{\mathbb{R}}$ *and* $\varepsilon \in (-\rho, \rho)$, *the function* $\theta \in \mathbb{T} \mapsto \tilde{u}_M(e^{2\pi i\omega})(\theta, \varepsilon)$ *has zero mean value and parametrizes through (3)–(4) an invariant graph of frequency* ω *for* T_ε.

The proof of theorem 1 will start in section 2.

Theorem 1 provides a function $\tilde{u}_M$ on K_M which is an extension of the function u_M that we had on $E(A_M^{\mathbb{R}})$. *This extension is unique and, if we consider* $M_1 > M$, *then* $\tilde{u}_{M_1}$ *is an extension of* $\tilde{u}_M$ (if we regard $\mathbb{B}_{R,\rho}$ as a subspace of $\mathbb{B}_{R,\rho_1}$ for $\rho_1 < \rho$); these facts are simple consequences of a quasianalyticity property, which is established in [MS11] for all functions $\mathscr{C}^1$-holomorphic on K_M and which we now recall.

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Denote by $\mathscr{H}^1$ the one-dimensional Hausdorff outer measure associated with the spherical metric in $\widehat{\mathbb{C}}$. Suppose that C is a subset of $\widehat{\mathbb{C}}$ and that $\mathscr{L}$ is a linear space of functions, all of which are defined on C. We say that $\mathscr{L}$ is $\mathscr{H}^1$-*quasianalytic relatively to* C if any subset of C of positive $\mathscr{H}^1$-measure is a uniqueness set for $\mathscr{L}$ (i.e. the only function of $\mathscr{L}$ vanishing identically on this subset of C is $\equiv 0$); in other words, a function of $\mathscr{L}$ is determined by its restriction to any subset of C of positive $\mathscr{H}^1$-measure.

Observe that the interior $\overset{\circ}{K}_M$ of the above compact set K_M has two connected components:

$$\overset{\circ}{K}_M^{(i)} := \overset{\circ}{K}_M \cap \{\, q \in \widehat{\mathbb{C}} \mid |q| < 1 \,\}, \qquad \overset{\circ}{K}_M^{(e)} := \overset{\circ}{K}_M \cap \{\, q \in \widehat{\mathbb{C}} \mid |q| > 1 \,\}.$$

It is proved in [MS11] that K_M has the property that, for any Banach space B, the Banach space

$$\mathscr{O}(K_M, B) = \{\, \varphi \colon K_M \to B \text{ continuous in } K_M \text{ and holomorphic in } \overset{\circ}{K}_M \,\}$$

is $\mathscr{H}^1$-quasianalytic relatively to K_M. Since $\mathscr{C}^1_{\mathrm{hol}}(K_M, B) \subset \mathscr{O}(K_M, B)$, this space inherits the $\mathscr{H}^1$-quasianalyticity property[7]. This is why $\tilde{u}_M$ is uniquely determined and must coincide with $\tilde{u}_{M_1|K_M}$ for $M_1 > M$: this function is in fact determined by its restriction u_M to $E(A_M^{\mathbb{R}})$, and even by its restriction to any subset of $E(A_M^{\mathbb{R}})$ of positive Haar measure of the unit circle.

This quasianalyticity property was our main motivation. What is striking is that, as we shall see in section 1.4, finding the complexified function $\tilde{u}_M$ in restriction to $\{\, |\Im m\omega| \geqslant h \,\}$, i.e. for $|q| \leqslant \mathrm{e}^{-2\pi h}$ or $|q| \geqslant \mathrm{e}^{2\pi h}$, is relatively easy because this can be done by solving an equation which does not involve any small divisor; still, the quasianalyticity property shows that all the real KAM curves determined by u_M can be obtained from this easy-to-find function by a kind of 'generalized analytic continuation' (the restriction of this function to a positive $\mathscr{H}^1$-measure subset of K_M is even sufficient).

On the other hand, resonances produce an obstruction to the analytic continuation in the Weierstraß sense through any point of the unit circle $\mathbb{S}$, no matter how small ρ is taken:

Theorem 2. *Suppose $\tau > 0$, $0 < R < R_0$, f real analytic and holomorphic in a neighbourhood of $\overline{S}_{R_0}$, with zero mean value but not identically zero. Let $M > 2\zeta(\tau + 1)$ and $\tilde{u}_M \in \mathscr{C}^1_{\mathrm{hol}}(K_M, \mathbb{B}_{R,\rho})$ as in theorem 1, possibly with a smaller $\rho > 0$. Consider the restriction of $\tilde{u}_M$ to $\overset{\circ}{K}_M^{(i)}$ or to $\overset{\circ}{K}_M^{(e)}$, which is a $\mathbb{B}_{R,\rho}$-valued holomorphic function on an open subset of $\widehat{\mathbb{C}} - \mathbb{S}$. Then, given a point q_* of the unit circle $\mathbb{S}$, this holomorphic function has no analytic continuation in any neighbourhood of q_*.*

The proof of theorem 2 is given in section 5.

Remark 1.1. For fixed M, one can also fix $\varepsilon \in \mathbb{D}_\rho$ and consider $(q, \theta) \mapsto \tilde{u}_M(q)(\theta, \varepsilon)$ as an element of $\mathscr{C}^1_{\mathrm{hol}}\big(K_M, H^\infty(S_R)\big)$, i.e. a function of θ which depends $\mathscr{C}^1$-holomorphically on q; the space $\mathscr{C}^1_{\mathrm{hol}}\big(K_M, H^\infty(S_R)\big)$ enjoys the aforementioned quasianalyticity property.

1.3. Monogenic extension

Gluing together the $\tilde{u}_M$'s given by theorem 1, we get a *monogenic function in the sense of Borel*. Here is the precise definition taken from [MS11] (which is an adaptation of the definition given in [He85]): suppose that $(\mathscr{K}_j)_{j \in \mathbb{N}}$ is a monotonic non-decreasing sequence of compact subsets of $\widehat{\mathbb{C}}$ and $(B_j)_{j \in \mathbb{N}}$ is a monotonic non-decreasing sequence of Banach spaces with continuous

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[7] Thus proving $\tilde{u} \in \mathscr{O}(K_M, \mathbb{B}_{R,\rho})$ is sufficient to get the $\mathscr{H}^1$-quasianalyticity; this can be achieved with $\rho = cM^{-4}$ by a simple adaptation of the proof of theorem 1.

injections $B_j \hookrightarrow B_{j+1}$; the corresponding space of monogenic functions is the Fréchet space obtained as the projective limit of Banach spaces

$$\mathfrak{M}\big((\mathscr{K}_j), (B_j)\big) = \varprojlim \mathcal{A}_J,$$

$$\mathcal{A}_J = \bigcap_{0 \leqslant j \leqslant J} \mathscr{C}^1_{\mathrm{hol}}(\mathscr{K}_j, B_j), \qquad \|\varphi\|_{\mathcal{A}_J} = \max_{0 \leqslant j \leqslant J} \|\varphi_{|\mathscr{K}_j}\|_{\mathscr{C}^1_{\mathrm{hol}}(\mathscr{K}_j, B_j)}.$$

As a set, $\mathfrak{M}\big((\mathscr{K}_j), (B_j)\big)$ thus consists of all the functions φ which are defined in $\bigcup_{j \in \mathbb{N}} \mathscr{K}_j$ and such that, for every $j \in \mathbb{N}$, the restriction $\varphi_{|\mathscr{K}_j}$ belongs to $\mathscr{C}^1_{\mathrm{hol}}(\mathscr{K}_j, B_j)$.

Let us apply this construction, under the hypotheses of theorem 1, with any increasing sequence of positive numbers $(M_j)_{j \in \mathbb{N}}$ tending to $+\infty$ (we suppose $M_j > 2\zeta(1 + \tau)$ for all j) and with $B_j = \mathbb{B}_{R, \rho_j}$, $\rho_j = c M_j^{-8}$. We use the notation

$$\mathscr{M} = \mathfrak{M}\big((K_{M_j}), (\mathbb{B}_{R, \rho_j})\big), \qquad \mathscr{F}_\tau = \bigcup_{j \in \mathbb{N}} K_{M_j}, \qquad \mathrm{DC}_\tau = \bigcup_{M > 0} A^\mathbb{R}_M$$

(DC_τ is the full-measure set of all real frequencies Diophantine with exponent $2 + \tau$). Observe that

$$\mathscr{F}_\tau = \{\, q \in \widehat{\mathbb{C}} \mid |q| < 1 \,\} \cup E(\mathrm{DC}_\tau) \cup \{\, q \in \widehat{\mathbb{C}} \mid |q| > 1 \,\},$$

with $\mathscr{F}_\tau \cap \mathbb{S} = E(\mathrm{DC}_\tau)$ a subset of full Haar measure of the unit circle $\mathbb{S}$, and that the elements of $\mathscr{M}$ may be viewed as functions from $\mathscr{F}_\tau$ to the space $H^\infty(S_R)\{\varepsilon\}$ (holomorphic germs in ε with values in $H^\infty(S_R)$). Theorem 1 immediately yields.

Corollary 3. *There is a function $\tilde{u} \in \mathscr{M}$ such that, for each $\omega \in \mathrm{DC}_\tau$ and for each real ε close enough to 0, the function $\theta \in \mathbb{T} \mapsto \tilde{u}(\mathrm{e}^{2\pi \mathrm{i}\omega})(\theta, \varepsilon)$ has zero mean value and parametrizes through (3)–(4) an invariant graph of frequency ω for T_ε.*

As explained in [MS11], the space $\mathscr{M}$ is $\mathscr{H}^1$-quasianalytic relatively to $\mathscr{F}_\tau$, so we can say that $\tilde{u}$ is determined by its restriction to any positive $\mathscr{H}^1$-measure subset of $\mathscr{F}_\tau$. In particular, we emphasize that the function $\tilde{u}_{|E(\mathrm{DC}_\tau)}$ which encodes the real KAM curves is determined by the restriction of $\tilde{u}$ to any such subset and we repeat that finding this restriction is easy when the subset is contained in $\{\, |q| \leqslant \mathrm{e}^{-2\pi h} \,\}$ or $\{\, |q| \geqslant \mathrm{e}^{2\pi h} \,\}$.

Thus, we have a single analytic object, $\tilde{u}$, which determines all the real KAM curves, as if there were only one KAM curve instead of separate invariant graphs, each of which with its own Diophantine frequency $\omega \in \mathrm{DC}_\tau$.

The interior $\overset{\circ}{\mathscr{F}}_\tau$ of $\mathscr{F}_\tau$ has two connected components, $\{\, q \in \widehat{\mathbb{C}} \mid |q| < 1 \,\}$ and $\{\, q \in \widehat{\mathbb{C}} \mid |q| > 1 \,\}$, and each function of $\mathscr{M}$ is holomorphic in $\overset{\circ}{\mathscr{F}}_\tau$. The $\mathscr{H}^1$-quasianalyticity thus implies a form of coherence: if two functions of this space coincide on one of the connected components of $\overset{\circ}{\mathscr{F}}_\tau$, then they coincide on the whole of $\mathscr{F}_\tau$. Given a function like $\tilde{u}$, we may think of the outside function $\tilde{u}_{|\{|q| > 1\}}$ as of a 'generalized analytic continuation' of the inside function $\tilde{u}_{|\{|q| < 1\}}$ (see [RS02]), although, according to theorem 2, classical analytic continuation in the sense of Weierstraß is impossible across the unit circle $\mathbb{S}$. These two holomorphic functions give rise to a boundary value function $\tilde{u}_{|E(\mathrm{DC}_\tau)}$ which encodes the real KAM curves.

Remark 1.2. Instead of keeping τ fixed as we did in the previous discussion, one can also let it vary so as to reach the set of all Diophantine real numbers

$$\mathrm{DC} := \bigcup_{\tau > 0} \mathrm{DC}_\tau.$$

One just needs to take any sequences $\tau_j \uparrow +\infty$ and $M_j \uparrow +\infty$. Observing that

$$\mathrm{DC} = \bigcup_{j \in \mathbb{N}} A_{\tau_j, M_j}^{\mathbb{R}}$$

(with $A_{\tau, M}^{\mathbb{R}}$ defined by (8) but we now make explicit the dependence on τ) and using correspondingly the space of monogenic functions associated with K_{τ_j, M_j} (instead of K_{τ, M_j} with fixed τ) and $\mathbb{B}_{R, \rho_j}$ with $\rho_j := c(\tau_j) M_j^{-8}$ (with $c(\tau)$ as in theorem 1), we get a monogenic $\tilde{u}$ defined on $\mathscr{F} := \bigcup K_{\tau_j, M_j} = \{ q \in \widehat{\mathbb{C}} \mid |q| < 1 \} \cup E(\mathrm{DC}) \cup \{ q \in \widehat{\mathbb{C}} \mid |q| > 1 \}$.

Remark 1.3. Stronger than $\mathscr{H}^1$-quasianalyticity is the following more classical property: if $\mathscr{L}$ is a linear space of functions defined in $C \subset \widehat{\mathbb{C}}$, all of which admit an asymptotic expansion at a point $q_0 \in C$, we say that $\mathscr{L}$ is quasianalytic at q_0 in the sense of Hadamard if the only function of $\mathscr{L}$ with zero asymptotic expansion at q_0 is $\equiv 0$, i.e. a non-tricial function of $\mathscr{L}$ cannot be flat at q_0 (see [MS11]). A convergent Taylor series is a particular case of asymptotic expansion, thus if the functions of $\mathscr{L}$ are analytic at q_0 the question of the quasianalyticity at q_0 makes sense; the question is not trivial when the functions are holomorphic in the interior of C but this interior is not connected.

