ONE-SIDED APPROXIMATION OF SETS OF FINITE PERIMETER

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Abstract. In this note we present a new proof of a one-sided approximation of sets of finite perimeter introduced in [2], in order to fill a gap in the original proof.

1. Introduction

It is a classical result in geometric measure theory that a set of finite perimeter E can be approximated with smooth sets E_k such that

$$\mathcal{L}^N(E_k) \to \mathcal{L}^N(E) \text{ and } P(E_k) \to P(E), \tag{1.1}$$

where $P(E)$ is the perimeter of E and $\mathcal{L}^N$ is the Lebesgue measure in $\mathbb{R}^N$. The approximating smooth sets (see for instance Ambrosio-Fusco-Pallara [1, Remark 3.42] and Maggi [6, Theorem 13.8]) are the superlevel sets of the convolutions of χ_E, which can be chosen for a.e. $t \in (0,1)$. The one-sided approximation refines the classical result in the sense that it distinguishes between the superlevel sets for a.e. $t \in (\frac{1}{2}, 1)$ from the ones corresponding to a.e. $t \in (0, \frac{1}{2})$, thus providing an *interior* and an *exterior* approximation of the set respectively (see Theorem 3.1 and Theorem 4.1). Indeed, in the first case, the difference between the level sets and the measure theoretic interior is asymptotically vanishing with respect to the $\mathcal{H}^{N-1}$-measure; in the latter, we obtain the same result for the measure theoretic exterior.

The main aim of this note is to fill a gap in the original proof of the main approximation Theorem 4.1 (see [2, Theorem 4.10]). Thus, the results of this note are not only interesting by themselves, but they also validate the application of Theorem 4.1 to the construction of interior and exterior normal traces of essentially bounded divergence-measure fields. This construction was developed in [2] and it was motivated by the study of systems of hyperbolic conservation laws with Lax entropy condition, where these vector fields naturally appear.

2. preliminaries

In what follows we will work in $\mathbb{R}^N$. We introduce now a few basic definitions and results on the theory of functions of bounded variations and sets of finite perimeter, for which we refer mainly to [1], [4] and [6] (see also [5] and [7]).

Definition 2.1. A function $u \in L^1(\mathbb{R}^N)$ is called a *function of bounded variation* if Du is a finite $\mathbb{R}^N$-vector valued Radon measure on $\mathbb{R}^N$. A measurable set $E \subset \mathbb{R}^N$ is called a *set of finite perimeter* in $\mathbb{R}^N$ (or a Caccioppoli set) if $\chi_E \in BV(\mathbb{R}^N)$. Consequently, $D\chi_E$ is an $\mathbb{R}^N$-vector valued Radon measure on $\mathbb{R}^N$ whose total variation is denoted as $\|D\chi_E\|$.

By the polar decomposition of measures, we can write $D\chi_E = \nu_E \|D\chi_E\|$, where ν_E is a $\|D\chi_E\|$-measurable function such that $|\nu_E(x)| = 1$ for $\|D\chi_E\|$-a.e. $x \in \mathbb{R}^N$.

We define the perimeter of E as

$$P(E) := \sup \left\{ \int_E \operatorname{div}(\varphi)\, dx : \varphi \in C^1_c(\mathbb{R}^N; \mathbb{R}^N), \|\varphi\|_\infty \le 1 \right\}$$

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and it can be proved that $P(E) = \|D\chi_E\|(\mathbb{R}^N)$.

The notion of perimeter generalizes the idea of $\mathcal{H}^{N-1}$-measure of the boundary of the set E. It is a well-known fact that the topological boundary of a set of finite perimeter can be very irregular, it can even have full Lebesgue measure. This suggests that for a set of finite perimeter is interesting to consider subsets of ∂E instead. In [3], De Giorgi considered a set of finite $\mathcal{H}^{N-1}$-measure on which $\|D\chi_E\|$ is concentrated, which he called reduced boundary.

Definition 2.2. We say that $x \in \partial^* E$, the *reduced boundary* of E, if

(1) $\|D\chi_E\|(B(x,r)) > 0, \forall r > 0$;
(2) $\lim_{r\to 0} \dfrac{1}{\|D\chi_E\|(B(x,r))} \displaystyle\int_{B(x,r)} \nu_E(y)\, d\|D\chi_E\|(y) = \nu_E(x)$;
(3) $|\nu_E(x)| = 1$.

It can be shown that this definition implies a geometrical characterization of the reduced boundary, by using the blow-up of the set E around a point of $\partial^* E$.

