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Corresponding Author:	stefano marmi, Ph.D. scuola normale superiore Pisa, PI ITALY
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	scuola normale superiore
Corresponding Author's Secondary Institution:	
First Author:	stefano marmi, Ph.D.
First Author Secondary Information:	
Order of Authors:	stefano marmi, Ph.D. Jean-Christophe Yoccoz
Order of Authors Secondary Information:	
Author Comments:	

HÖLDER REGULARITY OF THE SOLUTIONS OF THE COHOMOLOGICAL EQUATION FOR ROTH TYPE INTERVAL EXCHANGE MAPS

STEFANO MARMI AND JEAN-CHRISTOPHE YOCOZ

ABSTRACT. We prove that the solutions of the cohomological equation for Roth type interval exchange maps are Hölder continuous provided that the datum is of class C^r with $r > 1$ and belongs to a finite-codimension linear subspace.

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1. INTRODUCTION

1.1. Roth type interval exchange maps. An *interval exchange map* (i.e.m.) T on a (finite length) interval I is a one-to-one map which is locally a translation except at a finite number of discontinuities. Clearly T is orientation-preserving and preserves Lebesgue measure. When the number d of intervals of continuity of T is equal to 2 then (modulo identification of the endpoints of I) an i.e.m. corresponds to a rotation of the circle. It can be thought as the first return map of a linear flow on a two-dimensional torus on a transversal circle. Analogously when $d \geq 3$ by singular suspension any i.e.m. is related to the linear flow on a suitable translation surface (see, e.g. [Ve1] for details, or section 2.2 below) typically having genus higher than 1. All translation surfaces obtained by suspension from i.e.m. with the same combinatorics have the same genus g , and the same number s of marked points; these numbers are related to the number d of intervals of continuity of T by the formula $d = 2g + s - 1$.

Rauzy and Veech have defined an algorithm that generalizes the classical continued fraction algorithm (corresponding to the choice $d = 2$) and associates to an i.e.m. another i.e.m. which is its first return map to an appropriate subinterval [Ra, Ve2]. The Rauzy-Veech algorithm stops if and only if the i.e.m. has a *connection*, i.e. a finite orbit which starts and ends at a discontinuity. Both Rauzy-Veech "continued fraction" algorithm and its accelerated version due to Zorich [Zo2] (see also [MMY1]) are ergodic w.r.t. an absolutely continuous invariant measure in the space of normalized standard i.e.m.'s. The ergodic properties of these renormalization dynamics in parameter space have been studied in detail ([Ve3],[Ve4], [Zo3], [Zo4], [AGY], [B], [AB], [Y4]).

In [MMY1] we introduced a class of interval exchange maps, called *Roth type*, which has full measure in parameter space and for which the cohomological equation could be solved (after a finite-dimensional correction) with a loss of differentiability of about two derivatives (one obtains a continuous solution starting from a datum of class C^1 on each interval of continuity of T and with a derivative of bounded variation). Roth type i.e.m. generalize Roth type irrational numbers. Roth type numbers have a purely arithmetical definition but can also be characterized by means of the regularity of the solutions of the cohomological equation associated to the rotation $R_\alpha : x \mapsto x + \alpha$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Indeed an irrational number α is of Roth type if and only if for all $r, s \in \mathbb{R}$ with $r > s + 1 \geq 1$ and for all functions Φ of class C^r on \mathbb{T} with mean zero, there exists a unique function Ψ of class C^s on \mathbb{T} with mean zero such that $\Phi = \Psi \circ R_\alpha - \Psi$.

The goal of this paper is to prove Theorem 3.10, i.e. a stronger regularity result than in [MMY1] for the solutions of the cohomological equation and which is closer to the optimal result (in the case of circle rotations) quoted above. It is stated in full generality in section 3.4. Here we give a less precise statement:

Theorem 1.1. *Let T be an interval exchange map with no connection and of restricted Roth type. Let r be a real number > 1 . There exists $\bar{\delta} > 0$ such that given any function φ of class C^r on each interval of continuity of T which belongs to the kernel of the boundary operator (defined in section 2.6) one can find a piecewise constant function χ and a $\bar{\delta}$ -Hölder continuous function ψ such that*

$$\varphi = \chi + \psi \circ T - \psi .$$

In the theorem one has $\bar{\delta} < r - 1$ and the i.e.m. T are the Roth type i.e.m. for which the Lyapunov exponents of the KZ-cocycle (see section 2.5) are non zero (in [MMY2] we call this *restricted Roth type*). They still form a full measure set by Forni's theorem [For2].

The first step in the direction of extending small divisor results beyond the torus case was achieved by Forni's important paper [For1] on the cohomological equation associated to linear flows on surfaces of higher genus. He solved the cohomological equation on any translation surface for almost every direction and worked in the Sobolev scale obtaining a loss of differentiability of at most $3 + \varepsilon$ derivatives (for every $\varepsilon > 0$). The improved loss of regularity obtained in [MMY1] (w.r.t. [For1]) turned out to be decisive for the proof of a local conjugacy theorem for deformations of interval exchange maps (proven in [MMY2]). We recall that the cohomological equation is the linearization of the conjugacy equation $T \circ h = h \circ T_0$ for a generalized i.e.m. T (see [MMY2]) close to the standard i.e.m. T_0 . In [For3] sharper results on the loss of differentiability for solutions of the cohomological equations for translations surfaces have been proved: for almost all translation surface in every stratum of the moduli space and for almost all direction the loss of Sobolev regularity is $1 + \varepsilon$ (for any $\varepsilon > 0$). As in [For1] the i.e.m. case is not considered, nor an explicit diophantine condition is obtained. Our Theorem 3.10 deals with the loss of Hölder regularity (which turns out to be $1 + \varepsilon$ for *some* $\varepsilon > 0$) and we get an explicit diophantine condition, but we do not know if almost every direction on a given translation surface leads to a restricted Roth type i.e.m. .

The tools of the proof of Theorem 3.10 allow us to prove Theorem 3.11, a result of independent interest: when T is of restricted Roth type, for a function φ of class C^1 on each interval of continuity of T which belongs to the kernel of the boundary operator, there exists a piecewise constant function χ such that the growth of the Birkhoff sums of $(\varphi - \chi)$ is slower than any positive power of time.

1.2. Summary of the contents. In section 2 we introduce interval exchange maps and we develop the continued fraction algorithms to an extent which will allow us to introduce Roth type i.e.m.. Sections 2.1 and 2.2 are devoted to recalling the basic definitions and the construction of the suspension. The elementary step of the Rauzy-Veech continued fraction algorithm is described in section 2.3. When the i.e.m. has no connection the Rauzy-Veech algorithm (section 2.4) can be iterated indefinitely. In section 2.5 we introduce the discrete time Kontsevich-Zorich cocycle and we discuss its relationship with return times and with (special) Birkhoff sums of piecewise constant functions. After having introduced the boundary operator (section 2.6) and having briefly described its homological interpretation we conclude (section 2.7) with some consequences of the symplecticity of the Kontsevich-Zorich cocycle.

Section 3 is devoted to the study of the cohomological equation and to the proof of our main theorem. We first recall (section 3.1) the definition of i.e.m. of restricted Roth type which was given in [MMY2] and deduce (section 3.2) some consequences of the hyperbolicity of the Kontsevich-Zorich cocycle. After reviewing previous results (section 3.3) on the regularity of the solutions of the cohomological equation [For1, For3], and especially those obtained in [MMY1] and refined in [MMY2], in section 3.4 we give a precise statement of our main result Theorem 3.10.

The strategy of the proof is similar to our previous papers. In section 3.5 we prove Theorem 3.11 by estimating special Birkhoff sums of piecewise C^1 functions. In section 3.6 we obtain better estimates for piecewise C^r functions. In sections 3.7 and 3.8 we introduce time and space decompositions which relate in particular general Birkhoff sums to special Birkhoff sums. In section 3.9 we obtain the required Hölder estimate for pairs of points which are in special relative position; then the space decomposition allows us to get the general estimate (section 3.10). Section 3.11 deals with the case of higher differentiability ($r > 2$).

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2. BACKGROUND ON INTERVAL EXCHANGE MAPS

2.1. Interval exchange maps. Let \mathcal{A} be an alphabet with $d \geq 2$ symbols.

Definition 2.1. An *interval exchange map* (i.e.m.) T , acting upon an open bounded interval $I(T)$, is defined by two partitions mod.0 of $I(T)$ into d open subintervals indexed by \mathcal{A} (the *top* and *bottom* partitions):

$$I(T) = \sqcup I_\alpha^t(T) = \sqcup I_\alpha^b(T) \text{ mod } 0$$

such that $|I_\alpha^t(T)| = |I_\alpha^b(T)|$ for all $\alpha \in \mathcal{A}$. The restriction of T to $I_\alpha^t(T)$ is the translation onto $I_\alpha^b(T)$.

The domain of the map T is $\sqcup I_\alpha^t(T)$, the domain of T^{-1} is $\sqcup I_\alpha^b(T)$.

Definition 2.2. Let T be an i.e.m. The points $u_1(T) < \dots < u_{d-1}(T)$ separating the $I_\alpha^t(T)$ are called the *singularities of T* . The points $v_1(T) < \dots < v_{d-1}(T)$ separating the $I_\alpha^b(T)$ are called the *singularities of T^{-1}* . We also write $I(T) = (u_0(T), u_d(T)) = (v_0(T), v_d(T))$.

Definition 2.3. The *combinatorial data* of an i.e.m T is the pair $\pi = (\pi_t, \pi_b)$ of bijections from \mathcal{A} onto $\{1, \dots, d\}$ such that

$$I_\alpha^t(T) = (u_{\pi_t(\alpha)-1}(T), u_{\pi_t(\alpha)}(T)), \quad I_\alpha^b(T) = (v_{\pi_b(\alpha)-1}(T), v_{\pi_b(\alpha)}(T))$$

for each $\alpha \in \mathcal{A}$.

