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Hodge theory for twisted differentials

Abstract: We study cohomologies and Hodge theory for complex manifolds with twisted differentials. In particular, we get another cohomological obstruction for manifolds in class $\mathcal{C}$ of Fujiki. We give a Hodge-theoretical proof of the characterization of solvmanifolds in class $\mathcal{C}$ of Fujiki, first stated by D. Arapura.

Keywords: twisted differential; local system; Dolbeault cohomology; Bott-Chern cohomology; Hodge decomposition; solvmanifolds; class $\mathcal{C}$ of Fujiki

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Introduction

Let (M, J) be a complex manifold. We have the $\mathbb{K}$-valued de Rham complex $(A^\bullet(M)_\mathbb{K}, \mathrm{d})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Once fixed a d-closed 1-form $\phi \in A^1(M)_\mathbb{K}$, we can consider another complex. Namely, define

$$\mathrm{d}_\phi \;:=\; \mathrm{d} + L_\phi \;,$$

where $L_\phi := \phi \wedge \cdot$. The cochain complex

$$(A^\bullet(M)_\mathbb{K}, \mathrm{d}_\phi)$$

can be considered as the de Rham complex with values in the topologically trivial flat bundle $M \times \mathbb{K}$ with the connection form ϕ. Hence it is determined by the character $\rho_\phi \colon \pi_1(M) \to \mathrm{GL}_1(\mathbb{K})$ given by $\rho_\phi(\gamma) = \exp\left(\int_\gamma \phi\right)$. In particular, it is determined by the cohomology class $[\phi] \in H^1(M; \mathbb{K})$.

This de Rham complex with a twisted differential appears naturally in locally conformally Kähler geometry, see, e.g., [28].

On compact Kähler manifolds, the Hodge theory for local systems was developed by the theory of Higgs bundles, see [31]. In this note, we study cohomologies and Hodge theory for general complex manifolds (M, J) with twisted differentials. More precisely, denote by $H^{\bullet,\bullet}_{BC}(M) := \frac{\ker\partial \cap \ker\overline{\partial}}{\mathrm{im}\,\partial\overline{\partial}}$ the Bott-Chern cohomology of (M, J); in particular, $H^{1,0}_{BC}(M) = \{\theta \in A^{1,0}(M) | \partial\theta = \overline{\partial}\theta = 0\}$. For $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$, consider the bi-differential $\mathbb{Z}$-graded complex

$$\left(A^\bullet(M)_\mathbb{C},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) \;,$$

where

$$\partial_{(\theta_1,\theta_2)} \;:=\; \partial + L_{\theta_2} + L_{\overline{\theta_1}} \qquad \text{and} \qquad \overline{\partial}_{(\theta_1,\theta_2)} \;:=\; \overline{\partial} - L_{\overline{\theta_2}} + L_{\theta_1} \;.$$

The aim of this note is to investigate cohomological properties of this bi-differential complex.

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Several cohomologies can be defined, and the identity induces natural maps between them:

$$\begin{array}{ccccc} & & H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) & & \\ & \swarrow^{\iota_{BC,\partial}} & \downarrow^{\iota_{BC,dR}} & \searrow^{\iota_{BC,\overline{\partial}}} & \\ H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)}\right) & \iota_{BC,A} & H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \mathrm{d}_\phi\right) & & H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \overline{\partial}_{(\theta_1,\theta_2)}\right) \\ & \searrow_{\iota_{\partial,A}} & \downarrow^{\iota_{dR,A}} & \swarrow_{\iota_{\overline{\partial},A}} & \\ & & H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)}\right) & & \end{array}$$

Here, $H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \mathrm{d}_\phi\right)$, $H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)}\right)$, and $H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \overline{\partial}_{(\theta_1,\theta_2)}\right)$ denote the cohomology of the corresponding complex, and, in the notation of [16],

$$H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) := \frac{\ker \partial_{(\theta_1,\theta_2)} \cap \ker \overline{\partial}_{(\theta_1,\theta_2)}}{\operatorname{im} \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}},$$

$$H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)}\right) := \frac{\ker \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}}{\operatorname{im} \partial_{(\theta_1,\theta_2)} + \operatorname{im} \overline{\partial}_{(\theta_1,\theta_2)}},$$

are the counterpart of Bott-Chern [13] and Aeppli [1] cohomologies.

The above maps are in general neither injective nor surjective: so they do not allow a direct comparison of the cohomologies. Hence we are especially interested in studying the following properties:

- (M, J) is said to satisfy the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-*Lemma* if the natural map $\iota_{BC,A}$ is injective;
- (M, J) admits the (θ_1, θ_2)-*Hodge decomposition* if the natural maps $\iota_{BC,\partial}$ and $\iota_{BC,dR}$ and $\iota_{BC,\overline{\partial}}$ are isomorphisms.

Admitting the (θ_1, θ_2)-Hodge decomposition is a stronger property than satisfying the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma. For $(\theta_1, \theta_2) = (0, 0)$, the above properties are in fact equivalent. See [16] for a proof. If (M, J) admits Kähler metrics, then it satisfies the two properties for any $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$.

Corollary 2.3. *Let (M, J) be a compact complex manifold endowed with a Kähler metric. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$. Then (M, J) satisfies the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma and admits the (θ_1, θ_2)-Hodge decomposition.*

In Example 3.2, we study an explicit example on the completely-solvable Nakamura manifold $X = \Gamma\backslash G$. It is known that $\Gamma\backslash G$ satisfies the $\partial\overline{\partial}$-Lemma, see [4, Example 2.17]. For $\theta_1 := \frac{\mathrm{d}z_1}{2} \in H^{1,0}_{BC}(X)$ and $\theta_2 := 0$, (see page 83 for notation,) it follows that X does not admit the (θ_1, θ_2)-Hodge decomposition, see [23, §8]. Furthermore, we show that it does not satisfy the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma.

In particular, we are interested in studying the behaviour of $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma and (θ_1, θ_2)-Hodge decomposition under modifications. We recall that a *modification* $\mu\colon (\tilde{M}, \tilde{J}) \to (M, J)$ is a holomorphic map between compact complex manifolds of the same dimension that yields a biholomorphism $\mu\lfloor_{\tilde{M}\backslash\mu^{-1}(S)}\colon \tilde{M}\backslash\mu^{-1}(S) \to M\backslash S$ outside the preimage of an analytic subset $S \subset M$ of codimension greater than or equal to 1. We prove the following results.

Theorem 2.5 and Theorem 2.6. *Let $\mu\colon (\tilde{M}, \tilde{J}) \to (M, J)$ be a proper modification of a compact complex manifold (M, J). Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$.*

- *If $(\tilde{M}, \tilde{J})$ satisfies the $\partial_{(\mu^*\theta_1,\mu^*\theta_2)}\overline{\partial}_{(\mu^*\theta_1,\mu^*\theta_2)}$-Lemma, then (M, J) satisfies the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma.*
- *If $(\tilde{M}, \tilde{J})$ satisfies the $(\mu^*\theta_1, \mu^*\theta_2)$-Hodge decomposition, then (M, J) satisfies the (θ_1, θ_2)-Hodge decomposition.*

From this, we get another cohomological obstruction for complex manifolds belonging to class $\mathcal{C}$ of Fujiki. We recall that a compact complex manifold (M, J) is said to be in *class $\mathcal{C}$ of Fujiki* [18] if it admits a proper modification $\mu\colon \left(\tilde{M}, \tilde{J}\right) \to (M, J)$ with $\left(\tilde{M}, \tilde{J}\right)$ admitting Kähler metrics. (Note that, by Hironaka's Chow Lemma, every modification is just dominated by a locally-finite sequence of blowups.)

Corollary 2.7. *Let (M, J) be a compact complex manifold in class $\mathcal{C}$ of Fujiki. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$. Then (M, J) satisfies the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma and admits the (θ_1, θ_2)-Hodge decomposition.*

The previous results can be adapted to compact complex orbifolds of global-quotient type, namely, quotients of compact complex manifolds by finite groups of biholomorphisms, see Theorem 2.8 and Corollary 2.9.

The second author studied in [23] the property of satisfying the (θ_1, θ_2)-Hodge decomposition for any $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$. In [23, Theorem 1.7], he proved that a solvmanifold admitting hyper-strong-Hodge-decomposition admits a Kähler metric. Therefore, we get a more direct proof of the characterization of solvmanifolds in class $\mathcal{C}$ of Fujiki.

Theorem 3.3 (see also [7, Theorem 9], [10, Theorem 1.1]). *Let (M, J) be a solvmanifold endowed with a complex structure. If (M, J) belongs to class $\mathcal{C}$ of Fujiki, then it admits a Kähler metric.*

This result was firstly stated by D. Arapura [7]. His sketched proof uses the fact that "the classes of fundamental groups of compact [manifolds in class $\mathcal{C}$ of Fujiki] and compact Kähler manifolds coincide". This is a consequence of the Hironaka elimination of indeterminacies (see [10, Lemma 2.1]). But our proof does not rely on the Hironaka elimination of indeterminacies.

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1 Twisted differentials and cohomologies on complex manifolds

1.1 Twisted differentials

Let (M, J) be a complex manifold of complex dimension n. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, denote by $A^\bullet(M)_\mathbb{K}$ the space of $\mathbb{K}$-valued differential forms on M. Consider the cochain complex $\left(A^\bullet(M)_\mathbb{K}, \mathrm{d}\right)$.

For $\phi \in A^r(M)_\mathbb{K}$, we define the operator

$$L_\phi\colon A^\bullet(M)_\mathbb{K} \to A^{\bullet+r}(M)_\mathbb{K} , \qquad L_\phi(x) \;:=\; \phi \wedge x .$$

If ϕ is a d-closed 1-form, then the operator

$$\mathrm{d}_\phi \;:=\; \mathrm{d} + L_\phi$$

satisfies $\mathrm{d}_\phi \circ \mathrm{d}_\phi = 0$. Hence we have the cochain complex

$$\left(A^\bullet(M)_\mathbb{K}, \mathrm{d}_\phi\right) .$$

Note that d_ϕ satisfies the following Leibniz rule:

$$\text{for any } \alpha \in A^\bullet(M)_\mathbb{K} , \quad \left[\mathrm{d}_\phi, L_\alpha\right] \;=\; L_{\mathrm{d}\,\alpha} .$$

The cochain complex $(A^\bullet(M)_\mathbb{K}, \mathrm{d}_\phi)$ is considered as the de Rham complex with values in the topologically trivial flat bundle $M \times \mathbb{K}$ with the connection form ϕ. Hence the structure of the cochain complex $(A^\bullet(M)_\mathbb{K}, \mathrm{d}_\phi)$ is determined by the character $\rho_\phi\colon \pi_1(M) \to \mathrm{GL}_1(\mathbb{K})$ given by $\rho_\phi(\gamma) = \exp\left(\int_\gamma \phi\right)$. In particular, the cochain complex $(A^\bullet(M)_\mathbb{K}, \mathrm{d}_\phi)$ is determined by the cohomology class $[\phi] \in H^1(M; \mathbb{K})$.