This is what happens with our spaces $\mathscr{O}(K_M, B)$ and $\mathscr{M}$: given any $q_0 \in \widehat{\mathbb{C}}$ with $|q_0| \neq 1$, the function $\tilde{u}$ which encodes all the KAM curves has a convergent Taylor series at q_0 and this Taylor series determines $\tilde{u}$ everywhere. In the case of $q_0 = 0$, this Taylor series is particularly easy to compute inductively (see formula (12) in section 1.4)[8]. See the end of section 1.4 for an open question about the Hadamard quasianalyticity.

1.4. Comments

Invariant graphs with complex frequencies for the complexified map. The extension $\tilde{u}$ will be obtained as a solution of (5), viewed as a complexified difference equation involving the holomorphic extension of f to the strip S_{R_0} and a complex frequency $\omega \in A_M^{\mathbb{C}}$. This corresponds to determining a complex invariant curve for the holomorphic map $\tilde{T}_\varepsilon \colon S_{R_0} \times \mathbb{C} \to \mathbb{C}/\mathbb{Z} \times \mathbb{C}$ defined by (1). (In fact, the hypothesis that f be *real* analytic is not necessary: f holomorphic in S_{R_0} is sufficient.)

This might be surprising at first sight: if $R < R_0 < \infty$, $\theta \in S_R$ and $|\Im m \omega|$ is too large, we cannot prevent $\theta \pm \omega$ from lying out of the strip S_R where the solution $\tilde{u}$ is supposed to be defined; what is then the meaning of the left-hand side of (5)? The explanation is that, in fact, $\tilde{u}$ will be defined in the larger strip $S_{R+|\Im m \omega|}$.

This can be viewed as an effect of the regularizing effect of the operator E_q which is defined via Fourier series by the formulas

$$E_q \colon \varphi = \sum_{k \in \mathbb{Z}} \hat{\varphi}_k e_k \mapsto \psi = \sum_{k \in \mathbb{Z}^*} \hat{\psi}_k e_k, \qquad \hat{\psi}_k := \frac{1}{\mathrm{e}^{2\pi \mathrm{i} k \omega} - 2 + \mathrm{e}^{-2\pi \mathrm{i} k \omega}} \hat{\varphi}_k = \frac{1}{q^k - 2 + q^{-k}} \hat{\varphi}_k$$

(with the notation of section 2.3) and which has the property $\psi(\theta + \omega) - 2\psi(\theta) + \psi(\theta - \omega) = \varphi(\theta) - \langle \varphi \rangle$. Indeed, when $|\Im m \omega| \geqslant h > 0$, one can check that φ holomorphic in S_R implies $\psi = E_q \varphi$ holomorphic in S_{R+h}; on the other hand, equation (5) with the requirement $\langle u \rangle = 0$ is equivalent to

$$u = \varepsilon E_q \big(f \circ (\mathrm{id} + u) \big) \tag{11}$$

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[8] Observe that this Taylor series $\sum_{n \geqslant 1} q^n u_n$ also determines the function f defining the dynamical system T_ε too (the kth Fourier coefficient of εf coincides with the kth Fourier coefficient of $u_{|k|}$), which may be considered as a kind of inverse scattering.

and the vanishing of $\beta = \varepsilon \langle f \circ (\mathrm{id} + u) \rangle = 0$. For $|q| \neq 1$, it is easy to find a solution $u(q)$ of equation (11) by means of a fixed point method and to check that it depends holomorphically on q; the difference $\mathcal{E}(u)$ between the right-hand side and the left-hand side of (5) is then the constant β and it is $\equiv 0$ because of lemma 15 ($\mathcal{E}(u) = \beta$ constant implies $\beta = \langle (1 + \partial_\theta u) \mathcal{E}(u) \rangle$).

The Taylor series at $q = 0$ of the solution is obtained as follows: the operator E_q can be expanded as

$$E_q = \sum_{n \geqslant 1} q^n E^{(n)}, \qquad E^{(n)} \colon \sum_{k \in \mathbb{Z}} \hat{\varphi}_k e_k \mapsto \sum d \left(\hat{\varphi}_m e_m + \hat{\varphi}_{-m} e_{-m} \right),$$

where the last summation is over all factorizations $n = md$, with integers $m, d \geqslant 1$; then we have $u = \sum_{n \geqslant 1} q^n u_n$ with $u_n \in \mathbb{B}_{R,\rho_M}$, convergent for $|q| < \mathrm{e}^{-2\pi/M}$, with $u_1 = \varepsilon E^{(1)} f$ and

$$u_n = \varepsilon E^{(n)} f + \varepsilon \sum E^{(n_0)} \left(\frac{1}{r!} f^{(r)} u_{n_1} \cdots u_{n_r} \right), \tag{12}$$

where the last summation is over all decompositions $n = n_0 + \cdots + n_r$ with integers $r, n_0, \ldots, n_r \geqslant 1$. Observe that each u_n is a polynomial in ε and a trigonometric polynomial in θ, with Fourier coefficients $\mathcal{F}_k(u_n) = 0$ for $|k| > n$ and $\mathcal{F}_{\pm n}(u_n) = \varepsilon \mathcal{F}_{\pm n}(f)$.

Complex versus real Whitney differentiability. For a closed subset A of $\mathbb{R}^n$, a Banach space B and a function $\varphi \colon A \to B$, the definition of being $\mathscr{C}^1$ or $\mathscr{C}^\infty$ in the sense of Whitney is intrinsic: it involves only the set A on which φ is defined. Whitney's extension theorem yields an alternative definition (see [Wh34, St70]): φ is Whitney $\mathscr{C}^1$ on A if and only if there exists a function $\tilde{\varphi}$ which is $\mathscr{C}^1$ on $\mathbb{R}^n$ such that $\tilde{\varphi}_{|A} = \varphi$. Of course, the extension $\tilde{\varphi}$ is in general not unique.

Similarly, if K is a perfect subset of $\mathbb{C} \simeq \mathbb{R}^2$ (or $\widehat{\mathbb{C}}$), there is an intrinsic definition of being $\mathscr{C}^1$-holomorphic on K (see section 2.1) and Whitney's extension theorem[9] implies that φ is $\mathscr{C}^1$-holomorphic on K if and only if there exists a function $\tilde{\varphi}$ which is $\mathscr{C}^1$ in the real sense on $\mathbb{C}$ (or $\widehat{\mathbb{C}}$) such that $\tilde{\varphi}_{|K} = \varphi$ and $\bar{\partial} \varphi_{|K} = 0$.

The classical results of Pöschel [Pö82] and Lazutkin [Laz73] are concerned with *real* Whitney regularity[10]. Whitney's extension theorem is then useful for instance to estimate the measure of the union of the KAM tori, but the extension it yields is somewhat arbitrary and has no direct dynamical meaning, whereas in this paper we are interested in the uniqueness properties that one gets when $\mathscr{C}^1$-holomorphy is imposed on a sufficiently large set K_M or $\mathscr{F}_\tau$.

Possible generalizations and open questions. The Siegel-type Diophantine condition (8) that we use is not the optimal one for real frequencies: it is shown in [BG01] that a Bruno-type condition is sufficient for the existence of an invariant graph for $|\varepsilon|$ small enough. Maybe one could work with a set of complex frequencies larger than $A_M^{\mathbb{C}}$, built from the set of Bruno numbers instead of $A_M^{\mathbb{R}}$, as the one which is used in [Ri99] to prove $\mathscr{C}^\infty$-holomorphy results in the context of circle maps. (Besides, in the case of the Siegel-type condition, we claim no optimality for the radius $\rho = cM^{-8}$.)

Even sticking to the Siegel Diophantine condition, another issue to be considered is $\mathscr{C}^\infty$-holomorphy, i.e. the existence of infinitely many complex derivatives in the sense of Whitney. One can indeed expect that our function $\tilde{u}$ belong to a space of functions $\mathscr{C}^\infty$-holomorphic on

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[9] See [ALG95], remark III.4 and proposition III.8: our $\mathscr{C}^1$-holomorphic functions correspond to their W-Taylorian 1-fields; see also [Gl58], pp 65–6.

[10] Moreover they are dealing with the difficult case of finitely differentiable data rather than analytic ones. However, [Pö82] (section 5a, pp 690–1), prompted by a remark of Arnold, alludes to a complexified version of his result in the analytic case.

appropriate compact subsets of $\widehat{\mathbb{C}}$ containing $E(\mathrm{DC}_\tau)$; if so, one could then raise the following question, which is analogous to the open question formulated by Herman at the end of his paper on circle maps [He85]: given a point q_* of $E(DC_\tau)$, is $\tilde{u}$, or any function of this space of $\mathscr{C}^\infty$-holomorphic functions, determined by the sequence of the values of its derivatives at q_*? In other words, is this space of functions quasianalytic in the sense of Hadamard at q_*?

Finally, one should consider the adaptation of this circle of ideas to perturbations of higher-dimensional twist maps or to near-integrable Hamiltonian systems.

2. Functional spaces

The adaptation of Levi–Moser's method [LM01] which we employ in section 4 to reach the $\mathscr{C}^1$-holomorphy in the frequency requires the definition of appropriate functional spaces. The point is to deal as much as possible with Banach algebras[11] (even if this requires defining norms which differ from the usual ones), so as to ease all the processes of nonlinear analysis (in particular functional composition). We thus begin by a new definition.

2.1. An algebra norm on the Banach space of $\mathscr{C}^1$-holomorphic functions

Let B be a complex Banach space.

Case of a perfect subset of $\mathbb{C}$. Let K be a perfect subset of $\mathbb{C}$. For any two functions $\varphi, \psi \colon K \to B$, we set

$$\Omega_{\varphi,\psi} \colon K \times K \to B, \qquad \Omega_{\varphi,\psi}(q, q') = \begin{cases} 0 & \text{if } q = q', \\ \dfrac{\varphi(q') - \varphi(q)}{q' - q} - \psi(q) & \text{if } q \neq q'. \end{cases} \tag{13}$$

The usual definition of $\mathscr{C}^1$-holomorphy can be rephrased as follows:

> *A function* $\varphi \colon K \to B$ *is* $\mathscr{C}^1$*-holomorphic iff it is continuous and there exists a continuous* $\psi \colon K \to B$ *such that* $\Omega_{\varphi,\psi}$ *is continuous on* $K \times K$.

Since K has no isolated point, the function ψ is then unique; we usually denote it by φ' or $\varphi^{(1)}$. Observe that at any interior point of K the function φ is holomorphic and φ' gives the ordinary complex derivative. Notice also that, if K is contained in an open set U, then the restriction to K of any holomorphic function $U \to B$ is $\mathscr{C}^1$-holomorphic.

We shall impose furthermore that φ, φ' and $\Omega_{\varphi,\varphi'}$ be bounded. With the notation

$$\delta_\varphi \colon K \times K \to B, \qquad \delta_\varphi(q, q') = \varphi(q') - \varphi(q),$$

we get a Banach space by setting

$$\mathscr{C}^1_{\mathrm{hol}}(K, B) := \{\, \varphi \colon K \to B, \ \mathscr{C}^1\text{-holomorphic, such that } \|\varphi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)} < \infty \,\}, \tag{14}$$

$$\|\varphi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)} := n_0(\varphi) + n_1(\varphi) + n_2(\varphi), \qquad \left| \begin{aligned} &n_0(\varphi) := \sup_K |\varphi|, \\ &n_1(\varphi) := \max\left\{ \sup_K |\varphi'|, \sup_{K \times K} |\delta_\varphi| \right\}, \\ &n_2(\varphi) := \sup_{K \times K} |\Omega_{\varphi,\varphi'}|. \end{aligned} \right. \tag{15}$$

This norm is not the standard one (see for instance [MS03]) but is equivalent to it. An elementary property is

[11] We follow Rudin's convention [Ru91] and call 'Banach algebra' a complex Banach algebra possessing a unit element whose norm is 1.

Lemma 4. *If $\ell : B \to B_0$ is a bounded linear map of norm L between complex Banach spaces, then $\varphi \in \mathscr{C}^1_{\mathrm{hol}}(K, B) \Rightarrow \ell \circ \varphi \in \mathscr{C}^1_{\mathrm{hol}}(K, B_0)$ and*

$$\|\ell \circ \varphi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B_0)} \leqslant L\|\varphi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)}. \tag{16}$$

Moreover, if ℓ is an isometry, then the inequality in (16) is in fact an equality.

