A variational Analysis of the Toda System on Compact Surfaces

ANDREA MALCHIODI
SISSA, Trieste, Sector of Functional Analysis and Applications

AND

DAVID RUIZ
University of Granada

Abstract

In this paper we consider the following *Toda system* of equations on a compact surface:

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right), \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right). \end{cases}$$

We will give existence results by using variational methods in a non coercive case. A key tool in our analysis is a new Moser-Trudinger type inequality under suitable conditions on the center of mass and the scale of concentration of the two components u_1, u_2.

1 Introduction

Let Σ be a compact orientable surface without boundary, and g a Riemannian metric on Σ. Consider the following system of equations:

$$-\frac{1}{2}\Delta u_i(x) = \sum_{j=1}^{N} a_{ij} e^{u_j(x)}, \quad x \in \Sigma,\ i = 1, \dots, N, \tag{1.1}$$

where $\Delta = \Delta_g$ stands for the Laplace-Beltrami operator and $A = (a_{ij})_{ij}$ is the *Cartan matrix* of $SU(N+1)$,

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix}.$$

Equation (1.1) is known as the *Toda system*, and has been extensively studied in the literature. This problem has a close relationship with geometry, since it can be

Communications on Pure and Applied Mathematics, Vol. 000, 0001–0038 (2000)

seen as the Frenet frame of holomorphic curves in $\mathbb{CP}^N$ (see [11]). Moreover, it arises in the study of the non-abelian Chern-Simons theory in the self-dual case, when a scalar Higgs field is coupled to a gauge potential, see [10, 28, 31].

Let us assume, for the sake of simplicity, that Σ has total area equal to 1, i.e. $\int_\Sigma 1\, dV_g = 1$. In this paper we study the following version of the Toda system for $N = 2$:

$$\left\{ \begin{array}{l} -\Delta u_1 = 2\rho_1 \left(h_1 e^{u_1} - 1\right) - \rho_2 \left(h_2 e^{u_2} - 1\right), \\ -\Delta u_2 = 2\rho_2 \left(h_2 e^{u_2} - 1\right) - \rho_1 \left(h_1 e^{u_1} - 1\right), \end{array} \right. \tag{1.2}$$

where h_i are smooth and strictly positive functions defined on Σ. By integrating on Σ both equations, we obtain that any solution (u_1, u_2) of (1.2) satisfies:

$$\int_\Sigma h_i e^{u_i}\, dV_g = 1, \quad i = 1,\, 2.$$

Hence, problem (1.2) is equivalent to:

$$\left\{ \begin{array}{l} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right), \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right). \end{array} \right. \tag{1.3}$$

Problem (1.3) is variational, and solutions can be found as critical points of a functional $J_\rho : H^1(\Sigma) \times H^1(\Sigma) \to \mathbb{R}$ ($\rho = (\rho_1, \rho_2)$) given by

$$J_\rho(u_1, u_2) = \int_\Sigma Q(u_1, u_2)\, dV_g + \sum_{i=1}^{2} \rho_i \left(\int_\Sigma u_i dV_g - \log \int_\Sigma h_i e^{u_i} dV_g \right), \tag{1.4}$$

where $Q(u_1, u_2)$ is defined as:

$$Q(u_1, u_2) = \frac{1}{3} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \cdot \nabla u_2 \right). \tag{1.5}$$

Here and throughout the paper $\nabla u = \nabla_g u$ stands for the gradient of u with respect to the metric g, whereas $\cdot$ denotes the Riemannian scalar product.

Observe that both (1.3) and (1.4) are invariant under addition of constants to u_1, u_2. The structure of the functional J_ρ strongly depends on the parameters ρ_1, ρ_2. To start with, the following analogue of the Moser-Trudinger inequality has been given in [16]:

$$4\pi \sum_{i=1}^{2} \left(\log \int_\Sigma h_i e^{u_i} dV_g - \int_\Sigma u_i dV_g \right) \leq \int_\Sigma Q(u_1, u_2)\, dV_g + C, \tag{1.6}$$

for some $C = C(\Sigma)$. As a consequence, J_ρ is bounded from below for $\rho_i \leq 4\pi$ (see also [5, 25, 30] for related inequalities). In particular, if $\rho_i < 4\pi$ ($i = 1, 2$), J_ρ is coercive and a solution for (1.3) can be easily found as a minimizer.

If $\rho_i > 4\pi$ for some $i = 1,\, 2$, then J_ρ is unbounded from below and a minimization technique is no more possible. Let us point out that the Leray-Schauder degree associated to (1.3) is not known yet. Instead, for the scalar case (see (1.7) below) the Leray-Schauder has been computed in [4]. To our knowledge, the only result on the topological degree for Liouville systems is [18], but our case is not covered there. In this paper we use variational methods to obtain existence of critical points (generally of saddle type) for J_ρ.

Before stating our results, let us comment briefly on some aspects of the problem under consideration. When some of the parameters ρ_i equals 4π, the situation becomes more subtle. For instance, if we fix $\rho_1 < 4\pi$ and let $\rho_2 \nearrow 4\pi$, then u_2 could exhibit a blow-up behavior (see the proof of Theorem 1.1 in [14]). In this case, u_2 would become close to a function $U_{\lambda,x}$ defined as:

$$U_{\lambda,x}(y) = \log\left(\frac{4\lambda}{(1+\lambda\, d(x,y)^2)^2}\right),$$

where $y \in \Sigma$, $d(x,y)$ stands for the geodesic distance and λ is a large parameter. Those functions $U_{\lambda,x}$ are the unique entire solutions of the Liouville equation (see [3]):

$$-\Delta U = 2e^U, \quad \int_{\mathbb{R}^2} e^U\, dx < +\infty.$$

In [14] and [17] some conditions for existence are given when some of the ρ_i's equals 4π. The proofs involve a delicate analysis of the limit behavior of the solutions when ρ_i converge to 4π from below, in order to avoid bubbling of solutions. For that, some conditions on the functions h_i are needed.

The scalar counterpart of (1.3) is a Liouville-type problem in the form:

$$-\Delta u = 2\rho\left(\frac{h(x)e^u}{\int_\Sigma h(x)e^u dV_g} - 1\right), \tag{1.7}$$

with $\rho \in \mathbb{R}$. This equation has been very much studied in the literature; there are by now many results regarding existence, compactness of solutions, bubbling behavior, etc. We refer the interested reader to the reviews [20, 29].

Solutions of (1.7) correspond to critical points of the functional $I_\rho : H^1(\Sigma) \to \mathbb{R}$

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g + 2\rho\left(\int_\Sigma u dV_g - \log\int_\Sigma h(x)e^u dV_g\right). \tag{1.8}$$

The classical Moser-Trudinger inequality implies that I_ρ is bounded from below for $\rho \le 4\pi$. For larger values of ρ, variational methods were applied to (1.7) for the first time in [7], [27]. In [9] the Q-curvature prescription problem is addressed in a 4-dimensional compact manifold: however, the arguments of the proof can be easily translated to the Liouville problem (1.7), see [8].

Let us briefly describe the proof of [9] in the case $\rho \in (4\pi, 8\pi)$, for simplicity. In [9] it is shown that, whenever $I_\rho(u_n) \to -\infty$, then (up to a subsequence)

$$\frac{e^{u_n}}{\int_\Sigma e^{u_n} dV_g} \rightharpoonup \delta_x, \; x \in \Sigma,$$

in the sense of measures. Moreover, for $L > 0$ sufficiently large, one can define a homotopy equivalence (see also [21]):

$$I_\rho^{-L} = \{u \in H^1(\Sigma) : \; I_\rho(u) < -L\} \simeq \{\delta_x : \; x \in \Sigma\} \simeq \Sigma.$$

Therefore the sublevel I_ρ^{-L} is not contractible, and this allows us to use a min-max argument to find a solution. We point out that [9] also deals with the case of higher values of ρ, whenever $\rho \notin 4\pi\mathbb{N}$.

Coming back to system (1.3), there are very few results when $\rho_i > 4\pi$ for some $i = 1, 2$. One of them is given in [22] and concerns the case $\rho_1 < 4\pi$ and $\rho_2 \in (4\pi m, 4\pi(m+1))$, $m \in \mathbb{N}$. There, the situation is similar to [9]; in a certain sense, one can describe the set J_ρ^{-L} from the behavior of the second component u_2 as in [9].

In Theorem 1.4 of [14], an existence result is stated for $\rho_i \in (0, 4\pi) \cup (4\pi, 8\pi)$ on a compact surface Σ with positive genus: however, the min-max argument used in the proof seems not to be correct. The main problem is that a one-dimensional linking argument is used to obtain conditions on both the components of the system. In any case, the core of [14] is the blow-up analysis for the Toda system (see Remark 3.13 for more details). In particular, it is shown that if the ρ_i's are bounded away from $4\pi\mathbb{N}$, the set of solutions of (1.3) is compact (up to addition of constants). This is an essential tool for our analysis.

In this paper we deal with the case $\rho_i \in (4\pi, 8\pi)$, $i = 1, 2$. Our main result is the following:

Theorem 1.1. *Assume that $\rho_i \in (4\pi, 8\pi)$ and that h_1, h_2 are two positive C^1 functions on Σ. Then there exists a solution (u_1, u_2) of* (1.3).

Let us point out that we find existence of solutions also if Σ is a sphere. Moreover, our existence result is based on a detailed study of the topological properties of the low sublevels of J_ρ. This study is interesting in itself; in the scalar case an analogous study has been used to deduce generic multiplicity results (see [6]) and degree computation formulas (see [21]).

We shall see that the low sublevels of J_ρ contain couples in which at least one component is very concentrated around some point of Σ. Moreover, both components can concentrate at two points that could eventually coincide. However, we shall see that, in a certain sense,

(1.9) *if u_1, u_2 concentrate around the same point at the same rate,* then *J_ρ is bounded from below.*

To make this statement rigorous, we need several tools.

The first is a definition of a rate of concentration of a positive function $f \in \Sigma$, normalized in L^1, which is a refinement of the one given in [23]; this will be measured by a positive parameter called $\sigma = \sigma(f)$. In a sense, the smaller is σ, the higher is the rate of concentration of f. Compared to the classical concentration compactness arguments, our function σ has the property of being continuous with respect to the L^1 topology (see Remark 3.5). Second, we also need to define a continuous center of mass when $\sigma \leq \delta$ for some fixed $\delta > 0$: we will denote it by $\beta = \beta(f) \in \Sigma$. When $\sigma \geq \delta$, the function is not concentrated and the center of mass cannot be defined. Hence, we have a map:

$$\psi : H^1(\Sigma) \to \overline{\Sigma}_\delta, \ \psi(u_i) = (\beta(f_i), \sigma(f_i)), \ \text{ where } f_i = \frac{e^{u_i}}{\int_\Sigma e^{u_i}\, dV_g}.$$

Here $\overline{\Sigma}_\delta$ is the topological cone with base Σ, so that we make the identification to a point when $\sigma \geq \delta$.

Third, we need an improvement of the Moser-Trudinger inequality in the following form: if $\psi(f_1) = \psi(f_2)$, then $J_\rho(u_1, u_2)$ is bounded from below. In this sense, (1.9) is made precise. The proof uses local versions of the Moser-Trudinger inequality and applications of it to small balls (via a convenient dilation) and to annuli with small internal radius (via a Kelvin transform).

Roughly speaking, on low sublevels one of the following alternatives hold:

(1) one component concentrates at a point whereas the other does not concentrate ($\sigma_i < \delta \leq \sigma_j$), or
(2) the two components concentrate at different points ($\sigma_i < \delta, \ \beta_1 \neq \beta_2$), or
(3) the two components concentrate at the same point with different rates of concentration ($\sigma_i < \sigma_j < \delta$, $\beta_1 = \beta_2$).

With this at hand, for $L > 0$ large we are able to define a continuous map:

$$J_\rho^{-L} \quad \overset{\psi \oplus \psi}{\longrightarrow} \quad X := (\overline{\Sigma}_\delta \times \overline{\Sigma}_\delta) \setminus \overline{D},$$

where $\overline{D}$ is the diagonal of $\overline{\Sigma}_\delta \times \overline{\Sigma}_\delta$. We can also proceed in the opposite direction: in Section 4 we construct a family of test functions modeled on X on which J_ρ attains arbitrarily low values, see Lemma 4.3 for the precise result. Calling $\phi : X \to J_\rho^{-L}$ the corresponding map, we will prove that the composition

$$X \quad \overset{\phi}{\longrightarrow} \quad J_\rho^{-L} \quad \overset{\psi \oplus \psi}{\longrightarrow} \quad X \tag{1.10}$$

is homotopically equivalent to the identity map. In this situation it is said that J_ρ^{-L} *dominates* X (see [12], page 528). In a certain sense, those maps are natural since they describe properly the topological properties of J_ρ^{-L}.

We will see that for any compact orientable surface Σ, X is non-contractible; this is proved by estimating its cohomology groups. As a consequence, $\phi(X)$ is not contractible in J_ρ^{-L}. This allows us to use a min-max argument to find a critical point of J_ρ. Here, the compactness of solutions proved in [14] is an essential tool, since the Palais-Smale property for J_ρ is an open problem (as it is for the scalar case).

The rest of the paper is organized as follows. In Section 2 we present the notations that will be used in the paper, as well as some preliminary results. The definition of the map ψ, its properties, and the improvement of the Moser-Trudinger inequality will be exposed in Section 3. In Section 4 we define the map ϕ and prove that the composition (1.10) is homotopic to the identity. Here we also develop the min-max scheme that gives a critical point of J_ρ. The fact that X is not contractible is proved in a final Appendix.

2 Notations and preliminaries

In this section we collect some useful notation and preliminary facts. Throughout the paper, Σ is a compact orientable surface without boundary; for simplicity, we assume $|\Sigma| = \int_\Sigma 1 dV_g = 1$. Given $\delta > 0$, we define the topological cone:

$$\overline{\Sigma}_\delta = (\Sigma \times (0, +\infty)) \,|_{(\Sigma \times [\delta, +\infty))}. \tag{2.1}$$

For $x, y \in \Sigma$ we denote by $d(x,y)$ the metric distance between x and y on Σ. In the same way, for any $p \in \Sigma$, $\Omega, \Omega' \subseteq \Sigma$, we denote:

$$d(p,\Omega) = \inf\{d(p,x) \ : x \in \Omega\}, \quad d(\Omega,\Omega') = \inf\left\{d(x,y) \ : \ x \in \Omega, \ y \in \Omega'\right\}.$$

Moreover, the symbol $B_p(r)$ stands for the open metric ball of radius r and center p, and $A_p(r,R)$ the open annulus of radii r and R, $r < R$. The complement of a set Ω in Σ will be denoted by Ω^c.

