AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **56**, 052202 (2015); doi: 10.1063/1.4921265
View online: http://dx.doi.org/10.1063/1.4921265
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/56/5?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS **56**, 052202 (2015)

Normal form decomposition for Gaussian-to-Gaussian superoperators

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(Received 6 February 2015; accepted 7 May 2015; published online 20 May 2015)

In this paper, we explore the set of linear maps sending the set of quantum Gaussian states into itself. These maps are in general not positive, a feature which can be exploited as a test to check whether a given quantum state belongs to the convex hull of Gaussian states (if one of the considered maps sends it into a non-positive operator, the above state is certified not to belong to the set). Generalizing a result known to be valid under the assumption of complete positivity, we provide a characterization of these Gaussian-to-Gaussian (not necessarily positive) superoperators in terms of their action on the characteristic function of the inputs. For the special case of one-mode mappings, we also show that any Gaussian-to-Gaussian superoperator can be expressed as a concatenation of a phase-space dilatation, followed by the action of a completely positive Gaussian channel, possibly composed with a transposition. While a similar decomposition is shown to fail in the multi-mode scenario, we prove that it still holds at least under the further hypothesis of homogeneous action on the covariance matrix.
[http://dx.doi.org/10.1063/1.4921265]

I. INTRODUCTION

Gaussian Bosonic States (GBSs) play a fundamental role in the study of continuous-variable (CV) quantum information processing[1–4] with applications in quantum cryptography, quantum computation, and quantum communication where they are known to provide optimal ensembles for a large class of quantum communication lines (specifically the phase-invariant Gaussian Bosonic maps).[5–10] GBSs are characterized by the property of having Gaussian Wigner quasi-distribution and describe Gibbs states of Hamiltonians which are quadratic in the field operators of the system. Further, in quantum optics, they include coherent, thermal, and squeezed states of light and can be easily created via linear amplification and loss.

Directly related to the definition of GBSs is the notion of Gaussian transformations,[1,3,4] i.e., superoperators mapping the set $\mathfrak{G}$ of GBSs into itself. In the last two decades, a great deal of attention has been devoted to characterizing these objects. In particular, the community focused on Gaussian Bosonic Channels (GBCs),[5] i.e., Gaussian transformations which are completely positive (CP) and provide hence the proper mathematical representation of data-processing and quantum communication procedures which are physically implementable.[11] On the contrary, less attention has been devoted to the study of Gaussian superoperators which are not CP or even non-positive. A typical example of such mappings is provided by the phase-space dilatation, which, given the Wigner quasi-distribution $W_{\hat{\rho}}(\mathbf{r})$ of a state $\hat{\rho}$ of n Bosonic modes, yields the function $W_{\hat{\rho}}^{(\lambda)}(\mathbf{r}) \equiv W_{\hat{\rho}}(\mathbf{r}/\lambda)/\lambda^{2n}$ as an output, with the real parameter λ satisfying the condition $|\lambda| > 1$. On one hand, when acting on $\mathfrak{G}$, the mapping

$$W_{\hat{\rho}}(\mathbf{r}) \mapsto W_{\hat{\rho}}^{(\lambda)}(\mathbf{r}) , \tag{1}$$

always outputs proper (Gaussian) states. Specifically, given $\hat{\rho} \in \mathfrak{G}$, one can identify another Gaussian density operator $\hat{\rho}'$ which admits the function $W^{(\lambda)}_{\hat{\rho}}(\mathbf{r})$ as Wigner distribution, i.e., $W_{\hat{\rho}'}(\mathbf{r}) = W^{(\lambda)}_{\hat{\rho}}(\mathbf{r})$. On the other hand, there exist inputs $\hat{\rho}$ for which $W^{(\lambda)}_{\hat{\rho}}(\mathbf{r})$ is no longer interpretable as the Wigner quasi-distribution of *any* quantum state: in this case, in fact $W^{(\lambda)}_{\hat{\rho}}(\mathbf{r})$ is the Wigner quasi-distribution $W_{\hat{\theta}}(\mathbf{r})$ of an operator $\hat{\theta}$ which is not positive[12] (for example, any pure non-Gaussian state has this property for any $\lambda \neq \pm 1$[13]). Accordingly, phase-space dilatations (1) should be considered as "unphysical" transformations, i.e., mappings which do not admit implementations in the laboratory. Still dilatations and similar exotic Gaussian-to-Gaussian mappings turn out to be useful mathematical tools that can be employed to characterize the set of states of CV systems in a way which is not dissimilar to what happens for positive (but not completely positive) transformations in the analysis of entanglement.[14] In particular, Bröcker and Werner[12] used (1) to study the convex hull $\mathfrak{C}$ of Gaussian states (i.e., the set of density operators $\hat{\rho}$ which can be expressed as a convex combination of elements of $\mathfrak{G}$). The rationale of this analysis is that the set $\mathfrak{F}$ of density operators which are mapped into proper output states by this transformation includes $\mathfrak{C}$ as a proper subset, see Fig. 1. Accordingly if a certain input $\hat{\rho}$ yields a $W^{(\lambda)}_{\hat{\rho}}(\mathbf{r})$ which is not the Wigner distribution of a state, we can conclude that $\hat{\rho}$ is not an element of $\mathfrak{C}$. Finding mathematical and experimental criteria which help in identifying the boundaries of $\mathfrak{C}$ is indeed a timely and important issue which is ultimately related with the characterization of non-classical behavior in CV systems, see, e.g., Refs. 15–25 and also Refs. 26 and 27 for the fermionic case.

In this context, a classification of non-positive Gaussian-to-Gaussian operations is mandatory. This analysis has been initiated in Ref. 28, where Gaussian-to-Gaussian maps are characterized through their Choi-Jamiołkowski state, under the hypothesis that this state has a Gaussian characteristic function. One goal of this paper is proving this hypothesis: we prove that the action of such transformations on the covariance matrix and on the first moment must be linear, and we write explicitly the transformation properties of the characteristic function (Theorem 3.1). In the classical case, any probability measure can be written as a convex superposition of Dirac deltas, so the convex hull of the Gaussian measures coincides with the whole set of measures. A simple consequence of this fact is that a linear transformation sending Gaussian measures into Gaussian (and then positive) measures

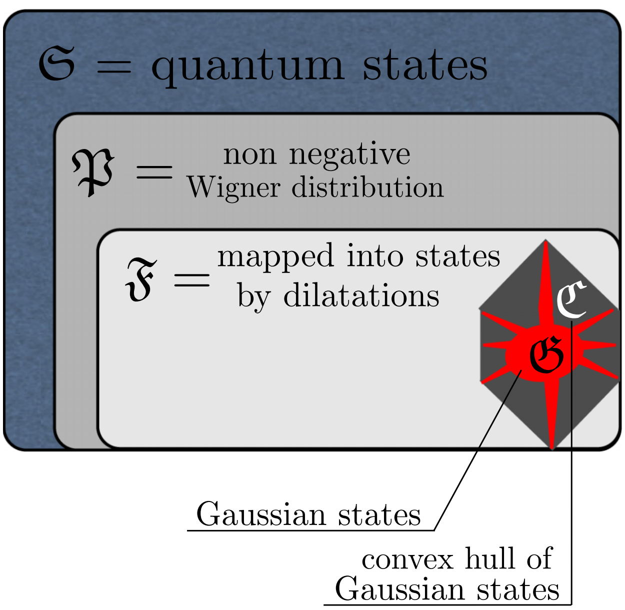

FIG. 1. Pictorial representation of the structure of the set of states $\mathfrak{S}$ of a CV system. $\mathfrak{P}$ is the subset of density operators $\hat{\rho}$ which have non-negative Wigner distribution (10). $\mathfrak{F}$ is set of states which instead are mapped into proper density operators by arbitrary dilatation (1). $\mathfrak{G}$ is the set of Gaussian states and $\mathfrak{C}$ its convex hull. $\mathfrak{S}$, $\mathfrak{P}$, $\mathfrak{F}$, and $\mathfrak{C}$ are closed under convex convolution, whereas $\mathfrak{G}$ is not. For a detailed study of the relations among these sets, see Ref. 12.

is always positive. Nothing of this holds in the more interesting quantum case, so we focus on it and use Theorem 3.1 to get a decomposition which, for single-mode operations, shows that any linear quantum Gaussian-to-Gaussian transformation can always be decomposed as a proper combination of dilatation (1) followed by a CP Gaussian mapping plus possibly a transposition. We also show that our decomposition theorem applies to the multi-mode case, as long as we restrict the analysis to Gaussian transformations which are homogeneous at the level of covariance matrix. For completeness, we finally discuss the case of contractions: these are mappings of form (1) with $|\lambda| < 1$. They are not proper Gaussian transformations because they map some Gaussian states into non-positive operators. Still some of the results which apply to the dilatations can be extended to this set.

