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Energy and Vorticity in Fast Rotating Bose-Einstein Condensates

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Abstract

We study a rapidly rotating Bose-Einstein condensate confined to a finite trap in the framework of two-dimensional Gross-Pitaevskii theory in the strong coupling (Thomas-Fermi) limit. Denoting the coupling parameter by $1/\varepsilon^2$ and the rotational velocity by Ω, we evaluate exactly the next to leading order contribution to the ground state energy in the parameter regime $|\log \varepsilon| \ll \Omega \ll 1/(\varepsilon^2 |\log \varepsilon|)$ with $\varepsilon \to 0$. While the TF energy includes only the contribution of the centrifugal forces the next order corresponds to a lattice of vortices whose density is proportional to the rotational velocity.

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1 Introduction

Bose-Einstein condensates respond to rotational motion of the enclosing container by the creation of quantized vortices. This remarkable manifestation of superfluidity has been studied, both experimentally and theoretically, in dilute, ultracold Bose gases since almost a decade and still offers a number of unsolved problems. We refer to the monograph [1], the review article [2], as well as the papers [3]-[17] for extensive lists of references. Most of the theoretical work has been carried out within the framework of the Gross-Pitaevskii (GP) equation for the wave function of the condensate. In the GP equation the interaction is encoded in a single parameter $g = 4\pi Na/L$, where a is the scattering length of the interaction potential, N the particle number and L the length scale associated with the external confining potential. For rotating gases in their ground state the GP equation was derived in [10] (upper bound) and [11] (lower bound) from the quantum mechanical many-body Hamiltonian with purely repulsive, short range interactions and fixed values of the rotational velocity and the coupling parameter as $N \to \infty$. The extension of this derivation to the case when the coupling parameter and the rotational velocity tend to infinity (or approach a critical value in the case of harmonic traps) has not yet been completed, but the leading order asymptotics of the many-body energy for large coupling and rotational velocity in anharmonic traps was computed in [12].

Detailed results on the emergence of vortices as the rotational velocity is increased have been obtained within two-dimensional GP theory when the GP interaction parameter is large ('Thomas-Fermi' limit) and the rotational velocity is of the order of the logarithm of this parameter [6, 7, 8, 9, 18]. In this case the number of vortices remains finite as the interaction parameter tends to infinity. The rotation has no effect on the energy to leading order in the coupling parameter but there is a logarithmic contribution due to the vortices in the next to leading order. By contrast, the papers [13]–[17] are mainly concerned with the situation when the rotation is so fast that the centrifugal energy and the interaction energy are comparable in magnitude. This holds when the rotational velocity increases like the square root of the interaction parameter. This case was also considered in the many-body context in [12], relying partly on estimates from [16]. The main result of [16] and [17] was the rigorous evaluation of the GP ground state energy to leading order in the interaction parameter in the regime just mentioned. This energy can be computed by minimizing a simple density functional that contains besides the interaction term another term of the same order corresponding to the centrifugal potential. This functional was first introduced in [13] where many of the basic insights about the physics of rapidly rotating condensates in anharmonic traps can be found, see also [14] and [15]. We note that, since the rotational velocity is unbounded, the confining potential must increase more rapidly than quadratically with the distance form the rotational axis in order that the centrifugal forces do not tear the condensate apart.

In the present paper we evaluate exactly the next term in the asymptotic expansion beyond the leading contribution in the parameter regime

$$|\log\varepsilon| \ll \Omega \ll 1/(\varepsilon^2|\log\varepsilon|) \tag{1.1}$$

where the coupling parameter g has been written as $1/\varepsilon^2$ with $\varepsilon \to 0$, and Ω is the rotational velocity. We remark that the dimensionless parameter ε can be interpreted as the ratio between the 'healing length' $(4\pi N/L^3)^{-1/2}$ and the extension L of the confining trap. The subleading term in the energy corresponds to the energy of a lattice of vortices of degree one such that the total vorticity is proportional to the rotational velocity. In order to bring out the salient points as simply as possible we restrict ourselves to the model considered in [16], i.e., the case of a flat, circular trap ('bucket') of finite radius.

When computing the upper bound on the energy we make a variational ansatz with a wave function that is essentially the product of a shape function, taking the deformation due to the centrifugal forces into account, and a function corresponding to a lattice of vortices uniformly distributed over the trap. The evaluation of the energy can be cast in the form of an electrostatic problem with the vortices playing the role of point charges while the vector potential due to the rotation can be regarded as an electric field generated by a uniform charge distribution. The optimal arrangement of the vortices is then determined by a minimization problem for the total electrostatic energy. When the rotational velocity reaches $O(1/(\varepsilon^2|\log\varepsilon|))$ a different trial function, with the vorticity concentrated in a region where the density is small ('giant vortex'), gives a lower energy. This transition was first noted in [13] and the estimates of the present paper corroborate it since our rigorous upper bound to the energy is smaller than the energy of the giant vortex if $\Omega \ll 1/(\varepsilon^2|\log\varepsilon|)$.

To prove the lower bound the problem is reformulated in such a way that results from Ginzburg-Landau (GL) theory obtained in [20] and [21] can be employed. The strong inhomogeneity of the density in fast rotating condensates causes problems that make the reduction to the GL case not entirely straightforward, but once these have been overcome a lower bound that matches the upper bound to subleading order in the asymptotic parameter range (1.1) can be derived. The techniques of [20] and [21] also turn out to be useful for the investigation of the vorticity of the minimizer.

2 The Mathematical Setting

We now recall the setting of [16] that will be used in the present paper. The condensate is confined to the two-dimensional unit disc $\mathcal{B}_1$ and the rotational axis is perpendicular to the disc and passes through its center. We note that this model can also be applied to the description of a three-dimensional rotating condensate confined to a long cylinder. The plane of the disc is the xy plane and $\vec{r} = (x, y)$ is the position vector with length r, while $\vec{e}_z$ denotes the unit vector in the z direction. The complex valued order parameter (the wave function of the condensate) is denoted by $\Psi(\vec{r})$. In the non-inertial rotating frame the GP energy functional can be written as

$$\mathcal{E}^{\mathrm{GP}}[\Psi] = \int_{\mathcal{B}_1} \mathrm{d}\vec{r} \left\{ \left| \left(\nabla - i\vec{A} \right) \Psi \right|^2 - \frac{\Omega^2 r^2 |\Psi|^2}{4} + \frac{|\Psi|^4}{\varepsilon^2} \right\}, \tag{2.1}$$

where the vector potential $\vec{A}$ is given by

$$\vec{A} \equiv \frac{\Omega}{2}\, \vec{e}_z \wedge \vec{r}. \tag{2.2}$$

For convenience we also introduce the abbreviation

$$\omega \equiv \varepsilon\Omega. \tag{2.3}$$

For fixed ω the centrifugal and the interaction terms in (2.1) are both $O(1/\varepsilon^2)$. The kinetic first term, containing $A \sim \Omega$, is formally also of order $1/\varepsilon^2$ if $\Omega \sim 1/\varepsilon$, but a complex phase factor in Ψ, due to vortices, can partly compensate the effect of $\vec{A}$. Indeed, in the ground state this term is of lower order as we shall see.

The ground state properties of the condensate are obtained by minimizing the GP functional over the domain

$$\mathcal{D}^{\mathrm{GP}} = \left\{ \Psi \in H^1(\mathcal{B}_1) \mid \|\Psi\|_2 = 1 \right\}. \tag{2.4}$$

Here $H^1(\mathcal{B}_1)$ denotes the Sobolev space of complex valued functions Ψ on $\mathcal{B}_1$ such that both Ψ and $\nabla\Psi$ are square integrable. The choice (2.4) naturally leads to (magnetic) Neumann boundary conditions for the minimizer on $\partial\mathcal{B}_1$. Alternatively one could impose Dirichlet boundary conditions. For $\Omega \sim 1/\varepsilon$ this would affect the energy to order $O(1/\varepsilon)$ that is negligible compared to the vortex contribution $O(\Omega|\log\varepsilon|)$ that we are interested in. In other parameter regions the effect of the boundary conditions can be more significant, and the same remark applies to an extension of our analysis to homogeneous potentials as in [17]. For simplicity we shall, however, in this paper stick to the choice (2.4) that highlights the vortex contributions.

We denote by E^{GP} the GP ground state energy and by Ψ^{GP} any corresponding minimizer. The existence of such minimizer(s) as well as the fact that any minimizer solves the GP differential equation

$$-\left(\nabla - i\vec{A} \right)^2 \Psi^{\mathrm{GP}} - A^2 \Psi^{\mathrm{GP}} + 2\varepsilon^{-2} \left| \Psi^{\mathrm{GP}} \right|^2 \Psi^{\mathrm{GP}} = \mu^{\mathrm{GP}} \Psi^{\mathrm{GP}}, \tag{2.5}$$

with boundary condition $\nabla_r \Psi^{\mathrm{GP}} = 0$ on $\partial\mathcal{B}_1$ can be deduced by standard techniques (see, e.g., [7]). The chemical potential μ^{GP} is fixed by the L^2−normalization of Ψ^{GP}, i.e.,

$$\mu^{\mathrm{GP}} = E^{\mathrm{GP}} + \varepsilon^{-2} \|\Psi^{\mathrm{GP}}\|_4^4. \tag{2.6}$$

In [16] we studied the asymptotics of E^{GP} as $\varepsilon \to 0$ and proved that the energy is well approximated to leading order by minimizing the 'Thomas-Fermi' (TF) functional

$$\mathcal{E}^{\mathrm{TF}}[\rho] = \frac{1}{\varepsilon^2} \int_{\mathcal{B}_1} \mathrm{d}\vec{r} \left\{ \rho^2 - \frac{\omega^2 r^2 \rho}{4} \right\}. \tag{2.7}$$

Note that, unlike in [16, 17], we have included the factor $1/\varepsilon^2$ in the definition of $\mathcal{E}^{\mathrm{TF}}$. The density $\rho(\vec{r}) \geq 0$ is the probability density associated with a condensate wave function Ψ, i.e., $\rho = |\Psi|^2$. The TF ground state energy,

$$E^{\mathrm{TF}} \equiv \min_{\|\rho\|_1 = 1, \rho \geq 0} \mathcal{E}^{\mathrm{TF}}[\rho] = \mathcal{E}^{\mathrm{TF}}[\rho^{\mathrm{TF}}], \tag{2.8}$$

and the corresponding normalized density ρ^{TF} can be explicitly calculated. The formulas and some properties of relevance for this paper are collected in the Appendix. We note in particular that the centrifugal forces may create a 'hole' in ρ^{TF}, i.e., the density ρ^{TF} vanishes on a disc centered at the origin if $\omega > \omega_{\mathrm{h}} \equiv 4/\sqrt{\pi}$.