As a matter of fact, $\mathscr{C}^1_{\mathrm{hol}}(K) := \mathscr{C}^1_{\mathrm{hol}}(K, \mathbb{C})$ is a Banach algebra and, if B is a Banach algebra, then $\mathscr{C}^1_{\mathrm{hol}}(K, B)$ is a Banach algebra on the algebra $\mathscr{C}^1_{\mathrm{hol}}(K)$:

Lemma 5. *For any perfect subset K of $\mathbb{C}$,*

(i) $\lambda \in \mathscr{C}^1_{\mathrm{hol}}(K)$ *and* $\varphi \in \mathscr{C}^1_{\mathrm{hol}}(K, B) \Rightarrow \lambda \cdot \varphi \in \mathscr{C}^1_{\mathrm{hol}}(K, B)$ *and*

$$\|\lambda \cdot \varphi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)} \leqslant \|\lambda\|_{\mathscr{C}^1_{\mathrm{hol}}(K)}\|\varphi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)}; \tag{17}$$

moreover, when viewed as an element of $\mathscr{C}^1_{\mathrm{hol}}(K)$, the constant function 1 has norm $\|1\|_{\mathscr{C}^1_{\mathrm{hol}}(K)} = 1$;

(ii) if B is a Banach algebra (not necessarily commutative), then $\varphi, \psi \in \mathscr{C}^1_{\mathrm{hol}}(K, B) \Rightarrow \varphi \cdot \psi \in \mathscr{C}^1_{\mathrm{hol}}(K, B)$ and

$$\|\varphi \cdot \psi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)} \leqslant \|\varphi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)}\|\psi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)}. \tag{18}$$

Proof. The proofs of the two inequalities are similar, we content ourselves with (18). We set $\chi = \varphi\psi$ and $\chi_1 = \varphi\psi' + \varphi'\psi$; simple computations yield

$$\delta_\chi(q, q') = \varphi(q') \cdot \delta_\psi(q, q') + \delta_\varphi(q, q') \cdot \psi(q),$$
$$\Omega_{\chi,\chi_1}(q, q') = \varphi(q') \cdot \Omega_{\psi,\psi'}(q, q') + \delta_\varphi(q, q') \cdot \psi'(q) + \Omega_{\varphi,\varphi'}(q, q') \cdot \psi(q),$$

whence the $\mathscr{C}^1$-holomorphy follows, with $\chi' = \chi_1$; moreover,

$$n_0(\chi) \leqslant n_0(\varphi) \cdot n_0(\psi),$$
$$n_1(\chi) \leqslant n_0(\varphi) \cdot n_1(\psi) + n_1(\varphi) \cdot n_0(\psi),$$
$$n_2(\chi) \leqslant n_0(\varphi) \cdot n_2(\psi) + n_1(\varphi) \cdot n_1(\psi) + n_2(\varphi) \cdot n_0(\psi),$$

whence

$$\|\chi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)} \leqslant n_0(\varphi)\big(n_0(\psi) + n_1(\psi) + n_2(\psi)\big) + n_1(\varphi)\big(n_0(\psi) + n_1(\psi)\big) + n_2(\varphi)n_0(\psi)$$
$$\leqslant \|\varphi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)}\|\psi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)}.$$

$\square$

Case of a perfect subset of $\widehat{\mathbb{C}}$.   To be able to apply the $\mathscr{H}^1$-quasianalyticity result of [MS11], we needed to add 0 and ∞ to $E(A_M^{\mathbb{C}})$ in the definition (10) of K_M, so as to have a compact subset of $\widehat{\mathbb{C}}$ of the appropriate form. We thus need to explain the definition of $\mathscr{C}^1_{\mathrm{hol}}(K, B)$ when K is a perfect subset of the Riemann sphere $\widehat{\mathbb{C}}$.

Let $\tilde{K} = \{ \xi \in \mathbb{C} \mid 1/\xi \in K \}$ (with the convention $1/0 = \infty$); both $\tilde{K}$ and $K \cap \mathbb{C}$ are perfect subsets of $\mathbb{C}$. Our definition of $\mathscr{C}^1$-holomorphy is:

A function $\varphi\colon K \to B$ is $\mathscr{C}^1$-holomorphic iff its restriction $\varphi_{|K \cap \mathbb{C}}$ is $\mathscr{C}^1$-holomorphic on $K \cap \mathbb{C}$ and the function $\tilde{\varphi}\colon \xi \in \tilde{K} \mapsto \varphi(1/\xi)$ is $\mathscr{C}^1$-holomorphic on $\tilde{K}$.

We then define

$$\|\varphi\|_{\mathscr{C}^1_{\mathrm{hol}}(K,B)} := \max \left\{ \|\varphi_{|K\cap\mathbb{C}}\|_{\mathscr{C}^1_{\mathrm{hol}}(K\cap\mathbb{C},B)}, \|\tilde{\varphi}\|_{\mathscr{C}^1_{\mathrm{hol}}(\tilde{K},B)} \right\} \tag{19}$$

(i.e. we cover $\widehat{\mathbb{C}}$ by two charts, using q as a complex coordinate in $\mathbb{C}$ and $\xi = 1/q$ as a complex coordinate in $\widehat{\mathbb{C}} - \{0\}$). Formula (19) defines a norm on the space $\mathscr{C}^1_{\mathrm{hol}}(K, B)$ of all B-valued $\mathscr{C}^1$-holomorphic functions on K, which makes it a Banach space. It is easy to check that lemma 4 is still valid, as well as lemma 5:

$\mathscr{C}^1_{\mathrm{hol}}(K) := \mathscr{C}^1_{\mathrm{hol}}(K, \mathbb{C})$ *is a Banach algebra and, if B is a Banach algebra, $\mathscr{C}^1_{\mathrm{hol}}(K, B)$ is a Banach algebra on the algebra $\mathscr{C}^1_{\mathrm{hol}}(K)$.*

2.2. Preliminary small divisor estimates for $q \in K_M$

Here is an example of a $\mathscr{C}^1$-holomorphic function which, in a sense, is the elementary brick of all the solutions of small divisor problems:

Proposition 6. *Let $\tau > 0$, $M > 2\zeta(1 + \tau)$ and K_M as in (10). Then, for any $k \in \mathbb{Z}^*$, the formula*

$$\lambda_k(q) = \frac{1}{q^k - 1} \tag{20}$$

defines a function $\lambda_k \in \mathscr{C}^1_{\mathrm{hol}}(K_M)$ satisfying

$$\|\lambda_k\|_{\mathscr{C}^1_{\mathrm{hol}}(K_M)} \leqslant 7M^2 |k|^{3+2\tau}. \tag{21}$$

Of course, the functions λ_k are just the restrictions to K_M of meromorphic functions but, considering them simultaneously, one easily gets non-meromorphic $\mathscr{C}^1$-holomorphic functions; for instance, for any s in the open unit disk, the series of functions $\varphi = \sum_{k \geqslant 1} s^k \lambda_k$ is convergent in $\mathscr{C}^1_{\mathrm{hol}}(K_M)$ and defines a $\mathscr{C}^1$-holomorphic function for which the unit circle is a natural boundary (because of the accumulation of poles at the roots of unity)—this is an example of Borel–Wolff–Denjoy series (see e.g. [MS03]).

The proof of proposition 6 will make use of use the Diophantine condition (8) in the form of

Lemma 7.

$$k \in \mathbb{Z}^* \qquad \text{and} \qquad q \in K_M \;\Rightarrow\; \left| \frac{1}{q^k - 1} \right| \leqslant \sqrt{2} M |k|^{1+\tau}. \tag{22}$$

The proof of lemma 7 is left to the reader.

Proof of proposition 6. We set $\tilde{K}_M = \{ \xi \in \mathbb{C} \mid 1/\xi \in K_M \}$ and observe that $\tilde{K}_M = K_M \cap \mathbb{C} = E(A_M^{\mathbb{C}}) \cup \{0\}$ (because $A_M^{\mathbb{C}}$ is symmetric with respect to 0). Let $\tilde{\lambda}_k : \xi \in \tilde{K}_M \mapsto \lambda_k(1/\xi)$. We must check that $\lambda_{k|\tilde{K}_M}$ and $\tilde{\lambda}_k \in \mathscr{C}^1_{\mathrm{hol}}(\tilde{K}_M)$ and control their norms.

Since $\lambda_k = \tilde{\lambda}_{-k}$, we can restrict ourselves to the case $k \geqslant 1$.

The function $\lambda := \lambda_{k|\tilde{K}_M}$, being the restriction of a meromorphic function of $\mathbb{C}$ with poles off $\tilde{K}_M$, is $\mathscr{C}^1$-holomorphic on $\tilde{K}_M$. Inequality (22) implies $n_0(\lambda) \leqslant \sqrt{2} M k^{1+\tau}$ and we compute

$$\lambda'(q) = -k \frac{q^{k-1}}{(q^k - 1)^2},$$

$$\delta_\lambda(q, q') = \lambda(q') - \lambda(q) = -\frac{q'^k - q^k}{(q^k - 1)(q'^k - 1)},$$

$$\Omega_{\lambda, \lambda'}(q, q') = k \frac{q^{k-1}}{(q^k - 1)^2} - \frac{q'^{k-1} + q'^{k-2}q + \cdots + q'q^{k-2} + q^{k-1}}{(q^k - 1)(q'^k - 1)}$$

$$= k \frac{q^{k-1}}{(q^k - 1)^2} - \sum_{k-1=a+b} \frac{q^a}{(q^k - 1)} \frac{q'^b}{(q'^k - 1)}.$$

13

We have $|\delta_\lambda(q, q')| \leqslant 2n_0(\lambda) \leqslant 2\sqrt{2}Mk^{1+\tau}$ and, noticing that

$$0 \leqslant \ell \leqslant k \qquad \text{and} \qquad q \in K_M \;\Rightarrow\; \left| \frac{q^\ell}{q^k - 1} \right| \leqslant \sqrt{2}M|k|^{1+\tau}$$

(this reinforcement of (22) is obtained by distinguishing the cases $|q| \leqslant 1$ and $|q| \geqslant 1$, and by writing $q^\ell/(q^k - 1) = -(q^{-(k-\ell)}/(q^{-k} - 1))$ in the latter case), we get $|\lambda'(q)| \leqslant 2M^2 k^{3+2\tau}$ and $|\Omega_{\lambda,\lambda'}(q, q')| \leqslant 4M^2 k^{3+2\tau}$. Since $\sqrt{2}Mk^{1+\tau} < M^2 k^{3+2\tau}$ (because $\tau > 0$ and $M > 2\zeta(1 + \tau) > 2$), we conclude that $\|\lambda\|_{\mathscr{C}^1_{\mathrm{hol}}(\tilde{K}_M)} \leqslant 7M^2 k^{3+2\tau}$.

On the other hand, $n_0(\tilde{\lambda}_k) \leqslant \sqrt{2}M|k|^{1+\tau}$ (still because of (22)) and $\tilde{\lambda}_k = -1 - \lambda$, thus $\tilde{\lambda}'_k = -\lambda'$ and $\delta_{\tilde{\lambda}_k} = -\delta_\lambda$, $\Omega_{\tilde{\lambda}_k, \tilde{\lambda}'_k} = -\Omega_{\lambda,\lambda'}$, hence $\|\tilde{\lambda}_k\|_{\mathscr{C}^1_{\mathrm{hol}}(\tilde{K}_M)} \leqslant 7M^2 k^{3+2\tau}$ and the proof is complete. □

2.3. Fourier analysis and functional composition in $\mathscr{C}^1_{\mathrm{hol}}\big(K, H^\infty(S_r, B)\big)$

We shall mainly deal with $\mathscr{C}^1_{\mathrm{hol}}(K_M, B)$ with target spaces $B = \mathbb{C}$ or $H^\infty(\mathbb{D}_\rho)$ or $\mathbb{B}_{r,\rho} = H^\infty(S_r \times \mathbb{D}_\rho)$, with the notation (6)–(7). Notice that $\mathbb{B}_{r,\rho}$ is canonically isomorphic to $H^\infty(S_r, B)$ with $B = H^\infty(\mathbb{D}_\rho)$, where we denote by $H^\infty(D, B)$ the Banach space of all B-valued bounded holomorphic functions on D endowed with the sup norm (for any complex manifold D and any complex Banach space B) and $H^\infty(D) := H^\infty(D, \mathbb{C})$.

Let B denote any complex Banach space. We shall use Fourier analysis in $H^\infty(S_r, B)$, as indicated in the classical

Lemma 8. *Let $r > 0$ and $k \in \mathbb{Z}$. The formula*

$$\mathcal{F}_k \colon \varphi \mapsto \hat{\varphi}_k := \int_0^1 \varphi(\theta)\, \mathrm{e}^{-2\pi \mathrm{i}k\theta}\, \mathrm{d}\theta$$

defines a bounded linear map $\mathcal{F}_k \colon H^\infty(S_r, B) \to B$, with

$$\|\hat{\varphi}_k\|_B \leqslant \mathrm{e}^{-2\pi r|k|} \|\varphi\|_{H^\infty(S_r, B)}.$$

We shall also use the notation

$$e_k(\theta) = \mathrm{e}^{2\pi \mathrm{i}k\theta}, \qquad \theta \in \mathbb{C},\; k \in \mathbb{Z}$$

and $\langle \varphi \rangle := \hat{\varphi}_0$. Observe that, if $\varphi \in H^\infty(S_r, B)$, the fact that $|e_k(\theta)| = \mathrm{e}^{-2\pi k \Im m\theta}$ implies that the series $\sum \hat{\varphi}_k e_k(\theta)$ converges to $\varphi(\theta)$ for every $\theta \in S_r$, while $\sum \hat{\varphi}_k e_{k|S_{r'}} = \varphi_{|S_{r'}}$ is absolutely convergent in $H^\infty(S_{r'}, B)$ in general only for $r' < r$.