The manuscript is organized as follows. We start in Sec. II by introducing the notation and recalling some basic facts about characteristic functions, Wigner distributions, and Gaussian states. In Sec. III, we state the problem and prove the Theorem 3.1 characterizing the action of Gaussian-to-Gaussian superoperators on the characteristic functions of quantum states and its variations, including the probabilistic analog. In Subsection III A, we consider the case of contractions. In Sec. IV, we present the main result of the manuscript, i.e., the decomposition theorem for single-mode Gaussian-to-Gaussian transformations. The multi-mode case is then analyzed in Sec. V. The paper ends hence with Sec. VI where we present a brief summary and discuss some possible future development. In Appendix, we prove the unboundedness of phase-space dilatations with respect to the trace norm.

II. NOTATION

In this section, we introduce the notation and some basic definitions which are useful to set the problem.

A. Symplectic structure

Consider a CV system constituted by n independent modes[1,2] with quadratures satisfying the canonical commutation relations

$$\left[\hat{Q}^j,\ \hat{P}^k\right] = i\delta^{jk}\ \hat{\mathbb{1}}\ , \qquad j, k = 1,\ldots,n\ . \tag{2}$$

Organizing these operators in the column vector $\hat{\mathbf{R}}$ of elements

$$\hat{\mathbf{R}} \equiv \left(\hat{Q}^1,\ \hat{P}^1,\ldots,\hat{Q}^n,\ \hat{P}^n\right)^T\ , \tag{3}$$

Equation (2) can be cast in the equivalent form,

$$\left[\hat{R}^i,\ \hat{R}^j\right] = i\Delta^{ij}\ \hat{\mathbb{1}}\ , \tag{4}$$

where Δ is the skew-symmetric matrix

$$\Delta = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ , \tag{5}$$

which defines the symplectic form of the problem. Accordingly, a real matrix S is said to be symplectic if it preserves Δ, i.e., if the following identity holds

$$S\Delta S^T = \Delta\ . \tag{6}$$

B. Characteristic and Wigner functions

For any density operator $\hat{\rho} \in \mathfrak{S}$, we define its symmetrically ordered characteristic function as

$$\chi(\mathbf{k}) \equiv \mathrm{Tr}\left(\hat{\rho}\ \exp\left(i\,\mathbf{k}^T\,\hat{\mathbf{R}}\right)\right)\ , \qquad \mathbf{k} \in \mathbb{R}^{2n}\ . \tag{7}$$

This formula makes sense for any trace-class operator $\hat\rho$, the associated $\chi(\mathbf{k})$ being a bounded continuous function. By using the Parceval-type formula[1]

$$\mathrm{Tr}\left(\hat\rho_1^\dagger\, \hat\rho_2\right) = \int \overline{\chi_{\hat\rho_1}(\mathbf{k})}\, \chi_{\hat\rho_2}(\mathbf{k})\; \frac{d\mathbf{k}}{(2\pi)^n}\ , \tag{8}$$

one can extend correspondence (7) to Hilbert-Schmidt operators, the associated $\chi(\mathbf{k})$ being a square-integrable function. The correspondence is the isomorphism between the Hilbert space $\mathfrak{H}$ of Hilbert-Schmidt operators with the inner product given by the left hand side of (8) and that of square-integrable functions of $\mathbf{k}$. The operator $\hat\rho$ can be expressed as

$$\hat\rho = \int \chi_{\hat\rho}(\mathbf{k})\; e^{-i\mathbf{k}^T\hat{\mathbf{R}}}\; \frac{d\mathbf{k}}{(2\pi)^n}\ . \tag{9}$$

We also define its Wigner function as the Fourier transform of the characteristic function, square integrable for any $\hat\rho \in \mathfrak{H}$,

$$W_{\hat\rho}(\mathbf{r}) = \int \chi_{\hat\rho}(\mathbf{k})\; e^{-i\mathbf{k}^T\mathbf{r}}\; \frac{d\mathbf{k}}{(2\pi)^{2n}}\ . \tag{10}$$

C. Heisenberg uncertainty

The covariance matrix of $\hat\rho$ is defined as

$$\sigma^{ij} \equiv \mathrm{Tr}\left(\hat\rho\ \left\{\hat R^i - \left\langle \hat R^i\right\rangle,\ \left(\hat R^j - \left\langle \hat R^j\right\rangle\right)\right\}\right)\ , \tag{11}$$

with $\left\langle \hat{\mathbf{R}}\right\rangle \equiv \mathrm{Tr}\left[\hat\rho\ \hat{\mathbf{R}}\right]$, provided the second moments of $\hat\rho$ are finite. The positivity of $\hat\rho$ together with commutation relations (4) implies the Robertson-Heisenberg uncertainty relation,

$$\sigma \geq \pm i\Delta\ , \tag{12}$$

where the inequality is meant to hold for both plus and minus signs in the right-hand-side. We also recall that the Williamson theorem[29] ensures that given a symmetric positive definite matrix σ, there exists a symplectic matrix S such that

$$S\sigma S^T = \bigoplus_{i=1}^{n} \nu_i \mathbb{1}_2\ , \qquad \nu_i > 0\ . \tag{13}$$

The ν_i are called symplectic eigenvalues of σ and can be computed as the positive ordinary eigenvalues of the matrix $i\Delta^{-1}\sigma$ (they come in couples of opposite sign). Heisenberg uncertainty principle (12) is hence satisfied iff they are all greater than one,

$$\nu_j \geq 1\ , \qquad j = 1, \ldots, n\ . \tag{14}$$

Symplectic condition (6) simplifies in the case of one mode. Indeed for any 2×2-matrix M, we have

$$M\Delta M^T = \Delta \det M\ ; \tag{15}$$

therefore, a 2×2-matrix S is symplectic iff

$$\det S = 1\ . \tag{16}$$

We recall also that a symmetric positive definite 2×2-matrix σ has a single symplectic eigenvalue, which is given by

$$\nu^2 = \det \sigma\ ; \tag{17}$$

then from (14) $\sigma \geq 0$ is the covariance matrix of a quantum state iff

$$\det \sigma \geq 1\ . \tag{18}$$

D. Gaussian states

States with positive Wigner function (10) form a convex subset $\mathfrak{P}$ in the space of the density operators $\mathfrak{S}$ of the system. The set $\mathfrak{G}$ of Gaussian states is a proper subset of $\mathfrak{P}$. A Gaussian state

$\hat{\rho}_{\mathbf{x},\,\sigma} \in \mathfrak{G}$ with covariance matrix σ and first moments $\mathbf{x}$ is defined by the property of having Gaussian characteristic function

$$\chi(\mathbf{k}) = e^{-\frac{1}{4}\mathbf{k}^T\sigma\mathbf{k}+i\mathbf{k}^T\mathbf{x}}\,, \tag{19}$$

i.e., Gaussian Wigner function

$$W(\mathbf{r}) = \frac{1}{\sqrt{\det\left(\pi\,\sigma\right)}}\; e^{-(\mathbf{r}-\mathbf{x})^T\sigma^{-1}(\mathbf{r}-\mathbf{x})}\,. \tag{20}$$

For $\sigma = \mathbb{1}_{2n}$, we obtain the family of coherent states $\hat{\rho}_{\mathbf{x},\,\mathbb{1}_{2n}}$, $\mathbf{x} \in \mathbb{R}^{2n}$. The Husimi function $\mathrm{Tr}\left(\hat{\rho}_{\mathbf{x},\,\mathbb{1}_{2n}}\hat{\rho}\right)$ of any bounded operator $\hat{\rho}$ uniquely defines $\hat{\rho}$. It follows that the linear span of the set of coherent states, and hence of all Gaussian states, is dense in the Hilbert space of Hilbert-Schmidt operators $\mathfrak{H}$. Similarly, these linear spans are dense in the Banach space of trace-class operators $\mathfrak{T}$.