We consider the bi-grading $A^\bullet(M)_\mathbb{C} = A^{\bullet,\bullet}(M)$ and the decomposition $\mathrm{d} = \partial + \overline{\partial}$. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$. Consider

$$\partial_{(\theta_1,\theta_2)} := \partial + L_{\theta_2} + L_{\overline{\theta_1}} \qquad \text{and} \qquad \overline{\partial}_{(\theta_1,\theta_2)} := \overline{\partial} - L_{\overline{\theta_2}} + L_{\theta_1} .$$

We have

$$\partial_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)} = \overline{\partial}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)} = \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)} + \overline{\partial}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)} = 0 .$$

Therefore we have the bi-differential $\mathbb{Z}$-graded complex

$$\left(A^\bullet(M)_\mathbb{C},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) .$$

Note that $\partial_{(\theta_1,\theta_2)}$ and $\overline{\partial}_{(\theta_1,\theta_2)}$ satisfy the following Leibniz rule:

$$\text{for any } \alpha \in A^\bullet(M)_\mathbb{C} , \quad \left[\partial_{(\theta_1,\theta_2)},\, L_\alpha\right] = L_{\partial\alpha} \quad \text{and} \quad \left[\overline{\partial}_{(\theta_1,\theta_2)},\, L_\alpha\right] = L_{\overline{\partial}\alpha} .$$

Note also that the associated total cochain complex is

$$\left(A^\bullet(M)_\mathbb{C},\, \mathrm{d}_{(\theta_1+\bar\theta_1)+(\theta_2-\bar\theta_2)}\right) .$$

1.2 Hodge theory with twisted differentials

Let (M, J) be a compact complex manifold of complex dimension n. Take a J-Hermitian metric g on M. We consider the ($\mathbb{R}$-linear, possibly $\mathbb{C}$-anti-linear) Hodge-$\star$-operator $\bar\star \colon A^\bullet(M)_\mathbb{K} \to A^{2n-\bullet}(M)_\mathbb{K}$ associated to g. Consider the inner product on $A^\bullet(M)_\mathbb{K}$ given by

$$(x, y) := \int_M x \wedge \bar\star y .$$

Consider the adjoint operators $\mathrm{d}^\star$, $\partial^\star$, and $\overline{\partial}^\star$ of the operators d, ∂, and $\overline{\partial}$, respectively, with respect to $(\cdot, \cdot\cdot)$. Then one has

$$\mathrm{d}^\star = -\bar\star\, \mathrm{d}\, \bar\star , \qquad \partial^\star = -\bar\star\, \partial\, \bar\star , \qquad \overline{\partial}^\star = -\bar\star\, \overline{\partial}\, \bar\star .$$

For $\phi \in A^r(M)_\mathbb{K}$, consider the operator L_ϕ and define its adjoint operator with respect to $(\cdot, \cdot\cdot)$:

$$\Lambda_\phi \colon A^\bullet(M)_\mathbb{K} \to A^{\bullet-r}(M)_\mathbb{K} \quad \text{given by} \quad \left(L_\phi\cdot, \cdot\cdot\right) = \left(\cdot, \Lambda_\phi\cdot\cdot\right) .$$

For $x \in A^{|y|-r}(M)_\mathbb{K}$ and $y \in A^{|y|}(M)_\mathbb{K}$, we compute

$$\begin{aligned}\left(L_\phi x, y\right) &= \int_M \phi \wedge x \wedge \bar\star y = (-1)^{r(|y|-r)} \int_M x \wedge \bar\star\bar\star^{-1} \phi \wedge \bar\star y \\ &= (-1)^{r(|y|-r)} \left(x, \bar\star^{-1} L_\phi \bar\star y\right) .\end{aligned}$$

Hence we get

$$\Lambda_\phi\lfloor_{A^{|y|}(M)_\mathbb{K}} = (-1)^{r(|y|-r)}\, \bar\star^{-1} L_\phi \bar\star .$$

In particular, since the real dimension of M is even, when ϕ is a 1-form, we have

$$\Lambda_\phi = \bar\star L_\phi \bar\star .$$

For a d-closed 1-form ϕ, by considering the differential $\mathrm{d}_\phi = \mathrm{d} + \phi$, the adjoint operator $\mathrm{d}_\phi^\star$ with respect to $(\cdot, \cdot\cdot)$ is given by

$$\mathrm{d}_\phi^\star = \mathrm{d}^\star + \Lambda_\phi = -\bar\star\, \mathrm{d}\, \bar\star + \bar\star L_\phi \bar\star = -\bar\star\, \mathrm{d}_{-\phi}\, \bar\star .$$

Analogously, for $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$, the adjoint operators $\partial^\star_{(\theta_1,\theta_2)}$ and $\overline{\partial}^\star_{(\theta_1,\theta_2)}$ of the operators $\partial_{(\theta_1,\theta_2)}$ and $\overline{\partial}_{(\theta_1,\theta_2)}$, respectively, with respect to $(\cdot, \cdot\cdot)$ are

$$\partial^\star_{(\theta_1,\theta_2)} = \partial^\star + \Lambda_{\theta_2} + \Lambda_{\bar\theta_1} = -\bar\star\, \partial_{(-\theta_2,-\theta_1)}\, \bar\star$$

and

$$\overline{\partial}^{*}_{(\theta_1,\theta_2)} = \overline{\partial}^{*} - \Lambda_{\bar{\theta}_2} + \Lambda_{\theta_1} = -\bar{*}\,\overline{\partial}_{(-\theta_2,-\theta_1)}\,\bar{*} .$$

Suppose g is a Kähler metric, with associated Kähler form ω. Then we have the Kähler identities

$$\Lambda_\omega\, \partial - \partial\, \Lambda_\omega = \sqrt{-1}\, \overline{\partial}^{*} \quad \text{and} \quad \Lambda_\omega\, \overline{\partial} - \overline{\partial}\, \Lambda_\omega = -\sqrt{-1}\, \partial^{*} .$$

For a $(1,0)$-form θ, as the local argument for the Kähler identities, see, e.g., [32, Lemma 6.6], we have

$$\begin{aligned} \Lambda_\omega\, L_\theta - L_\theta\, \Lambda_\omega &= \sqrt{-1}\, \Lambda_{-\bar{\theta}} = -\sqrt{-1}\, \Lambda_{\bar{\theta}} , \\ \Lambda_\omega L_{\bar{\theta}} - L_{\bar{\theta}} \Lambda_\omega &= -\sqrt{-1}\, \Lambda_{-\theta} = \sqrt{-1} \Lambda_\theta . \end{aligned}$$

Hence, for $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$, we have

$$\begin{aligned} \Lambda_\omega \partial_{(\theta_1,\theta_2)} - \partial_{(\theta_1,\theta_2)} \Lambda_\omega &= \sqrt{-1}\, \overline{\partial}^{*}_{(\theta_1,\theta_2)} , \\ \Lambda_\omega \overline{\partial}_{(\theta_1,\theta_2)} - \overline{\partial}_{(\theta_1,\theta_2)} \Lambda_\omega &= -\sqrt{-1}\, \partial^{*}_{(\theta_1,\theta_2)} . \end{aligned}$$

1.3 Cohomologies with twisted differentials

Let (M, J) be a complex manifold; suppose also that M is compact. For $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$ and $\phi := \theta_1 + \overline{\theta_1} + \theta_2 - \overline{\theta_2}$, we consider the (bi-)differential $\mathbb{Z}$-graded algebras

$$\left(A^\bullet(M)_{\mathbb{C}},\, \mathrm{d}_\phi\right) \qquad \text{and} \qquad \left(A^\bullet(M)_{\mathbb{C}},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right)$$

as above. We consider the following cohomologies:

$$\begin{aligned} H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \mathrm{d}_\phi\right) &:= \frac{\ker \mathrm{d}_\phi}{\operatorname{im} \mathrm{d}_\phi} , \\ H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)}\right) &:= \frac{\ker \partial_{(\theta_1,\theta_2)}}{\operatorname{im} \partial_{(\theta_1,\theta_2)}} , \\ H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \overline{\partial}_{(\theta_1,\theta_2)}\right) &:= \frac{\ker \overline{\partial}_{(\theta_1,\theta_2)}}{\operatorname{im} \overline{\partial}_{(\theta_1,\theta_2)}} , \\ H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) &:= \frac{\ker \partial_{(\theta_1,\theta_2)} \cap \ker \overline{\partial}_{(\theta_1,\theta_2)}}{\operatorname{im} \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}} , \\ H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) &:= \frac{\ker \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}}{\operatorname{im} \partial_{(\theta_1,\theta_2)} + \operatorname{im} \overline{\partial}_{(\theta_1,\theta_2)}} . \end{aligned}$$

The identity induces natural maps

$$\begin{array}{ccccc} & & H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) & & \\ & \swarrow & \downarrow & \searrow & \\ H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)}\right) & & H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \mathrm{d}_\phi\right) & & H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \overline{\partial}_{(\theta_1,\theta_2)}\right) \\ & \searrow & \downarrow & \swarrow & \\ & & H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) & & \end{array}$$

of $\mathbb{Z}$-graded vector spaces.

By [6, Theorem 2.4], and using the finite-dimensionality of $H^\bullet(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)})$ and $H^\bullet(A^\bullet(M)_\mathbb{C};\, \overline{\partial}_{(\theta_1,\theta_2)})$, see the next subsection or [31, page 22], we have the following inequality *à la* Frölicher.

Theorem 1.1. *Let (M, J) be a compact complex manifold of complex dimension n. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$, and $\phi := \theta_1 + \overline{\theta_1} + \theta_2 - \overline{\theta_2}$. Then the inequality*

$$\begin{aligned}&\dim_\mathbb{C} H^\bullet(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}) + \dim_\mathbb{C} H^\bullet(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}, \overline{\partial}_{(\theta_1,\theta_2)})\\ &\geq \dim_\mathbb{C} H^\bullet(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}) + \dim_\mathbb{C} H^\bullet(A^\bullet(M)_\mathbb{C};\, \overline{\partial}_{(\theta_1,\theta_2)})\end{aligned}$$

holds.

Remark 1.2. Note that, if $\theta_1 = 0$, then $\left(A^\bullet(M)_\mathbb{C},\, \partial_{(0,\theta_2)},\, \overline{\partial}_{(0,\theta_2)}\right)$ has in fact a structure of double complex. Hence we have the Frölicher inequalities, [17, Theorem 2],

$$\dim_\mathbb{C} H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \partial_{(0,\theta_2)}\right) \geq \dim_\mathbb{C} H^\bullet(A^\bullet(M)_\mathbb{C};\, \mathrm{d}_\phi)$$

and

$$\dim_\mathbb{C} H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \overline{\partial}_{(0,\theta_2)}\right) \geq \dim_\mathbb{C} H^\bullet(A^\bullet(M)_\mathbb{C};\, \mathrm{d}_\phi)\ .$$

Hence we get the following inequality *à la* Frölicher, [6, Corollary 2.6],

$$\begin{aligned}&\dim_\mathbb{C} H^\bullet(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}) + \dim_\mathbb{C} H^\bullet(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}, \overline{\partial}_{(\theta_1,\theta_2)})\\ &\geq 2\, \dim_\mathbb{C} H^\bullet(A^\bullet(M)_\mathbb{C};\, \mathrm{d}_\phi)\ .\end{aligned}$$

But in general, when $\theta_1 \neq 0$, one does not have a double complex structure on $\left(A^\bullet(M)_\mathbb{C},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right)$. In fact, Example 3.2 shows that the inequality

$$\dim_\mathbb{C} H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \overline{\partial}_{(\theta_1,\theta_2)}\right) \geq \dim_\mathbb{C} H^\bullet(A^\bullet(M)_\mathbb{C};\, \mathrm{d}_\phi)$$

may fail.