3 The Main Results

The main results proved in [16] are, in a slightly different notation, contained in the following

Theorem 3.1 (Leading order ground state energy asymptotics [16])
If $\Omega \sim 1/\varepsilon$ as $\varepsilon \to 0$, then

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + O\left(\varepsilon^{-1}|\log\varepsilon|\right), \tag{3.1}$$

whereas if $1/\varepsilon \ll \Omega$,

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + O(\varepsilon^{-2}) + O((\varepsilon\Omega)^2|\log\varepsilon|). \tag{3.2}$$

In this paper we investigate the correction beyond the leading order TF term for the parameter range $|\log\varepsilon| \ll \Omega \ll 1/(\varepsilon^2|\log\varepsilon|)$. Our main result is as follows (the notation $\Omega \lesssim 1/\varepsilon$ means that $\Omega \leq C/\varepsilon$ as $\varepsilon \to 0$, with some $C < \infty$):

Theorem 3.2 (Improved ground state energy asymptotics)
If $|\log\varepsilon| \ll \Omega \lesssim \varepsilon^{-1}$ as $\varepsilon \to 0$, then

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \frac{\Omega|\log(\varepsilon^2\Omega)|}{2}(1+o(1)), \tag{3.3}$$

whereas, if $\varepsilon^{-1} \lesssim \Omega \ll \varepsilon^{-2}|\log\varepsilon|^{-1}$,

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \frac{\Omega|\log\varepsilon|}{2}(1+o(1)). \tag{3.4}$$

In [16] we have shown that as a consequence of the energy asymptotics $|\Psi^{\mathrm{GP}}|^2$ converges as $\varepsilon \to 0$ to ρ^{TF} in L^1-norm. Inside the hole, if present, it is exponentially small, i.e., bounded by $\exp(-\mathrm{const.}/\varepsilon^\beta)$ for a $\beta > 0$. See [16], Propositions 2.4 and 2.5.

The energy bounds also allow to prove a result about the uniform distribution of the vorticity of Ψ^{GP} outside the hole, at least for $\Omega \lesssim 1/\varepsilon$:

Theorem 3.3 (Uniform distribution of vorticity)
Let Ψ^{GP} be any GP minimizer and $\varepsilon > 0$ sufficiently small. If $|\log\varepsilon| \ll \Omega \lesssim \varepsilon^{-1}$, there exists a finite family of disjoint balls $\left\{\mathcal{B}_\varepsilon^i\right\} \subset \mathrm{supp}\left(\rho^{\mathrm{TF}}\right)$ such that

1. *the radius of any ball is smaller than $\Omega^{-1/2}$,*
2. *the sum of all the radii is much smaller than $\Omega^{1/2}$,*
3. *on $\partial\mathcal{B}_\varepsilon^i$, $\left|\Psi^{\mathrm{GP}}\right| \geq C|\log(\varepsilon^2\Omega)|^{-1}$ with $C > 0$*

and, denoting by $\vec{r}_{i,\varepsilon}$ the center of each ball $\mathcal{B}_\varepsilon^i$ and by $d_{i,\varepsilon}$ the winding number of $|\Psi^{\mathrm{GP}}|^{-1}\Psi^{\mathrm{GP}}$ on $\partial\mathcal{B}_\varepsilon^i$,

$$\frac{2\pi}{\Omega}\sum d_{i,\varepsilon}\delta\left(\vec{r}-\vec{r}_{i,\varepsilon}\right) \underset{\varepsilon\to 0}{\xrightarrow{\mathrm{w}}} \chi^{\mathrm{TF}}(\vec{r})\,\mathrm{d}\vec{r}, \tag{3.5}$$

in the sense of measures, where $\chi^{\mathrm{TF}}(\vec{r})$ stands for the characteristic function of $\mathrm{supp}\left(\rho^{\mathrm{TF}}\right)$.

For $\Omega \gg 1/\varepsilon$ the vorticity distribution is still an open question. In this regime $\Psi^{\rm GP}$ is not uniformly bounded in ε and is essentially supported in an annulus of very small width $\sim \omega^{-1}$ close to the boundary. As we shall see, a trial function with a uniform distribution of vortices still gives the right energy to subleading order for $\Omega \ll 1/(\varepsilon^2|\log\varepsilon|)$, but for larger Ω a trial function without any vortices in the support of $\rho^{\rm TF}$ (a 'giant vortex') has lower energy. It can be expected that for the true minimizers the vortices are gradually expelled from the essential support of the density as Ω approaches $1/(\varepsilon^2|\log\varepsilon|)$ but there are so far no rigorous results on the details of this phenomenon. The numerical investigations of [22], however, support this picture.

4 Energy Upper Bound

For an upper bound we test the functional (2.1) with a trial function of the form

$$\Psi(\vec{r}) = c\sqrt{\rho(\vec{r})}\,\xi(\vec{r})g(\vec{r}), \tag{4.1}$$

where c is a normalization constant, ρ a suitable regularization of $\rho^{\rm TF}$, g is a phase factor, and ξ a function that vanishes at the vortices, i.e., the singularities of g. To define the functions precisely we first introduce some notation.

We denote by $\mathcal{L}$ a finite, regular lattice (triangular, rectangular or hexagonal) of points $\vec{r}_i \in \mathcal{B}_1$. Each lattice point $\vec{r}_i$ lies at the center of a lattice cell Q^i and the lattice constant ℓ is chosen so that the area of Q^i is

$$|Q^i| = \frac{2\pi}{\Omega}. \tag{4.2}$$

Thus,

$$\ell = (\text{const.})\Omega^{-1/2} \tag{4.3}$$

and the total number of lattice points in the unit disc is

$$\mathcal{N} = \frac{\Omega}{2}(1 + O(\Omega^{-1/2})). \tag{4.4}$$

For large $\omega = \varepsilon\Omega$ the support of $\rho^{\rm TF}$ has an area of the order $(\omega+1)^{-1}$ and the number of lattice points on the support of $\rho^{\rm TF}$ is of the order

$$\mathcal{N}' = (\omega+1)^{-1}\Omega. \tag{4.5}$$

In particular, for $\Omega \gg 1/\varepsilon$, $\mathcal{N}' = O(1/\varepsilon)$.

Using complex notation $\zeta = x + iy$ for the points $\vec{r} = (x, y) \in \mathbb{R}^2$ the phase factor g is defined as

$$g(\vec{r}) = \prod_{\zeta_i \in \mathcal{L}} \frac{\zeta - \zeta_i}{|\zeta - \zeta_i|}. \tag{4.6}$$

The phase factor is singular at the lattice points but these singularities are compensated by the function

$$\xi(\vec{r}) = \begin{cases} 1 & \text{if } |\zeta - \zeta_i| \geq t, \\ t^{-1}|\zeta - \zeta_i| & \text{if } |\zeta - \zeta_i| \leq t. \end{cases} \tag{4.7}$$

Here t, with

$$\min\{\varepsilon, (\varepsilon/\Omega)^{1/2}\} \leq t \ll \Omega^{-1/2}, \tag{4.8}$$

is a variational parameter that will be fixed later. Thus $\xi(\vec{r})$ vanishes at the lattice points $\vec{r}_i$ and is equal to 1 outside of the union of the discs $\mathcal{B}_t^i$ of radius t centered at those points.

The size of t can be estimated by the following heuristic argument. The kinetic energy of a vortex in a cell is of the order $e_{\rm kin} \sim \int_t^\ell (1/r)^2 r\,dr \sim \log(\ell/t)$. Creating a vortex also causes an excess interaction energy because the density depletion in the vortex core of radius t has to be compensated by an increase in density elsewhere. This leads to the additional interaction energy $e_{\rm int} \sim \rho\,(t/\varepsilon)^2$, and by minimizing $e_{\rm kin} + e_{\rm int}$ we obtain $t \sim \varepsilon\rho^{-1/2}$. For slow rotations where $\rho = O(1)$ this gives $t \sim \varepsilon$, while for rapid rotation, where $\rho = O(\omega)$, we obtain $t \sim (\varepsilon/\Omega)^{1/2}$. These heuristic considerations are confirmed by the rigorous estimates below.

The density $\rho(\vec{r})$ can for $\omega \le \omega_{\rm h}$ (see Eq. (A.1)) simply be taken to be equal to the TF density $\rho^{\rm TF}(\vec{r})$ (A.3) (note that $\rho^{\rm TF}$ depends on ω). For $\omega > \omega_{\rm h}$, however, $\rho^{\rm TF}$ vanishes in a 'hole' of radius $R_{\rm h} = 1 - {\rm const.}(\omega)^{-1}$ (see Eq. (A.6)) and $\sqrt{\rho^{\rm TF}}$ does not have finite kinetic energy. Hence it is necessary in this case to regularize $\rho^{\rm TF}$ near the boundary of the hole. At the same time one has to take care that the TF energy of the regularized density remains close to $E^{\rm TF}$. Both conditions are met if we put

$$\rho(r) = \begin{cases} 0 & \text{if } r \le R_{\rm h}, \\ \rho^{\rm TF}(R_{\rm h} + \Omega^{-1})\Omega^2(r - R_{\rm h})^2 & \text{if } R_{\rm h} \le r \le R_{\rm h} + \Omega^{-1}, \\ \rho^{\rm TF}(r) & \text{otherwise.} \end{cases} \tag{4.9}$$

Thus the regularized density is equal to $\rho^{\rm TF}$ except in an annulus of thickness Ω^{-1} around the hole, where it increases quadratically with the distance from the hole. The latter property ensures finiteness of the kinetic energy. We also note that, by (A.6) and (A.7), $\rho^{\rm TF}(R_{\rm h} + \Omega^{-1}) = O(\varepsilon^2\Omega)$, so that, for any $\vec{r} \in \mathcal{B}_1$,

$$\rho(r) = \rho^{\rm TF}(r) + O(\varepsilon^2\Omega). \tag{4.10}$$

We now collect some simple estimates that are needed in the proof of the upper bound. In the following C will stand for a positive, finite constant that may differ from line to line but is independent of Ω and ε.