E. The Wigner-positive states and the convex hull of Gaussian states

Starting from the vacuum, devices as simple as beam-splitters combined with one-mode squeezers permit (at least in principle) to realize all the elements of $\mathfrak{G}$. Then, choosing randomly according to a certain probability distribution which Gaussian state to produce, it is in principle possible to realize all the states in the convex hull $\mathfrak{C}$ of the Gaussian ones, i.e., all the states $\hat{\rho}$ that can be written as

$$\hat{\rho} = \int \hat{\rho}_{\mathbf{x},\,\sigma}\; d\mu(\mathbf{x},\sigma)\,, \tag{21}$$

where μ is the associated probability measure of the process.

It is easy to verify that $\mathfrak{C}$ is strictly larger than $\mathfrak{G}$, i.e., there exist states (21) which are not Gaussian. On the other hand, one can observe that (21) implies

$$W_{\hat{\rho}}(\mathbf{r}) = \int \frac{1}{\sqrt{\det\left(\pi\,\sigma\right)}}\; e^{-(\mathbf{r}-\mathbf{x})^T\sigma^{-1}(\mathbf{r}-\mathbf{x})}\; d\mu(\mathbf{x},\sigma) > 0\,, \tag{22}$$

so also $\mathfrak{C}$ is included into $\mathfrak{P}$, see Fig. 1. There are however elements of $\mathfrak{P}$ which are not contained in $\mathfrak{C}$:, for example, any finite mixture of Fock states

$$\hat{\rho} = \sum_{n=0}^{N} p_n|n\rangle\langle n| \qquad N < \infty \qquad p_n \geq 0 \qquad \sum_{n=0}^{N} p_n = 1 \tag{23}$$

is not even contained in the weak closure of $\mathfrak{C}$, even if some of them have positive Wigner function.[12]

III. CHARACTERIZATION OF GAUSSIAN-TO-GAUSSIAN MAPS

Determining whether a given state $\hat{\rho}$ belongs to the convex hull $\mathfrak{C}$ of the Gaussian set is a difficult problem.[15–18] Then, there comes the need to find criteria to certify that $\hat{\rho}$ *cannot* be written in form (21). A possible idea is to consider a non-positive superoperator Φ sending any Gaussian state into a state.[12] By linearity, Φ will also send any state of $\mathfrak{C}$ into a state; therefore, if $\Phi(\hat{\rho})$ is not a state, $\hat{\rho}$ cannot be an element of $\mathfrak{C}$: in other words, the transformation Φ acts as a mathematical *probe* for $\mathfrak{C}$. In what follows, we will focus on those probes which are also Gaussian transformations, i.e., which not only send $\mathfrak{G}$ into states but which ensure that the output states $\Phi(\hat{\rho})$ are again elements of $\mathfrak{G}$. Then the following characterization theorem holds

Theorem 3.1. *Let Φ be a linear bounded map of the space $\mathfrak{H}$ of Hilbert-Schmidt operators, sending the set of Gaussian states $\mathfrak{G}$ into itself. Then, its action in terms of characteristic function (7), the first moments, and covariance matrix (11) is of the form*

$$\Phi : \chi(\mathbf{k}) \to \chi\left(K^T\mathbf{k}\right)\; e^{-\frac{1}{4}\mathbf{k}^T\alpha\mathbf{k}+i\mathbf{k}^T\mathbf{y}_0}\,, \tag{24}$$

$$\Phi : \mathbf{x} \to K\mathbf{x} + \mathbf{y}_0 \tag{25}$$

$$\Phi : \sigma \to K\sigma K^T + \alpha \,, \tag{26}$$

where $\mathbf{y}_0$ *is an* $\mathbb{R}^n$ *vector, and* K *and* α *are* $2n \times 2n$ *real matrices such that* α *is symmetric, and for any* $\sigma \geq \pm i\Delta$,

$$K\sigma K^T + \alpha \geq \pm i\Delta \,, \tag{27}$$

where the inequalities are meant to hold for both plus and minus signs in the right-hand-sides.

Condition (27) imposes that $\Phi(\hat{\rho})$ is a Gaussian state for any Gaussian $\hat{\rho}$. It is weaker than the condition which guarantees complete positivity (see Eq. (45) below), which also ensures the mapping of Gaussian states into Gaussian states. An example of not completely positive mapping fulfilling (27) is provided by the dilatations defined in Eq. (1). Such mappings in fact, while explicitly not CP,[12] correspond to transformations (24) where we set $\mathbf{y}_0 = \mathbf{0}$ and take

$$K = \lambda \mathbb{1}_{2n} \,, \qquad \alpha = 0 \,, \tag{28}$$

with $|\lambda| > 1$. At the level of covariance matrices (26), this implies $\sigma' = \lambda^2\sigma$ which clearly still preserve Heisenberg inequality (12) (indeed $\lambda^2\sigma \geq \sigma \geq \pm i\Delta$), ensuring hence condition (27). Dilatations are not bounded with respect to the trace norm (see Theorem 1.1 of Appendix). This explains why Theorem 3.1 is formulated on the space of Hilbert-Schmidt operators. Indeed, via the Parceval formula, we can prove that dilatations are bounded in this space,

$$\begin{aligned} \|\Phi(\hat{\rho})\|^2 &= \int \left|\chi_{\hat{\rho}}(\lambda\mathbf{k})\right|^2 \frac{d\mathbf{k}}{(2\pi)^n} = \\ &= \int \left|\chi_{\hat{\rho}}(\mathbf{k})\right|^2 \frac{d\mathbf{k}}{(2\pi\lambda^2)^n} = \frac{1}{\lambda^{2n}}\|\hat{\rho}\|^2 \,. \end{aligned}$$

For $\lambda = \frac{1}{\mu}$ with $|\mu| > 1$, transformation (28) yields a contraction of the output Wigner quasi-distribution. In the Hilbert space $\mathfrak{H}$, the contraction by λ is λ^{2n} times the adjoint of the dilatation by $\mu = \frac{1}{\lambda}$, as follows from Parceval formula (8). As different from the dilatations, these mappings no longer ensure that all Gaussian states will be transformed into proper density operators. For instance, the vacuum state is mapped into a non-positive operator (this shows in particular that the contractions and hence the adjoint dilatations are non-positive maps).

Another example of transformation not fulfilling CP requirement (45) but respecting (27) is the (complete) transposition

$$K = T_{2n} \qquad \alpha = 0 \,, \tag{29}$$

that is well-known not to be CP. Unfortunately, being positive, it cannot be used to certify that a given state is not contained in the convex hull $\mathfrak{C}$ of the Gaussian ones. Is there anything else? We will prove that for one mode, any channel satisfying (27) can be written as a dilatation composed with a completely positive channel, possibly composed with transposition (29), see Fig. 2. We will also show that in the multi-mode case, this simple decomposition does not hold in general; however, it still holds if we restrict to the channels that do not add noise, i.e., with $\alpha = 0$.

Proof. Let the Gaussian state $\hat{\rho}_{\mathbf{x},\,\sigma}$ be sent into the Gaussian state $\hat{\rho}_{\mathbf{y},\,\tau}$ with covariance matrix $\tau(\mathbf{x}, \sigma)$ and first moment $\mathbf{y}(\mathbf{x}, \sigma)$ with the characteristic function

$$\chi_{\Phi(\hat{\rho}_{\mathbf{x},\,\sigma})}(\mathbf{k}) \equiv \chi_{y,\tau}(\mathbf{k}) = e^{-\frac{1}{4}\mathbf{k}^T\,\tau\,\mathbf{k} + i\mathbf{k}^T\,\mathbf{y}} \,. \tag{30}$$

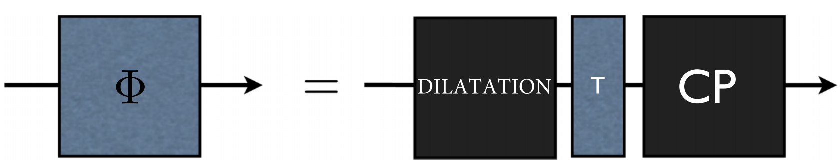

FIG. 2. Pictorial representation of the decomposition of a generic (not necessarily positive) Gaussian single-mode transformation Φ in terms of a dilatation, CP mapping, and (possibly) a transposition. The same decomposition applies also to the case of n-mode transformations when no extra noise is added to the system, see Sec. V.