1.4 Hodge theory and cohomologies with twisted differentials

Take a Hermitian metric g on (M, J). We consider the adjoint operators $\mathrm{d}_\phi^\star$, $\partial_{(\theta_1,\theta_2)}^\star$, and $\overline{\partial}_{(\theta_1,\theta_2)}^\star$ of the operators d_ϕ, $\partial_{(\theta_1,\theta_2)}$, and $\overline{\partial}_{(\theta_1,\theta_2)}$, respectively, with respect to the inner product $(\cdot, \cdot\cdot)$ induced by g. We define the Laplacian operators

$$\begin{aligned}
\Delta_{\mathrm{d}_\phi} &:= \left[\mathrm{d}_\phi, \mathrm{d}_\phi^\star\right] := \mathrm{d}_\phi \mathrm{d}_\phi^\star + \mathrm{d}_\phi^\star \mathrm{d}_\phi\ ,\\
\Delta_{\partial_{(\theta_1,\theta_2)}} &:= \left[\partial_{(\theta_1,\theta_2)}, \partial_{(\theta_1,\theta_2)}^\star\right] := \partial_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}^\star + \partial_{(\theta_1,\theta_2)}^\star\partial_{(\theta_1,\theta_2)}\ ,\\
\Delta_{\overline{\partial}_{(\theta_1,\theta_2)}} &:= \left[\overline{\partial}_{(\theta_1,\theta_2)}, \overline{\partial}_{(\theta_1,\theta_2)}^\star\right] := \overline{\partial}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}^\star + \overline{\partial}_{(\theta_1,\theta_2)}^\star\overline{\partial}_{(\theta_1,\theta_2)}\ ,\\
\Delta_{BC,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} &:= \left(\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)\left(\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)^\star + \left(\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)^\star\left(\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)\\
&\quad + \left(\overline{\partial}_{(\theta_1,\theta_2)}^\star\partial_{(\theta_1,\theta_2)}\right)\left(\overline{\partial}_{(\theta_1,\theta_2)}^\star\partial_{(\theta_1,\theta_2)}\right)^\star + \left(\overline{\partial}_{(\theta_1,\theta_2)}^\star\partial_{(\theta_1,\theta_2)}\right)^\star\left(\overline{\partial}_{(\theta_1,\theta_2)}^\star\partial_{(\theta_1,\theta_2)}\right)\\
&\quad + \overline{\partial}_{(\theta_1,\theta_2)}^\star\overline{\partial}_{(\theta_1,\theta_2)} + \partial_{(\theta_1,\theta_2)}^\star\partial_{(\theta_1,\theta_2)}\ ,\\
\Delta_{A,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} &:= \partial_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}^\star + \overline{\partial}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}^\star
\end{aligned}$$

$$+ \left(\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)^* \left(\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) + \left(\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) \left(\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)^*$$

$$+ \left(\overline{\partial}_{(\theta_1,\theta_2)}\partial^*_{(\theta_1,\theta_2)}\right)^* \left(\overline{\partial}_{(\theta_1,\theta_2)}\partial^*_{(\theta_1,\theta_2)}\right) + \left(\overline{\partial}_{(\theta_1,\theta_2)}\partial^*_{(\theta_1,\theta_2)}\right) \left(\overline{\partial}_{(\theta_1,\theta_2)}\partial^*_{(\theta_1,\theta_2)}\right)^* .$$

Note that the principal parts of the above operators are equal to the principal parts of the corresponding operators with $\phi = 0$ and $(\theta_1, \theta_2) = (0, 0)$. In particular, Δ_{d_ϕ}, and $\Delta_{\partial_{(\theta_1,\theta_2)}}$, and $\Delta_{\overline{\partial}_{(\theta_1,\theta_2)}}$ are 2nd order self-adjoint elliptic differential operators, see [31, page 22]. In particular, one has the orthogonal decompositions

$$A^\bullet(M)_\mathbb{C} = \ker \Delta_{\mathrm{d}_\phi} \overset{\perp}{\oplus} \operatorname{im} \Delta_{\mathrm{d}_\phi} ,$$

$$A^\bullet(M)_\mathbb{C} = \ker \Delta_{\partial_{(\theta_1,\theta_2)}} \overset{\perp}{\oplus} \operatorname{im} \Delta_{\partial_{(\theta_1,\theta_2)}} ,$$

$$A^\bullet(M)_\mathbb{C} = \ker \Delta_{\partial_{(\theta_1,\theta_2)}} \overset{\perp}{\oplus} \operatorname{im} \Delta_{\partial_{(\theta_1,\theta_2)}} ,$$

with respect to the inner product induced by g, and hence the isomorphisms

$$H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \mathrm{d}_\phi\right) \simeq \ker \Delta_{\mathrm{d}_\phi} ,$$

$$H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}\right) \simeq \ker \Delta_{\partial_{(\theta_1,\theta_2)}} ,$$

$$H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \overline{\partial}_{(\theta_1,\theta_2)}\right) \simeq \ker \Delta_{\overline{\partial}_{(\theta_1,\theta_2)}}$$

of vector spaces, depending on the metric, see [31, page 22].

Furthermore, M. Schweitzer proved that $\tilde{\Delta}_{BC} := \Delta_{BC,\partial_{(0,0)},\overline{\partial}_{(0,0)}}$ and $\tilde{\Delta}_{A} := \Delta_{A,\partial_{(0,0)},\overline{\partial}_{(0,0)}}$ are 4th order self-adjoint elliptic differential operators, in [30, §2.b, §2.c], see also [25, Proposition 5]. Hence we have the following result.

Theorem 1.3. *Let (M, J) be a compact complex manifold endowed with a Hermitian metric g. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$. Then the operators $\Delta_{BC,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}}$ and $\Delta_{A,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}}$ are 4th order self-adjoint elliptic differential operators.*

For the classical theory of self-adjoint elliptic differential operators, see, e.g., [24, page 450], we get the following corollaries.

Corollary 1.4. *Let (M, J) be a compact complex manifold endowed with a Hermitian metric g. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$.*

– *There is an orthogonal decomposition*

$$A^\bullet(M)_\mathbb{C} = \ker \Delta_{BC,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} \overset{\perp}{\oplus} \operatorname{im} \Delta_{BC,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} ,$$

with respect to the inner product induced by g. Hence there is an isomorphism

$$H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) = \ker \Delta_{BC,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} ,$$

depending on the metric.

– *There is an orthogonal decomposition*

$$A^\bullet(M)_\mathbb{C} = \ker \Delta_{A,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} \overset{\perp}{\oplus} \operatorname{im} \Delta_{A,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} ,$$

with respect to the inner product induced by g. Hence there is an isomorphism

$$H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) = \ker \Delta_{A,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} ,$$

depending on the metric.

In particular, it follows that the Hodge-$\star$-operator induces isomorphisms between cohomologies. (With abuse of notation, for $\zeta_1, \zeta_2 \in H^{0,1}_{BC}(X)$, denote $\partial_{(\zeta_1,\zeta_2)} := \partial + L_{\zeta_2} + L_{\overline{\zeta_1}}$ and $\overline{\partial}_{(\zeta_1,\zeta_2)} := \overline{\partial} - L_{\overline{\zeta_2}} + L_{\zeta_1}$.)

Corollary 1.5. *Let (M, J) be a compact complex manifold, of complex dimension n, endowed with a Hermitian metric g. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$, and $\phi := \theta_1 + \overline{\theta_1} + \theta_2 - \overline{\theta_2}$. Fix a J-Hermitian metric g, and consider the associated $\mathbb{C}$-anti-linear Hodge-$\star$-operator $\bar{\star}\colon A^\bullet(M)_\mathbb{C} \to A^{2n-\bullet}(M)_\mathbb{C}$. It induces the isomorphisms*

$$\bar{\star}\colon H^\bullet\left(A^\bullet(M)_\mathbb{C}; \mathrm{d}_\phi\right) \stackrel{\simeq}{\to} H^{2n-\bullet}\left(A^\bullet(M)_\mathbb{C}; \mathrm{d}_{-\phi}\right) ,$$

$$\bar{\star}\colon H^\bullet\left(A^\bullet(M)_\mathbb{C}; \partial_{(\theta_1,\theta_2)}\right) \stackrel{\simeq}{\to} H^{2n-\bullet}\left(A^\bullet(M)_\mathbb{C}; \partial_{(-\theta_2,-\theta_1)}\right) ,$$

$$\bar{\star}\colon H^\bullet\left(A^\bullet(M)_\mathbb{C}; \overline{\partial}_{(\theta_1,\theta_2)}\right) \stackrel{\simeq}{\to} H^{2n-\bullet}\left(A^\bullet(M)_\mathbb{C}; \overline{\partial}_{(-\theta_2,-\theta_1)}\right) ,$$

$$\bar{\star}\colon H^\bullet\left(A^\bullet(M)_\mathbb{C}; \partial_{(\theta_1,\theta_2)}, \overline{\partial}_{(\theta_1,\theta_2)}; \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)$$
$$\stackrel{\simeq}{\to} H^{2n-\bullet}\left(A^\bullet(M)_\mathbb{C}; \partial_{(-\theta_2,-\theta_1)}\overline{\partial}_{(-\theta_2,-\theta_1)}; \partial_{(-\theta_2,-\theta_1)}, \overline{\partial}_{(-\theta_2,-\theta_1)}\right) .$$

Proof. Note that

$$\begin{aligned}
\bar{\star}\,\Delta_{\mathrm{d}_\phi} &= \Delta_{\mathrm{d}_{-\phi}}\,\bar{\star} , \\
\bar{\star}\,\Delta_{\partial_{(\theta_1,\theta_2)}} &= \Delta_{\partial_{(-\theta_2,-\theta_1)}}\,\bar{\star} , \\
\bar{\star}\,\Delta_{\overline{\partial}_{(\theta_1,\theta_2)}} &= \Delta_{\overline{\partial}_{(-\theta_2,-\theta_1)}}\,\bar{\star} , \\
\bar{\star}\,\Delta_{BC_{(\theta_1,\theta_2)}} &= \Delta_{A_{(-\theta_2,-\theta_1)}}\,\bar{\star} .
\end{aligned}$$

The statement follows from [31, page 22] and Corollary 1.4. □

1.5 Hodge theory on Kähler manifolds with twisted differentials

Consider the case of a compact complex manifold endowed with a Kähler metric. Thanks to the Kähler identities for twisted differentials, we have the following analogue of the classical Hodge decomposition theorem.

Proposition 1.6. *Let (M, J) be a compact complex manifold endowed with a Kähler metric g. Take $[\phi] \in H^1(M;\mathbb{C})$. Consider $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$ such that $\phi = \theta_1 + \overline{\theta}_1 + \theta_2 - \overline{\theta}_2$. Then*

$$\Delta_{\mathrm{d}_\phi} = 2\,\Delta_{\partial_{(\theta_1,\theta_2)}} = 2\,\Delta_{\overline{\partial}_{(\theta_1,\theta_2)}}$$

and

$$\Delta_{BC,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} = \Delta^2_{\overline{\partial}_{(\theta_1,\theta_2)}} + \partial^\star_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)} + \overline{\partial}^\star_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$$

and

$$\Delta_{A,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} = \Delta^2_{\overline{\partial}_{(\theta_1,\theta_2)}} + \partial_{(\theta_1,\theta_2)}\partial^\star_{(\theta_1,\theta_2)} + \overline{\partial}_{(\theta_1,\theta_2)}\overline{\partial}^\star_{(\theta_1,\theta_2)} .$$

Proof. For the sake of completeness, we detail the proof.

Take $[\phi] \in H^1(M;\mathbb{C})$. Then $[\phi] = [\mathrm{Re}\phi] + \sqrt{-1}\,[\mathrm{Im}\phi]$ where $[\mathrm{Re}\phi] \in H^1(M;\mathbb{R}) \subset H^1(M;\mathbb{C})$ and $[\mathrm{Im}\phi] \in H^1(M;\mathbb{R}) \subset H^1(M;\mathbb{C})$. By the Hodge decomposition theorem for compact Kähler manifolds, one has that the identity induces the isomorphism $H^{1,0}_{BC}(M) \oplus H^{0,1}_{BC}(M) \stackrel{\simeq}{\to} H^1(M;\mathbb{C})$. Hence there exists $\theta_1 \in H^{1,0}_{BC}(M)$ such that $\mathrm{Re}\phi = \theta_1 + \overline{\theta}_1$, and there exists $\theta_2 \in H^{1,0}_{BC}(M)$ such that $\sqrt{-1}\mathrm{Im}\Phi = \theta_2 - \overline{\theta}_2$.

Note that

$$\mathrm{d}_\phi = \mathrm{d} + L_\phi = \partial + L_{\theta_2} + L_{\overline{\theta}_1} + \overline{\partial} - L_{\overline{\theta}_2} + L_{\theta_1} = \partial_{(\theta_1,\theta_2)} + \overline{\partial}_{(\theta_1,\theta_2)} .$$

Firstly, note that, by the Kähler identities for twisted differentials, we have:

$$\begin{aligned}\left[\partial_{(\theta_1,\theta_2)},\, \overline{\partial}^{\star}_{(\theta_1,\theta_2)}\right] &= -\sqrt{-1}\,\left[\partial_{(\theta_1,\theta_2)},\, \Lambda_\omega\,\partial_{(\theta_1,\theta_2)} - \partial_{(\theta_1,\theta_2)}\,\Lambda_\omega\right] \\ &= -\sqrt{-1}\,\left(\partial_{(\theta_1,\theta_2)}\,\Lambda_\omega\,\partial_{(\theta_1,\theta_2)} - \partial^2_{(\theta_1,\theta_2)}\,\Lambda_\omega + \Lambda_\omega\,\partial^2_{(\theta_1,\theta_2)} - \partial_{(\theta_1,\theta_2)}\,\Lambda_\omega\,\partial_{(\theta_1,\theta_2)}\right) \\ &= 0\end{aligned}$$

and, by conjugation,

$$\left[\overline{\partial}_{(\theta_1,\theta_2)},\, \partial^{\star}_{(\theta_1,\theta_2)}\right] = 0\,,$$

where ω denotes the $(1,1)$-form associated to g.