Given a function $u \in L^1(\Sigma)$ and $\Omega \subset \Sigma$, we consider the average of u on Ω:

$$\fint_\Omega u \, dV_g = \frac{1}{|\Omega|} \int_\Omega u \, dV_g.$$

We denote by $\overline{u}$ the average of u in Σ: since we are assuming $|\Sigma| = 1$, we have

$$\overline{u} = \int_\Sigma u \, dV_g = \fint_\Sigma u \, dV_g.$$

Throughout the paper we will denote by C large constants which are allowed to vary among different formulas or even within lines. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C, as C_δ, etc.. Also constants with subscripts are allowed to vary. Moreover, sometimes we will write $o_\alpha(1)$ to denote quantities that tend to 0 as $\alpha \to 0$ or $\alpha \to +\infty$, depending on the case. We will similarly use the symbol $O_\alpha(1)$ for bounded quantities.

We begin by recalling the following compactness result from [14].

Theorem 2.1. *([14]) Let m_1, m_2 be two non-negative integers, and suppose Λ_1, Λ_2 are two compact sets of the intervals $(4\pi m_1, 4\pi(m_1+1))$ and $(4\pi m_2, 4\pi(m_2+1))$ respectively. Then if $\rho_1 \in \Lambda_1$ and $\rho_2 \in \Lambda_2$ and if we impose $\int_\Sigma u_i dV_g = 0$, $i = 1, 2$, the solutions of (1.3) stay uniformly bounded in $L^\infty(\Sigma)$ (actually in every $C^l(\Sigma)$ with $l \in \mathbb{N}$).*

Next, we also recall some Moser-Trudinger type inequalities. As commented in the introduction, problem (1.3) is the Euler-Lagrange equation of the energy functional J_ρ given in (1.4). This functional is bounded below only for certain values of ρ_1, ρ_2, as has been proved by Jost and Wang (see also (1.6)):

Theorem 2.2. *([16]) The functional J_ρ is bounded from below if and only if $\rho_i \leq 4\pi$, $i = 1, 2$.*

The next proposition can be thought of as a local version of Theorem 2.2, and will be of use in Section 3. Let us recall the definition of the quadratic form Q in (1.5).

Proposition 2.3. *Fix $\delta > 0$, and let $\Omega_1 \subset \Omega_2 \subset \Sigma$ be such that $d(\Omega_1, \partial\Omega_2) \geq \delta$. Then, for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, \delta)$ such that for all $u \in H^1(\Sigma)$*

$$\begin{aligned} 4\pi \left(\log \int_{\Omega_1} e^{u_1} dV_g + \log \int_{\Omega_1} e^{u_2} dV_g - \fint_{\Omega_2} u_1 dV_g - \fint_{\Omega_2} u_2 dV_g \right) \\ \leq (1+\varepsilon) \int_{\Omega_2} Q(u_1, u_2) dV_g + C. \end{aligned} \tag{2.2}$$

Proof. We can assume without loss of generality that $\fint_{\Omega_2} u_i dV_g = 0$ for $i = 1, 2$. Let us write

$$u_i = v_i + w_i, \qquad \int_{\Omega_2} v_i \, dV_g = \int_{\Omega_2} w_i \, dV_g = 0,$$

where $v_i \in L^\infty(\Omega_2)$ and $w_i \in H^1(\Omega_2)$ will be fixed later. We have

$$\begin{aligned} \log \int_{\Omega_1} e^{u_1} dV_g + \log \int_{\Omega_1} e^{u_2} dV_g \leq \|v_1\|_{L^\infty(\Omega_1)} + \|v_2\|_{L^\infty(\Omega_1)} \\ + \log \int_{\Omega_1} e^{w_1} dV_g + \log \int_{\Omega_1} e^{w_2} dV_g. \end{aligned} \tag{2.3}$$

We next consider a smooth cutoff function χ with values into $[0, 1]$ satisfying

$$\begin{cases} \chi(x) = 1 & \text{for } x \in \Omega_1, \\ \chi(x) = 0 & \text{if } d(x, \Omega) > \delta/2, \end{cases}$$

and then define

$$\tilde{w}_i(x) = \chi(x) w_i(x); \quad i = 1, 2.$$

Clearly $\tilde{w}_i$ belongs to $H^1(\Sigma)$ and is supported in a compact set of the interior of Ω_2. Hence we can apply Theorem 2.2 to $\tilde{w}_i$ on Σ, finding

$$\log \int_{\Omega_1} e^{w_1} dV_g + \log \int_{\Omega_1} e^{w_2} dV_g \leq \log \int_\Sigma e^{\tilde{w}_1} dV_g + \log \int_\Sigma e^{\tilde{w}_2} dV_g$$

$$\leq \frac{1}{4\pi}\int_\Sigma Q(\tilde{w}_1,\tilde{w}_2)dV_g + \fint_\Sigma (\tilde{w}_1+\tilde{w}_2)\,dV_g + C.$$

Using the Leibnitz rule and Hölder's inequality we obtain

$$\int_\Sigma Q(\tilde{w}_1,\tilde{w}_2)dV_g \leq (1+\varepsilon)\int_{\Omega_2} Q(w_1,w_2)dV_g + C_\varepsilon \int_{\Omega_2}(w_1^2+w_2^2)dV_g.$$

Moreover, we can estimate the mean value of $\tilde{w}_i$ in the following way:

$$\fint_\Sigma \tilde{w}_i\,dV_g \leq C\left(\int_\Sigma |\nabla \tilde{w}_i|^2\,dV_g\right)^{1/2} \leq C_\varepsilon + \varepsilon \int_{\Omega_2}|\nabla\tilde{w}_i|^2\,dV_g$$
$$\leq C_\varepsilon + C\varepsilon\left(\int_{\Omega_2}|\nabla w_i|^2\,dV_g + C\int_{\Omega_2} w_i^2\,dV_g\right).$$

From (2.3) and the last formulas we find

$$\log\int_{\Omega_1} e^{u_1}dV_g + \log\int_{\Omega_1} e^{u_2}dV_g \leq \|v_1\|_{L^\infty(\Omega_1)} + \|v_2\|_{L^\infty(\Omega_1)}$$
$$+\frac{1+\varepsilon}{4\pi}\int_{\Omega_2} Q(w_1,w_2)dV_g + C_\varepsilon\int_{\Omega_2}(w_1^2+w_2^2)dV_g + C. \tag{2.4}$$

To control the latter terms we use truncations in Fourier modes. Define V_ε to be the direct sum of the eigenspaces of the Laplacian on Ω_2 (with Neumann boundary conditions) with eigenvalues less or equal than $C_\varepsilon \varepsilon^{-1}$. Take now v_i to be the orthogonal projection of u_i onto V_ε. In V_ε the L^∞ norm is equivalent to the L^2 norm: by using Poincaré's inequality we get

$$C_\varepsilon\int_{\Omega_2}(w_1^2+w_2^2)dV_g \leq \varepsilon\int_{\Omega_2} Q(u_1,u_2)dV_g,$$

$$\|v_i\|_{L^\infty(\Omega_1)} \leq C_\varepsilon\|v_i\|_{L^2(\Omega_2)} \leq C_\varepsilon\left(\int_{\Omega_2}|\nabla u_i|^2 dV_g\right)^{\frac{1}{2}} \leq \varepsilon\int_{\Omega_2} Q(u_1,u_2)dV_g + C_\varepsilon.$$

Hence, from (2.4) and the above inequalities we derive (2.2) by renaming ε properly.

□

Remark 2.4. While the Fourier decomposition used in the above proof depends on Ω_2, the constants only depend on Σ, δ and ε. In fact, one can replace Ω_2 by a domain $\check{\Omega}_2$, $\Omega_2 \subseteq \check{\Omega}_2 \subseteq B_{\Omega_2}(\delta/2)$ with boundary curvature depending only on δ and satisfying a uniform interior sphere condition with spheres of radius δ^3. For example, one can obtain such a domain $\check{\Omega}_2$ triangulating Σ by simplexes with diameters of order δ^2, take suitable union of triangles and smoothing the corners. For these domains, which are finitely many, the eigenvalue estimates will only depend on δ.

We next prove a criterion which gives us a first insight on the properties of the low sublevels of J_ρ. This result is in the spirit of an improved inequality in [2], and we use an extra covering argument to track the concentration properties of both components of the system. We need first an auxiliary lemma.

Lemma 2.5. *Let $\delta_0 > 0$, $\gamma_0 > 0$, and let $\Omega_{i,j} \subseteq \Sigma$, $i,j = 1,2$, satisfy $d(\Omega_{i,j}, \Omega_{i,k}) \geq \delta_0$ for $j \neq k$. Suppose that $u_1, u_2 \in H^1(\Sigma)$ are two functions verifying*

$$\frac{\int_{\Omega_{i,j}} e^{u_i} dV_g}{\int_\Sigma e^{u_i} dV_g} \geq \gamma_0, \quad i,j = 1,2. \tag{2.5}$$

Then there exist positive constants $\tilde{\gamma}_0$, $\tilde{\delta}_0$, depending only on γ_0, δ_0, and two sets $\tilde{\Omega}_1, \tilde{\Omega}_2 \subseteq \Sigma$, depending also on u_1, u_2 such that

$$d(\tilde{\Omega}_1, \tilde{\Omega}_2) \geq \tilde{\delta}_0; \quad \frac{\int_{\tilde{\Omega}_i} e^{u_1} dV_g}{\int_\Sigma e^{u_1} dV_g} \geq \tilde{\gamma}_0, \quad \frac{\int_{\tilde{\Omega}_i} e^{u_2} dV_g}{\int_\Sigma e^{u_2} dV_g} \geq \tilde{\gamma}_0; \quad i = 1,2. \tag{2.6}$$

Proof. First, we fix a number $r_0 < \frac{\delta_0}{80}$. Then we cover Σ with a finite union of metric balls $(B_{x_l}(r_0))_l$, whose number can be bounded by an integer N_{r_0} which depends only on r_0 (and Σ).

Next we cover $\overline{\Omega}_{i,j}$ by a finite number of these balls, and we choose $y_{i,j} \in \cup_l (x_l)$ such that

$$\int_{B_{y_{i,j}}(r_0)} e^{u_i} dV_g = \max \left\{ \int_{B_{x_l}(r_0)} e^{u_i} dV_g \ : \ B_{x_l}(r_0) \cap \overline{\Omega}_{i,j} \neq \emptyset \right\}.$$

Since the total number of balls is bounded by N_{r_0} and since by our assumption the (normalized) integral of e^{u_i} over $\Omega_{i,j}$ is greater or equal than γ_0, it follows that

$$\frac{\int_{B_{y_{i,j}}(r_0)} e^{u_i} dV_g}{\int_\Sigma e^{u_i} dV_g} \geq \frac{\gamma_0}{N_{r_0}}. \tag{2.7}$$

By the properties of the sets $\Omega_{i,j}$, we have that:

$$B_{y_{i,j}}(2r_0) \cap B_{y_{i,k}}(r_0) = \emptyset \qquad \text{for } j \neq k.$$

Now, one of the following two possibilities occurs:

(A): $B_{y_{1,1}}(5r_0) \cap \left(B_{y_{2,1}}(5r_0) \cup B_{y_{2,2}}(5r_0)\right) \neq \emptyset$ or
$B_{y_{1,2}}(5r_0) \cap \left(B_{y_{2,1}}(5r_0) \cup B_{y_{2,2}}(5r_0)\right) \neq \emptyset$;

(B): $B_{y_{1,1}}(5r_0) \cap \left(B_{y_{2,1}}(5r_0) \cup B_{y_{2,2}}(5r_0)\right) = \emptyset$ and
$B_{y_{1,2}}(5r_0) \cap \left(B_{y_{2,1}}(5r_0) \cup B_{y_{2,2}}(5r_0)\right) = \emptyset$.

In case **(a)** we define the sets $\tilde{\Omega}_i$ as

$$\tilde{\Omega}_1 = B_{y_{1,1}}(30r_0), \quad \tilde{\Omega}_2 = B_{y_{1,1}}(40r_0)^c,$$

while in case **(b)** we define

$$\tilde{\Omega}_1 = B_{y_{1,1}}(r_0) \cup B_{y_{2,1}}(r_0); \quad \tilde{\Omega}_2 = B_{y_{1,2}}(r_0) \cup B_{y_{2,2}}(r_0)).$$

We also set $\tilde{\gamma}_0 = \frac{\gamma_0}{N_{r_0}}$ and $\tilde{\delta}_0 = r_0$. We notice that $\tilde{\gamma}_0$ and $\tilde{\delta}_0$ depend only on γ_0 and δ_0, as claimed, and that the sets $\tilde{\Omega}_i$ satisfy the required conditions. This concludes the proof of the lemma.

□

We next derive the improvement of the constants in Theorem 2.2, in the spirit of [2].

Proposition 2.6. *Let $u_1, u_2 \in H^1(\Sigma)$ be a couple of functions satisfying the assumptions of Lemma 2.5 for some positive constants δ_0, γ_0. Then for any $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$, depending on ε, δ_0, and γ_0 such that*

$$8\pi \left(\log \int_\Sigma e^{u_1 - \bar{u}_1} dV_g + \log \int_\Sigma e^{u_2 - \bar{u}_2} dV_g \right) \leq (1+\varepsilon) \int_\Sigma Q(u_1,u_2) dV_g + C.$$

Proof. Let $\tilde{\delta}_0, \tilde{\gamma}_0$ and $\tilde{\Omega}_1, \tilde{\Omega}_2$ be as in Lemma 2.5, and assume without loss of generality that $\bar{u}_1 = \bar{u}_2 = 0$. Let us define $U_i = \{x \in \Omega : \ d(x, \tilde{\Omega}_i) < \tilde{\delta}_0/2\}$. By applying Proposition 2.3, we get:

$$\begin{aligned} 4\pi \left(\log \int_{\tilde{\Omega}_i} e^{u_1} dV_g + \log \int_{\tilde{\Omega}_i} e^{u_2} dV_g - \fint_{U_i} (u_1+u_2) dV_g \right) \\ \leq (1+\varepsilon) \int_{U_i} Q(u_1,u_2) dV_g + C. \end{aligned} \tag{2.8}$$

Observe that:

$$\log \int_{\tilde{\Omega}_i} e^{u_j} dV_g \geq \log \left(\int_\Sigma e^{u_j} dV_g \right) + \log \tilde{\gamma}_0.$$

Since $U_1 \cap U_2 = \emptyset$, we can sum (2.8) for $i = 1,2$, to obtain

$$\begin{aligned} 8\pi \left(\log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g - \sum_{i=1}^{2} \fint_{U_i} (u_1+u_2) dV_g \right) \\ \leq (1+\varepsilon) \int_\Sigma Q(u_1,u_2) dV_g + C. \end{aligned}$$

It suffices now to estimate the term $\fint_{U_i} (u_1+u_2) dV_g$. By using Poincaré's inequality and the estimate $|U_i| \geq \tilde{\delta}_0^2$, we have:

$$\fint_{U_i} u_j \, dV_g \leq \tilde{\delta}_0^{-2} \int_{U_i} u_j \, dV_g \leq C \left(\int_\Sigma |\nabla u_j|^2 \, dV_g \right)^{1/2} \leq C + \varepsilon \int_\Sigma |\nabla u_j|^2 \, dV_g.$$

To finish the proof it suffices to properly rename ε.