We first remark that the functions $\tau(\mathbf{x}, \sigma)$ and $\mathbf{y}(\mathbf{x}, \sigma)$ are continuous. The map Φ is bounded and hence continuous in the Hilbert-Schmidt norm. The required continuity follows from

Lemma 3.2. The bijection $(\mathbf{x}, \sigma) \to \hat{\rho}_{\mathbf{x},\, \sigma}$ *is bicontinuous in the Hilbert-Schmidt norm.*

The proof of the lemma follows from the Parceval formula by direct computation of the Gaussian integral,

$$\int \left|\chi_{\hat{\rho}_{\mathbf{x},\,\sigma}}(\mathbf{k}) - \chi_{\hat{\rho}_{\mathbf{x}',\,\sigma'}}(\mathbf{k})\right|^2 \frac{d\mathbf{k}}{(2\pi)^n} \ .$$

Next, we have the identity,

$$\int \hat{\rho}_{\mathbf{x}',\,\sigma'}\, \mu_{\mathbf{x},\,\sigma}(d\mathbf{x}') = \hat{\rho}_{\mathbf{x},\,\sigma'+\sigma} \ , \tag{31}$$

where $\mu_{\mathbf{x},\,\sigma}$ is Gaussian probability measure with the first moments $\mathbf{x}$ and covariance matrix σ, which is verified by comparing the quantum characteristic functions of both sides.

Applying to both sides of this identity the continuous map Φ, we obtain

$$\int \hat{\rho}_{\mathbf{y}(\mathbf{x}',\,\sigma'),\ \tau(\mathbf{x}',\,\sigma')}\, \mu_{\mathbf{x},\,\sigma}(d\mathbf{x}') = \hat{\rho}_{\mathbf{y}(\mathbf{x},\,\sigma'+\sigma),\ \tau(\mathbf{x},\,\sigma'+\sigma)} \ .$$

By taking the quantum characteristic functions of both sides, we obtain

$$\int \chi_{\mathbf{y}(\mathbf{x}',\,\sigma'),\ \tau(\mathbf{x}',\,\sigma')}(\mathbf{k})\ \mu_{\mathbf{x},\,\sigma}(d\mathbf{x}') = \\ = \chi_{\mathbf{y}(\mathbf{x},\,\sigma'+\sigma),\ \tau(\mathbf{x},\,\sigma'+\sigma)}(\mathbf{k}) \ , \qquad \boldsymbol{k} \in \mathbb{R}^n \ . \tag{32}$$

Now notice that $\mu_{\mathbf{x},\,\sigma}$ is the fundamental solution of the diffusion equation,

$$du = \frac{1}{4}\partial_i d\sigma^{ij}\partial_j u \ , \tag{33}$$

where d is the differential with respect to σ, i.e.,

$$d = \sum_{i,\,j=1}^{m} d\sigma^{ij}\frac{\partial}{\partial\sigma^{ij}} \tag{34}$$

and

$$\partial_i = \frac{\partial}{\partial x^i} \ , \tag{35}$$

with the sum over the repeated indices. Relation (32) means that for any fixed $\mathbf{k}$, the function $u(\mathbf{x}, \sigma) = \chi_{\mathbf{y}(\mathbf{x},\,\sigma'+\sigma),\ \tau(\mathbf{x},\,\sigma'+\sigma)}(\mathbf{k})$ is the solution of the Cauchy problem for Eq. (32) with the initial condition $u(\mathbf{x}, 0) = \chi_{\mathbf{y}(\mathbf{x},\,\sigma'),\ \tau(\mathbf{x},\,\sigma')}(\mathbf{k})$. Since the last function is bounded and continuous, the solution of the Cauchy problem is infinitely differentiable in $(\mathbf{x}, \sigma)$ for $\sigma > 0$. Substituting

$$u(\mathbf{x}, \sigma) = \exp\left[-\frac{1}{4}\mathbf{k}^T\, \tau(\mathbf{x}, \sigma' + \sigma)\, \mathbf{k} + i\mathbf{k}^T\, \mathbf{y}(\mathbf{x}, \sigma' + \sigma)\right]$$

into (33) and differentiating the exponent, we obtain the identity

$$-\frac{1}{4}\mathbf{k}^T\ d\tau\ \mathbf{k} + i\mathbf{k}^T d\mathbf{y} = \\ = \frac{1}{4}\left(\frac{1}{4}\mathbf{k}^T\partial_i\tau\,\mathbf{k} - i\mathbf{k}^T\partial_i\mathbf{y}\right) d\sigma^{ij} \left(\frac{1}{4}\mathbf{k}^T\partial_j\tau\,\mathbf{k} - i\mathbf{k}^T\partial_j\mathbf{y}\right) \\ -\frac{1}{16}\mathbf{k}^T\left(\partial_i d\sigma^{ij}\partial_j\tau\right)\mathbf{k} + \frac{i}{4}\mathbf{k}^T\partial_i d\sigma^{ij}\partial_j\mathbf{y} \ .$$

We can now compare the two expressions. Since the left hand side contains only terms at most quadratic in $\mathbf{k}$, we get

$$\partial_i\tau = 0 \ , \tag{36}$$

i.e., τ does not depend on $\mathbf{x}$. Then, the right hand side simplifies into

$$-\frac{1}{4}\mathbf{k}^T\left(\partial_i\mathbf{y}\, d\sigma^{ij}\partial_j\mathbf{y}^T\right)\mathbf{k} + \frac{i}{4}\mathbf{k}^T\partial_i d\sigma^{ij}\partial_j\mathbf{y}\ . \tag{37}$$

Comparing again with the left hand side, we get

$$d\tau(\sigma) = \partial_i\mathbf{y}\, d\sigma^{ij}\partial_j\mathbf{y}^T, \tag{38}$$

$$d\mathbf{y}(\mathbf{x},\sigma) = \frac{1}{4}\partial_i d\sigma^{ij}\partial_j\mathbf{y}\ . \tag{39}$$

Since $d\tau(\sigma)$ does not depend on $\mathbf{x}$, also $\partial_i\mathbf{y}$ cannot, i.e., $\mathbf{y}$ is a linear function of $\mathbf{x}$,

$$\mathbf{y}(\mathbf{x},\sigma) = K(\sigma)\,\mathbf{x} + \mathbf{y}_0(\sigma)\ , \tag{40}$$

where $K(\sigma)$ and $\mathbf{y}_0(\sigma)$ are still arbitrary functions. But now (39) becomes

$$d\mathbf{y}(\mathbf{x},\sigma) = 0\ , \tag{41}$$

i.e., $\mathbf{y}$ does not depend on σ, i.e.,

$$\mathbf{y} = K\mathbf{x} + \mathbf{y}_0\ , \tag{42}$$

with K and $\mathbf{y}_0$ constant. Finally, (38) becomes

$$d\tau(\sigma) = K\, d\sigma\, K^T\ , \tag{43}$$

that can be integrated into

$$\tau(\sigma) = K\,\sigma\,K^T + \alpha\ . \tag{44}$$

Thus, we get that the transformation rules for the first and second moments are given by Eqs. (25) and (26). The positivity condition for quantum Gaussian states implies (27). The map defined by (24) correctly reproduces (25) and (26), so it coincides with Φ on the Gaussian states. Since it is linear and continuous, and the linear span of Gaussian states is dense in $\mathfrak{H}$, it coincides with Φ on the whole $\mathfrak{H}$. □

Remark 3.3. A similar argument can be used to prove that any linear positive map Φ of the Banach space $\mathfrak{T}$ of trace-class operators, leaving the set of Gaussian states globally invariant, has form (24). By Lemma 2.2.1 of Ref. 30, any such map is bounded, and the proof of Theorem 3.1 can be repeated, with $\mathfrak{H}$ replaced by $\mathfrak{T}$. In addition, since the trace of operator is continuous on $\mathfrak{T}$, formula (24) implies preservation of trace. However, the positivity condition is difficult to express in terms of the map parameters $\mathbf{y}_0, K, \alpha$.

On the other hand, if Φ is completely positive, then the necessary and sufficient condition is

$$\alpha \geq \pm i(\Delta - \Delta_K)\ , \tag{45}$$

where

$$\Delta_K \equiv K\Delta K^T\ . \tag{46}$$

Thus, Φ is a quantum Gaussian channel,[5] and condition Eq. (27) is replaced by the more stringent constraint (45).

For automorphisms of the C^*-algebra of the canonical commutation relations, a similar characterization, based on a different proof using partial ordering of Gaussian states, was first given in Refs. 31 and 32.