Therefore

$$\begin{aligned}\Delta_{\mathrm{d}_\phi} &= \left[\mathrm{d}_\phi,\, \mathrm{d}_\phi^{\star}\right] = \left[\partial_{(\theta_1,\theta_2)} + \overline{\partial}_{(\theta_1,\theta_2)},\, \partial^{\star}_{(\theta_1,\theta_2)} + \overline{\partial}^{\star}_{(\theta_1,\theta_2)}\right] \\ &= \left[\partial_{(\theta_1,\theta_2)},\, \partial^{\star}_{(\theta_1,\theta_2)}\right] + \left[\partial_{(\theta_1,\theta_2)},\, \overline{\partial}^{\star}_{(\theta_1,\theta_2)}\right] + \left[\overline{\partial}_{(\theta_1,\theta_2)},\, \partial^{\star}_{(\theta_1,\theta_2)}\right] + \left[\overline{\partial}_{(\theta_1,\theta_2)},\, \overline{\partial}^{\star}_{(\theta_1,\theta_2)}\right] \\ &= \Delta_{\partial_{(\theta_1,\theta_2)}} + \Delta_{\overline{\partial}_{(\theta_1,\theta_2)}}\,.\end{aligned}$$

Hence we have to show that

$$\Delta_{\partial_{(\theta_1,\theta_2)}} = \Delta_{\overline{\partial}_{(\theta_1,\theta_2)}}\,.$$

Indeed, by using the Kähler identities, we have

$$\begin{aligned}\Delta_{\partial_{(\theta_1,\theta_2)}} &= \left[\partial_{(\theta_1,\theta_2)},\, \partial^{\star}_{(\theta_1,\theta_2)}\right] = \sqrt{-1}\,\left[\partial_{(\theta_1,\theta_2)},\, \left[\Lambda_\omega,\, \overline{\partial}_{(\theta_1,\theta_2)}\right]\right] \\ &= \sqrt{-1}\,\left[\Lambda_\omega,\, \left[\partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right]\right] + \sqrt{-1}\,\left[\overline{\partial}_{(\theta_1,\theta_2)},\, \left[\partial_{(\theta_1,\theta_2)},\, \Lambda_\omega\right]\right] \\ &= \left[\overline{\partial}_{(\theta_1,\theta_2)},\, \overline{\partial}^{\star}_{(\theta_1,\theta_2)}\right] = \Delta_{\overline{\partial}_{(\theta_1,\theta_2)}}\,.\end{aligned}$$

Again by the Kähler identities, we have (compare [25, Proposition 6], [30, Proposition 2.4])

$$\begin{aligned}\Delta^2_{\overline{\partial}_{(\theta_1,\theta_2)}} &= \Delta_{\overline{\partial}_{(\theta_1,\theta_2)}}\,\Delta_{\partial_{(\theta_1,\theta_2)}} \\ &= \overline{\partial}_{(\theta_1,\theta_2)}\overline{\partial}^{\star}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\partial^{\star}_{(\theta_1,\theta_2)} + \overline{\partial}^{\star}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\partial^{\star}_{(\theta_1,\theta_2)} \\ &\quad + \overline{\partial}_{(\theta_1,\theta_2)}\overline{\partial}^{\star}_{(\theta_1,\theta_2)}\partial^{\star}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)} + \overline{\partial}^{\star}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\partial^{\star}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)} \\ &= -\overline{\partial}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\overline{\partial}^{\star}_{(\theta_1,\theta_2)}\partial^{\star}_{(\theta_1,\theta_2)} - \overline{\partial}^{\star}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\partial^{\star}_{(\theta_1,\theta_2)} \\ &\quad - \overline{\partial}_{(\theta_1,\theta_2)}\partial^{\star}_{(\theta_1,\theta_2)}\overline{\partial}^{\star}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)} - \overline{\partial}^{\star}_{(\theta_1,\theta_2)}\partial^{\star}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)} \\ &= \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\overline{\partial}^{\star}_{(\theta_1,\theta_2)}\partial^{\star}_{(\theta_1,\theta_2)} + \overline{\partial}^{\star}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\partial^{\star}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)} \\ &\quad + \partial^{\star}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\overline{\partial}^{\star}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)} + \overline{\partial}^{\star}_{(\theta_1,\theta_2)}\partial^{\star}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)} \\ &= \Delta_{BC,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} - \overline{\partial}^{\star}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)} - \partial^{\star}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\,.\end{aligned}$$

Analogously,

$$
\begin{aligned}
\Delta^2_{\overline{\partial}_{(\theta_1,\theta_2)}} &= \Delta_{\overline{\partial}_{(\theta_1,\theta_2)}}\,\Delta_{\partial_{(\theta_1,\theta_2)}} \\
&= \overline{\partial}_{(\theta_1,\theta_2)}\overline{\partial}^{*}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)} + \overline{\partial}^{*}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)} \\
&\quad + \overline{\partial}_{(\theta_1,\theta_2)}\overline{\partial}^{*}_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)} + \overline{\partial}^{*}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)} \\
&= -\overline{\partial}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\overline{\partial}^{*}_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)} - \overline{\partial}^{*}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)} \\
&\quad - \overline{\partial}_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)}\overline{\partial}^{*}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)} - \overline{\partial}^{*}_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)} \\
&= \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\overline{\partial}^{*}_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)} + \partial_{(\theta_1,\theta_2)}\overline{\partial}^{*}_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)} \\
&\quad + \overline{\partial}_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\overline{\partial}^{*}_{(\theta_1,\theta_2)} + \overline{\partial}^{*}_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)}\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)} \\
&= \Delta_{A,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} - \overline{\partial}_{(\theta_1,\theta_2)}\overline{\partial}^{*}_{(\theta_1,\theta_2)} - \partial_{(\theta_1,\theta_2)}\partial^{*}_{(\theta_1,\theta_2)}\,.
\end{aligned}
$$

These identities conclude the proof. □

In particular, it follows that, on compact Kähler manifolds, all the above cohomologies are isomorphic.

Corollary 1.7. *Let (M, J) be a compact complex manifold endowed with a Kähler metric g. Take $[\phi] \in H^1(M;\mathbb{C})$. Consider $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$ such that $\phi = \theta_1 + \overline{\theta}_1 + \theta_2 - \overline{\theta}_2$. Then there are isomorphisms*

$$
\begin{array}{ccccc}
 & & H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) & & \\
 & \swarrow \simeq & \downarrow \simeq & \simeq \searrow & \\
H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)}\right) & & H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \mathrm{d}_\phi\right) & & H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \overline{\partial}_{(\theta_1,\theta_2)}\right) \\
 & \searrow \simeq & \downarrow \simeq & \simeq \swarrow & \\
 & & H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) & &
\end{array}
$$

of $\mathbb{Z}$-graded vector spaces.

Since the isomorphisms in [31, page 22] and Corollary 1.4 depend on the Kähler metric, also the isomorphisms in Corollary 1.7, *a priori*, depend on the Kähler metric. In fact, the following result holds, analogous to the $\partial\overline{\partial}$-Lemma.

Theorem 1.8. *Let (M, J) be a compact complex manifold endowed with a Kähler metric g. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$. Then the natural map*

$$
H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) \to H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right)
$$

induced by the identity is injective.

Proof. We detail the proof, which follows the argument in [16, pages 266–267].

We prove that

$$
\ker \partial_{(\theta_1,\theta_2)} \cap \ker \overline{\partial}_{(\theta_1,\theta_2)} \cap \left(\operatorname{im} \partial_{(\theta_1,\theta_2)} + \operatorname{im} \overline{\partial}_{(\theta_1,\theta_2)}\right) = \operatorname{im} \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\,.
$$

Consider $\alpha = \partial_{(\theta_1,\theta_2)}\beta + \overline{\partial}_{(\theta_1,\theta_2)}\gamma \in A^k(M)_{\mathbb{C}}$ such that $\partial_{(\theta_1,\theta_2)}\alpha = \overline{\partial}_{(\theta_1,\theta_2)}\alpha = 0$, where $\beta \in A^{k-1}(M)_{\mathbb{C}}$ and $\gamma \in A^{k-1}(M)_{\mathbb{C}}$.

Fix a Hermitian metric g. By Corollary 1.4 and Proposition 1.6, one has

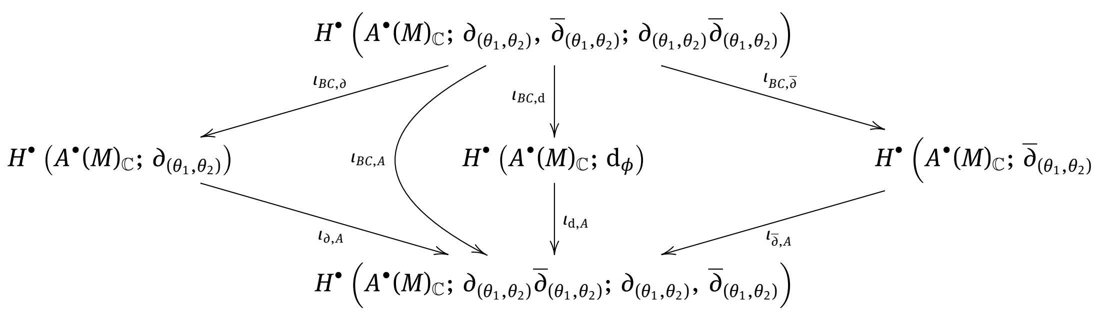
$$\begin{aligned}\alpha &\in \operatorname{im}\partial_{(\theta_1,\theta_2)} + \operatorname{im}\overline{\partial}_{(\theta_1,\theta_2)} \subseteq \operatorname{im}\Delta_{A,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} \\ &\perp \ker\Delta_{A,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} = \ker\Delta_{BC,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} ,\end{aligned}$$

therefore, again by Corollary 1.4, one has

$$\alpha \in \operatorname{im}\Delta_{BC,\partial_{(\theta_1,\theta_2)},\overline{\partial}_{(\theta_1,\theta_2)}} = \operatorname{im}\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)} \overset{\perp}{\oplus} \left(\operatorname{im}\partial^{*}_{(\theta_1,\theta_2)} + \operatorname{im}\overline{\partial}^{*}_{(\theta_1,\theta_2)}\right) .$$

Since $\partial_{(\theta_1,\theta_2)}\alpha = 0$, then $\alpha \perp \operatorname{im}\partial^{*}_{(\theta_1,\theta_2)}$. Since $\overline{\partial}_{(\theta_1,\theta_2)}\alpha = 0$, then $\alpha \perp \operatorname{im}\overline{\partial}^{*}_{(\theta_1,\theta_2)}$. Hence $\alpha \in \operatorname{im}\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$. This concludes the proof. □

Corollary 1.9. *Let (M, J) be a compact complex manifold endowed with a Kähler metric. Take $[\phi] \in H^1(M;\mathbb{C})$. Consider $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$ such that $\phi = \theta_1 + \overline{\theta}_1 + \theta_2 - \overline{\theta}_2$. Then the natural maps*



induced by the identity are isomorphisms.

