Rapidly Rotating Bose-Einstein Condensates in Strongly Anharmonic Traps

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Abstract

We study a rotating Bose-Einstein Condensate in a strongly anharmonic trap (flat trap with a finite radius) in the framework of 2D Gross-Pitaevskii theory. We write the coupling constant for the interactions between the gas atoms as $1/\varepsilon^2$ and we are interested in the limit $\varepsilon \to 0$ (TF limit) with the angular velocity Ω depending on ε. We derive rigorously the leading asymptotics of the ground state energy and the density profile when Ω tends to infinity as a power of $1/\varepsilon$. If $\Omega(\varepsilon) = \Omega_0/\varepsilon$ a "hole" (i.e., a region where the density becomes exponentially small as $1/\varepsilon \to \infty$) develops for Ω_0 above a certain critical value. If $\Omega(\varepsilon) \gg 1/\varepsilon$ the hole essentially exhausts the container and a "giant vortex" develops with the density concentrated in a thin layer at the boundary. While we do not analyse the detailed vortex structure we prove that rotational symmetry is broken in the ground state for const.$|\log\varepsilon| < \Omega(\varepsilon) \lesssim$ const.$/\varepsilon$.

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1 Introduction

In recent years much effort, both experimental and theoretical, has been put into the study of vortices in rotating Bose-Einstein condensates, see, e.g., [CD], the review [FS] and the monograph [A] where extensive lists of references can be found. Most of the theoretical research is carried out in the framework of Gross-Pitaevskii theory, whose status as an approximation of the quantum mechanical many-body problem was established in [LSY] for the non-rotating case and in [LS] for rotating systems. On the mathematical physics side an important topic has been the vortex structure in the strong coupling (Thomas-Fermi, TF) regime in harmonic traps when the rotational velocity is scaled with the coupling in such a way that the number of vortices remains finite [AD, IM1, IM2]. General results on symmetry breaking for sufficiently large interactions or rotational velocities and in traps of arbitrary shape were proved in the papers [S1, S2] that are not limited to the TF regime.

Recently, attention has focused on rapidly rotating condensates where the number of vortices is much larger than unity (see, e.g., [ECHSC, SCEMC] for experimental results). Much of this research has been for harmonic traps, e.g., [AB, ABD, ABN1, ABN2, WBP1, WGBP], where "rapid rotation" means a velocity close to the limiting velocity beyond which the centrifugal forces destabilize the condensate, but anharmonic traps (mostly quartic plus harmonic) have also been discussed [AAB, B, BP, F, FB, FZ, KB, KF, KTU]. For harmonic traps the eigenstates of a noninteracting rotating gas fall into Landau levels and rapid rotation implies, also for an interacting gas, that essentially only the lowest Landau level (LLL) is occupied. Using this fact detailed informations about the lattice of vortices have been obtained in [ABD, ABN1, ABN2, WBP1]. Some results for harmonic traps going beyond the LLL approximation

are discussed in [WBP1, WGBP]. In an anharmonic trap a restriction to the LLL is not adequate. The reason is that the energy gap between Landau levels is proportional to the angular velocity while the centrifugal energy is proportional to the angular velocity squared. In an anharmonic trap the latter can be much larger than the former, while in a harmonic trap close to the limiting angular velocity the potential and centrifugal energies almost cancel each other and the LLL energy is the dominating contribution.

In the present paper we study a rapidly rotating gas in a trap that is as far from being harmonic as possible: The gas is confined within a finite radius R and the trap is "flat", i.e., the confining potential is constant (zero) inside the trap. Formally this trapping potential can be regarded as a limit of a homogeneous potential $V(r) \sim (r/R)^s$ with $s \to \infty$. Such a limit naturally leads to Dirichlet conditions at the boundary, but it is mathematically somewhat simpler to consider the case of Neumann (or free) boundary conditions and this is what we shall do. In this way the interplay between rotational effects and the nonlinear interaction terms are brought out in a particularly clean way. Dirichlet boundary conditions lead, in fact, to exactly the same results in the TF limit as we shall also show. Generalizations to homogeneous potentials with $s < \infty$ are in principle straightforward but the case $s = \infty$ merits a special treatment because it brings out clearly the essential differences between harmonic and anharmonic traps and also because of some special features with respect to the breaking of rotational symmetry. This will be discussed in Section 2.1.

Our main results concern the density profile and the ground state energy in the asymptotic limit when the coupling constant $1/\varepsilon^2$ (see below) tends to infinity (TF limit) and the rotational velocity $\Omega(\varepsilon)$ is at the same time scaled with ε. (The TF limit of the 2D GP functional without rotation is discussed in [LSY].) Our estimates are not sharp enough to rigorously uncover the vortex structure of the condensate, but the variational functions that we use and which give the correct energy to leading order in ε provide important hints about this structure. In particular, the regimes $\Omega(\varepsilon) \sim 1/\varepsilon$ and $\Omega(\varepsilon) \gg 1/\varepsilon$ require different variational functions, the former with a lattice of vortices distributed over the trap and the latter with a "giant vortex" in the region where the density is exponentially small.

When considering the TF limit there is an important difference between traps that confine the gas strictly to a bounded region and traps where the gas can spread out indefinitely. If one considers for instance a trap given by an homogeneous potential $V(r) = r^s$, for some $0 < s < \infty$ and performs the TF limit in a naive way, the result is trivial, namely the minimizer goes to zero and the energy to infinity. In order to obtain a non-trivial limit it is necessary to rescale all lengths by $\varepsilon^{\frac{4}{2+s}}$. In the case of infinitely high walls considered here the characteristic length of the problem is fixed from the outset and therefore no rescaling is needed in the TF limit. Consequences of this difference for the question of symmetry breaking in the rotating case will be discussed in Section 2.1.

Rapidly rotating condensates in a flat trap have been previously studied by Fischer and Baym in [FB] and the paper of these authors triggered, in fact, the present investigation. Our analysis underpins and extends their general picture by rigorous estimates. We do not, however, confirm that the transition to the "giant vortex" state takes place for $\Omega(\varepsilon) \sim 1/(\varepsilon^2|\log\varepsilon|)$ as implied by Eq. (20) in [FB]. Our conclusion is rather that such a state emerges asymptotically at all rotational velocities $\Omega(\varepsilon) \gg 1/\varepsilon$. The reasons for this difference are discussed in Section 2.4.

We now define the setting more precisely. The starting point is the 2D Gross-Pitaevskii (GP) energy functional

$$\mathcal{E}_R^{\mathrm{GP}}[\Psi] \equiv \int_{\mathcal{B}_R} d\vec{r} \left\{ |\nabla\Psi|^2 - \Omega(\varepsilon)\Psi^* L\Psi + \frac{|\Psi|^4}{\varepsilon^2} \right\} \tag{1.1}$$

where $\mathcal{B}_R$ denotes a ball (disc) of radius R centered at the origin, L the third component of the angular momentum (i.e. $L = -i\partial/\partial\vartheta$ in polar coordinates (r, ϑ)), $\Omega(\varepsilon)$ the angular velocity and ε is a nonnegative, small parameter.
The functional is defined on the domain[1]

$$\mathcal{D}_R^{\mathrm{GP}} = H^1(\mathcal{B}_R). \tag{1.2}$$

We also define

$$E_\varepsilon^{\mathrm{GP}}(R) \equiv \min_{\substack{\Psi \in \mathcal{D}_R^{\mathrm{GP}} \\ \|\Psi\|_2 = 1}} \mathcal{E}_R^{\mathrm{GP}}[\Psi] \tag{1.3}$$

[1]By Sobolev immersion $H^1(\mathcal{B}_R)$ is contained in $L^4(\mathcal{B}_R)$, so the functional is well defined on $\mathcal{D}_R^{\mathrm{GP}}$.

and denote by $\Psi_\varepsilon^{\rm GP}$ a corresponding minimizer[2]. Indeed, for any $\Omega(\varepsilon) < \infty$, one can prove (see for instance [S1]) that the functional is bounded from below and there exists at least one minimizer.

From the physics point of view a minimizer of (1.1) describes the macroscopic wave function (the wave function of the condensate) of a Bose-Einstein condensate in the rotating reference frame. The 2D description is a simplification that is justified either in the limit of thin ("disc shaped") 3D traps, or traps that are very elongated along the rotational axis ("cigar shaped" traps) so that the 3D wave function is essentially constant along this axis. In both cases the coupling $1/\varepsilon^2$ is proportional to Na/h where a is the scattering length of the two-body potential (for its definition see, e.g., the Appendix in [LY]) , N the particle number and h the extension of the 3D trap along the rotational axis.[3]

In the non-rotating case, $\Omega(\varepsilon) = 0$, the minimizer is actually unique, by the strict convexity of the functional, and it is given by the (normalized) constant function,

$$\Psi_\varepsilon^{\rm GP}\big|_{\Omega=0} = \frac{1}{\sqrt{\pi}R}.$$

The ground state energy is then

$$E_\varepsilon^{\rm GP}(R)\big|_{\Omega=0} = \frac{1}{\pi R^2\varepsilon^2}.$$

If the angular velocity is different from zero the minimizer may not be unique since a rotational symmetry breaking occurs for $\Omega(\varepsilon)$ above a certain threshold value as will be shown in Section 2.1.

We point out that the dependence on the radius R of the trap can be scaled out:

$$E_\varepsilon^{\rm GP} \equiv E_\varepsilon^{\rm GP}(1) = R^2 E_\varepsilon^{\rm GP}(R) \tag{1.4}$$

so that, without loss of generality, we can choose $R = 1$ and denote the functional by $\mathcal{E}^{\rm GP}[\Psi]$.

The GP functional can be rewritten in the following form that we are going to use:

$$\mathcal{E}^{\rm GP}[\Psi] = \int_{\mathcal{B}_1} d\vec{r}\,\left\{\left|\left(\nabla - i\vec{A}_\varepsilon\right)\Psi\right|^2 - \frac{\Omega(\varepsilon)^2 r^2|\Psi|^2}{4} + \frac{|\Psi|^4}{\varepsilon^2}\right\} \tag{1.5}$$

where $\vec{A}_\varepsilon$ is the vector potential associated with the rotation, i.e.,

$$\vec{A}_\varepsilon(\vec{r}) \equiv \frac{\Omega(\varepsilon)}{2}\hat{z}\times\vec{r}, \tag{1.6}$$

with $\hat{z}$ the unit vector in the z-direction. In (1.5) one can recognize an analogy with the Ginzburg-Landau (GL) functional (see, e.g., [BR]) in the theory of superconductivity. The vector potential (1.6) is in this context due to a uniform magnetic field, while the wave function of the condensate is the GL order parameter (density of Cooper pairs). Using the L^2-normalization of the minimizer, the analogy can be made even closer, namely the minimization problem in (1.3) is equivalent to the minimization of the functional

$$\mathcal{E}^{\rm GP\prime}[\Psi] = \int_{\mathcal{B}_1} d\vec{r}\,\left\{\left|\left(\nabla - i\vec{A}_\varepsilon\right)\Psi\right|^2 - \frac{\Omega(\varepsilon)^2 r^2|\Psi|^2}{4} + \frac{\left(1-|\Psi|^2\right)^2}{\varepsilon^2}\right\}$$

over L^2-normalized functions. At this point, however, an important difference becomes evident, namely the presence of the centrifugal energy (the second term in the expression above), which in the GL context could be interpreted as an electric field. This contribution, usually not present in the GL functional, is proportional to the square of the angular velocity and we are going to see that, in the regimes we are considering, it is responsible for a rather different behavior of the minimizer compared to GL theory. Another important difference between the GP and the GL minimization problems is the L^2-normalization condition, that prevents, for instance, the minimizer from being identically zero, as it can be in the GL case. It also gives rise to an additional term (chemical potential) in the variational equation associated to (1.3).

In the next Section 2 we introduce some notations and state the main results of this paper. We first discuss the problem of spontaneous symmetry breaking in the ground state, then we study the regimes $\Omega(\varepsilon) \ll 1/\varepsilon$ (Section 2.2), $\Omega(\varepsilon) \sim 1/\varepsilon$ (Section 2.3) and $\Omega(\varepsilon) \gg 1/\varepsilon$ (Section 2.4). Section 3 is devoted to the proofs, while in Section 4 we comment on the results and perspectives.

[2] Any result for $\Psi_\varepsilon^{\rm GP}$ stated in the following is meant to be true for any minimizer, if it is not unique.

[3] In "thin" traps a different formula applies at extreme dilution where the coupling becomes independent of a and depends logarithmically on the average density (see [SY]).

2 Main Results

2.1 Spontaneous Symmetry Breaking in the Ground State

The GP functional for a rotating 2D condensate in a general trap has already been studied in [S1]. A very interesting phenomenon generated by the rotation is the spontaneous breaking of rotational symmetry in the ground state. If the trap potential is polynomially bounded at infinity one can prove (see Theorem 4 in [S1]) that for any fixed angular velocity Ω, there exists ε_Ω such that, if $\varepsilon < \varepsilon_\Omega$, no ground state of the GP functional is an eigenfunction of the angular momentum. The rotational symmetry of the functional is then spontaneously broken at the level of the ground state. An important consequence is that the minimizer is no longer unique, since a rotation by an arbitrary angle gives rise to a state with the same energy.

A crucial ingredient of the proof in [S1] is that, in a polynomially bounded potential trap, the density of the minimizer tends to zero as $\varepsilon \to 0$. In fact, Theorem 4 in [S1] is not true in the case of a trap with infinitely high walls as we are considering and we actually expect the opposite behavior: If Ω is kept fixed, then for ε sufficiently small the ground state is an eigenfunction of the angular momentum, and after an appropriate choice of a constant phase factor, a unique, strictly positive radial function. This difference can be understood by noting that in a trap of radius R the kinetic energy of a vortex is of the order $R^{-2}|\log\varepsilon|$ for small ε. Thus, if R is fixed, an angular velocity of order $|\log\varepsilon|$ is needed in order to create vortices. In a polynomially bounded trap, on the other hand, the effective radius of the condensate increases as $\varepsilon \to 0$ and the critical velocity for the creation of a vortex behaves as $\varepsilon^{4/(s+2)}|\log\varepsilon|$ for a trap potential $\sim r^s$, cf. the remark at the end of Section 3 in [S1]. Any fixed Ω thus exceeds the critical velocity as $\varepsilon \to 0$ if $s < \infty$.

Despite this difference, our proof of symmetry breaking is obtained partly by a modification of the arguments of Theorem 2 in [S1]. The following Proposition 2.1 states that for angular velocities smaller than $1/\sqrt{\pi}\varepsilon$, symmetric vortices of degree higher than 1 are unstable[4].

Proposition 2.1 (Instability for Higher Vorticity)
Let $\Psi_n(\vec{r})$, $n \geq 2$, be the unique minimizer of $\mathcal{E}^{\mathrm{GP}}[\Psi]$ on the subspace of functions with angular momentum n, i.e., on $\{\Psi \in \mathcal{D}^{\mathrm{GP}} \,|\, L\Psi = n\Psi\}$. For any $\Omega(\varepsilon) \leq 1/\sqrt{\pi}\varepsilon$, Ψ_n is unstable, i.e., it is not a local minimizer of $\mathcal{E}^{\mathrm{GP}}[\Psi]$.

Proof: From the variational equation satisfied by the radial part of $\Psi_n \equiv \xi_n(r)e^{in\vartheta}$,

$$-\Delta\xi_n + \frac{n^2\xi_n}{r^2} - \Omega(\varepsilon)n\xi_n + \frac{2\xi_n^3}{\varepsilon^2} = \mu_n(\varepsilon)\xi_n$$

where the chemical potential $\mu_n(\varepsilon)$ is fixed by the L^2-normalization of Ψ_n, it is not hard to prove by a rearrangement argument (see, e.g., Lemma 1 in [S1]) that ξ_n is a positive non-decreasing function, $\xi_n(r) = O(r^n)$ as $r \to 0$ and $\xi_n'(1) = 0$, i.e., ξ_n satisfies Neumann boundary conditions. Moreover by a subharmonicity argument[5] we can also prove the bound:

$$\|\Psi_n\|^2_{L^\infty(\mathcal{B}_1)} \leq \frac{\varepsilon^2}{2}\left\{\mu_n(\varepsilon) + \Omega(\varepsilon)n - n^2\right\} \tag{2.1}$$

Indeed, suppose that $n \geq 1$ and the opposite is true, then setting

$$\mathcal{B}^> \equiv \left\{ r \in (0,1) \,\middle|\, \xi_n^2(r) > \varepsilon^2\left(\mu_n(\varepsilon) + \Omega(\varepsilon)n - n^2\right)/2 \right\}$$

we can have two possibilities: Either $\mathcal{B}^> = \emptyset$, and then the result easily follows, or it is an open interval, $\mathcal{B}^> \equiv (R^>, 1)$, by monotonicity of ξ_n, and $\Delta\xi_n|_{\mathcal{B}^>} > 0$. In this case, by integrating $\Delta\xi_n$ over $\mathcal{B}^>$ and using Neuman boundary conditions, one has

$$\int_{\mathcal{B}^>} \Delta\xi_n r dr = -R^> \xi_n'(R^>) > 0$$

[4] In the opposite regime of weak coupling and fixed Ω, vortices of degree 2 or higher may be energetically favorable in anharmonic traps [Lu].