Remark 3.4. There is a counterpart of Theorem 3.1 in probability theory.

Theorem 3.5. *Let Φ be an endomorphism (linear bounded transformation) of the Banach space $\mathcal{M}(\mathbb{R}^n)$ of finite signed Borel measures on $\mathbb{R}^n$ (equipped with the total variation norm) having the Feller property (the dual Φ^* leaves invariant the space of bounded continuous functions on $\mathbb{R}^n$). Then, if Φ sends the set of Gaussian probability measures into itself, Φ is a Markov operator whose action in terms of characteristic functions is of form (24), with the condition (27) replaced by $\alpha \geq 0$.*

Proof. The proof is parallel to the proof of Theorem 3.1, with replacement of (31) by the corresponding identity for Gaussian probability measures. As a result, we obtain that the action of Φ in terms of characteristic functions is given by (24) for any measure μ which is a linear combination of Gaussian probability measures. For arbitrary measure $\mu \in \mathcal{M}(\mathbb{R}^n)$, the characteristic function of $\Phi(\mu)$ is

$$\chi_{\Phi(\mu)}(\mathbf{k}) = \int \mathbf{e}^{i\,\mathbf{k}^T\mathbf{x}}\, \Phi(\mu)(d\mathbf{x}) = \int \Phi^*\left(\mathbf{e}^{i\,\mathbf{k}^T\mathbf{x}}\right)\, \mu(d\mathbf{x})\,,$$

where $\Phi^*\left(\mathbf{e}^{i\,\mathbf{k}^T\mathbf{x}}\right)$ is continuous bounded function by the Feller property. Since the linear span of Gaussian probability measures is dense in $\mathcal{M}(\mathbb{R}^n)$ in the weak topology defined by continuous bounded functions (it suffices to take Dirac's deltas, i.e, probability measures degenerated at the points of $\mathbb{R}^n$), formula (24) extends to characteristic function of arbitrary finite signed Borel measure on $\mathbb{R}^n$. The action of Φ on the moments is given by (25) and (26). The positivity of the output covariance matrix when the input is a Dirac delta implies $\alpha \geq 0$. □

A. Contractions

A contraction by $\lambda = \frac{1}{\mu}$ behaves properly on the restricted subset $\mathfrak{G}^{(>)}_{\mu^2}$ of $\mathfrak{G}$ formed by the Gaussian states whose covariance matrix admits symplectic eigenvalues larger than μ^2. Indeed, all elements of $\mathfrak{G}^{(>)}_{\mu^2}$ will be mapped into proper Gaussian output states by the contraction (and by linearity also the convex hull of $\mathfrak{G}^{(>)}_{\mu^2}$ will be mapped into proper output density operators). We will prove that any transformation with this property can be written as a contraction of $1/\mu$, followed by a transformation of the kind of Theorem 3.1. Let us first notice that:

Lemma 3.6. A set (K, α) satisfies (27) for any σ with symplectic eigenvalues greater than μ^2 iff $(\mu K,\ \alpha)$ satisfies (27) for any $\sigma \geq \pm i\Delta$.

Proof. σ has all the symplectic eigenvalues greater than μ^2 iff $\sigma \geq \pm i\mu^2\Delta$, i.e., iff $\sigma' = \sigma/\mu^2$ is a state. Then, (27) is satisfied for any $\sigma \geq \pm i\mu^2\Delta$ iff

$$\mu^2 K\sigma' K^T + \alpha \geq \pm i\Delta \qquad \forall\, \sigma' \geq \pm i\Delta\,, \tag{47}$$

i.e., iff $(\mu K,\ \alpha)$ satisfies (27) for any $\sigma \geq \pm i\Delta$. □

Then, we can state the following result:

Corollary 3.7. Any transformation associated with (K, α) satisfying (27) for any state in $\mathfrak{G}^{(>)}_{\mu^2}$ (i.e., for any $\sigma \geq \pm i\mu^2\Delta$) can be written as a contraction of $1/\mu$, followed by a transformation satisfying (27) for any state in $\mathfrak{G}$ (i.e., for any $\sigma \geq \pm i\Delta$).

IV. ONE MODE

Here, we will give a complete classification of all one-mode maps (24) satisfying (27).
We will need the following:

Lemma 4.1. A set $(K,\ \alpha)$ satisfies (27) iff

$$\sqrt{\det\alpha} \geq 1 - |\det K|\,. \tag{48}$$

Proof. For one mode, $\sigma \geq 0$ satisfies $\sigma \geq \pm i\Delta$ iff $\det\sigma \geq 1$, and condition (27) can be rewritten as

$$\det\left(K\sigma K^T + \alpha\right) \geq 1, \qquad \forall\, \sigma \geq 0,\ \det\sigma \geq 1\,. \tag{49}$$

To prove (49) $\Longrightarrow$ (48), consider first the case $\det K \neq 0$. Choosing σ such that $K\sigma K^T = \frac{|\det K|}{\sqrt{\det \alpha}}\alpha$, we have $\sigma \geq 0, \det \sigma \geq 1$. Inserting this into (49), we obtain

$$\left(1 + \frac{|\det K|}{\sqrt{\det \alpha}}\right)^2 \det \alpha \geq 1$$

or, taking square root,

$$\left(1 + \frac{|\det K|}{\sqrt{\det \alpha}}\right) \sqrt{\det \alpha} \geq 1 \ .$$

hence (48) follows.

If $\det K = 0$, then there is a unit vector $\mathbf{e}$ such that $K\mathbf{e} = 0$. Choose $\sigma = \epsilon^{-1}\mathbf{e}\mathbf{e}^T + \epsilon \mathbf{e}_1\mathbf{e}_1^T$, where $\epsilon > 0$, and $\mathbf{e}_1$ is a unit vector orthogonal to $\mathbf{e}$. Then $\sigma \geq 0, \det \sigma = 1$, and $K\sigma K^T = \epsilon A$, where $A = K\mathbf{e}_1\mathbf{e}_1^T K^T \geq 0$. Inserting this into (49), we obtain

$$\det(\epsilon A + \alpha) \geq 1, \quad \forall\, \epsilon \geq 0,$$

hence (48) follows.

To prove (48) $\Longrightarrow$ (49), we use Minkowski's determinant inequality

$$\sqrt{\det(A+B)} \geq \sqrt{\det A} + \sqrt{\det B} \qquad \forall\, A, B \geq 0 \ . \tag{50}$$

We have for all $\sigma \geq 0, \det \sigma \geq 1$,

$$\begin{aligned} &\sqrt{\det(K\sigma K^T + \alpha)} \geq \\ &\geq |\det K| \sqrt{\det \sigma} + \sqrt{\det \alpha} \geq \\ &\geq |\det K| + \sqrt{\det \alpha} \ \geq 1, \end{aligned} \tag{51}$$

where in the last step we have used (48). □

To compare transformations satisfying (48) with CP ones, we need also

Lemma 4.2. A set (K, α) characterizes a completely positive transformation (i.e., satisfies (45)) iff

$$\sqrt{\det \alpha} \geq |1 - \det K| \ . \tag{52}$$

Proof. For one mode, using (15),

$$\Delta_K = K\Delta K^T = \det K\ \Delta \ , \tag{53}$$

and (45) becomes

$$\alpha \geq \pm i(1 - \det K)\Delta \ . \tag{54}$$

Recalling (18), for linearity, (54) becomes exactly

$$\det \alpha \geq (1 - \det K)^2 \ . \tag{55}$$

□

We recall here that a complete classification of single mode CP maps has been provided in Refs. 33 and 34.

We are now ready to prove the main result of this section.

Theorem 4.3. *Any map Φ satisfying (27) can be written as a dilatation possibly composed with the transposition, followed by a completely positive map. In more detail, given a pair (K, α) satisfying (27),*

a1: *If*

$$0 \leq \det K \leq 1 \ , \tag{56}$$

Φ is completely positive.

a2: *If*

$$\det K > 1 , \tag{57}$$

Φ can be written as a phase-space dilatation of parameter $\lambda = \sqrt{\det K} > 1$, composed with the symplectic transformation given by

$$S = \frac{K}{\sqrt{\det K}} , \tag{58}$$

composed with the addition of Gaussian noise given by α.

b1: *If*

$$-1 \leq \det K < 0 , \tag{59}$$

Φ can be written as a transposition composed with a completely positive map.

b2: *If*

$$\det K < -1 , \tag{60}$$

Φ can be written as a dilatation of $\sqrt{|\det K|}$ composed with the transposition, followed by the symplectic transformation given by

$$S = \frac{K}{\sqrt{|\det K|}} , \tag{61}$$

composed with the addition of Gaussian noise given by α.