We always assume that the combinatorial data are *irreducible*: for $1 \leq k < d$, we have

$$\pi_t^{-1}(\{1, \dots, k\}) \neq \pi_b^{-1}(\{1, \dots, k\}).$$

2.2. Suspension and genus. Let T be an i.e.m. with combinatorial data $\pi = (\pi_t, \pi_b)$ acting on an interval $I(T) = (u_0(T), u_d(T))$. For $\alpha \in \mathcal{A}$ define

$$\lambda_\alpha = |I_\alpha^t(T)| = |I_\alpha^b(T)|, \quad \zeta_\alpha = \lambda_\alpha + \sqrt{-1}(\pi_b(\alpha) - \pi_t(\alpha)),$$

and then, for $0 \leq i \leq d$

$$U_i = u_0(T) + \sum_{\pi_t \alpha \leq i} \zeta_\alpha, \quad V_i = u_0(T) + \sum_{\pi_b \alpha \leq i} \zeta_\alpha.$$

One has $U_0 = V_0 = u_0(T)$, $U_d = V_d = u_d(T)$, and $\Im U_i > 0 > \Im V_i$ for $0 < i < d$. The $2d$ segments $[U_{i-1}, U_i]$, $[V_{i-1}, V_i]$ ($0 < i \leq d$) form the boundary of a polygon. Gluing the parallel top and bottom ζ_α sides of this polygon produces a translation surface M_T ([Zo1]), in which the vertices of the polygon define a set Σ of marked points.

The cardinality s of Σ , the genus g of M_T and the number d of intervals are related by

$$d = 2g + s - 1.$$

Both g and s only depend on π .

The genus g can be computed from the combinatorial data as follows. Define an anti-symmetric matrix $\Omega = \Omega(\pi)$ by

$$\Omega_{\alpha\beta} = \begin{cases} +1 & \text{if } \pi_t(\alpha) < \pi_t(\beta), \pi_b(\alpha) > \pi_b(\beta), \\ -1 & \text{if } \pi_t(\alpha) > \pi_t(\beta), \pi_b(\alpha) < \pi_b(\beta), \\ 0 & \text{otherwise.} \end{cases}$$

Then the rank of Ω is $2g$. Actually ([Y1],[Y4]), if one identifies \mathbb{R}^A with the relative homology group $H_1(M_T, \Sigma, \mathbb{R})$ via the basis defined by the sides ζ_α of the polygon, the image of Ω coincides with the absolute homology group $H_1(M_T, \mathbb{R})$. Another way to compute s (and thus g) consists in going around the marked points, as explained in section 2.6.

2.3. The elementary step of the Rauzy-Veech algorithm.

Definition 2.4. Let T be an interval exchange map. A *connection* is a triple (u_i, v_j, m) consisting of a singularity of T , a singularity of T^{-1} and a nonnegative integer which satisfy

$$T^m(v_j) = u_i.$$

Keane has proved [Kea1] that an i.e.m. with no connection is minimal. Having no connection means in particular that no point is both a singularity of T and T^{-1} .

Definition 2.5. Let T be an i.e.m. such that $u_{d-1}(T) \neq v_{d-1}(T)$. This is the case if T has no connection. Define $\hat{u}_d = \hat{v}_d := \max(u_{d-1}(T), v_{d-1}(T))$, $\hat{I} := (u_0(T), \hat{u}_d)$. The first return map of T to \hat{I} , denoted by \hat{T} , is an i.e.m whose combinatorial data $\hat{\pi}$ are canonically labeled by the same alphabet \mathcal{A} than π (cf.[MMY1] p.829). The return time is 1 or 2. The process $T \mapsto \hat{T}$ is the *elementary step of the Rauzy-Veech renormalization algorithm*.

The combinatorial data $\hat{\pi}$ of \hat{T} are irreducible.

If T has no connection, then \hat{T} has no connection. This allows to iterate indefinitely the elementary step of the algorithm when T has no connection.

2.4. The Rauzy-Veech algorithm. Let T be an i.e.m. with no connection acting on a bounded open interval $I(T)$.

Starting from $T(0) := T$, we iterate the elementary step of the Rauzy-Veech algorithm described in section 2.3. We obtain a sequence $(T(n))_{n \geq 0}$ of i.e.m acting on a decreasing sequence of intervals $I(n) := I(T(n))$ with the same left endpoint $u_0(T)$. For $n \geq m$, the i.e.m. $T(n)$ is the first return map of $T(m)$ to $I(n)$. We denote by $\pi(n)$ the combinatorial data of $T(n)$.

2.5. The discrete time Kontsevich-Zorich cocycle. Let T be an i.e.m..

Definition 2.6. We denote by $\Gamma(T)$ the space of functions on $\sqcup I_\alpha^t(T)$ which are constant on each $I_\alpha^t(T)$. It is canonically identified with \mathbb{R}^A . For a real number $r \geq 0$ we denote by $C^r(\mathbf{u}(T))$ the product $\prod_1^d C^r([u_{i-1}(T), u_i(T)])$.

Definition 2.7. Assume that T has no connection. Let $(T(n))_{n \geq 0}$ be the sequence defined by the Rauzy-Veech algorithm. For $n \geq m \geq 0$ and $\varphi \in C^0(\mathbf{u}(T(m)))$ the *special Birkhoff sum* $S(m, n)\varphi$ is the function in $C^0(\mathbf{u}(T(n)))$ defined by

$$S(m, n)\varphi(x) = \sum_{0 \leq j < r(x)} \varphi \circ T(m)^j(x),$$

where $x \in [u_{i-1}(T(n)), u_i(T(n))] \subset I(n)$ and $r(x)$ is the return time of x in $I(n)$ under $T(m)$.

For $m \leq n \leq p$, one has the cocycle relation

$$S(m, p) = S(n, p) \circ S(m, n).$$

The operator $S(m, n)$ maps $\Gamma(T(m))$ onto $\Gamma(T(n))$. We denote by $B(m, n)$ the matrix of the restriction of $S(m, n)$ to $\Gamma(T(m))$ w.r.t. the canonical bases of $\Gamma(T(m))$ and $\Gamma(T(n))$. The coefficient $B_{\alpha\beta}(m, n)$ is therefore equal to the number of visits under $T(m)$ of $I_\alpha(n)$ into $I_\beta(m)$ before returning to $I(n)$. For $x \in I_\alpha(n)$, the return time $r(x)$ is equal to $B_\alpha(m, n) := \sum_\beta B_{\alpha\beta}(m, n)$. We have the partition

$$(2.1) \quad I(m) = \bigsqcup_{\alpha \in \mathcal{A}} \bigsqcup_{0 \leq j < B_\alpha(m, n)} T(m)^j (I_\alpha^t(n)) \bmod{0}.$$

The matrices $B(m, n)$ satisfy the cocycle relation

$$B(m, p) = B(n, p)B(m, n).$$

They define the (*extended*) *Kontsevich-Zorich cocycle*.

Let $m \geq 0$, and let α_t, α_b be the letters of \mathcal{A} such that $\pi_t(m)\alpha_t = \pi_b(m)\alpha_b = d$. For $\alpha, \beta \in \mathcal{A}$, let $E_{\alpha\beta}$ be the elementary matrix whose only nonzero coefficient is equal to 1 in position $\alpha\beta$. The matrix $B(m, m+1)$ is equal to $\mathbf{1} + E_{\alpha_t\alpha_b}$ if $u_{d-1}(T(m)) > v_{d-1}(T(m))$ and to $\mathbf{1} + E_{\alpha_b\alpha_t}$ if $u_{d-1}(T(m)) < v_{d-1}(T(m))$.

It follows that $B(m, n)$ belongs to $SL(\mathbb{Z}^{\mathcal{A}})$ and has nonnegative coefficients. Moreover, for each $\alpha, \beta \in \mathcal{A}$, the sequence $B_{\alpha\beta}(m, n)$ is a nondecreasing function of n . It follows from [MMY1] p. 832-833 that for fixed m and n large enough all coefficients $B_{\alpha\beta}(m, n)$ are strictly positive.

We define a sequence of integers $(n_k)_{k \geq 0}$ as follows: $n_0 = 0$ and n_{k+1} is the smallest integer such that all coefficients of $B(n_k, n_{k+1})$ are strictly positive.

2.6. The boundary operator. Let π be irreducible combinatorial data over the alphabet \mathcal{A} .

Define a $2d$ -element set \mathcal{S} by

$$\mathcal{S} = \{U_0 = V_0, U_1, V_1, \dots, U_{d-1}, V_{d-1}, U_d = V_d\}.$$

These symbols correspond to the vertices of the polygon produced by the suspension of an i.e.m T with combinatorial data π (section 2.2).

Going anticlockwise around the vertices (taking the gluing into account) produces a permutation σ of \mathcal{S} :

$$\begin{aligned} \sigma(U_i) &= V_j & \text{with } \pi_b^{-1}(j+1) &= \pi_t^{-1}(i+1) & \text{for } 0 \leq i < d \\ \sigma(V_{j'}) &= U_{i'} & \text{with } \pi_t^{-1}(i') &= \pi_b^{-1}(j') & \text{for } 0 < j' \leq d. \end{aligned}$$

The cycles of σ in \mathcal{S} are canonically associated to the marked points on the translation surface M_T obtained by suspension of T . We will denote by Σ the set of cycles of σ , by s the cardinality of Σ (cf. section 2.2).