First, note that $\rho^{\rm TF} \le C(\omega+1)$ while the area of the support of $\rho^{\rm TF}$ is $C(\omega+1)^{-1}$. The density of vortices is $\Omega/2\pi$ and the area of each vortex disc is πt^2. Also, $\varepsilon^2\Omega = o(1)$ by assumption. We thus have, using (4.5) and (4.10),

$$\int \rho\xi^2 = \int \rho - \int \rho(1-\xi^2) \ge \int \rho^{\rm TF} - \varepsilon^2 - C\,\Omega \cdot t^2 \ge 1 - O(t^2\Omega). \tag{4.11}$$

Hence the normalization constant satisfies

$$c \le 1 + C\,t^2\Omega. \tag{4.12}$$

Likewise, using that $|\nabla\xi| = t^{-1}$ in each vortex disc and zero outside the union of the discs, while the number of vortices in the support of ρ is $\le (\omega+1)^{-1}\Omega$,

$$\|\sqrt{\rho}\,\nabla\xi\|_2^2 \le C(\omega+1) \cdot t^{-2} \cdot (\omega+1)^{-1}\Omega \cdot t^2 = C\,\Omega. \tag{4.13}$$

Next we consider, for $\omega > \omega_{\rm h}$, i.e., $R_{\rm h} > 0$,

$$\|\xi\nabla\sqrt{\rho}\|_2^2 \le \frac{1}{4}\int \frac{|\nabla\rho|^2}{\rho} \le \frac{1}{4}\int_{r<R_{\rm h}+\Omega^{-1}} \frac{|\nabla\rho|^2}{\rho} + \frac{1}{4}\int_{r\ge R_{\rm in}+\Omega^{-1}} \frac{|\nabla\rho^{\rm TF}|^2}{\rho^{\rm TF}}. \tag{4.14}$$

By (4.9) and (4.10) first term is bounded by $C\,\Omega\cdot(\varepsilon^2\Omega)$. Using (A.3) we obtain

$$\int_{r\ge R_{\rm h}+\Omega^{-1}} \frac{|\nabla\rho^{\rm TF}|^2}{\rho^{\rm TF}} = C\,(\varepsilon\Omega)^2 \int_{R_{\rm h}+\Omega^{-1}}^{1} \frac{r^3\,dr}{r^2 - {R_{\rm h}}^2} \le C\,(\varepsilon\Omega)^2 \int_{\Omega^{-1}}^{\omega^{-1}} \frac{du}{u} \le C\,(\varepsilon^2\Omega)\cdot\Omega|\log\varepsilon|. \tag{4.15}$$

The above estimate shows that the closer Ω is to ε^{-2}, the larger is the kinetic contribution of the profile $\sqrt{\rho^{\rm TF}}$ and for $\Omega \sim (\varepsilon^2|\log\varepsilon|)^{-1}$ is becomes of the same order as the other remainders, i.e., $\sim \Omega$. For $\omega \le \omega_{\rm h}$, on the other hand, $\rho = \rho^{\rm TF}$, and $\nabla\sqrt{\rho}$ is uniformly bounded in ω, so $\|\xi\nabla\sqrt{\rho}\|_2^2 \le C$ in this case.

Because g is a phase factor while $\sqrt{\rho}$ and ξ are real-valued functions we have

$$\left|\left(\nabla - i\vec{A}\right)(\sqrt{\rho}\,\xi\, g)\right|^2 = |\nabla(\sqrt{\rho}\,\xi)|^2 + \xi^2\rho\left|\left(\nabla - i\vec{A}\right)g\right|^2 \leq \\ 2\xi^2\,|\nabla\sqrt{\rho}|^2 + 2\rho\,|\nabla\xi|^2 + \xi^2\rho\left|\left(\nabla - i\vec{A}\right)g\right|^2. \quad (4.16)$$

We now obtain, using Eqs. (4.12)–(4.15),

$$\mathcal{E}^{\rm GP}[\Psi] - \mathcal{E}^{\rm TF}[|\Psi|^2] = c^2\int_{\mathcal{B}_1}{\rm d}\vec{r}\,\left|\left(\nabla - i\vec{A}\right)(\sqrt{\rho}\,\xi\,g)\right|^2 \leq \\ c^2\int_{\mathcal{B}_1}{\rm d}\vec{r}\,\xi^2\rho\left|\left(\nabla - i\vec{A}\right)g\right|^2 + 2c^2\int_{\mathcal{B}_1}{\rm d}\vec{r}\,\xi^2\,|\nabla\sqrt{\rho}|^2 + 2c^2\int_{\mathcal{B}_1}{\rm d}\vec{r}\,\rho\,|\nabla\xi|^2 \leq \\ \left(1 + O(t^2\Omega)\right)\int_{\mathcal{B}_1}{\rm d}\vec{r}\,\xi^2\rho^{\rm TF}\left|\left(\nabla - i\vec{A}_\varepsilon\right)g\right|^2 + C\left\{\Omega + \varepsilon^2\Omega^2|\log\varepsilon|\right\}. \quad (4.17)$$

The estimate for the vortex kinetic energy $\int_{\mathcal{B}_1}{\rm d}\vec{r}\xi^2\,\rho^{\rm TF}|(\nabla - i\vec{A})\,g|^2$ is given in the following Proposition.

Proposition 4.1 (Vortex kinetic energy)
If $\varepsilon \to 0$, and $1 \ll \Omega \ll 1/\varepsilon^2$, then

$$\int_{\mathcal{B}_1}{\rm d}\vec{r}\,\xi^2\,\rho^{\rm TF}\left|\left(\nabla - i\vec{A}\right)\,g\right|^2 \leq \tfrac{1}{2}\Omega\,|\log(t^2\Omega)| + O(\Omega) + O(\Omega\,(\varepsilon^2\Omega)^{1/2}|\log(t^2\Omega)|). \quad (4.18)$$

Proof: The idea behind the proof is the electrostatic analogy that was already mentioned in the Introduction and is made precise in Eq. (4.22) below and the considerations following it. The vortices, i.e., the singularities of the phase factor g, play the role of unit charges while the vector potential corresponds, after a conformal transformation, to the electric field of a uniform charge distribution. The density of the vortices is chosen in such a way that the field from the uniform charge distribution is compensated as far as possible and this requires in particular that each unit cell Q^i has total charge zero. If the unit cells were rotationally symmetric there would be no interaction between them by Newton's theorem. Complete rotational symmetry is, of course, not possible, but the closest approximation to it among the regular lattices is a lattice with hexagonal cells, i.e., a triangular arrangement of the vortices, that gives the lowest electrostatic interaction energy. However, the difference between the three possible types of unit cells, triangular, rectangular and hexagonal does not show up in the term of order $\Omega\,|\log(t^2\Omega)|$ but only in higher order corrections to this contribution.

To formalize these ideas we note first that $|(\nabla - i\vec{A})\,g|^2 = |\nabla\phi - \vec{A}|^2$ where $\phi = \sum_i \arg(\zeta - \zeta_i)$ is the phase of g. The conjugate harmonic function

$$\tilde{\phi}(\vec{r}) = \sum_i \log|\vec{r} - \vec{r}_i| \quad (4.19)$$

satisfies

$$\nabla\phi = \nabla_r\tilde{\phi}\,\vec{e}_\vartheta - \nabla_\vartheta\tilde{\phi}\,\vec{e}_r \quad (4.20)$$

where $\nabla_r = \vec{e}_r\cdot\nabla = \partial/\partial r$ and $\nabla_\vartheta = \vec{e}_\vartheta\cdot\nabla = r^{-1}\partial/\partial\vartheta$. With $\vec{A} = A(r)\vec{e}_\vartheta$ we thus have

$$\left|\nabla\phi - \vec{A}\right|^2 = \left|\nabla_\vartheta\tilde{\phi}\right|^2 + \left|\nabla_r\tilde{\phi} - A\right|^2 = \left|\nabla_\vartheta\tilde{\phi}\,\vec{e}_\vartheta + \nabla_r\tilde{\phi}\,\vec{e}_r - A\,\vec{e}_r\right|^2 = \left|\nabla\tilde{\phi} - A\,\vec{e}_r\right|^2. \quad (4.21)$$

We now define

$$\vec{E}(\vec{r}) = \nabla\tilde{\phi}(\vec{r}) - A(r)\vec{e}_r \quad (4.22)$$

and note that $\nabla\tilde{\phi} = \sum_i (\vec{r}-\vec{r}_i)/|\vec{r}-\vec{r}_i|^2$ can be regarded as the electric field generated by point charges localized at the positions of the vortices, while $A(r)\vec{e}_r = (\Omega/2)r\vec{e}_r$ is the field generated by a uniform charge density of magnitude $\Omega/2\pi = |Q^i|^{-1}$. We can thus write $\vec{E}(\vec{r}) = \sum_i \vec{E}_i(\vec{r}) = \sum_i \nabla\Phi_i(\vec{r})$ with

$$\Phi_i(\vec{r}) = \int_{\mathcal{B}_1} \mathrm{d}\vec{r}'\, \sigma_i(\vec{r}')\log|\vec{r}-\vec{r}'| \tag{4.23}$$

and

$$\sigma_i(\vec{r}') = \delta(\vec{r}'-\vec{r}_i) - |Q^i|^{-1}\chi_i(\vec{r}') \tag{4.24}$$

where χ_i is the characteristic function of the cell Q^i.
By a transformation of variables, writing $\vec{r} = \Omega^{-1/2}\vec{x}$, we map the cells Q^i of side length $\ell \sim \Omega^{-1/2}$ onto cells Q_1^i of side length $O(1)$. The characteristic function of Q_1^i is denoted by $\chi_{i,1}(\vec{x}')$ and we use the index 1 also for the charge densities, electric fields and potentials generated by the cells Q_1^i. We can then write

$$\sigma_i(\vec{r}) = \Omega\left[\delta(\vec{x}'-\vec{x}_i) - |Q_1^i|^{-1}\chi_{i,1}(\vec{x}')\right] = \Omega\,\sigma_{i,1}(\vec{x}') \tag{4.25}$$

and

$$E_i(\vec{r}) = \Omega^{1/2}E_{i,1}(\vec{x}) \tag{4.26}$$

where

$$E_{i,1}(\vec{x}) = \nabla\int_{\mathcal{B}_1}\mathrm{d}\vec{x}'\,\sigma_{i,1}(\vec{x}')\log|\vec{x}-\vec{x}'| = \nabla\Phi_{i,1}(\vec{x}). \tag{4.27}$$