Proof. **a:** Let us start from the case

$$\det K \geq 0 . \tag{62}$$

a1: If

$$0 \leq \det K \leq 1 , \tag{63}$$

(48) and (52) coincide, so Φ is completely positive.

a2: If

$$\det K > 1 , \tag{64}$$

we can write K as

$$K = S \sqrt{\det K}\, \mathbb{1}_2 , \tag{65}$$

where

$$S = \frac{K}{\sqrt{\det K}} \tag{66}$$

is symplectic since $\det S = 1$. Then, Φ can be written as a dilatation of $\sqrt{\det K} > 1$, followed by the symplectic transformation given by S, composed with the addition of the Gaussian noise given by α.

b: If

$$\det K < 0 , \tag{67}$$

we can write K as

$$K = K'T , \tag{68}$$

where T is the one-mode transposition

$$T = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tag{69}$$

and

$$\det K' = -\det K > 0 . \tag{70}$$

From (48), we can see that also K' satisfies

$$\sqrt{\det \alpha} \geq 1 - |\det K'| \ , \tag{71}$$

and we can exploit the classification with positive determinant, ending with the same decomposition with the addition of the transposition after (or before, since they commute) the eventual dilatation.

□

V. MULTI-MODE CASE

In the multi-mode case, a classification as simple as the one of Theorem 4.3 does not exist. However, we will prove that if Φ does not add any noise, i.e $\alpha = 0$, the only solution to (27) is a dilatation possibly composed with a (total) transposition, followed by a symplectic transformation. We will also provide examples that do not fall in any classification like 4.3, i.e., that are not composition of a dilatation, possibly followed by a (total) transposition, and a completely positive map.

We will need the following lemma:

Lemma 5.1.

$$\inf_{\sigma \geq \pm i\Delta} \mathbf{w}^\dagger \sigma \mathbf{w} = \left|\mathbf{w}^\dagger \Delta \mathbf{w}\right| \qquad \forall \ \ \mathbf{w} \in \mathbb{C}^{2n} \ . \tag{72}$$

Proof. a. Lower bound. The lower bound for the LHS is straightforward: for any $\sigma \geq \pm i\Delta$ and $\mathbf{w} \in \mathbb{C}^{2n}$, we have

$$\mathbf{w}^\dagger \sigma \mathbf{w} \geq \pm i \mathbf{w}^\dagger \Delta \mathbf{w} \ , \tag{73}$$

and then

$$\inf_{\sigma \geq \pm i\Delta} \mathbf{w}^\dagger \sigma \mathbf{w} \geq \left|\mathbf{w}^\dagger \Delta \mathbf{w}\right| \ . \tag{74}$$

b. Upper bound. To prove the converse, let

$$\mathbf{w} = \mathbf{w}_1 + i\mathbf{w}_2 \ , \qquad \mathbf{w}_i \in \mathbb{R}^{2n} \ ,$$

where without lost of generality, we assume $\mathbf{w}_1 \neq \mathbf{0}$. Then,

$$\mathbf{w}^\dagger \sigma \mathbf{w} = \mathbf{w}_1^T \sigma \mathbf{w}_1 + \mathbf{w}_2^T \sigma \mathbf{w}_2, \quad \left|\mathbf{w}^\dagger \Delta \mathbf{w}\right| = 2\left|\mathbf{w}_1^T \Delta \mathbf{w}_2\right| .$$

Assume first $\mathbf{w}_1^T \Delta \mathbf{w}_2 \equiv \epsilon \neq 0$. Then, we can introduce the symplectic basis $\{\mathbf{e}_j, \ \mathbf{h}_j\}_{j=1,\ldots,n}$, where

$$\mathbf{e}_1 = \frac{\mathbf{w}_1}{\sqrt{|\epsilon|}} \ , \qquad \mathbf{h}_1 = \frac{\mathrm{sign}(\epsilon)\ \mathbf{w}_2}{\sqrt{|\epsilon|}} \ .$$

Expressed in this basis, question (72) reduces to the first mode, and the infimum is attained by the matrix of the form,

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \sigma_{n-1} \ ,$$

where σ_{n-1} is any quantum correlation matrix in the rest $n-1$ modes.

Consider next the case where $\mathbf{w}_1^T \Delta \mathbf{w}_2 = 0$ and $\mathbf{w}_2$ is not proportional to $\mathbf{w}_1$. In this context, we introduce the symplectic basis $\{\mathbf{e}_j, \ \mathbf{h}_j\}_{j=1,\ldots,n}$, where

$$\mathbf{e}_1 = \mathbf{w}_1 \ , \qquad \mathbf{e}_2 = \mathbf{w}_2 \ .$$

Accordingly, identity (72) reduces to the first two modes, and the infimum is attained by the matrices of the form,

$$\sigma(\epsilon) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \oplus \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \oplus \sigma_{n-2} \ ,$$

where σ_{n-2} is any quantum correlation matrix in the rest $n-2$ modes and $\epsilon \to 0$.

Finally, if $\mathbf{w}_2 = c\ \mathbf{w}_1, \ c \in \mathbb{R}$, we introduce the symplectic basis $\{\mathbf{e}_j, \ \mathbf{h}_j\}_{j=1,\ldots,n}$, where $\mathbf{e}_1 = \mathbf{w}_1$. Question (72) reduces to the first mode, and the infimum is attained by the matrices of the form

$$\sigma(\epsilon) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \oplus \sigma_{n-1} ,$$

where σ_{n-1} is any quantum correlation matrix in the rest $n-1$ modes, and $\epsilon \to 0$. □

A simple consequence of Lemma 5.1 is

Lemma 5.2. Any α satisfying (27) for some K is positive semidefinite.

Proof. Constraint (27) implies

$$\left(K^T \mathbf{k}\right)^T \sigma \left(K^T \mathbf{k}\right) + \mathbf{k}^T \alpha \mathbf{k} \geq 0 \tag{75}$$

for any $\sigma \geq \pm i\Delta$ and $\mathbf{k} \in \mathbb{R}^{2n}$. Taking the inf over $\sigma \geq \pm i\Delta$ and exploiting Lemma 5.1 with $\mathbf{w} = K^T \mathbf{k}$, we get

$$\mathbf{k}^T \alpha \mathbf{k} \geq 0 , \tag{76}$$

i.e., α is positive semidefinite. In deriving (76), we have used that, since Δ is antisymmetric, $\mathbf{k}\Delta\mathbf{k}^T = 0$ for any real $\mathbf{k}$. □

The Lemma 5.1 allows us to rephrase the problem: indeed, constraint (27) can be written as

$$(K^T \mathbf{w})^\dagger \sigma (K^T \mathbf{w}) + \mathbf{w}^\dagger \alpha \mathbf{w} \geq \left|\mathbf{w}^\dagger \Delta \mathbf{w}\right| , \tag{77}$$

$\forall\ \sigma \geq \pm i\Delta$, $\forall \mathbf{w} \in \mathbb{C}^{2n}$. Taking the inf over σ in the LHS, we hence get

$$\left|\mathbf{w}^\dagger \Delta_K \mathbf{w}\right| + \mathbf{w}^\dagger \alpha \mathbf{w} \geq \left|\mathbf{w}^\dagger \Delta \mathbf{w}\right| , \quad \forall\ \mathbf{w} \in \mathbb{C}^{2n} , \tag{78}$$

with Δ_K as in Eq. (46). Notice that, as for complete positivity constraint (45), since K enters in (78) only through $\left|\mathbf{w}^\dagger \Delta_K \mathbf{w}\right|$, whether given K and α satisfy (27) depends not on the entire K but only on Δ_K.

The easiest way to give a general classification of the channels satisfying (78) (and then (27)) would seem choosing a basis in which Δ is as in (5), and then try to put the antisymmetric matrix Δ_K in some canonical form using symplectic transformations preserving Δ. However, the complete classification of antisymmetric matrices under symplectic transformations is very involved,[35] and in the multi-mode case, the problem simplifies only if we consider maps Φ that do not add noise, since in this case constraint (78) rules out almost all the equivalence classes. In the general case, we will provide examples showing the other possibilities.

A. No noise

The main result of this section is the classification of the maps Φ that do not add noise ($\alpha = 0$) and satisfy (27).