The Cauchy inequalities also yield

Lemma 9. *If $0 < r' < r$, then the derivation with respect to θ induces a bounded linear map $\partial_\theta \colon H^\infty(S_r, B) \to H^\infty(S_{r'}, B)$, with*

$$\|\partial_\theta^p \varphi\|_{H^\infty(S_{r'}, B)} \leqslant \frac{p!}{(r - r')^p} \|\varphi\|_{H^\infty(S_r, B)}, \qquad p \in \mathbb{N}.$$

Applying lemma 4 with $\ell = \mathcal{F}_k$ or ∂_θ^p, we get

Corollary 10. *Let K be a perfect subset of $\mathbb{C}$ or $\widehat{\mathbb{C}}$, B a complex Banach space, $r > 0$ and $\varphi \in \mathscr{C}^1_{\mathrm{hol}}(K, H^\infty(S_r, B))$. Then:*

1. *For all $k \in \mathbb{Z}$, the Fourier coefficients $\hat{\varphi}_k := \mathcal{F}_k \circ \varphi$ belong to $\mathscr{C}^1_{\mathrm{hol}}(K, B)$ and satisfy*

$$\|\hat{\varphi}_k\|_{\mathscr{C}^1_{\mathrm{hol}}(K, B)} \leqslant \mathrm{e}^{-2\pi r|k|} \|\varphi\|_{\mathscr{C}^1_{\mathrm{hol}}(K, H^\infty(S_r, B))}. \tag{23}$$

2. *If $0 < r' < r$, then $\partial_\theta^p \varphi$ belongs to $\mathscr{C}_{\text{hol}}^1(K, H^\infty(S_{r'}, B))$ and satisfies*

$$\|\partial_\theta^p \varphi\|_{\mathscr{C}_{\text{hol}}^1(K, H^\infty(S_{r'}, B))} \leqslant \frac{p!}{(r - r')^p} \|\varphi\|_{\mathscr{C}_{\text{hol}}^1(K, H^\infty(S_r, B))} \tag{24}$$

for all $p \in \mathbb{N}$.

We now consider composition with respect to the variable θ:

Lemma 11. *Let K be a perfect subset of $\mathbb{C}$ or $\widehat{\mathbb{C}}$, B a complex Banach algebra, $r > r' > 0$ and $\kappa \in (0, 1)$. Then, for any $\varphi \in \mathscr{C}_{\text{hol}}^1(K, H^\infty(S_r, B))$ and $\psi \in \mathscr{C}_{\text{hol}}^1(K, H^\infty(S_{r'}, B))$ such that*

$$\|\psi\|_{\mathscr{C}_{\text{hol}}^1(K, H^\infty(S_{r'}))} \leqslant \kappa(r - r'),$$

the series

$$\varphi \circ (\text{id} + \psi) = \sum_{p \geqslant 0} \frac{1}{p!} (\partial_\theta^p \varphi) \psi^p$$

is absolutely convergent in $\mathscr{C}_{\text{hol}}^1(K, H^\infty(S_{r'}, B))$ and defines a function which satisfies

$$\|\varphi \circ (\text{id} + \psi)\|_{\mathscr{C}_{\text{hol}}^1(K, H^\infty(S_{r'}, B))} \leqslant (1 - \kappa)^{-1} \|\varphi\|_{\mathscr{C}_{\text{hol}}^1(K, H^\infty(S_r, B))}.$$

Proof. Use the Cauchy inequalities (24) and the product inequalities (18). $\qquad\square$

We shall use lemma 11 with $B = H^\infty(\mathbb{D}_\rho)$, in which case

$$\big(\varphi \circ (\text{id} + \psi)\big)(q)(\theta, \varepsilon) = \varphi(q)\big(\theta + \psi(q)(\theta, \varepsilon), \varepsilon\big)$$

if we use the identification $\mathscr{C}_{\text{hol}}^1(K, H^\infty(S_r, B)) = \mathscr{C}_{\text{hol}}^1\big(K, H^\infty(S_r \times \mathbb{D}_\rho)\big)$.

3. $\mathscr{C}^1$-holomorphy of the solution of the cohomological equation

For $\omega \in \mathbb{R} - \mathbb{Q}$ and φ analytic on $\mathbb{T}$ of zero mean value, the 'cohomological equation' is the linear equation

$$\psi(\theta + \omega) - \psi(\theta) = \varphi(\theta) \tag{25}$$

which will appear in Levi–Moser's scheme in section 4. Its solution is formally given by the Fourier series

$$\psi = \sum_{k \in \mathbb{Z}^*} \frac{1}{e^{2\pi i k\omega} - 1} \hat{\varphi}_k e_k = \sum_{k \in \mathbb{Z}^*} \lambda_k(q) \hat{\varphi}_k e_k, \qquad q = e^{2\pi i \omega}. \tag{26}$$

As is well known, this defines an analytic function ψ when $\omega \in A_M^{\mathbb{R}}$ (use e.g. (22) with q on the unit circle). We shall see that this is still true when one considers the complexified equation associated with $q \in K_M$, i.e. $\omega \in A_M^{\mathbb{C}}$ or $\omega = \pm i\infty$, and that the solution ψ depends $\mathscr{C}^1$-holomorphically on q when φ does.

Notice however that, if $\Im m\,\omega \neq 0$, equation (25) requires that the unknown ψ be defined in a strip which is larger than the strip where φ admits a holomorphic extension. Suppose for instance $\Im m\,\omega > 0$ and φ holomorphic in S_r; then we seek a solution ψ holomorphic for all $\theta \in \mathbb{C}/\mathbb{Z}$ such that $-r < \Im m\,\theta < r + \Im m\,\omega$, so that equation (25) makes sense for all $\theta \in S_r$; it turns out that (26) defines such a function. This will be part of proposition 12; before stating it, we introduce a notation for the functional spaces we shall systematically deal with from now on.

Notation 3.1. *We fix* $\tau > 0$, $M > 2\zeta(1+\tau)$ *and* $\rho > 0$ *and define* K_M *as in* (10) *and*

$$\mathbb{B}_\rho := H^\infty(\mathbb{D}_\rho). \tag{27}$$

For any $r > 0$, *as in* (6)–(7) *we define* $\mathbb{B}_{r,\rho} = H^\infty(S_r \times \mathbb{D}_\rho)$, *which is thus is canonically isomorphic to* $H^\infty(S_r, \mathbb{B}_\rho)$, *and we set*

$$\mathscr{C}_{r,\rho} := \mathscr{C}^1_{\mathrm{hol}}(K_M, \mathbb{B}_{r,\rho}) = \mathscr{C}^1_{\mathrm{hol}}\big(K_M, H^\infty(S_r, \mathbb{B}_\rho)\big). \tag{28}$$

For any $\varphi \in \mathscr{C}_{r,\rho}$, *we also use the notation* $\|\varphi\|_r := \|\varphi\|_{\mathscr{C}^1_{\mathrm{hol}}(K_M, \mathbb{B}_{r,\rho})}$ *and*

$$\varphi(q,\theta) := \varphi(q)(\theta) \in \mathbb{B}_\rho, \qquad q \in K_M,\ \theta \in S_r,$$

as well as $\varphi(q)(\theta, \cdot)$ *or* $\varphi(q, \theta, \cdot)$.

Definition 3.2. *For* $r > 0$, *we define* $\mathscr{C}^+_{r,\rho}$, *respectively,* $\mathscr{C}^-_{r,\rho}$, *to be the space of all functions* $\varphi = \sum_{k \in \mathbb{Z}} \hat\varphi_k e_k \in \mathscr{C}_{r,\rho}$ *such that each function*

$$q^k \hat\varphi_k(q), \qquad \text{respectively} \qquad q^{-k}\hat\varphi_k(q)$$

has limits for $q \to 0$ *and* $q \to \infty$, *and the series*

$$\varphi^+(q,\theta) = \sum_{k\in\mathbb{Z}} q^k \hat\varphi_k(q) e_k(\theta), \qquad \text{respectively} \qquad \varphi^-(q,\theta) = \sum_{k\in\mathbb{Z}} q^{-k}\hat\varphi_k(q)e_k(\theta),$$

converges in $\mathbb{B}_\rho$ *for all* $(q,\theta) \in K_M \times S_r$ *and defines an element of* $\mathscr{C}_{r,\rho}$. *For any such function, we define*

$$\|\varphi\|_{r,+} := \max\big(\|\varphi\|_r, \|\varphi^+\|_r\big), \qquad \text{respectively} \qquad \|\varphi\|_{r,-} := \max\big(\|\varphi\|_r, \|\varphi^-\|_r\big),$$

$$\nabla\varphi := \varphi^+ - \varphi \in \mathscr{C}_{r,\rho}, \qquad \text{respectively} \qquad \nabla^-\varphi := \varphi - \varphi^- \in \mathscr{C}_{r,\rho}.$$

We also set

$$\mathscr{C}^{(\pm)}_{r,\rho} := \mathscr{C}^+_{r,\rho} \cap \mathscr{C}^-_{r,\rho}, \qquad \|\varphi\|_{r,(\pm)} := \max\big(\|\varphi\|_{r,+}, \|\varphi\|_{r,-}\big).$$

It is easy to check that $\big(\mathscr{C}^+_{r,\rho}, \|\cdot\|_{r,+}\big)$, $\big(\mathscr{C}^-_{r,\rho}, \|\cdot\|_{r,-}\big)$ and $\big(\mathscr{C}^{(\pm)}_{r,\rho}, \|\cdot\|_{r,(\pm)}\big)$ are Banach spaces. Notice that, for $q = \mathrm{e}^{2\pi\mathrm{i}\omega} \in K_M - \{0, \infty\}$,

$$\varphi^+(q,\theta) = \varphi(q, \theta + \omega), \qquad \varphi^-(q,\theta) = \varphi(q, \theta - \omega) \tag{29}$$

and, if $\Im m\omega > 0$ [respectively if $\Im m\omega < 0$],

$$\varphi \in \mathscr{C}^+_{r,\rho} \ \Rightarrow\ \varphi(q) \in H^\infty(S_{-r, r+\Im m\omega}, \mathbb{B}_\rho) \quad \big[\text{respectively } H^\infty(S_{-r+\Im m\omega, r}, \mathbb{B}_\rho)\big]$$

$$\varphi \in \mathscr{C}^-_{r,\rho} \ \Rightarrow\ \varphi(q) \in H^\infty(S_{-r-\Im m\omega, r}, \mathbb{B}_\rho) \quad \big[\text{respectively } H^\infty(S_{-r, r-\Im m\omega}, \mathbb{B}_\rho)\big]$$

with the notation $S_{h_1,h_2} = \{\theta \in \mathbb{C} \mid h_1 < \Im m(\theta) < h_2\}$.

Thus, the cohomological equation (25) can be rephrased as $\nabla\psi = \varphi$, where φ is given in $\mathscr{C}_{r,\rho}$ and ψ is sought in $\mathscr{C}^+_{r,\rho}$. We shall also need to deal with the shifted equation $\nabla^-\psi = \varphi$ for which ψ is sought in $\mathscr{C}^-_{r,\rho}$. But this is asking too much: we'll have to content ourselves with $\psi \in \mathscr{C}^+_{r',\rho}$, respectively $\mathscr{C}^-_{r',\rho}$, with any $r' < r$.

Proposition 12. *Suppose that* $0 < r' < r$ *and* $\varphi = \sum_{k\in\mathbb{Z}} \hat\varphi_k e_k \in \mathscr{C}_{r,\rho}$. *Then the formulas*

$$(\Gamma\varphi)(q) = \sum_{k\in\mathbb{Z}^*} \lambda_k(q)\hat\varphi_k(q)e_k, \qquad (\Gamma^-\varphi)(q) = -\sum_{k\in\mathbb{Z}^*} \lambda_{-k}(q)\hat\varphi_k(q)e_k \tag{30}$$

(still with $\lambda_k(q) = 1/(q^k - 1)$*) define two functions* $\Gamma\varphi \in \mathscr{C}^+_{r',\rho}$ *and* $\Gamma^-\varphi \in \mathscr{C}^-_{r',\rho}$, *which satisfy*

$$\nabla(\Gamma\varphi) = \nabla^-(\Gamma^-\varphi) = \varphi - \hat\varphi_0 \qquad \text{in } \mathscr{C}_{r',\rho}. \tag{31}$$

Moreover,

$$(\Gamma\varphi)^+ = \Gamma^-\varphi, \qquad (\Gamma^-\varphi)^- = \Gamma\varphi \tag{32}$$

and there exists a positive constant C_1 which depends only on τ such that, if $r - r' \leqslant 1$, then

$$\|\Gamma\varphi\|_{r',+}, \ \|\Gamma^-\varphi\|_{r',-} \leqslant \frac{C_1 M^2}{(r - r')^\sigma} \|\varphi\|_r, \tag{33}$$

$$\|\partial_\theta\Gamma\varphi\|_{r',+}, \ \|\partial_\theta\Gamma^-\varphi\|_{r',-} \leqslant \frac{C_1 M^2}{(r - r')^{\sigma+1}} \|\varphi\|_r, \tag{34}$$

where $\sigma := 4 + 2\tau$.