Proof. By [16, Lemma 5.15], see also [6, Lemma 1.4], the maps $\iota_{BC,A}$, $\iota_{BC,\partial}$, $\iota_{BC,\overline{\partial}}$, and $\iota_{BC,\mathrm{d}}$ are injective, and the maps $\iota_{BC,A}$, $\iota_{\partial,A}$, $\iota_{\overline{\partial},A}$, and $\iota_{\mathrm{d},A}$ are surjective. By Corollary 1.7, they are in fact isomorphisms, being either injective or surjective maps between finite-dimensional vector spaces of the same dimension. □

1.6 Homologies with twisted differentials

Consider the space $D^\bullet(M)_\mathbb{C} = D^{\bullet,\bullet}(M)$ of currents, where we denote by $D^p(M)_\mathbb{C}$ the space of complex $(\dim_\mathbb{R} M - p)$-dimensional currents. Then we also have the bi-differential ($\mathbb{Z}$-graded) algebra

$$\left(D^\bullet(M)_\mathbb{C},\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)}\right) .$$

We consider the inclusion

$$T\colon A^\bullet(M)_\mathbb{C} \to D^\bullet(M)_\mathbb{C} , \qquad T_\eta := \int_M \eta \wedge \cdot .$$

Then we have the following result.

Theorem 1.10. *Let (M, J) be a compact complex manifold endowed with a Hermitian metric g. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$, and $\phi := \theta_1 + \overline{\theta_1} + \theta_2 - \overline{\theta_2}$. The inclusion $T\colon A^\bullet(M)_\mathbb{C} \to D^\bullet(M)_\mathbb{C}$ induces the cohomology isomorphisms*

$$\begin{aligned} H^\bullet\left(A^\bullet(M)_\mathbb{C};\ \mathrm{d}_\phi\right) &\overset{\simeq}{\to} H^\bullet\left(D^\bullet(M)_\mathbb{C};\ \mathrm{d}_\phi\right) , \\ H^\bullet\left(A^\bullet(M)_\mathbb{C};\ \partial_{(\theta_1,\theta_2)}\right) &\overset{\simeq}{\to} H^\bullet\left(D^\bullet(M)_\mathbb{C};\ \partial_{(\theta_1,\theta_2)}\right) , \\ H^\bullet\left(A^\bullet(M)_\mathbb{C};\ \overline{\partial}_{(\theta_1,\theta_2)}\right) &\overset{\simeq}{\to} H^\bullet\left(D^\bullet(M)_\mathbb{C};\ \overline{\partial}_{(\theta_1,\theta_2)}\right) ,\end{aligned}$$

$$H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) \stackrel{\simeq}{\to} H^\bullet\left(D^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) ,$$
$$H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) \stackrel{\simeq}{\to} H^\bullet\left(D^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) .$$

Proof. Consider each case. For a fixed Hermitian metric, consider the corresponding Laplacian operator Δ. Then, by [34, Theorem 4.12], we have the operators

$$G\colon A^\bullet(M)_\mathbb{C} \to A^\bullet(M)_\mathbb{C} \quad \text{and} \quad H\colon A^\bullet(M)_\mathbb{C} \to A^\bullet(M)_\mathbb{C} ,$$

where H is given by the projection $A^\bullet(M)_\mathbb{C} \to \ker\Delta$ and G is given by the inverse of the restriction of Δ on $A^\bullet(M)_\mathbb{C} \cap (\ker\Delta)^\perp$, such that

$$\Delta \circ G + H = G \circ \Delta + H = \mathrm{id} .$$

Since Δ is self-adjoint, G and H are also self-adjoint. We can define the operators Δ, G, and H on $D^\bullet(M)_\mathbb{C}$. They still satisfy $\Delta \circ G + H = G \circ \Delta + H = \mathrm{id}$. By the regularity of the kernel of elliptic differential operators in Sobolev spaces, see, e.g., [34, Theorem 4.8], we have

$$\ker\Delta\lfloor_{A^\bullet(M)_\mathbb{C}} = \ker\Delta\lfloor_{D^\bullet(M)_\mathbb{C}} .$$

Hence we have

$$D^\bullet(M)_\mathbb{C} = \ker\Delta\lfloor_{A^\bullet(M)_\mathbb{C}} + \Delta\left(D^\bullet(M)_\mathbb{C}\right) .$$

For $T \in D^\bullet(M)_\mathbb{C}$ and $h \in \ker\Delta\lfloor_{A^\bullet(M)_\mathbb{C}}$, suppose that $\Delta\lfloor_{D^\bullet(M)_\mathbb{C}} T = h$. Then, by [34, Theorem 4.9], we have $T \in A^\bullet(M)_\mathbb{C}$, and hence $h = 0$. Thus we have

$$D^\bullet(M)_\mathbb{C} = \ker\Delta\lfloor_{A^\bullet(M)_\mathbb{C}} \oplus \Delta\left(D^\bullet(M)_\mathbb{C}\right) .$$

This completes the proof. □

2 Hodge decomposition with twisted differentials and modifications

Let $f\colon M_1 \to M_2$ be a holomorphic map between compact complex manifolds M_1 and M_2. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M_2)$, and $\phi := \theta_1 + \overline{\theta_1} + \theta_2 - \overline{\theta_2}$.

2.1 Modifications and cohomologies with twisted differentials

Consider the pull-back $f^\star\colon A^{\bullet,\bullet}(M_2) \to A^{\bullet,\bullet}(M_1)$. We have $f^\star\theta_1, f^\star\theta_2 \in H^{1,0}_{BC}(M_1)$. Since $f^\star$ commutes with ∂ and $\overline{\partial}$, then

$$f^\star\colon \left(A^\bullet(M_2)_\mathbb{C},\, \mathrm{d}_\phi\right) \to \left(A^\bullet(M_1)_\mathbb{C},\, \mathrm{d}_{f^\star\phi}\right)$$

is a morphism of differential $\mathbb{Z}$-graded complexes, and

$$f^\star\colon \left(A^\bullet(M_2)_\mathbb{C},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) \to \left(A^\bullet(M_1)_\mathbb{C},\, \partial_{(f^\star\theta_1,f^\star\theta_2)},\, \overline{\partial}_{(f^\star\theta_1,f^\star\theta_2)}\right)$$

is a morphism of bi-differential $\mathbb{Z}$-graded complexes. In particular, f induces the maps

$$\begin{aligned}
f^{\star}_{dR,\phi} &: H^{\bullet}\left(A^{\bullet}(M_2)_{\mathbb{C}};\, \mathrm{d}_{\phi}\right) \to H^{\bullet}\left(A^{\bullet}(M_1)_{\mathbb{C}};\, \mathrm{d}_{f^*\phi}\right) , \\
f^{\star}_{\overline{\partial},(\theta_1,\theta_2)} &: H^{\bullet}\left(A^{\bullet}(M_2)_{\mathbb{C}};\, \overline{\partial}_{(\theta_1,\theta_2)}\right) \to H^{\bullet}\left(A^{\bullet}(M_1)_{\mathbb{C}};\, \overline{\partial}_{(f^*\theta_1,f^*\theta_2)}\right) , \\
f^{\star}_{BC,(\theta_1,\theta_2)} &: H^{\bullet}\left(A^{\bullet}(M_2)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) \\
&\to H^{\bullet}\left(A^{\bullet}(M_1)_{\mathbb{C}};\, \partial_{(f^*\theta_1,f^*\theta_2)},\, \overline{\partial}_{(f^*\theta_1,f^*\theta_2)};\, \partial_{(f^*\theta_1,f^*\theta_2)}\overline{\partial}_{(f^*\theta_1,f^*\theta_2)}\right) , \\
f^{\star}_{A,(\theta_1,\theta_2)} &: H^{\bullet}\left(A^{\bullet}(M_2)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) \\
&\to H^{\bullet}\left(A^{\bullet}(M_1)_{\mathbb{C}};\, \partial_{(f^*\theta_1,f^*\theta_2)}\overline{\partial}_{(f^*\theta_1,f^*\theta_2)};\, \partial_{(f^*\theta_1,f^*\theta_2)},\, \overline{\partial}_{(f^*\theta_1,f^*\theta_2)}\right) .
\end{aligned}$$

2.2 Modifications and homologies with twisted differentials

Suppose that f is proper. Then we have the map $f_\star\colon D^{\bullet,\bullet}(M_1) \to D^{\bullet,\bullet}(M_2)$, called the direct image map, such that $f_\star$ commutes with ∂ and $\overline{\partial}$, and $f_\star\left(f^*\alpha \wedge C\right) = \alpha \wedge f_\star C$ for any $\alpha \in A^{\bullet,\bullet}(M_2)$ and $C \in D^{\bullet,\bullet}(M_1)$. Hence the map

$$f_\star\colon \left(D^{\bullet}(M_1)_{\mathbb{C}},\, \mathrm{d}_{f^*\phi}\right) \to \left(D^{\bullet}(M_2)_{\mathbb{C}},\, \mathrm{d}_{\phi}\right)$$

is a morphism of differential $\mathbb{Z}$-graded complexes, and the map

$$f_\star\colon \left(D^{\bullet}(M_1)_{\mathbb{C}},\, \partial_{(f^*\theta_1,f^*\theta_2)},\, \overline{\partial}_{(f^*\theta_1,f^*\theta_2)}\right) \to \left(D^{\bullet}(M_2)_{\mathbb{C}},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right)$$

is a morphism of bi-differential $\mathbb{Z}$-graded complexes. In particular, f induces the maps

$$\begin{aligned}
f_{\star}^{dR,\phi} &: H^{\bullet}\left(D^{\bullet}(M_1)_{\mathbb{C}};\, \mathrm{d}_{f^*\phi}\right) \to H^{\bullet}\left(D^{\bullet}(M_2)_{\mathbb{C}};\, \mathrm{d}_{\phi}\right) , \\
f_{\star}^{\overline{\partial},(\theta_1,\theta_2)} &: H^{\bullet}\left(D^{\bullet}(M_1)_{\mathbb{C}};\, \overline{\partial}_{(f^*\theta_1,f^*\theta_2)}\right) \to H^{\bullet}\left(D^{\bullet}(M_2)_{\mathbb{C}};\, \overline{\partial}_{(\theta_1,\theta_2)}\right) , \\
f_{\star}^{BC,(\theta_1,\theta_2)} &: H^{\bullet}\left(D^{\bullet}(M_1)_{\mathbb{C}};\, \partial_{(f^*\theta_1,f^*\theta_2)},\, \overline{\partial}_{(f^*\theta_1,f^*\theta_2)};\, \partial_{(f^*\theta_1,f^*\theta_2)}\overline{\partial}_{(f^*\theta_1,f^*\theta_2)}\right) \\
&\to H^{\bullet}\left(D^{\bullet}(M_2)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) , \\
f_{\star}^{A,(\theta_1,\theta_2)} &: H^{\bullet}\left(D^{\bullet}(M_1)_{\mathbb{C}};\, \partial_{(f^*\theta_1,f^*\theta_2)}\overline{\partial}_{(f^*\theta_1,f^*\theta_2)};\, \partial_{(f^*\theta_1,f^*\theta_2)},\, \overline{\partial}_{(f^*\theta_1,f^*\theta_2)}\right) \\
&\to H^{\bullet}\left(D^{\bullet}(M_2)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) .
\end{aligned}$$

2.3 Hodge decomposition and $\partial\overline{\partial}$-Lemma with twisted differentials

As in [16], we consider the following definitions in the case of twisted differentials.

Definition 2.1. Let (M, J) be a compact complex manifold of complex dimension n. For $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$, and $\phi := \theta_1 + \overline{\theta_1} + \theta_2 - \overline{\theta_2}$, consider the bi-differential $\mathbb{Z}$-graded complex

$$\left(A^{\bullet}(M)_{\mathbb{C}},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) .$$

We say that (M, J):

(i) satisfies the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-*Lemma* if

$$\ker \partial_{(\theta_1,\theta_2)} \cap \ker \overline{\partial}_{(\theta_1,\theta_2)} \cap \left(\mathrm{im}\, \partial_{(\theta_1,\theta_2)} + \mathrm{im}\, \overline{\partial}_{(\theta_1,\theta_2)}\right) = \mathrm{im}\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)} ,$$

i.e., if the natural map

$$H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)\to H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)}\right)$$

induced by the identity is injective;

(ii) admits the (θ_1,θ_2)-*Hodge decomposition* if the natural maps

$$H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)\to H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \mathrm{d}_\phi\right)$$

and

$$H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)\to H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)}\right)$$

and

$$H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)\to H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \overline{\partial}_{(\theta_1,\theta_2)}\right)$$

induced by the identity are isomorphisms.