□

Proposition 2.6 implies that on low sublevels, at least one of the components must be very concentrated around a certain point. A more precise description of the topological properties of J_ρ^{-L} will be given later on.

3 Volume concentration and improved inequality

In this section we give the main tools for the description of the sublevels of the energy functional J_ρ. Those will be contained in Propositions 3.1 and 3.2, whose proof will be given in the subsequent subsections.

First, we give continuous definitions of center of mass and scale of concentration of positive functions normalized in L^1, which are adequate for our purposes. Those are a refinement of [23].

Consider the set

$$A = \left\{ f \in L^1(\Sigma) \, : \, f > 0 \text{ a. e. and } \int_\Sigma f dV_g = 1 \right\},$$

endowed with the topology inherited from $L^1(\Sigma)$.

Moreover, let us recall the definition (2.1) for the cone $\overline{\Sigma}_\delta$.

Proposition 3.1. *Let us fix a constant $R > 1$. Then there exists $\delta = \delta(R) > 0$ and a continuous map:*

$$\psi : A \to \overline{\Sigma}_\delta, \quad \psi(f) = (\beta, \sigma),$$

satisfying the following property: for any $f \in A$ there exists $p \in \Sigma$ such that

a) $d(p, \beta) \le C' \sigma$ *for* $C' = \max\{3R+1, \delta^{-1} diam(\Sigma)\}$.

b) *There holds:*

$$\int_{B_p(\sigma)} f \, dV_g > \tau, \quad \int_{B_p(R\sigma)^c} f \, dV_g > \tau,$$

where $\tau > 0$ depends only on R and Σ.

Roughly speaking, the above map $\psi(f) = (\beta, \sigma)$ gives us a center of mass of f and its scale of concentration around that point. Indeed, the smaller is σ, the bigger is the rate of concentration. Moreover, if σ exceeds a certain positive constant, β could not be defined; so, it is natural to make the identification in $\overline{\Sigma}_\delta$.

Next, we state an improved Moser-Trudinger inequality for couples (u_1, u_2) such that e^{u_i} are centered at the same point with the same rate of concentration. Being more specific, we have the following:

Proposition 3.2. *Given any $\varepsilon > 0$, there exist $R = R(\varepsilon) > 1$ and ψ as given in Proposition 3.1, such that for any $(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ with:*

$$\psi\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1} dV_g} \right) = \psi\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2} dV_g} \right),$$

the following inequality holds:

$$(1+\varepsilon)\int_\Sigma Q(u_1,u_2)\,dV_g \geq 8\pi\left(\log\int_\Sigma e^{u_1-\overline{u}_1}dV_g + \log\int_\Sigma e^{u_2-\overline{u}_2}dV_g\right)+C,$$

for some $C = C(\varepsilon)$.

The rest of the section is devoted to the proof of those propositions.

3.1 Proof of Proposition 3.1

Take $R_0 = 3R$, and define $\sigma : \Sigma \times A \to (0,+\infty)$ such that:

$$\int_{B_x(\sigma(x,f))} f\,dV_g = \int_{B_x(R_0\sigma(x,f))^c} f\,dV_g. \tag{3.1}$$

It is easy to check that $\sigma(x,f)$ is uniquely determined and continuous. Moreover, σ satisfies:

$$d(x,y) \leq R_0 \max\{\sigma(x,f),\sigma(y,f)\} + \min\{\sigma(x,f),\sigma(y,f)\}. \tag{3.2}$$

Otherwise, $B_x(R_0\sigma(x,f)) \cap B_y(\sigma(y,f)+\varepsilon) \neq \emptyset$ for some $\varepsilon > 0$. Moreover, since $B_y(R_0\sigma(y,f))$ does not fulfil the whole space Σ, $A_y(\sigma(y,f),\sigma(y,f)+\varepsilon)$ is a nonempty open set. Then:

$$\int_{B_x(\sigma(x,f))} f\,dV_g = \int_{B_x(R_0\sigma(x,f))^c} f\,dV_g \geq \int_{B_y(\sigma(y,f)+\varepsilon)} f\,dV_g > \int_{B_y(\sigma(y,f))} f\,dV_g.$$

By interchanging the roles of x and y, we would also obtain the reverse inequality. This contradiction proves (3.2).

We now define:

$$T : A\times\Sigma \to \mathbb{R}, \quad T(x,f) = \int_{B_x(\sigma(x,f))} f dV_g.$$

Lemma 3.3. *If $x_0 \in \Sigma$ is such that $T(x_0,f) = \max_{y\in\Sigma} T(y,f)$, then $\sigma(x_0,f) < 3\sigma(x,f)$ for any other $x \in \Sigma$.*

Proof. Choose any $x \in \Sigma$ and $\varepsilon > 0$. First, observe that $B_x(R_0\sigma(x,f)+\varepsilon)$ must intersect $B_{x_0}(\sigma(x_0,f))$. Otherwise, reasoning as above, we would obtain that the annulus $A_x(R_0\sigma(x,f),R_0\sigma(x,f)+\varepsilon)$ is an open nonempty set. Then

$$T(x_0,f) = \int_{B_{x_0}(\sigma(x_0,f))} f\,dV_g < \int_{B_x(R_0\sigma(x,f))^c} f\,dV_g = T(x,f),$$

a contradiction. Arguing in the same way, we can conclude that $B_{x_0}(R_0\sigma(x_0,f))$ cannot contain the ball $B_x(R_0\sigma(x,f)+\varepsilon)$.

By the triangular inequality, we obtain that:

$$2(R_0\sigma(x,f)+\varepsilon) > (R_0-1)\sigma(x_0,f).$$

Since $\varepsilon > 0$ is arbitrary, there follows:

$$\sigma(x,f) \geq \frac{R_0 - 1}{2R_0}\sigma(x_0,f).$$

Recalling that $R_0 > 3$, we are done.

□

As a consequence of the previous lemma, we obtain the following:

Lemma 3.4. *There exists a fixed $\tau > 0$ such that*

$$\max_{x\in\Sigma} T(x,f) > \tau > 0 \quad \textit{for all } f \in A.$$

Proof. Let us fix $x_0 \in \Sigma$ such that $T(x_0,f) = \max_{x\in\Sigma} T(x,f)$. By Lemma 3.3 we have that, for any $x \in A_{x_0}(\sigma(x_0,f), R\sigma(x_0,f))$:

$$\int_{B_x(\sigma(x_0,f)/3)} f\, dV_g \leq \int_{B_x(\sigma(x,f))} f\, dV_g \leq T(x_0,f).$$

Let us take a finite covering:

$$A_{x_0}(\sigma(x_0,f), R\sigma(x_0,f)) \subset \cup_{i=1}^k B_{x_i}(\sigma(x_0,f)/3).$$

Observe that k is independent of f or $\sigma(x_0,f)$, and depends only on Σ and R. Therefore:

$$1 = \int_\Sigma f\, dV_g \leq \int_{B_{x_0}(\sigma(x_0,f))} f\, dV_g + \int_{B_{x_0}(R\sigma(x_0,f))^c} f\, dV_g + \sum_{i=1}^k \int_{B_{x_i}(\sigma(x_0,f)/3)} f\, dV_g$$

$$\leq (k+2)T(x_0,f).$$

□

Let us define:

$$\sigma : A \to \mathbb{R}, \quad \sigma(f) = 3\min\{\sigma(x,f) : \ x \in \Sigma\},$$

which is obviously a continuous function.

Remark 3.5. In [23] (see Section 3 there) a sort of concentration parameter is defined, but it does not depend continuously on f. Moreover, the definition of barycenter given below has been modified compared to [23]. Finally, the application ψ is mapped to a cone; this interpretation, which is crucial in our framework, was missing in [23].

Given τ as in Lemma 3.4, consider the set:

$$S(f)=\{x\in\Sigma\,:\,T(x,f)>\tau,\ \sigma(x,f)<\sigma(f)\}. \tag{3.3}$$

If $x_0\in\Sigma$ is such that $T(x_0,f)=\max_{x\in\Sigma}T(x,f)$, then Lemmas 3.3 and 3.4 imply that $x_0\in S(f)$. Therefore, $S(f)$ is a nonempty open set for any $f\in A$. Moreover, from (3.2), we have that:

$$diam(S(f))\leq(R_0+1)\sigma(f). \tag{3.4}$$

By the Nash embedding theorem, we can assume that $\Sigma\subset\mathbb{R}^N$ isometrically, $N\in\mathbb{N}$. Take an open tubular neighborhood $\Sigma\subset U\subset\mathbb{R}^N$ of Σ, and $\delta>0$ small enough so that:

$$co\,[B_x((R_0+1)\delta)\cap\Sigma]\subset U\ \forall x\in\Sigma, \tag{3.5}$$

where co denotes the convex hull in $\mathbb{R}^N$.

We define now

$$\eta(f)=\frac{\displaystyle\int_\Sigma(T(x,f)-\tau)^+\,(\sigma(f)-\sigma(x,f))^+\,x\,dV_g}{\displaystyle\int_\Sigma(T(x,f)-\tau)^+\,(\sigma(f)-\sigma(x,f))^+\,dV_g}\in\mathbb{R}^N.$$

The map η yields a sort of center of mass in $\mathbb{R}^N$. Observe that the integrands become nonzero only on the set $S(f)$. However, whenever $\sigma(f)\leq\delta$, (3.4) and (3.5) imply that $\eta(f)\in U$, and so we can define:

$$\beta:\{f\in A:\ \sigma(f)\leq\delta\}\to\Sigma,\ \ \beta(f)=P\circ\eta(f),$$

where $P:U\to\Sigma$ is the orthogonal projection.

Now, let us check that $\psi(f)=(\beta(f),\sigma(f))$ satisfies the conditions given by Proposition 3.1. If $\sigma(f)\leq\delta$, then $\beta(f)\in co[S(f)]\cap\Sigma$. Therefore, $d(\beta(f),S(f))<(R_0+1)\sigma(f)$. Take any $p\in S(f)$. Recall that $R_0=3R$ and that $\sigma(f)\leq3\sigma(x,f)<3\sigma(f)$ for any $x\in S(f)$: it is easy to conclude then *a)* and *b)*.

If $\sigma(f)\geq\delta$, β is not defined. Observe that *a)* is then satisfied for any $\beta\in\Sigma$.

3.2 Proof of Proposition 3.2

First of all, we will need the following technical lemma:

Lemma 3.6. *There exists $C>0$ such that for any $x\in\Sigma$, $d>0$ small,*

$$\left|\fint_{B_x(d)}u\,dV_g-\fint_{\partial B_x(d)}u\,dS_g\right|\leq C\left(\int_{B_x(d)}|\nabla u|^2\,dV_g\right)^{1/2}.$$

Moreover, given $r\in(0,1)$, there exists $C=C(r,\Sigma)>0$ such that for any x_1, $x_2\in\Sigma$, $d>0$ with $B_1=B_{x_1}(rd)\subset B_2=B_{x_2}(d)$, then:

$$\left| \fint_{B_1} u \, dV_g - \fint_{B_2} u \, dV_g \right| \leq C \left(\int_{B_2} |\nabla u|^2 \, dV_g \right)^{1/2}.$$

Proof. The existence of such a constant C is given just by the L^1 embedding of H^1 and trace inequalities. Moreover, C is independent of d since both inequalities above are dilation invariant.

□

In view of the statement of Proposition 3.1, we now deduce a Moser-Trudinger type inequality for small balls, and also for annuli with small internal radius. Those inequalities are in the core of the proof of Proposition 3.2, and are contained in the following two lemmas. The first one uses a dilation argument:

Lemma 3.7. *For any $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that*

$$(1+\varepsilon) \int_{B_p(s)} Q(u_1, u_2) \, dV_g + C \geq 4\pi \left(\log \int_{B_p(s/2)} e^{u_1} dV_g + \log \int_{B_p(s/2)} e^{u_2} dV_g \right)$$
$$- \, 4\pi(\bar{u}_1(s) + \bar{u}_2(s) + 4 \log s),$$

for any $u_1, u_2 \in H^1(\Sigma)$, $p \in \Sigma$, $s > 0$ small and for $\bar{u}_i(s) = \fint_{B_p(s)} u_i \, dV_g$.

Proof. For $s > 0$ smaller than the injectivity radius, the result follows easily from Proposition 2.3, for some $C = C(s, \varepsilon)$. Then, we need to prove that the constant C can be taken independent of s as $s \to 0$. Notice that, as $s \to 0$ we consider quantities defined on smaller and smaller geodesic balls $B_q(\varsigma)$ on Σ. Working in normal geodesic coordinates at q, gradients, averages and the volume element will resemble Euclidean ones. If we assume that near q the metric of Σ is flat, we will get negligible error terms which will be omitted for reasons of brevity.

To prove the lemma, we simply make a dilation of the pair (u_1, u_2) of the form:

$$v_i(x) = u_i(sx + p).$$

From easy computations there follows:

$$\int_{B(p,s)} Q(u_1, u_2) \, dV_g = \int_{B(0,1)} Q(v_1, v_2) \, dV_g,$$

$$\bar{u}_i(s) = \fint_{B(0,1)} v_i \, dV_g,$$

$$\int_{B_p(s/2)} e^{u_i} \, dV_g = s^2 \int_{B(0,1/2)} e^{v_i} \, dV_g.$$

Applying Proposition 2.3 to the pair (v_1, v_2), we conclude the proof of the lemma.

□

The next lemma gives us an estimate of the quadratic form Q on annuli by using the Kelvin transform. This transformation is indeed very natural in this framework, see Remark 3.10 for a more detailed discussion.

Lemma 3.8. *Given $\varepsilon > 0$, there exists a fixed $r_0 > 0$ (depending only on Σ and ε) satisfying the following property: for any $r \in (0, r_0)$ fixed, there exists $C = C(r, \varepsilon) > 0$ such that, for any $(u_1, u_2) \in H^1(\Sigma)$ with $u_i = 0$ in $\partial B_p(2r)$,*

$$\int_{A_p(s/2,2r)} Q(u_1,u_2)\, dV_g + \varepsilon \int_{B_p(2r)} Q(u_1,u_2)\, dV_g + C$$
$$\geq 4\pi \left(\log \int_{A_p(s,r)} e^{u_1} dV_g + \log \int_{A_p(s,r)} e^{u_2} dV_g + (\bar{u}_1(s) + \bar{u}_2(s) + 4\log s)(1+\varepsilon) \right),$$
with $p \in \Sigma$, $s \in (0,r)$ and $\bar{u}_i(s) = \fint_{B_p(s)} u_i\, dV_g$.