Definition 2.8. Let $\varphi \in C^0(\mathbf{u}(T))$. We write $\varphi(u_i(T) - 0)$ (resp. $\varphi(u_i(T) + 0)$) for its value at $u_i(T)$ considered as a point in $[u_{i-1}(T), u_i(T)]$ (resp. $[u_i(T), u_{i+1}(T)]$). We will use the convention that $\varphi(u_0(T) - 0) = \varphi(u_d(T) + 0) = 0$.

Definition 2.9. Let T be an i.e.m. with combinatorial data π , and let Σ be the set of cycles of the associated permutation σ of \mathcal{S} . The *boundary operator* ∂ is the linear operator from

$C^0(\mathbf{u}(T))$ to \mathbb{R}^Σ defined by

$$(2.2) \quad (\partial\varphi)_C := \sum_{0 \leq i \leq d, U_i \in C} (\varphi(u_i(T) - 0) - \varphi(u_i(T) + 0)),$$

for any $\varphi \in C^0(\mathbf{u}(T))$, $C \in \Sigma$.

Remark 2.10. The presentation of the permutation σ is different from [MMY2].

Remark 2.11. If $\varphi \in C^0(\mathbf{u}(T))$ satisfies $\varphi(u_i(T) + 0) = \varphi(u_i(T) - 0)$ for $0 \leq i \leq d$ then $\partial\varphi = 0$.

Remark 2.12. Let $\varphi \in C^1(\mathbf{u}(T))$. One has

$$\sum_{C \in \Sigma} (\partial\varphi)_C = \int_{u_0(T)}^{u_d(T)} D\varphi(x) dx.$$

Remark 2.13. The intersection of every cycle of σ with the $(d+1)$ -subset $\{U_0, \dots, U_d\}$ of \mathbb{S} has at least two elements.

Remark 2.14. The name boundary operator is due to the following homological interpretation. The space $\Gamma(T) \subset C^0(\mathbf{u}(T))$ is naturally isomorphic to the first relative homology group $H_1(M_T, \Sigma, \mathbb{R})$ of the translation surface M_T : the characteristic function of $I_\alpha^t(T)$ corresponds to the homology class $[\zeta_\alpha]$ of the side ζ_α (oriented rightwards) of the polygon giving rise to M_T . Through this identification, the restriction of the boundary operator to $\Gamma(T)$ is the usual boundary operator

$$\partial : H_1(M_T, \Sigma, \mathbb{R}) \longrightarrow H_0(\Sigma, \mathbb{R}) \simeq \mathbb{R}^\Sigma.$$

We recall the following statement from [MMY2] (Proposition 3.2, p. 1597):

Proposition 2.15. *Let T be an i.e.m. with combinatorial data π .*

- (1) *Let $\psi \in C^0([u_0(T), u_d(T)])$. One has $\partial\psi = \partial(\psi \circ T)$.*
- (2) *The kernel $\Gamma_\partial(T)$ of the restriction of ∂ to $\Gamma(T)$ is the image of $\Omega(\pi)$; the image of this restriction is the hyperplane $\mathbb{R}_0^\Sigma := \{x \in \mathbb{R}^\Sigma, \sum_C x_C = 0\}$.*
- (3) *The boundary operator $\partial : C^0(\mathbf{u}(T)) \rightarrow \mathbb{R}^\Sigma$ is onto.*
- (4) *Let $n \geq 0$, let $T(n)$ be the i.e.m. obtained from T by n steps of the Rauzy–Veech algorithm. For $\varphi \in C^0(\mathbf{u}(T))$ one has*

$$\partial(S(0, n)\varphi) = \partial\varphi,$$

where the left-hand side boundary operator is defined using the combinatorial data $\pi(n)$ of $T(n)$.

2.7. Symplecticity of the Kontsevich-Zorich cocycle. Let T be an i.e.m. with no connection. We use the notations of section 2.5. For $0 \leq m \leq n$, one has (see [Y4], p. 28)

$$(2.3) \quad B(m, n)\Omega(\pi(m))^t B(m, n) = \Omega(\pi(n)).$$

The above relation has several consequences:

- The operator ${}^t B(m, n)^{-1}$ maps the kernel $\text{Ker } \Omega(\pi(m))$ of $\Omega(\pi(m))$ onto the kernel $\text{Ker } \Omega(\pi(n))$ of $\Omega(\pi(n))$.
- The operator $B(m, n)$ maps the image $\text{Im } \Omega(\pi(m))$ of $\Omega(\pi(m))$ onto the image $\text{Im } \Omega(\pi(n))$ of $\Omega(\pi(n))$.

- The formula

$$\langle \Omega(\pi(m))v, \Omega(\pi(m))w \rangle := {}^t v \Omega(\pi(m))w$$

defines a symplectic structure on $\text{Im } \Omega(\pi(m))$, and similarly on $\text{Im } \Omega(\pi(n))$. The operator $B(m, n)$ is symplectic with respect to these structures.

Moreover (see [Y4], p. 28), one can choose for every irreducible combinatorial data π a basis of $\text{Ker } \Omega(\pi)$ such that the matrix of

$${}^t B(m, n)^{-1} : \text{Ker } \Omega(\pi(m)) \rightarrow \text{Ker } \Omega(\pi(n))$$

w.r.t. the corresponding bases is the identity matrix.

3. THE COHOMOLOGICAL EQUATION

3.1. Interval exchange maps of restricted Roth Type. We recall the diophantine condition on an i.e.m. introduced in [MMY1] and slightly modified in [MMY2].

For a matrix $B = (B_{\alpha\beta})_{\alpha, \beta \in \mathcal{A}}$, we define $\|B\| := \max_{\alpha} \sum_{\beta} |B_{\alpha\beta}|$, which is the operator norm for the ℓ_{∞} norm on $\mathbb{R}^{\mathcal{A}}$.

Let T be an i.e.m. with combinatorial data π . In the space $\Gamma(T) \simeq \mathbb{R}^{\mathcal{A}}$ of piecewise constant functions on $I(T)$, we define the hyperplane

$$\Gamma_0(T) := \left\{ \chi \in \Gamma(T), \int \chi(x) dx = 0 \right\}.$$

Assuming that T has no connection, we also define the stable subspace

$$\Gamma_s(T) := \left\{ \chi \in \Gamma(T), \exists \sigma > 0, \|B(0, n)\chi\| = \mathcal{O}(\|B(0, n)\|^{-\sigma}) \right\}.$$

As $\Gamma_s(T)$ is finite-dimensional, there exists an exponent $\sigma > 0$ which works for every $\chi \in \Gamma_s(T)$. We fix such an exponent in the rest of the paper.

The subspace $\Gamma_s(T)$ is contained in $\text{Im } \Omega(\pi)$, and is an isotropic subspace of this symplectic space. We have $\Gamma_s(T(n)) = B(0, n)\Gamma_s(T)$ for $n \geq 0$.

We introduce four conditions. The sequence (n_k) in the first condition has been defined in section 2.5.

(a) For all $\tau > 0$, $\|B(n_k, n_{k+1})\| = \mathcal{O}(\|B(0, n_k)\|^{\tau})$.

(b) There exists $\theta > 0$ such that

$$\|B(0, n)|_{\Gamma_0(T)}\| = \mathcal{O}(\|B(0, n)\|^{1-\theta}).$$

(c) For $0 \leq m \leq n$, let

$$\begin{aligned} B_s(m, n) &: \Gamma_s(T(m)) \rightarrow \Gamma_s(T(n)), \\ B_b(m, n) &: \Gamma(T(m))/\Gamma_s(T(m)) \rightarrow \Gamma(T(n))/\Gamma_s(T(n)) \end{aligned}$$

be the operators induced by $B(m, n)$. We ask that, for all $\tau > 0$,

$$\|B_s(m, n)\| = \mathcal{O}(\|B(0, n)\|^{\tau}), \quad \|(B_b(m, n))^{-1}\| = \mathcal{O}(\|B(0, n)\|^{\tau}).$$

(d) $\dim \Gamma_s(T) = g$.

Definition 3.1. An i.e.m. T with no connection is of *restricted Roth type* if the four conditions (a), (b), (c), (d) are satisfied.

Remark 3.2. In [MMY1], condition (a) had a slightly different (but equivalent) formulation. Interval exchange maps satisfying the first three conditions (a),(b),(c) were said to be of *Roth type*.

Remark 3.3. An i.e.m. T with no connection satisfying condition (b) is uniquely ergodic.

Remark 3.4. For any combinatorial data, the set of i.e.m of restricted Roth type has full measure. First, it is obvious that almost all i.e.m. have no connection. That condition (c) is almost surely satisfied is a consequence of Oseledets theorem applied to the Kontsevich-Zorich cocycle. Condition (d) follows from the hyperbolicity of the (restricted) Kontsevich-Zorich cocycle, proved by Forni ([For2]). A proof that condition (a) has full measure is provided in [MMY1], but much better diophantine estimates were later obtained in [AGY]. Finally, the fact that condition (b) has full measure is a consequence from the fact that the larger Lyapunov exponent of the Kontsevich-Zorich cocycle is simple (see [Ve4]).

3.2. Some consequences of the hyperbolicity of the Kontsevich-Zorich cocycle. Assume that T satisfies condition (d) of the previous section. Then the subspace $\Gamma_s(T)$ is a lagrangian subspace of the symplectic space $\text{Im } \Omega(\pi)$. Choose a basis \mathfrak{B}_0 of $\Gamma(T)$ such that the first g vectors of \mathfrak{B}_0 form a basis of $\Gamma_s(T)$ and the first $2g$ vectors of \mathfrak{B}_0 form a symplectic basis of $\text{Im } \Omega(\pi)$.