Consider now the cell Q_1^0 centered at the origin. The multipole expansion of $\Phi_{0,1}(\vec{x})$ for $\vec{x}\notin Q_1^0$ is

$$\Phi_{0,1}(\vec{x}) = q\log|\vec{x}| - \sum_{k=1}^{\infty}\frac{C_k\cos(k\vartheta)+S_k\sin(k\vartheta)}{|\vec{x}|^k} \tag{4.28}$$

with

$$q = \int_{Q_1^0}\mathrm{d}\vec{x}'\,\sigma_{0,1}(\vec{x}'),$$

$$C_k = k^{-1}\int_{Q_1^0}\mathrm{d}\vec{x}'\,\sigma_{0,1}(\vec{x}')|\vec{x}'|^k\cos(k\vartheta'), \qquad S_k = k^{-1}\int_{Q_1^0}\mathrm{d}\vec{x}'\,\sigma_{0,1}(\vec{x}')|\vec{x}'|^k\sin(k\vartheta'). \tag{4.29}$$

By neutrality of the charge distribution (4.24) it is clear that $q=0$ and by symmetry of the unit cell it is also clear that there is no dipole moment, i.e., $C_1 = S_1 = 0$. We conclude that $\Phi_{0,1}(\vec{x})$ decays at least as $|\vec{x}|^{-2}$ and the corresponding field $\vec{E}_{0,1}(\vec{x})$ decays at least as $|\vec{x}|^{-3}$. For square or hexagonal cells it decreases even faster.

All cells Q^i are obtained by translations and scaling from the cell Q_1^0. From the considerations above (note, in particular, Eq. (4.26)) we can thus conclude that if two of the original cells, Q^i and Q^j have distance $O(\Omega^{-1/2}n)$ from each other, then the strength of the field $\vec{E}_j(\vec{r})$ for $\vec{r}\in Q^i$ is at most $O(\Omega^{1/2}n^{-3})$. Since, for a fixed cell Q^i, there are at most $O(n)$ cells at distance $O(\Omega^{-1/2}n)$ from it, we can estimate for $\vec{r}\in Q^i$

$$\left|\vec{E}(\vec{r})-\vec{E}_i(\vec{r})\right| \le \sum_{j\ne i}|\vec{E}_j(\vec{r})| \le \mathrm{const.}\,\Omega^{1/2}\sum_n n\cdot n^{-3} = O(\Omega^{1/2}). \tag{4.30}$$

Writing

$$|\vec{E}|^2 = |\vec{E}_i|^2 + 2(\vec{E}-\vec{E}_i)\cdot\vec{E}_i + |\vec{E}-\vec{E}_i|^2 \tag{4.31}$$

and using the simple bound $\vec{E}_i(\vec{r}) \le |\vec{r}-\vec{r}_i|^{-1}$, we conclude that for $\vec{r}\in Q^i$

$$|\vec{E}(\vec{r})|^2 \le |\vec{E}_i(\vec{r})|^2 + \mathrm{const.}(\Omega^{1/2}|\vec{r}-\vec{r}_i|^{-1}+\Omega) \tag{4.32}$$

and hence

$$\int_{Q^i\setminus\mathcal{B}_t^i} \mathrm{d}\vec{r}\,|\vec{E}(\vec{r})|^2 - \int_{Q^i\setminus\mathcal{B}_t^i} \mathrm{d}\vec{r}\,|\vec{E}_i(\vec{r})|^2 \leq \text{const.} \int_t^{C\,\Omega^{-1/2}} \mathrm{d}r\, r\, \left(\Omega^{1/2}r^{-1} + \Omega\right) = O(1) \tag{4.33}$$

while

$$\int_{\mathcal{B}_t^i} \mathrm{d}\vec{r}\,\xi(\vec{r})^2|\vec{E}(\vec{r})|^2 - \int_{\mathcal{B}_t^i} \mathrm{d}\vec{r}\,\xi(\vec{r})^2|\vec{E}_i(\vec{r})|^2 \leq \text{const.} \int_0^t \mathrm{d}r\, r\,(r/t)^2(\Omega^{1/2}r^{-1} + \Omega) = O((t^2\Omega)^{1/2}). \tag{4.34}$$

On the other hand, since $\vec{E}_i(\vec{r}) \leq |\vec{r} - \vec{r}_i|^{-1}$,

$$\int_{Q^i\setminus\mathcal{B}_t^i} \mathrm{d}\vec{r}\,|\vec{E}_i(\vec{r})|^2 \leq 2\pi \int_t^{C\,\Omega^{1/2}} \mathrm{d}r\, r\, r^{-2} = \pi|\log(t^2\Omega)| + O(1) \tag{4.35}$$

and

$$\int_{\mathcal{B}_t^i} \mathrm{d}\vec{r}\,\xi(\vec{r})^2|\vec{E}_i(\vec{r})|^2 \leq 2\pi \int_0^t \mathrm{d}r\, r\,(r(\Omega/\varepsilon)^{1/2})^2 r^{-2} = O(1). \tag{4.36}$$

Putting all the estimates above together we obtain

$$\int_{\mathcal{B}_1} \mathrm{d}\vec{r}\,\rho^{\mathrm{TF}}(\vec{r})\,\xi(\vec{r})^2|\vec{E}(\vec{r})|^2 \leq \left(1 + O((t^2\Omega)^{1/2})\right) \sum_i \sup_{\vec{r}\in Q^i} \rho^{\mathrm{TF}}(\vec{r}) \left(\pi|\log(t^2\Omega)| + O(1)\right). \tag{4.37}$$

It remains to estimate the Riemann approximation error

$$\mathcal{R} \equiv |Q^0| \sum_i \sup_{\vec{r}\in Q^i} \rho^{\mathrm{TF}}(\vec{r}) - \int_{\mathcal{B}_1} \mathrm{d}\vec{r}\,\rho^{\mathrm{TF}}(\vec{r}) \leq |Q^0| \sum_i \left\{ \sup_{\vec{r}\in Q^i} \rho^{\mathrm{TF}}(\vec{r}) - \inf_{\vec{r}\in Q^i} \rho^{\mathrm{TF}}(\vec{r}) \right\}. \tag{4.38}$$

We use here that $\|d\rho^{\mathrm{TF}}/dr\|_\infty \leq C(\varepsilon\Omega)^2$ and that the number of cells Q^i that intersect the support of ρ^{TF} is bounded by $C\,\varepsilon^{-1}(1 + \Omega^{-1/2})$. Hence

$$\mathcal{R} \leq C\Omega^{-1} \cdot \Omega^{-1/2}(\varepsilon\Omega)^2 \cdot \varepsilon^{-1}(1 + \Omega^{-1/2}) = C\,(\varepsilon^2\Omega)^{1/2}(1 + \Omega^{-1/2}). \tag{4.39}$$

It now follows that the right hand side of (4.37) is bounded by

$$(1 + O((t^2\Omega)^{1/2}))\,(1 + \mathcal{R})\,|Q^0|^{-1} \left(\pi|\log(t^2\Omega)| + O(1)\right) = \tfrac{1}{2}\Omega\,|\log(t^2\Omega)| + O(\Omega) + O(\Omega\,(\varepsilon^2\Omega)^{1/2}|\log(t^2\Omega)|). \tag{4.40}$$

□

To complete the proof of the upper bound we still need to estimate the difference between $\mathcal{E}^{\mathrm{TF}}[|\Psi|^2]$ and $E^{\mathrm{TF}} = \mathcal{E}^{\mathrm{TF}}[\rho^{\mathrm{TF}}]$ and choose the radius t of the vortex discs.

The TF functional is

$$\mathcal{E}^{\mathrm{TF}}[|\Psi|^2] = \varepsilon^{-2} \int_{\mathcal{B}_1} \mathrm{d}\vec{r} \left\{ |\Psi|^4 - \frac{(\varepsilon\Omega)^2 r^2|\Psi|^2}{4} \right\}. \tag{4.41}$$

We consider the two terms separately. For the nonlinear interaction term we use that $c = 1 + O(t^2\Omega)$ and $\xi^2\rho \leq \rho^{\mathrm{TF}}$ to obtain

$$\varepsilon^{-2} \int_{\mathcal{B}_1} \mathrm{d}\vec{r}|\Psi|^4 = \varepsilon^{-2} \int_{\mathcal{B}_1} \mathrm{d}\vec{r}(c^2\xi^2\rho)^2 \leq \frac{1 + C\,t^2\Omega}{\varepsilon^2} \int_{\mathcal{B}_1} \mathrm{d}\vec{r} \left(\rho^{\mathrm{TF}}\right)^2 = \varepsilon^{-2} \int_{\mathcal{B}_1} \mathrm{d}\vec{r} \left(\rho^{\mathrm{TF}}\right)^2 + \text{remainder}. \tag{4.42}$$

Since $\rho^{\rm TF} \le C\,(\varepsilon\Omega+1)$ and $\int \rho^{\rm TF} = 1$, the remainder is

$$\frac{t^2\Omega}{\varepsilon^2}\int_{\mathcal{B}_1} {\rm d}\vec{r}\ \left(\rho^{\rm TF}\right)^2 \le C\frac{t^2\Omega}{\varepsilon^2}(\varepsilon\Omega+1) \tag{4.43}$$

and this has to be small compared to $\Omega|\log(t^2\Omega)|$. If $\Omega \lesssim 1/\varepsilon$ this is clearly satisfied for $t=\varepsilon$. On the other hand, if $\varepsilon\Omega \gg 1$ we can take $t = (\varepsilon/\Omega)^{1/2}$ Note that for $\Omega \sim 1/\varepsilon$ both choices coincide.