Proof. As in the proof of Lemma 3.7, we need to show that C is independent of s as $s \to 0$. By taking r_0 small enough, also here the metric becomes close to the Euclidean one. Reasoning as in the proof of Lemma 3.7, we can then assume that the metric is flat around p.

We can define the Kelvin transform:

$$K : A_p(s/2, 2r) \to A_p(s/2, 2r), \quad K(x) = p + rs \frac{x-p}{|x-p|^2}.$$

Observe that K maps the interior boundary of $A_p(s/2, 2r)$ onto the exterior one and viceversa, and fixes the set $\partial B_p(\sqrt{sr})$. Let us define the functions $\hat{u}_i \in H^1(B_p(2r))$ as:

$$\hat{u}_i(x) = \begin{cases} u_i(K(x)) - 4\log|x-p| & \text{if } |x-p| \geq s/2, \\ -4\log(s/2) & \text{if } |x-p| \leq s/2. \end{cases}$$

Our goal is to apply the Moser-Trudinger inequality given by Proposition 2.3 to $(\hat{u}_1, \hat{u}_2)$. In order to do so, let us compute:

$$\int_{A_p(s,r)} e^{\hat{u}_i}\, dV_g = \int_{A_p(s,r)} e^{u_i(K(x))} |x-p|^{-4}\, dV_g = \frac{1}{s^2 r^2} \int_{A_p(s,r)} e^{u_i(x)}\, dV_g, \tag{3.6}$$

since the Jacobian of K is $J(K(x)) = -r^2 s^2 |x-p|^{-4}$.

Moreover, by Lemma 3.6, we have that:

$$\left| \fint_{B_p(2r)} \hat{u}_i\, dV_g - \fint_{\partial B_p(r)} \hat{u}_i\, dS_x \right| \leq C \left(\int_{B_p(2r)} |\nabla \hat{u}_i|^2\, dV_g \right)^{1/2}$$
$$\leq C + \varepsilon \int_{B_p(2r)} |\nabla \hat{u}_i|^2\, dV_g.$$

By using again a change of variables,

$$\fint_{\partial B_p(r)} \hat{u}_i \, dS_x = \fint_{\partial B_p(s)} u_i \, dS_x - 8\pi r \log r.$$

Therefore,

$$\left| \fint_{B_p(2r)} \hat{u}_i \, dV_g - \bar{u}_i(s) \right| \leq C + \varepsilon \int_{B_p(2r)} |\nabla \hat{u}_i|^2 \, dV_g + \varepsilon \int_{B_p(s)} |\nabla u_i|^2 \, dV_g. \tag{3.7}$$

Let us now estimate the gradient terms. For $|x-p| \geq s/2$,

$$|\nabla \hat{u}_i(x)|^2 = |\nabla u_i(K(x))|^2 \frac{s^2 r^2}{|x-p|^4} + \frac{16}{|x-p|^2} + 8 \nabla u(K(x)) \cdot \frac{x-p}{|x-p|^4} sr.$$

Therefore,

$$\begin{aligned}
\int_{B_p(2r)} |\nabla \hat{u}_i(x)|^2 \, dV_g &= \int_{A_p(s/2,2r)} |\nabla \hat{u}_i(x)|^2 \, dV_g \\
= \int_{A_p(s/2,2r)} &|\nabla u_i(K(x))|^2 \frac{s^2 r^2}{|x-p|^4} \, dV_g + 16 \int_{A_p(s/2,2r)} \frac{dV_g}{|x-p|^2} \\
&+ 8 \int_{A_p(s/2,2r)} \nabla u_i(K(x)) \cdot \frac{x-p}{|x-p|^4} \, s r \, dV_g \\
&= \int_{A_p(s/2,2r)} |\nabla u_i(x)|^2 \, dV_g + 32\pi \log(4r/s) \\
&+ 8 \int_{A_p(s/2,2r)} \nabla u_i(K(x)) \cdot \frac{K(x)-p}{|K(x)-p|^2} \, \frac{s^2 r^2}{|x-p|^4} \, dV_g \\
= \int_{A_p(s/2,2r)} &|\nabla u_i(x)|^2 \, dV_g + 32\pi \log(4r/s) + 8 \int_{A_p(s/2,2r)} \nabla u_i(x) \cdot \frac{x-p}{|x-p|^2} \, dV_g \\
&= \int_{A_p(s/2,2r)} |\nabla u_i(x)|^2 \, dV_g + 32\pi \log(4r/s) - 16\pi \fint_{\partial B_p(s/2)} u_i \, dS_x.
\end{aligned}$$

In the last equality we have used integration by parts. By using again Lemma 3.6,

$$\begin{aligned}
\left| \int_{B_p(2r)} |\nabla \hat{u}_i(x)|^2 \, dV_g - \int_{A_p(s/2,2r)} |\nabla u_i(x)|^2 \, dV_g + 32\pi \log s + 16\pi \bar{u}_i(s) \right| \\
\leq C + \varepsilon \int_{B_p(s)} |\nabla u_i|^2 \, dV_g.
\end{aligned} \tag{3.8}$$

Regarding the mixed term $\nabla \hat{u}_1 \cdot \nabla \hat{u}_2$, we have that for $|x-p| \geq s/2$,

$$\nabla \hat{u}_1(x) \cdot \nabla \hat{u}_2(x) = \nabla u_1(K(x)) \cdot \nabla u_2(K(x)) \frac{s^2 r^2}{|x-p|^4} + \frac{16}{|x-p|^2}$$

$$+\frac{4sr}{|x-p|^4}(\nabla u_1(K(x))+\nabla u_2(K(x))\cdot(x-p).$$

Reasoning as above, we obtain the estimate:

$$\begin{aligned}(3.9)\qquad &\Big|\int_{B_p(2r)}\nabla\hat{u}_1(x)\cdot\nabla\hat{u}_2(x)\,dV_g-\int_{A_p(s/2,2r)}\nabla u_1(x)\cdot\nabla u_2(x)\,dV_g\\ &+32\pi\log s+8\pi\bar{u}_1(s)+8\pi\bar{u}_2(s)\Big|\leq C+\varepsilon\int_{B_p(s)}\left(|\nabla u_1|^2+|\nabla u_2|^2\right)dV_g.\end{aligned}$$

We now apply Proposition 2.3 to $(\hat{u}_1,\hat{u}_2)$ and use the estimates (3.6), (3.7), (3.8) and (3.9), to obtain:

$$\begin{aligned}&4\pi\left[\log\int_{A_p(s,r)}e^{u_1}\,dV_g+\log\int_{A_p(s,r)}e^{u_2}\,dV_g-(4\log s+\bar{u}_1(s)+\bar{u}_2(s))\right]\\ &\leq 4\pi\left[\log\int_{A_p(s,r)}e^{\hat{u}_1}\,dV_g+\log\int_{A_p(s,r)}e^{\hat{u}_2}\,dV_g-\fint_{B_p(2r)}(\hat{u}_1+\hat{u}_2)\right]\\ &+\varepsilon\int_{B_p(2r)}\left(|\nabla\hat{u}_1|^2+|\nabla\hat{u}_2|^2\right)dV_g+\varepsilon\int_{B_p(s)}\left(|\nabla u_1|^2+|\nabla u_2|^2\right)dV_g+C\\ &\leq(1+C\varepsilon)\int_{B_p(2r)}Q(\hat{u}_1,\hat{u}_2)\,dV_g+\varepsilon\int_{B_p(s)}\left(|\nabla u_1|^2+|\nabla u_2|^2\right)dV_g+C\\ &\leq(1+C\varepsilon)\left[\int_{A_p(s/2,2r)}Q(u_1,u_2)\,dV_g-8\pi(4\log s+\bar{u}_1(s)+\bar{u}_2(s))\right]\\ &\qquad+\varepsilon\int_{B_p(s)}\left(|\nabla u_1|^2+|\nabla u_2|^2\right)dV_g+C.\end{aligned}$$

By renaming ε conveniently, we conclude the proof.

□

Remark 3.9. The term $\bar{u}(s)+2\log s$ has an easy interpretation; by the Jensen inequality we have the estimate

$$\log\int_{B_p(s)}e^u\,dV_g=\log\left(|B_p(s)|\fint_{B_p(s)}e^u\,dV_g\right)\geq\bar{u}(s)+2\log s-C.$$

Remark 3.10. The transformation K is used to exploit the geometric properties of the problem, in order to gain as much control as possible on the exponential terms. From the formulas in [15] one has that both components of the entire solutions of the Toda system in $\mathbb{R}^2$ decay at infinity at the rate $-4\log|x|$. In this way, the Kelvin transform brings these functions to (nearly) constants at the origin, giving a sort of optimization in the Dirichlet part. The minimal value of Dirichlet energy to obtain concentration of volume at a scale s (as in the statement of Lemma 3.8) is then transformed into a boundary integral which cancels exactly the extra terms in Lemma 3.7 due to the s-dilation.

Remark 3.11. Lemmas 3.7 and 3.8, together with Proposition 3.1, give a precise idea of the proof. Indeed, assume that for some $p \in \Sigma$, $\sigma > 0$:

$$\int_{B_p(\sigma)} e^{u_i} dV_g \geq \tau \int_\Sigma e^{u_i} dV_g, \; i = 1,2; \tag{3.10}$$

$$\int_{B_p(R\sigma)^c} e^{u_i} dV_g \geq \tau \int_\Sigma e^{u_i} dV_g, \; i = 1,2. \tag{3.11}$$

If we sum the inequalities given by Lemmas 3.7 and 3.8, the term $\bar{u}_1(\sigma) + \bar{u}_2(\sigma) + 4\log\sigma$ cancels and we deduce the estimate of Proposition 3.2.

The problem is that when $\psi\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1} dV_g}\right) = \psi\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2} dV_g}\right)$ we do not really have (3.10), (3.11) around the same point p. Moreover, u_i needs not be zero on the boundary of a ball, as requested in Proposition 3.8. Some technical work is needed to deal with those difficulties.

We now prove Proposition 3.2. Fixed $\varepsilon > 0$, take $R > 1$ (depending only on ε) and let ψ be the continuous map given by Proposition 3.1. Fix also $\delta > 0$ small (which will depend only on ε, too).

Let u_1 and u_2 be two functions in $H^1(\Sigma)$ with $\int_\Sigma u_i \, dV_g = 0$, such that:

$$\psi\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1} dV_g}\right) = \psi\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2} dV_g}\right) = (\beta, \sigma) \in \overline{\Sigma}_\delta.$$

If $\sigma \geq \frac{\delta}{R^2}$, then Proposition 2.6 yields the result. Therefore, assume $\sigma < \frac{\delta}{R^2}$; Proposition 3.1 implies the existence of $\tau > 0$, $p_1, \; p_2 \in \Sigma$ satisfying:

$$\int_{B_{p_i}(\sigma)} e^{u_i} dV_g \geq \tau \int_\Sigma e^{u_i} dV_g, \; i = 1,2; \tag{3.12}$$

$$\int_{B_{p_i}(R\sigma)^c} e^{u_i} dV_g \geq \tau \int_\Sigma e^{u_i} dV_g \;\; i = 1,2; \tag{3.13}$$

$$d(p_1, p_2) \leq (6R + 2)\sigma;$$

The proof will be divided into two cases:

CASE 1: Assume that:

$$\int_{A_{p_i}(R\sigma, \delta)} e^{u_i} dV_g \geq \tau/2 \int_\Sigma e^{u_i} dV_g. \tag{3.14}$$

In order to be able to apply Lemma 3.8, we need to modify our functions outside a certain ball. Choose $k \in \mathbb{N}$, $k \leq 2\varepsilon^{-1}$, such that:

$$\int_{A_{p_1}(2^{k-1}\delta,2^{k+1}\delta)} \left(|\nabla u_1|^2+|\nabla u_2|^2\right) dV_g \leq \varepsilon \int_{\Sigma} \left(|\nabla u_1|^2+|\nabla u_2|^2\right) dV_g.$$

We define $\tilde{u}_i \in H^1(\Sigma)$ by:

$$\begin{cases} \tilde{u}_i(x) = u_i(x) & x \in B_{p_1}(2^k\delta), \\ \Delta\tilde{u}_i(x) = 0 & x \in A_{p_1}(2^k\delta, 2^{k+1}\delta), \\ \tilde{u}_i(x) = 0 & x \notin B_{p_1}(2^{k+1}\delta). \end{cases}$$

Since we plan to apply Lemma 3.8 to $(\tilde{u}_1,\tilde{u}_2)$, we need to choose δ small enough so that $2^{3\varepsilon^{-1}}\delta < r_0$, where r_0 is given by that Lemma.

It is easy to check, by using Lemma 3.6, that

$$\int_{A_{p_1}(2^k\delta,2^{k+1}\delta)} \left(|\nabla \tilde{u}_1|^2+|\nabla \tilde{u}_2|^2\right) dV_g$$

$$\leq C\int_{A_{p_1}(2^{k-1}\delta,2^{k}\delta)} \left(|\nabla u_1|^2+|\nabla u_2|^2\right) dV_g \leq C\varepsilon \int_{\Sigma} \left(|\nabla u_1|^2+|\nabla u_2|^2\right) dV_g,$$

where C is a universal constant.

Case 1.1: $d(p_1,p_2) \leq R^{\frac{1}{2}}\sigma$.

By applying Lemma 3.7 to u_i for $p=p_1$ and $s = 2(R^{1/2}+1)\sigma$, and taking into account (3.12), we obtain:

$$\frac{1+\varepsilon}{4\pi}\int_{B_p(s)} Q(u_1,u_2)\, dV_g + C$$

$$\geq \log \int_{B_p(s/2)} e^{u_1} dV_g + \log \int_{B_p(s/2)} e^{u_2} dV_g - (\bar{u}_1(\sigma)+\bar{u}_2(\sigma)+4\log\sigma) \tag{3.15}$$

$$\geq \log \int_{\Sigma} e^{u_1} dV_g + \log \int_{\Sigma} e^{u_2} dV_g - (\bar{u}_1(\sigma)+\bar{u}_2(\sigma)+4\log\sigma) - C,$$

where $\bar{u}_i(\sigma) = ⨍_{B_p(\sigma)} u_i\, dV_g$.