Theorem 5.3. *A map Φ with $\alpha = 0$ satisfying (27) can always be decomposed as a dilatation (28), possibly composed with the transposition, followed by a symplectic S transformation, i.e.,*

$$K = S\ \kappa \mathbb{1}_{2n} \qquad or \qquad K = S\ T\ \kappa \mathbb{1}_{2n} , \tag{79}$$

with $\kappa \geq 1$.

Proof. With $\alpha = 0$ and

$$\mathbf{w} = \mathbf{w}_1 + i\mathbf{w}_2 , \qquad \mathbf{w}_i \in \mathbb{R}^{2n} , \tag{80}$$

(78) becomes

$$\left|\mathbf{w}_1^T \Delta_K \mathbf{w}_2\right| \geq \left|\mathbf{w}_1^T \Delta \mathbf{w}_2\right| , \tag{81}$$

i.e., all the matrix elements of Δ_K are in modulus bigger than the corresponding ones of Δ in *any* basis. In particular, if some matrix element Δ_K^{ij} vanishes, also Δ^{ij} must vanish. Let us choose a basis in which Δ_K has the canonical form,

$$\Delta_K = \bigoplus_{i=1}^{\frac{r}{2}} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \oplus 0_{2n-r} \,, \tag{82}$$

where

$$r \equiv \operatorname{rank} \Delta_K \,. \tag{83}$$

For (81), in this basis, Δ must be of the form,

$$\Delta = \bigoplus_{i=1}^{\frac{r}{2}} \begin{pmatrix} & \lambda_i \\ -\lambda_i & \end{pmatrix} \oplus 0_{2n-r} \,, \qquad |\lambda_i| \leq 1 \,. \tag{84}$$

Since Δ has full rank, there cannot be zeroes in its decomposition, so r must be $2n$.

We will prove that all the eigenvalues λ_i must be equal. Let us take two eigenvalues λ and μ and consider the restriction of Δ and Δ_K to the subspace associated to them,

$$\Delta_K = \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right) \qquad \Delta = \left(\begin{array}{cc|cc} & \lambda & & \\ -\lambda & & & \\ \hline & & & \mu \\ & & -\mu & \end{array}\right) . \tag{85}$$

If we change basis with the rotation matrix

$$R = \begin{pmatrix} \cos\theta \; \mathbb{1}_2 & -\sin\theta \; \mathbb{1}_2 \\ \sin\theta \; \mathbb{1}_2 & \cos\theta \; \mathbb{1}_2 \end{pmatrix} ,$$
$$\Delta \mapsto R\Delta R^T \,, \qquad \Delta_K \mapsto R\Delta_K R^T \,, \tag{86}$$

Δ_K remains of the same form, while Δ acquires off-diagonal elements proportional to $\lambda - \mu$. Since for (81) the off-diagonal elements of Δ must vanish also in the new basis, the only possibility is $\lambda = \mu$. Then all the λ_i must be equal, and Δ_K must then be proportional to Δ,

$$\Delta_K = \frac{1}{\lambda}\Delta \,, \qquad 0 < |\lambda| \leq 1 \,, \tag{87}$$

where we have put all the λ_i equal to $\lambda \neq 0$ (since Δ is nonsingular they cannot vanish). Relation (87) means

$$K\Delta K^T = \frac{1}{\lambda}\Delta \,, \tag{88}$$

i.e.,

$$\left(\sqrt{|\lambda|}\; K\right) \; \Delta \; \left(\sqrt{|\lambda|}\; K\right)^T = \operatorname{sign}(\lambda)\Delta \,. \tag{89}$$

If $0 < \lambda \leq 1$, we can write K as a dilatation of

$$\kappa = \frac{1}{\sqrt{\lambda}} \,, \tag{90}$$

composed with a symplectic transformation given by

$$S = \sqrt{\lambda}\; K \,, \tag{91}$$

i.e.,

$$K = S\; \kappa \mathbb{1}_{2n} \,, \qquad S\Delta S^T = \Delta \,. \tag{92}$$

If $-1 \leq \lambda < 0$, since the total transposition T changes the sign of Δ,

$$T\Delta T^T = -\Delta \,, \tag{93}$$

we can write K as a dilatation of

$$\kappa = \frac{1}{\sqrt{|\lambda|}}\ , \tag{94}$$

composed with T followed by a symplectic transformation,

$$K = S\ T\ \kappa \mathbb{1}_{2n}\ , \qquad S\Delta S^T = \Delta\ . \tag{95}$$

□

B. Examples with nontrivial decomposition

If $\alpha \neq 0$, a decomposition as simple as the one of Theorem 5.3 does no more exist: here, we will provide some examples in which the canonical form of Δ_K is less trivial and that do not fall in any classification like the precedent one. Essentially, they are all based on this observation.

Proposition 5.4. If α is the covariance matrix of a quantum state, i.e., $\alpha \geq \pm i\Delta$, constraint (27) is satisfied by any K.

Since for one mode the decomposition of Theorem 4.3 holds, we will provide examples with two-mode systems.

We will always consider bases in which

$$\Delta = \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right)\ . \tag{96}$$

1. Partial transpose

The first example is the partial transpose of the second subsystem, composed with a dilatation of $\sqrt{\nu}$ and the addition of the covariance matrix of the vacuum as noise,

$$K = \sqrt{\nu}\begin{pmatrix} \mathbb{1}_2 & \\ & T_2 \end{pmatrix}\ , \qquad \nu > 0\ , \qquad \alpha = \mathbb{1}_4\ . \tag{97}$$

In this case, we have

$$\Delta_K = \left(\begin{array}{cc|cc} & \nu & & \\ -\nu & & & \\ \hline & & & -\nu \\ & & \nu & \end{array}\right)\ , \tag{98}$$

and $i(\Delta - \Delta_K)$ has eigenvalues,

$$\pm(1+\nu) \qquad \pm(1-\nu)\ , \tag{99}$$

so that one of them is $|1 + |\nu|| > 1$, and complete positivity requirement (45)

$$\mathbb{1}_4 \geq \pm i(\Delta - \Delta_K) \tag{100}$$

cannot be fulfilled by any $\nu \neq 0$.

We will prove that this map cannot be written as a dilatation, possibly composed with the transposition, followed by a completely positive map. Indeed, suppose we can write K as

$$K = K' \, \lambda \mathbb{1}_4 \qquad \text{or} \qquad K = K' \, T_4 \, \lambda \mathbb{1}_4 \, , \qquad \lambda \geq 1 \, . \tag{101}$$

Then,

$$\Delta_{K'} = \pm \frac{1}{\lambda^2} \Delta_K \tag{102}$$

is always of form (98) with

$$\nu' = \pm \frac{\nu}{\lambda^2} \, , \tag{103}$$

and also the transformation with K' cannot be completely positive.

2. Q exchange

As second example, we take for the added noise α still the covariance matrix of the vacuum, and for the matrix K the partial transposition of the first mode composed with the exchange of Q^1 and Q^2 followed by a dilatation of $\sqrt{\nu}$,

$$\alpha = \mathbb{1}_4 \geq \pm i \Delta \, , \quad K = \sqrt{\nu} \left(\begin{array}{cc|cc} & & 1 & \\ & -1 & & \\ \hline 1 & & & \\ & & & 1 \end{array} \right) , \quad \nu > 0 \, . \tag{104}$$

With this choice,

$$\Delta_K = \left(\begin{array}{cc|cc} & & & \nu \\ & & \nu & \\ \hline & -\nu & & \\ -\nu & & & \end{array} \right) . \tag{105}$$

The transformation is completely positive iff

$$\mathbb{1}_4 \geq \pm i (\Delta - \Delta_K) \, , \tag{106}$$

and since the eigenvalues of $i(\Delta - \Delta_K)$ are

$$\pm \sqrt{1 + \nu^2} \, , \tag{107}$$

(106) is never fulfilled for any $\nu \neq 0$.

As before, we will prove that this map cannot be written as a dilatation, possibly composed with the transposition, followed by a completely positive map. Indeed, suppose we can write K as

$$K = K' \, \lambda \mathbb{1}_4 \qquad \text{or} \qquad K = K' \, T_4 \, \lambda \mathbb{1}_4 \, , \qquad \lambda \geq 1 \, . \tag{108}$$

Then,

$$\Delta_{K'} = \pm \frac{1}{\lambda^2} \Delta_K \tag{109}$$

is always of form (105) with

$$\nu' = \pm \frac{\nu}{\lambda^2} \, , \tag{110}$$

and also the transformation with K' cannot be completely positive.