[5] For a similar proof see, e.g., Lemma 2.1 in [LSY].

which is a contradiction because ξ_n in non-decreasing.
The rest of the proof coincides with the proof of Theorem 2 in [S1]. Using the estimate (2.1), we can extract, as in (2.33) in [S1], a sufficient condition on the chemical potential for instability of the corresponding vortex: The symmetric vortex of degree[6] $n \geq d \in \mathbb{N}$ is unstable if

$$-(d-1)^2\mu_\varepsilon + (d^2-1)\Omega(\varepsilon)n - (d-1)^2n^2 < 0$$

or, choosing $d = 2$,

$$-\mu_n(\varepsilon) + 3\Omega(\varepsilon)n - n^2 < 0.$$

From the definition of the chemical potential and Schwarz's inequality it also follows that

$$\mu_n(\varepsilon) = \int_{\mathcal{B}_1} d\vec{r}\,\left\{(\nabla\xi_n)^2 + \frac{n^2\xi_n^2}{r^2} - \Omega(\varepsilon)n\xi_n^2 + \frac{2\xi_n^4}{\varepsilon^2}\right\} \geq n^2 - \Omega(\varepsilon)n + \frac{2}{\pi\varepsilon^2}.$$

Inserting this bound in the condition above, we have instability if

$$n^2 - 2\Omega(\varepsilon)n + \frac{1}{\pi\varepsilon^2} > 0.$$

Hence any vortex of order $n \geq 2$ is unstable, provided $\Omega(\varepsilon) \leq 1/\sqrt{\pi}\varepsilon$.

□

From Prop. 2.1 it follows that in order to prove the symmetry breaking in the ground state for a given $\Omega(\varepsilon) \leq 1/\sqrt{\pi}\varepsilon$ it is sufficient to show that a rotationally symmetric vortex of degree smaller or equal to 1 cannot be a minimizer of the GP functional at this angular velocity. This in turn can be achieved by exploiting some energy estimates. Let us first define

$$E_n(\varepsilon) \equiv \min_{\|\xi\|_{L^2(\mathcal{B}_1)}=1} \int_{\mathcal{B}_1} d\vec{r}\,\left\{|\nabla\xi|^2 + \frac{n^2\xi^2}{r^2} + \frac{\xi^2}{\varepsilon^2}\right\}$$

and

$$\Omega_n(\varepsilon) \equiv E_{n+1}(\varepsilon) - E_n(\varepsilon)$$

so for $\Omega(\varepsilon) > \Omega_{\bar{n}}(\varepsilon)$ no symmetric vortex of degree $n \leq \bar{n}$ can be a global minimizer of the GP functional. Since $\Omega_n(\varepsilon) \leq (2n+1)\Omega_0(\varepsilon)$ (see Lemma 3 in [S1]) we can use an upper bound on $E_1(\varepsilon) - E_0(\varepsilon)$ to prove the symmetry breaking.

Proposition 2.1 applies only for $\Omega(\varepsilon) \leq 1/\sqrt{\pi}\varepsilon$ but symmetry breaking can, in fact, be proved for $\Omega(\varepsilon) \lesssim C/\varepsilon$ with an arbitrary constant C by using Theorem 2.1 that is proved later in the paper. Hence a part of the proof of the next proposition will be postponed to the end of Section 2.3.

Proposition 2.2 (Symmetry Breaking in the Ground State)
For ε sufficiently small, no minimizer of $\mathcal{E}^{\mathrm{GP}}[\Psi]$ is an eigenfunction of the angular momentum, if

$$6|\log\varepsilon| + 3 < \Omega(\varepsilon) \lesssim \frac{C}{\varepsilon}$$

for any constant $C \in \mathbb{R}^+$.

Proof: Using the normalized trial function

$$\xi(r) \equiv c_\varepsilon \begin{cases} \dfrac{r}{\varepsilon} & \text{if} \quad 0 \leq r \leq \varepsilon \\ 1 & \text{otherwise} \end{cases}$$

[6]The variational parameter $d \in \mathbb{N}$ is involved in the definition of a suitable trial function used in Theorem 2 in [S1]. The requirement $n \geq d$ is necessary, otherwise such a trial function does not belong to $H^1(\mathcal{B}_1)$.

we can prove the upper bound

$$E_1(\varepsilon) \leq \frac{1}{\pi\varepsilon^2} + \log\left(\frac{1}{\varepsilon^2}\right) + 1.$$

Since $\Omega_n(\varepsilon) \leq (2n+1)\Omega_0(\varepsilon)$ (see Lemma 3 in [S1]) and $E_0(\varepsilon) = \frac{1}{\pi\varepsilon^2}$, we get

$$\Omega_1(\varepsilon) \leq 3\left[\log\left(\frac{1}{\varepsilon^2}\right) + 1\right].$$

Hence no symmetric vortex of degree ≤ 1 can be a global minimizer of the GP functional if $\Omega(\varepsilon) \geq 6|\log\varepsilon| + 3$. On the other hand vortices of degree higher or equal to 2 are excluded by Proposition 2.1 provided that $\Omega(\varepsilon) \lesssim 1/\sqrt{\pi}\varepsilon$. The proof of symmetry breaking for general $\Omega(\varepsilon) \sim 1/\varepsilon$ will be given at the end of Section 2.3.

□

2.2 Energy and density for $\Omega(\varepsilon) \ll 1/\varepsilon$

If $\Omega(\varepsilon) \ll 1/\varepsilon$, the rotation has no leading order effect in the TF regime. More precisely the energy asymptotics is the same as for a non-rotating condensate, and the density profile, $|\Psi_\varepsilon^{\mathrm{GP}}|^2$, converges to the normalized constant function, namely the minimizer of the GP functional without rotation:

Proposition 2.3 (Energy and Density Asymptotics)
For any $\Omega(\varepsilon)$ such that $\lim_{\varepsilon\to 0}\varepsilon\Omega(\varepsilon) = 0$ and for ε sufficiently small

$$\varepsilon^2 E_\varepsilon^{\mathrm{GP}} = \frac{1}{\pi} - O(\varepsilon^2\Omega(\varepsilon)^2) \tag{2.2}$$

$$\left\||\Psi_\varepsilon^{\mathrm{GP}}|^2 - 1/\pi\right\|_{L^1(\mathcal{B}_1)} = O(\varepsilon\Omega(\varepsilon)). \tag{2.3}$$

Proof: Since $\|\Psi_\varepsilon^{\mathrm{GP}}\|_{L^2(\mathcal{B}_1)} = 1$ and $R = 1$, we have

$$\int_{\mathcal{B}_1} d\vec{r}\, r^2 |\Psi_\varepsilon^{\mathrm{GP}}|^2 \leq 1$$

and $\|\Psi_\varepsilon^{\mathrm{GP}}\|^4_{L^4(\mathcal{B}_1)} \geq 1/\pi$ by Schwarz's inequality. Hence (1.5) leads to the lower bound,

$$\varepsilon^2 E_\varepsilon^{\mathrm{GP}} \geq \frac{1}{\pi} - \frac{\varepsilon^2\Omega(\varepsilon)^2}{4}.$$

The upper bound is obtained by evaluating the functional on the trial function $1/\sqrt{\pi}$, namely

$$\varepsilon^2 E_\varepsilon^{\mathrm{GP}} \leq \frac{1}{\pi}.$$

Moreover, since $\|\Psi_\varepsilon^{\mathrm{GP}}\|_{L^2(\mathcal{B}_1)} = 1$, this estimate implies

$$\left\||\Psi_\varepsilon^{\mathrm{GP}}|^2 - 1/\pi\right\|^2_{L^2(\mathcal{B}_1)} \leq \frac{\varepsilon^2\Omega(\varepsilon)^2}{4}$$

The L^1-convergence of the density profile now follows by Schwarz's inequality.

□

We stress that the result above says nothing about the fine structure of the minimizer and also nothing about its uniqueness. As far as the density profile is concerned, the first critical velocity at which some new effect comes into play is $\Omega(\varepsilon) \sim 1/\varepsilon$ as it will be discussed in the next subsection. On the other hand, the fine structure of $\Psi_\varepsilon^{\mathrm{GP}}$ depends on the angular velocity, even if $\Omega(\varepsilon) \ll 1/\varepsilon$. If $\Omega(\varepsilon)$ is simply a constant and ε is sufficiently small, it is not hard to see that the minimizer is unique. More precisely, it

is a radial function (and hence an eigenfunction of the angular momentum) which can be chosen strictly positive. In this case the result in (2.3) can be improved and the convergence can be extended to $L^\infty(\mathcal{B}_1)$.

According to the discussion in [AD] and the rigorous analysis in [IM1, IM2] of rotating Bose-Einstein condensates in harmonic traps, the first critical velocity for the occurrence of vortices, i.e. isolated zeros of the minimizer, is in that case[7] of the order $\Omega(\varepsilon) \sim \varepsilon|\log\varepsilon|$. More precisely, if $\tilde{\Omega}_d(\varepsilon) < \Omega(\varepsilon) < \tilde{\Omega}_{d+1}(\varepsilon)$, where

$$\tilde{\Omega}_d(\varepsilon) \equiv c\,\varepsilon\,[|\log\varepsilon| + (d-1)\log|\log\varepsilon|]\,, \tag{2.4}$$

the minimizer has exactly d vortices of degree 1. A similar behavior was shown in [Se] for a slightly different model of superfluids.
Such results together with the considerations in Section 2.1 suggest that in a flat trap vortices start to occur if $\Omega(\varepsilon) \sim |\log\varepsilon|$ and the rotational symmetry can be broken. The spontaneous symmetry breaking cannot be seen at the level of the density profile $|\Psi_\varepsilon^{\mathrm{GP}}|^2$ however, because the average size of each vortex is very small (area of the core of order ε) in the TF limit. The total vorticity of the minimizer is proportional to the angular velocity, provided that $\Omega(\varepsilon) \gg |\log\varepsilon|$, and therefore, as long as $\Omega(\varepsilon) \ll 1/\varepsilon$, the region covered by the vortex cores has Lebesgue measure zero in the limit $\varepsilon \to 0$, in accord with (2.3).

2.3 The Regime $\Omega(\varepsilon) \sim 1/\varepsilon$

In the regime $\Omega(\varepsilon) \sim 1/\varepsilon$ the rotation is so fast that it modifies the density profile itself: Since the centrifugal energy in (1.5) is of the same order of the non-linear term, it is no longer convenient for the condensate to be uniformly distributed over the trap, like in the non-rotating case. Such an effect can be seen at a macroscopic level, namely the density profile converges to a non-constant function, which minimizes a TF-like functional.

Before stating the main results we first need some new notations. Without loss of generality we can assume that $\Omega_0 \equiv \varepsilon\Omega(\varepsilon)$ is a constant independent of ε. Moreover, for any $\Omega_0 > 0$, we introduce the *TF functional*,

$$\mathcal{E}^{\mathrm{TF}}[\rho] \equiv \int_{\mathcal{B}_1} d\vec{r}\,\left\{\rho^2 - \frac{\Omega_0^2 r^2 \rho}{4}\right\}, \tag{2.5}$$

defined on the domain

$$\mathcal{D}^{\mathrm{TF}} = \left\{\rho \in L^2(\mathcal{B}_1)\,|\,\rho \geq 0\right\}. \tag{2.6}$$

The functional above has a unique minimizer, ρ^{TF}, and we denote

$$E^{\mathrm{TF}} \equiv \min_{\substack{\rho\,\in\,\mathcal{D}^{TF} \\ \int\rho = 1}} \mathcal{E}^{\mathrm{TF}}[\rho] = \mathcal{E}^{\mathrm{TF}}[\rho^{\mathrm{TF}}]. \tag{2.7}$$

The minimizer ρ^{TF} can be explicitly calculated:

$$\rho^{\mathrm{TF}}(r) = \begin{cases} \dfrac{1}{\pi} - \dfrac{\Omega_0^2}{16}(1-2r^2) & \text{if} \quad \Omega_0 \leq \dfrac{4}{\sqrt{\pi}} \\[2ex] \left[\dfrac{\Omega_0^2}{8}(r^2-1) + \dfrac{\Omega_0}{2\sqrt{\pi}}\right]_+ & \text{if} \quad \Omega_0 > \dfrac{4}{\sqrt{\pi}} \end{cases} \tag{2.8}$$

where $[\,\cdot\,]_+$ stands for the positive part, and the ground state energy is

$$E^{\mathrm{TF}} = \begin{cases} \dfrac{1}{\pi} - \dfrac{\Omega_0^2}{8} - \dfrac{\pi\Omega_0^4}{768} & \text{if} \quad \Omega_0 \leq \dfrac{4}{\sqrt{\pi}} \\[2ex] \dfrac{\Omega_0}{4}\left(\dfrac{8}{3\sqrt{\pi}} - \Omega_0\right) & \text{if} \quad \Omega_0 > \dfrac{4}{\sqrt{\pi}}. \end{cases} \tag{2.9}$$

[7] The overall factor ε in the critical velocities (2.4) is due to the scaling mentioned in Section 1 and Section 2.1.

We point out that, if $\Omega_0 > \frac{4}{\sqrt{\pi}}$, $\rho^{\rm TF}$ has a "hole", i.e., a macroscopic region where it is identically zero, centered at the origin: With

$$R_0 \equiv \sqrt{1 - \frac{4}{\sqrt{\pi}\Omega_0}} \tag{2.10}$$

we have $\rho^{\rm TF}(r) = 0$, for any $r \leq R_0$. We also define

$$\mathcal{D}_0 \equiv \mathrm{supp}(\rho^{\rm TF}) = \{\vec{r} \in \mathcal{B}_1 \mid r \geq R_0\}. \tag{2.11}$$

The first result concerns the energy asymptotics:

Theorem 2.1 (Energy Asymptotics)
For any $\Omega_0 > 0$ and for ε sufficiently small

$$\varepsilon^2\, E_\varepsilon^{\rm GP} = E^{\rm TF} + O(\varepsilon|\log\varepsilon|). \tag{2.12}$$

The leading order term, proportional to $1/\varepsilon^2$, in the asymptotic expansion of $E_\varepsilon^{\rm GP}$ is due to the centrifugal bending of the profile while the remainder (of the order $|\log\varepsilon|/\varepsilon$) is the contribution coming from the fine structure of the minimizer. Indeed, $\Psi_\varepsilon^{\rm GP}$ is expected to carry a very large number (of the same order as $\Omega(\varepsilon)$) of vortices of degree 1. As suggested by the trial function (3.2) used in the proof of Theorem 2.1 (see also [FB]), such vortices should be distributed over a lattice with a spacing of order $\sqrt{\varepsilon}$, so that the average vortex core covers an area proportional to ε. A simple argument (see, for instance, [BBH2]) shows that the kinetic energy of each vortex is of the order $|\log\varepsilon|$. This explains why the total energy contribution of vortices produces a remainder of the order $|\log\varepsilon|/\varepsilon$.

As stated in the Introduction, the results proved in Theorem 2.1 and in the rest of this Section also hold in the case of Dirichlet boundary conditions (see the Remark 3.1 in Section 3.1). The crucial point is that the limiting functional (2.5) contains no kinetic energy and hence boundary conditions become irrelevant in the TF limit, at least to the leading order.