We now apply Lemma 3.8 to $\tilde{u}_i$ for $p=p_1$, $s' = 4(R^{1/2}+1)\sigma$ and $r = 2^{k+1}\delta$:

(3.16)

$$\frac{1}{4\pi}\int_{A_p(s'/2,2r)} Q(\tilde{u}_1,\tilde{u}_2)\, dV_g + \varepsilon\int_{\Sigma}\left(|\nabla u_1|^2+|\nabla u_2|^2\right) dV_g + C$$

$$\geq \log \int_{A_p(s',r)} e^{\tilde{u}_1} dV_g + \log \int_{A_p(s',r)} e^{\tilde{u}_2} dV_g + (\bar{u}_1(\sigma)+\bar{u}_2(\sigma)+4\log\sigma)(1+\varepsilon).$$

Taking into account (3.14), we conclude:

$$
\begin{aligned}
&\frac{1}{4\pi}\int_{A_p(s'/2,2r)} Q(\tilde{u}_1,\tilde{u}_2)\, dV_g + \varepsilon \int_\Sigma \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) dV_g + C \\
&\geq \log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g + (\bar{u}_1(\sigma) + \bar{u}_2(\sigma) + 4\log\sigma)(1+\varepsilon).
\end{aligned}
\tag{3.17}
$$

Combining (3.15) and (3.17) we obtain our result (after properly renaming ε).

Case 1.2: $d(p_1,p_2) \geq R^{\frac{1}{2}}\sigma$ and $\int_{B_{p_1}(R^{\frac{1}{3}}\sigma)} e^{u_2} dV_g \geq \tau/4 \int_\Sigma e^{u_2} dV_g$.

Here we argue as in Case 1.1: as a first step, we apply Lemma 3.7 to (u_1,u_2) for $p = p_1$ and $s = 2(R^{1/3}+1)\sigma$. Then, we use Lemma 3.8 with $(\tilde{u}_1,\tilde{u}_2)$ for $p = p_1$, $s' = 4(R^{1/3}+1)\sigma$ and $r = 2^{k+1}\delta$.

Case 1.3: $d(p_1,p_2) \geq R^{\frac{1}{2}}\sigma$ and $\int_{B_{p_2}(R^{\frac{1}{3}}\sigma)} e^{u_1} dV_g \geq \tau/4 \int_\Sigma e^{u_1} dV_g$.

This case can be treated as in Case 1.2, by just interchanging the indices 1 and 2.

Case 1.4: $d(p_1,p_2) \geq R^{\frac{1}{2}}\sigma$, $\int_{B_{p_j}(R^{\frac{1}{3}}\sigma)} e^{u_i} dV_g \leq \tau/4 \int_\Sigma e^{u_i} dV_g$, $i,\ j = 1,2,\ i \neq j$.

Here we need to use again some harmonic lifting of our functions. Take $n \in \mathbb{N}$, $n \leq 2\varepsilon^{-1}$ so that

$$
\sum_{i=1}^{2} \int_{A_{p_i}(2^{n-1}\sigma,2^{n+1}\sigma)} \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) dV_g \leq \varepsilon \int_\Sigma \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) dV_g,
$$

where we have chosen R so that $2^{3\varepsilon^{-1}} < R^{1/3}$.

We define the function v_i of class H^1 by:

$$
\begin{cases}
v_i(x) = u_i(x) & x \in B_{p_1}(2^n\sigma) \cup B_{p_2}(2^n\sigma), \\
\Delta v_i(x) = 0 & x \in A_{p_1}(2^n\sigma, 2^{n+1}\sigma) \cup A_{p_2}(2^n\sigma, 2^{n+1}\sigma), \\
v_i(x) = \bar{u}_i(\sigma) & x \notin B_{p_1}(2^{n+1}\sigma) \cup B_{p_2}(2^{n+1}\sigma).
\end{cases}
$$

Again,

$$
\begin{aligned}
&\sum_{i=1}^{2} \int_{A_{p_i}(2^n\sigma,2^{n+1}\sigma)} \left(|\nabla v_1|^2 + |\nabla v_2|^2\right) dV_g \\
&\leq C \sum_{i=1}^{2} \int_{A_{p_i}(2^{n-1}\sigma,2^{n+1}\sigma)} \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) dV_g \leq C\varepsilon \int_\Sigma \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) dV_g,
\end{aligned}
$$

where C is a universal constant.

We now apply Lemma 3.7 to (v_1,v_2) with $p=p_1$ and $s=2(6R+2)\sigma$, and take into account (3.12):

(3.18)
$$\int_{B_{p_1}(2^n\sigma)\cup B_{p_2}(2^n\sigma)} Q(u_1,u_2)\,dV_g + C\varepsilon\int_\Sigma \left(|\nabla u_1|^2+|\nabla u_2|^2\right) dV_g + C$$
$$\geq (1+\varepsilon)\int_{B_p(s)} Q(v_1,v_2)\,dV_g + C$$
$$\geq 4\pi\left(\log\int_{B_p(s/2)} e^{v_1}dV_g + \log\int_{B_p(s/2)} e^{v_2}dV_g - (\bar u_1(s)+\bar u_2(s)+4\log s)\right)$$
$$\geq 4\pi\left(\log\int_\Sigma e^{u_1}dV_g + \log\int_\Sigma e^{u_2}dV_g - (\bar u_1(s)+\bar u_2(s)+4\log s)\right) - C.$$

Now, we define $w_i\in H^1(\Sigma)$ as:

$$\begin{cases} w_i(x)=\bar u_i(\sigma) & x\in B_{p_1}(2^n\sigma)\cup B_{p_2}(2^n\sigma),\\ \Delta w_i(x)=0 & x\in A_{p_1}(2^n\sigma,2^{n+1}\sigma)\cup A_{p_2}(2^n\sigma,2^{n+1}\sigma),\\ w_i(x)=\tilde u_i & x\notin B_{p_1}(2^{n+1}\sigma)\cup B_{p_2}(2^{n+1}\sigma). \end{cases}$$

As before,

$$\sum_{i=1}^2\int_{A_{p_i}(2^n\sigma,2^{n+1}\sigma)}\left(|\nabla w_1|^2+|\nabla w_2|^2\right) dV_g$$
$$\leq C\sum_{i=1}^2\int_{A_{p_i}(2^{n-1}\sigma,2^{n+1}\sigma)}\left(|\nabla u_1|^2+|\nabla u_2|^2\right) dV_g \leq C\varepsilon\int_\Sigma\left(|\nabla u_1|^2+|\nabla u_2|^2\right) dV_g,$$

where also here C is a universal constant.

We apply Lemma 3.8 to (w_1,w_2) for any point p' such that $d(p',p_1)=\frac{1}{2}R^{1/3}\sigma$, $s'=\sigma$ and $r=2^{k+1}\delta$:

(3.19)
$$\int_{(B_{p_1}(2^{n+1}\sigma)\cup B_{p_2}(2^{n+1}\sigma))^c} Q(u_1,u_2)\,dV_g + C\varepsilon\int_\Sigma\left(|\nabla u_1|^2+|\nabla u_2|^2\right) dV_g + C$$
$$\geq (1+\varepsilon)\int_{A_{p'}(s'/2,2r)} Q(w_1,w_2)\,dV_g + C$$
$$\geq 4\pi\left((\bar u_1(\sigma)+\bar u_2(\sigma)+4\log\sigma)(1+\varepsilon) + \sum_{i=1}^2\log\int_{A_{p'}(s',r)} e^{w_i}dV_g\right).$$

Taking into account (3.14) and the hypothesis of Case 1.4,

(3.20)
$$\int_{(B_{p_1}(2^n\sigma)\cup B_{p_2}(2^n\sigma))^c} Q(u_1,u_2)\,dV_g + C\varepsilon\int_\Sigma \left(|\nabla u_1|^2+|\nabla u_2|^2\right)\,dV_g + C$$
$$\geq 4\pi\left(\log\int_\Sigma e^{u_1}dV_g+\log\int_\Sigma e^{u_2}dV_g+(\bar{u}_1(\sigma)+\bar{u}_2(\sigma)+4\log\sigma)(1+\varepsilon)\right).$$

Combining inequalities (3.18) and (3.20), we obtain our result.

CASE 2: Assume that for some $i=1,2$:

$$\int_{B_{p_i}(\delta)^c} e^{u_i}dV_g \geq \tau/2\int_\Sigma e^{u_i}dV_g.$$

Without loss of generality, let us consider $i=1$.

Take $\delta'=\frac{\delta}{2^{3/\varepsilon}}$. If moreover:

$$\int_{B_{p_2}(\delta')^c} e^{u_2}dV_g \geq \tau/2,$$

then Proposition 2.6 implies the desired inequality. So, we can assume that:

$$\int_{A_{p_2}(R\sigma,\delta')} e^{u_2}dV_g \geq \tau/2. \tag{3.21}$$

We now apply the whole procedure of Case 1 to u_1, u_2, replacing δ with δ'.

For instance, as in Case 1.1, we would get (3.15) and (3.16). However, here (3.17) does not follow immediately since now we do not know whether:

$$\int_{A_p(s',r)} e^{u_1}dV_g \geq \alpha\int_\Sigma e^{u_1}dV_g,$$

for some fixed $\alpha>0$. This is needed to estimate:

$$\log\int_{A_p(s',r)} e^{\tilde{u}_1}dV_g \geq \log\int_\Sigma e^{u_1}dV_g - C,$$

which allows us to obtain (3.17).

By applying the Jensen inequality and Lemma 3.6, we get:

$$\log\int_{A_p(s',r)} e^{\tilde{u}_1}dV_g \geq \log\int_{A_p(r/8,r/4)} e^{u_1}dV_g$$
$$\geq \log\fint_{A_p(r/8,r/4)} e^{u_1}dV_g - C \geq \fint_{A_p(r/8,r/4)} u_1 dV_g - C \geq -\varepsilon\int_\Sigma |\nabla u_1|^2dV_g - C.$$

Therefore, from (3.21) and (3.16) we get:

$$
(3.22)\quad \begin{aligned} & \int_{A_p(s'/2,2r)} Q(\tilde{u}_1,\tilde{u}_2)\, dV_g + C\varepsilon \int_\Sigma \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) dV_g + C \\ & \geq 4\pi \left(\log \int_\Sigma e^{u_2} dV_g + (\bar{u}_1(\sigma) + \bar{u}_2(\sigma) + 4\log\sigma)(1+\varepsilon) \right). \end{aligned}
$$

Now, we apply Proposition 2.3, to find:

$$
(1+C\varepsilon) \int_{B_{p_1}(\delta/2)^c} Q(u_1,u_2) dV_g + C \geq 4\pi \sum_{i=1}^{2} \log \int_{B_{p_1}(\delta)^c} e^{u_i} dV_g.
$$

Again here we can use Jensen inequality and the hypothesis of Case 2 to deduce:

$$
(3.23)\quad \begin{aligned} & \int_{B_{p_1}(\delta/2)^c} Q(u_1,u_2) dV_g + C\varepsilon \int_\Sigma \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) dV_g + C \\ & \geq 4\pi \left(\log \int_\Sigma e^{u_1} dV_g \right). \end{aligned}
$$

We conclude now by combining (3.23), (3.22) and (3.15).

We can argue in the same way if we are under the conditions of Cases 1.2, 1.3 or 1.4.

Remark 3.12. From the proof of Proposition 3.2 it is clear that the condition $\psi(f_1) = \psi(f_2)$ can be relaxed. Indeed, let us fix two constants $M > 1$, $M' > 0$, and define $\psi(f_i) = (\beta_i, \sigma_i)$. Then, the assumption

$$
\frac{1}{M} \leq \frac{\sigma_1}{\sigma_2} \leq M, \;\; d(\beta_1,\beta_2) \leq M'\sigma_1,
$$

is enough for the conclusion of the Proposition 3.2. In such case, the constant C would depend also on M, M'.

However, we believe that the topological description of the low sub-levels of J_ρ (see next Section), which is based on Proposition 3.2, is optimal.

Remark 3.13. The improved inequality in Proposition 3.2 is consistent with the asymptotic analysis in [14]. Here the authors prove that when both u_1, u_2 blow-up at the same rate at the same point, then the corresponding quantization of conformal volume is $(8\pi, 8\pi)$. On the other hand when the blow-up rates are different, but occur at the same point, then the quantization values are $(4\pi, 8\pi)$ or $(8\pi, 4\pi)$.

4 Min-max scheme

Let $\overline{\Sigma}_\delta$ be as in (2.1), and let us set

$$
(4.1)\quad \overline{D}_\delta = diag(\overline{\Sigma}_\delta \times \overline{\Sigma}_\delta) = \left\{ (\vartheta_1, \vartheta_2) \in \overline{\Sigma}_\delta \times \overline{\Sigma}_\delta : \vartheta_1 = \vartheta_2 \right\}; \quad X = \overline{\Sigma}_\delta \times \overline{\Sigma}_\delta \setminus \overline{D}_\delta.
$$

Let $\varepsilon > 0$ be such that $\rho_i + \varepsilon < 8\pi$ for $i = 1,2$, and let R, δ, ψ be as in Proposition 3.1. Consider then the map $\Psi : H^1(\Sigma) \times H^1(\Sigma)$ defined in the following way

$$\Psi(u_1,u_2) = \left(\psi\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1} dV_g} \right), \psi\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2} dV_g} \right) \right). \tag{4.2}$$

By Proposition 3.2, and since $C \geq h_i(x) \geq \frac{1}{C} > 0$ for any $x \in \Sigma$, we have that $I_\rho(u_1,u_2)$ is bounded from below for any $(u_1,\, u_2)$ such that $\Psi(u_1,u_2) \in \overline{D}_\delta$. Therefore, there exists a large $L > 0$ such that

$$J_\rho(u_1,u_2) \leq -L \quad \Rightarrow \quad \Psi(u_1,u_2) \in X. \tag{4.3}$$

By our definition of $\overline{\Sigma}_\delta$, the set X is not compact: however it retracts to some compact subset $\mathscr{X}_\nu$, as it is shown in the next result.

Lemma 4.1. *For $\nu \ll \delta$, define*

$$\mathscr{X}_{\nu,1} = \Big\{ ((x_1,t_1),(x_2,t_2)) \in X \,:\, |t_1 - t_2|^2 + d(x_1,x_2)^2 \geq \delta^4,$$
$$\max\{t_1,t_2\} < \delta,\ \min\{t_1,t_2\} \in \left[\nu^2, \nu\right] \Big\};$$
$$\mathscr{X}_{\nu,2} = \left\{ ((x_1,t_1),(x_2,t_2)) \in X \,:\, \max\{t_1,t_2\} = \delta, \min\{t_1,t_2\} \in \left[\nu^2,\nu\right] \right\},$$

and set

$$\mathscr{X}_\nu = (\mathscr{X}_{\nu,1} \cup \mathscr{X}_{\nu,2}) \subseteq X. \tag{4.4}$$

Then there is a retraction R_ν of X onto $\mathscr{X}_\nu$.

Proof. We proceed in two steps. First, we define a deformation of X in itself satisfying that:

a) either $\max\{t_1,t_2\} < \delta$ and $|t_1 - t_2|^2 + d(x_1,x_2)^2 \geq \delta^4$,
b) or $\max\{t_1,t_2\} = \delta$.

Then another deformation will provide us with the condition $\min\{t_1,t_2\} \in \left[\nu^2,\nu\right]$.

