3.36pt

Spectral Multipliers on Stratified Groups

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Invariant Differential Operators & Fourier Transform

Study of translation-invariant differential operators on $\mathbb{R}^n$ $\rightsquigarrow$ Fourier transform

G Lie group

Study of left-invariant differential operators on G $\rightsquigarrow$

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Study of left-invariant differential operators on G $\rightsquigarrow$ Fourier transform ?

The Fourier transform is operator-valued, hence less manageable.

An Alternative Way

$\mathcal{L}_1, \dots, \mathcal{L}_n$: left-invariant differential operators such that $(\mathcal{L}_j, C_c^\infty)$ are essentially self-adjoint with commuting closures.
E: associated spectral measure,

$$\mathcal{L}_j = \int_{\mathbb{R}^n} \lambda_j \,\mathrm{d}E(\lambda).$$

For every bounded measurable function $m\colon \mathbb{R}^n \to \mathbb{C}$ there is a unique distribution $\mathcal{K}(m)$ such that

$$\int_{\mathbb{R}^n} m \,\mathrm{d}E \cdot \varphi = \varphi * \mathcal{K}(m) \qquad \forall \varphi \in C_c^\infty.$$

Questions

- Does $\mathcal{K}$ induce a Plancherel isomorphism?

(RL) $\mathcal{K}(m) \in L^1(G) \implies m \in C_0(\mathbb{R}^n)$?

- $\mathcal{K}\colon \mathcal{S}(\mathbb{R}^n) \to ?$

(S) If $\mathcal{K}\colon \mathcal{S}(\mathbb{R}^n) \to$ '$\mathcal{S}(G)$,' does the converse hold?

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Stratified Groups

Definition

G is a stratified group iff:

- G is nilpotent and simply connected, so that $\exp\colon \mathfrak{g} \xrightarrow{\cong} G$;
- $\mathfrak{g} = \bigoplus_{k\geqslant 1} \mathfrak{g}_k$, with $\mathfrak{g}_{k+1} = [\mathfrak{g}_1, \mathfrak{g}_k] \quad \forall k \geqslant 1$.

Natural dilations: $\bigoplus_{k\geqslant 1} \mathfrak{g}_k \ni (x_k) \mapsto (r^k x_k) \in \bigoplus_{k\geqslant 1} \mathfrak{g}_k$, $r > 0$.

A differential operator $\mathcal{L}$ is homogeneous of degree $\delta \in \mathbb{C}$ if

$$\mathcal{L}[\varphi(r\,\cdot\,)] = r^\delta (\mathcal{L}\varphi)(r\,\cdot\,)$$

for every $r > 0$ and $\varphi \in C^\infty$.

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Hypoellipticity and Schwartz Functions

Theorem

Let $\mathcal{L}_1, \dots, \mathcal{L}_n$ be formally self-adjoint, commuting, homogeneous, left-invariant differential operators without constant terms.
Then, the following assertions are equivalent:

1. $\mathcal{L}_1, \dots, \mathcal{L}_n$ *are jointly hypoelliptic;*
2. *the algebra generated by* $\mathcal{L}_1, \dots, \mathcal{L}_n$ *contains a Rockland (i.e., homogeneous and hypoelliptic) operator;*
3. $\mathrm{d}\pi(\mathcal{L}_1), \dots, \mathrm{d}\pi(\mathcal{L}_n)$ *are jointly injective on* $C^\infty(\pi)$ *for every non-trivial irreducible unitary representation* π *of* G*;*
4. $\mathcal{K} \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(G)$.

(Helffer and Nourrigat (1979), Hulanicki (1984), Veneruso (2003), Martini (2010), C.)

Plancherel Measure

Proposition

$\exists! \beta$ *positive Radon measure on* $\mathbb{R}^n$ *such that*

$$\mathcal{K}\colon L^2(\beta) \to L^2(G) \qquad \textit{isometrically.}$$

(Christ (1991), Martini (2010).)

Corollary

$\mathcal{K}\colon L^1(\beta) \to C_0(G)$.
(Martini (2010).)

Proposition

$\mathcal{K}(L^\infty(\beta)) \cap \mathcal{S}(G)$ *is closed in* $\mathcal{S}(G)$.
Analogously for $C_c^\infty(G)$, $L^1(G)$, *etc.*

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2-Step Groups

Setting: $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}, \mathfrak{g}]$.