The convergence of the profile $|\Psi_\varepsilon^{\rm GP}|^2$ to $\rho^{\rm TF}$ is a straightforward consequence of Theorem 2.1:

Corollary 2.1 (Density Asymptotics)
For any $\Omega_0 > 0$ and for ε sufficiently small,

$$\big\| |\Psi_\varepsilon^{\rm GP}|^2 - \rho^{\rm TF} \big\|_{L^1(\mathcal{B}_1)} = O(\sqrt{\varepsilon|\log\varepsilon|}). \tag{2.13}$$

If $\Omega_0 > \frac{4}{\sqrt{\pi}}$ the estimate above can be improved and we can prove that the profile $|\Psi_\varepsilon^{\rm GP}|^2$ is exponentially small in ε inside the "hole", i.e. where $\rho^{\rm TF}$ is zero:

Proposition 2.4 (Exponential Smallness of the Density in the "Hole")
Denote

$$\mathcal{T}_\varepsilon \equiv \left\{ \vec{r} \in \mathcal{B}_1 \,\middle|\, r \leq R_0 - \varepsilon^{\frac{1}{3}} \right\} \tag{2.14}$$

where R_0 is defined in (2.10)*. For any $\Omega_0 > \frac{4}{\sqrt{\pi}}$ and ε sufficiently small, there exist two constants C_{Ω_0} and C'_{Ω_0} such that, for $\vec{r} \in \mathcal{T}_\varepsilon$,*

$$\left|\Psi_\varepsilon^{\rm GP}(\vec{r})\right|^2 \leq C_{\Omega_0}\varepsilon^{\frac{1}{6}}\sqrt{|\log\varepsilon|}\,\exp\left[-\frac{C'_{\Omega_0}\mathrm{dist}(\vec{r},\partial\mathcal{T}_\varepsilon)^2}{\varepsilon^{\frac{2}{3}}}\right]. \tag{2.15}$$

The results stated above allow us to complete the proof of Proposition 2.2:

Proof of Proposition 2.2
It remains to prove the statement for any $\Omega(\varepsilon) = \Omega_0/\varepsilon$. Suppose that the opposite statement is true, namely the ground state energy $E_\varepsilon^{\rm GP}$ is reached on a symmetric vortex of the form $\xi_n e^{in\vartheta}$. Then there must be some $\bar{n}_\varepsilon \in \mathbb{N}$ such that

$$E_{\bar{n}_\varepsilon}(\varepsilon) - \Omega(\varepsilon)\bar{n}_\varepsilon = E_\varepsilon^{\rm GP} \leq \frac{E^{\rm TF}}{\varepsilon^2} + \frac{C|\log\varepsilon|}{\varepsilon} \tag{2.16}$$

where we have used the upper bound for $E_\varepsilon^{\rm GP}$ proved in Theorem 2.1. Using the rough lower bound $E_n(\varepsilon) \geq n^2$, we immediately get the upper bound $\bar{n}_\varepsilon \leq C/\varepsilon$ for some constant C independent of ε. The right hand side of (2.16) can be bounded below by

$$E_n - \Omega(\varepsilon)n \geq \int_{\mathcal{B}_1} d\vec{r}\, \left[\frac{n}{r} - \frac{\Omega(\varepsilon)r}{2}\right]^2 \xi_n^2(r) + \frac{\mathcal{E}^{\rm TF}[\xi_n^2]}{\varepsilon^2} \geq \int_{\mathcal{B}_1} d\vec{r}\, \left[n - \frac{\Omega(\varepsilon)r^2}{2}\right]^2 \xi_n^2(r) + \frac{E^{\rm TF}}{\varepsilon^2}.$$

Therefore one has the estimate

$$\int_{\mathcal{B}_1} d\vec{r}\, \left[\bar{n}_\varepsilon - \frac{\Omega(\varepsilon)r^2}{2}\right]^2 \xi_{\bar{n}_\varepsilon}^2(r) \leq \frac{C|\log\varepsilon|}{\varepsilon}$$

Since $\xi_{\bar{n}_\varepsilon} e^{i\bar{n}_\varepsilon\vartheta}$ is a ground state, it must satisfy the estimate (2.13), and then

$$\frac{C|\log\varepsilon|}{\varepsilon} \geq \int_{\mathcal{B}_1} d\vec{r}\, \left[\bar{n}_\varepsilon - \frac{\Omega(\varepsilon)r^2}{2}\right]^2 \xi_{\bar{n}_\varepsilon}^2(r) \geq \int_{\mathcal{B}_1} d\vec{r}\, \left[\bar{n}_\varepsilon - \frac{\Omega(\varepsilon)r^2}{2}\right]^2 \rho^{\rm TF}(r) - \frac{C\sqrt{|\log\varepsilon|}}{\varepsilon^{\frac{3}{2}}}$$

or

$$\int_{\mathcal{B}_1} d\vec{r}\, \left[\bar{n}_\varepsilon - \frac{\Omega(\varepsilon)r^2}{2}\right]^2 \rho^{\rm TF}(r) \leq \frac{C_{\Omega_0}\sqrt{|\log\varepsilon|}}{\varepsilon^{\frac{3}{2}}} \tag{2.17}$$

where we have used the bound

$$\int_{\mathcal{B}_1} d\vec{r}\, \left[\bar{n}_\varepsilon - \frac{\Omega(\varepsilon)r^2}{2}\right]^2 \left(\xi_{\bar{n}_\varepsilon}^2(r) - \rho^{\rm TF}(r)\right) \geq -\left\|\bar{n}_\varepsilon - \frac{\Omega(\varepsilon)r^2}{2}\right\|^2_{L^\infty(\mathcal{B}_1)} \left\|\xi_{\bar{n}_\varepsilon}^2 - \rho^{\rm TF}\right\|_{L^1(\mathcal{B}_1)} \geq -\frac{C\sqrt{|\log\varepsilon|}}{\varepsilon^{\frac{3}{2}}}.$$

The left hand side of (2.17) can be explicitly calculated: If $\Omega_0 \leq 4/\sqrt{\pi}$, one has

$$\int_{\mathcal{B}_1} d\vec{r}\, \left[\bar{n}_\varepsilon - \frac{\Omega(\varepsilon)r^2}{2}\right]^2 \rho^{\rm TF}(r) = \bar{n}_\varepsilon^2 - \frac{\Omega(\varepsilon)\bar{n}_\varepsilon}{2}\left[1 + \frac{\pi\Omega_0^2}{48}\right] + \frac{\Omega^2(\varepsilon)}{12}\left[1 + \frac{\pi\Omega_0^2}{32}\right]$$

while, if $\Omega_0 > 4/\sqrt{\pi}$,

$$\int_{\mathcal{B}_1} d\vec{r}\, \left[\bar{n}_\varepsilon - \frac{\Omega(\varepsilon)r^2}{2}\right]^2 \rho^{\rm TF}(r) = \bar{n}_\varepsilon^2 - \Omega(\varepsilon)\bar{n}_\varepsilon\left[1 - \frac{4}{3\sqrt{\pi}\Omega_0}\right] + \frac{\Omega^2(\varepsilon)}{2}\left[1 - \frac{8}{3\sqrt{\pi}\Omega_0} + \frac{8}{3\pi\Omega_0^2}\right].$$

By minimizing over $\bar{n}_\varepsilon$, i.e., taking

$$\bar{n}_\varepsilon = \frac{\Omega(\varepsilon)}{4}\left[1 + \frac{\pi\Omega_0^2}{48}\right]$$

in the first case and

$$\bar{n}_\varepsilon = \frac{\Omega(\varepsilon)}{2}\left[1 - \frac{4}{3\sqrt{\pi}\Omega_0}\right]$$

in the second, we get

$$\int_{\mathcal{B}_1} d\vec{r}\, \left[\bar{n}_\varepsilon - \frac{\Omega(\varepsilon)r^2}{2}\right]^2 \rho^{\rm TF}(r) \geq \frac{\Omega^2(\varepsilon)}{72} = \frac{\Omega_0^2}{72\varepsilon^2} \equiv \frac{C'_{\Omega_0}}{\varepsilon^2},$$

if $\Omega_0 \leq 4/\sqrt{\pi}$, and

$$\int_{\mathcal{B}_1} d\vec{r}\, \left[\bar{n}_\varepsilon - \frac{\Omega(\varepsilon)r^2}{2}\right]^2 \rho^{\rm TF}(r) \geq \frac{\Omega^2(\varepsilon)}{4}\left[\frac{1}{3} + \frac{32}{9\pi\Omega_0^2}\right] \equiv \frac{C'_{\Omega_0}}{\varepsilon^2},$$

if $\Omega_0 > 4/\sqrt{\pi}$. Therefore in both cases (2.17) implies that

$$0 < C'_{\Omega_0} \leq C_{\Omega_0}\sqrt{\varepsilon|\log\varepsilon|}$$

for some strictly positive constant C'_{Ω_0}. For ε sufficiently small this is a contradiction and then no symmetric vortex can be a ground state of the GP functional.

□

2.4 The Regime $\Omega(\varepsilon) \gg 1/\varepsilon$

In order to present the results in a transparent way we assume that the angular velocity is a power of $1/\varepsilon$, namely

$$\Omega(\varepsilon) = \frac{\Omega_1}{\varepsilon^{1+\alpha}}$$

for some $\alpha > 0$[8]. In this case the limiting functional is analogous to the one introduced in Section 2.3, provided Ω_0 is replaced by $\Omega_1/\varepsilon^\alpha$ and the energy scale by $\varepsilon^{2\alpha}$, i.e.

$$\mathcal{E}_\varepsilon^{\mathrm{TF}}[\rho] \equiv \varepsilon^{2\alpha} \int_{\mathcal{B}_1} d\vec{r} \left\{ \rho^2 - \frac{\Omega_1^2 r^2 \rho}{4\varepsilon^{2\alpha}} \right\}. \tag{2.18}$$

The ground state energy of this functional, i.e.,

$$E_\varepsilon^{\mathrm{TF}} \equiv \min_{\substack{\rho \in \mathcal{D}^{TF} \\ \int \rho = 1}} \mathcal{E}_\varepsilon^{\mathrm{TF}}[\rho] = \mathcal{E}_\varepsilon^{\mathrm{TF}}[\rho_\varepsilon^{\mathrm{TF}}] \tag{2.19}$$

is given by

$$E_\varepsilon^{\mathrm{TF}} = -\frac{\Omega_1^2}{4}\left(1 - \frac{8\varepsilon^\alpha}{3\sqrt{\pi}\Omega_1}\right) \tag{2.20}$$

and the corresponding minimizer is

$$\rho_\varepsilon^{\mathrm{TF}} = \frac{\Omega_1^2}{8\varepsilon^{2\alpha}} \left[r^2 - R_\varepsilon^2\right]_+ \tag{2.21}$$

where

$$R_\varepsilon \equiv \sqrt{1 - \frac{4\varepsilon^\alpha}{\sqrt{\pi}\Omega_1}}. \tag{2.22}$$

Hence the function $\rho_\varepsilon^{\mathrm{TF}}$ is supported in a very thin layer near the boundary and, as $\varepsilon \to 0$, it converges as a distribution to a radial delta function supported at $r = 1$: If $F(r)$ is a continuous function, one has

$$\int_{\mathcal{B}_1} d\vec{r}\, \rho_\varepsilon^{\mathrm{TF}}(r) F(r) = \frac{\pi\Omega_1^2}{8\varepsilon^{2\alpha}} \int_0^{1-R_\varepsilon^2} dz\, z\, F\left(\sqrt{z + R_\varepsilon^2}\right) = \int_0^1 dz\, z\, F\left(\sqrt{\frac{2\varepsilon^\alpha z}{\sqrt{\pi}\Omega_1} + R_\varepsilon^2}\right) \underset{\varepsilon\to 0}{\longrightarrow} F(1)$$

We can now state the main results for this regime, starting with the energy asymptotics for $\varepsilon \to 0$:

Theorem 2.2 (Energy Asymptotics)
For any $\Omega_1 > 0$, $\alpha > 0$, and for ε sufficiently small

$$\varepsilon^{2+2\alpha}\, E_\varepsilon^{\mathrm{GP}} = E_\varepsilon^{\mathrm{TF}} + O(\varepsilon^{2\alpha}) + O(\varepsilon^2 |\log \varepsilon|). \tag{2.23}$$

The two remainders in the asymptotic estimation of the GP energy $E_\varepsilon^{\mathrm{GP}}$ have different sources: The second one, of the order $|\log\varepsilon|/\varepsilon^{2\alpha}$, is due to the convergence of the density to a delta function and the radial kinetic energy that is ignored in the TF functional. The other term, of order $1/\varepsilon^2$, is due to the approximation of the vortex structure in the region where the density is exponentially small by a trial function with a single vortex located at the origin. It is clear that the estimate (2.12) is not the $\alpha \to 0$ limit of (2.23). We also note that for $\alpha \geq 2$ the last error term in (2.23) is larger than the second term in (2.20).

Since the function $\rho_\varepsilon^{\mathrm{TF}}$ does not converge in any L^p-space, a result analogous to Corollary 2.1 does not hold. We are able to show, however, that the L^2-norm of $\Psi_\varepsilon^{\mathrm{GP}}$ converges to zero almost everywhere, except for a thin region (with a size of order of a suitable power of ε) near the boundary.

[8] Our analysis applies, in fact, to arbitrary angular velocites $\Omega(\varepsilon) \gg 1/\varepsilon$, one just has to replace $\Omega_1/\varepsilon^\alpha$ by $\varepsilon\Omega(\varepsilon)$.

Corollary 2.2 (Density Asymptotics) *If $\Omega_1 > 0$ and ε is sufficiently small,*

$$\left\|\Psi_\varepsilon^{\mathrm{GP}}\right\|^2_{L^2(\mathcal{B}_{R_\varepsilon})} = O(\varepsilon^\alpha) + O(\varepsilon^{2-\alpha}|\log\varepsilon|) \tag{2.24}$$

for $0 < \alpha < 2$, while for $\alpha \geq 2$,

$$\left\|\Psi_\varepsilon^{\mathrm{GP}}\right\|^2_{L^2(\mathcal{B}_{R_{\varepsilon,\beta}})} = O(\varepsilon^{2-\beta}|\log\varepsilon|) \tag{2.25}$$

with $R_{\varepsilon,\beta} = (1-\varepsilon^\beta)^{1/2}$, for any $1 \leq \beta < 2$.

The above estimate is strengthened in the following proposition. The reason why we state Corollary 2.2 separately is the analogy with the previous Corollary 2.1. It is also used in the proof of the following.

Proposition 2.5 (Exponential Smallness of the Density)
Denote

$$\mathcal{T}'_\varepsilon = \left\{\vec{r} \in \mathcal{B}_1 \,\middle|\, r \leq 1 - \varepsilon^{\alpha'/4}\right\} \tag{2.26}$$

and

$$\mathcal{T}''_\varepsilon = \left\{\vec{r} \in \mathcal{B}_1 \,\middle|\, r \leq 1 - \varepsilon^{(2-\beta)/4}\right\} \tag{2.27}$$

where $\alpha' = \min[\alpha, 2-\alpha]$ and β is any number such that $1 \leq \beta < 2$.
For any $\Omega_1 > 0$ there exist constants C_{Ω_1} and C'_{Ω_1}, such that for ε sufficiently small,

$$\left|\Psi_\varepsilon^{\mathrm{GP}}(\vec{r})\right|^2 \leq C_{\Omega_1}\varepsilon^{\alpha'/3}|\log\varepsilon| \,\exp\left[-\frac{C'_{\Omega_1}\mathrm{dist}(\vec{r},\partial\mathcal{T}'_\varepsilon)^2}{\varepsilon^{1+\frac{\alpha'}{2}}}\right] \tag{2.28}$$

if $0 < \alpha < 2$ and $\vec{r} \in \mathcal{T}'_\varepsilon$, and

$$\left|\Psi_\varepsilon^{\mathrm{GP}}(\vec{r})\right|^2 \leq C_{\Omega_1}\varepsilon^{(2-\beta)/3}|\log\varepsilon| \,\exp\left[-\frac{C'_{\Omega_1}\mathrm{dist}(\vec{r},\partial\mathcal{T}''_\varepsilon)^2}{\varepsilon^{1+\frac{\alpha}{2}}}\right] \tag{2.29}$$

if $\alpha \geq 2$ and $\vec{r} \in \mathcal{T}''_\varepsilon$.

A straightforward consequence of the above estimates together with the normalization of $\Psi_\varepsilon^{\mathrm{GP}}$ is that the density of any minimizer of the GP functional converges to $\delta(1-r)$ in a distributional sense, in accord with the discussion in [FB].