By [16, Lemma 5.15], see also [6, Lemma 1.4], we have the following equivalent characterizations of $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma. (In case of double complex, a further characterization is proven in [5].)

Lemma 2.2. Let (M, J) be a compact complex manifold of complex dimension n. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$, and $\phi := \theta_1 + \overline{\theta_1} + \theta_2 - \overline{\theta_2}$. The following conditions are equivalent:

- (M, J) satisfies the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma, i.e., the natural map
$$H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)\to H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)}\right)$$
induced by the identity is injective;
- the natural map
$$H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)\to H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)}\right)$$
induced by the identity is an isomorphism;
- the natural maps
$$H^\bullet(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)})\to H^\bullet(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)})$$
and
$$H^\bullet(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)})\to H^\bullet(A^\bullet(M)_{\mathbb{C}};\ \overline{\partial}_{(\theta_1,\theta_2)})$$
induced by the identity are injective;
- the natural maps
$$H^\bullet(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)})\to H^\bullet(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)})$$
and
$$H^\bullet(A^\bullet(M)_{\mathbb{C}};\ \overline{\partial}_{(\theta_1,\theta_2)})\to H^\bullet(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)})$$
induced by the identity are surjective.

Furthermore, they imply the following conditions:

- the natural map
$$H^\bullet(A^\bullet(M)_{\mathbb{C}};\ \partial_{(\theta_1,\theta_2)},\ \overline{\partial}_{(\theta_1,\theta_2)};\ \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)})\to H^\bullet(A^\bullet(M)_{\mathbb{C}};\ \mathrm{d}_\phi)$$
induced by the identity is injective;

– the natural map

$$H^\bullet(A^\bullet(M)_\mathbb{C};\, \mathrm{d}_\phi) \to H^\bullet(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)})$$

induced by the identity is surjective.

In particular, we have that admitting the (θ_1, θ_2)-Hodge decomposition is a stronger condition than satisfying the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma. We wonder whether it is strictly stronger, namely, whether there exists an example of a compact complex manifold M satisfying the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma but not admitting the (θ_1, θ_2)-Hodge decomposition, for some $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$.

In the Kähler case, we can summarize Theorem 1.8 and Theorem 1.9 in the following.

Corollary 2.3. *Let (M, J) be a compact complex manifold endowed with a Kähler metric. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$. Then (M, J) satisfies the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma and admits the (θ_1, θ_2)-Hodge decomposition.*

2.4 Modifications and cohomologies with twisted differentials

We recall that a *modification* of a compact complex manifold (M, J) of complex dimension n is a holomorphic map

$$\mu\colon\ (\tilde{M}, \tilde{J}) \to (M, J)$$

such that:

- $(\tilde{M}, \tilde{J})$ is a compact n-dimensional complex manifold;
- there exists an analytic subset $S \subset M$ of codimension greater than or equal to 1 such that $\mu\lfloor_{\tilde{M}\setminus\mu^{-1}(S)}\colon \tilde{M}\setminus\mu^{-1}(S) \to M\setminus S$ is a biholomorphism.

For the following, compare [33, Theorem 3.1].

Theorem 2.4. *Let $\mu\colon\ (\tilde{M}, \tilde{J}) \to (M, J)$ be a proper modification of a compact complex manifold (M, J). Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$, and $\phi := \theta_1 + \overline{\theta_1} + \theta_2 - \overline{\theta_2}$. Then the map μ induces the injective maps*

$$\begin{aligned}
&\mu^\star_{dR,\phi}\colon H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \mathrm{d}_\phi\right) \to H^\bullet\left(A^\bullet(\tilde{M})_\mathbb{C};\, \mathrm{d}_{\mu^\star\phi}\right)\ ,\\
&\mu^\star_{\partial,(\theta_1,\theta_2)}\colon H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}\right) \to H^\bullet\left(A^\bullet(\tilde{M})_\mathbb{C};\, \partial_{(\mu^\star\theta_1,\mu^\star\theta_2)}\right)\ ,\\
&\mu^\star_{\overline{\partial},(\theta_1,\theta_2)}\colon H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \overline{\partial}_{(\theta_1,\theta_2)}\right) \to H^\bullet\left(A^\bullet(\tilde{M})_\mathbb{C};\, \overline{\partial}_{(\mu^\star\theta_1,\mu^\star\theta_2)}\right)\ ,\\
&\mu^\star_{BC,(\theta_1,\theta_2)}\colon H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right)\\
&\qquad \to H^\bullet\left(A^\bullet(\tilde{M})_\mathbb{C};\, \partial_{(\mu^\star\theta_1,\mu^\star\theta_2)},\, \overline{\partial}_{(\mu^\star\theta_1,\mu^\star\theta_2)};\, \partial_{(\mu^\star\theta_1,\mu^\star\theta_2)}\overline{\partial}_{(\mu^\star\theta_1,\mu^\star\theta_2)}\right)\ ,\\
&\mu^\star_{A,(\theta_1,\theta_2)}\colon H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right)\\
&\qquad \to H^\bullet\left(A^\bullet(\tilde{M})_\mathbb{C};\, \partial_{(\mu^\star\theta_1,\mu^\star\theta_2)}\overline{\partial}_{(\mu^\star\theta_1,\mu^\star\theta_2)};\, \partial_{(\mu^\star\theta_1,\mu^\star\theta_2)},\, \overline{\partial}_{(\mu^\star\theta_1,\mu^\star\theta_2)}\right)\ .
\end{aligned}$$

and the surjective maps

$$\begin{aligned}
&\mu_\star^{dR,\phi}\colon H^\bullet\left(A^\bullet(\tilde{M})_\mathbb{C};\, \mathrm{d}_{\mu^\star\phi}\right) \to H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \mathrm{d}_\phi\right)\ ,\\
&\mu_\star^{\partial,(\theta_1,\theta_2)}\colon H^\bullet\left(A^\bullet(\tilde{M})_\mathbb{C};\, \partial_{(\mu^\star\theta_1,\mu^\star\theta_2)}\right) \to H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \partial_{(\theta_1,\theta_2)}\right)\ ,\\
&\mu_\star^{\overline{\partial},(\theta_1,\theta_2)}\colon H^\bullet\left(A^\bullet(\tilde{M})_\mathbb{C};\, \overline{\partial}_{(\mu^\star\theta_1,\mu^\star\theta_2)}\right) \to H^\bullet\left(A^\bullet(M)_\mathbb{C};\, \overline{\partial}_{(\theta_1,\theta_2)}\right)\ ,
\end{aligned}$$

$$\mu_\star^{BC,(\theta_1,\theta_2)} \colon H^\bullet\left(A^\bullet(\tilde{M})_{\mathbb{C}};\, \partial_{(\mu^*\theta_1,\mu^*\theta_2)},\, \overline{\partial}_{(\mu^*\theta_1,\mu^*\theta_2)};\, \partial_{(\mu^*\theta_1,\mu^*\theta_2)}\overline{\partial}_{(\mu^*\theta_1,\mu^*\theta_2)}\right)$$
$$\to H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) ,$$
$$\mu_\star^{A,(\theta_1,\theta_2)} \colon H^\bullet\left(A^\bullet(\tilde{M})_{\mathbb{C}};\, \partial_{(\mu^*\theta_1,\mu^*\theta_2)}\overline{\partial}_{(\mu^*\theta_1,\mu^*\theta_2)};\, \partial_{(\mu^*\theta_1,\mu^*\theta_2)},\, \overline{\partial}_{(\mu^*\theta_1,\mu^*\theta_2)}\right)$$
$$\to H^\bullet\left(A^\bullet(M)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) .$$

Proof. We follow closely the proof by R. O. Wells in [33, Theorem 3.1].

Consider the diagram

$$\begin{array}{ccc} A^\bullet(\tilde{M})_{\mathbb{C}} & \xrightarrow{\;T\;} & D^\bullet(\tilde{M})_{\mathbb{C}} \\ {\scriptstyle \mu^*}\uparrow & & \downarrow{\scriptstyle \mu_\star} \\ A^\bullet(M)_{\mathbb{C}} & \xrightarrow[\;T\;]{} & D^\bullet(M)_{\mathbb{C}} \, . \end{array}$$

There exists a proper analytic subset $\tilde{S} \subset \tilde{M}$ such that

$$\mu\lfloor_{\tilde{M}\setminus\tilde{S}} \colon \tilde{M}\setminus\tilde{S} \to M \setminus \mu(S)$$

is a finitely-sheeted covering map of sheeting number $\ell \in \mathbb{N}\setminus\{0\}$. Let $\mathfrak{U} := \{U_\alpha\}_{j\in J}$ be an open covering of $M\setminus\mu(S)$, and let $\{\rho_\alpha\}_{j\in J}$ be an associated partition of unity. For every $\alpha, \beta \in A^\bullet(M)_{\mathbb{C}}$, one has that

$$\begin{aligned} \left\langle \mu_\star T\mu^*\alpha, \beta\right\rangle &= \left\langle T\mu^*\alpha, \mu^*\beta\right\rangle = \int_{\tilde{M}} \mu^*\alpha\wedge\mu^*\beta = \int_{\tilde{M}} \mu^*(\alpha\wedge\beta) = \int_{\tilde{M}\setminus\tilde{S}} \mu^*(\alpha\wedge\beta) \\ &= \sum_{j\in J}\int_{\mu^{-1}(U_j)} \mu^*\left(\rho_j\cdot(\alpha\wedge\beta)\right) = \sum_{j\in J}\sum_{\sharp\{U\in\mathfrak{U}\,:\,\mu(U)=\mu(U_j)\}}\int_{U_j}\rho_j\cdot(\alpha\wedge\beta) \\ &= \ell\cdot\sum_{j\in J}\int_{U_j}\rho_j\cdot(\alpha\wedge\beta) = \ell\cdot\int_{M\setminus\mu(\tilde{S})}\alpha\wedge\beta = \ell\cdot\int_M \alpha\wedge\beta = \langle \ell\cdot T\alpha,\beta\rangle \, , \end{aligned}$$

and hence one gets that

$$\mu_\star T\mu^* = \ell\cdot T \, .$$

In particular, one gets, for $\sharp \in \left\{\partial, \overline{\partial}, BC, A\right\}$,

$$\mu_\star^{dR,\phi}T\mu^*_{dR,\phi} = \ell\cdot T \qquad \text{and} \qquad \mu_\star^{\sharp,(\theta_1,\theta_2)}T\mu^*_{\sharp,(\theta_1,\theta_2)} = \ell\cdot T \, .$$

Hence, in particular, for $\sharp \in \left\{\partial, \overline{\partial}, BC, A\right\}$, the maps $\mu^*_{dR,\phi}$ and $\mu^*_{\sharp,(\theta_1,\theta_2)}$ are injective, and the maps $\mu_\star^{dR,\phi}$ and $\mu_\star^{\sharp,(\theta_1,\theta_2)}$ are surjective. □

2.5 Modifications and $\partial\overline{\partial}$-Lemma with twisted differentials

As a consequence, we get the following two results. The first one concerns the behaviour of $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma under proper modifications.