Proof. For $k \in \mathbb{Z}^*$, by virtue of (21) and (23), the fact that $\|e_k\|_{H^\infty(S_{r'})} = e^{2\pi|k|r'}$ and $\|\partial_\theta e_k\|_{H^\infty(S_{r'})} = 2\pi|k|e^{2\pi|k|r'}$ implies

$$\|\lambda_k\hat\varphi_k e_k\|_{r'}, \ \|\lambda_{-k}\hat\varphi_k e_k\|_{r'} \leqslant 7M^2|k|^{3+2\tau}e^{-2\pi(r-r')|k|}\|\varphi\|_r,$$
$$\|\lambda_k\hat\varphi_k \partial_\theta e_k\|_{r'}, \ \|\lambda_{-k}\hat\varphi_k \partial_\theta e_k\|_{r'} \leqslant 7M^2(2\pi)|k|^{4+2\tau}e^{-2\pi(r-r')|k|}\|\varphi\|_r.$$

The series in (30) can thus be viewed as an absolutely convergent series in $\mathscr{C}_{r',\rho}$, defining functions $\Gamma\varphi$, $\Gamma^-\varphi$ which satisfy

$$\|\Gamma\varphi\|_{r'}, \ \|\Gamma^-\varphi\|_{r'} \leqslant 14M^2\Sigma(\alpha, 3 + 2\tau)\|\varphi\|_r,$$
$$\|\partial_\theta\Gamma\varphi\|_{r'}, \ \|\partial_\theta\Gamma^-\varphi\|_{r'} \leqslant 14M^2(2\pi)\Sigma(\alpha, 4 + 2\tau)\|\varphi\|_r,$$

where

$$\Sigma(\alpha, \beta) = \sum_{k\geqslant 1} k^\beta e^{-\alpha k}, \qquad \alpha = 2\pi(r - r'), \qquad \beta > 1.$$

If $r - r' \leqslant 1$, then $\alpha \leqslant 2\pi$. Since $x \mapsto x^\beta e^{-\alpha x}$ is increasing on $(0, \beta/\alpha)$, decreasing on $(\beta/\alpha, +\infty)$ and bounded by $(\beta/e)^\beta \alpha^{-\beta}$, with $\int_0^\infty x^\beta e^{-\alpha x}\,\mathrm{d}x = \beta!\alpha^{-\beta-1}$, we have

$$\Sigma(\alpha, \beta) < \beta!\alpha^{-\beta-1} + \left(\frac{\beta}{e}\right)^\beta \alpha^{-\beta} \leqslant \left[\beta! + 2\pi\left(\frac{\beta}{e}\right)^\beta\right]\left(2\pi(r - r')\right)^{-\beta-1},$$

hence

$$\|\Gamma\varphi\|_{r'}, \ \|\Gamma^-\varphi\|_{r'} \leqslant \frac{C_1 M^2}{(r - r')^\sigma}\|\varphi\|_r, \qquad \|\partial_\theta\Gamma\varphi\|_{r'}, \ \|\partial_\theta\Gamma^-\varphi\|_{r'} \leqslant \frac{C_1 M^2}{(r - r')^{\sigma+1}}\|\varphi\|_r,$$

with $\sigma = 4 + 2\tau$ and $C_1 := 14[\sigma! + 2\pi(\frac{\sigma}{e})^\sigma](2\pi)^{-\sigma}$.

We now observe that, for each $k \in \mathbb{Z}^*$, $q^k\lambda_k(q) = q^k/(q^k - 1) = -\lambda_{-k}(q)$ defines an element of $\mathscr{C}^1_{\mathrm{hol}}(K_M, \mathbb{B}_\rho)$, thus the Fourier series $(\Gamma\varphi)^+(q) = \sum q^k\lambda_k(q)\hat\varphi_k(q)e_k$ coincides with $(\Gamma^-\varphi)(q)$ and defines an element of $\mathscr{C}_{r',\rho}$, hence $\Gamma\varphi \in \mathscr{C}^+_{r',\rho}$ with $\|\Gamma\varphi\|_{r',+} \leqslant C_1 M^2(r - r')^{-\sigma}\|\varphi\|_r$ and $\|\partial_\theta\Gamma\varphi\|_{r',+} \leqslant C_1 M^2(r - r')^{-\sigma-1}\|\varphi\|_r$. Similarly, $(\Gamma^-\varphi)^- = \Gamma\varphi$ (hence (32)), $\Gamma^-\varphi \in \mathscr{C}^-_{r',\rho}$ with $\|\Gamma^-\varphi\|_{r',-} \leqslant C_1 M^2(r - r')^{-\sigma}\|\varphi\|_r$, $\|\partial_\theta\Gamma^-\varphi\|_{r',-} \leqslant C_1 M^2(r - r')^{-\sigma-1}\|\varphi\|_r$.

The identities (31) stem from the relations $(q^k - 1)\lambda_k(q) = -(1 - q^{-k})\lambda_{-k}(q) = 1$ valid for all $k \in \mathbb{Z}^*$. $\qquad\square$

Remark 3.1. Observe that the kernels of ∇ or ∇^- do not depend on r and consist of the functions which are constant in θ, i.e. they coincide with

$$\mathscr{C}_\rho := \mathscr{C}^1_{\mathrm{hol}}(K_M, \mathbb{B}_\rho).$$

Remark 3.2. Observe that

$$\varphi \in \mathscr{C}^-_{r,\rho} \ \Rightarrow \ \Gamma\varphi = \Gamma^-(\varphi^-) \in \mathscr{C}^{(\pm)}_{r',\rho}$$

(because $\lambda_k(q) = -\lambda_{-k}(q)q^{-k}$ for each $k \in \mathbb{Z}^*$), with

$$\|\Gamma\varphi\|_{r',(\pm)} \leqslant \frac{C_1 M^2}{(r - r')^\sigma}\|\varphi\|_{r,-}, \qquad \|\partial_\theta\Gamma\varphi\|_{r',(\pm)} \leqslant \frac{C_1 M^2}{(r - r')^{\sigma+1}}\|\varphi\|_{r,-}.$$

Lastly, as we shall have to keep track of real analyticity with respect to the variables $\omega, \theta, \varepsilon$, we introduce

Definition 3.3. *We denote by* $\mathscr{C}_{r,\rho,\mathbb{R}}$ *the subspace of* $\mathscr{C}_{r,\rho}$ *consisting of all functions* φ *whose Fourier coefficients satisfy*

$$\text{conj}\big(\hat{\varphi}_k(q)\big) = \hat{\varphi}_{-k}(1/\bar{q}), \qquad k \in \mathbb{Z}, \ q \in K_M,$$

where conj *denotes complex conjugacy in* $\mathbb{B}_\rho$ *(i.e.* $(\text{conj}\,\psi)(\varepsilon) := \overline{\psi(\bar{\varepsilon})}$ *for any* $\psi \in \mathbb{B}_\rho$*). We also set*

$$\mathscr{C}_{r,\rho,\mathbb{R}}^+ := \mathscr{C}_{r,\rho}^+ \cap \mathscr{C}_{r,\rho,\mathbb{R}}, \quad \mathscr{C}_{r,\rho,\mathbb{R}}^- := \mathscr{C}_{r,\rho}^- \cap \mathscr{C}_{r,\rho,\mathbb{R}}, \quad \mathscr{C}_{r,\rho,\mathbb{R}}^{(\pm)} := \mathscr{C}_{r,\rho}^{(\pm)} \cap \mathscr{C}_{r,\rho,\mathbb{R}}$$

and $\mathscr{C}_{\rho,\mathbb{R}} := \{ \varphi \in \mathscr{C}_\rho \mid \text{conj}\big(\varphi(q)\big) = \varphi(1/\bar{q}) \ for \ all \ q \in K_M \}$.

The functions in these spaces have the property

$$|q| = 1, \ \theta \in \mathbb{R}/\mathbb{Z}, \ \varepsilon \in \mathbb{R} \ \Rightarrow \ \varphi(q)(\theta, \varepsilon) \in \mathbb{R}.$$

Since $\text{conj}\big(\lambda_k(q)\big) = \lambda_{-k}(1/\bar{q})$, we see that the operators Γ and Γ^- preserve real analyticity, so that they induce operators

$$\Gamma : \mathscr{C}_{r,\rho,\mathbb{R}} \to \mathscr{C}_{r',\rho,\mathbb{R}}^+, \quad \Gamma^- : \mathscr{C}_{r,\rho,\mathbb{R}} \to \mathscr{C}_{r',\rho,\mathbb{R}}^-.$$

4. Levi–Moser's modified Newton scheme

4.1. Reduction of the problem

Suppose $f \in H^\infty(S_{R_0})$ real analytic with zero mean value, with f'' bounded in S_{R_0} and $0 < R < R_0$. We shall see that this is enough to prove theorem 1.

In view of definition 3.2 and formulas (29), we can define the operator

$$\Delta = \nabla - \nabla^- = \nabla\nabla^- = \nabla^-\nabla, \qquad \Delta : \mathscr{C}_{r,\rho}^{(\pm)} \to \mathscr{C}_{r,\rho}, \qquad r > 0,$$

so that, for $q = \mathrm{e}^{2\pi\mathrm{i}\omega} \in K_M - \{0, \infty\}$ and $\theta \in S_r$,

$$(\Delta u)(q, \theta) = u(q, \theta + \omega) - 2u(q, \theta) + u(q, \theta - \omega) \in \mathbb{B}_\rho$$

appears as a complexification of the left-hand side of equation (5). Therefore, the equation

$$\Delta u = \varepsilon f \circ (\mathrm{id} + u) \tag{35}$$

boils down to equation (5) when $q = \mathrm{e}^{2\pi\mathrm{i}\omega}$ is on the unit circle and θ and ε are real.

Proposition 13. *The right-hand side of (35) makes sense for any* $r, \rho > 0$ *and* $u \in \mathscr{C}_{r,\rho}^{(\pm)}$ *such that* $\|u\|_r < R_0 - r$.

Let $r' \in (0, r)$. *If a function* $u \in \mathscr{C}_{r,\rho,\mathbb{R}}^{(\pm)}$ *is a solution of (35) which satisfies* $\|u\|_r < \min(R_0 - r, r)$ *and* $\|u\|_{r'} < r - r'$, *then the formula*

$$\tilde{u}_M(q, \theta) := u\big(q, \theta - \hat{u}_0(q)\big) - \hat{u}_0(q)$$

defines a function $\tilde{u}_M \in \mathscr{C}^1(K_M, \mathbb{B}_{r',\rho})$ *such that, for each* $\omega \in A_M^{\mathbb{R}}$ *and* $\varepsilon \in (-\rho, \rho)$, *the function* $\theta \in \mathbb{T} \mapsto \tilde{u}_M(\mathrm{e}^{2\pi\mathrm{i}\omega}, \theta, \varepsilon)$ *has zero mean value and parametrizes through (3)–(4) an invariant graph of frequency* ω *for* T_ε.

Proof. The right-hand side of (35) makes sense because the hypothesis $\|u\|_r < R_0 - r$ allows us to interpret it according to lemma 11, viewing εf as an element of $\mathbb{B}_{R_0,\rho} = H^\infty(S_{R_0}, \mathbb{B}_\rho)$, or even as a function of $\mathscr{C}_{R_0,\rho} = \mathscr{C}_{\text{hol}}^1(K_M, \mathbb{B}_{R_0,\rho})$ which is constant in q.

The assumption $\|u\|_r < r$ and the Cauchy inequalities imply $|\partial_\theta u(q, \theta, \varepsilon)| < 1$ for real θ, hence $\theta \mapsto U(\theta) = \theta + u(q, \theta, \varepsilon)$ defines a homeomorphism of $\mathbb{R}$ for every $q = \mathrm{e}^{2\pi\mathrm{i}\omega}$

on the unit circle (i.e. $\omega \in A_M^{\mathbb{R}}$) and $\varepsilon \in (-\rho, \rho)$; according to section 1.1, this yields an invariant graph for every $\omega \in A_M^{\mathbb{R}}$, we just need to shift the parametrization and consider $U(\theta - \hat{u}_0) = \theta + \tilde{u}_M(q, \theta, \varepsilon)$ to get the zero mean value normalization.

The fact that $\tilde{u}_M \in \mathscr{C}^1(K_M, \mathbb{B}_{r',\rho})$ follows from lemma 11 if we view $\hat{u}_0$ as a function of $\mathscr{C}_{r',\rho}$ (constant in θ) satisfying $\|\hat{u}_0\|_{r'} \leqslant \|u\|_{r'} < r - r'$ and observe that $\tilde{u}_M = u \circ (\mathrm{id} - \hat{u}_0) - \hat{u}_0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Therefore, to prove theorem 1, we only need to find $c > 0$ independent of M and a solution $u \in \mathscr{C}_{R_\infty,\rho,\mathbb{R}}^{(\pm)}$ of (35) with $\rho = cM^{-8}$ and some $R_\infty \in (R, R_0)$, such that

$$\|u\|_{R_\infty} < \min(R_0 - R_\infty, R_\infty - R) \tag{36}$$

(applying the previous proposition with $r = R_\infty$ and $r' = R$).

To obtain this solution u, we shall inductively construct a sequence $(u_n)_{n \geqslant 1}$ with

$$u_n \in \mathscr{C}_{R_n,\rho,\mathbb{R}}^{(\pm)}, \qquad R_0 > R_1 > R_2 > \ldots, \qquad R_n \underset{n \to \infty}{\longrightarrow} R_\infty > R$$

in such a way that the restrictions $u_{n|K_M \times S_{R_\infty} \times \mathbb{D}_\rho}$ converge to the desired solution in $\mathscr{C}_{R_\infty,\rho,\mathbb{R}}^{(\pm)}$, at least if ρ is small enough. We shall see that the constant c will depend on f only through $\|f\|_{R_0}$ and $\|f''\|_{R_0}$.