Another important difference compared to the regime $\Omega(\varepsilon) \sim 1/\varepsilon$ is the form of the trial function (3.36) used in the proof of Theorem 2.2. This function is an eigenfunction of the angular momentum, i.e., the whole vorticity is concentrated at the origin. On the other hand, Proposition 2.2 implies that the true minimizer cannot be an eigenfunction of the angular momentum, at least as long as $\Omega(\varepsilon) \lesssim 1/\varepsilon$. Nevertheless we expect that the number of vortices contained in the region where $\Psi_\varepsilon^{\mathrm{GP}}$ is not exponentially small is negligible compared to the total vorticity of the function (see, e.g., the numerical simulations contained in [KTU]). A wave function of this kind is often referred to in the physics literature as a "giant vortex". Since the vortex contribution to the energy depends essentially only on the winding number at the boundary of the thin region where the wave function of the condensate is not exponentially small, a trial function with the vorticity concentrated at the origin can lead to a good approximation to the energy.
This behavior is also suggested by the fact that the minimization of a modified GP functional over the subspace of functions with fixed angular momentum with a subsequent minimization over the value of the angular momentum gives a ground state energy with the same leading order asymptotics as $E_\varepsilon^{\mathrm{TF}}$ and $E_\varepsilon^{\mathrm{GP}}$. Indeed, if we define (c.f. [FB])

$$\mathcal{E}^{\mathrm{TF}\prime}_{\varepsilon,\nu}[\rho] \equiv \int_{\mathcal{B}_1} d\vec{r}\,\left\{\frac{\nu^2\rho}{r^2} - \Omega_1\nu\rho + \varepsilon^{2\alpha}\rho^2\right\} \tag{2.30}$$

and

$$E_\varepsilon^{\mathrm{TF}\prime} \equiv \min_{\nu\in\mathbb{R}^+} \min_{\substack{\rho\in\mathcal{D}^{TF\prime} \\ \int\rho = 1}} \mathcal{E}^{\mathrm{TF}\prime}_{\varepsilon,\nu}[\rho] \tag{2.31}$$

where $\mathcal{D}^{TF'}$ is the natural domain for the functional (2.30), then it is not hard to see that

$$\lim_{\varepsilon\to 0} E_\varepsilon^{\mathrm{TF}'} = \lim_{\varepsilon\to 0} E_\varepsilon^{\mathrm{TF}} = \lim_{\varepsilon\to 0} \varepsilon^{2+2\alpha} E_\varepsilon^{\mathrm{GP}} = -\frac{\Omega_1^2}{4} \tag{2.32}$$

even though $E_\varepsilon^{\mathrm{TF}'} > E_\varepsilon^{\mathrm{TF}}$ for any $\varepsilon > 0$. The functional (2.30) is obtained from (1.5) by neglecting the radial part of the kinetic energy and restricting it to eigenfunctions with fixed angular momentum[9] $\nu/\varepsilon^{1+\alpha}$. In other words the TF functional (2.30) describes the asymptotic behavior of the GP functional almost as well as (2.18).

However, while the heuristic discussion in [FB] suggests that a giant vortex occurs only for angular velocities larger than $1/(\varepsilon^2|\log\varepsilon|)$, we found no evidence in our rigorous analysis that a change in the minimizer occurs above that threshold. In fact such a critical angular velocity for the transition to the giant vortex is estimated in [FB] by imposing the condition that the ground state energy $E_\varepsilon^{\mathrm{TF}'}$ equals, to leading order, a certain upper bound for $\varepsilon^{2+2\alpha}E_\varepsilon^{\mathrm{GP}}$. This upper bound, however, is calculated in [FB] with a trial function of the form (3.2) and is not optimal. In fact, the same comparison with a better upper bound[10] for $\varepsilon^{2+2\alpha}E_\varepsilon^{\mathrm{GP}}$ gives the correct answer, namely that $\varepsilon^{2+2\alpha}E_\varepsilon^{\mathrm{GP}}$ is close to $E_\varepsilon^{\mathrm{TF}'}$ for any $\alpha > 0$ (see (2.32)). Hence the transition to the giant vortex should occur for any angular velocity $\Omega(\varepsilon)$ of order higher than $1/\varepsilon$.

3 Proofs

3.1 The Regime $\Omega(\varepsilon) \sim 1/\varepsilon$

The main result concerning the regime $\Omega(\varepsilon) \sim 1/\varepsilon$ is Theorem 2.1 and we start by proving it. Some technical but crucial details of the proof are contained in Subsection 3.1.1 (Proposition 3.1 and Theorem 3.1), where we present some estimates for the kinetic energy of the trial function.

Proof of Theorem 2.1
We are going to prove the result by comparing an upper bound for the ground state energy with a suitable lower bound.

Lower Bound: The lower bound for $E_\varepsilon^{\mathrm{GP}}$ is actually trivial. By simply neglecting the positive contribution of the magnetic kinetic energy in (1.5) we immediately get

$$\mathcal{E}^{\mathrm{GP}}[\Psi] \geq \frac{\mathcal{E}^{\mathrm{TF}}[|\Psi|^2]}{\varepsilon^2} \geq \frac{E^{\mathrm{TF}}}{\varepsilon^2}. \tag{3.1}$$

Upper Bound: We prove the upper bound by testing the functional on a trial function of the following form

$$\tilde{\Psi}(\vec{r}) = c_\varepsilon f_\varepsilon(r)\chi_\varepsilon(\vec{r})g_\varepsilon(\vec{r}). \tag{3.2}$$

The radial part is given by

$$f_\varepsilon(r) = \begin{cases} \sqrt{\rho^{\mathrm{TF}}} & \text{if} \quad \Omega_0 \leq \dfrac{4}{\sqrt{\pi}} \\[2ex] j_\varepsilon\sqrt{\rho^{\mathrm{TF}}} & \text{if} \quad \Omega_0 > \dfrac{4}{\sqrt{\pi}} \end{cases} \tag{3.3}$$

where j_ε is a suitable cut-off function to regularize $\sqrt{\rho^{\mathrm{TF}}}$ at the boundary of the hole. Our choice is

$$j_\varepsilon(r) = \begin{cases} 0 & \text{if} \quad r \leq R_0 \\[2ex] \dfrac{r-R_0}{\varepsilon} & \text{if} \quad R_0 \leq r \leq R_0+\varepsilon \\[2ex] 1 & \text{otherwise.} \end{cases} \tag{3.4}$$

[9] More precisely, the correct value of the angular momentum should be the integer part of $\nu/\varepsilon^{1+\alpha}$ but the difference between these two quantities produces a correction of smaller order in (2.31).

[10] For instance the one calculated using the trial function (3.36) in the proof of Theorem 2.2.

The function g_ε is a phase factor that can be expressed in complex coordinates $z = x + iy$ as

$$g_\varepsilon(z) = \prod_{i\in\mathcal{L}} \frac{z - z_i}{|z - z_i|} \tag{3.5}$$

where $\mathcal{L}$ is a square lattice of spacing ℓ_ε defined in the following way:

$$\mathcal{L} = \left\{ \vec{r} = (m\ell_\varepsilon, n\ell_\varepsilon),\ \ m, n \in \mathbb{Z} \,\middle|\, r \le 1 - 2\sqrt{2}\ell_\varepsilon \right\}. \tag{3.6}$$

We assume that the spacing is of order $\sqrt{\varepsilon}$, i.e. $\ell_\varepsilon = \delta\sqrt{\varepsilon}$, for some $\delta > 0$ independent of ε, so that the number of lattice points, denoted by N_ε, is proportional to $1/\varepsilon$.
We note that the phase g_ε carries vortices of degree 1 centered at the lattice points. Moreover, each vortex core is contained inside the fundamental cell and its radius is of order $\sqrt{\varepsilon}$. This choice is suggested by previous works (see e.g. [BBH2, IM1, IM2]) on rotating condensates, where it is shown that vortices of higher degree than 1 are energetically unfavorable.
Since g_ε is not differentiable at the points of the lattice we need to multiply it by a function, χ_ε, that vanishes at these points and we take

$$\chi_\varepsilon(\vec{r}) = \begin{cases} 1 & \text{if} \quad |\vec{r} - \vec{r}_i| \ge \varepsilon^\eta \\ \dfrac{|\vec{r} - \vec{r}_i|}{\varepsilon^\eta} & \text{if} \quad |\vec{r} - \vec{r}_i| \le \varepsilon^\eta \end{cases} \tag{3.7}$$

for some $\eta > 1/2$.
Finally the constant c_ε is fixed by the normalization condition and it can be easily checked that, for ε sufficiently small,

$$1 \le c_\varepsilon^2 \le 1 + C\varepsilon. \tag{3.8}$$

Setting

$$\Lambda = \mathcal{B}_1 \backslash \bigcup_{i\in\mathcal{L}} \mathcal{B}_\varepsilon^i \tag{3.9}$$

where $\mathcal{B}_\varepsilon^i$ is a ball of radius ε^η centered at $\vec{r}_i$, the functional evaluated on the trial function (3.2) is given by

$$\begin{aligned}
\mathcal{E}^{\mathrm{GP}}[\tilde{\Psi}] &= c_\varepsilon^2 \int_\Lambda d\vec{r}\, |\nabla f_\varepsilon|^2 + c_\varepsilon^2 \int_\Lambda d\vec{r}\, f_\varepsilon^2 \left| \left(\nabla - i\vec{A}_\varepsilon\right) g_\varepsilon \right|^2 + \\
&+ \int_{\cup_{i\in\mathcal{L}}\mathcal{B}_\varepsilon^i} d\vec{r} \left| \left(\nabla - i\vec{A}_\varepsilon\right) \tilde{\Psi} \right|^2 + \frac{\mathcal{E}^{\mathrm{TF}}[|\tilde{\Psi}|^2]}{\varepsilon^2} = \\
&= c_\varepsilon^2 \int_\Lambda d\vec{r}\, |\nabla f_\varepsilon|^2 + c_\varepsilon^2 \int_\Lambda d\vec{r}\, f_\varepsilon^2 \left| \left(\nabla - i\vec{A}_\varepsilon\right) g_\varepsilon \right|^2 + c_\varepsilon^2 \int_{\cup_{i\in\mathcal{L}}\mathcal{B}_\varepsilon^i} d\vec{r}\, |\nabla (\chi_\varepsilon f_\varepsilon)|^2 + \\
&+ c_\varepsilon^2 \int_{\cup_{i\in\mathcal{L}}\mathcal{B}_\varepsilon^i} d\vec{r}\, \chi_\varepsilon^2 f_\varepsilon^2 \left| \left(\nabla - i\vec{A}_\varepsilon\right) g_\varepsilon \right|^2 + \frac{\mathcal{E}^{\mathrm{TF}}[|\tilde{\Psi}|^2]}{\varepsilon^2} \le \\
&\le c_\varepsilon^2 \int_{\mathcal{B}_1} d\vec{r}\, |\nabla f_\varepsilon|^2 + c_\varepsilon^2 \int_\Lambda d\vec{r}\, f_\varepsilon^2 \left| \left(\nabla - i\vec{A}_\varepsilon\right) g_\varepsilon \right|^2 + c_\varepsilon^2 \int_{\cup_{i\in\mathcal{L}}\mathcal{B}_\varepsilon^i} d\vec{r}\, \chi_\varepsilon^2 f_\varepsilon^2 \left| \left(\nabla - i\vec{A}_\varepsilon\right) g_\varepsilon \right|^2 + \\
&+ \frac{\mathcal{E}^{\mathrm{TF}}[|\tilde{\Psi}|^2]}{\varepsilon^2} + \frac{C}{\varepsilon} \le \\
&\le c_\varepsilon^2 \int_{\mathcal{B}_1} d\vec{r}\, |\nabla f_\varepsilon|^2 + C_1 \int_\Lambda d\vec{r} \left| \left(\nabla - i\vec{A}_\varepsilon\right) g_\varepsilon \right|^2 + C_2 \int_{\cup_{i\in\mathcal{L}}\mathcal{B}_\varepsilon^i} d\vec{r}\, \chi_\varepsilon^2 |\nabla g_\varepsilon|^2 + \\
&+ C_3 \int_{\cup_{i\in\mathcal{L}}\mathcal{B}_\varepsilon^i} d\vec{r} \left| \vec{A}_\varepsilon \right|^2 + \frac{\mathcal{E}^{\mathrm{TF}}[|\tilde{\Psi}|^2]}{\varepsilon^2} + \frac{C}{\varepsilon} \le
\end{aligned}$$

$$\leq c_\varepsilon^2 \int_{\mathcal{B}_1} d\vec{r}\, |\nabla f_\varepsilon|^2 + C_1 \int_\Lambda d\vec{r}\, \left|\left(\nabla - i\vec{A}_\varepsilon\right) g_\varepsilon\right|^2 + C_2 \int_{\cup_{i\in\mathcal{L}}\mathcal{B}_\varepsilon^i} d\vec{r}\, \chi_\varepsilon^2 \, |\nabla g_\varepsilon|^2 + \\ + \frac{\mathcal{E}^{\mathrm{TF}}[|\tilde{\Psi}|^2]}{\varepsilon^2} + \frac{C_4}{\varepsilon^{3-4\eta}} + \frac{C}{\varepsilon}$$

where we have used the uniform boundedness of f_ε, the estimate (3.8), and the fact that the number of lattice points N_ε is bounded by C/ε.
The gradient of the phase g_ε can be bounded from above inside any ball $\mathcal{B}_\varepsilon^i$:

$$|\nabla g_\varepsilon| \leq \sum_{j\in\mathcal{L}} \frac{1}{|\vec{r}-\vec{r}_j|} \leq \frac{1}{|\vec{r}-\vec{r}_i|} + \frac{N_\varepsilon}{\inf_{j\neq i} |\vec{r}-\vec{r}_j|} \leq \frac{1}{|\vec{r}-\vec{r}_i|} + \frac{N_\varepsilon}{\ell_\varepsilon} \tag{3.10}$$

for any $\vec{r} \in \mathcal{B}_\varepsilon^i$, so that

$$\int_{\cup_{i\in\mathcal{L}}\mathcal{B}_\varepsilon^i} d\vec{r}\, \chi_\varepsilon^2 \, |\nabla g_\varepsilon|^2 \leq \frac{C\left|\cup_{i\in\mathcal{L}}\mathcal{B}_\varepsilon^i\right|}{\varepsilon^{2\eta}} + \frac{C'\left|\cup_{i\in\mathcal{L}}\mathcal{B}_\varepsilon^i\right|}{\varepsilon^3} \leq \frac{C}{\varepsilon} + \frac{C'}{\varepsilon^{4-2\eta}}$$

and hence

$$\mathcal{E}^{\mathrm{GP}}[\tilde{\Psi}] \leq \int_{\mathcal{B}_1} d\vec{r}\, |\nabla f_\varepsilon|^2 + C_1 \int_\Lambda d\vec{r}\, \left|\left(\nabla - i\vec{A}_\varepsilon\right) g_\varepsilon\right|^2 + \frac{\mathcal{E}^{\mathrm{TF}}[|\tilde{\Psi}|^2]}{\varepsilon^2} + \frac{C_2}{\varepsilon}$$

for any $\eta > 3/2$.
Moreover, the radial part of the kinetic energy can be bounded by a constant if $\Omega_0 \leq \frac{4}{\sqrt{\pi}}$, and by

$$\int_{\mathcal{B}_1} d\vec{r}\, |\nabla f_\varepsilon|^2 \leq C_1 \int_{\mathcal{B}_1} d\vec{r}\, |\nabla j_\varepsilon|^2 + C_2 \int_{R_0}^1 dr\, r\, \frac{j_\varepsilon^2}{\rho^{\mathrm{TF}}} \leq C_3 + C_4 \int_{R_0+\varepsilon}^1 dr\, \frac{r}{r^2 - R_0^2} \leq C|\log\varepsilon|$$

if $\Omega_0 > \frac{4}{\sqrt{\pi}}$. Then we get

$$\mathcal{E}^{\mathrm{GP}}[\tilde{\Psi}] \leq C_1 \int_\Lambda d\vec{r}\, \left|\left(\nabla - i\vec{A}_\varepsilon\right) g_\varepsilon\right|^2 + \frac{\mathcal{E}^{\mathrm{TF}}[|\tilde{\Psi}|^2]}{\varepsilon^2} + \frac{C_2}{\varepsilon}$$

for a possibly different constant C_2.
The upper bound[11] now follows using Proposition 3.1 and Theorem 3.1 in the next section, choosing $\delta = \sqrt{\frac{2\pi}{\Omega_0}}$ and $5/2 < \eta < \infty$:

$$\mathcal{E}^{\mathrm{GP}}[\tilde{\Psi}] \leq \frac{E^{\mathrm{TF}}}{\varepsilon^2} + \frac{C_{\Omega_0}|\log\varepsilon|}{\varepsilon}. \tag{3.11}$$

□

Remark 3.1 (Dirichlet Problem)
The proof in the case of Dirichlet boundary conditions, i.e., if (1.2) is replaced with $H_0^1(\mathcal{B}_1)$, looks exactly the same: The trial function has simply to be multiplied by a cut-off function, which is 1 everywhere except for a very thin region in the neighborhood of the boundary where it goes to 0, in order to satisfy the required boundary conditions. The error coming from such a cut-off function can then be included in the remainder in (3.11).