Theorem 2.5. *Let $\mu\colon \left(\tilde{M}, \tilde{J}\right) \to (M, J)$ be a proper modification of a compact complex manifold (M, J). Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$. If $\left(\tilde{M}, \tilde{J}\right)$ satisfies the $\partial_{(\mu^*\theta_1,\mu^*\theta_2)}\overline{\partial}_{(\mu^*\theta_1,\mu^*\theta_2)}$-Lemma, then (M, J) satisfies the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H^{\bullet}_{BC}\left(M;(\theta_1,\theta_2)\right) & \xrightarrow{\mu^{\star}_{BC,(\theta_1,\theta_2)}} & H^{\bullet}_{BC}\left(\tilde{M};(\mu^{\star}\theta_1,\mu^{\star}\theta_2)\right) \\ \downarrow{\scriptstyle \mathrm{id}^{\star}_{M}} & & \downarrow{\scriptstyle \mathrm{id}^{\star}_{\tilde{M}}} \\ H^{\bullet}_{A}\left(M;(\theta_1,\theta_2)\right) & \xrightarrow[\mu^{\star}_{A,(\theta_1,\theta_2)}]{} & H^{\bullet}_{A}\left(\tilde{M};(\mu^{\star}\theta_1,\mu^{\star}\theta_2)\right) \end{array}$$

where, for simplicity, we have denoted, e.g.,

$$H^{\bullet}_{BC}\left(\tilde{M};(\mu^{\star}\theta_1,\mu^{\star}\theta_2)\right) := H^{\bullet}\left(A^{\bullet}(\tilde{M})_{\mathbb{C}};\ \partial_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)},\ \overline{\partial}_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)};\ \partial_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)}\overline{\partial}_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)}\right)$$

and

$$H^{\bullet}_{A}\left(\tilde{M};(\mu^{\star}\theta_1,\mu^{\star}\theta_2)\right) := H^{\bullet}\left(A^{\bullet}(\tilde{M})_{\mathbb{C}};\ \partial_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)}\overline{\partial}_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)};\ \partial_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)},\ \overline{\partial}_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)}\right) .$$

Suppose that $(\tilde{M},\tilde{J})$ satisfies the $\partial_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)}\overline{\partial}_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)}$-Lemma. Then, by definition, the map $\mathrm{id}^{\star}_{\tilde{M}}$ is injective. Furthermore, the map $\mu^{\star}_{BC,(\theta_1,\theta_2)}$ is injective by Theorem 2.4. Hence the map $\mathrm{id}^{\star}_{M}$ is injective, that is, (M, J) satisfies the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma. □

2.6 Modifications and Hodge decomposition with twisted differentials

The second result concerns Hodge decomposition with twisted differential.

Theorem 2.6. *Let $\mu\colon (\tilde{M},\tilde{J}) \to (M, J)$ be a proper modification of a compact complex manifold (M, J). Take $\theta_1,\theta_2 \in H^{1,0}_{BC}(M)$. If $(\tilde{M},\tilde{J})$ satisfies the $(\mu^{\star}\theta_1,\mu^{\star}\theta_2)$-Hodge decomposition, then (M, J) satisfies the (θ_1,θ_2)-Hodge decomposition.*

Proof. For simplicity, we denote, e.g.,

$$H^{\bullet}_{BC}\left(\tilde{M};(\mu^{\star}\theta_1,\mu^{\star}\theta_2)\right) := H^{\bullet}\left(A^{\bullet}(\tilde{M})_{\mathbb{C}};\ \partial_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)},\ \overline{\partial}_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)};\ \partial_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)}\overline{\partial}_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)}\right)$$

and

$$H^{\bullet}_{A}\left(\tilde{M};(\mu^{\star}\theta_1,\mu^{\star}\theta_2)\right) := H^{\bullet}\left(A^{\bullet}(\tilde{M})_{\mathbb{C}};\ \partial_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)}\overline{\partial}_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)};\ \partial_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)},\ \overline{\partial}_{(\mu^{\star}\theta_1,\mu^{\star}\theta_2)}\right) .$$

Consider the commutative diagram

$$\begin{array}{ccc} H^{\bullet}_{BC}\left(M;(\theta_1,\theta_2)\right) & \xrightarrow{\mu^{\star}_{BC,(\theta_1,\theta_2)}} & H^{\bullet}_{BC}\left(\tilde{M};(\mu^{\star}\theta_1,\mu^{\star}\theta_2)\right) . \\ \downarrow{\scriptstyle \mathrm{id}^{\star}_{M}} & & \downarrow{\scriptstyle \mathrm{id}^{\star}_{\tilde{M}}} \\ H^{\bullet}_{dR}\left(M;\phi\right) & \xrightarrow[\mu^{\star}_{dR,\phi}]{} & H_{dR}\left(\tilde{M};\mu^{\star}\phi\right) \end{array}$$

Then by Theorem 2.4, $\mu^{\star}_{BC,(\theta_1,\theta_2)}$ and $\mu^{\star}_{dR,\phi}$ are injective. Hence by the injectivity of $\mathrm{id}^{\star}_{\tilde{M}}$, the map $\mathrm{id}^{\star}_{M} : H^{\bullet}_{BC}\left(M;(\theta_1,\theta_2)\right) \to H^{\bullet}_{dR}\left(M;\phi\right)$ is injective.

Considering the direct image maps, we have the commutative diagram

$$\begin{array}{ccc} H^{\bullet}_{BC}\left(M;(\theta_1,\theta_2)\right) & \xleftarrow[\mu_{\star}^{BC,(\theta_1,\theta_2)}]{} & H^{\bullet}_{BC}\left(\tilde{M};(\mu^{\star}\theta_1,\mu^{\star}\theta_2)\right) . \\ \downarrow{\scriptstyle \mathrm{id}^{\star}_{M}} & & \downarrow{\scriptstyle \mathrm{id}^{\star}_{\tilde{M}}} \\ H^{\bullet}_{dR}\left(M;\phi\right) & \xleftarrow{\mu_{\star}^{dR,\phi}} & H_{dR}\left(\tilde{M};\mu^{\star}\phi\right) \end{array}$$

By Theorem 2.4, $\mu_{\star}^{BC,(\theta_1,\theta_2)}$ and $\mu_{\star}^{dR,\phi}$ are surjective. By the surjectivity of $\mathrm{id}^{\star}_{\tilde{M}}$, the map $\mathrm{id}^{\star}_{M} : H^{\bullet}_{BC}\left(M;(\theta_1,\theta_2)\right) \to H^{\bullet}_{dR}\left(M;\phi\right)$ is surjective.

Arguing in the same way with the Dolbeault cohomologies, we get that M satisfies the (θ_1,θ_2)-Hodge decomposition. □

We recall that a compact complex manifold (M, J) is said to be in *class* $\mathcal{C}$ *of Fujiki*, [18], if it admits a proper modification $\mu\colon \left(\tilde{M}, \tilde{J}\right) \to (M, J)$ with $\left(\tilde{M}, \tilde{J}\right)$ admitting Kähler metrics. In particular, a *Moĭšhezon manifold*, [26], (that is, a compact complex manifold of complex dimension n such that the degree of transcendence over $\mathbb{C}$ of the field of meromorphic functions is equal to n,) admits a proper modification from a projective manifold, [26, Theorem 1], and therefore belongs to class $\mathcal{C}$ of Fujiki.

Corollary 2.7. *Let (M, J) be a compact complex manifold in class $\mathcal{C}$ of Fujiki. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$. Then (M, J) satisfies the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma and admits the (θ_1, θ_2)-Hodge decomposition.*

Proof. Consider a proper modification $\mu\colon \left(\tilde{M}, \tilde{J}\right) \to (M, J)$ with $\left(\tilde{M}, \tilde{J}\right)$ admitting Kähler metrics. From Corollary 2.3, (see also Theorem 1.8 and Theorem 1.9,) the compact Kähler manifold $\left(\tilde{M}, \tilde{J}\right)$ satisfies the $\partial_{(\mu^*\theta_1,\mu^*\theta_2)}\overline{\partial}_{(\mu^*\theta_1,\mu^*\theta_2)}$-Lemma, and admits the $(\mu^*\theta_1, \mu^*\theta_2)$-Hodge decomposition. Therefore, from Theorem 2.5 and Theorem 2.6, the compact complex manifold (M, J) satisfies the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma and admits the (θ_1, θ_2)-Hodge decomposition. □

2.7 Complex orbifolds of global-quotient type

I. Satake introduce in [29] the notion of *orbifold*, also called *V-manifold*; see also [8, 9]. It is a singular complex space whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^n/G$, for finite subgroups $G \subset \mathrm{GL}_n(\mathbb{C})$. In particular, we are interested in compact *complex orbifolds of global-quotient type*, namely, compact complex orbifolds given by $\hat{M} = M/G$ where M is a compact complex manifold and G is a finite group of biholomorphisms of M. See [3] and the references therein for motivations.

From the cohomological point of view, one can adapt both the sheaf-theoretic and the analytic tools to complex orbifolds, see [3, 8, 9, 29]. In particular, let $\hat{M} = M/G$ be a compact complex orbifold of global-quotient type. Consider the double-complex $\left(\wedge^{\bullet,\bullet}\hat{M}, \partial, \overline{\partial}\right)$, where the space $\wedge^{\bullet,\bullet}\hat{M}$ of differential forms on $\hat{M}$ is defined as the space of G-invariant differential forms on M. Consider the associated cohomologies. Fix a Hermitian metric on $\hat{M}$, namely, a G-invariant Hermitian metric on M. Consider the Laplacian operators defined as in the smooth case. Then Hodge theory applies, [8, Theorem H, Theorem K], [3, Theorem 1].

By considering objects on $\hat{M}$ as G-invariant objects on M, one can adapt all the definitions and results in the previous sections in a straightforward way. In particular, as an analogue of [3, Theorem 2], we can restate Theorem 2.5 and Theorem 2.6 as follows.

Theorem 2.8. *Let $\mu\colon \hat{N} \to \hat{M}$ be a proper modification between compact complex orbifolds of global-quotient type. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(\hat{M})$.*

- *If $\hat{N}$ satisfies the $\partial_{(\mu^*\theta_1,\mu^*\theta_2)}\overline{\partial}_{(\mu^*\theta_1,\mu^*\theta_2)}$-Lemma, then $\hat{M}$ satisfies the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma.*
- *If $\hat{N}$ satisfies both the $\left(\mu^*\theta_1, \mu^*\theta_2\right)$-Hodge decomposition, then $\hat{M}$ satisfies the (θ_1, θ_2)-Hodge decomposition.*

Therefore, we have the following corollary. (As usual, by compact complex orbifold $\hat{M}$ of global-quotient type in class $\mathcal{C}$ of Fujiki, we mean that there exists a proper modification $\mu\colon \hat{N} \to \hat{M}$ where $\hat{N}$ is a compact complex orbifold of global-quotient type admitting Kähler metrics.)

Corollary 2.9. *Let $\hat{M}$ be a compact complex orbifold of global-quotient type in class $\mathcal{C}$ of Fujiki. Take $\theta_1, \theta_2 \in H^{1,0}_{BC}(\hat{M})$. Then $\hat{M}$ satisfies the $\partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}$-Lemma and admits the (θ_1, θ_2)-Hodge decomposition.*

3 Solvmanifolds

In this section, we consider solvmanifolds, i.e., compact quotients $\Gamma\backslash G$ where G is a connected simply-connected solvable Lie group and Γ is a co-compact discrete subgroup (such subgroup is called a lattice).

3.1 Cohomology computations for solvmanifolds

Let G be a connected simply-connected solvable Lie group endowed with a left-invariant complex structure J and admitting a lattice Γ. Its associated Lie algebra is denoted by $\mathfrak{g}$, and its complexification by $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Then we consider the sub-double complex

$$\left(\wedge^{\bullet,\bullet}\mathfrak{g}_{\mathbb{C}}^{*},\, \partial,\, \overline{\partial}\right) \hookrightarrow \left(A^{\bullet,\bullet}(\Gamma\backslash G)_{\mathbb{C}},\, \partial,\, \overline{\partial}\right) .$$

Take $\theta_1, \theta_2 \in H^{1,0}\left(\wedge^{\bullet,\bullet}\mathfrak{g}_{\mathbb{C}}^{*}; \partial, \overline{\partial}; \partial\overline{\partial}\right) \hookrightarrow H^{1,0}\left(A^{\bullet,\bullet}(\Gamma\backslash G)_{\mathbb{C}}; \partial, \overline{\partial}; \partial\overline{\partial}\right)$. (For the injectivity, see [2, Lemma 3.6], see also [14, Lemma 9].) Then we have the bi-differential $\mathbb{Z}$-graded sub-complex

$$\left(\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) \hookrightarrow \left(A^{\bullet}(\Gamma\backslash G)_{\mathbb{C}},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) .$$

We firstly prove the following result, which generalizes [14, Lemma 9] and [2, Lemma 3.6] to the case of twisted differentials. (Consider also the F. A. Belgun symmetrization trick, [11, Theorem 7], as a different argument.)