As in [LM01], the passage from u_n to u_{n+1} will be a variant of the Newton method, which we now explain.

4.2. The inductive step

Let us define an 'error functional' as

$$u \mapsto \mathcal{E}(u) := -\Delta u + \varepsilon f \circ (\mathrm{id} + u), \tag{37}$$

so that equation (35) amounts to $\mathcal{E}(u) = 0$. As noticed earlier,

$$u \in \mathscr{C}_{r,\rho,\mathbb{R}}^{(\pm)}, \ \|u\|_r \leqslant R_0 - r \ \Rightarrow \ \mathcal{E}(u) \in \mathscr{C}_{r,\rho,\mathbb{R}}. \tag{38}$$

The Taylor formula yields

$$\mathcal{E}(u + h) = \mathcal{E}(u) + \mathcal{E}'(u)[h] + Q(u, h)$$

with a map

$$\mathcal{E}'(u)[h] := -\Delta h + \big(\varepsilon f' \circ (\mathrm{id} + u)\big) h$$

which is linear in h and a remainder term

$$Q(u, h) := \left(\int_0^1 \varepsilon f'' \circ (\mathrm{id} + u + th)(1 - t)\, \mathrm{d}t \right) h^2$$

which has a norm of the same order of magnitude as $\|h\|^2$.

The classical Newton method would consist in defining $u_{n+1} = u_n + h$ with h chosen so that $\mathcal{E}(u_n) + \mathcal{E}'(u_n)[h] = 0$ and $\|h\|$ comparable to $\|\mathcal{E}(u_n)\|$, hence a new error $\mathcal{E}(u_n + h) = Q(u_n, h)$ which would be quadratically smaller than $\|\mathcal{E}(u_n)\|$. Unfortunately, our operator $\mathcal{E}'(u)$ is hard to invert because it is the sum of a constant coefficient difference operator and a multiplication operator.

Levi–Moser's trick consists in adding a term which does not affect the quadratic gain but makes it possible to determine easily the increment h: let $u = u_n$ and $A = 1 + \partial_\theta u_n$; if, instead of requiring $\mathcal{E}'(u)[h] = -\mathcal{E}(u)$, we require

$$A\mathcal{E}'(u)[h] - h\mathcal{E}'(u)[A] = -A\mathcal{E}(u) \tag{39}$$

and manage to get a solution h of size comparable to $\|\mathcal{E}(u)\|$, then we get $A\mathcal{E}(u + h) = h\mathcal{E}'(u)[A] + AQ(u, h)$, hence

$$\mathcal{E}(u + h) = \frac{h}{A}\partial_\theta\big(\mathcal{E}(u)\big) + Q(u, h), \tag{40}$$

which might be sufficient to ensure the convergence of the scheme. Now, equation (39) is tractable because the multiplication operator part in $\mathcal{E}'(u)$ cancels out from the left-hand side: equation (39) is equivalent to

$$A\Delta h - h\Delta A = A\mathcal{E}(u) \tag{41}$$

and lemma 14 shows how to factorize the left-hand side, while lemma 15 shows that the right-hand side has zero mean value, which turns out to be sufficient to obtain a solution h, as stated in lemma 16.

Lemma 14. *For any $r > 0$ and $A, h \in \mathscr{C}_{r,\rho}^{(\pm)}$ such that A is invertible,*

$$A\Delta h - h\Delta A = \nabla^-\Big(AA^+\nabla\Big(\frac{h}{A}\Big)\Big) \tag{42}$$

with the notation of definition 3.2.

Proof. Let $w = h/A$ and $a = AA^+$. Since $\varphi \mapsto \varphi^+$ and $\varphi \mapsto \varphi^-$ are algebra maps, the left-hand side of (42) is

$$Ah^+ + Ah^- - hA^+ - hA^- = aw^+ + a^-w^- - aw - a^-w = a(\nabla w) - a^-(\nabla w)^-,$$

whence the result follows. $\qquad\square$

Lemma 15. *Let $u \in \mathscr{C}_{r,\rho}^{(\pm)}$ satisfy $\|u\|_r < R_0 - r$, so that $\mathcal{E}(u) = -\Delta u + \varepsilon f \circ (\mathrm{id} + u) \in \mathscr{C}_{r,\rho}$ is well-defined, and let $A = 1 + \partial_\theta u$. Then $A\mathcal{E}(u)$, which belongs to $\mathscr{C}_{r',\rho}$ for every $r' \in (0, r)$, has zero mean value:*

$$\langle A\mathcal{E}(u)\rangle = 0.$$

Proof. Since $\langle f\rangle = 0$, we can write f as the θ-derivative of a periodic function $F \in H^\infty(S_{R_0})$. The function $A \cdot \big(\varepsilon f \circ (\mathrm{id} + u)\big)$ clearly has zero mean value, as the θ-derivative of $\varepsilon F \circ (\mathrm{id} + u)$. As for the remaining part, adding and subtracting $(\partial_\theta u)^+\nabla u$, we can write it as

$$-A\Delta u = -\Delta u - (\partial_\theta u)(\nabla u - \nabla^- u) = -\Delta u + \big((\partial_\theta u)^+ - \partial_\theta u\big)\nabla u - (\partial_\theta u)^+\nabla u + \partial_\theta u(\nabla u)^-$$

$$= -\Delta u + (\nabla\partial_\theta u)\nabla u - \nabla\big(\partial_\theta u \cdot (\nabla^- u)\big) = -\Delta u + \partial_\theta\big(\tfrac{1}{2}(\nabla u)^2\big) - \nabla\big(\partial_\theta u \cdot (\nabla^- u)\big),$$

whence the claim follows since the difference operators Δ and ∇ as well as the differential operator ∂_θ kill the constant terms. $\qquad\square$

The next lemma shows how to find a solution of an equation like (41) in the form $h = Aw$, $w = \mathcal{F}\big(A, \mathcal{E}(u)\big)$, with a functional $\mathcal{F}$ which involves the operators Γ and Γ^- of proposition 12 and the operator of division by AA^+.

Lemma 16. *Suppose $0 < r' < r$, $E \in \mathscr{C}_{r,\rho,\mathbb{R}}$, $A \in \mathscr{C}_{r,\rho,\mathbb{R}}^{(\pm)}$ and $\|A - 1\|_{r,(\pm)} \leqslant 1/6$. Then there exists a unique $w \in \mathscr{C}_{r',\rho,\mathbb{R}}^{(\pm)}$, which we denote $\mathcal{F}(A, E)$, such that $\langle w\rangle = 0$ and*

$$A\Delta(Aw) - (Aw)\Delta A = AE - \langle AE\rangle. \tag{43}$$

Moreover, $\partial_\theta w \in \mathscr{C}_{r',\rho,\mathbb{R}}^{(\pm)}$ as well and there exists a positive constant C_2 which depends only on τ such that

$$\|w\|_{r',(\pm)} \leqslant \frac{C_2 M^4}{(r - r')^{2\sigma}}\|E\|_r, \quad \|\partial_\theta w\|_{r',(\pm)} \leqslant \frac{C_2 M^4}{(r - r')^{2\sigma+1}}\|E\|_r, \tag{44}$$

where $\sigma = 4 + 2\tau$.

Proof. Let $\delta = \|A - 1\|_{r,(\pm)}$, thus $0 \leqslant \delta \leqslant 1/6$, and $r'' = (r'+r)/2$, whence $0 < r' < r'' < r$.

1. We first observe that $\alpha := \frac{1}{AA^+} \in \mathscr{C}_{r,\rho,\mathbb{R}}^-$, with

$$\|\alpha - 1\|_{r,-} \leqslant \frac{1}{(1-\delta)^2} - 1 \leqslant \frac{11}{25} < \frac{1}{2},$$

because $(1/A) - 1$ can be written as the absolutely convergent series $\sum_{n \geqslant 1} (1-A)^n$ of $\mathscr{C}_{r,\rho,\mathbb{R}}^{(\pm)}$ with $\|(1/A) - 1\|_{r,(\pm)} \leqslant (\delta/(1-\delta))$ and $\|\alpha - 1\|_{r,-} \leqslant \|1/A\|_{r,-}\|(1/A^+) - 1\|_{r,-} + \|(1/A) - 1\|_{r,-}$.

As a consequence, if we denote simply by $\|\cdot\|$ the norm in $\mathscr{C}_\rho = \mathscr{C}_{\mathrm{hol}}^1(K_M, \mathbb{B}_\rho)$ or $\mathscr{C}_{\rho,\mathbb{R}}$, we get $\|1 - \langle\alpha\rangle\| \leqslant \frac{1}{2}$, hence $(1/\langle\alpha\rangle) = \sum_{n \geqslant 0}(1 - \langle\alpha\rangle)^n$ absolutely convergent in $\mathscr{C}_{\rho,\mathbb{R}}$ and

$$\frac{1}{\langle\alpha\rangle} \in \mathscr{C}_{\rho,\mathbb{R}}, \qquad \|\frac{1}{\langle\alpha\rangle}\| \leqslant 2.$$

2. Formula (42) of lemma 14 shows that

$$(43) \quad \Leftrightarrow \quad \nabla^-\left(\frac{1}{\alpha}\nabla w\right) = AE - \langle AE\rangle,$$

where $AE \in \mathscr{C}_{r,\rho,\mathbb{R}}$. Now, in view of proposition 12 and remark 3.1,

$$(43) \quad \Leftrightarrow \quad \exists \mu \in \mathscr{C}_\rho \text{ such that } \frac{1}{\alpha}\nabla w = \psi + \mu,$$

where $\psi = \Gamma^-(AE) \in \mathscr{C}_{r'',\rho,\mathbb{R}}^-$.

3. The equation $\nabla w = \alpha\psi + \mu\alpha$ leaves no choice but

$$\mu = \mu_0 := -\frac{1}{\langle\alpha\rangle}\langle\alpha\psi\rangle$$

(only possibility for having $\langle\alpha\psi + \mu\alpha\rangle = 0$). Notice that $\mu_0 \in \mathscr{C}_{\rho,\mathbb{R}}$. We end up with

$$\big(\langle w\rangle = 0\big) \ \& \ (43) \quad \Leftrightarrow \quad w = \Gamma\chi,$$

where $\chi = \alpha\psi + \mu_0\alpha$. We notice that $\chi \in \mathscr{C}_{r'',\rho,\mathbb{R}}^-$, thus, in view of remark 3.2, the unique solution to our problem satisfies

$$w \in \mathscr{C}_{r',\rho,\mathbb{R}}^{(\pm)}, \quad \|w\|_{r',(\pm)} \leqslant \frac{C_1 M^2}{(r'' - r')^\sigma}\|\chi\|_{r'',-}, \quad \|\partial_\theta w\|_{r',(\pm)} \leqslant \frac{C_1 M^2}{(r'' - r')^{\sigma+1}}\|\chi\|_{r'',-}.$$

4. We obtain the bounds (44) by observing that, on the one hand,

$$\|\psi\|_{r'',-} \leqslant \frac{7}{6}\frac{C_1 M^2}{(r - r'')^\sigma}\|E\|_{r,-}$$

and, on the other hand, $\|\mu_0\| \leqslant 2\|\alpha\psi\|_{r'',-} \leqslant 3\|\psi\|_{r'',-}$ and $\|\chi\|_{r'',-} \leqslant \frac{3}{2}\|\psi + \mu_0\|_{r'',-}$, thus $\|\chi\|_{r'',-} \leqslant 7(C_1 M^2/(r - r'')^\sigma)\|E\|_{r,-}$ and the result follows with $C_2 = 7C_1^2 \cdot 2^{2\sigma+1}$. $\qquad\square$

Putting lemma 15 and lemma 16 together, taking into account the fact that (41) implies (40) and working out the appropriate estimates, we can summarize the inductive step in

Proposition 17. *Suppose that* $0 < r' < r < R_0$, $r - r' \leqslant 1$, $\rho \leqslant 1$, *and that* $u \in \mathscr{C}_{r,\rho,\mathbb{R}}^{(\pm)}$ *satisfies* $\partial_\theta u \in \mathscr{C}_{r,\rho,\mathbb{R}}^{(\pm)}$ *and*

$$\|u\|_{r,(\pm)} \leqslant R_0 - r, \qquad \|\partial_\theta u\|_{r,(\pm)} \leqslant 1/6. \tag{45}$$

Let $\sigma = 4 + 2\tau$. *Then*

$$A := 1 + \partial_\theta u \in \mathscr{C}_{r,\rho,\mathbb{R}}^{(\pm)} \qquad \text{and} \qquad w := \mathcal{F}\big(A, \mathcal{E}(u)\big) \in \mathscr{C}_{r',\rho,\mathbb{R}}^{(\pm)}$$

are well-defined and:

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1. *The function $\partial_\theta w$ belongs to $\mathscr{C}^{(\pm)}_{r',\rho,\mathbb{R}}$ and*

$$\|Aw\|_{r',(\pm)} \leqslant 2(r-r')\xi, \qquad \|\partial_\theta(Aw)\|_{r',(\pm)} \leqslant 4\xi, \qquad \text{where}$$
$$\xi := (C_2 M^4/(r-r')^{2\sigma+1})\|\mathcal{E}(u)\|_r.$$

2. *If $\|u\|_{r,(\pm)} + 2(r-r')\xi \leqslant \frac{1}{2}(R_0 - r')$ then $\mathcal{E}(u+Aw) \in \mathscr{C}_{r',\rho,\mathbb{R}}$ and*

$$\|\mathcal{E}(u+Aw)\|_{r'} \leqslant \frac{C_3 M^8}{(r-r')^{4\sigma}}\big(\|\mathcal{E}(u)\|_r\big)^2, \tag{46}$$

where C_3 is a positive constant which depends only on τ and $\|f''\|_{R_0}$.