Proof of Corollary 2.1
Let us first consider the case $\Omega_0 < \frac{4}{\sqrt{\pi}}$. Using the explicit form of the TF minimizer ρ^{TF} (see (2.8)) and the estimate (3.11), one can calculate

$$\int_{\mathcal{B}_1} d\vec{r}\, \left(|\Psi_\varepsilon^{\mathrm{GP}}|^2 - \rho^{\mathrm{TF}}\right)^2 = \mathcal{E}^{\mathrm{TF}}[|\Psi_\varepsilon^{\mathrm{GP}}|^2] - \frac{2}{\pi} + \frac{\Omega_0^2}{8} + \int_{\mathcal{B}_1} d\vec{r}\, \left(\rho^{\mathrm{TF}}\right)^2 \leq$$

[11] Note that the constant C_{Ω_0} actually depends linearly on η (see the proof of Lemma 3.1, in particular Eq. (3.32)).

$$\leq E^{\rm TF} - \frac{2}{\pi} + \frac{\Omega_0^2}{8} + \int_{\mathcal{B}_1} d\vec{r}\, \left(\rho^{\rm TF}\right)^2 + C_{\Omega_0}\varepsilon|\log\varepsilon| = C_{\Omega_0}\varepsilon|\log\varepsilon|$$

and then the L^1-bound follows from Schwarz's inequality.
On the other hand, if $\Omega_0 \geq \frac{4}{\sqrt{\pi}}$, one has

$$\int_{\mathcal{D}_0} d\vec{r}\, \left(|\Psi_\varepsilon^{\rm GP}|^2 - \rho^{\rm TF}\right)^2 = \mathcal{E}^{\rm TF}[|\Psi_\varepsilon^{\rm GP}|^2] - E^{\rm TF} - \int_{\mathcal{B}_1\setminus\mathcal{D}_0} d\vec{r}\, |\Psi_\varepsilon^{\rm GP}|^4 +$$

$$+\frac{\Omega_0^2}{4}\int_{\mathcal{B}_1\setminus\mathcal{D}_0} d\vec{r}\,(r^2 - R_0^2)|\Psi_\varepsilon^{\rm GP}|^2 \leq \mathcal{E}^{\rm TF}[|\Psi_\varepsilon^{\rm GP}|^2] - E^{\rm TF} \leq C_{\Omega_0}\varepsilon|\log\varepsilon|$$

where we have used again (2.8) and (3.11). From the same inequality one also has

$$\int_{\mathcal{D}_0} d\vec{r}\, \left(|\Psi_\varepsilon^{\rm GP}|^2 - \rho^{\rm TF}\right)^2 + \int_{\mathcal{B}_1\setminus\mathcal{D}_0} d\vec{r}\, |\Psi_\varepsilon^{\rm GP}|^4 = \mathcal{E}^{\rm TF}[|\Psi_\varepsilon^{\rm GP}|^2] - E^{\rm TF} + \frac{\Omega_0^2}{4}\int_{\mathcal{B}_1\setminus\mathcal{D}_0} d\vec{r}\,(r^2 - R_0^2)|\Psi_\varepsilon^{\rm GP}|^2 \leq$$

$$\leq \mathcal{E}^{\rm TF}[|\Psi_\varepsilon^{\rm GP}|^2] - E^{\rm TF} \leq C_{\Omega_0}\varepsilon|\log\varepsilon|,$$

so that

$$\int_{\mathcal{B}_1\setminus\mathcal{D}_0} d\vec{r}\, |\Psi_\varepsilon^{\rm GP}|^4 \leq C_{\Omega_0}\varepsilon|\log\varepsilon|. \tag{3.12}$$

□

Proof of Proposition 2.4
The proof is similar to the one of Proposition 2.5 in [AAB].
The variational equation satisfied by $\Psi_\varepsilon^{\rm GP}$ is

$$-\Delta\Psi_\varepsilon^{\rm GP} - \frac{\Omega_0}{\varepsilon}L\Psi_\varepsilon^{\rm GP} + \frac{2}{\varepsilon^2}|\Psi_\varepsilon^{\rm GP}|^2\Psi_\varepsilon^{\rm GP} = \mu_\varepsilon\Psi_\varepsilon^{\rm GP} \tag{3.13}$$

where the chemical potential μ_ε is fixed by the L^2-normalization of $\Psi_\varepsilon^{\rm GP}$:

$$\mu_\varepsilon = E_\varepsilon^{\rm GP} + \frac{\|\Psi_\varepsilon^{\rm GP}\|_4^4}{\varepsilon^2}. \tag{3.14}$$

Setting $U_\varepsilon \equiv |\Psi_\varepsilon^{\rm GP}|^2$ and using the simple estimate

$$\frac{\Omega_0}{\varepsilon}\left|\Psi_\varepsilon^{{\rm GP}*}L\Psi_\varepsilon^{\rm GP}\right| \leq \left|\nabla\Psi_\varepsilon^{\rm GP}\right|^2 + \frac{\Omega_0^2 r^2|\Psi_\varepsilon^{\rm GP}|^2}{4\varepsilon^2}$$

one can easily check that

$$-\frac{1}{2}\Delta U_\varepsilon \leq \left[\frac{\Omega_0^2 r^2}{4} + \varepsilon^2\mu_\varepsilon - 2U_\varepsilon\right]\frac{U_\varepsilon}{\varepsilon^2}.$$

Moreover, thanks to Theorem 2.1 and Corollary 2.1,

$$\varepsilon^2\mu_\varepsilon \leq E^{\rm TF} + \|\rho^{\rm TF}\|_2^2 + C_{\Omega_0}\sqrt{\varepsilon|\log\varepsilon|}$$

so that

$$-\frac{1}{2}\Delta U_\varepsilon \leq \left[\frac{\Omega_0^2(r^2 - R_0^2)}{4} - 2U_\varepsilon + C_{\Omega_0}\sqrt{\varepsilon|\log\varepsilon|}\right]\frac{U_\varepsilon}{\varepsilon^2}.$$

If we define

$$\tilde{\mathcal{T}}_\varepsilon \equiv \left\{\vec{r}\in\mathcal{B}_1 \,\middle|\, r \leq R_0 - \frac{\varepsilon^{\frac{1}{3}}}{2}\right\}$$

then, for ε sufficiently small, the function U_ε is subharmonic in $\tilde{\mathcal{T}}_\varepsilon$ and therefore, for any point $\vec{r}\in\tilde{\mathcal{T}}_\varepsilon$ with $\mathrm{dist}(\vec{r},\partial\tilde{\mathcal{T}}_\varepsilon) \geq \varrho$,

$$U_\varepsilon(\vec{r}) \leq \frac{1}{\pi\varrho^2}\int_{\mathcal{B}_\varrho(\vec{r})} d\vec{x}\, U_\varepsilon(\vec{x}) \leq \frac{C\|U_\varepsilon\|_{L^2(\mathcal{B}_1\setminus\mathcal{D}_0)}}{\varrho}.$$

Hence, using the estimate (3.12) and choosing, for instance, $\varrho = \varepsilon^{\frac{1}{3}}/2$, we can conclude that

$$U_\varepsilon(\vec{r}) \leq C_{\Omega_0}\varepsilon^{\frac{1}{6}}\sqrt{|\log\varepsilon|}$$

for any $\vec{r} \in \mathcal{T}_\varepsilon$.
Let us now define

$$U'_\varepsilon \equiv \frac{U_\varepsilon}{C_{\Omega_0}\varepsilon^{\frac{1}{6}}\sqrt{|\log\varepsilon|}}.$$

For any $\vec{r} \in \mathcal{T}_\varepsilon$,

$$-\Delta U'_\varepsilon + \frac{CU'_\varepsilon}{\varepsilon^{\frac{4}{3}}} \leq 0$$

and

$$0 \leq U'_\varepsilon \leq 1$$

i.e., U'_ε is a subsolution in $\mathcal{T}_\varepsilon$ of

$$\begin{cases} -\Delta u + \dfrac{Cu}{\varepsilon^{\frac{4}{3}}} = 0 \\ \\ u(\partial\mathcal{T}_\varepsilon) = 1. \end{cases}$$

On the other hand it is not so hard to verify (see e.g. Lemma 2 in [BBH1]) that the function

$$\exp\left\{\frac{\sqrt{C}\left[r^2 - (R_0 - \varepsilon^{\frac{1}{3}})^2\right]}{4\varepsilon^{\frac{2}{3}}(R_0 - \varepsilon^{\frac{1}{3}})}\right\}$$

is a supersolution for the same problem if

$$\varepsilon^{\frac{2}{3}} \leq \frac{3(R_0 - \varepsilon^{\frac{1}{3}})}{4}$$

and hence for any ε sufficiently small. The result now follows from the comparison principle.

□

3.1.1 Technical Estimates

In this section we want to present some estimates involving the function (3.2). We start by stating a simple but important result:

Proposition 3.1 (Upper Bound on the TF Energy)
Let $\tilde{\Psi}$ be the function defined in (3.2)*, with $\eta > 1$ and $\Omega_0 > 0$, then, for ε sufficiently small,*

$$\mathcal{E}^{\mathrm{TF}}[|\tilde{\Psi}|^2] \leq E^{\mathrm{TF}} + C_{\Omega_0}\varepsilon. \tag{3.15}$$

Proof: A simple estimate using (3.8) shows that

$$\mathcal{E}^{\mathrm{TF}}[|\tilde{\Psi}|^2] - \mathcal{E}^{\mathrm{TF}}[\rho^{\mathrm{TF}}] \leq \int_\Lambda d\vec{r}\left\{\left[c_\varepsilon^4 f_\varepsilon^4 - (\rho^{\mathrm{TF}})^2\right] - \frac{\Omega_0^2 r^2\left[c_\varepsilon^2 f_\varepsilon^2 - \rho^{\mathrm{TF}}\right]}{4}\right\} +$$

$$+C\varepsilon^{2\eta-1} \leq C'\left\|c_\varepsilon^2 f_\varepsilon^2 - \rho^{\mathrm{TF}}\right\|_{L^1(\mathcal{B}_1)} + C\varepsilon^{2\eta-1}$$

where Λ is the region defined in (3.9) and we have used the fact that ρ^{TF} and $\tilde{\Psi}$ are both uniformly bounded and the area $\left|\cup_{i\in\mathcal{L}}\mathcal{B}_\varepsilon^i\right|$ is bounded by $C\varepsilon^{2\eta-1}$.
Now

$$\left\|c_\varepsilon^2 f_\varepsilon^2 - \rho^{\mathrm{TF}}\right\|_{L^1(\mathcal{B}_1)} \leq \int_\Lambda d\vec{r}\,\left|\rho^{\mathrm{TF}} - c_\varepsilon^2 f_\varepsilon^2\right| + C\varepsilon^{2\eta-1}$$

and, setting $d_\varepsilon = c_\varepsilon^2 - 1 \leq C\varepsilon$, the first term on the right-hand side is bounded by

$$d_\varepsilon \int_{\mathcal{B}_1} d\vec{r}\, \rho^{\mathrm{TF}} \leq C_{\Omega_0}\varepsilon$$

if $\Omega_0 \leq \frac{4}{\sqrt{\pi}}$. On the other hand, if $\Omega_0 \geq \frac{4}{\sqrt{\pi}}$,

$$\int_\Lambda d\vec{r}\, \left|\rho^{\mathrm{TF}} - c_\varepsilon^2 f_\varepsilon^2\right| \leq d_\varepsilon \int_{\mathcal{D}_0} d\vec{r}\, \rho^{\mathrm{TF}} + C_{\Omega_0}\varepsilon$$

where the last term is due to the cut-off function j_ε and $\mathcal{D}_0$ is defined in (2.11).

□

The main result contained in this Section is the following

Theorem 3.1 (Upper Bound on the Vortex Contribution)
Let g_ε be the function defined in (3.5), $\ell_\varepsilon = \delta\sqrt{\varepsilon}$ *and* $\Omega_0 > 0$. *There exists a constant $C_{\Omega_0,\delta}$ independent of ε such that for ε sufficiently small and* $5/2 < \eta < \infty$

$$\int_\Lambda d\vec{r}\, \left|\left(\nabla - i\vec{A}_\varepsilon\right) g_\varepsilon\right|^2 \leq \frac{\pi}{2\varepsilon^2}\left(\frac{\Omega_0}{2} - \frac{\pi}{\delta^2}\right)^2 + \frac{C_{\Omega_0,\delta}|\log\varepsilon|}{\varepsilon} \tag{3.16}$$

where Λ (depending on ε^η) is defined in (3.9).

Remark 3.2 (Vortex Lattice)
As far as the leading order of the GP energy is concerned, the vortex structure of the minimizer $\Psi_\varepsilon^{\mathrm{GP}}$ is not so important: The choice of a regular square lattice in (3.6) is just the simplest for computational purposes but the result in Theorem 3.1 is expected to hold for any trial function with vortices on a regular lattice, provided that δ^2 in (3.16) is replaced with the volume of the rescaled fundamental cell, which is the relevant parameter in the estimate.

Proof: Expanding the expression in (3.16), we get

$$\int_\Lambda d\vec{r}\, \left|\left(\nabla - i\vec{A}_\varepsilon\right) g_\varepsilon\right|^2 = \int_\Lambda d\vec{r}\, |\nabla g_\varepsilon|^2 + \frac{i\Omega_0}{\varepsilon}\int_\Lambda d\vec{r}\, g_\varepsilon^*\left(\vec{r}\times\nabla g_\varepsilon\right) + \tag{3.17}$$
$$+\frac{\Omega_0^2}{4\varepsilon^2}\int_\Lambda d\vec{r}\, r^2|g_\varepsilon|^2.$$

The last term can be easily bounded from above by

$$\frac{\Omega_0^2}{4\varepsilon^2}\int_{\mathcal{B}_1} d\vec{r}\, r^2 = \frac{\pi\Omega_0^2}{8\varepsilon^2}.$$

Using the fact that $g_\varepsilon = e^{i\phi}$, where

$$\phi(\vec{r}) = \sum_{i\in\mathcal{L}} \arctan\left[\frac{y-y_i}{x-x_i}\right] \tag{3.18}$$

the second term can be explicitly calculated: By applying Stokes's theorem,

$$\frac{i\Omega_0}{\varepsilon}\int_\Lambda d\vec{r}\, g_\varepsilon^*\left(\vec{r}\times\nabla g_\varepsilon\right) = -\frac{\Omega_0}{\varepsilon}\int_\Lambda d\vec{r}\, \vec{r}\times\nabla\phi = -\frac{\Omega_0}{2\varepsilon}\int_\Lambda d\vec{r}\, \nabla\times\left(r^2\nabla\phi\right) =$$
$$= -\frac{\Omega_0}{2\varepsilon}\int_{\partial\mathcal{B}_1} d\vec{s}\cdot\nabla\phi + \frac{\Omega_0}{2\varepsilon}\sum_{i\in\mathcal{L}}\int_{\partial\mathcal{B}_\varepsilon^i} d\vec{s}\cdot r^2\nabla\phi =$$
$$= -\frac{\pi\Omega_0 N_\varepsilon}{\varepsilon} + \frac{\pi\Omega_0}{\varepsilon}\sum_{i\in\mathcal{L}} r_i^2 + \frac{\Omega_0}{2\varepsilon}\sum_{i\in\mathcal{L}}\int_{\partial\mathcal{B}_\varepsilon^i} d\vec{s}\left(r^2 - r_i^2\right)\cdot\nabla\phi. \tag{3.19}$$

Since for any $\vec{r} \in \partial\mathcal{B}^i_\varepsilon$,

$$\left|r^2 - r_i^2\right| \leq C\varepsilon^\eta$$

the last term in the expression above can easily be bounded by

$$\left|\frac{\Omega_0}{2\varepsilon}\sum_{i\in\mathcal{L}}\int_{\partial\mathcal{B}^i_\varepsilon} d\vec{s}\,\left(r^2 - r_i^2\right)\cdot\,\nabla\phi\right| \leq \frac{C_{\Omega_0}N_\varepsilon}{\varepsilon^{1-\eta}}\int_{\partial\mathcal{B}^i_\varepsilon} d\vec{s}\,|\nabla\phi| \leq \frac{C_{\Omega_0}N_\varepsilon^2}{\varepsilon^{1-\eta}}$$

where we have used the estimate (3.10)

$$|\nabla\phi(\vec{r})| \leq \sum_{i\in\mathcal{L}}\frac{1}{|\vec{r}-\vec{r_i}|} \leq \frac{N_\varepsilon}{\inf_{i\in\mathcal{L}}|\vec{r}-\vec{r_i}|} \leq \frac{N_\varepsilon}{\varepsilon^\eta}. \tag{3.20}$$

Since the lattice spacing ℓ_ε is chosen to be equal to $\delta\sqrt{\varepsilon}$, the number of lattice points satisfies the bound

$$N_\varepsilon \leq \frac{\pi\left(1-\frac{3\ell_\varepsilon}{\sqrt{2}}\right)^2}{\ell_\varepsilon^2} \leq \frac{C_\delta}{\varepsilon} \tag{3.21}$$

and then

$$\left|\frac{\Omega_0}{2\varepsilon}\sum_{i\in\mathcal{L}}\int_{\partial\mathcal{B}^i_\varepsilon} d\vec{s}\,\left(r^2 - r_i^2\right)\cdot\,\nabla\phi\right| \leq \frac{C_{\Omega_0,\delta}}{\varepsilon^{3-\eta}}. \tag{3.22}$$