For each $n \geq 0$, choose a basis \mathfrak{B}_n of $\Gamma(T(n))$ such that

- the first g vectors of \mathfrak{B}_n form a basis of $\Gamma_s(T(n))$;
- the first $2g$ vectors of \mathfrak{B}_n form a symplectic basis of $\text{Im } \Omega(\pi(n))$;
- consider the operator from $\Gamma(T)/\text{Im } \Omega(\pi)$ to $\Gamma(T(n))/\text{Im } \Omega(\pi(n))$ induced by $B(0, n)$. Its matrix representation w.r.t. the bases given by the last $(s - 1)$ vectors of \mathfrak{B}_0 , \mathfrak{B}_n is the identity matrix (see section 2.7);
- The sequence of bases \mathfrak{B}_n has bounded distortion: there exists a constant $c > 0$ independent of n such that, for all $n \geq 0$, $a \in \mathbb{R}^{\mathfrak{B}_n}$

$$c^{-1} \|a\|_\infty \leq \left\| \sum_{\chi \in \mathfrak{B}_n} a_\chi \chi \right\|_\infty \leq c \|a\|_\infty .$$

Achieving this is possible since the orthogonal group acts transitively on the set of lagrangian subspaces of \mathbb{R}^{2n} .

For $m \leq n$, the operator $B(m, n)$ sends $\text{Im } \Omega(\pi(m))$ onto $\text{Im } \Omega(\pi(n))$ and is symplectic for the natural symplectic forms on these two spaces. The matrix of this operator with respect to the bases $\mathfrak{B}_m, \mathfrak{B}_n$ has the block triangular form

$$\begin{pmatrix} b_s(m, n) & \star & \star \\ 0 & b_u(m, n) & \star \\ 0 & 0 & \mathbf{1} \end{pmatrix}$$

The block $\begin{pmatrix} b_u(m, n) & \star \\ 0 & \mathbf{1} \end{pmatrix}$ is the matrix of the operator $B_b(m, n)$, with respect to the bases induced by $\mathfrak{B}_m, \mathfrak{B}_n$. Let us write $\begin{pmatrix} b_u^{-1}(m, n) & \mathfrak{b}(m, n) \\ 0 & \mathbf{1} \end{pmatrix}$ for the inverse of this block.

From the symplecticity of the restriction of $B(m, n)$ to $\text{Im } \Omega(\pi(m))$, one has

$$(3.1) \quad b_u^{-1}(m, n) = {}^t b_s(m, n).$$

When $m = 0$, this implies that we have, from the definition of $\Gamma_s(T)$

$$(3.2) \quad \|b_u^{-1}(0, n)\| \leq C \|B(0, n)\|^{-\sigma}.$$

The cocycle relation $B(m, p) = B(n, p)B(m, n)$ ($m \leq n \leq p$) gives

$$(3.3) \quad \mathbf{b}(m, p) = \mathbf{b}(m, n) + {}^t b_s(m, n)\mathbf{b}(n, p).$$

Taking $m = 0$ this leads to

$$(3.4) \quad \mathbf{b}(0, p) = \sum_{n=0}^{p-1} {}^t b_s(0, n)\mathbf{b}(n, n+1).$$

which gives $\|\mathbf{b}(0, p)\| \leq C$.

More generally, we also get

$$(3.5) \quad \mathbf{b}(m, p) = \sum_{n=m}^{p-1} {}^t b_s(m, n)\mathbf{b}(n, n+1).$$

Assuming the first half of condition (c) in the last section

$$\|b_s(m, n)\| = \mathcal{O}(\|B(0, n)\|^\tau), \quad \forall \tau > 0,$$

we obtain

$$(3.6) \quad \|\mathbf{b}(m, p)\| = \mathcal{O}\left(\sum_{n=m}^{p-1} \|B(0, n)\|^\tau\right) = \mathcal{O}(\|B(0, p)\|^\tau), \quad \forall \tau > 0.$$

As we have also $\|b_u^{-1}(m, p)\| = \|{}^t b_s(m, p)\| = \mathcal{O}(\|B(0, p)\|^\tau)$, we conclude that

Proposition 3.5. *When T satisfies condition (d) of section 3.1, the first part of condition (c) implies the second part.*

3.3. Previous results on the cohomological equation. To state results on the cohomological equation, we consider Hölder regularity conditions, like in Appendix A of [MMY2], rather than the bounded variation setting of [MMY1] and of the main text of [MMY2].

Definition 3.6. Let T be an i.e.m. and let $r \geq 0$ be a real number. We denote by $C_0^r(\mathbf{u}(T))$ the kernel of the boundary operator ∂ in $C^r(\mathbf{u}(T))$.

Theorem 3.7. *Let T be an i.e.m. of restricted Roth type. Let $\Gamma_u(T)$ be a subspace of $\Gamma_\partial(T)$ which is supplementing $\Gamma_s(T)$ in $\Gamma_\partial(T)$. Let r be a real number > 1 . There exist bounded linear operators $L_0 : \varphi \mapsto \psi$ from $C_0^r(\mathbf{u}(T))$ to $C^0([u_0(T), u_d(T)])$ and $L_1 : \varphi \mapsto \chi$ from $C_0^r(\mathbf{u}(T))$ to $\Gamma_u(T)$ such that any $\varphi \in C_0^r(\mathbf{u}(T))$ satisfies*

$$\varphi = \chi + \psi \circ T - \psi, \quad \int_{u_0(T)}^{u_d(T)} \psi(x) dx = 0.$$

The proof is a combination of the proof of Theorem A in [MMY1], of Remark 3.11 of [MMY2], and of Appendix A of the same paper. The main steps of the proof will be recalled in the next sections, where a stronger result, stated below, is proved.

Remark 3.8. By Proposition 2.15 part (a), the coboundary of a continuous function belongs to the kernel of the boundary operator.

Remark 3.9. The operators L_0 and L_1 are uniquely defined by the conclusions of the theorem. For L_1 , it follows from Remark 3.12 below. Then, as T is minimal, L_0 is also uniquely defined.

3.4. Hölder regularity of the solutions of the cohomological equation. Our main result is an improvement on the regularity of the solutions of the cohomological equation.

Theorem 3.10. *In the setting of Theorem 3.7, there exists $\bar{\delta} > 0$ such that the operator L_0 takes its value in the space $C^{\bar{\delta}}([u_0(T), u_d(T)])$ of Hölder continuous functions of exponent $\bar{\delta}$. The operator L_0 is bounded from $C_0^r(\mathbf{u}(T))$ to $C^{\bar{\delta}}([u_0(T), u_d(T)])$. The exponent $\bar{\delta}$ depends only on r and the constants θ, σ appearing in section 3.1.*

The rest of the paper is devoted to the proof of this theorem.

3.5. Estimates for special Birkhoff sums with C^1 data. The following result is of independent interest.

Theorem 3.11. *Let T be an i.e.m. which satisfies conditions (a), (c), (d) of section 3.1. Let $\Gamma_u(T)$ be a subspace of $\Gamma_{\partial}(T)$ which is supplementing $\Gamma_s(T)$ in $\Gamma_{\partial}(T)$. The operator $L_1 : \varphi \mapsto \chi$ of Theorem 3.7 extends to a bounded operator from $C_0^1(\mathbf{u}(T))$ to $\Gamma_u(T)$. For $\varphi \in C_0^1(\mathbf{u}(T))$, the function $\chi = L_1(\varphi) \in \Gamma_u(T)$ is characterized by the property that the special Birkhoff sums of $\varphi - \chi$ satisfy, for any $\tau > 0$*

$$\|S(0, n)(\varphi - \chi)\|_{C^0} \leq C(\tau) \|\varphi\|_{C^1} \|B(0, n)\|^\tau.$$

Remark 3.12. The function $\chi \in \Gamma_u(T)$ is characterized by the inequality of the theorem. We have to show that if a function $\chi \in \Gamma_u(T)$ satisfies

$$\|B(0, n)\chi\| \leq C_\tau \|B(0, n)\|^\tau \|\chi\|,$$

for all $\tau > 0$, then $\chi = 0$.

Indeed, choose bases \mathfrak{B}_n of $\Gamma(T(n))$ as in section 3.2, and write $\chi = \begin{pmatrix} \star \\ \chi_u \\ 0 \end{pmatrix}$ in \mathfrak{B}_0 .

Then the image $B(0, n)\chi$ has the form $\begin{pmatrix} \star \\ b_u(0, n)\chi_u \\ 0 \end{pmatrix}$ in \mathfrak{B}_n . Therefore, we have, for all $\tau > 0$

$$\|b_u(0, n)\chi_u\| \leq C_\tau \|B(0, n)\|^\tau \|\chi\|.$$

From inequality (3.2), we must have $\chi_u = 0$ thus, as $\chi \in \Gamma_u(T)$, $\chi = 0$.

Remark 3.13. The inequality of the theorem for special Birkhoff sums implies a similar inequality for general Birkhoff sums: for all $\tau > 0$,

$$\left\| \sum_{j=0}^{N-1} (\varphi - \chi) \circ T^j \right\|_{C^0} \leq C(\tau) \|\varphi\|_{C^1} N^\tau.$$

This follows from the time decomposition in section 3.7.

Proof. The proof of the Theorem 3.11 is a variant of the proof of the Theorem at p. 845 of [MMY1].

Recall the following result from [MMY1], p. 835.