Let us consider the following ODE in $(\Sigma \times (0,\delta])^2$:

$$\frac{d}{ds} \begin{pmatrix} x_1(s) \\ t_1(s) \\ x_2(s) \\ t_2(s) \end{pmatrix} = \begin{pmatrix} (\delta - \max_i\{t_i(s)\}) \nabla_{x_1} d(x_1(s),x_2(s))^2 \\ (t_1(s) - t_2(s)) t_1(s)(\delta - t_1(s)) \\ (\delta - \max_i\{t_i(s)\}) \nabla_{x_2} d(x_1(s),x_2(s))^2 \\ (t_2(s) - t_1(s)) t_2(s)(\delta - t_2(s)) \end{pmatrix}.$$

Notice that if $|t_1 - t_2|^2 + d(x_1,x_2) < \delta^4$ (and $\max\{t_1,t_2\} < \delta$) then $d(x_1,x_2)$ is small so $d(x_1,x_2)^2$ is a smooth function on $(\Sigma \times \mathbb{R})^2$, and the above vector field is well defined. For each initial datum $(\vartheta_1,\vartheta_2) \in X$ we define $s_{\vartheta_1,\vartheta_2} \geq 0$ as the smallest value of s for which the above flow satisfies either a) or b).

To define the first homotopy $H_1(s,\cdot)$ then one can use the above flow, rescaling in the evolution variable (depending on the initial datum) as $s \mapsto \tilde{s} = s_{\vartheta_1,\vartheta_2} s$.

To define the second homotopy, we introduce two cutoff functions χ_1, χ_2:

$$\begin{cases} \chi_1(t) = 1 & \text{for } t \leq \nu^2, \\ \chi \text{ is non increasing,} & \\ \chi_1(t) = -1 & \text{for } t \geq \nu, \end{cases} \qquad \begin{cases} \chi_2(t) = 1 & \text{for } t \leq \delta/2, \\ \chi_2(t) = 2\left(1 - \frac{t}{\delta}\right) & t \in (\delta/2, \delta), \\ \chi_2(t) = 0 & \text{for } t \geq \delta, \end{cases}$$

and consider the following ODE

$$\frac{d}{ds}\begin{pmatrix} t_1(s) \\ t_2(s) \end{pmatrix} = \begin{pmatrix} \chi_1(\min_i\{t_i(s)\})\chi_2(t_1(s)) \\ \chi_1(\min_i\{t_i(s)\})\chi_2(t_2(s)) \end{pmatrix}.$$

As in the previous case, there exists $\hat{s}_{\vartheta_1,\vartheta_2}$ such that the condition $\min_i t_i \in [\nu^2, \nu]$ is reached for $s = \hat{s}_{\vartheta_1,\vartheta_2}$, and one can define the homotopy H_2 rescaling in s correspondingly. Observe that along the homotopy H_2 the distance $|t_1 - t_2|$ is non decreasing if $|t_1 - t_2| \leq \delta/4$.

The concatenation of the homotopies H_1 and H_2 gives the desired conclusion. Note that both H_1 and H_2, by the way they are constructed, preserve the quotient relations in the definition of X. □

We next construct a family of test functions parameterized by $\mathscr{X}_\nu$ on which J_ρ attains large negative values. For $(\vartheta_1, \vartheta_2) = ((x_1,t_1),(x_2,t_2)) \in \mathscr{X}_\nu$ define

$$\varphi_{(\vartheta_1,\vartheta_2)}(y) = (\varphi_1(y), \varphi_2(y)), \tag{4.5}$$

where we have set

$$\varphi_1(y) = \log \frac{1+\tilde{t}_2^2 d(x_2,y)^2}{\left(1+\tilde{t}_1^2 d(x_1,y)^2\right)^2}, \quad \varphi_2(y) = \log \frac{1+\tilde{t}_1^2 d(x_1,y)^2}{\left(1+\tilde{t}_2^2 d(x_2,y)^2\right)^2}, \tag{4.6}$$

with

$$\tilde{t}_i = \tilde{t}_i(t_i) = \begin{cases} \frac{1}{t_i}, & \text{for } t_i \leq \frac{\delta}{2}, \\ -\frac{4}{\delta^2}(t_i - \delta) & \text{for } t_i \geq \frac{\delta}{2}; \end{cases}, \; i = 1,\, 2. \tag{4.7}$$

Notice that, by our choices of $\tilde{t}_1, \tilde{t}_2$, this map is well defined on $\mathscr{X}_\nu$ (especially for what concerns the identifications in $\overline{\Sigma}_\delta$). We have then the following result.

Lemma 4.2. *For ν sufficiently small, and for $(\vartheta_1, \vartheta_2) \in \mathscr{X}_\nu$, there exists a constant $C = C(\delta, \Sigma) > 0$, depending only on Σ and δ, such that*

$$\frac{1}{C}\frac{t_i^2}{t_j^2} \leq \int_\Sigma e^{\varphi_i(y)} dV_g(y) \leq C\frac{t_i^2}{t_j^2}, \quad i \neq j; \tag{4.8}$$

Proof. First, we notice that by an elementary change of variables

$$\int_{\mathbb{R}^2} \frac{1}{\left(1+\lambda^2|x|^2\right)^2} dx = \frac{C_0}{\lambda^2}; \quad \lambda > 0 \tag{4.9}$$

for some fixed positive constant C_0. We distinguish next the two cases

$$|t_1 - t_2| \geq \delta^3 \quad \text{and} \quad |t_1 - t_2| < \delta^3. \tag{4.10}$$

In the first alternative, by the definition of $\mathscr{X}_\nu$ and by the fact that $\nu \ll \delta$, one of the t_i's belongs to $[\nu^2, \nu]$, while the other is greater or equal to $\frac{\delta^3}{2}$.

If $t_1 \in [\nu^2, \nu]$ and if $t_2 \geq \frac{\delta^3}{2}$ then the function $1 + \tilde{t}_2^2 d(x_2, y)^2$ is bounded above and below by two positive constants depending only on Σ and δ. Therefore, working in geodesic normal coordinates centered at x_1 and using (4.9) we obtain

$$\frac{t_1^2}{C} \leq \frac{1}{C\tilde{t}_1^2} \leq \int_\Sigma e^{\varphi_1(y)} dV_g(y) \leq \frac{C}{\tilde{t}_1^2} \leq C t_1^2.$$

If instead $t_2 \in [\nu^2, \nu]$ and if $t_1 \geq \frac{\delta^3}{2}$ then the function $1 + \tilde{t}_1^2 d(x_1, y)^2$ is bounded above and below by two positive constants depending only on Σ and δ, hence one finds

$$\int_\Sigma e^{\varphi_1(y)} dV_g(y) \geq \frac{1}{C} \int_\Sigma (1 + \tilde{t}_2^2 d(x_2, y)^2) dV_g(y) \geq \frac{\tilde{t}_2^2}{C} = \frac{1}{C t_2^2},$$

and similarly

$$\int_\Sigma e^{\varphi_1(y)} dV_g(y) \leq C \int_\Sigma (1 + \tilde{t}_2^2 d(x_2, y)^2) dV_g(y) \leq C \tilde{t}_2^2 = \frac{C}{t_2^2}.$$

In both the last two cases we then obtain the conclusion.

Suppose now that $|t_1 - t_2| < \delta^3$: then by the definition of $\mathscr{X}_\nu$ we have that $d(x_1, x_2) \geq \frac{\delta^2}{2}$ and that $t_1, t_2 \leq \nu + \delta^3$. Then, from (4.9) and some elementary estimates we derive

$$\int_\Sigma e^{\varphi_1(y)} dV_g(y) \geq \int_{B_{x_1}(\delta^3)} e^{\varphi_1(y)} dV_g(y) \geq \frac{1}{C} \frac{1 + \tilde{t}_2^2 d(x_1, x_2)^2}{\tilde{t}_1^2} \geq \frac{1}{C} \frac{t_1^2}{t_2^2}.$$

By the same argument we obtain

$$\int_{B_{x_1}(\delta^3)} e^{\varphi_1(y)} dV_g(y) \leq C \frac{1 + \tilde{t}_2^2 d(x_1, x_2)^2}{\tilde{t}_1^2} \leq C \frac{t_1^2}{t_2^2}.$$

Moreover, we have

$$\int_{(B_{x_1}(\delta^3))^c} e^{\varphi_1(y)} dV_g(y) \leq \frac{C}{\tilde{t}_1^4} \int_{(B_{x_1}(\delta^3))^c} (1 + \tilde{t}_2^2 d(x_2, y)^2) dV_g(y) \leq C \frac{t_1^4}{t_2^2}.$$

This concludes the proof. □

Lemma 4.3. *For $(\vartheta_1, \vartheta_2) \in \mathscr{X}_\nu$, let $\varphi_{(\vartheta_1, \vartheta_2)}$ be defined as in the above formula. Then*

$$J_\rho(\varphi_{(\vartheta_1, \vartheta_2)}) \to -\infty \quad \text{as } \nu \to 0 \quad \text{uniformly for } (\vartheta_1, \vartheta_2) \in \mathscr{X}_\nu.$$

Proof. The statement follows from Lemma 4.2 once the following three estimates are shown

$$\int_\Sigma Q\left(\varphi_{(\vartheta_1, \vartheta_2)}\right) dV_g \leq 8\pi(1 + o_\delta(1)) \log \frac{1}{t_1} + 8\pi(1 + o_\delta(1)) \log \frac{1}{t_2}; \tag{4.11}$$

$$\fint_{\Sigma} \varphi_1 dV_g = 4(1+o_\delta(1))\log t_1 - 2(1+o_\delta(1))\log t_2; \tag{4.12}$$

$$\fint_{\Sigma} \varphi_2 dV_g = 4(1+o_\delta(1))\log t_2 - 2(1+o_\delta(1))\log t_1. \tag{4.13}$$

In fact, these yield the inequality

$$J_\rho(\varphi_{(\vartheta_1,\vartheta_2)}) \leq \sum_{i=1}^{2}(2\rho_i - 8\pi + o_\delta(1))\log t_i \to -\infty \quad \text{ as } \nu \to 0$$

uniformly for $(\vartheta_1, \vartheta_2) \in \mathscr{X}_\nu$, since $\rho_1, \rho_2 > 4\pi$. Here again we are using that $C \geq h_i(x) \geq \frac{1}{C} > 0$ for any $x \in \Sigma$.

We begin by showing (4.12), whose proof clearly also yields (4.13). It is convenient to write

$$\varphi_1 = \log\left(1+\tilde{t}_2^2 d(x_2,y)^2\right) - 2\log\left(1+\tilde{t}_1^2 d(x_1,y)^2\right),$$

and to divide Σ into the two subsets

$$\mathscr{A}_1 = B_{x_1}(\delta) \cup B_{x_2}(\delta); \quad \mathscr{A}_2 = \Sigma \setminus \mathscr{A}_1.$$

For $y \in \mathscr{A}_2$ we have that

$$\frac{1}{C_{\delta,\Sigma} t_1^2} \leq 1+\tilde{t}_1^2 d(x_1,y)^2 \leq \frac{C_{\delta,\Sigma}}{t_1^2}; \quad \frac{1}{C_{\delta,\Sigma} t_2^2} \leq 1+\tilde{t}_2^2 d(x_2,y)^2 \leq \frac{C_{\delta,\Sigma}}{t_2^2},$$

which implies

$$\frac{1}{|\Sigma|}\int_{\mathscr{A}_2} \varphi_1 dV_g = 4(1+o_\delta(1))\log t_1 - 2(1+o_\delta(1))\log t_2. \tag{4.14}$$

On the other hand, working in normal geodesic coordinates at x_i one also finds

$$\int_{B_\delta(x_i)} \log\left(1+\tilde{t}_i^2 d(x_i,y)^2\right) dV_g = o_\delta(1)\log t_i.$$

Using (4.14) and the last formula we then obtain (4.12).

Let us now show (4.11). We clearly have that

$$\begin{aligned} \nabla\varphi_1 &= \nabla\log\left(1+\tilde{t}_2^2 d(x_2,y)^2\right) - 2\nabla\log\left(1+\tilde{t}_1^2 d(x_1,y)^2\right) \\ &= \frac{2\tilde{t}_2^2 d(x_2,y)\nabla_y d(x_2,y)}{1+\tilde{t}_2^2 d(x_2,y)^2} - \frac{4\tilde{t}_1^2 d(x_1,y)\nabla_y d(x_1,y)}{1+\tilde{t}_1^2 d(x_1,y)^2}, \end{aligned}$$

and similarly

$$\begin{aligned} \nabla\varphi_2 &= \nabla\log\left(1+\tilde{t}_1^2 d(x_1,y)^2\right) - 2\nabla\log\left(1+\tilde{t}_2^2 d(x_2,y)^2\right) \\ &= \frac{2\tilde{t}_1^2 d(x_1,y)\nabla_y d(x_1,y)}{1+\tilde{t}_1^2 d(x_1,y)^2} - \frac{4\tilde{t}_2^2 d(x_2,y)\nabla_y d(x_2,y)}{1+\tilde{t}_2^2 d(x_2,y)^2}. \end{aligned}$$

From now on we will assume, without loss of generality, that $t_1 \leq t_2$. We distinguish between the case $t_2 \geq \delta^3$ and $t_2 \leq \delta^3$.