Proof. By (38), we have a well-defined $\mathcal{E}(u) \in \mathscr{C}_{r,\rho,\mathbb{R}}$ and then, by lemma 16, a well-defined $w \in \mathscr{C}^{(\pm)}_{r',\rho,\mathbb{R}}$ which satisfies

$$\|w\|_{r',(\pm)} \leqslant (r-r')\xi, \qquad \|\partial_\theta w\|_{r',(\pm)} \leqslant \xi.$$

Since $\|A\|_{r,(\pm)} \leqslant 7/6 < 2$, we get $\|Aw\|_{r',(\pm)} \leqslant 2(r-r')\xi$ and the Cauchy inequalities yield $\|\partial_\theta A\|_{r',(\pm)} \leqslant (1/r-r')\|A\|_{r,(\pm)}$, whence

$$\|\partial_\theta(Aw)\|_{r',(\pm)} \leqslant \|A\|_{r,(\pm)}\left(\frac{1}{r-r'}\|w\|_{r',(\pm)} + \|\partial_\theta w\|_{r',(\pm)}\right) \leqslant 4\xi.$$

Supposing now that $\|u\|_{r,(\pm)} + 2(r-r')\xi \leqslant \frac{1}{2}(R_0 - r')$, we have a well-defined $\mathcal{E}(u+Aw) \in \mathscr{C}_{r',\rho,\mathbb{R}}$ still by (38), but the fact that $\langle A\mathcal{E}(u)\rangle = 0$ (lemma 15) implies that $h = Aw$ solves (41), we can thus take advantage of (40), which takes the form

$$\mathcal{E}(u+Aw) = w\partial_\theta\big(\mathcal{E}(u)\big) + (Aw)^2 \int_0^1 \varepsilon f'' \circ (\mathrm{id} + u + tAw)(1-t)\,\mathrm{d}t. \tag{47}$$

Since $\|u+tAw\|_{r'} \leqslant \frac{1}{2}(R_0 - r')$, by lemma 11, we have a continuous curve $t \in [0,1] \mapsto \varepsilon f'' \circ (\mathrm{id} + u + tAw) \in \mathscr{C}_{r',\rho,\mathbb{R}}$, bounded in norm by $2\rho\|f''\|_{R_0}$ and the second term in the right-hand side of (47) is bounded in norm by $8(r-r')^2\xi^2\rho\|f''\|_{R_0}$. For the first term, we use the cauchy inequalities: $\|w\partial_\theta\big(\mathcal{E}(u)\big)\|_{r'} \leqslant (r-r')\xi \cdot (1/(r-r'))\|\mathcal{E}(u)\|_r$. Using $\rho \leqslant 1$, $M \geqslant 1$ and $r-r' \leqslant 1$, we get the desired bound with $C_3 := C_2 + 8C_2^2\|f''\|_{R_0}$. $\qquad\square$

To simplify further the estimates, we can take into account the fact that $2\sigma + 1 < 4\sigma$ and $M > 1$ (because $\tau > 0$ and $M > 2\zeta(1+\tau)$), thus $(M^4/(r-r')^{2\sigma+1}) < (M^8/(r-r')^{4\sigma})$ and, setting $C := \max(4C_2, C_3)$, content ourselves with

Corollary 18. *Suppose $0 < r' < r < R_0$, $r-r' \leqslant 1$, $\rho \leqslant 1$ and $\kappa \leqslant \min(R_0 - r, \frac{1}{3})$. Then*

$$\|u\|_{r,(\pm)}, \|\partial_\theta u\|_{r,(\pm)} \leqslant \delta \leqslant \frac{\kappa}{2} \quad \Rightarrow \quad w := \mathcal{F}\big(A, \mathcal{E}(u)\big) \in \mathscr{C}^{(\pm)}_{r',\rho,\mathbb{R}} \text{ and}$$

$$\|Aw\|_{r',(\pm)}, \|\partial_\theta(Aw)\|_{r',(\pm)} \leqslant \frac{CM^8}{(r-r')^{4\sigma}}\|\mathcal{E}(u)\|_r \tag{48}$$

(still with $A := 1 + \partial_\theta u$) and, if the right-hand side in (48) is $\leqslant (\kappa/2) - \delta$, then

$$\|\mathcal{E}(u+Aw)\|_{r'} \leqslant \frac{CM^8}{(r-r')^{4\sigma}}\big(\|\mathcal{E}(u)\|_r\big)^2. \tag{49}$$

4.3. The iterative scheme

Following [LM01], as alluded to at the end of section 4.1, we now prove theorem 1 by means of the modified Newton method on a scale of Banach spaces corresponding to strips S_{R_n} which go on shrinking around a limiting strip S_{R_∞}.

We suppose that we are given $0 < R < R_0$ and $f \in H^\infty(S_{R_0})$ real analytic with $f'' \in H^\infty(S_{R_0})$. Without loss of generality, we also suppose $R_0 - R \leqslant \frac{4}{3}$. We set

$$R_\infty := \frac{R + R_0}{2}, \quad \kappa := \frac{R_0 - R_\infty}{2} = \frac{R_\infty - R}{2} \leqslant \frac{1}{3}, \quad R_n := R_\infty + 2^{-n-1}\kappa \quad \text{for } n \geqslant 1.$$

We observe that $R_0 > R_n > R_{n+1}$,

$$R_n - R_{n+1} = 2^{-n}\kappa < 1$$

and $R_0 - R_n \geqslant R_0 - R_1 = \kappa$ for $n \geqslant 1$. Corollary 18 with $r = R_n$ and $r' = R_{n+1}$ motivates

Lemma 19. *Suppose* $0 < \varepsilon_1 \leqslant (\kappa^{4\sigma}/2^{8\sigma+1}CM^8)$. *Then the induction* $\varepsilon_{n+1} := 2^{4\sigma n}\kappa^{-4\sigma}CM^8(\varepsilon_n)^2$ *determines a sequence of positive numbers which satisfies* $\sum_{n \geqslant 1} 2^{4\sigma n}\kappa^{-4\sigma}CM^8\varepsilon_n \leqslant (\kappa/2)$.

Proof. Let $\bar{\varepsilon}_n := 2^{4\sigma(n+1)}\kappa^{-4\sigma}CM^8\varepsilon_n$ for $n \geqslant 1$, so that $\bar{\varepsilon}_{n+1} = (\bar{\varepsilon}_n)^2$ and $\bar{\varepsilon}_1 = 2^{8\sigma}\kappa^{-4\sigma}CM^8\varepsilon_1 \leqslant \frac{1}{2}$. We have $\bar{\varepsilon}_n = (\bar{\varepsilon}_1)^{2^{n-1}} \leqslant (\bar{\varepsilon}_1)^n \leqslant 2^{-(n-1)}\bar{\varepsilon}_1$ and the result follows from $\sum_{n \geqslant 1} \bar{\varepsilon}_n \leqslant 2\bar{\varepsilon}_1 < 2^{4\sigma}\frac{\kappa}{4}$. $\qquad\square$

Let $0 < \rho \leqslant 1$. We set $u_1 := 0 \in \mathscr{C}_{R_1,\rho,\mathbb{R}}^{(\pm)}$ and $\delta_1 := 0$, so that $\mathcal{E}(u_1) = \varepsilon f \in \mathscr{C}_{R_1,\rho,\mathbb{R}}$ with $\|\mathcal{E}(u_1)\|_{R_1} = \rho\|f\|_{R_1}$. From now on, we suppose that

$$\varepsilon_1 := \rho\|f\|_{R_1} \leqslant \frac{\kappa^{4\sigma}}{2^{8\sigma+1}CM^8} \tag{50}$$

and we define inductively

$$\delta_{n+1} := \delta_n + 2^{4\sigma n}\kappa^{-4\sigma}CM^8\varepsilon_n, \qquad \varepsilon_{n+1} := 2^{4\sigma n}\kappa^{-4\sigma}CM^8(\varepsilon_n)^2, \tag{51}$$

so that, by lemma 19, $\delta_n \leqslant (\kappa/2)$ for all $n \geqslant 1$. This is sufficient to apply inductively corollary 18: the formula

$$u_{n+1} := u_n + (1 + \partial_\theta u_n)\mathcal{F}\big(1 + \partial_\theta u_n, \mathcal{E}(u_n)\big), \qquad n \geqslant 1 \tag{52}$$

yields a sequence of functions $u_n \in \mathscr{C}_{R_n,\rho,\mathbb{R}}^{(\pm)}$ satisfying

$$\|u_n\|_{R_n,(\pm)}, \|\partial_\theta u_n\|_{R_n,(\pm)} \leqslant \delta_n, \qquad \|\mathcal{E}(u_n)\|_{R_n} \leqslant \varepsilon_n.$$

Moreover, $\|u_{n+1} - u_n\|_{R_\infty,(\pm)} \leqslant \|u_{n+1} - u_n\|_{R_{n+1},(\pm)} \leqslant 2^{4\sigma n}\kappa^{-4\sigma}CM^8\varepsilon_n$, hence there is a limit

$$u_\infty := \lim_{n \to \infty} u_{n|K_M \times S_{R_\infty} \times \mathbb{D}_\rho} \in \mathscr{C}_{R_\infty,\rho,\mathbb{R}}^{(\pm)}$$

with $\|u_\infty\|_{R_\infty,(\pm)} \leqslant (\kappa/2) < \min(R_0 - R_\infty, R_\infty - R)$.

This limit u_∞ is indeed a solution of the equation $\mathcal{E}(u) = 0$ because $\varepsilon_n \xrightarrow[n \to \infty]{} 0$ and $\mathcal{E}$ is Lipschitz when viewed as a map from $\{u \in \mathscr{C}_{R_\infty,\rho,\mathbb{R}}^{(\pm)} \mid \|u\|_{R_\infty,(\pm)} \leqslant R_0 - R_\infty\}$ to $\mathscr{C}_{R_\infty,\rho,\mathbb{R}}$ (the Lipschitz constant is $\leqslant 4 + \rho\|f'\|_{R_0}$).

5. The unit circle as a natural boundary

This section is devoted to the proof of theorem 2. It will be a consequence of

Proposition 20. *Let* $\omega_* = p/m$ *with* $p \in \mathbb{Z}$ *and* $m \in \mathbb{N}^*$. *Suppose that* f *is a non-zero trigonometric polynomial with zero mean value, or that* f *is an analytic function on* $\mathbb{T}$ *with zero mean value whose Fourier expansion contains at least one non-zero coefficient with index* $k \in m\mathbb{Z}$. *Then the difference equation*

$$u(\theta + \omega_*, \varepsilon) - 2u(\theta, \varepsilon) + u(\theta - \omega_*, \varepsilon) = \varepsilon f\left(\theta + u(\theta, \varepsilon)\right) \tag{53}$$

has no solution formal in ε *and analytic* θ, *i.e. no formal solution* $u = \sum_{n \geqslant 1} \varepsilon^n u_n(\theta)$ *with all coefficients* u_n *analytic on* $\mathbb{T}$.

Proposition 20 implies theorem 2. Suppose that the assumptions of theorem 2 hold and that the restriction of $\tilde{u}_M$ to $\overset{\circ}{K}{}_M^{(i)}$ (or its restriction to $\overset{\circ}{K}{}_M^{(e)}$) has an analytic continuation in an open connected set U which intersects the unit circle $\mathbb{S}$ and $\overset{\circ}{K}{}_M^{(i)}$ (or $\overset{\circ}{K}{}_M^{(e)}$); denote this analytic continuation $u \in \mathscr{O}(U, \mathbb{B}_{R,\rho})$. We shall reach a contradiction.

Let us choose an open disc $D \subset \mathbb{C}$ such that $E(D) \subset U$ and $D \cap \mathbb{R} \neq \emptyset$. Let us take $M_1 > M$ large enough so that $D \cap A_{M_1}^{\mathbb{R}} \neq \emptyset$ (this is possible since $D \cap \mathbb{R}$ is an open interval and the Lebesgue measure of $(D \cap \mathbb{R}) - A_{M_1}^{\mathbb{R}}$ tends to 0 as $M_1 \to \infty$). We observe that, perhaps at the price of diminishing ρ, we can consider u as an analytic continuation of the restriction of $\tilde{u}_{M_1}$ to $\overset{\circ}{K}{}_M^{(i)}$ (or $\overset{\circ}{K}{}_M^{(e)}$) as well. Therefore, for each $\omega \in D \cap A_{M_1}^{\mathbb{R}}$ and $\varepsilon \in (-\rho, \rho)$, the function $\theta \mapsto u(e^{2\pi i \omega})(\theta, \varepsilon)$ is a solution of the difference equation (5); moreover $u(e^{2\pi i \omega})(\theta, 0) \equiv 0$ (because, for ω irrational, (5) implies $u(e^{2\pi i \omega})_{|\varepsilon=0}$ constant and we have normalized $\tilde{u}_{M_1}$ by imposing zero mean value). Since $D \cap A_{M_1}^{\mathbb{R}}$ is not discrete, by analytic continuation with respect to ω, we also have $\theta \in \mathbb{T} \mapsto u(e^{2\pi i \omega})(\theta, \varepsilon)$ analytic solution of (5) for each $\omega \in D \cap \mathbb{R}$ and $\varepsilon \in (-\rho, \rho)$, with $u(e^{2\pi i \omega})(\theta, 0) \equiv 0$.