The centrifugal contribution can by partial integration and using the normalization of Ψ be written

$$-\frac{\Omega^2}{4}\int_{\mathcal{B}_1} {\rm d}\vec{r}\, r^2|\Psi|^2 = -\frac{\pi\Omega^2}{2} + \pi\Omega^2\int_0^1 {\rm d}r\ r\ \Phi(r), \tag{4.44}$$

with

$$\Phi(r) = \int_0^r {\rm d}r'\ r'\ |\Psi(r')|^2. \tag{4.45}$$

Likewise,

$$-\frac{\Omega^2}{4}\int_{\mathcal{B}_1} {\rm d}\vec{r}\, r^2\rho^{\rm TF} = -\frac{\pi\Omega^2}{2} + \pi\Omega^2\int_0^1 {\rm d}r\ r\ \Phi^{\rm TF}(r), \tag{4.46}$$

with

$$\Phi^{\rm TF}(r) = \int_0^r {\rm d}r'\ r'\ \rho^{\rm TF}(r').$$

From (4.9) and (4.12) we obtain

$$\Phi(r) \le \Phi^{\rm TF}(r) + C\,t^2\Omega. \tag{4.47}$$

Moreover, the support of Φ as well as $\Phi^{\rm TF}$ has area $\le (\varepsilon\Omega+1)^{-1}$. Hence

$$\Omega^2\int_0^1 {\rm d}r\ \left\{\Phi^{\rm TF}(r) - \Phi(r)\right\} \le C\,\Omega^2\cdot t^2\Omega(\varepsilon\Omega+1)^{-1}. \tag{4.48}$$

If $\varepsilon\Omega$ is bounded and $t=\varepsilon$, this is bounded by $C\,\Omega\cdot(\varepsilon\Omega)^2 \le C\,\Omega$. If $\varepsilon\Omega \gg 1$, we take $t^2 = \varepsilon/\Omega$ and obtain again $C\,\Omega$ as bound.
We summarize the findings in the following

Proposition 4.2 (Energy upper bound)
For $\varepsilon \to 0$ and $1 \ll \Omega \lesssim 1/\varepsilon$ we have

$$E^{\rm GP} \le E^{\rm TF} + \tfrac{1}{2}\Omega|\log(\varepsilon^2\Omega)| + O(\Omega), \tag{4.49}$$

and for $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$

$$E^{\rm GP} \le E^{\rm TF} + \tfrac{1}{2}\Omega|\log\varepsilon| + O(\Omega) + O(\Omega\,(\varepsilon^2\Omega)^{1/2}|\log\varepsilon|). \tag{4.50}$$

As we will see in the next section, the upper bounds are matched by corresponding lower bounds only in the parameter range $|\log\varepsilon| \ll \Omega \ll 1/(\varepsilon^2|\log\varepsilon|)$. In fact, for $\Omega \lesssim |\log\varepsilon|$ there are only finitely many vortices [6, 8, 9, 18] and the upper bound (4.49) is too large. For $1/(\varepsilon^2|\log\varepsilon|) \lesssim \Omega \ll 1/\varepsilon^2$, on the other hand, a trial function different from (4.1) gives lower energy than (4.50). This is the trial function considered in [16] Eq. (3.36) for $\Omega \gg \varepsilon^{-1}$ that corresponds to a *'giant vortex'* where all the vorticity is concentrated at the center and the support of $\rho^{\rm TF}$ is vortex free. In fact, for such a trial function the next correction to the TF energy is $O(1/\varepsilon^2)$, cf. (3.2), and this is smaller than $\Omega|\log\varepsilon|$ in the parameter range. This transition at $\Omega \sim 1/(\varepsilon^2|\log\varepsilon|)$ can also be understood by the following heuristic argument, employing the electrostatic analogy: For $\Omega \gg 1/\varepsilon$ the number of cells in the support of $\rho^{\rm TF}$ is

$\sim 1/\varepsilon$. Without vortices each cell has unit 'charge', originating from the vector potential, and the mutual interaction energy of the cells is of the order $1/\varepsilon^2$. Putting a vortex in each cell neutralizes the charge so that the interaction energy becomes negligible, but instead there is an energy cost of order $\Omega|\log\varepsilon|$ due to the vortices. Equating these two energies leads to $\Omega \sim 1/(\varepsilon^2|\log\varepsilon|)$ as the limiting rotational velocity above which the ansatz (4.1) is definitely not optimal.

It should be noted that also for $1/\varepsilon \lesssim \Omega \ll 1/(\varepsilon^2|\log\varepsilon|)$ one could for the upper bound replace the distribution of the vorticity on a lattice within the 'hole' by a single phase factor corresponding to a giant vortex at the origin, but in order to obtain the correction beyond the TF term the support of ρ^{TF} can not be vortex free. The detailed vortex distribution of the true minimizer of the GP energy functional is, however, an open question.

5 Energy Lower Bound

The lower bound to the GP ground state energy E^{GP} will be proved by a step-by-step reduction to the lower bound of the energy of a Ginzburg-Landau (GL) energy function for which results of [19, 20, 21] can be employed. As a preparation we first prove a bound on the GP minimizers in terms of the TF density:

Lemma 5.1 (Upper Bound for $\left|\Psi^{\mathrm{GP}}\right|$)
For $\varepsilon \to 0$ and $|\log\varepsilon| \ll \Omega \ll (\varepsilon^2|\log\varepsilon|)^{-1}$,

$$\left\|\Psi^{\mathrm{GP}}\right\|_\infty^2 \leq \rho^{\mathrm{TF}}(1)(1+o(1)). \tag{5.1}$$

Proof: Setting $U \equiv \left|\Psi^{\mathrm{GP}}\right|^2$, we first note that the upper bound (4.50) and the trivial lower bound $E^{\mathrm{GP}} \geq E^{\mathrm{TF}}$ imply the convergence of U to ρ^{TF} in L^2-norm. Indeed, using the simple bound $2\rho^{\mathrm{TF}} \geq \varepsilon^2\mu^{\mathrm{TF}} + \omega^2 r^2/4$, the L^1 normalization of U, and the identity $\mu^{\mathrm{TF}} = E^{\mathrm{TF}} + \varepsilon^{-2}\|\rho^{\mathrm{TF}}\|_2^2$, we have

$$\int_{\mathcal{B}_1} \mathrm{d}\vec{r}\, \left(U - \rho^{\mathrm{TF}}\right)^2 \leq \int_{\mathcal{B}_1} \mathrm{d}\vec{r}\, \left[U^2 - \mu^{\mathrm{TF}} U - \frac{\omega^2 r^2 U}{4} + {\rho^{\mathrm{TF}}}^2\right] = \varepsilon^2\left(\mathcal{E}^{\mathrm{TF}}[U] - E^{\mathrm{TF}}\right) \tag{5.2}$$

$$\leq \varepsilon^2\left(E^{\mathrm{GP}} - E^{\mathrm{TF}}\right) \leq C\,\Omega\varepsilon^2|\log\varepsilon| = o(1), \tag{5.3}$$

by (4.49) and (4.50) and the conditions on Ω. As a consequence

$$\|U\|_2^2 - \left\|\rho^{\mathrm{TF}}\right\|_2^2 = 2\int_{\mathcal{B}_1} \mathrm{d}\vec{r}\, \rho^{\mathrm{TF}}\left(U - \rho^{\mathrm{TF}}\right) + \int \mathrm{d}\vec{r}\, \left(U - \rho^{\mathrm{TF}}\right)^2 \leq \rho^{\mathrm{TF}}(1)^{1/2}\, o(1) \tag{5.4}$$

where we have used the Schwarz inequality and the trivial bound $\|\rho^{\mathrm{TF}}\|_2^2 \leq \|\rho^{\mathrm{TF}}\|_\infty = \rho^{\mathrm{TF}}(1)$, which follows from the L^1 normalization of ρ^{TF}. Now $\rho^{\mathrm{TF}}(1) \geq C(\omega+1)$ and theqrefore

$$\|U\|_2^2 - \left\|\rho^{\mathrm{TF}}\right\|_2^2 \leq o(1)\,\rho^{\mathrm{TF}}(1). \tag{5.5}$$

Since $\varepsilon^2(\mu^{\mathrm{GP}} - \mu^{\mathrm{TF}}) = \varepsilon^2(E^{\mathrm{GP}} - E^{\mathrm{TF}}) + \|U\|_2^2 - \left\|\rho^{\mathrm{TF}}\right\|_2^2$ we thus have

$$\varepsilon^2(\mu^{\mathrm{GP}} - \mu^{\mathrm{TF}}) \leq o(1)\,\rho^{\mathrm{TF}}(1). \tag{5.6}$$

Now acting as in the proof of Proposition 2.4 in [16], we obtain from the variational equation (2.5)

$$-\frac{1}{2}\Delta U \leq \left[\varepsilon^2\mu^{\mathrm{GP}} + \frac{\omega^2}{4} - 2U\right]\frac{U}{\varepsilon^2} \leq \left[\varepsilon^2\left(\mu^{\mathrm{GP}} - \mu^{\mathrm{TF}}\right) + 2\left(\rho^{\mathrm{TF}}(1) - U\right)\right]\frac{U}{\varepsilon^2} \leq$$

$$2\left[(1+o(1))\rho^{\mathrm{TF}}(1) - U\right]\frac{U}{\varepsilon^2}, \tag{5.7}$$

by (A.5) and the above estimate for $\varepsilon^2(\mu^{\mathrm{GP}} - \mu^{\mathrm{TF}})$. At the maximum of U the left hand side of (5.7) is nonnegative and thus (5.1) holds.