VI. CONCLUSIONS

In this paper, we have explored both at the classical and quantum levels the set of linear transformations sending the set of Gaussian states into itself without imposing any further requirement, such as positivity. We have proved that the action on the covariance matrix and on the first moment must be linear, and we have found the form of the action on the characteristic function. Focusing on the quantum case, for one mode, we have obtained a complete classification, stating that the only not CP transformations in the set are actually the total transposition and the dilatations (and their compositions with CP maps). The same result holds also in the multi-mode scenario, but it needs the further hypothesis of homogeneous action on the covariance matrix, since we have shown the existence of non-homogeneous transformations belonging to the set but not falling into our classification.

Despite the set $\mathfrak{F}$ of quantum states that are sent into positive operators by any dilatation is known to strictly contain the convex hull of Gaussian states $\mathfrak{C}$ even in the one-mode case,[12] the dilatations are then confirmed to be (at least in the single mode or in the homogeneous action cases) the only transformation in class (24) that can act as a probe for $\mathfrak{C}$.

ACKNOWLEDGMENTS

G.D.P. thanks F. Poloni for useful discussions. This work was supported in part by the ERC through the Advanced Grant No. 321122 SouLMan.

APPENDIX: UNBOUNDEDNESS OF DILATATIONS

Theorem 1.1. *For any $\lambda \neq \pm 1$, the phase-space dilatation by λ is not bounded in the Banach space $\mathfrak{T}$ of trace-class operators.*

Proof. Fix $\lambda \neq \pm 1$, and let Θ be the phase-space dilatation by λ. Suppose Θ to be bounded, i.e.,

$$\left\|\Theta\left(\hat{X}\right)\right\|_1 \leq \|\Theta\| \; \left\|\hat{X}\right\|_1 \qquad \forall\, \hat{X} \in \mathfrak{T}\,. \tag{A1}$$

Let also

$$p_n^{(m)} := \langle n|\Theta\left(|m\rangle\langle m|\right)|n\rangle\,. \tag{A2}$$

Equation (A1) implies

$$\sum_{n=0}^{\infty} \left|p_n^{(m)}\right| \leq \|\Theta\| \qquad \forall\, m \in \mathbb{N}\,. \tag{A3}$$

The moment generating function of $p^{(m)}$ is[12]

$$g_m(q) := \sum_{n=0}^{\infty} p_n^{(m)}\, e^{-i\,n\,q} = \frac{1-\tau}{1-\tau\, e^{-i\,q}} \left(\frac{1-\tau\, e^{i\,q}}{e^{i\,q}-\tau}\right)^m , \tag{A4}$$

where $q \in \mathbb{R}$ and

$$\tau := \frac{\lambda^2-1}{\lambda^2+1}\,. \tag{A5}$$

Define

$$a_m := \frac{1-\tau}{\sqrt[3]{m\,\tau(1+\tau)}}\,. \tag{A6}$$

Let $\phi \in C_c^{\infty}(\mathbb{R})$ be an infinitely differentiable test function with compact support. We must then have

$$\sum_{n=0}^{\infty} \phi\left(a_m\left(n-\lambda^2 m\right)\right)\; p_n^{(m)} \leq \|\phi\|_\infty\, \|\Theta\|\,. \tag{A7}$$

Expressed in terms of the Fourier transform of ϕ

$$\widetilde{\phi}(k) = \int_{-\infty}^{\infty} \phi(x)\, e^{i\,k\,x}\, dx\,, \tag{A8}$$

(A7) becomes

$$\sum_{n=0}^{\infty}\left(\int_{-\infty}^{\infty}\widetilde{\phi}(k)\; e^{i\,\lambda^2\,m\,a_m\,k}\; e^{-i\,k\,a_m\,n}\;\frac{dk}{2\pi}\right)p_n^{(m)}\leq$$

$$\leq \|\phi\|_\infty\;\|\Theta\|\;. \tag{A9}$$

Since the sum of the integrands is dominated by the integrable function

$$\frac{\|\Theta\|}{2\pi}\left|\widetilde{\phi}(k)\right|\;,$$

we can bring the sum inside the integral, getting

$$\int_{-\infty}^{\infty}\widetilde{\phi}(k)\;g_m\left(a_m k\right)\;e^{i\,\lambda^2\,m\,a_m\,k}\;\frac{dk}{2\pi}\leq \|\phi\|_\infty\;\|\Theta\|\;. \tag{A10}$$

Since for any k

$$\lim_{m\to\infty}\left(g_m\left(a_m\,k\right)\;e^{i\,\lambda^2\,m\,a_m\,k}\right)=e^{\frac{i\,k^3}{3}} \tag{A11}$$

(see subsection 1 of the Appendix), by the dominated convergence theorem

$$\lim_{m\to\infty}\int_{-\infty}^{\infty}\widetilde{\phi}(k)\;g_m\left(a_m k\right)\;e^{i\,\lambda^2\,m\,a_m\,k}\;\frac{dk}{2\pi}=$$

$$=\int_{-\infty}^{\infty}\widetilde{\phi}(k)\;e^{\frac{i\,k^3}{3}}\;\frac{dk}{2\pi}=\int_{-\infty}^{\infty}\phi(x)\;\mathrm{Ai}(x)\;dx\;, \tag{A12}$$

where $\mathrm{Ai}(x)$ is the Airy function. Now, we get

$$\int_{-\infty}^{\infty}\mathrm{Ai}(x)\;\phi(x)\;dx\leq\|\Theta\|\;\|\phi\|_\infty\qquad\forall\;\phi\in C_c^\infty(\mathbb{R})\;. \tag{A13}$$

Since the Airy function is continuous and the set of its zeroes has no accumulation points (except $-\infty$), there exists a sequence of test functions $\phi_r\in C_c^\infty(\mathbb{R})$, $r\in\mathbb{N}$ with $\|\phi_r\|_\infty=1$ approximating $\mathrm{sign}\,(\mathrm{Ai}(x))$, i.e., such that

$$\lim_{r\to\infty}\int_{-\infty}^{\infty}\mathrm{Ai}(x)\;\phi_r(x)\;dx=\int_{-\infty}^{\infty}|\mathrm{Ai}(x)|\;dx=\infty\;, \tag{A14}$$

implying

$$\|\Theta\|=\infty\;. \tag{A15}$$

□

1. Computation of the limit in (A11)

Here, we compute explicitly the limit in (A11). It is better to rephrase it in terms of

$$q:=a_m\,k\;,\qquad q\to 0 \tag{A16}$$

(remember that $a_m\sim 1/\sqrt[3]{m}$). Putting together (A11), (A4)–(A6), we have to compute

$$\lim_{q\to 0}\left(\frac{1-\tau}{1-\tau\,e^{-i\,q}}\left(\frac{1-\tau\,e^{i\,q}}{e^{i\,q}-\tau}\;e^{i\,\frac{1+\tau}{1-\tau}\,q}\right)^{\frac{k^3(1-\tau)^3}{q^3\,\tau(1+\tau)}}\right)\stackrel{?}{=}e^{\frac{i\,k^3}{3}}\;. \tag{A17}$$

The first term on the left-hand-side tends to one. The second term on the left-hand-side instead can be treated via Taylor expansion, i.e.,

$$\frac{1-\tau\,e^{i\,q}}{e^{i\,q}-\tau}\;e^{i\,\frac{1+\tau}{1-\tau}\,q}=1+\frac{i\,q^3\,\tau(1+\tau)}{3(1-\tau)^3}+O\left(q^5\right) \tag{A18}$$

for $q \to 0$. This gives

$$\lim_{q\to 0}\left(\frac{1-\tau\, e^{i\,q}}{e^{i\,q}-\tau}\; e^{i\,\frac{1+\tau}{1-\tau}\,q}\right)^{\frac{k^3(1-\tau)^3}{q^3\,\tau(1+\tau)}} =$$

$$= \lim_{q\to 0}\left(1+\frac{i\,q^3\,\tau(1+\tau)}{3(1-\tau)^3}+O\left(q^5\right)\right)^{\frac{k^3(1-\tau)^3}{q^3\,\tau(1+\tau)}} =$$

$$= e^{\frac{i\,k^3}{3}}\,, \tag{A19}$$

which proves the identity of (A17).

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