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Lemma 3.14. *Let T be an i.e.m. with no connection and let $n \geq m \geq 0$.*

(1) *One has*

$$\min_{\alpha} |I_{\alpha}^t(T(n))| \leq |I(T(m))| \|B(m, n)\|^{-1} \leq d \max_{\alpha} |I_{\alpha}^t(T(n))|.$$

(2) *Assume that T satisfies condition (a) of section 3.1. Then, for any $\tau > 0$, there exists C_{τ} such that*

$$\max_{\alpha} |I_{\alpha}^t(T(n))| \leq C_{\tau} \|B(0, n)\|^{\tau} \min_{\alpha} |I_{\alpha}^t(T(n))|.$$

Let $\varphi \in C_0^1(\mathbf{u}(T))$. Let $n \geq 0$. From the trivial estimate

$$\|S(0, n)D\varphi\|_{C^0} \leq \|D\varphi\|_{C^0} \|B(0, n)\|$$

and Lemma 3.14, we get, for $\alpha \in \mathcal{A}$, $x, y \in I_{\alpha}^t(T(n))$

$$(3.7) \quad |S(0, n)\varphi(y) - S(0, n)\varphi(x)| \leq C_{\tau} \|D\varphi\|_{C^0} \|B(0, n)\|^{\tau}.$$

There exists a function $\Phi(n) \in \Gamma_{\partial}(T(n))$ such that $S(0, n)\varphi - \Phi(n)$ is continuous on $I(T(n))$ and vanishes at both endpoints of this interval. Indeed, let $\Phi(n)$ be the function in $\Gamma(T(n))$ such that $S(0, n)\varphi - \Phi(n)$ is continuous on $I(T(n))$ and vanishes at $u_0(T(n))$. As $\varphi \in C_0^1(\mathbf{u}(T))$, $D\varphi$ has mean zero, hence $S(0, n)(D\varphi) = D(S(0, n)\varphi)$ has also mean zero; thus $S(0, n)\varphi - \Phi(n)$ vanishes at $u_d(T(n))$. Then $S(0, n)\varphi - \Phi(n)$ belongs to $C_0^1(\mathbf{u}(T(n)))$. As $S(0, n)\varphi$ also belongs to $C_0^1(\mathbf{u}(T(n)))$, we conclude that $\Phi(n) \in \Gamma_{\partial}(T(n))$.

For $\alpha \in \mathcal{A}$, $x \in I_{\alpha}^t(T(n))$, we get from (3.7)

$$(3.8) \quad |S(0, n)\varphi(x) - \Phi_{\alpha}(n)| \leq C_{\tau} \|D\varphi\|_{C^0} \|B(0, n)\|^{\tau}.$$

We will show that there exists $\chi \in \Gamma_u(T)$ such that

$$(3.9) \quad \|\Phi(n_{\ell}) - B(0, n_{\ell})\chi\| \leq C'_{\tau} \|\varphi\|_{C^1} \|B(0, n_{\ell})\|^{\tau},$$

and that it satisfies $\|\chi\| \leq C\|\varphi\|_{C^1}$.

We write C_{τ} for various constants depending only on τ (through conditions (a), (c) of section 3.1). From $S(0, n_{\ell+1}) = S(n_{\ell}, n_{\ell+1}) \circ S(0, n_{\ell})$ and (3.8), we get, for $\ell \geq 0$

$$(3.10) \quad \|\Phi(n_{\ell+1}) - B(n_{\ell}, n_{\ell+1})\Phi(n_{\ell})\| \leq C_{\tau} \|D\varphi\|_{C^0} \|B(0, n_{\ell})\|^{\tau}.$$

Define $\Lambda_{\ell+1}$ to be the class mod. $\Gamma_s(T(n_{\ell+1}))$ of $\Phi(n_{\ell+1}) - B(n_{\ell}, n_{\ell+1})\Phi(n_{\ell})$. We define χ to be the unique element in $\Gamma_u(T)$ which is equal mod. $\Gamma_s(T)$ to

$$\Phi(0) + \sum_{\ell > 0} B_b^{-1}(0, n_{\ell})\Lambda_{\ell}.$$

We will show shortly that the series is convergent. Notice that, if χ is defined in this way, the image $B(0, n_k)\chi$ is equal mod. $\Gamma_s(T(n_k))$ to

$$\Phi(n_k) + \sum_{\ell > k} B_b^{-1}(n_k, n_{\ell})\Lambda_{\ell}.$$

To estimate the norm of the series $\sum_{\ell > k} B_b^{-1}(n_k, n_{\ell})\Lambda_{\ell}$, we split the sum into two parts

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- As long as $\|B(0, n_\ell)\|^{\sigma/2} \leq \|B(0, n_k)\|$, we will use (from condition (c) in section 3.1 and (3.10))

$$(3.11) \quad \|B_b^{-1}(n_k, n_\ell)\Lambda_\ell\| \leq C_\tau \|D\varphi\|_{C^0} \|B(0, n_\ell)\|^\tau.$$

The total contribution of this part of the series is at most

$$C_\tau \|D\varphi\|_{C^0} \|B(0, n_k)\|^\tau.$$

- When $\|B(0, n_\ell)\|^{\sigma/2} > \|B(0, n_k)\|$, we proceed differently. We use notations of section 3.2. As $\Phi(n)$ belongs to $\Gamma_\partial(T(n))$, the function $\Phi(n_{\ell+1}) - B(n_\ell, n_{\ell+1})\Phi(n_\ell)$ belongs to $\Gamma_\partial(T(n_{\ell+1}))$. Therefore we have, mod. $\Gamma_s(T(n_k))$

$$\begin{aligned} \|B_b^{-1}(n_k, n_\ell)\Lambda_\ell\| &= \|b_u^{-1}(n_k, n_\ell)\Lambda_\ell\| \\ &\leq \|b_u(0, n_k)\| \|b_u^{-1}(0, n_\ell)\| \|\Lambda_\ell\| \\ &\leq \|B(0, n_k)\| \|b_s(0, n_\ell)\| \|\Lambda_\ell\| \\ &\leq \|B(0, n_k)\| \|B(0, n_\ell)\|^{-\sigma} \|\Lambda_\ell\| \\ &\leq C \|B(0, n_\ell)\|^{-\sigma/3} \|D\varphi\|_{C^0}. \end{aligned}$$

This shows that the series $\sum_{\ell>k} B_b^{-1}(n_k, n_\ell)\Lambda_\ell$ is convergent and that we have, mod. $\Gamma_s(T(n_k))$

$$(3.12) \quad \|B(0, n_k)\chi - \Phi(n_k)\| \leq C_\tau \|D\varphi\|_{C^0} \|B(0, n_k)\|^\tau.$$

Taking $k = 0$ and observing that $\|\Phi(0)\| \leq C\|\varphi\|_{C^0}$, one obtains $\|\chi\| \leq C\|\varphi\|_{C^1}$.

To show that (3.9) holds in $\Gamma_\partial(T(n_k))$ (and not only mod. $\Gamma_s(T(n_k))$) we may assume that $\chi = 0$ and proceed as follows. Write $S(0, n_k)\varphi = \varphi_k - \chi_k$ with $\chi_k \in \Gamma_s(T(n_k))$ and

$$(3.13) \quad \|\varphi_k\|_{C^0} \leq C_\tau \|B(0, n_k)\|^\tau \|D\varphi\|_{C^0}.$$

Define

$$\Delta_0 := \chi_0, \quad \Delta_k := \chi_k - B(n_{k-1}, n_k)\chi_{k-1} = \varphi_k - S(n_{k-1}, n_k)\varphi_{k-1} \in \Gamma_s(T(n_k)).$$

Using condition (a) of section 3.1, one obtains

$$(3.14) \quad \|\Delta_0\| \leq C\|\varphi\|_{C^0}, \quad \|\Delta_k\| \leq C_\tau \|B(0, n_k)\|^\tau \|D\varphi\|_{C^0} \text{ for } k > 0.$$

In the formula

$$S(0, n_k)\varphi = \varphi_k - \sum_{\ell \leq k} B(n_\ell, n_k)\Delta_\ell$$

one has from condition (c) in section 3.1

$$\|B(n_\ell, n_k)\Delta_\ell\| \leq C_\tau \|B(0, n_k)\|^\tau \|\varphi\|_{C^1}.$$

We conclude that (3.9) holds. The proof of the theorem is complete. \square

3.6. **Estimates for special Birkhoff sums with C^r data.** The proof of the following lemma can be found in Appendix A of [MMY2].

Lemma 3.15. *Let T be an i.e.m. with no connection. Assume that T satisfies conditions (a) and (b) of section 3.1. Let ρ be a positive real number. There exists $\delta > 0$, depending on ρ and on the constant θ in condition (b), such that, for any function $\varphi \in C^\rho(\mathbf{u}(T))$ of mean 0 and any $k \geq 0$, the special Birkhoff sum $S(0, n_k)\varphi$ satisfies,*

$$\|S(0, n_k)\varphi\|_{C^0} \leq C\|B(0, n_k)\|^{1-\delta}\|\varphi\|_{C^\rho}.$$

The constant C depends on ρ , θ and the other implied constants in conditions (a) and (b) of 3.1.

The following lemma improves the conclusion of Theorem 3.11 for functions in $C_0^r(\mathbf{u}(T))$.

Lemma 3.16. *Let $r = 1 + \rho \in (1, 2)$, T and $\Gamma_u(T)$ be as in the statement of Theorem 3.7. There exists $\delta_1 > 0$, depending only on r and the constants θ , σ of section 3.1 such that for any $\varphi \in C_0^r(\mathbf{u}(T))$, the function $\chi = L_1(\varphi)$ of Theorem 3.11 satisfies*

$$\|S(0, n_k)(\varphi - \chi)\|_{C^0} \leq C\|B(0, n_k)\|^{-\delta_1}\|\varphi\|_{C^r}.$$

The constant C depends on ρ , θ , σ and the other implied constants in section 3.1.

Proof. We adapt the proof of Theorem 3.11. Observe that, if $\varphi \in C_0^r(\mathbf{u}(T))$, the function $D\varphi \in C^\rho(\mathbf{u}(T))$ has mean 0 (cf. Remark 2.12).