□

We now proceed with the proof of the lower bound. The first step is the extraction of the TF profile ρ^{TF} from the GP minimizer Ψ^{GP}, i.e., the ansatz $\Psi^{\mathrm{GP}} = \sqrt{\rho^{\mathrm{TF}}} u$, which, one the one hand, allows to get rid of the leading order term in the energy asymptotics and, on the other hand, implies that u minimizes a weighted GL functional. Unfortunately such a factorization is well defined only if the TF profile ρ^{TF} does not vanish inside $\mathcal{B}_1$, i.e., for $\omega < \omega_{\mathrm{h}}$. In order to get rid of this problem we first restrict the integration domain in the GP functional and set

$$\mathcal{T} \equiv \left\{ \vec{r} \in \mathcal{B}_1 \;\middle|\; \rho^{\mathrm{TF}}(r) \geq \omega |\log \delta|^{-1} \right\}, \tag{5.8}$$

with

$$\delta \equiv \varepsilon^2 \Omega |\log \varepsilon| \ll 1, \tag{5.9}$$

by (1.1). Note that

$$|\log \delta| \leq C |\log \varepsilon|, \tag{5.10}$$

since $\delta \gg \varepsilon^2 |\log \varepsilon|^2$, by (1.1), so that $0 \geq \log \delta \geq \log(\varepsilon^2 |\log \varepsilon|^2) \geq C \log \varepsilon$. Note that $R_{\mathrm{h}}^2 + \omega^{-1} |\log \delta|^{-1} < 1$, since $R_{\mathrm{h}}^2 = 1 - C\omega^{-1}$ and $|\log \delta| \gg 1$, so, by (A.7), the set $\mathcal{T}$ is not empty. Moreover, if $\omega/\omega_{\mathrm{h}}$ is sufficiently small then the set $\mathcal{T}$ coincides with the whole trap $\mathcal{B}_1$ since, in that case, $\rho^{\mathrm{TF}}(r) \geq C > 0$ for any $\vec{r} \in \mathcal{B}_1$.
For Ω and ε satisfying (1.1) we now define for $\vec{r} \in \mathcal{T}$

$$u(\vec{r}) \equiv \Psi^{\mathrm{GP}}(\vec{r}) \, \rho^{\mathrm{TF}}(r)^{-1/2}. \tag{5.11}$$

This is a smooth function with $|u|^2 \leq C |\log \delta|$ because of Lemma 5.1. Adding the kinetic energy term to both sides of (5.2) we obtain, exploiting the nonnegativity of the integrand,

$$E^{\mathrm{GP}} \geq E^{\mathrm{TF}} + \int_{\mathrm{supp}(\rho^{\mathrm{TF}})} \mathrm{d}\vec{r} \left\{ \left| \left(\nabla - i\vec{A} \right) \Psi^{\mathrm{GP}} \right|^2 + \varepsilon^{-2} \left(\rho^{\mathrm{TF}} - \left| \Psi^{\mathrm{GP}} \right|^2 \right)^2 \right\}. \tag{5.12}$$

Introducing the weighted GL-type functional

$$\tilde{\mathcal{E}}^{\mathrm{GP}}[u] \equiv \int_{\mathcal{T}} \mathrm{d}\vec{r} \, \rho^{\mathrm{TF}}(r) \left\{ \left| \left(\nabla - i\vec{A} \right) u \right|^2 + \varepsilon^{-2} \rho^{\mathrm{TF}}(r) \left(1 - |u|^2 \right)^2 \right\}, \tag{5.13}$$

we thus obtain, since $\mathcal{T} \subset \mathrm{supp}\left(\rho^{\mathrm{TF}}\right)$,

$$\mathcal{E}^{\mathrm{GP}}\left[\Psi^{\mathrm{GP}}\right] - E^{\mathrm{TF}} - \tilde{\mathcal{E}}^{\mathrm{GP}}[u] \geq \frac{1}{2} \int_{\mathcal{T}} \mathrm{d}\vec{r} \, \nabla \rho^{\mathrm{TF}} \cdot \nabla |u|^2 \geq C\omega^2 \int_{\mathcal{T}} \mathrm{d}\vec{r} \, \vec{r} \cdot \nabla |u|^2 \geq \\ - C\omega^2 \int_{\mathcal{T}} \mathrm{d}\vec{r} \, |u|^2 \geq -C\omega |\log \delta|, \tag{5.14}$$

which yields, by (5.10),

$$E^{\mathrm{GP}} \geq E^{\mathrm{TF}} + \tilde{\mathcal{E}}^{\mathrm{GP}}[u] - C\omega |\log \varepsilon|. \tag{5.15}$$

According to (5.15) the correction to the leading term E^{TF} can thus be estimated from below by a weighted GL energy $\tilde{\mathcal{E}}^{\mathrm{GP}}[u]$, where the Lebesgue measure is replaced by $\rho^{\mathrm{TF}}(\vec{r}) \, \mathrm{d}\vec{r}$. Compared with the usual GL setting there are two differences, however: The internal magnetic field $\vec{A}$ is in our case fixed from the outset and the coupling parameter is $\rho^{\mathrm{TF}}(\vec{r}) \, \varepsilon^{-2}$, i.e., it depends on the TF density at each position.

To deal with the latter point we decompose the integration domain into small cells: Let $\hat{\mathcal{L}}$ be the square regular lattice

$$\hat{\mathcal{L}} \equiv \left\{ \vec{r}_i = (m\hat{\ell}, n\hat{\ell}), m, n \in \mathbb{Z} \;\middle|\; \mathcal{Q}^i \subset \mathcal{T} \right\}, \tag{5.16}$$

where $\mathcal{Q}^i$ denotes the lattice cell centered at $\vec{r}_i \in \hat{\mathcal{L}}$ and

$$\sqrt{\frac{|\log \varepsilon|}{\Omega}} \ll \hat{\ell} \ll \min\left[1, \frac{1}{\omega |\log \delta|}\right]. \tag{5.17}$$

Note that the above conditions are compatible: Multiplying both side by ω and assuming that $\omega \geq |\log \delta|^{-1}$, (5.17) becomes $\sqrt{\delta} \ll \omega \hat{\ell} \ll |\log \delta|^{-1}$, which can always be fulfilled since $\delta \ll 1$ by definition. Note also that the lattice spacing is much larger than the one chosen in the upper bound proof, where the lattice constant was $\ell \sim \Omega^{-1/2}$. By the lower bound on $\hat{\ell}$ each lattice cell can be expected to contain a large number of vortices which turns out to be helpful for estimating the energy. The upper bound on $\hat{\ell}$ guarantees that $\hat{\ell}$ is much smaller than the width of $\mathcal{T}$, which is of order $(\omega+1)^{-1}$. This is useful for the extraction of the TF profile.

By (A.3), $\rho^{\mathrm{TF}}(r) \geq \rho^{\mathrm{TF}}(r_i)(1 - O(\hat{\ell}\omega|\log\delta|))$, for any $\vec{r} \in \mathcal{Q}^i$, so that the above inequalities imply

$$\tilde{\mathcal{E}}^{\mathrm{GP}}[u] \geq \sum_{\vec{r}_i \in \mathcal{L}} \int_{\mathcal{Q}^i} \mathrm{d}\vec{r}\, \rho^{\mathrm{TF}}(r) \left\{ \left| \left(\nabla - i\vec{A} \right) u \right|^2 + \varepsilon^{-2} \rho^{\mathrm{TF}}(r) \left(1 - |u|^2 \right)^2 \right\} \geq \\ (1 - o(1)) \sum_{\vec{r}_i \in \mathcal{L}} \rho^{\mathrm{TF}}(r_i)\, \mathcal{E}^{(i)}[u], \tag{5.18}$$

with

$$\mathcal{E}^{(i)}[u] \equiv \int_{\mathcal{Q}^i} \mathrm{d}\vec{r} \left\{ \left| \left(\nabla - i\vec{A} \right) u \right|^2 + \varepsilon^{-2} \rho^{\mathrm{TF}}(r_i) \left(1 - |u|^2 \right)^2 \right\}. \tag{5.19}$$

Now the analogy with the GL functional is made explicit, since, except for the coupling parameter which still contains ρ^{TF}, the functional $\mathcal{E}^{(i)}$ is precisely the GL energy functional

$$\mathcal{E}^{\mathrm{GL}}[u, \vec{A}'] = \int_{\mathcal{Q}^i} \mathrm{d}\vec{r} \left\{ \left| \left(\nabla - i\vec{A}' \right) u \right|^2 + \left| \nabla \wedge \vec{A}' - \vec{h}_{\mathrm{ex}} \right|^2 + \varepsilon^{-2} \rho^{\mathrm{TF}}(r_i) \left(1 - |u|^2 \right)^2 \right\} \tag{5.20}$$

evaluated at $(u, \vec{A})$ with an external magnetic field $\vec{h}_{\mathrm{ex}} = \Omega \vec{e}_z$. It is clear that the GL energy

$$E^{\mathrm{GL}} \equiv \inf_{u, \vec{A}'} \mathcal{E}^{\mathrm{GL}}[u, \vec{A}'] \tag{5.21}$$

is a lower bound to the ground state energy of $\mathcal{E}^{(i)}$ because the configuration with the uniform internal magnetic field corresponding to $\vec{A}' = \vec{A} = \Omega \vec{e}_z \wedge \vec{r}/2$ is only one among all possible configurations considered in the minimization of the GL functional.

We can now state the main estimate needed for the proof of the lower bound.

Proposition 5.1 (Lower bound inside cells)
For any Ω satisfying (1.1) *and ε sufficiently small, it is possible to find $\hat{\ell}$ in such a way that* (5.17) *is fulfilled and*

$$\mathcal{E}^{(i)}[u] \geq \frac{\Omega \hat{\ell}^2 |\log \gamma|}{2} (1 - o(1)), \tag{5.22}$$

where $\gamma \equiv \min[\varepsilon, \varepsilon^2 \Omega]$.