Moreover the sum appearing in (3.19) can be replaced by the integral over $\mathcal{B}_1$: let $\mathcal{Q}_\varepsilon$ and $\mathcal{Q}^i_\varepsilon$ be the fundamental cell centered at the origin and at $\vec{r}_i$ respectively,

$$r_i^2 - \frac{1}{\ell_\varepsilon^2}\int_{\mathcal{Q}^i_\varepsilon} d\vec{r}\,r^2 = \frac{1}{\ell_\varepsilon^2}\int_{\mathcal{Q}_\varepsilon} d\vec{r}\,r^2 = \frac{\ell_\varepsilon^2}{6}$$

so that, setting $\mathcal{A}_\varepsilon \equiv \mathcal{B}_1 \setminus (\cup_{i\in\mathcal{L}}\mathcal{Q}^i_\varepsilon)$,

$$\sum_{i\in\mathcal{L}} r_i^2 = \frac{1}{\ell_\varepsilon^2}\int_{\cup_{i\in\mathcal{L}}\mathcal{Q}^i_\varepsilon} d\vec{r}\,r^2 + \frac{N_\varepsilon\ell_\varepsilon^2}{6} \leq \frac{1}{\ell_\varepsilon^2}\int_{\mathcal{B}_1} d\vec{r}\,r^2 - \frac{1}{\ell_\varepsilon^2}\int_{\mathcal{A}_\varepsilon} d\vec{r}\,r^2 + C \leq$$

$$\leq \frac{\pi}{2\ell_\varepsilon^2} - \frac{(1-C'\ell_\varepsilon)^2(\pi - N_\varepsilon\ell_\varepsilon^2)}{\ell_\varepsilon^2} + C \leq -\frac{\pi}{2\ell_\varepsilon^2} + N_\varepsilon + \frac{C'(\pi - N_\varepsilon\ell_\varepsilon^2)}{\ell_\varepsilon} + C \tag{3.23}$$

because the lattice is chosen in such a way that, for any $i \in \mathcal{L}$, $r_i \leq 1 - 2\sqrt{2}\ell_\varepsilon$.
From inequalities (3.22) and (3.23) we then get (for any $\eta > 5/2$)

$$\frac{i\Omega_0}{\varepsilon}\int_\Lambda d\vec{r}\,g^*_\varepsilon\,(\vec{r}\times\nabla g_\varepsilon) \leq -\frac{\pi^2\Omega_0}{2\varepsilon\ell_\varepsilon^2} + \frac{C'_{\Omega_0,\delta}(\pi - N_\varepsilon\ell_\varepsilon^2)}{\varepsilon\ell_\varepsilon} + \frac{C_{\Omega_0}}{\varepsilon}$$

but the number of points in the lattice can be estimated below as

$$\left|N_\varepsilon - \frac{\pi(1-2\sqrt{2}\ell_\varepsilon)^2}{\ell_\varepsilon^2}\right| \leq \frac{C}{\ell_\varepsilon^{2/3}} \tag{3.24}$$

(see for instance Theorem 7.7.16 in [H]) so that

$$\frac{i\Omega_0}{\varepsilon}\int_\Lambda d\vec{r}\,g^*_\varepsilon\,(\vec{r}\times\nabla g_\varepsilon) \leq -\frac{\pi^2\Omega_0}{2\varepsilon\ell_\varepsilon^2} + \frac{C_{\Omega_0,\delta}}{\varepsilon} \leq -\frac{\pi^2\Omega_0}{2\delta^2\varepsilon^2} + \frac{C_{\Omega_0,\delta}}{\varepsilon}. \tag{3.25}$$

The first term in (3.17) is the most difficult to estimate and we deal with it in the following Lemma 3.1. Altogether the three upper bounds then give the result for a possibly different constant $C_{\Omega_0,\delta}$.

□

Lemma 3.1 (Kinetic Energy of Vortices)
Let g_ε be the function defined in (3.5), $\ell_\varepsilon = \delta\sqrt{\varepsilon}$ *and* $\eta > \frac{5}{2}$. *There exists a constant C_δ independent of ε such that for ε sufficiently small*

$$\int_\Lambda d\vec{r}\ |\nabla g_\varepsilon|^2 \leq \frac{\pi^3}{2\delta^4\varepsilon^2} + \frac{C_\delta|\log\varepsilon|}{\varepsilon}. \tag{3.26}$$

Proof: We first notice the useful fact that

$$\int_\Lambda d\vec{r}\ |\nabla g_\varepsilon|^2 = \int_\Lambda d\vec{r}\ |\nabla\phi|^2 = \int_\Lambda d\vec{r}\ \left|\nabla\tilde{\phi}\right|^2$$

where ϕ is defined in (3.18) and $\tilde{\phi}$ is the function

$$\tilde{\phi}(\vec{r}) = \sum_{i\in\mathcal{L}} \ln|\vec{r}-\vec{r}_i|. \tag{3.27}$$

Indeed, $\tilde{\phi}$ and ϕ are conjugate harmonic functions (the real and imaginary parts of $\ln\prod_i(z-z_i)$)) so that $\partial_x\tilde{\phi} = -\partial_y\phi$, $\partial_y\tilde{\phi} = \partial_x\phi$.
Since $\tilde{\phi}$ is harmonic, the last integral can be explicitly evaluated by means of partial integration:

$$\int_\Lambda d\vec{r}\ \left|\nabla\tilde{\phi}\right|^2 = \int_{\partial\mathcal{B}_1} d\vec{s}\ \cdot\ \frac{\partial\tilde{\phi}}{\partial\vec{n}}\ \tilde{\phi} - \sum_{i\in\mathcal{L}}\int_{\partial\mathcal{B}^i_\varepsilon} d\vec{s}\ \cdot\ \frac{\partial\tilde{\phi}}{\partial\vec{n}}\ \tilde{\phi}$$

where $\vec{n}$ stands for the outer normal to integration path.
We are going to consider the two terms separately.

Outer boundary: The contribution at the outer boundary is given by

$$\int_{\partial\mathcal{B}_1} d\vec{s}\ \cdot\ \frac{\partial\tilde{\phi}}{\partial\vec{n}}\ \tilde{\phi} = \frac{1}{2}\sum_{i,j\in\mathcal{L}}\int_0^{2\pi} d\vartheta\ \frac{2 - z_j e^{-i\vartheta} - z_j^* e^{i\vartheta}}{|e^{i\vartheta}-z_j|^2}\ \ln\left|e^{i\vartheta}-z_i\right| \tag{3.28}$$

where we have used the complex coordinate notation, $z = x + iy$.
The first step in the proof is the replacement of the sum over i with an integral over $\mathcal{B}_{1-\frac{3\ell_\varepsilon}{\sqrt{2}}}$:

$$\sum_{i\in\mathcal{L}}\ln\left|e^{i\vartheta}-z_i\right|^2 - \frac{1}{\ell_\varepsilon^2}\int_{\cup_{i\in\mathcal{L}}\mathcal{Q}^i_\varepsilon} dz\ \ln\left|e^{i\vartheta}-z\right|^2 = -\frac{1}{\ell_\varepsilon^2}\sum_{i\in\mathcal{L}}\int_{\mathcal{Q}_\varepsilon} dz\ \ln\frac{\left|e^{i\vartheta}-z_i-z\right|^2}{\left|e^{i\vartheta}-z_i\right|^2}.$$

Thanks to the choice of the lattice (3.6),

$$\frac{\left|e^{i\vartheta}-z_i-z\right|}{|e^{i\vartheta}-z_i|} > \frac{1}{2}$$

because, for any $\vartheta\in[0,2\pi]$, $\left|e^{i\vartheta}-z_i\right| \geq 2\sqrt{2}\ell_\varepsilon$ and $|z| \leq \ell_\varepsilon/\sqrt{2}$. Using therefore the bound

$$\ln(1+t) \geq t - t^2$$

which holds true for any $t > -1/2$, we get

$$-\frac{1}{\ell_\varepsilon^2}\sum_{i\in\mathcal{L}}\int_{\mathcal{Q}_\varepsilon} dz\ \ln\frac{1-\left|e^{i\vartheta}-z_i-z\right|^2}{\left|e^{i\vartheta}-z_i\right|^2} \leq -\frac{1}{\ell_\varepsilon^2}\sum_{i\in\mathcal{L}}\int_{\mathcal{Q}_\varepsilon} d\vec{r}\left\{\frac{-2\vec{r}\cdot((\cos\vartheta,\sin\vartheta)-\vec{r}_i)+r^2}{|(\cos\vartheta,\sin\vartheta)-\vec{r}_i|^2}+\right.$$

$$\left.-\frac{\left[-2\vec{r}\cdot((\cos\vartheta,\sin\vartheta)-\vec{r}_i)+r^2\right]^2}{|(\cos\vartheta,\sin\vartheta)-\vec{r}_i|^4}\right\} \leq$$

$$\leq \frac{1}{\ell_\varepsilon^2}\sum_{i\in\mathcal{L}}\int_{\mathcal{Q}_\varepsilon} d\vec{r}\ \frac{4\left[\vec{r}\cdot((\cos\vartheta,\sin\vartheta)-\vec{r}_i)\right]^2+r^4}{|(\cos\vartheta,\sin\vartheta)-\vec{r}_i|^4} \leq$$

$$\leq \frac{1}{\ell_\varepsilon^2}\sum_{i\in\mathcal{L}}\int_{\mathcal{Q}_\varepsilon} d\vec{r}\,\left\{\frac{4r^2}{|(\cos\vartheta,\sin\vartheta)-\vec{r}_i|^2}+\frac{r^4}{|(\cos\vartheta,\sin\vartheta)-\vec{r}_i|^4}\right\}\leq$$

$$\leq \sum_{i\in\mathcal{L}}\left\{\frac{C_1\ell_\varepsilon^2}{|(\cos\vartheta,\sin\vartheta)-\vec{r}_i|^2}+\frac{C_2\ell_\varepsilon^4}{|(\cos\vartheta,\sin\vartheta)-\vec{r}_i|^4}\right\}.$$

Since the functions $1/r^2$ and $1/r^4$ are positive and subharmonic we can easily bound the expression above by

$$\sum_{i\in\mathcal{L}}\left\{\frac{C_1\ell_\varepsilon^2}{|(\cos\vartheta,\sin\vartheta)-\vec{r}_i|^2}+\frac{C_2\ell_\varepsilon^4}{|(\cos\vartheta,\sin\vartheta)-\vec{r}_i|^4}\right\}\leq$$

$$\leq \int_{\cup_{i\in\mathcal{L}}\mathcal{Q}_\varepsilon^i} d\vec{r}\,\left\{\frac{C_1\ell_\varepsilon^2}{|(\cos\vartheta,\sin\vartheta)-\vec{r}|^2}+\frac{C_2\ell_\varepsilon^4}{|(\cos\vartheta,\sin\vartheta)-\vec{r}|^4}\right\}\leq$$

$$\leq \int_{\mathcal{B}_{1-\frac{3\ell_\varepsilon}{\sqrt{2}}}} d\vec{r}\,\left\{\frac{C_1\ell_\varepsilon^2}{|(\cos\vartheta,\sin\vartheta)-\vec{r}|^2}+\frac{C_2\ell_\varepsilon^4}{|(\cos\vartheta,\sin\vartheta)-\vec{r}|^4}\right\}\leq C$$

so that

$$\sum_{i\in\mathcal{L}}\ln\left|e^{i\vartheta}-z_i\right|\leq\frac{1}{\ell_\varepsilon^2}\int_{\cup_{i\in\mathcal{L}}\mathcal{Q}_\varepsilon^i} dz\,\ln\left|e^{i\vartheta}-z\right|+C.$$

On the other hand

$$\frac{1}{\ell_\varepsilon^2}\int_{\cup_{i\in\mathcal{L}}\mathcal{Q}_\varepsilon^i} dz\,\ln\left|e^{i\vartheta}-z\right|=\frac{1}{\ell_\varepsilon^2}\int_{\mathcal{B}_{1-\frac{3\ell_\varepsilon}{\sqrt{2}}}} dz\,\ln\left|e^{i\vartheta}-z\right|-\frac{1}{\ell_\varepsilon^2}\int_{\mathcal{B}_{1-\frac{3\ell_\varepsilon}{\sqrt{2}}}\setminus\cup_{i\in\mathcal{L}}\mathcal{Q}_\varepsilon^i} dz\,\ln\left|e^{i\vartheta}-z\right|=$$

$$=-\frac{1}{\ell_\varepsilon^2}\int_{\mathcal{B}_{1-\frac{3\ell_\varepsilon}{\sqrt{2}}}\setminus\cup_{i\in\mathcal{L}}\mathcal{Q}_\varepsilon^i} dz\,\ln\left|e^{i\vartheta}-z\right|\leq\frac{C|\ln\ell_\varepsilon|\left|\mathcal{B}_{1-\frac{3\ell_\varepsilon}{\sqrt{2}}}\setminus\cup_{i\in\mathcal{L}}\mathcal{Q}_\varepsilon^i\right|}{\ell_\varepsilon^2}\leq$$

$$\leq\frac{C|\ln\ell_\varepsilon|\left[\pi\left(1-\frac{3\ell_\varepsilon}{\sqrt{2}}\right)^2-N_\varepsilon\ell_\varepsilon^2\right]}{\ell_\varepsilon^2}\leq\frac{C|\ln\ell_\varepsilon|}{\ell_\varepsilon}$$

where we have used the estimate (3.24) for the number of points and the fact that

$$\int_{\mathcal{B}_R} dz\,\ln\left|e^{i\vartheta}-z\right|=0$$

for any $0<R<1$.
Since the function

$$a(z)=\frac{2-ze^{-i\vartheta}-z^*e^{i\vartheta}}{|e^{i\vartheta}-z|^2}$$

is positive for any $\vartheta\in[0,2\pi]$ and $z\in\mathcal{B}_{1-3\varepsilon/\sqrt{2}}$, the initial expression in (3.28) is bounded from above by

$$\int_{\partial\mathcal{B}_1} d\vec{s}\cdot\frac{\partial\tilde{\phi}}{\partial\vec{n}}\,\tilde{\phi}\leq\frac{1}{2}\sum_{j\in\mathcal{L}}\int_0^{2\pi} d\vartheta\,\frac{2-z_je^{-i\vartheta}-z_j^*e^{i\vartheta}}{|e^{i\vartheta}-z_j|^2}\,\mathcal{R}_\varepsilon(\vartheta)$$

where

$$\mathcal{R}_\varepsilon(\vartheta)\equiv-\frac{1}{\ell_\varepsilon^2}\int_{\mathcal{B}_{1-\frac{3\ell_\varepsilon}{\sqrt{2}}}\setminus\cup_{i\in\mathcal{L}}\mathcal{Q}_\varepsilon^i} dz\,\ln\left|e^{i\vartheta}-z\right|+C$$

is easily proved to satisfy the upper bound

$$|\mathcal{R}_\varepsilon(\vartheta)|\leq\frac{C|\ln\ell_\varepsilon|\left|\mathcal{B}_{1-\frac{3\ell_\varepsilon}{\sqrt{2}}}\setminus\cup_{i\in\mathcal{L}}\mathcal{Q}_\varepsilon^i\right|}{\ell_\varepsilon^2}\leq\frac{C|\ln\ell_\varepsilon|}{\ell_\varepsilon}.$$

We need now to replace the sum over j with an integral over $\mathcal{B}_{1-2\sqrt{2}\ell_\varepsilon}$: Since the function $a(z)$ is harmonic, we can apply the mean value theorem to get

$$a(z_j) - \frac{1}{\ell_\varepsilon^2}\int_{\mathcal{Q}_\varepsilon} dz\, a(z_j+z) = \frac{1}{\ell_\varepsilon^2}\int_{\mathcal{Q}_\varepsilon\setminus\mathcal{B}_{\frac{\ell_\varepsilon}{2}}} dz\ [a(z_j)-a(z_j+z)] \equiv b_\varepsilon(z_j).$$

For any $j\in\mathcal{L}$, the right hand side can be easily estimated using Harnack's inequality:

$$b_\varepsilon(z_j) \le \frac{\left|\mathcal{Q}_\varepsilon\setminus\mathcal{B}_{\frac{\ell_\varepsilon}{2}}\right|}{\ell_\varepsilon^2}\left[a(z_j) - \frac{1-\frac{\sqrt{2}\ell_\varepsilon}{2}}{1+\frac{\sqrt{2}\ell_\varepsilon}{2}}a(z_j)\right] \le \frac{C\left|\mathcal{Q}_\varepsilon\setminus\mathcal{B}_{\frac{\ell_\varepsilon}{2}}\right| a(z_j)}{\ell_\varepsilon} \le C\ell_\varepsilon a(z_j).$$