From Lemma 3.15 and Lemma 3.14 we get, for $\alpha \in \mathcal{A}$, $x, y \in I_\alpha^t(T(n))$

$$|S(0, n)\varphi(y) - S(0, n)\varphi(x)| \leq C\|D\varphi\|_{C^\rho}\|B(0, n)\|^{-\delta/2}.$$

Define the sequence $\Phi(n)$ as in the proof of Theorem 3.11. Instead of (3.8) we get

$$(3.15) \quad |S(0, n)\varphi(x) - \Phi_\alpha(n)| \leq C\|D\varphi\|_{C^\rho}\|B(0, n)\|^{-\delta/2}.$$

Then the sequence $\Phi(n_\ell)$ satisfies (instead of (3.10))

$$(3.16) \quad \|\Phi(n_{\ell+1}) - B(n_\ell, n_{\ell+1})\Phi(n_\ell)\| \leq C\|D\varphi\|_{C^\rho}\|B(0, n_\ell)\|^{-\delta/3}.$$

Define then Λ_ℓ as in the proof of Theorem 3.11. We have

$$(3.17) \quad \|\Lambda_\ell\| \leq C\|D\varphi\|_{C^\rho}\|B(0, n_\ell)\|^{-\delta/3}.$$

The series $\sum_{\ell > k} B_b^{-1}(n_k, n_\ell)\Lambda_\ell$ is again split into two parts:

- as long as $\|B(0, n_\ell)\|^{\sigma/2} \leq \|B(0, n_k)\|$, we will use (from condition (c) in section 3.1 and (3.17))

$$(3.18) \quad \|B_b^{-1}(n_k, n_\ell)\Lambda_\ell\| \leq C\|D\varphi\|_{C^\rho}\|B(0, n_\ell)\|^{-\delta/4}.$$

The total contribution of this part of the series is at most

$$C\|D\varphi\|_{C^\rho}\|B(0, n_k)\|^{-\delta/5}.$$

- When $\|B(0, n_\ell)\|^{\sigma/2} > \|B(0, n_k)\|$, we obtain as in the proof of Theorem 3.11

$$(3.19) \quad \|B_b^{-1}(n_k, n_\ell)\Lambda_\ell\| \leq C\|B(0, n_\ell)\|^{-\sigma/3}\|D\varphi\|_{C^0}.$$

We may assume that $\delta < \sigma$. We obtain instead of (3.12)

$$(3.20) \quad \|B(0, n_k)\chi - \Phi(n_k)\| \leq C\|D\varphi\|_{C^\rho} \|B(0, n_k)\|^{-\delta/5}.$$

This gives the required estimate mod. $\Gamma_s(T(n_k))$. To get the full estimate we assume that $\chi = 0$ and define φ_k and Δ_k as before. Instead of (3.13) we have

$$(3.21) \quad \|\varphi_k\|_{C^0} \leq C\|D\varphi\|_{C^\rho} \|B(0, n_k)\|^{-\delta/5}.$$

The vector Δ_k satisfies (instead of (3.14))

$$(3.22) \quad \|\Delta_0\| \leq C\|\varphi\|_{C^0}, \quad \|\Delta_k\| \leq C\|D\varphi\|_{C^\rho} \|B(0, n_k)\|^{-\delta/6} \text{ for } k > 0.$$

In the formula

$$S(0, n_k)\varphi = \varphi_k - \sum_{\ell \leq k} B(n_\ell, n_k)\Delta_\ell$$

we estimate again the terms in two different ways.

- if $\|B(0, n_\ell)\| \leq \|B(0, n_k)\|^{\sigma/2}$, we have, as $\Delta_\ell \in \Gamma_s(T(n_\ell))$

$$\begin{aligned} \|B(n_\ell, n_k)\Delta_\ell\| &\leq C\|B(0, n_k)\|^{-\sigma} \|B(0, n_\ell)\|^{-1} \|\Delta_\ell\| \\ &\leq C\|B(0, n_k)\|^{-\sigma} \|B(0, n_\ell)\| \|\varphi\|_{C^r} \\ &\leq C\|B(0, n_k)\|^{-\sigma/2} \|\varphi\|_{C^r}. \end{aligned}$$

- if $\|B(0, n_\ell)\| > \|B(0, n_k)\|^{\sigma/2}$, we get from (3.22) and condition (c) in section 3.1

$$\|B(n_\ell, n_k)\Delta_\ell\| \leq C\|D\varphi\|_{C^\rho} \|B(0, n_k)\|^{-\delta/7}.$$

With $\delta < \sigma$ we obtain the conclusion of the Lemma with $\delta_1 = \delta/8$. \square

3.7. Time decomposition. We recall from [MMY1] p. 840 how to decompose orbits in order to estimate general Birkhoff sums from special Birkhoff sums.

Let T be an i.e.m. with no connection and let $x \in I(T)$, $N > 0$. We will decompose the finite orbit $(T^j(x))_{0 \leq j < N}$.

Let y be the point of this orbit which is closest to 0. We divide the orbit into a positive part $(T^j(y))_{0 \leq j < N^+}$ and a negative part $(T^j(y))_{N^- \leq j < 0}$ (with $N^+ - N^- = N$).

Let $k \geq 0$ be the largest integer such that at least one of the points $T^j(y)$, $0 < j < N^+$, belongs to $I(T(n_k))$. Because $T(n_k)$ is the first return map of T into $I(T(n_k))$, the points $T^j(y)$, $0 < j < N^+$ which belong to $I(T(n_k))$ are precisely $T(n_k)(y), \dots, (T(n_k))^{b(k)}(y)$ for some $b(k) > 0$.

Define by decreasing induction $y(\ell)$, $b(\ell)$ for $0 \leq \ell < k$ as follows. Define $y(k-1) = (T(n_k))^{b(k)}(y)$. The induction hypothesis is that $y(\ell) = T^{N(\ell)}(y)$ is the last point of the orbit $(T^j(y))_{0 \leq j < N^+}$ which belongs to $I(T(n_{\ell+1}))$. The points of the orbit $(T^j(y))_{N(\ell) \leq j < N^+}$ which belong to $I(T(n_\ell))$ are $y(\ell), \dots, (T(n_\ell))^{b(\ell)}(y(\ell)) =: y(\ell-1)$ for some $b(\ell) \geq 0$.

At the end, the point $y(0) = T^{N(0)}(y)$ satisfies $N(0) < N^+$ and $T^j(y) \notin I(T(n_1))$ for $N(0) < j < N^+$. We set $b(0) = N^+ - N(0)$ and $y(k) = y$.

For a function φ on $I(T)$ the Birkhoff sum of order N^+ of φ at y decomposes as

$$(3.23) \quad \sum_{0 \leq j < N^+} \varphi(T^j(y)) = \sum_{\ell=0}^k \sum_{0 \leq b < b(\ell)} S(0, n_\ell) \varphi((T(n_\ell))^b(y(\ell))).$$

As $b(\ell)$, for $0 \leq \ell \leq k$, is at most equal to the maximal return time into $I(T(n_{\ell+1}))$ under $T(n_\ell)$ one has

$$(3.24) \quad b(\ell) \leq \|B(n_\ell, n_{\ell+1})\|.$$

The negative part of the orbit $(T^j(y))_{N^- \leq j < 0}$ is decomposed in a similar way.

Remark 3.17. Formulas (3.23) and (3.24) imply the estimate in Remark 3.13.

3.8. Space decomposition. Recall from section 2.5, equation (2.1), the partition (for $k \geq 0$):

$$(\mathcal{P}(k)) \quad I(T) = \bigsqcup_{\alpha \in \mathcal{A}} \bigsqcup_{0 \leq j < B_\alpha(0, n_k)} T^j(I_\alpha^t(T(n_k))) \bmod 0.$$

Let $x_- < x_+$ be two distinct points in $I(T)$. Let k be the smallest integer such that (x_-, x_+) contains at least one interval of the partition $\mathcal{P}(k)$. Let $J^{(k)}(1), \dots, J^{(k)}(b(k))$ (with $b(k) > 0$) be the intervals of $\mathcal{P}(k)$ contained in (x_-, x_+) . The complement mod.0 in (x_-, x_+) of their union is the union of two intervals $(x_+(k), x_+)$ and $(x_-, x_-(k))$, which may be degenerate.

We define $x_+(\ell)$, for $\ell > k$, as the largest endpoint $< x_+$ of an interval of $\mathcal{P}(\ell)$. One has $x_+(\ell) \geq x_+(\ell-1)$. The interval $(x_+(\ell-1), x_+(\ell))$ is the union mod.0 of intervals $J_+^{(\ell)}(1), \dots, J_+^{(\ell)}(b_+(\ell))$ of $\mathcal{P}(\ell)$ for some $b_+(\ell) \geq 0$.

One has $\lim_{\ell \rightarrow +\infty} x_+(\ell) = x_+$.

One defines similarly $x_-(\ell)$ for $\ell > k$ and $b_-(\ell) \geq 0$ with $\lim_{\ell \rightarrow +\infty} x_-(\ell) = x_-$.

The decomposition of (x_-, x_+) is

$$(3.25) \quad (x_-, x_+) = \bigsqcup_{1 \leq b \leq b(k)} J^{(k)}(b) \bigsqcup_{\ell > k} \bigsqcup_{\varepsilon = \pm} \bigsqcup_{1 \leq b \leq b_\varepsilon(\ell)} J_\varepsilon^{(\ell)}(b) \bmod 0.$$

The number of intervals of $\mathcal{P}(\ell+1)$ contained in a single interval of $\mathcal{P}(\ell)$ is at most

$$\max_\beta \sum_\alpha B_{\alpha\beta}(n_\ell, n_{\ell+1}).$$

One has therefore

$$\begin{aligned} b(k) &\leq \|{}^t B(n_{k-1}, n_k)\| \\ b_\varepsilon(\ell) &\leq \|{}^t B(n_{\ell-1}, n_\ell)\|, \end{aligned}$$

for $\varepsilon = \pm$, $\ell > k$.