Proof: The key point in the proof of (5.22) is a rescaling of $\mathcal{Q}^i$ (together with the choice (5.17) of the lattice spacing), which allows to reduce the problem to the minimization of a GL functional in a different regime. We thus set $\vec{x} \equiv \hat{\ell}^{-1}(\vec{r} - \vec{r}_i)$,

$$\tilde{u}(\vec{x}) \equiv u\left(\vec{r}_i + \hat{\ell}\vec{x}\right), \qquad \vec{B}(\vec{x}) \equiv \hat{\ell}\vec{A}\left(\vec{r}_i + \hat{\ell}\vec{x}\right), \tag{5.23}$$

where $\vec{r}_i$ stands for the center of $\mathcal{Q}^i$. By such a change of coordinates in (5.19), we obtain

$$\mathcal{E}^{(i)}[u] = \tilde{\mathcal{E}}^{(i)}\left[\tilde{u}\right] = \int_{\mathcal{Q}_1} \mathrm{d}\vec{x} \left\{ \left| \left(\nabla - i\vec{B}\right)\tilde{u} \right|^2 + \varepsilon^{-2}\hat{\ell}^2 \rho^{\mathrm{TF}}(r_i)\left(1 - |\tilde{u}|^2\right)^2 \right\}, \tag{5.24}$$

where $\mathcal{Q}_1$ is a unitary square centered at the origin. Note that the rescaled vector potential $\vec{B}$ is explicitly given by

$$\vec{B}(\vec{x}) = \frac{\Omega\hat{\ell}\vec{e}_z \wedge \vec{r}_i}{2} + \frac{\Omega\hat{\ell}^2\vec{e}_z \wedge \vec{x}}{2}, \tag{5.25}$$

and the corresponding magnetic field is

$$\tilde{h} \equiv \mathrm{curl}\vec{B} = \Omega\hat{\ell}^2. \tag{5.26}$$

In the following we investigate the minimization of the functional $\tilde{\mathcal{E}}^{(i)}$: We first notice that, by gauge invariance, one can get rid of the constant term $\Omega\hat{\ell}\vec{e}_z \wedge \vec{r}_i/2$ in (5.25):

$$\inf_{\tilde{u}\in H^1(\mathcal{Q}_1)} \tilde{\mathcal{E}}^{(i)}[\tilde{u}] \geq \inf_{\tilde{u}\in H^1(\mathcal{Q}_1)} \int_{\mathcal{Q}_1} \mathrm{d}\vec{x} \left\{ \left| \left(\nabla - i\hat{\ell}^2\vec{A}(\vec{x})\right)\tilde{u} \right|^2 + \varepsilon^{-2}\hat{\ell}^2\rho^{\mathrm{TF}}(r_i)\left(1 - |\tilde{u}|^2\right)^2 \right\}. \tag{5.27}$$

We now introduce a new infinitesimal parameter ϵ defined as

$$\epsilon \equiv \frac{\varepsilon}{\hat{\ell}\sqrt{\rho^{\mathrm{TF}}(r_i)}} \leq C\varepsilon\sqrt{\frac{\Omega|\log\delta|}{\omega|\log\varepsilon|}} \leq C\sqrt{\varepsilon} \ll 1, \tag{5.28}$$

by (5.9), (5.8) and (5.17). It follows that

$$\mathcal{E}^{(i)}[u] \geq \inf_{\tilde{u}\in H^1(\mathcal{Q}_1)} \int_{\mathcal{Q}_1} \mathrm{d}\vec{x} \left\{ \left| \left(\nabla - \frac{i\tilde{h}_{\mathrm{ex}}\vec{e}_z \wedge \vec{x}}{2}\right)\tilde{u} \right|^2 + \epsilon^{-2}\left(1 - |\tilde{u}|^2\right)^2 \right\}, \tag{5.29}$$

for a magnetic field $\tilde{h}_{\mathrm{ex}}$ satisfying the conditions

$$|\log\epsilon| \ll \tilde{h}_{\mathrm{ex}} = \Omega\hat{\ell}^2 \ll \frac{1}{\epsilon^2}. \tag{5.30}$$

Indeed, by (5.8) and (5.28),

$$\Omega\hat{\ell}^2 = \frac{\Omega\varepsilon^2}{\epsilon^2\rho^{\mathrm{TF}}(r_i)} \leq \frac{\varepsilon|\log\delta|}{\epsilon^2} \ll \frac{1}{\epsilon^2}, \qquad \Omega\hat{\ell}^2 \gg |\log\varepsilon| \geq |\log\epsilon|, \tag{5.31}$$

because $0 \geq \log\epsilon = \log\varepsilon - \log(\hat{\ell}\sqrt{\rho^{\mathrm{TF}}(r_i)})$ and

$$\hat{\ell}\sqrt{\rho^{\mathrm{TF}}(r_i)} \ll \min\left[\frac{1}{\sqrt{|\log\delta|}}, \frac{1}{\sqrt{\omega}|\log\delta|}\right] \ll 1, \tag{5.32}$$

which implies $\log\epsilon \geq \log\varepsilon$ and $|\log\epsilon| \leq |\log\varepsilon|$.
The functional on the right hand side of (5.29) is precisely the GL functional on $\mathcal{Q}_1$ with external magnetic field $\tilde{h}_{\mathrm{ex}}\vec{e}_z$ and parameter ϵ, i.e.,

$$\tilde{\mathcal{E}}^{\mathrm{GL}}\left[\tilde{u},\vec{A}'\right] \equiv \int_{\mathcal{Q}_1} \mathrm{d}\vec{x}\,\left\{\left|\left(\nabla - i\vec{A}'\right)\tilde{u}\right|^2 + \left|\mathrm{curl}\vec{A}' - \tilde{h}_{\mathrm{ex}}\vec{e}_z\right|^2 + \epsilon^{-2}\left(1-|\tilde{u}|^2\right)^2\right\}, \tag{5.33}$$

evaluated on the configuration

$$\left(\tilde{u}\,,\vec{A}'\right) = \left(u\,,\tilde{h}_{\mathrm{ex}}(\epsilon)\vec{e}_z \wedge \vec{x}/2\right). \tag{5.34}$$

and (5.30) corresponds to the GL regime where the external magnetic field is between the first and the second critical fields. We can thus apply the lower bound for the GL functional proven in [20], Theorem 1.1 (note that in the definition of the GL functional given in [20] there is overall factor 1/2), to get

$$\mathcal{E}^{(i)}[u] \geq (1-o(1))h_{\mathrm{ex}}\log\frac{1}{\epsilon\sqrt{h_{\mathrm{ex}}}} = (1-o(1))\frac{\Omega\hat{\ell}^2}{2}\log\frac{\rho^{\mathrm{TF}}(r_i)}{\varepsilon^2\Omega} \geq (1-o(1))\,\frac{\Omega\hat{\ell}^2|\log\gamma|}{2}, \tag{5.35}$$

since $\rho^{\mathrm{TF}}(r_i) \geq \omega|\log\delta|^{-1}$ inside $\mathcal{T}$, if $\Omega \gtrsim \varepsilon^{-1}$, and $\rho^{\mathrm{TF}}(r_i) \geq C$, if $\Omega \ll \varepsilon^{-1}$.

□

The proof of the lower bound to the GP energy E^{GP} is now almost complete. Collecting the lower bounds inside all cells proven in the proposition above, we have

$$\tilde{\mathcal{E}}^{\mathrm{GP}}[u] \geq \frac{\Omega\hat{\ell}^2|\log\gamma|}{2}\sum_{\vec{r}_i\in\mathcal{L}}\rho^{\mathrm{TF}}(r_i)(1-o(1)). \tag{5.36}$$

The replacement of the Riemann sum by the integral can be done exactly as in (5.18): By the symmetry of the lattice cell and the L^1-normalization of ρ^{TF},

$$\sum_{\vec{r}_i\in\mathcal{L}}\rho^{\mathrm{TF}}(r_i) \geq \frac{1}{\hat{\ell}^2}\left(\int_{\cup_i\mathcal{Q}^i}\mathrm{d}\vec{r}\,\rho^{\mathrm{TF}}(r) - C\max[\hat{\ell},\omega\hat{\ell}]\right) \geq \frac{1-o(1)}{\hat{\ell}^2}, \tag{5.37}$$

and we finally obtain

$$E^{\mathrm{GP}} \geq E^{\mathrm{TF}} + \frac{\Omega|\log\gamma|}{2}(1-o(1)). \tag{5.38}$$

Since $\gamma = \min[\varepsilon,\varepsilon^2\Omega]$ this gives the lower bounds in (3.3) and (3.4). Note that the condition $\Omega \ll 1/(\varepsilon^2|\log\varepsilon|)$ entered in (5.9).

6 Vorticity of GP Minimizers

In this Section we prove Theorem 3.3, which is a consequence of the energy asymptotics in (3.3) together with a similar result in GL theory. Indeed we shall prove that one can associate to any GP minimizer a GL configuration satisfying certain energy bounds, which, by exploiting a result proven in [20], yield the uniform distribution of vorticity. The proof closely follows the analysis performed in Section 5 in [20] and relies on Proposition 5.1 in this reference as a key ingredient.

It is appropriate to point out that Theorem 3.3 is a statement about the uniform distribution of the local winding numbers of Ψ in the support of ρ^{TF} but not about the nature of the singularties. The energy considerations behind this result are not sufficient to exclude the occurrence of singularities that are not pointlike, e.g., lines of zeros of Ψ^{GP}. While we expect that Ψ^{GP} contains only isolated vortices in the parameter range (1.1), a proof of this has not been accomplished. The same is true for the corresponding question in GL theory (see, e.g., [20, 21]).