In the same way it is possible to show that for ε sufficiently small, there exists a possibly different constant C such that

$$b_\varepsilon(z_j) \ge -C\ell_\varepsilon a(z_j)$$

so that

$$\frac{1}{(1+C\ell_\varepsilon)\ell_\varepsilon^2}\int_{\mathcal{Q}_\varepsilon} dz\, a(z_j+z) \le a(z_j) \le \frac{1}{(1-C\ell_\varepsilon)\ell_\varepsilon^2}\int_{\mathcal{Q}_\varepsilon} dz\, a(z_j+z)$$

and then

$$\left|a(z_j) - \frac{1}{\ell_\varepsilon^2}\int_{\mathcal{Q}_\varepsilon} dz\, a(z_j+z)\right| \le C\ell_\varepsilon.$$

Since

$$\sum_{j\in\mathcal{L}}\int_{\mathcal{Q}_\varepsilon} dz\, a(z_j+z) = \int_{\cup_j\mathcal{Q}_\varepsilon^j} dz\, a(z)$$

and

$$0 \le \int_{\mathcal{B}_{1-2\sqrt{2}\ell_\varepsilon}\setminus\cup_j\mathcal{Q}_\varepsilon^j} dz\, a(z) \le \int_{1-2\sqrt{2}\ell_\varepsilon}^{1-\frac{3\ell_\varepsilon}{\sqrt{2}}} dr\, r\int_0^{2\pi} d\gamma\, a(re^{i\gamma}) \le C\ell_\varepsilon$$

we conclude that

$$|\mathcal{R}'_\varepsilon(\vartheta)| \equiv \left|\sum_{j\in\mathcal{L}}\frac{2-z_je^{-i\vartheta}-z_j^*e^{i\vartheta}}{|e^{i\vartheta}-z_j|^2} - \frac{1}{\ell_\varepsilon^2}\int_{\mathcal{B}_{1-2\sqrt{2}\ell_\varepsilon}} dz\,\frac{2-ze^{-i\vartheta}-z^*e^{i\vartheta}}{|e^{i\vartheta}-z|^2}\right| \le C\ell_\varepsilon N_\varepsilon. \tag{3.29}$$

On the other hand, using again the harmonicity of a, one has

$$\frac{1}{\ell_\varepsilon^2}\int_{\mathcal{B}_{1-2\sqrt{2}\ell_\varepsilon}} dz\,\frac{2-ze^{-i\vartheta}-z^*e^{i\vartheta}}{|e^{i\vartheta}-z|^2} = \frac{2\pi\left(1-2\sqrt{2}\ell_\varepsilon\right)^2}{\ell_\varepsilon^2}.$$

Since for the fundamental solution of the Laplace equation

$$\frac{1}{2\pi}\int_0^{2\pi} d\vartheta\ \ln\left|Re^{i\vartheta}-\vec{x}\right| = \begin{cases}\ln R & \text{if } |\vec{x}|\le R\\ \ln|\vec{x}| & \text{if } |\vec{x}|\ge R\end{cases} \tag{3.30}$$

we then get altogether

$$\int_{\partial\mathcal{B}_1} d\vec{s}\cdot\frac{\partial\tilde{\phi}}{\partial\vec{n}}\,\tilde{\phi} \le -\frac{2\pi\left(1-2\sqrt{2}\ell_\varepsilon\right)^2}{\ell_\varepsilon^4}\int_{\mathcal{B}_{1-\frac{3\ell_\varepsilon}{\sqrt{2}}}\setminus\cup_j\mathcal{Q}_\varepsilon^j} dz\int_0^{2\pi} d\vartheta\ \ln\left|e^{i\vartheta}-z\right| + \int_0^{2\pi} d\vartheta\ |\mathcal{R}'_\varepsilon(\vartheta)|\,|\mathcal{R}_\varepsilon(\vartheta)| + \frac{C}{\ell_\varepsilon^2} \le$$

$$\le 2\pi\sup_{\vartheta\in[0,2\pi]}|\mathcal{R}'_\varepsilon(\vartheta)|\,|\mathcal{R}_\varepsilon(\vartheta)| + \frac{C}{\ell_\varepsilon^2} \le \frac{C|\ln\ell_\varepsilon|}{\ell_\varepsilon^2} + \frac{C}{\ell_\varepsilon^2} \le \frac{C_\delta|\log\varepsilon|}{\varepsilon}. \tag{3.31}$$

Inner boundary: A straightforward calculation gives

$$-\sum_{i\in\mathcal{L}}\int_{\partial\mathcal{B}_\varepsilon^i} d\vec{s}\cdot\frac{\partial\tilde{\phi}}{\partial\vec{n}}\,\tilde{\phi} = 2\pi\eta N_\varepsilon|\ln\varepsilon| - \sum_{\substack{i,j\in\mathcal{L}\\ i\ne j}}\int_0^{2\pi} d\vartheta\ \ln\left|\varepsilon^\eta e^{i\vartheta}+z_i-z_j\right| +$$

$$-\frac{\varepsilon^\eta}{2}\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\int_0^{2\pi} d\vartheta\ \tilde\phi(z_i+\varepsilon^\eta e^{i\vartheta})\ \frac{\varepsilon^\eta-(z_i-z_j)e^{-i\vartheta}-(z_i-z_j)^* e^{i\vartheta}}{|\varepsilon^\eta e^{i\vartheta}+z_i-z_j|^2}\leq$$

$$\leq 2\pi\eta N_\varepsilon|\ln\varepsilon|-2\pi\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\ln|z_i-z_j|+\frac{C\varepsilon^\eta N_\varepsilon}{\ell_\varepsilon^2}\sum_{i\in\mathcal{L}}\int_0^{2\pi} d\vartheta\ |\tilde\phi(z_i+\varepsilon^\eta e^{i\vartheta})|\leq$$

$$\leq 2\pi\eta N_\varepsilon|\ln\varepsilon|-2\pi\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\ln|z_i-z_j|+\frac{C\varepsilon^\eta N_\varepsilon^2|\log\varepsilon|}{\ell_\varepsilon^2}\leq\frac{C_\delta|\log\varepsilon|}{\varepsilon}-2\pi\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\ln|z_i-z_j| \tag{3.32}$$

where we have used (3.20).
In order to get the desired estimate we need now to replace the sum over one of the two indices in the expression above with the integration on a suitable domain. Therefore the quantity which has to be estimated is the difference

$$\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\left\{-\ln|\vec r_i-\vec r_j|+\frac{1}{\ell_\varepsilon^4}\int_{\mathcal{Q}_\varepsilon}d\vec r\int_{\mathcal{Q}_\varepsilon}d\vec r'\ \ln|\vec r_i-\vec r_j+\vec r-\vec r'|\right\}. \tag{3.33}$$

Using the estimate $\ln(t)\leq\frac{1}{2}(t^2-1)$, which holds for any $t>0$, we can bound the expression under the sum in the following way

$$\frac{1}{\ell_\varepsilon^4}\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\int_{\mathcal{Q}_\varepsilon}d\vec r\int_{\mathcal{Q}_\varepsilon}d\vec r'\ \ln\frac{|\vec r_i-\vec r_j+\vec r-\vec r'|}{|\vec r_i-\vec r_j|}\leq\frac{1}{2\ell_\varepsilon^4}\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\int_{\mathcal{Q}_\varepsilon}d\vec r\int_{\mathcal{Q}_\varepsilon}d\vec r'\ \left\{\frac{|\vec r_i-\vec r_j+\vec r-\vec r'|^2}{|\vec r_i-\vec r_j|^2}-1\right\}=$$

$$=\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\frac{1}{2\ell_\varepsilon^4|\vec r_i-\vec r_j|^2}\int_{\mathcal{Q}_\varepsilon}d\vec r\int_{\mathcal{Q}_\varepsilon}d\vec r'\ \left[|\vec r-\vec r'|^2+2\,(\vec r-\vec r')\cdot(\vec r_i-\vec r_j)\right]=$$

$$=\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\frac{1}{2\ell_\varepsilon^4|\vec r_i-\vec r_j|^2}\int_{\mathcal{Q}_\varepsilon}d\vec r\int_{\mathcal{Q}_\varepsilon}d\vec r'\ \left(r^2+r'^2\right)\leq\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\frac{C\ell_\varepsilon^2}{|\vec r_i-\vec r_j|^2}$$

where we have used the central symmetry of the fundamental cell $\mathcal{Q}_\varepsilon$ and the lattice $\mathcal{L}$.
On the other hand, since the function $1/r^2$ is subharmonic and positive, one can easily prove that

$$\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\frac{C\ell_\varepsilon^2}{|\vec r_i-\vec r_j|^2}\leq\frac{C}{\ell_\varepsilon^2}\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\int_{\mathcal{Q}_\varepsilon}d\vec r\int_{\mathcal{Q}_\varepsilon}d\vec r'\ \frac{1}{|\vec r_i-\vec r_j+\vec r-\vec r'|^2}\leq$$

$$\leq\frac{C}{\ell_\varepsilon^2}\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\int_{\mathcal{Q}_\varepsilon^i}d\vec r\int_{\mathcal{Q}_\varepsilon^j}d\vec r'\ \frac{1}{|\vec r-\vec r'|^2}\leq\frac{C}{\ell_\varepsilon^2}\int_{\frac{\ell_\varepsilon}{2}}^2\frac{dr}{r}\leq\frac{C_\delta|\log\varepsilon|}{\varepsilon}$$

so that the difference in (3.33) is bounded by $C_\delta|\log\varepsilon|/\varepsilon$.
In order to extend the integration to the whole disc $\mathcal{B}_1$, we observe that

$$-\frac{1}{\ell_\varepsilon^4}\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\int_{\mathcal{Q}_\varepsilon^i}d\vec r\int_{\mathcal{Q}_\varepsilon^j}d\vec r'\ \ln|\vec r-\vec r'|\leq-\frac{1}{\ell_\varepsilon^4}\int_{\cup_{i\in\mathcal{L}}\mathcal{Q}_\varepsilon^i}d\vec r\int_{\cup_{j\in\mathcal{L}}\mathcal{Q}_\varepsilon^j}d\vec r'\ \ln|\vec r-\vec r'|=$$

$$=-\frac{1}{\ell_\varepsilon^4}\int_{\mathcal{B}_1}d\vec r\int_{\cup_{i\in\mathcal{L}}\mathcal{Q}_\varepsilon^i}d\vec r'\ \ln|\vec r-\vec r'|+\frac{1}{\ell_\varepsilon^4}\int_{\mathcal{A}_\varepsilon}d\vec r\int_{\mathcal{B}_1}d\vec r'\ \ln|\vec r-\vec r'|-\frac{1}{\ell_\varepsilon^4}\int_{\mathcal{A}_\varepsilon}d\vec r\int_{\mathcal{A}_\varepsilon}d\vec r'\ \ln|\vec r-\vec r'| \tag{3.34}$$

where $\mathcal{A}_\varepsilon$ stands for the domain $\mathcal{B}_1\setminus\cup_{i\in\mathcal{L}}\mathcal{Q}_\varepsilon^i$.
The last term in the expression above is bounded by

$$-\frac{1}{\ell_\varepsilon^4}\int_{\mathcal{A}_\varepsilon}d\vec r\int_{\mathcal{A}_\varepsilon}d\vec r'\ \ln|\vec r-\vec r'|\leq-\frac{1}{\ell_\varepsilon^4}\int_{\tilde{\mathcal{A}}_\varepsilon}d\vec r\int_{\tilde{\mathcal{A}}_\varepsilon}d\vec r'\ \ln|r-r'|\leq-\frac{C\ln\ell_\varepsilon}{\ell_\varepsilon^2}\leq\frac{C_\delta|\log\varepsilon|}{\varepsilon}$$

where $\tilde{\mathcal{A}}_\varepsilon \equiv \mathcal{B}_1 \setminus \mathcal{B}_{1-2\sqrt{2}\ell_\varepsilon}$.
For the second term in (3.34), we can use (3.30) to get

$$\frac{1}{\ell_\varepsilon^4}\int_{\mathcal{A}_\varepsilon} d\vec{r}\int_{\mathcal{B}_1} d\vec{r}'\ \ln|\vec{r}-\vec{r}'| = \frac{2\pi}{\ell_\varepsilon^4}\int_{\mathcal{A}_\varepsilon} d\vec{r}\left\{\int_0^r dr' r'\ \ln r + \int_r^1 dr' r'\ \ln r'\right\} \le 0.$$

Therefore one has from (3.34)

$$-\frac{1}{\ell_\varepsilon^4}\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\int_{\mathcal{Q}_\varepsilon^i} d\vec{r}\int_{\mathcal{Q}_\varepsilon^j} d\vec{r}'\ \ln|\vec{r}-\vec{r}'| \le -\frac{1}{\ell_\varepsilon^4}\int_{\mathcal{B}_1} d\vec{r}\int_{\cup_{i\in\mathcal{L}}\mathcal{Q}_\varepsilon^i} d\vec{r}'\ \ln|\vec{r}-\vec{r}'| + \frac{C_\delta|\log\varepsilon|}{\varepsilon} \le$$

$$\le \frac{\pi}{2\ell_\varepsilon^4}\int_{\mathcal{B}_1} d\vec{r}'\ \left(1-r'^2\right) + \frac{C_\delta|\log\varepsilon|}{\varepsilon} \le \frac{\pi^2}{4\ell_\varepsilon^4} + \frac{C_\delta|\log\varepsilon|}{\varepsilon}$$

so that

$$-2\pi\sum_{\substack{i,j\in\mathcal{L}\\ i\neq j}}\ln|z_i-z_j| \le \frac{\pi^3}{2\ell_\varepsilon^4} + \frac{C_\delta|\log\varepsilon|}{\varepsilon}$$

and finally

$$-\sum_{i\in\mathcal{L}}\int_{\partial\mathcal{B}_\varepsilon^i} d\vec{s}\ \cdot\ \frac{\partial\tilde{\phi}}{\partial\vec{n}}\,\tilde{\phi} \le \frac{\pi^3}{2\ell_\varepsilon^4} + \frac{C_\delta|\log\varepsilon|}{\varepsilon}. \tag{3.35}$$

Combining this result with the estimate for the contribution at the outer boundary, we complete the proof.

□

3.2 The Regime $\Omega(\varepsilon) \gg 1/\varepsilon$

Proof of Theorem 2.2
The lower bound can be proved in the same way as in the proof of Theorem 2.1, so that one easily gets

$$E_\varepsilon^{\rm GP} \ge \frac{E_\varepsilon^{\rm TF}}{\varepsilon^{2+2\alpha}}.$$

For the upper bound we evaluate the functional on the following trial function

$$\tilde{\Psi}(\vec{r}) = \tilde{c}_\varepsilon j_\varepsilon(r)\,\sqrt{\rho_\varepsilon^{\rm TF}(r)}\ \exp\left\{i\left[\frac{\Omega_1}{2\varepsilon^{1+\alpha}}\right]\vartheta\right\} \tag{3.36}$$

where we used polar coordinates, $\vec{r}=(r,\vartheta)$, $[\ \cdot\ ]$ stands for the integer part, j_ε is the cut-off function

$$j_\varepsilon(r) = \begin{cases} 0 & \text{if}\quad r\le R_\varepsilon \\ \dfrac{r^2-R_\varepsilon^2}{\varepsilon^\beta} & \text{if}\quad R_\varepsilon^2\le r^2\le R_\varepsilon^2+\varepsilon^\beta \\ 1 & \text{otherwise}\end{cases} \tag{3.37}$$

with some $\beta>\alpha$, and $\tilde{c}_\varepsilon$ is a normalization constant satisfying the following bounds

$$1 < \tilde{c}_\varepsilon^2 \le 1 + C\varepsilon^{2\beta-2\alpha}. \tag{3.38}$$

A simple calculation shows that

$$\mathcal{E}^{\rm GP}[\tilde{\Psi}] = \tilde{c}_\varepsilon^2\int_{\mathcal{B}_1} d\vec{r}\,\left|\partial_r\left(j_\varepsilon\sqrt{\rho_\varepsilon^{\rm TF}}\right)\right|^2 + \tilde{c}_\varepsilon^2\int_{\mathcal{B}_1} d\vec{r}\,j_\varepsilon^2\,\rho_\varepsilon^{\rm TF}\,\left\{\frac{1}{r}\left[\frac{\Omega_1}{2\varepsilon^{1+\alpha}}\right] - \frac{\Omega_1 r}{2\varepsilon^{1+\alpha}}\right\}^2 +$$

$$+\frac{\mathcal{E}^{\rm TF}[\tilde{c}_\varepsilon^2 j_\varepsilon^2 \rho_\varepsilon^{\rm TF}]}{\varepsilon^{2+2\alpha}}. \tag{3.39}$$