3.9. Special Hölder estimate. In this section and the next one, T is an i.e.m of restricted Roth type, r is a real number > 1 and φ is a function in $C_0^r(\mathbf{u}(T))$. Substracting $L_1(\varphi)$ if necessary, we assume that $L_1(\varphi) = 0$. From Lemma 3.16, one has the estimate

$$\|S(0, n_k) \varphi\|_{C^0} \leq C \|B(0, n_k)\|^{-\delta_1} \|\varphi\|_{C^r}.$$

We introduce this estimate into the time decomposition formula (3.23), using also condition (a) of section 3.1 in (3.24), to obtain that the Birkhoff sum of φ of any order are uniformly bounded by $C \|\varphi\|_{C^r}$.

It follows from the Gottschalk-Hedlund theorem applied to a continuous model of T (cf. [MMY1] p.838-839) that there exists a bounded function ψ on $I(T)$ such that

$$\varphi = \psi \circ T - \psi.$$

In [MMY2] p.1598-1599, it is proven that ψ is continuous on $[u_0(T), u_d(T)]$. We normalize ψ by $\int_{u_0(T)}^{u_d(T)} \psi(x)dx = 0$. One has

$$\|\psi\|_{C^0} \leq C\|\varphi\|_{C^r}.$$

For an interval $J = (a, b)$, write $\Delta(J) := \psi(b) - \psi(a)$.

For $k \geq 0$, we define a vector $V(k) \in \mathbb{R}^{\mathcal{A}}$ by

$$V_\alpha(k) := \Delta(I_\alpha^t(T(n_k))), \quad \alpha \in \mathcal{A}$$

The vectors $V(k)$ are inductively related.

Lemma 3.18. *For $k \geq 0$, one has*

$$\|V(k+1) - ({}^tB(n_k, n_{k+1}))^{-1}V(k)\| \leq C\|B(0, n_k)\|^{-\delta_1/2}\|\varphi\|_{C^r}.$$

Proof. Let $\alpha \in \mathcal{A}$. One has

$$(3.26) \quad V_\alpha(k) = \sum_{\beta \in \mathcal{A}} \sum_{j \in N(\beta, \alpha)} \Delta(T(n_k)^j(I_\beta^t(T(n_{k+1}))))$$

with

$$N(\beta, \alpha) = \{j \in [0, B_\beta(n_k, n_{k+1}), T(n_k)^j(I_\beta^t(T(n_{k+1}))) \subset I_\alpha^t(T(n_k))\}.$$

Recall from section 2.5 that $B_\beta(n_k, n_{k+1})$ is the return time of $I_\beta^t(T(n_{k+1}))$ into $I(T(n_{k+1}))$ under $T(n_k)$. The cardinality of $N(\beta, \alpha)$ is equal to $B_{\beta\alpha}(n_k, n_{k+1})$.

For $0 \leq j < B_\beta(n_k, n_{k+1})$, we compare $\Delta(T(n_k)^{j+1}(I_\beta^t(T(n_{k+1}))))$ and $\Delta(T(n_k)^j(I_\beta^t(T(n_{k+1}))))$. Writing $T(n_k)^j(I_\beta^t(T(n_{k+1}))) = (a, b)$, we have, as $\varphi = \psi \circ T - \psi$

$$\begin{aligned} \Delta(T(n_k)^{j+1}(I_\beta^t(T(n_{k+1})))) &= \Delta(T(n_k)^j(I_\beta^t(T(n_{k+1})))) \\ &= (\psi(T(n_k)b) - \psi(b)) - (\psi(T(n_k)a) - \psi(a)) \\ &= S(0, n_k)\varphi(b) - S(0, n_k)\varphi(a). \end{aligned}$$

We use Lemma 3.16 and sum over j to obtain, for $0 \leq j < B_\beta(n_k, n_{k+1})$

$$|\Delta(T(n_k)^j(I_\beta^t(T(n_{k+1})))) - V_\beta(k+1)| \leq CB_\beta(n_k, n_{k+1})\|B(0, n_k)\|^{-\delta_1}\|\varphi\|_{C^r}.$$

Bringing this estimate into (3.26) gives, as $N(\beta, \alpha)$ has $B_{\beta\alpha}(n_k, n_{k+1})$ elements

$$\|V(k) - {}^tB(n_k, n_{k+1})V(k+1)\| \leq C\|B(n_k, n_{k+1})\|^2\|B(0, n_k)\|^{-\delta_1}\|\varphi\|_{C^r}.$$

Taking condition (a) of section 3.1 into account, this gives the estimate of the lemma. \square

Lemma 3.19. *If a sequence of vectors $(W(k))_{k \geq 0}$ in $\mathbb{R}^{\mathcal{A}}$ satisfies for $k \geq 0$*

$$\|W(k+1) - ({}^tB(n_k, n_{k+1}))^{-1}W(k)\| \leq C\|B(0, n_k)\|^{-\delta_1/2}\|\varphi\|_{C^r},$$

and $\lim_{k \rightarrow \infty} W(k) = 0$, then one has

$$\|W(k)\| \leq C'\|B(0, n_k)\|^{-\delta_2}\|\varphi\|_{C^r}$$

where the constant δ_2 depends only on r , θ and σ .

Proof. For $k \geq 0$, define $\bar{W}(k) := \Omega(\pi(n_k))W(k)$, where $\Omega(\pi(n_k))$ is the antisymmetric matrix associated to the combinatorial data of $T(n_k)$. From the hypothesis of the lemma and equation (2.3) in section 2.7, we have

$$(3.27) \quad \|\bar{W}(k+1) - B(n_k, n_{k+1})\bar{W}(k)\| \leq C \|B(0, n_k)\|^{-\delta_1/2} \|\varphi\|_{C^r}.$$

We will first deal with the size of $\bar{W}(k) \bmod \Gamma_s(T(n_k))$, then with the size of $\bar{W}(k)$, and finally with the size of $W(k)$ itself.

Define $W_b(k) := \bar{W}(k) \bmod \Gamma_s(T(n_k)) \in \Gamma_{\partial}(T(n_k))/\Gamma_s(T(n_k))$ and

$$r_b(k+1) := W_b(k+1) - B_b(n_k, n_{k+1})W_b(k).$$

We have

$$\|r_b(k+1)\| \leq C \|B(0, n_k)\|^{-\delta_1/2} \|\varphi\|_{C^r}.$$

For $k < \ell$, we write $W_b(\ell) = B_b(n_k, n_\ell)(W_b(k) + R_b(k, \ell))$ with

$$R_b(k, \ell) = \sum_{k < j \leq \ell} (B_b(n_k, n_j))^{-1} r_b(j).$$

We control the norm of $(B_b(n_k, n_j))^{-1}$ from condition (c) of section 3.1 and get

$$\|R_b(k, \ell)\| \leq C \|B(0, n_k)\|^{-\delta_1/3} \|\varphi\|_{C^r}.$$

We claim that we have

$$(3.28) \quad \|W_b(k)\| \leq 2C \|B(0, n_k)\|^{-\delta_1/3} \|\varphi\|_{C^r}.$$

Otherwise, the norm of the norm of the vector $w(k, \ell) := W_b(k) + R_b(k, \ell)$ would be bounded from below by $C \|B(0, n_k)\|^{-\delta_1/3} \|\varphi\|_{C^r}$. But the nondegenerate symplectic pairing between the lagrangian subspace Γ_s and the quotient $\Gamma_{\partial}/\Gamma_s$ (cf. section 2.7) gives, for any vector $v \in \Gamma_s(T(n_k))$

$$\langle v, w(k, \ell) \rangle = \langle v, (B_b(n_k, n_\ell))^{-1} W_b(\ell) \rangle = \langle B_b(n_k, n_\ell)v, W_b(\ell) \rangle.$$

As v belongs to the stable subspace and $W_b(\ell)$ converges to 0, the right-hand side converges to 0. By the nondegeneracy of the pairing, $w(k, \ell)$ converges as ℓ goes to $+\infty$ to 0 mod. $\Gamma_s(T(n_k))$, a contradiction. The claim is proved.

We may therefore write $\bar{W}(k) = W_u(k) + W_s(k)$, with $W_s(k) \in \Gamma_s(T(n_k))$ and $\|W_u(k)\| \leq 2C \|B(0, n_k)\|^{-\delta_1/3} \|\varphi\|_{C^r}$. Define

$$r_s(k+1) := W_s(k+1) - B(n_k, n_{k+1})W_s(k).$$

From (3.27), one has,

$$\begin{aligned} \|r_s(k+1)\| &\leq \|W_u(k+1)\| + \|B(n_k, n_{k+1})W_u(k)\| + C \|B(0, n_k)\|^{-\delta_1/2} \|\varphi\|_{C^r} \\ &\leq C \|\varphi\|_{C^r} (\|B(0, n_{k+1})\|^{-\delta_1/3} + \|B(0, n_k)\|^{-\delta_1/3} \|B(n_k, n_{k+1})\|) \\ &\leq C \|B(0, n_k)\|^{-\delta_1/4} \|\varphi\|_{C^r}. \end{aligned}$$

The vectors $W_s(k)$ satisfy

$$W_s(k) = B(0, n_k)W_s(0) + \sum_{\ell=1}^k B(n_\ell, n_k)r_s(\ell).$$

We estimate the sum in the same way than in the last lines of the proof of Lemma 3.16 to get

$$\|W_s(k)\| \leq C \|B(0, n_k)\|^{-\sigma\delta_1/12} \|\varphi\|_{C^r}.$$

We now have proven that

$$(3.29) \quad \|\bar{W}(k)\| \leq C \|B(0, n_k)\|^{-\sigma\delta_1/12} \|\varphi\|_{C^r}.$$