Choose $m_* \in \mathbb{N}^*$ larger that $1/|D \cap \mathbb{R}|$: for every $m \geqslant m_*$ there is a $p_m \in \mathbb{Z}$ such that $(p_m/m) \in D \cap \mathbb{R}$. If f is not a trigonometric polynomial, we then take $m \geqslant m_*$ such that the mth Fourier coefficient of f is non-zero; if f is a trigonometric polynomial, we take any $m \geqslant m_*$. In both cases, equation (5) with $\omega = (p_m/m)$ shows that the Taylor expansion of $u(e^{2\pi i p_m/m}) \in \mathbb{B}_{R,\rho}$ with respect to ε is a formal solution of (53) with coefficients analytic on $\mathbb{T}$ and proposition 20 yields a contradiction.　　□

Proof of proposition 20. Let ω_* and f be as in the assumptions of proposition 20. Let $\mathscr{C}$ denote the differential $\mathbb{C}$-algebra $\mathscr{C}^\omega(\mathbb{T})$ of all analytic functions of $\mathbb{T}$. Equation (53) can be written

$$\Delta_* u = \varepsilon f \circ (\mathrm{id} + u), \tag{54}$$

where the right-hand side is defined for any $u \in \varepsilon \mathscr{C}[[\varepsilon]]$, i.e. any formal series in ε with coefficients analytic on $\mathbb{T}$ and without 0th order term, by the Taylor formula

$$f \circ (\mathrm{id} + u) = \sum_{r \geqslant 0} \frac{1}{r!} f^{(r)} u^r \in \mathscr{C}[[\varepsilon]] \tag{55}$$

(formally convergent for the complete metric space structure induced by the ε−adic valuation in $\mathscr{C}[[\varepsilon]]$), while the left-hand side involves the $\mathbb{C}[[\varepsilon]]$-linear operator

$$\Delta_* : \mathscr{C}[[\varepsilon]] \to \mathscr{C}[[\varepsilon]]$$

which extends the second-order difference operator

$$\Delta_* : \mathscr{C} \to \mathscr{C}, \qquad (\Delta_* \varphi)(\theta) := \varphi(\theta + \omega_*) - 2\varphi(\theta) + \varphi(\theta - \omega_*).$$

To prove proposition 20, we argue by contradiction and suppose that we are given a formal solution $u = \sum_{n \geqslant 1} \varepsilon^n u_n \in \varepsilon \mathscr{C}[[\varepsilon]]$ of equation (54).

Let us view $\mathscr{C}$ as a linear subspace of the Hilbert space $L^2(\mathbb{T})$. The Fourier transform associates with any φ its coordinates $(\hat{\varphi}_k)_{k \in \mathbb{Z}}$, $\hat{\varphi}_k = \mathcal{F}_k(\varphi)$, on the Hilbert basis $(e_k)_{k \in \mathbb{Z}}$ (notation of section 2.3). We have an orthogonal decomposition

$$\mathscr{C} = V_0 \oplus V_1 \oplus \cdots \oplus V_{m-1},$$

where $V_j := \{ \varphi \in \mathscr{C} \mid \hat{\varphi}_k = 0 \text{ for } k \notin j + m\mathbb{Z} \}$; let us denote by $\Pi_0, \ldots, \Pi_{m-1}$ the corresponding orthogonal projectors. For instance, the Fourier expansions of f and $\Pi_0 f$ read

$$f = \sum_{k \in \mathbb{Z}^*} \hat{f}_k e_k, \qquad \Pi_0 f = \sum_{k \in m\mathbb{Z}^*} \hat{f}_k e_k. \tag{56}$$

The spectral decomposition of Δ_* is obtained from the relations $\Delta_* e_k = D_k e_k$, $k \in \mathbb{Z}$, with

$$D_k = e^{2\pi i k \omega_*} - 2 + e^{-2\pi i k \omega_*} = -4 \sin^2(k\pi\omega_*). \tag{57}$$

Since D_k depends only on $k + m\mathbb{Z}$, we get

$$\Delta_* = D_0 \Pi_0 + D_1 \Pi_1 + \cdots + D_{m-1} \Pi_{m-1}. \tag{58}$$

Observe that $D_0 = 0$ and $D_1, \ldots, D_{m-1} < 0$. Therefore

$$\ker(\Delta_*) = V_0, \qquad \text{range}(\Delta_*) = V_0^\perp = V_1 \oplus \cdots \oplus V_{m-1}. \tag{59}$$

This is enough to conclude the proof when f is not a trigonometric polynomial: indeed, our assumption on the Fourier coefficients of f in this case together with (56) imply that $\Pi_0 f$ is not identically zero, hence $f \notin \text{range}(\Delta_*)$, whereas expanding equation (54) in powers of ε yields $\Delta_* u_1 = f$, a contradiction.

From now on, we thus suppose that f is a trigonometric polynomial of the form

$$f = \sum_{|k| \leqslant K} \hat{f}_k e_k, \qquad \hat{f}_K = A \neq 0, \tag{60}$$

with a certain $K \in \mathbb{N}^*$ (the case $\hat{f}_{-K} \neq 0$ is reduced to (60) by changing θ into $-\theta$). It is sufficient to reach a contradiction in this case.

Let us first check that one can suppose that all the coefficients u_n of u belong to $V_0^\perp$.

Lemma 21. *For any solution $u \in \varepsilon \mathscr{C}[[\varepsilon]]$ of (54), the formal series* $\text{id} + \Pi_0 u$ *has a composition inverse of the form* $\text{id} + a$ *with $a \in \varepsilon V_0[[\varepsilon]]$ and the formula*

$$u_* = (u - \Pi_0 u) \circ (\text{id} + a)$$

defines a formal solution of equation (54) which has all its coefficients in $V_0^\perp$.

Proof of lemma 21. Of course, composition is to be understood 'with respect to θ at fixed ε', as in (55), i.e. we are inverting $(\varepsilon, \theta) \mapsto \big(\varepsilon, \theta + (\Pi_0 u)(\theta, \varepsilon)\big)$. The 0th order coefficient of $\Pi_0 u$ being zero, $\text{id} + \Pi_0 u$ has a composition inverse which is given by the (formally convergent) Lagrange inversion formula

$$(\text{id} + \Pi_0 u)^{-1} = \text{id} + a, \qquad a = \sum_{s \geqslant 1} \frac{(-1)^s}{s!} \Big(\frac{\mathrm{d}}{\mathrm{d}\theta} \Big)^{s-1} \big[(\Pi_0 u)^s \big] \in \varepsilon \mathscr{C}[[\varepsilon]].$$

Since V_0 consists of all $\frac{1}{m}$-periodic analytic functions of θ, it is a differential subalgebra of $\mathscr{C}$ and each subspace V_j is a V_0-module; since $\Pi_0 u \in \varepsilon V_0[[\varepsilon]]$, we deduce that $a \in \varepsilon V_0[[\varepsilon]]$,

that the composition operator $\varphi \mapsto \varphi \circ (\mathrm{id} + a)$ leaves $V_j[[\varepsilon]] \subset \mathscr{C}[[\varepsilon]]$ invariant for each j (beware that V_j is not a ring for $j \neq 0$, thus $V_j[[\varepsilon]]$ is just a linear subspace of $\mathscr{C}[[\varepsilon]]$, in fact a $V_0[[\varepsilon]]$-submodule) and that

$$\Delta_* \big(\varphi \circ (\mathrm{id} + a) \big) = (\Delta_* \varphi) \circ (\mathrm{id} + a). \tag{61}$$

The composition by $\mathrm{id} + a$ leaves also $V_0^\perp[[\varepsilon]]$ invariant, hence $u_* = (u - \Pi_0 u) \circ (\mathrm{id} + a)$ is a well-defined formal series with all its coefficients in $V_0^\perp$ and without 0th order term; the fact that it is a solution of (54) stems from (61), the associativity of composition and the relations $\Delta_*(u - \Pi_0 u) = \Delta_* u = \varepsilon f \circ (\mathrm{id} + u)$ and $(\mathrm{id} + u) \circ (\mathrm{id} + a) = (\mathrm{id} + \Pi_0 u + u - \Pi_0 u) \circ (\mathrm{id} + a) = \mathrm{id} + u_*$. $\qquad\square$

Lemma 22. *The formula*

$$E = \lambda_1 \Pi_1 + \cdots + \lambda_{m-1} \Pi_{m-1},$$

with $\lambda_j := -(1/4 \sin^2(j\pi\omega_*)) < 0$ *for* $j = 1, \ldots, m-1$*, defines an operator* $E \colon \mathscr{C} \to \mathscr{C}$ *such that, for any* $\varphi, \psi \in \mathscr{C}$,

$$\Delta_* \varphi = \psi \qquad \text{and} \qquad \varphi \in V_0^\perp \quad \Leftrightarrow \quad \psi \in V_0^\perp \qquad \text{and} \qquad \varphi = E\psi.$$

Proof of lemma 22. Immediate consequence of (57)–(58). $\qquad\square$

End of the proof of proposition 20. We assume that f is of the form (60) and we have a formal solution u of equation (54). By lemma 21, at the price of replacing u with u_*, we can suppose that all the coefficients u_n of u belong to $V_0^\perp$. Let

$$g = \varepsilon f \circ (\mathrm{id} + u) = \sum_{n \geqslant 1} \varepsilon^n g_n, \qquad g_n \in \mathscr{C}. \tag{62}$$

Lemma 22 allows us to rewrite (54) as

$$g_n \in V_0^\perp, \quad u_n = E g_n, \qquad n \geqslant 1. \tag{63}$$

A simple computation yields

$$g_n = \sum_{|k| \leqslant nK} \mathcal{F}_k(g_n) e_k, \quad u_n = \sum_{|k| \leqslant nK} \mathcal{F}_k(u_n) e_k, \qquad n \geqslant 1,$$

with $\gamma_n := \mathcal{F}_{nK}(g_n)$ and $\alpha_n := \mathcal{F}_{nK}(u_n)$ inductively determined by $\gamma_1 = A$ and

$$\alpha_n = \lambda_{[nK]} \gamma_n, \qquad\qquad\qquad\qquad n \geqslant 1,$$

$$\gamma_n = \sum_{r=1}^{n-1} \frac{(2\pi i K)^r A}{r!} \sum_{\substack{n_1, \ldots, n_r \geqslant 1 \\ n_1 + \cdots + n_r = n-1}} \alpha_{n_1} \cdots \alpha_{n_r}, \, n \geqslant 2,$$

with the notation $\lambda_{[k]} := 0$ if $k \in m\mathbb{Z}$ and $\lambda_{[k]} := \lambda_j$ with $j \in \{1, \ldots, m-1\}$ such that $k \in j + m\mathbb{Z}$ if $k \notin m\mathbb{Z}$.

Defining inductively $\beta_1 := 1$ and

$$\beta_n = \sum_{r=1}^{n-1} \frac{1}{r!} \sum_{\substack{n_1, \ldots, n_r \geqslant 1 \\ n_1 + \cdots + n_r = n-1}} (-\lambda_{[n_1 K]}) \beta_{n_1} \cdots (-\lambda_{[n_r K]}) \beta_{n_r}, \qquad n \geqslant 2,$$

we get

$$\gamma_n = (-2\pi i K)^{n-1} A^n \beta_n, \qquad n \geqslant 1.$$

Let n_* denote the smallest integer in $\{1, \ldots, m\}$ such that $n_* K \in m\mathbb{Z}$. We see that $\beta_1, \ldots, \beta_{n_*} > 0$. Since $A \neq 0$, we reach a contradiction when comparing the requirement $g_{n_*} \in V_0^\perp$ (according to (63)) and the relation $\mathcal{F}_{n_* K}(\Pi_0 g_{n_*}) = \mathcal{F}_{n_* K}(g_{n_*}) = \gamma_{n_*} \neq 0$ which implies $\Pi_0 g_{n_*} \neq 0$. $\qquad\square$

Remark 5.1. When f is assumed to be a real analytic function of zero mean value which does not belong to V_0 (i.e. there exists $k \in m\mathbb{Z}$ such that the kth Fourier coefficient of f is non-zero), one can prove by a more geometric method that there exists $\rho_* = \rho_*(m, f) > 0$ such that, for any non-zero $\varepsilon \in (-\rho_*, \rho_*)$, the difference equation (53) has no solution u real analytic on $\mathbb{T}$ (observing that, associated with such a solution, there would be a curve $\gamma(\theta) = \big(\theta + u(\theta), \omega_* + u(\theta) - u(\theta - \omega_*)\big)$ consisting of fixed points of T_ε^m, and showing that these fixed points are isolated).

Acknowledgments

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