Proposition 3.1. *Let $\Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure, and with associated Lie algebra $\mathfrak{g}$. Take $\theta_1, \theta_2 \in H^{1,0}\left(\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*}; \partial, \overline{\partial}; \partial\overline{\partial}\right)$, and $\phi := \theta_1 + \overline{\theta_1} + \theta_2 - \overline{\theta_2}$.*

The maps

$$\begin{aligned}
H^{\bullet}\left(\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*}; \mathrm{d}_{\phi}\right) &\to H^{\bullet}\left(A^{\bullet}(\Gamma\backslash G)_{\mathbb{C}}; \mathrm{d}_{\phi}\right) ,\\
H^{\bullet}\left(\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*}; \partial_{(\theta_1,\theta_2)}\right) &\to H^{\bullet}\left(A^{\bullet}(\Gamma\backslash G)_{\mathbb{C}}; \partial_{(\theta_1,\theta_2)}\right) ,\\
H^{\bullet}\left(\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*}; \overline{\partial}_{(\theta_1,\theta_2)}\right) &\to H^{\bullet}\left(A^{\bullet}(\Gamma\backslash G)_{\mathbb{C}}; \overline{\partial}_{(\theta_1,\theta_2)}\right) ,\\
H^{\bullet}\left(\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*}; \partial_{(\theta_1,\theta_2)}, \overline{\partial}_{(\theta_1,\theta_2)}; \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) &\to H^{\bullet}\left(A^{\bullet}(\Gamma\backslash G)_{\mathbb{C}}; \partial_{(\theta_1,\theta_2)}, \overline{\partial}_{(\theta_1,\theta_2)}; \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}\right) ,\\
H^{\bullet}\left(\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*}; \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}; \partial_{(\theta_1,\theta_2)}, \overline{\partial}_{(\theta_1,\theta_2)}\right) &\to H^{\bullet}\left(A^{\bullet}(\Gamma\backslash G)_{\mathbb{C}}; \partial_{(\theta_1,\theta_2)}\overline{\partial}_{(\theta_1,\theta_2)}; \partial_{(\theta_1,\theta_2)}, \overline{\partial}_{(\theta_1,\theta_2)}\right)
\end{aligned}$$

induced by the inclusion $\left(\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right) \hookrightarrow \left(A^{\bullet}(\Gamma\backslash G)_{\mathbb{C}},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right)$ are injective.

Proof. Consider each case. Fix g a G-left-invariant Hermitian metric on $\Gamma\backslash G$. The metric g and the forms θ_1 and θ_2 being left-invariant, the associated Laplacian operator $\Delta_{\sharp}^{g}$ satisfies $\Delta_{\sharp}^{g}\lfloor_{\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*}} : \wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*} \to \wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*}$. In particular, Hodge theory applies both to $\left(\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right)$ and to $\left(A^{\bullet}(\Gamma\backslash G)_{\mathbb{C}},\, \partial_{(\theta_1,\theta_2)},\, \overline{\partial}_{(\theta_1,\theta_2)}\right)$. Hence we have the commutative diagram

$$\begin{array}{ccc}
\Delta_{\sharp}^{g}\lfloor_{\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*}} & \hookrightarrow & \Delta_{\sharp}^{g} \\
\simeq\downarrow & & \downarrow\simeq \\
H_{\sharp}^{\bullet}\left(\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}^{*}\right) & \longrightarrow & H_{\sharp}^{\bullet}\left(A^{\bullet}(\Gamma\backslash G)_{\mathbb{C}}\right)
\end{array} ,$$

where $H^\bullet_\sharp\left(\wedge^\bullet \mathfrak{g}^\star_{\mathbb{C}}\right)$ and $H^\bullet_\sharp\left(A^\bullet(\Gamma\backslash G)_{\mathbb{C}}\right)$ denote the corresponding cohomologies. It yields the injectivity of the map $H^\bullet_\sharp\left(\wedge^\bullet \mathfrak{g}^\star_{\mathbb{C}}\right) \to H^\bullet_\sharp\left(A^\bullet(\Gamma\backslash G)_{\mathbb{C}}\right)$. Compare [4, Proposition 2.2]. $\square$

Example 3.2. Take $G := \mathbb{C} \ltimes_\phi \mathbb{C}^2$ where

$$\phi(z_1) := \begin{pmatrix} e^{\frac{z_1+\bar z_1}{2}} & 0 \\ 0 & e^{-\frac{z_1+\bar z_1}{2}} \end{pmatrix} \in \mathrm{GL}(\mathbb{C}^2) .$$

Then for some $a \in \mathbb{R}$ the matrix $\begin{pmatrix} e^{a} & 0 \\ 0 & e^{-a} \end{pmatrix}$ is conjugate to an element of $\mathrm{SL}(2;\mathbb{Z})$. Hence, for any $b \in \mathbb{R}\setminus\{0\}$, we have a lattice $\Gamma := \left(a\,\mathbb{Z} + b\sqrt{-1}\,\mathbb{Z}\right) \ltimes \Gamma''$ of G, where Γ'' is a lattice of $\mathbb{C}^2$. The solvmanifold $\Gamma\backslash G$ is called *completely-solvable Nakamura manifold*, [27, page 90]; see also, e.g., [15, §3], [21, Example 1], [4, Example 2.17]. If $b \notin \pi\,\mathbb{Z}$, then $\Gamma\backslash G$ satisfies the $\partial\bar\partial$-Lemma, see [4, Example 2.17] (see also [22]).

Consider local holomorphic coordinates (z_1, z_2, z_3) for $\mathbb{C} \ltimes_\phi \mathbb{C}^2$. We have

$$\wedge^{\bullet,\bullet}\mathfrak{g}^\star_{\mathbb{C}} = \wedge^{\bullet,\bullet}\left(\left\langle \mathrm{d}z_1,\, e^{-\frac{z_1+\bar z_1}{2}}\,\mathrm{d}z_2,\, e^{\frac{z_1+\bar z_1}{2}}\,\mathrm{d}z_3 \right\rangle \otimes \left\langle \mathrm{d}\bar z_1,\, e^{-\frac{z_1+\bar z_1}{2}}\,\mathrm{d}\bar z_2,\, e^{\frac{z_1+\bar z_1}{2}}\,\mathrm{d}\bar z_3 \right\rangle\right) .$$

Take

$$\theta_1 := \frac{\mathrm{d}z_1}{2} \qquad \text{and} \qquad \theta_2 := 0 ,$$

and set

$$\phi := \theta_1 + \overline{\theta_1} + \theta_2 - \overline{\theta_2} = \frac{\mathrm{d}z_1 + \mathrm{d}\bar z_1}{2} .$$

In [23, §8], the second author computed

$$H^\bullet(\Gamma\backslash G\,;\mathrm{d}_\phi) \neq \{0\}$$

and

$$H^\bullet(\Gamma\backslash G\,;\bar\partial_{(\theta_1,\theta_2)}) = \{0\} .$$

Hence $\Gamma\backslash G$ does not admit the (θ_1, θ_2)-Hodge decomposition.

We show now that also the $\partial_{(\theta_1,\theta_2)}\bar\partial_{(\theta_1,\theta_2)}$-Lemma does not hold on $\Gamma\backslash G$. Consider

$$\frac{1}{2}\, e^{\frac{z_1+\bar z_1}{2}}\,(\mathrm{d}z_1 + \mathrm{d}\bar z_1)\wedge \mathrm{d}\bar z_3 \in \wedge^2\mathfrak{g}^\star_{\mathbb{C}} .$$

We have

$$0 \neq \left[\frac{1}{2}\, e^{\frac{z_1+\bar z_1}{2}}\,(\mathrm{d}z_1 + \mathrm{d}\bar z_1)\wedge \mathrm{d}\bar z_3\right] \in H^2\left(\wedge^\bullet\mathfrak{g}^\star_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)},\, \bar\partial_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\bar\partial_{(\theta_1,\theta_2)}\right) .$$

On the other hand, we have

$$\frac{1}{2}\, e^{\frac{z_1+\bar z_1}{2}}\,(\mathrm{d}z_1 + \mathrm{d}\bar z_1)\wedge \mathrm{d}\bar z_3 = \bar\partial_{(\theta_1,\theta_2)}\left(e^{\frac{z_1+\bar z_1}{2}}\,\mathrm{d}\bar z_3\right) .$$

Therefore the map

$$\begin{aligned} H^2\left(\wedge^\bullet\mathfrak{g}^\star_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)},\, \bar\partial_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\bar\partial_{(\theta_1,\theta_2)}\right) &\to H^2\left(A^\bullet(\Gamma\backslash G)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)},\, \bar\partial_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\bar\partial_{(\theta_1,\theta_2)}\right) \\ &\to H^2\left(A^\bullet(\Gamma\backslash G)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)}\bar\partial_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \bar\partial_{(\theta_1,\theta_2)}\right) \end{aligned}$$

is not injective. Since the first map is injective by Proposition 3.1, it follows that the natural map $H^2\left(A^\bullet(\Gamma\backslash G)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)},\, \bar\partial_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)}\bar\partial_{(\theta_1,\theta_2)}\right) \to H^2\left(A^\bullet(\Gamma\backslash G)_{\mathbb{C}};\, \partial_{(\theta_1,\theta_2)}\bar\partial_{(\theta_1,\theta_2)};\, \partial_{(\theta_1,\theta_2)},\, \bar\partial_{(\theta_1,\theta_2)}\right)$ induced by the identity is not injective. It follows that $\Gamma\backslash G$ does not satisfy the $\partial_{(\theta_1,\theta_2)}\bar\partial_{(\theta_1,\theta_2)}$-Lemma.

3.2 Solvmanifolds and $\partial\overline{\partial}$-Lemma with twisted differentials

The Weinstein and Thurston problem, concerning the characterization of nilmanifolds admitting Kähler structures, was solved by Ch. Benson and C. S. Gordon, [12, Theorem A]. In fact, in [19, Theorem 1, Corollary], K. Hasegawa proved that non-tori nilmanifold are not formal in the sense of Sullivan, and hence do not belong to class $\mathcal{C}$ of Fujiki.

As regards the characterization of solvmanifolds admitting Kähler structure, K. Hasegawa proved the following in [20, Main Theorem]. Let X be a compact homogeneous space of solvable Lie group, that is, a compact differentiable manifold on which a connected solvable Lie group acts transitively. Then X admits a Kähler structure if and only if it is a finite quotient of a complex torus which has a structure of a complex torus-bundle over a complex torus. In particular, a completely-solvable solvmanifold has a Kähler structure if and only if it is a complex torus.

As regards a characterization of solvmanifolds in class $\mathcal{C}$ of Fujiki, it is stated in [7, Theorem 9] by D. Arapura. More precisely, [7, Theorem 3, Theorem 9] say that, for solvmanifolds endowed with complex structures, the properties of admitting Kähler metrics and of belonging to class $\mathcal{C}$ of Fujiki are equivalent. The proof is only sketched, at [7, page 136], and is based on the fact that a finitely-presented group is a Fujiki group if and only if it is a Kähler group, see also [10, Theorem 1.1] by G. Bharali, I. Biswas, and M. Mj. In fact, their result founds on the Hironaka elimination of indeterminacies, [10, §2]. By using the results by the second author in [23] and the above results, we can provide a different and more direct proof, of cohomological flavour.

Theorem 3.3. *Let (M, J) be a solvmanifold endowed with a complex structure. If (M, J) belongs to class $\mathcal{C}$ of Fujiki, then it admits a Kähler metric.*

Proof. Take any $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$. By Corollary 2.7, the manifold (M, J) admits the (θ_1, θ_2)-Hodge decomposition. In [23], the property of satisfying the Hodge-decomposition with respect to any $\theta_1, \theta_2 \in H^{1,0}_{BC}(M)$ is called hyper-strong-Hodge-decomposition. The second author proved in [23, Theorem 1.7] that a solvmanifold admitting hyper-strong-Hodge-decomposition admits a Kähler metric. □

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