The first term in (3.39) is bounded by (using (3.38))

$$\tilde{c}_\varepsilon^2 \int_{\mathcal{B}_1} d\vec{r}\, \left|\partial_r \left(j_\varepsilon \sqrt{\rho_\varepsilon^{\rm TF}}\right)\right|^2 \leq 2\tilde{c}_\varepsilon^2 \int_{\mathcal{B}_1} d\vec{r}\, |\partial_r j_\varepsilon|^2 \rho_\varepsilon^{\rm TF} + \tilde{c}_\varepsilon^2 \int_{\mathcal{B}_1} d\vec{r}\, \frac{j_\varepsilon^2}{2\rho_\varepsilon^{\rm TF}} \left(\frac{\partial \rho_\varepsilon^{\rm TF}}{\partial r}\right)^2 \leq$$

$$\leq \frac{C_1}{\varepsilon^{2\beta}} \int_{R_\varepsilon}^{\sqrt{R_\varepsilon^2+\varepsilon^\beta}} dr r^3\, \rho_\varepsilon^{\rm TF}(r) + \frac{C_2}{\varepsilon^{\beta-2\alpha}} \int_{R_\varepsilon}^{\sqrt{R_\varepsilon^2+\varepsilon^\beta}} dr r (r^2 - R_\varepsilon^2) \left(\frac{\partial \rho_\varepsilon^{\rm TF}}{\partial r}\right)^2 +$$

$$+C_3 \varepsilon^{2\alpha} \int_{\sqrt{R_\varepsilon^2+\varepsilon^\beta}}^1 dr \frac{r}{r^2 - R_\varepsilon^2} \left(\frac{\partial \rho_\varepsilon^{\rm TF}}{\partial r}\right)^2 \leq$$

$$\leq \frac{C_1}{\varepsilon^{2\alpha}} + \frac{C_2}{\varepsilon^{2\alpha}} + \frac{C_3 |\log \varepsilon|}{\varepsilon^{2\alpha}} \leq \frac{C_{\Omega_1} |\log \varepsilon|}{\varepsilon^{2\alpha}}.$$

(The constants depend on the choice of β and $C_3 \to \infty$ if $\beta \to \infty$.) Moreover, using (3.38) and the fact that

$$\left[\frac{\Omega_1}{2\varepsilon^{1+\alpha}}\right] = \frac{\Omega_1}{2\varepsilon^{1+\alpha}} - \kappa_\varepsilon$$

for some $0 \leq \kappa_\varepsilon < 1$, we can estimate the second term in (3.39) as follows

$$\tilde{c}_\varepsilon^2 \int_{\mathcal{B}_1} d\vec{r}\, j_\varepsilon^2\, \rho_\varepsilon^{\rm TF} \left\{\frac{1}{r}\left[\frac{\Omega_1}{2\varepsilon^{1+\alpha}}\right] - \frac{\Omega_1 r}{2\varepsilon^{1+\alpha}}\right\}^2 \leq \frac{\tilde{c}_\varepsilon^2 \pi \Omega_1^2}{2\varepsilon^{2+2\alpha}} \int_{R_\varepsilon}^1 dr r\, \rho_\varepsilon^{\rm TF} \left(\frac{1}{r} - r\right)^2 +$$

$$+\tilde{c}_\varepsilon^2 \kappa_\varepsilon^2 2\pi \int_{R_\varepsilon}^1 dr\, \frac{\rho_\varepsilon^{\rm TF}}{r} \leq \frac{C_1 \left(1 - R_\varepsilon^2\right)^2}{\varepsilon^{2+2\alpha}} + C_2 \leq \frac{C_{\Omega_1}}{\varepsilon^2}.$$

In a similar way one can prove that

$$\mathcal{E}_\varepsilon^{\rm TF}[\tilde{c}_\varepsilon^2 j_\varepsilon^2 \rho_\varepsilon^{\rm TF}] \leq \mathcal{E}_\varepsilon^{\rm TF}[\rho_\varepsilon^{\rm TF}] + 2\pi \left(\tilde{c}_\varepsilon^4 - 1\right) \int_{R_\varepsilon}^1 dr\, r \rho_\varepsilon^{{\rm TF}\,2}(r) + \frac{\pi \Omega_1^2}{2} \int_{R_\varepsilon}^{\sqrt{R_\varepsilon^2+\varepsilon^\beta}} dr\, r^3 \left(1 - j_\varepsilon^2\right) \rho_\varepsilon^{\rm TF}(r) \leq$$

$$\leq E_\varepsilon^{\rm TF} + C_1 \varepsilon^{2\beta-4\alpha} \int_{R_\varepsilon^2}^1 dz\, \left(z - R_\varepsilon^2\right)^2 + \frac{C_2}{\varepsilon^{2\alpha}} \int_{R_\varepsilon^2}^{R_\varepsilon^2+\varepsilon^\beta} dz\, z \left[1 - \left(\frac{z - R_\varepsilon^2}{\varepsilon^\beta}\right)^2\right] \left(z - R_\varepsilon^2\right) \leq$$

$$\leq E_\varepsilon^{\rm TF} + C_1 \varepsilon^{2\beta-\alpha} + \frac{C_2}{\varepsilon^{2\alpha}} \int_0^{\varepsilon^\beta} dz\, z \left(1 - \frac{z^2}{\varepsilon^{2\beta}}\right) \leq E_\varepsilon^{\rm TF} + C_{\Omega_1} \varepsilon^{2\beta-2\alpha}$$

and then the result follows if we choose a finite $\beta > 2\alpha$.

□

Proof of Corollary 2.2

Let us start by considering the case $0 < \alpha < 2$. Defining $\mathcal{D}_\varepsilon = \mathcal{B}_1 \setminus \mathcal{B}_{R_\varepsilon}$ we first notice that for any non negative function $\rho \in L^2(\mathcal{D}_\varepsilon)$, normalized to 1 in $L^1(\mathcal{D}_\varepsilon)$,

$$\mathcal{E}_\varepsilon^{\rm TF}\,[\rho, \mathcal{D}_\varepsilon] \geq E_\varepsilon^{\rm TF}$$

where $\mathcal{E}_\varepsilon^{\rm TF}[\rho, \mathcal{D}_\varepsilon]$ denotes the functional

$$\mathcal{E}_\varepsilon^{\rm TF}[\rho, \mathcal{D}_\varepsilon] \equiv \varepsilon^{2\alpha} \int_{\mathcal{D}_\varepsilon} d\vec{r} \left\{\rho^2 - \frac{\Omega_1^2 r^2 \rho}{4\varepsilon^{2\alpha}}\right\}.$$

Hence, setting $\rho_\varepsilon \equiv |\Psi_\varepsilon^{\rm GP}|^2$ and

$$\tilde{\rho}_\varepsilon \equiv \frac{\rho_\varepsilon}{\|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)}}$$

we get

$$\mathcal{E}_\varepsilon^{\mathrm{TF}}[\rho_\varepsilon] = \|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)} \mathcal{E}_\varepsilon^{\mathrm{TF}}[\tilde{\rho}_\varepsilon, \mathcal{D}_\varepsilon] + \mathcal{E}_\varepsilon^{\mathrm{TF}}[\rho_\varepsilon, \mathcal{B}_1 \setminus \mathcal{D}_\varepsilon] +$$

$$+\varepsilon^{2\alpha} \|\rho_\varepsilon\|^2_{L^2(\mathcal{D}_\varepsilon)} \left(1 - \frac{1}{\|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)}}\right) \geq$$

$$\geq E_\varepsilon^{\mathrm{TF}} \|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)} + \mathcal{E}_\varepsilon^{\mathrm{TF}}[\rho_\varepsilon, \mathcal{B}_1 \setminus \mathcal{D}_\varepsilon] + \varepsilon^{2\alpha} \|\rho_\varepsilon\|^2_{L^2(\mathcal{D}_\varepsilon)} \left(1 - \frac{1}{\|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)}}\right) \geq$$

$$\geq E_\varepsilon^{\mathrm{TF}} \|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)} - \frac{\Omega_1^2 R_\varepsilon^2}{4} \left(1 - \|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)}\right) + \varepsilon^{2\alpha} \|\rho_\varepsilon\|^2_{L^2(\mathcal{D}_\varepsilon)} \left(1 - \frac{1}{\|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)}}\right) \geq$$

$$\geq E_\varepsilon^{\mathrm{TF}} \|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)} - \frac{\Omega_1^2 R_\varepsilon^2}{4} \left(1 - \|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)}\right) + \varepsilon^{2\alpha} \left(\|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)} - 1\right) \frac{\|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)}}{|\mathcal{D}_\varepsilon|}$$

where in the last step we have used Schwarz's inequality and the fact that $\|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)} \leq 1$.
On the other hand from the upper bound in the proof of Theorem 2.2 one has

$$\mathcal{E}_\varepsilon^{\mathrm{TF}}[\rho_\varepsilon] \leq \varepsilon^{2+2\alpha} \mathcal{E}^{\mathrm{GP}}\left[\Psi_\varepsilon^{\mathrm{GP}}\right] \leq E_\varepsilon^{\mathrm{TF}} + C_1 \varepsilon^{2\alpha} + C_2 \varepsilon^2 |\log \varepsilon|$$

and then (omitting for simplicity the subscript $L^1(\mathcal{D}_\varepsilon)$),

$$E_\varepsilon^{\mathrm{TF}} \|\rho_\varepsilon\| - \left[\frac{\Omega_1^2 R_\varepsilon^2}{4} + \frac{\varepsilon^{2\alpha} \|\rho_\varepsilon\|}{|\mathcal{D}_\varepsilon|}\right] (1 - \|\rho_\varepsilon\|) \leq E_\varepsilon^{\mathrm{TF}} + C_1 \varepsilon^{2\alpha} + C_2 \varepsilon^2 |\log \varepsilon|$$

and therefore

$$\left[-\frac{\Omega_1 \varepsilon^\alpha}{3\sqrt{\pi}} + \frac{\Omega_1 \varepsilon^\alpha}{4\sqrt{\pi}} \|\rho_\varepsilon\|\right] (1 - \|\rho_\varepsilon\|) + C_1 \varepsilon^{2\alpha} + C_2 \varepsilon^2 |\log \varepsilon| \geq 0$$

or

$$\|\rho_\varepsilon\|^2 - \frac{7}{3} \|\rho_\varepsilon\| + \frac{4}{3} - C_1 \varepsilon^\alpha - C_2 \varepsilon^{2-\alpha} |\log \varepsilon| \leq 0$$

which implies that, for ε sufficiently small,

$$\|\rho_\varepsilon\|_{L^1(\mathcal{D}_\varepsilon)} \geq 1 - C_{\Omega_1} \varepsilon^\alpha - C'_{\Omega_1} \varepsilon^{2-\alpha} |\log \varepsilon|.$$

The result therefore follows from the normalization of ρ_ε in $L^1(\mathcal{B}_1)$.
If $\alpha \geq 2$, the upper bound contained in the proof of Theorem 2.2 gives immediately the following bound

$$\int_{\mathcal{B}_1} d\vec{r}\, r^2 \left|\Psi_\varepsilon^{\mathrm{GP}}\right|^2 \geq 1 - C\varepsilon^2 |\log \varepsilon|$$

and then, using the normalization of $\Psi_\varepsilon^{\mathrm{GP}}$, the result is proved.

□

Proof of Proposition 2.5
Let us first consider the case $0 < \alpha < 2$: as in the proof of Eq. (2.15), we first need to prove a pointwise estimate for $U_\varepsilon \equiv \left|\Psi_\varepsilon^{\mathrm{GP}}\right|^2$ and, in order to find such an upper bound, we have to estimate the chemical potential, appearing in the variational equation satisfied by $\Psi_\varepsilon^{\mathrm{GP}}$. From the definition of the chemical potential and the upper bound contained in the proof of Theorem 2.2, we immediately get

$$\varepsilon^{2+2\alpha} \mu_\varepsilon \leq E_\varepsilon^{\mathrm{TF}} + C_1 \varepsilon^{2\alpha} + C_2 \varepsilon^2 |\log \varepsilon| + \varepsilon^{2\alpha} \left\|\Psi_\varepsilon^{\mathrm{GP}}\right\|^4_{L^4(\mathcal{B}_1)}$$

but, from the same upper bound we obtain

$$\varepsilon^{2\alpha} \left\|\Psi_\varepsilon^{\mathrm{GP}}\right\|^4_{L^4(\mathcal{B}_1)} \leq \frac{\Omega_1^2}{4} \int_{\mathcal{B}_1} d\vec{r}\, r^2 \left|\Psi_\varepsilon^{\mathrm{GP}}\right|^2 + E^{\mathrm{TF}} + C_1 \varepsilon^{2\alpha} + C_2 \varepsilon^2 |\log \varepsilon| \leq$$

$$\leq \frac{\Omega_1^2}{4} + E^{\mathrm{TF}} + C_1\varepsilon^{2\alpha} + C_2\varepsilon^2|\log\varepsilon|$$

and then

$$\varepsilon^{2+2\alpha}\mu_\varepsilon \leq 2E^{\mathrm{TF}} + \frac{\Omega_1^2}{4} + C_1\varepsilon^{2\alpha} + C_2\varepsilon^2|\log\varepsilon| \leq$$

$$\leq -\frac{\Omega_1^2}{4} + \frac{4\Omega_1\varepsilon^\alpha}{3\sqrt{\pi}} + C_1\varepsilon^{2\alpha} + C_2\varepsilon^2|\log\varepsilon|. \tag{3.40}$$

By replacing the above bound in the variational equation, we get

$$-\frac{1}{2}\Delta U_\varepsilon \leq \frac{\Omega_1^2}{4}\left[r^2 - 1 + \frac{16\varepsilon^\alpha}{3\sqrt{\pi}\Omega_1} + C_1\varepsilon^{2\alpha} + C_2\varepsilon^2|\log\varepsilon| - C_3\varepsilon^{2\alpha}U_\varepsilon\right]\frac{U_\varepsilon}{\varepsilon^{2+2\alpha}}$$

and we can conclude that there exists a constant c (depending on Ω_1), such that the function $U_\varepsilon(\vec{r})$ is subharmonic for any $r^2 \leq 1 - c\varepsilon^\alpha$. Following again the proof of Eq. (2.15), we can therefore obtain the following estimate

$$U_\varepsilon(\vec{r}) \leq \frac{C\|U_\varepsilon\|_{L^1(\mathcal{B}_1\setminus\mathcal{D}_\varepsilon)}}{\varrho^2}$$

for any $\vec{r} \in \mathcal{B}_1$ such that

$$r \leq \sqrt{1 - c\varepsilon^\alpha} - \varrho.$$

Choosing for instance $\varrho = \varepsilon^{\alpha'}/3$ and using (2.24), we can conclude that there exists a constant C_{Ω_1} such that

$$U_\varepsilon(\vec{r}) \leq C_{\Omega_1}\varepsilon^{\frac{\alpha'}{3}}|\log\varepsilon|$$

for any $\vec{r} \in \mathcal{T}_\varepsilon'$. The result then follows from the application of the comparison principle to the variational equation satisfied in $\mathcal{T}_\varepsilon'$ by

$$U_\varepsilon' \equiv \frac{U_\varepsilon}{C_{\Omega_1}\varepsilon^{\frac{\alpha'}{3}}|\log\varepsilon|}.$$

The case $\alpha \geq 2$ can be treated in a similar way. The only difference is in the upper bound for the chemical potential (3.40), which in this case becomes

$$\varepsilon^{2+2\alpha}\mu_\varepsilon \leq -\frac{\Omega_1^2}{4} + C\varepsilon^2|\log\varepsilon|.$$

The subharmonicity of U_ε can now be proved in the region $r \leq 1 - \varepsilon^\beta$, for any $1 \leq \beta < 2$. As a straightforward consequence of (2.25), we then get the pointwise estimate

$$U_\varepsilon(\vec{r}) \leq C_{\Omega_1}\varepsilon^{\frac{2-\beta}{3}}|\log\varepsilon|$$

for any $\vec{r} \in \mathcal{T}_\varepsilon''$. The result is again obtained by means of the comparison principle.

□

4 Conclusions and Perspectives

We have analyzed rigorously the leading order asymptotics for the ground state energy and the density profile of a rapidly rotating Bose-Einstein condensate in a flat trap with a finite radius in the limit where the coupling parameter is large. Depending on the scaling of the rotational velocity with the coupling parameter, different asymptotic density functionals emerge.

Our estimates are based on trial functions that capture the essential features of the expected vortex structure and show the possible formation of "holes" where the density is exponentially small as a function of the inverse coupling parameter. The error terms in our estimates are of the expected order but the bounds are not sharp enough to exhibit the details of the fine vortex structure. Nevertheless, we can prove that rotational symmetry is broken in the ground state for const.$|\log\varepsilon| < \Omega(\varepsilon) \lesssim$ const.$/\varepsilon$.

An important open problem is to carry the analysis further to the next to leading order and investigate the transition of the vortex lattice to a "giant vortex" at high rotational velocities.

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