We may therefore write $W(k) = W_0(k) + W_*(k)$, with $W_0(k) \in \text{Ker } \Omega(\pi(n_k))$ and $\|W_*(k)\| \leq C \|B(0, n_k)\|^{-\sigma\delta_1/12} \|\varphi\|_{C^r}$. Define

$$r_0(k+1) := W_0(k+1) - {}^t B(n_k, n_{k+1})^{-1} W_0(k).$$

The hypothesis of the lemma gives the estimate

$$\begin{aligned} \|r_0(k+1)\| &\leq \|W_*(k+1)\| + \|{}^t B(n_k, n_{k+1})^{-1} W_*(k)\| + C \|B(0, n_k)\|^{-\delta_1/2} \|\varphi\|_{C^r} \\ &\leq C (\|B(0, n_{k+1})\|^{-\sigma\delta_1/12} + \|{}^t B(n_k, n_{k+1})^{-1}\| \|B(0, n_k)\|^{-\sigma\delta_1/12}) \|\varphi\|_{C^r} \\ &\leq C \|B(0, n_k)\|^{-\sigma\delta_1/16} \|\varphi\|_{C^r}. \end{aligned}$$

As the extended Kontsevich-Zorich cocycle acts trivially on the kernels of the antisymmetric matrices $\Omega(\pi)$ (cf section 2.7) and $\lim_{\ell \rightarrow \infty} W_0(\ell) = 0$, we obtain

$$\begin{aligned} \|W_0(k)\| &\leq \sum_{\ell > k} \|r_0(\ell)\| \\ &\leq C \|B(0, n_k)\|^{-\sigma\delta_1/16} \|\varphi\|_{C^r}. \end{aligned}$$

This is the estimate of the lemma, with $\delta_2 = \sigma\delta_1/16$. \square

By the continuity of ψ and Lemma 3.18, the hypotheses of Lemma 3.19 are satisfied by the sequence $V(k)$ of Lemma 3.18. We obtain, for all $\alpha \in \mathcal{A}$

$$(3.30) \quad \|\Delta(I_\alpha^t(T(n_k)))\| \leq C \|B(0, n_k)\|^{-\delta_2} \|\varphi\|_{C^r}.$$

3.10. General Hölder estimate. The first step is to extend inequality (3.30).

Lemma 3.20. *For $k \geq 0$, $\alpha \in \mathcal{A}$, $0 \leq N < B_\alpha(0, n_k)$, one has*

$$\|\Delta(T^N(I_\alpha^t(T(n_k))))\| \leq C \|B(0, n_k)\|^{-\delta_2} \|\varphi\|_{C^r}.$$

Proof. From $\varphi = \psi \circ T - \psi$, one obtains

$$(3.31) \quad \Delta(T^N(I_\alpha^t(T(n_k)))) - \Delta(I_\alpha^t(T(n_k))) = \int_{I_\alpha^t(T(n_k))} \sum_{i=0}^{N-1} D\varphi \circ T^i(x) dx.$$

Observe that, for $x \in I_\alpha^t(T(n_k))$, the points $T^i(x)$, $0 < i < B_\alpha(0, n_k)$ are farther from 0 than x . According to (3.23), we write

$$\sum_{i=0}^{N-1} D\varphi \circ T^i(x) = \sum_{\ell=0}^{k-1} \sum_{0 \leq b < b(\ell)} S(0, n_\ell) D\varphi((T(n_\ell))^b(y(\ell))).$$

Here, the integers $b(\ell)$ satisfy, according to (3.24)

$$b(\ell) \leq \|B(n_\ell, n_{\ell+1})\|.$$

We use Lemma 3.15 to estimate the special Birkhoff sums of $D\varphi$. This gives

$$\left| \sum_{i=0}^{N-1} D\varphi \circ T^i(x) \right| \leq C \|D\varphi\|_{C^r} \sum_{\ell=0}^{k-1} \|B(n_\ell, n_{\ell+1})\| \|B(0, n_\ell)\|^{1-\delta}.$$

From Lemma 3.14 and condition (a), one obtains, as $\delta_2 < \delta$

$$\left| \int_{I_\alpha^t(T(n_k))} \sum_{i=0}^{N-1} D\varphi \circ T^i(x) dx \right| \leq C \|D\varphi\|_{C^r} \|B(0, n_k)\|^{-\delta_2}.$$

The estimate of the lemma now follows from (3.30) and (3.31). \square

We are now ready to complete the proof of the theorem. Let $x_- < x_+$ be distinct points in $I(T)$. Let k be the smallest integer such that x_-, x_+ contains at least one element of the partition $\mathcal{P}(k)$ of section 3.8. According to (3.25), we write

$$(x_-, x_+) = \bigsqcup_{1 \leq b \leq b(k)} J^{(k)}(b) \bigsqcup_{\ell > k} \bigsqcup_{\varepsilon = \pm 1} \bigsqcup_{1 \leq b \leq b_\varepsilon(\ell)} J_\varepsilon^{(\ell)}(b) \pmod{0}.$$

Here, $J^{(k)}(b)$ is an element of $\mathcal{P}(k)$ and $J_\varepsilon^{(\ell)}(b)$ is an element of $\mathcal{P}(\ell)$. One has from section 3.8

$$\begin{aligned} b(k) &\leq \|{}^t B(n_{k-1}, n_k)\|, \\ b_\varepsilon(\ell) &\leq \|{}^t B(n_{\ell-1}, n_\ell)\|. \end{aligned}$$

One has

$$\psi(x_+) - \psi(x_-) = \sum_{1 \leq b \leq b(k)} \Delta(J^{(k)}(b)) \sum_{\ell > k} \sum_{\varepsilon = \pm 1} \sum_{1 \leq b \leq b_\varepsilon(\ell)} \Delta(J_\varepsilon^{(\ell)}(b)).$$

Using Lemma 3.20, one obtains

$$\begin{aligned} |\psi(x_+) - \psi(x_-)| &\leq C \|\varphi\|_{C^r} \sum_{\ell \geq k} \|{}^t B(n_{\ell-1}, n_\ell)\| \|B(0, n_\ell)\|^{-\delta_2} \\ &\leq C \|\varphi\|_{C^r} \|B(0, n_k)\|^{-\delta_2/2}, \end{aligned}$$

where we have used condition (a) and the fact that $\|{}^t B(n_{\ell-1}, n_\ell)\|$ and $\|B(n_{\ell-1}, n_\ell)\|$ have the same order. On the other hand, by the definition of k and Lemma 3.14, one has, for any $\tau > 0$

$$\begin{aligned} |x_+ - x_-| &\geq \min_{\alpha \in \mathcal{A}} |I_\alpha^t(n_k)| \\ &\geq C_\tau^{-1} \|B(0, n_k)\|^{-\tau} \max_{\alpha \in \mathcal{A}} |I_\alpha^t(n_k)| \\ &\geq C'_\tau^{-1} |I(T)| \|B(0, n_k)\|^{-1-\tau}. \end{aligned}$$

We thus obtain

$$|\psi(x_+) - \psi(x_-)| \leq C \|\varphi\|_{C^r} \left(\frac{|x_+ - x_-|}{|I(T)|} \right)^{\delta_2/3}.$$

The proof of Theorem 3.10 is complete.

3.11. Higher differentiability. To formulate a result allowing for smoother solutions of the cohomological equation, we introduce the same finite-dimensional spaces than in [MMY2] (although the notations are slightly different). In the following definitions, T is an i.e.m., D is a nonnegative integer, and r is a real number ≥ 0 .

Definition 3.21. We denote by

- $C_D^{D+r}(\mathbf{u}(T))$ the subspace of $C^{D+r}(\mathbf{u}(T))$ consisting of functions φ which satisfy $\partial D^i \varphi = 0$ for $0 \leq i \leq D$.
- $\Gamma(D, T)$ the subspace of $C_D^\infty(\mathbf{u}(T))$ consisting of functions χ whose restriction to each $[u_{i-1}(T), u_i(T)]$ is a polynomial of degree $\leq D$.
- $\Gamma_s(D, T)$ the subspace of $\Gamma(D, T)$ consisting of functions χ which can be written as $\chi = \psi \circ T - \psi$, for some function ψ of class C^D on the closure of $[u_0(T), u_d(T)]$.

It is proven in [MMY2] p.1602 that the dimension of $\Gamma(D, T)$ is equal to $2g + D(2g - 1)$, and that the dimension of $\Gamma_s(D, T)$ is equal to $g + D$ when T is of restricted Roth type.

Theorem 3.22. *Let T be an i.e.m. of restricted Roth type. Let D be a nonnegative integer, and let r be a real number > 1 . Let $\Gamma_u(D, T)$ be a subspace of $\Gamma(D, T)$ supplementing $\Gamma_s(D, T)$. There exist a real number $\bar{\delta} > 0$, and bounded linear operators $L_0 : \varphi \mapsto \psi$ from $C_D^{D+r}(\mathbf{u}(T))$ to $C^{D+\bar{\delta}}([u_0(T), u_d(T)])$ and $L_1 : \varphi \mapsto \chi$ from $C_D^{D+r}(\mathbf{u}(T))$ to $\Gamma_u(D, T)$ such that any $\varphi \in C_D^{D+r}(\mathbf{u}(T))$ satisfies*

$$\varphi = \chi + \psi \circ T - \psi.$$

The number $\bar{\delta}$ is the same than in Theorem 3.10.

The derivation of this theorem from Theorem 3.10 (the case $D = 0$) is easy and done in [MMY2] p.1602-1603.

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SCUOLA NORMALE SUPERIORE AND C.N.R.S. UMI 3483 LABORATORIO FIBONACCI, PIAZZA DEI CAVALIERI 7, 56126 PISA, ITALY

E-mail address: s.marmi(at)sns.it

COLLÈGE DE FRANCE, 3, RUE D'ULM, 75005 PARIS, FRANCE

E-mail address: jean-c.yoccoz(at)college-de-france.fr