In the first case the function $1+\tilde{t}_2^2 d(x_2,y)^2$ is uniformly Lipschitz with bounds depending only on δ, and therefore we can write that

$$\nabla \varphi_1 = -\frac{4\tilde{t}_1^2 d(x_1,y)\nabla_y d(x_1,y)}{1+\tilde{t}_1^2 d(x_1,y)^2} + O_\delta(1);$$

$$\nabla \varphi_2 = \frac{2\tilde{t}_1^2 d(x_1,y)\nabla_y d(x_1,y)}{1+\tilde{t}_1^2 d(x_1,y)^2} + O_\delta(1).$$

Given a large but fixed constant $C_1 > 0$, we divide the surface Σ into the three regions

$$\mathscr{B}_1 = B_{x_1}(C_1 t_1); \quad \mathscr{B}_2 = B_{x_2}(C_1 t_2); \quad \mathscr{B}_3 = \Sigma \setminus (\mathscr{B}_1 \cup \mathscr{B}_2). \tag{4.15}$$

In $\mathscr{B}_1$ we have that $|\nabla \varphi_i| \leq C\tilde{t}_1$, while

$$\frac{\tilde{t}_1^2 d(x_1,y)\nabla_y d(x_1,y)}{1+\tilde{t}_1^2 d(x_1,y)^2} = (1+o_{C_1}(1))\frac{\nabla_y d(x_1,y)}{d(x_1,y)} \quad \text{in } \Sigma \setminus \mathscr{B}_1. \tag{4.16}$$

The last gradient estimates imply that

$$\begin{aligned} \int_\Sigma Q(\varphi_{(\vartheta_1,\vartheta_2)})dV_g &= \int_{\Sigma\setminus\mathscr{B}_1} Q(\varphi_{(\vartheta_1,\vartheta_2)})dV_g + o_\delta(1)\log\frac{1}{t_1} + O_\delta(1) \\ &= 8\pi \int_{C_1 t_1}^1 \frac{dt}{t} + o_\delta(1)\log\frac{1}{t_1} + O_\delta(1) \\ &= 8\pi(1+o_\delta(1))\left(\log\frac{1}{t_1} + \log\frac{1}{t_2}\right) + O_\delta(1) \quad \text{since } t_2 \geq \delta^3. \end{aligned} \tag{4.17}$$

Assume now that $t_2 \leq \delta^3$. Then by the definition of $\mathscr{X}_\nu$ we have that $d(x_1,x_2) \geq \frac{\delta^2}{2}$, and therefore $\mathscr{B}_1 \cap \mathscr{B}_2 = \emptyset$. Similarly to (4.16) we find

$$\begin{cases} \frac{\tilde{t}_1^2 d(x_1,y)\nabla_y d(x_1,y)}{1+\tilde{t}_1^2 d(x_1,y)^2} = (1+o_{C_1}(1))\frac{\nabla_y d(x_1,y)}{d(x_1,y)}; \\ \frac{\tilde{t}_2^2 d(x_2,y)\nabla_y d(x_2,y)}{1+\tilde{t}_2^2 d(x_2,y)^2} = (1+o_{C_1}(1))\frac{\nabla_y d(x_2,y)}{d(x_2,y)} \end{cases} \quad \text{in } \mathscr{B}_3.$$

Moreover we have the estimates

$$|\nabla \varphi_i| \leq C\tilde{t}_i \ \text{ in } \mathscr{B}_i, \ i = 1,\ 2; \quad |\nabla \varphi_i| \leq C \ \text{ in } \mathscr{B}_j, \ i \neq j.$$

Then, there follows:

$$\begin{aligned} \int_\Sigma Q(\varphi_{(\vartheta_1,\vartheta_2)})dV_g &= \int_{\mathscr{B}_3} Q(\varphi_{(\vartheta_1,\vartheta_2)})dV_g + o_\delta(1)\log\frac{1}{t_1} + o_\delta(1)\log\frac{1}{t_2} + O_\delta(1) \\ &= 8\pi(1+o_\delta(1))\left(\log\frac{1}{t_1} + \log\frac{1}{t_2}\right) + O_\delta(1). \end{aligned} \tag{4.18}$$

With formulas (4.17) and (4.18), we conclude the proof of (4.11) and hence that of the lemma. □

Since the functional J_ρ attains large negative values on the above test functions $\varphi_{(\vartheta_1,\vartheta_2)}$, these are mapped to X by Ψ. We next evaluate the image of Ψ with more precision, beginning with the following technical lemma.

Lemma 4.4. *Let φ_1, φ_2 be as in* (4.6)*: then, for some $C = C(\delta, \Sigma) > 0$, the following estimates hold uniformly in $(\vartheta_1, \vartheta_2) \in \mathscr{X}_\nu$:*

$$\sup_{x\in\Sigma} \int_{B_x(rt_i)} e^{\varphi_i} dV_g \leq Cr^2 \frac{t_i^2}{t_j^2} \quad \forall r > 0, \ i \neq j. \tag{4.19}$$

Moreover, given any $\varepsilon > 0$ there exists $C = C(\varepsilon, \delta, \Sigma)$, depending only on ε, δ and Σ (but not on ν), such that

$$\int_{B_{x_i}(Ct_i)} e^{\varphi_i} dV_g \geq (1-\varepsilon) \int_\Sigma e^{\varphi_i(y)} dV_g, \ i = 1, \ 2. \tag{4.20}$$

uniformly in $(\vartheta_1, \vartheta_2) \in \mathscr{X}_\nu$.

Proof. We prove the case $i = 1$. Observe that $1 + \tilde{t}_2^2 d(x_2,y)^2 \leq \frac{C}{t_2^2}$ and that $1 + \tilde{t}_1^2 d(x_1,y)^2 \geq 1$. Therefore we immediately find

$$\int_{B_x(t_1 r)} e^{\varphi_1} dV_g \leq \frac{C}{t_2^2} \int_{B_x(t_1 r)} \frac{1}{\left(1 + \tilde{t}_1^2 d(x_1,y)^2\right)^2} dV_g(y) \leq Cr^2 \frac{t_1^2}{t_2^2} \qquad \text{for all } x \in \Sigma,$$

which gives the first inequality in (4.19).

We now show (4.20), by evaluating the integral in the complement of $B_{x_1}(Rt_1)$ for some large R. Using again the fact that $1 + \tilde{t}_2^2 d(x_2,y)^2 \leq \frac{C}{t_2^2}$ we clearly have that

$$\int_{\Sigma \setminus B_{x_1}(Rt_1)} e^{\varphi_1(y)} dV_g(y) \leq \frac{C}{t_2^2} \int_{\Sigma \setminus B_{x_1}(Rt_1)} \frac{1}{\left(1 + \tilde{t}_1^2 d(x_1,y)^2\right)^2} dV_g(y). \tag{4.21}$$

To evaluate the last integral one can use normal geodesic coordinates centered at x_1 and (4.9) with a change of variable to find that

$$\lim_{t_1 \to 0^+} t_1^{-2} \int_{\Sigma \setminus B_{x_1}(Rt_1)} \frac{1}{\left(1 + \tilde{t}_1^2 d(x_1,y)^2\right)^2} dV_g = o_R(1) \qquad \text{as } R \to +\infty.$$

This and (4.21), jointly with the second inequality in (4.8), conclude the proof of the (4.20), by choosing R sufficiently large, depending on ε, δ and Σ. □

We next show that, parameterizing the test functions on $\mathscr{X}_\nu$ and composing with $R_\nu \circ \Psi$, we obtain a map homotopic to the identity on $\mathscr{X}_\nu$. This step will be fundamental for us in order to run the variational scheme later in this section.

Lemma 4.5. *Let $L > 0$ be so large that $\Psi(\{J_\rho \leq -L\}) \in X$, and let ν be so small that $J_\rho(\varphi_{(\vartheta_1,\vartheta_2)}) < -L$ for $(\vartheta_1,\vartheta_2) \in \mathscr{X}_\nu$ (see Lemma 4.3). Let R_ν be the retraction given in Lemma 4.1. Then the map from $T_\nu : \mathscr{X}_\nu \to \mathscr{X}_\nu$ defined as*

$$T_\nu((\vartheta_1,\vartheta_2)) = R_\nu(\Psi(\varphi_{(\vartheta_1,\vartheta_2)}))$$

is homotopic to the identity on $\mathscr{X}_\nu$.

Proof. Let us denote $\vartheta_i = (x_i,t_i)$,

$$f_i = \frac{e^{\varphi_i}}{\int_\Sigma e^{\varphi_i} dV_g}, \quad \psi(f_i) = (\beta_i,\sigma_i),$$

where ψ is given in Proposition 3.1. First, we claim that there is a constant $C = C(\delta,\Sigma) > 0$, depending only on Σ and δ, such that:

$$\frac{1}{C} \leq \frac{\sigma_i}{t_i} \leq C, \quad d(\beta_i,x_i) \leq Ct_i. \tag{4.22}$$

By (4.20), we have that

$$\sigma(x_i,f_i) \leq Ct_i,$$

where $\sigma(x,f)$ is the continuous map defined in (3.1). From that, we get that $\sigma_i \leq Ct_i$. Using now (4.19), we get the relation $t_i \leq C\sigma_i$.

Taking into account that $\sigma(x_i,f) \leq Ct_i$ and (3.2), we obtain that

$$d(x_i,S(f_i)) \leq Ct_i,$$

where $S(f)$ is the set defined in (3.3). But since the inequality

$$d(\beta_i,S(f_i)) \leq C\sigma_i$$

is always satisfied, we conclude the proof of (4.22).

We are now ready to prove the lemma. Let us define a first deformation H_1 in the following form:

$$\left(\begin{pmatrix} (\beta_1,\sigma_1) \\ (\beta_2,\sigma_2) \end{pmatrix}, s\right) \overset{H_1}{\longmapsto} \begin{pmatrix} (\beta_1,\ (1-s)\sigma_1 + s\kappa_1) \\ \\ (\beta_2,\ (1-s)\sigma_2 + s\kappa_2) \end{pmatrix},$$

where $\kappa_i = \min\{\delta, \frac{\sigma_i}{\sqrt{\nu}}\}$.

A second deformation H_2 is defined in the following way:

$$\left(\begin{pmatrix} (\beta_1,\kappa_1) \\ (\beta_2,\kappa_2) \end{pmatrix}, s\right) \overset{H_2}{\longmapsto} \begin{pmatrix} ((1-s)\beta_1 + sx_1,\ \kappa_1) \\ \\ ((1-s)\beta_2 + sx_2,\ \kappa_2) \end{pmatrix},$$

where $(1-s)\beta_i + sx_i$ stands for the geodesic joining β_i and x_i in unit time. A comment is needed here. If $\kappa_i < \delta$, then we have that $\sigma_i < \sqrt{\nu}\delta$. By choosing

ν small enough, this implies that β_i and x_i are close to each other (recall (4.22)). Instead, if $\kappa_i = \delta$, the identification in $\overline{\Sigma}_\delta$ makes the above deformation trivial.

We also use a third deformation H_3:

$$\left(\left(\begin{array}{c} (x_1, \kappa_1) \\ (x_2, \kappa_2) \end{array} \right), s \right) \stackrel{H_3}{\longmapsto} \left(\begin{array}{c} (x_1,\ (1-s)\kappa_1 + st_1) \\ \\ (x_2,\ (1-s)\kappa_2 + st_2) \end{array} \right).$$

We define H as the concatenation of those three homotopies. Then,

$$((\vartheta_1, \vartheta_2), s) \mapsto R_\nu \circ H(\Psi(\varphi_{(\vartheta_1, \vartheta_2)}), s)$$

gives us the desired homotopy to the identity.

Observe that, since $\nu \ll \delta$, $H(\Psi(\varphi_{(\vartheta_1,\vartheta_2)}), s)$ always stays in X, so that R_ν can be applied.

□

We now introduce the variational scheme which yields existence of solutions: this remaining part follows the ideas of [7] (see also [21]).

Let $\overline{\mathscr{X}}_\nu$ denote the (contractible) cone over $\mathscr{X}_\nu$, which can be represented as

$$\overline{\mathscr{X}}_\nu = (\mathscr{X}_\nu \times [0,1])\,|_\sim,$$

where the equivalence relation $\sim$ identifies $\mathscr{X}_\nu \times \{1\}$ to a single point. We choose $L > 0$ so large that (4.3) holds, and then ν so small that

$$J_\rho(\varphi_{(\vartheta_1,\vartheta_2)}) \leq -4L \quad \text{uniformly for } (\vartheta_1, \vartheta_2) \in \mathscr{X}_\nu,$$

the last claim being possible by Lemma 4.3. Fixing this value of ν, consider the following class of functions

(4.23)
$$\Gamma = \left\{ \eta : \overline{\mathscr{X}}_\nu \to H^1(\Sigma) \times H^1(\Sigma) \,:\, \eta \text{ is continuous and } \eta\big((\vartheta_1, \vartheta_2), 0\big) = \varphi_{(\vartheta_1,\vartheta_2)} \right\}.$$

Then we have the following properties.

Lemma 4.6. *The set* Γ *is non-empty and moreover, letting*

$$\alpha = \inf_{\eta \in \Gamma} \sup_{m \in \overline{\mathscr{X}}_\nu} J_\rho(\eta(m)), \text{ one has } \alpha > -2L.$$

Proof. To prove that $\Gamma \neq \emptyset$, we just notice that the map

$$\tilde{\eta}(\vartheta, s) = (1-s)\varphi_{(\vartheta_1,\vartheta_2)}, \quad (\vartheta, s) \in \overline{\mathscr{X}}_\nu, \tag{4.24}$$

belongs to Γ.

Suppose by contradiction that $\alpha \leq -2L$: then there would exist a map $\eta \in \Gamma$ satisfying the condition $\sup_{m \in \overline{\mathscr{X}}_\nu} J_\rho(\eta(m)) \leq -L$. Then, since Lemma 4.5 applies, writing $m = (\vartheta, s)$, with $\vartheta \in \mathscr{X}_\nu$, the map

$$s \mapsto R_\nu \circ \Psi \circ \eta(\cdot, s)$$

would be a homotopy in $\mathscr{X}_\nu$ between $R_\nu \circ \Psi \circ \varphi_{(\vartheta_1,\vartheta_2)}$ and a constant map. But this is impossible since $\mathscr{X}_\nu$ is non-contractible (by the results in Section A and by the fact that $\mathscr{X}_\nu$ is a retract of X) and since $R_\nu \circ \Psi \circ \varphi_{(\vartheta_1,\vartheta_2)}$ is homotopic to the identity on $\mathscr{X}_\nu$. Therefore we deduce $\alpha > -2L$, which is the desired conclusion. □

From the above Lemma, the functional J_ρ satisfies suitable structural properties for min-max theory. However, we cannot directly conclude the existence of a critical point, since it is not known whether the Palais-Smale condition holds or not. The conclusion needs a different argument, which has been used intensively (see for instance [7, 9]), so we will be sketchy.

We take $\mu > 0$ such that $\mathscr{J}_i := [\rho_i - \mu, \rho_i + \mu]$ is contained in $(4\pi, 8\pi)$ for both $i = 1, 2$. We then consider $\tilde{\rho}_i \in \mathscr{J}_i$ and the functional $J_{\tilde{\rho}}$ corresponding to these values of the parameters.

Following the estimates of the previous sections, one easily checks that the above min-max scheme applies uniformly for $\tilde{\rho}_i \in \mathscr{J}_i$ for ν sufficiently small. More precisely, given any large number $L > 0$, there exists ν so small that for $\tilde{\rho}_i \in \mathscr{J}_i$

$$\sup_{m \in \partial \overline{\mathscr{X}}_\nu} J_{\tilde{\rho}}(m) < -4L; \quad \alpha_{\tilde{\rho}} := \inf_{\eta \in \Gamma} \sup_{m \in \overline{\mathscr{X}}_\nu} J_{\tilde{\rho}}(\eta(m)) > -2L, \ \ (\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2)). \tag{4.25}$$

where Γ is defined in (4.23). Moreover, using for example the test map (4.24), one shows that for μ sufficiently small there exists a large constant $\overline{L}$ such that

$$\alpha_{\tilde{\rho}} \leq \overline{L} \quad \text{for } \tilde{\rho}_i \in \mathscr{J}_i. \tag{4.26}$$

Under these conditions, the following Lemma is well-known, usually taking the name "monotonicity trick". This technique was first introduced by Struwe in [26], and made general in [13] (see also [7, 19]).