- $\forall \omega \in \mathfrak{g}_2^*$, define $B_\omega \colon \mathfrak{g}_1 \times \mathfrak{g}_1 \ni (X, Y) \mapsto \langle \omega, [X, Y] \rangle$;
- d: minimum dimension of the radical of B_ω;
- $W := \{ \omega \colon \text{the dimension of the radical of } B_\omega \text{ is} > d \}$;

For example,

- $d = 0 \iff G$ is a Moore-Wolf group;
- $d = 0$ and $W = \{ 0 \} \iff G$ is a Métivier group.

Theorem

$\mathcal{L}$: homogeneous sub-Laplacian; $T_1, \dots, T_n$: basis of $\mathfrak{g}_2$; $n' \leqslant n$. Then $(\mathcal{L}, iT_1, \dots, iT_{n'})$ satisfies:

	$d = 0$	$d > 0$
$W = \{ 0 \}$		(RL) and (S)
$n' < n$		(RL)
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Free Groups & General Sub-Laplacians

Theorem

If G is a free 2-step nilpotent Lie group on an odd number of generators, $\mathcal{L}$ is a homogeneous sub-Laplacian on G and $T_1, \ldots, T_n$ form a basis of $\mathfrak{g}_2$, then $(\mathcal{L}, iT_1, \ldots, iT_n)$ satisfies property (RL).

Theorem

If $\mathcal{L}'$ is a (not necessarily homogeneous) sub-Laplacian on G, then it satisfies properties (RL) and (S).

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Groups of Heisenberg Type – 1

Definition

G is a group of Heisenberg type iff:

- $\mathfrak{g}$ is endowed with a scalar product such that $\mathfrak{g}_1 \perp \mathfrak{g}_2$;
- the map J_Z such that $\langle [X, Y] | Z \rangle = \langle X | J_Z(Y) \rangle$ $(X, Y \in \mathfrak{g}_1)$ satisfies $|J_Z| = |Z|\, I$ for every $Z \in \mathfrak{g}_2$.

Setting:

- assume that $\mathfrak{g}_1$ is the orthogonal sum of k J-invariant subspaces $\mathfrak{v}_1, \dots, \mathfrak{v}_k$;
- $\mathcal{L}_1, \dots, \mathcal{L}_k$ are the sum of the squares of an orthonormal basis of $\mathfrak{v}_1, \dots, \mathfrak{v}_k$, respectively;
- $T_1, \dots, T_n$ form a basis of $\mathfrak{g}_2$;
- for every linear mapping $\mu \colon \mathbb{R}^k \to \mathbb{R}^h$ which is proper on $\mathbb{R}^k_+$, define $L_\mu := (\mu(\mathcal{L}_1, \dots, \mathcal{L}_k), iT_1, \dots, iT_n)$.

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Groups of Heisenberg Type – 2

Theorem

The following assertions are equivalent:

- L_μ *satisfies property* (RL);
- L_μ *satisfies property* (S);
- *if* $m(L_\mu)$ *is a differential operator, then* m *is a polynomial;*
- $\dim_{\mathbb{Q}} \mu(\mathbb{Q}^k) = \dim_{\mathbb{R}} \mu(\mathbb{R}^k)$.

In addition, there is a linear mapping $\mu' : \mathbb{R}^k \to \mathbb{R}^p$ *such that* $L_{\mu'}$ *satisfies the preceding conditions and*

$$L_\mu = m(L_{\mu'}) \qquad \text{and} \qquad L_{\mu'} = m'(L_\mu)$$

for some measurable $m \colon \mathbb{R}^{p+n} \to \mathbb{R}^{h+n}$ *and* $m' : \mathbb{R}^{h+n} \to \mathbb{R}^{p+n}$.

Products of Heisenberg Groups

Setting:

- $G = G_1 \times \cdots \times G_n$, with $G_1, \dots, G_n$ Heisenberg groups ($n \geqslant 2$);
- $\mathcal{L}_1, \dots, \mathcal{L}_n$ are homogeneous sub-Laplacians on $G_1, \dots, G_n$, respectively;
- $\mathcal{L} := \mathcal{L}_1 + \cdots + \mathcal{L}_n$;
- $T_1, \dots, T_n$ form a basis of the centre of $\mathfrak{g}$.

Theorem

$(\mathcal{L}, iT_1, \dots, iT_{n'})$ satisfies properties (RL) and (S) iff $n' < n$.

Theorem

If G' is a stratified group and $\mathcal{L}'$ a positive Rockland operator on G' which satisfies property (S), then the family $(\mathcal{L} + \mathcal{L}', iT_1, \dots, iT_{n'})$, on $G \times G'$, satisfies properties (RL) and (S) for every $n' \leqslant n$.

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Thank you for your attention!