Proof of Theorem 3.3:
Exploiting the energy bounds proved before (see (4.49), (5.15), (5.18) and (5.22)), we obtain, with $\hat{\ell}$ and $\hat{\mathcal{L}}$ as in (5.16)-(5.17),

$$- o(1)\, \Omega \hat{\ell}^2 |\log(\varepsilon^2 \Omega)| \sum_{\vec{r}_i \in \hat{\mathcal{L}}} \rho^{\mathrm{TF}}(r_i) \leq \sum_{\vec{r}_i \in \hat{\mathcal{L}}} \rho^{\mathrm{TF}}(r_i) \left[\mathcal{E}^{(i)}[u] - \frac{\Omega \hat{\ell}^2 |\log(\varepsilon^2 \Omega)|}{2} \right] \leq C\Omega, \tag{6.1}$$

so that, since the sum is performed over the lattice $\hat{\mathcal{L}} \subset \mathcal{T}$, i.e., where $\rho^{\mathrm{TF}} \geq \omega |\log \delta|^{-1}$,

$$\sum_{\vec{r}_i \in \hat{\mathcal{L}}} \rho^{\mathrm{TF}}(r_i) \left| \mathcal{E}^{(i)}[u] - \frac{\Omega \hat{\ell}^2 |\log(\varepsilon^2 \Omega)|}{2} \right| \leq g(\varepsilon)\, \Omega \hat{\ell}^2 |\log(\varepsilon^2 \Omega)| \sum_{\vec{r}_i \in \hat{\mathcal{L}}} \rho^{\mathrm{TF}}(r_i), \tag{6.2}$$

for some $g(\varepsilon) \to 0$, as $\varepsilon \to 0$.
Now we can distinguish, as in [20], between good and bad cells, where the above inequality yields an upper (resp. lower) bound. The key point is that, if the definition of such cells is done in the appropriate way, the upper bound can be used to prove a uniform distribution of vorticity (inside good cells) and, at the same time, there are only few bad cells, i.e., their number is only a remainder with respect to the total number of cells. The final result would then be a simple consequence of the fact that cells cover $\mathrm{supp}\left(\rho^{\mathrm{TF}}\right)$ in the limit $\varepsilon \to 0$.
We say that a cell $\mathcal{Q}^i$ is a *good cell*, if

$$\mathcal{E}^{(i)}[u] - \frac{\Omega \hat{\ell}^2 |\log(\varepsilon^2 \Omega)|}{2} \leq \sqrt{g(\varepsilon)}\, \Omega \hat{\ell}^2 |\log(\varepsilon^2 \Omega)|, \tag{6.3}$$

while inside bad cells the inequality is reversed.
We can thus apply to any good cell Proposition 5.1 in [20], which implies the existence of a finite family of disjoint discs $\mathcal{B}_\varepsilon^i$, $i = 1, \ldots, k$, such that the sum of all the radii is bounded by $\Omega^{-1/2}$ and $|u| > 1/2$ on $\partial \mathcal{B}_\varepsilon^i$. Points 1, 2 and 3 in Proposition 3.3 then easily follows. In particular point 2 follows from a simple bound on the total number of cells, i.e., $N \ll \Omega^{-1}$, which is a consequence of the conditions (5.17) on $\hat{\ell}$. Furthermore, setting d^i equal to the winding number of u on $\partial \mathcal{B}_\varepsilon^i$, which is also the winding number of $|\Psi^{\mathrm{GP}}|^{-1} \Psi^{\mathrm{GP}}$ because $\rho^{\mathrm{TF}} > 0$ inside $\mathcal{T}$, we have

$$2\pi \sum d_\varepsilon^i \geq \Omega \hat{\ell}^2 (1 - o(1)), \qquad 2\pi \sum |d_\varepsilon^i| \leq \Omega \hat{\ell}^2 (1 + o(1)). \tag{6.4}$$

The second estimate above in particular implies that, by (5.17), the measure on the left hand side of (3.5) is uniformly bounded in ε, which guarantees its weak convergence. It remains only to show that it converges to the uniform measure on $\mathrm{supp}\left(\rho^{\mathrm{TF}}\right)$. To this purpose we first have to show that the number of bad cells included in any given open set $\mathcal{S} \subset \mathrm{supp}\left(\rho^{\mathrm{TF}}\right)$ (independent of ε, i.e., such that $|\mathcal{S}| \geq C > 0$) is, for $\varepsilon \to 0$, much smaller than the total number of cells in $\mathcal{S}$. In fact, since the area of $\mathcal{S}$ is positive and the diameter of the cells tends to zero for $\varepsilon \to 0$, it suffices to shows that this is true for $\mathcal{S} = \mathrm{supp}\left(\rho^{\mathrm{TF}}\right)$. Denote by $\mathcal{I}$ the sets of indices $i \in \mathbb{N}$ such that $\mathcal{Q}^i$ is a good (resp. bad) cell, then, by definition of bad cell and (6.2),

$$N^{\mathrm{B}} \Omega \hat{\ell}^2 |\log(\varepsilon^2 \Omega)| \sqrt{g(\varepsilon)} \leq \sum_{i \in \mathcal{J}} \left[\mathcal{E}^{(i)}[u] - \frac{\Omega \hat{\ell}^2 |\log(\varepsilon^2 \Omega)|}{2} \right] \leq C N \Omega \hat{\ell}^2 |\log(\varepsilon^2 \Omega)| g(\varepsilon), \tag{6.5}$$

where N^{B} denotes the number of bad cells and N the total number of cells. As a consequence

$$N^{\mathrm{B}} \leq C\sqrt{g(\varepsilon)}N, \tag{6.6}$$

and, since $g(\varepsilon) = o(1)$, the number of bad cells is always much smaller than the total number of cells. The result can be easily extended to any set $\mathcal{S} \subset \mathcal{B}_1$ by observing that the upper and lower bounds to the energy applies to any open subset of $\mathcal{B}_1$.
Theqrefore, for any given open subset $\mathcal{S} \subset \mathrm{supp}\left(\rho^{\mathrm{TF}}\right)$, good cells exhaust the whole of $\mathcal{S}$ as $\varepsilon \to 0$, i.e., $N^{\mathrm{G}}\hat{\ell}^2 \to |\mathcal{S}|$. Now, collecting all the disc families inside good cells and setting

$$\mu \equiv \frac{2\pi}{\Omega} \sum d_{i,\varepsilon} \delta\left(\vec{r} - \vec{r}_{i,\varepsilon}\right), \tag{6.7}$$

where $\vec{r}_{i,\varepsilon}$ stands for the center of $\mathcal{B}^i_\varepsilon$, one has, by the first estimate in (6.4),

$$\mu(\mathcal{S}) \geq N^{\mathrm{G}}\hat{\ell}^2 \geq (1 - o(1))|\mathcal{S}|, \tag{6.8}$$

and similarly, by the second estimate in (6.4),

$$\mu(\mathcal{S}) \leq (1 + o(1))N^{\mathrm{G}}\hat{\ell}^2 \leq (1 + o(1))|\mathcal{S}|, \tag{6.9}$$

which implies (3.5), since $\mathcal{S}$ is arbitrary.

□

7 Conclusions

Within the framework of two-dimensional GP theory we have evaluated exactly to subleading order the contributions of vorticity to the energy of a rapidly rotating Bose-Einstein condensate in a finite, flat trap. The results of the mathematical analysis lend support to the physical picture of a large number of vortices that are arranged in a triangular lattice at not too high rotational velocities but are eventually replaced by a 'giant vortex' with all the vorticity located outside the bulk of the density at sufficiently fast rotation. It would be desirable to substantiate this picture even further by generalizing Theorem 3.3 for $\Omega \gg \varepsilon^{-1}$ and by proving a lower bound to the energy matching the 'giant vortex' upper bound of [16] to subleading order for $\Omega \gtrsim \varepsilon^{-2}|\log\varepsilon|^{-1}$. Further interesting open problems concern the nature of the singularities of the GP minimizer, in particular the exclusion of line singularities and the precise arrangement of the vortices. This would in particular require energy estimates beyond the subleading order considered here.

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A The TF Energy and Density

We collect here from [16] some formulas for the TF energy and density (in a slightly different notation). Defining

$$\omega_{\mathrm{h}} \equiv 4/\sqrt{\pi}, \tag{A.1}$$

we have

$$\varepsilon^2 E^{\mathrm{TF}} = \begin{cases} \dfrac{1}{\pi} - \dfrac{\omega^2}{8} - \dfrac{\pi\omega^4}{768}, & \text{if } \omega \le \omega_{\mathrm{h}}, \\[2ex] -\dfrac{\omega^2}{4}\left[1 - \dfrac{8}{3\sqrt{\pi}\omega}\right], & \text{if } \omega > \omega_{\mathrm{h}}, \end{cases} \tag{A.2}$$

$$\rho^{\mathrm{TF}}(r) = \begin{cases} \dfrac{1}{\pi} + \dfrac{\omega^2}{16} - \dfrac{\omega^2}{8}(1-r^2), & \text{if } \omega \le \omega_{\mathrm{h}}, \\[2ex] \left[\dfrac{\omega}{2\sqrt{\pi}} - \dfrac{\omega^2}{8}(1-r^2)\right]_+, & \text{if } \omega > \omega_{\mathrm{h}}, \end{cases} \tag{A.3}$$

where $[t]_+ = t$, if $t \ge 0$, and 0 otherwise. The TF density ρ^{TF} can be as well expressed as

$$\rho^{\mathrm{TF}}(r) = \frac{1}{2}\left[\varepsilon^2\mu^{\mathrm{TF}} + \frac{\omega^2 r^2}{4}\right]_+, \tag{A.4}$$

where the chemical potential $\mu^{\mathrm{TF}} = E^{\mathrm{TF}} + \varepsilon^{-2}\|\rho^{\mathrm{TF}}\|_2^2$ is fixed by the normalization of ρ^{TF} and it is explicitly given by

$$\varepsilon^2\mu^{\mathrm{TF}} = \begin{cases} \dfrac{2}{\pi} - \dfrac{\omega^2}{8}, & \text{if } \omega \le \omega_{\mathrm{h}}, \\[2ex] -\dfrac{\omega^2}{4}\left[1 - \dfrac{4}{\sqrt{\pi}\omega}\right], & \text{if } \omega > \omega_{\mathrm{h}}. \end{cases} \tag{A.5}$$

Note that, if $\omega > \omega_{\mathrm{h}}$, a 'hole' centered at the origin occurs in the TF minimizer, i.e., $\rho^{\mathrm{TF}}(r) = 0$ for all $r \le R_{\mathrm{h}}$ with

$$R_{\mathrm{h}} \equiv \left(1 - \frac{\omega_{\mathrm{h}}}{\omega}\right)^{1/2} \tag{A.6}$$

the radius of the hole. For $\omega \ge \omega_{\mathrm{h}}$ and $R_{\mathrm{h}} \le r \le 1$ we can also write the density as

$$\rho^{\mathrm{TF}}(r) = \frac{\omega^2}{8}(r^2 - R_{\mathrm{h}}^2). \tag{A.7}$$

Note also that the behaviour of E^{TF} in the regimes $\Omega \ll \varepsilon^{-1}$ $(\omega \to 0)$ and $\Omega \gg \varepsilon^{-1}$ $(\omega \to \infty)$ is respectively

$$\varepsilon^2 E^{\mathrm{TF}} = \frac{1}{\pi} - O(\omega^2), \qquad \varepsilon^2 E^{\mathrm{TF}} = -\frac{\omega^2}{4}(1 - O(\omega^{-1})). \tag{A.8}$$

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