Lemma 4.7. *Let ν be so small that* (4.25) *holds. Then the functional $J_{t\rho}$ possesses a bounded Palais-Smale sequence $(u_l)_l$ at level $\tilde{\alpha}_{t\rho}$ for almost every $t \in \left[1 - \frac{\mu}{16\pi}, 1 + \frac{\mu}{16\pi}\right]$.*

Proof of Theorem 1.1. The existence of a bounded Palais-Smale sequence for $J_{t\rho}$ implies by standard arguments that this functional possesses a critical point. Let now $t_j \to 1$, $t_j \in \Lambda$ and let $(u_{1,j}, u_{2,j})$ denote the corresponding solutions. It is then sufficient to apply the compactness result in Theorem 2.1, which yields convergence of $(u_{1,j}, u_{2,j})_j$ by the fact that ρ_1, ρ_2 are not multiples of 4π. □

Appendix: The set $X = \overline{\Sigma}_\delta \times \overline{\Sigma}_\delta \setminus \overline{D}_\delta$ is not contractible.

Without loss of generality, we consider the case $\delta = 1$ (see (2.1)). Let us denote $\overline{\Sigma} = \overline{\Sigma}_1$.

If $\Sigma = \mathbb{S}^2$, we have a complete description of X. Indeed, in this case $\overline{\Sigma}$ can be identified with $B(0,1) \subset \mathbb{R}^3$. Therefore, we have:

$$X = (B(0,1) \times B(0,1)) \setminus E,$$

where $E = \{x \in \mathbb{R}^6 : \ x_i = x_{i+3}, \ i = 1, \ 2, \ 3\}$. By taking the orthogonal projection onto $E^\perp$, we have that $X \simeq U \setminus \{0\}$ ($\simeq$ stands for homotopical equivalence), where $U \subset E^\perp$ is a convex neighborhood of 0. And, clearly, $U \setminus \{0\} \simeq \mathbb{S}^2$.

The case of positive genus is not so easy and we have a less complete description of X. However, we will prove that it is non-contractible by studying its cohomology groups $H^*(X)$, where coefficients will be taken in $\mathbb{R}$. Indeed, we will show that:

Proposition A.1. *If the genus of Σ is positive, then $H^4(X)$ is nontrivial.*

Proof. In what follows, the elements of $\overline{\Sigma}$ will be written as (x,t), where $x \in \Sigma$, $t \in (0,1]$.

Clearly, $X = Y \cup Z$, where Y, Z are open sets defined as:

$$Y = \{((x_1,t_1),(x_2,t_2)) \in \overline{\Sigma} \times \overline{\Sigma} : \ t_1 \neq t_2\},$$
$$Z = \{((x_1,t_1),(x_2,t_2)) \in \overline{\Sigma} \times \overline{\Sigma} : \ t_1 < 1, \ t_2 < 1, \ x_1 \neq x_2\}.$$

Then, the Mayer-Vietoris Theorem gives the exactness of the sequence:

$$\cdots \to H^3(X) \to H^3(Y) \oplus H^3(Z) \to H^3(Y \cap Z) \to H^4(X) \to \cdots$$

Since our coefficients are real, the above cohomology groups are indeed real vector spaces. The exactness of the sequence then gives:

$$dim(H^3(Y \cap Z)) \leq dim(H^4(X)) + dim(H^3(Y) \oplus H^3(Z)). \tag{A.1}$$

Let us describe the sets involved above. First of all, observe that $Y = Y_1 \cup Y_2$ has two connected components:

$$Y_i = \{((x_1,t_1),(x_2,t_2)) \in \overline{\Sigma} \times \overline{\Sigma} : \ t_i > t_j, \ j \neq i\}.$$

To study Y_1, we define the following deformation retraction:

$$r_1 : Y_1 \to Y_1, \quad r_1((x_1,t_1),(x_2,t_2)) = ((x_1,1),(x_2,1/2)).$$

Clearly, $r_1(Y_1) = 0 \times (\Sigma \times \{1/2\})$, which is homeomorphic to Σ. Analogously, $Y_2 \simeq \Sigma$, and so $Y \simeq \Sigma \uplus \Sigma$ (here $\uplus$ stands for the disjoint union).

For what concerns Z, we can define a deformation retraction:

$$r : Z \to Z, \quad r((x_1,t_1),(x_2,t_2)) = ((x_1,1/2),(x_2,1/2)).$$

Observe that $r(Z) = (\Sigma \times \{1/2\} \times \Sigma \times \{1/2\}) \setminus \overline{D}$ which is homeomorphic to $\Sigma \times \Sigma \setminus D$, where D is the diagonal of $\Sigma \times \Sigma$. Let us set

$$A = \Sigma \times \Sigma \setminus D,$$

since it will appear many times in what follows.

Moreover, $Y \cap Z = (Y_1 \cap Z) \cup (Y_2 \cap Z)$, and so this has two connected components. Also here we have a deformation retraction:

$$r_1' : Y_1 \cap Z \to Y_1 \cap Z, \quad r_1'((x_1,t_1),(x_2,t_2)) = ((x_1,1/2),(x_2,1/3)).$$

It is clear that $r_1'(Y_1 \cap Z)$ is homeomorphic to $A = \Sigma \times \Sigma \setminus D$. Analogously we can argue for $Y_2 \cap Z$; therefore, $Y \cap Z \simeq A \,\dot\cup\, A$.

Hence, from (A.1) we obtain:

$$dim(H^4(X)) \geq dim(H^3(A)). \tag{A.2}$$

Let us now compute the cohomology of $A = \Sigma \times \Sigma \setminus D$. Given $\varepsilon > 0$, let us define:

$$B = \{(x,y) \in \Sigma \times \Sigma : \; d(x,y) < \varepsilon\},$$

which is an open neighborhood of D. Clearly, we can use the local contractibility of Σ to retract B onto D. Moreover, $A \cup B = \Sigma \times \Sigma$. The Mayer-Vietoris Theorem yields the exact sequence:

$$\cdots \to H^2(A \cap B) \to H^3(\Sigma \times \Sigma) \to H^3(A) \oplus H^3(B) \to H^3(A \cap B) \to \cdots \tag{A.3}$$

Therefore, in order to study $H^3(A)$ we need some information about $H^*(A \cap B)$.

By using the exponential map, we can define a homeomorphism:

$$h : A \cap B = \{(x,y) \in \Sigma \times \Sigma : \; 0 < d(x,y) < \varepsilon\} \to \{(x,v) \in T\Sigma : \; 0 < \|v\| < \varepsilon\},$$

$$h(x,y) = (x,v) \in T\Sigma \text{ such that } \exp_x(v) = y,$$

where $T\Sigma$ is the tangent bundle of Σ. Therefore, $A \cap B$ is homotopically equivalent to the unit tangent bundle $UT\Sigma$.

The cohomology groups of $UT\Sigma$ must be well known, but we have not been able to find a precise reference. We state and prove the following lemma:

Lemma A.2. *Let us denote by $g = g(\Sigma)$ the genus of Σ. Then:*

(1) *if $g = 1$, $H^0(UT\Sigma) \cong H^3(UT\Sigma) \cong \mathbb{R}$ and $H^1(UT\Sigma) \cong H^2(UT\Sigma) \cong \mathbb{R}^3$.*
(2) *if $g \neq 1$, $H^0(UT\Sigma) \cong H^3(UT\Sigma) \cong \mathbb{R}$ and $H^1(UT\Sigma) \cong H^2(UT\Sigma) \cong \mathbb{R}^{2g}$.*

Proof. We only need to compute $H^1(UT\Sigma)$ and $H^2(UT\Sigma)$.

If $g=1$, that is, $\Sigma \simeq \mathbb{T}^2$, then $T\Sigma$ is trivial and hence $UT\Sigma \simeq \mathbb{T}^2 \times \mathbb{S}^1 \simeq \mathbb{T}^3$. The Künneth formula gives us the result.

If $g \neq 1$, we use the Gysin exact sequence (see Proposition 14.33 of [1]):

$$0 \to H^1(\Sigma) \to H^1(UT\Sigma) \to H^0(\Sigma) \stackrel{\wedge e}{\to} H^2(\Sigma) \to H^2(UT\Sigma) \to H^1(\Sigma) \to H^3(\Sigma) = 0.$$

In the above sequence, $\wedge e$ is the wedge product with the Euler class e. Since we are working with real coefficients and the Euler characteristic of Σ is different from zero, then $\wedge e$ is an isomorphism. Therefore, we conclude:

$$H^1(UT\Sigma) \cong H^1(\Sigma) \cong \mathbb{R}^{2g}, \quad H^2(UT\Sigma) \cong H^1(\Sigma) \cong \mathbb{R}^{2g}.$$

□

Remark A.3. We have chosen real coefficients since they simplify our arguments and are enough for our purposes. As a counterpart, the above computations do not take into account the torsion part. For instance, it is known that $UT\mathbb{S}^2 = \mathbb{RP}^3$ (see [24]).

We now come back to the proof of Proposition A.1. With our information, (A.3) becomes:

$$\cdots \to H^2(A \cap B) \to \mathbb{R}^{4g} \to H^3(A) \to \mathbb{R} \to \cdots$$

In the above sequence we computed $H^3(\Sigma \times \Sigma)$ using the Künneth formula. Then, $4g \leq dim(H^2(UT\Sigma)) + dim(H^3(A))$. Therefore, $dim(H^3(A)) \geq 2g$, if $g > 1$, or $dim(H^3(A)) \geq 1$, if $g = 1$. In any case we conclude by (A.2).

□

Acknowledgment. The authors would like to thank the anonymous referee for suggesting Remark 3.12.

A. M. has been partially supported by GENIL (SPR) for a stay in Granada in 2011, and is supported by the FIRB project *Analysis and Beyond* from MIUR. D.R has been supported by the Spanish Ministry of Science and Innovation under Grant MTM2008-00988 and by J. Andalucia (FQM 116). Moreover, both authors have been supported by the Spanish Ministry of Science and Innovation under Grant MTM2011-26717.

Bibliography

[1] R. Bott and L. W. Tu, Differential forms in algebraic topology, Graduate Texts in Mathematics, 82. Springer-Verlag, New York-Berlin, 1982.

[2] W. X. Chen and C. Li, Prescribing Gaussian curvatures on surfaces with conical singularities, J. Geom. Anal. 1-4, (1991), 359-372.

[3] W. X. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63 (1991), no. 3, 615-622.

[4] C.C Chen and C.S. Lin, Topological degree for a mean field equation on Riemann surfaces, Comm. Pure Appl. Math. 56-12 (2003), 1667-1727.

[5] M. Chipot, I. Shafrir and G. Wolansky, On the solutions of Liouville systems, J. Differential Equations 140 (1997), no. 1, 59-105.

[6] F. De Marchis, Generic multiplicity for a scalar field equation on compact surfaces, J. Funct. Anal. 259 (2010) 2165-2192.

[7] W. Ding, J. Jost, J. Li and G. Wang, Existence results for mean field equations, Ann. Inst. Henri Poincaré, Anal. Non Linèaire 16-5 (1999), 653-666.

[8] Z. Djadli, Existence result for the mean field problem on Riemann surfaces of all genus, Comm. Contemp. Math. 10 (2008), no. 2, 205-220.

[9] Z. Djadli and A. Malchiodi, Existence of conformal metrics with constant Q-curvature, Annals of Math., 168 (2008), no. 3, 813-858.

[10] G. Dunne, Self-dual Chern-Simons Theories, Lecture Notes in Physics, vol. 36, Berlin: Springer-Verlag, 1995.

[11] M. A. Guest, Harmonic maps, loops groups, and integrable systems. London Mathematical Society Student Texts, 38. Cambridge University Press, Cambridge, 1997.

[12] A. Hatcher, Algebraic Topology, Cambridge University Press 2002.

[13] L. Jeanjean, On the existence of bounded Palais-Smale sequences and applications to a Landesman-Lazer type problem set on R^N, Proc. Roy. Soc. Edinburgh A 129 (1999) 787-809.

[14] J. Jost, C. S. Lin and G. Wang, Analytic aspects of the Toda system II. Bubbling behavior and existence of solutions, Comm. Pure Appl. Math. 59 (2006), 526-558.

[15] J. Jost and G. Wang, Classification of solutions of a Toda system in $\mathbb{R}^2$, Int. Math. Res. Not., 2002 (2002), 277-290.

[16] J. Jost and G. Wang, Analytic aspects of the Toda system I. A Moser-Trudinger inequality, Comm. Pure Appl. Math. 54 (2001), 1289-1319.

[17] J. Li and Y. Li, Solutions for Toda systems on Riemann surfaces, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4 (2005), no. 4, 703-728.

[18] C.S. Lin and L. Zhang, A Topological Degree Counting for Some Liouville Systems of Mean Field Type, Comm. Pure Appl. Math. 64 (2011), 556-590.

[19] M. Lucia, A deformation lemma with an application to a mean field equation, Topol. Methods Nonlinear Anal. 30 (2007), no. 1, 113-138.

[20] A. Malchiodi, Topological methods for an elliptic equation with exponential nonlinearities, Discrete Contin. Dyn. Syst. 21 (2008), no. 1, 277-294.

[21] A. Malchiodi, Morse theory and a scalar field equation on compact surfaces, Adv. Diff. Eq., 13 (2008), 1109-1129.

[22] A. Malchiodi and C. B. Ndiaye, Some existence results for the Toda system on closed surfaces, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 18 (2007), no. 4, 391-412.

[23] A. Malchiodi and D. Ruiz, New improved Moser-Trudinger inequalities and singular Liouville equations on compact surfaces, GAFA 21 (2011), 1196-1217.

[24] J. M. Montesinos, Classical Tesselations and Three-Manifolds, Springer-Verlag Berlin-Heidelberg, 1987.

[25] I. Shafrir and G. Wolansky, Moser-Trudinger and logarithmic HLS inequalities for systems, J. Eur. Math. Soc. 7-4 (2005), 413-448.

[26] M. Struwe, On the evolution of harmonic mappings of Riemannian surfaces, Comment. Math. Helv. 60 (1985) 558-581.

[27] M. Struwe and G. Tarantello, On multivortex solutions in Chern-Simons gauge theory, Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. (8)-1, (1998), 109-121.

[28] G. Tarantello, Self-Dual Gauge Field Vortices: An Analytical Approach, PNLDE 72, Birkhäuser Boston, Inc., Boston, MA, 2007.

[29] G. Tarantello, Analytical, geometrical and topological aspects of a class of mean field equations on surfaces, Discrete Contin. Dyn. Syst. 28 (2010), no. 3, 931-973.

[30] G. Wang, Moser-Trudinger inequalities and Liouville systems, C. R. Acad. Sci. Paris. Sér. I Math. 328 (1999), no. 10, 895-900.

[31] Y. Yang, Solitons in Field Theory and Nonlinear Analysis, Springer-Verlag, 2001.

Received Month 200X.