Theorem 2.3. *If $x \in \partial^* E$, then*

$$\frac{E - x}{\varepsilon} \to H^+_{\nu_E}(x) := \{y \in \mathbb{R}^N : y \cdot \nu_E(x) \geq 0\} \quad in \quad L^1_{\mathrm{loc}}(\mathbb{R}^N) \quad as \quad \varepsilon \to 0$$

and

$$\frac{(\mathbb{R}^N \setminus E) - x}{\varepsilon} \to H^-_{\nu_E}(x) := \{y \in \mathbb{R}^N : y \cdot \nu_E(x) \leq 0\} \quad in \quad L^1_{\mathrm{loc}}(\mathbb{R}^N) \quad as \quad \varepsilon \to 0.$$

The proof can be found in [4, Section 5.7.2, Theorem 1]. Formulated in another way, for $\varepsilon > 0$ small enough, $E \cap B(x,\varepsilon)$ is asymptotically close to the half ball $H^-_{\nu_E}(x) \cap B(x,\varepsilon)$.

Because of this result, we call $\nu_E(x)$ *measure theoretic unit interior normal* to E at $x \in \partial^* E$, since it is a generalization of the concept of unit interior normal.

In addition, De Giorgi proved that $\|D\chi_E\| = \mathcal{H}^{N-1} \llcorner \partial^* E$, so that $D\chi_E = \nu_E \mathcal{H}^{N-1} \llcorner \partial^* E$ and $P(E) = \mathcal{H}^{N-1}(\partial^* E)$ (see [4, Section 5.7.3, Theorem 2]).

For every $\alpha \in [0,1]$ we set

$$E^\alpha := \{x \in \mathbb{R}^N : D(E,x) = \alpha\},$$

where

$$D(E,x) := \lim_{r\to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|},$$

and we give the following definitions:

(1) E^1 is called the *measure theoretic interior* of E.
(2) E^0 is called the *measure theoretic exterior* of E.

We recall (see Maggi [6, Example 5.17]) that every Lebesgue measurable set is equivalent to the set of its points of density one; that is,

$$\mathcal{L}^N(E \Delta E^1) = \mathcal{L}^N((\mathbb{R}^N \setminus E)\Delta E^0) = 0. \tag{2.1}$$

It is also a well-know result due to Federer that there exists a set $\mathcal{N}$ with $\mathcal{H}^{N-1}(\mathcal{N}) = 0$ such that $\mathbb{R}^N = E^1 \cup \partial^* E \cup E^0 \cup \mathcal{N}$ (see [1, Theorem 3.61]).

The perimeter $P(E)$ of E is invariant under modifications by a set of $\mathcal{L}^N$-measure zero, even though these modifications might largely increase the size of the topological boundary. In this paper we consider the following representative

$$E := E^1 \cup \partial^* E. \tag{2.2}$$

Given a smooth nonnegative radially symmetric mollifier $\rho \in C^\infty_c(B(0,1))$, we denote the mollification of χ_E by $u_k(x) := (\chi_E * \rho_{\varepsilon_k})(x)$ for some positive sequence $\varepsilon_k \to 0$. We define, for $t \in (0,1)$,

$$A_{k;t} := \{u_k > t\}. \tag{2.3}$$

By Sard's theorem (for which we refer to [6, Lemma 13.15]), we know that, since $u_k : \mathbb{R}^N \to \mathbb{R}$ is C^∞, $\mathcal{L}^1$-a.e. $t \in (0,1)$ is not the image of a critical point for u_k and so $A_{k;t}$ has a smooth boundary for these values of t. Thus, for each k there exists a set $Z_k \subset (0,1)$, with $\mathcal{L}^1(Z_k) = 0$, which is the set of values of t for which $A_{k;t}$ has not a smooth boundary. If we set $Z := \bigcup_{k=1}^{+\infty} Z_k$, then $\mathcal{L}^1(Z) = 0$ and, for each $t \in (0,1) \setminus Z$ and for each k, $A_{k;t}$ has a smooth boundary.

It is a well-known result from BV theory (see for instance [1, Corollary 3.80]) that every function of bounded variations u admits a representative which is the pointwise limit $\mathcal{H}^{N-1}$-a.e. of any mollification of u and which coincides $\mathcal{H}^{N-1}$-a.e. with the precise representative u^*:

$$u^*(x) := \begin{cases} \lim_{r \to 0} \dfrac{1}{|B(x,r)|} \displaystyle\int_{B(x,r)} u(y)\, dy & \text{if this limit exists} \\ 0 & \text{otherwise} \end{cases}.$$

For any set of finite perimeter E, we denote the precise representative of the function χ_E by u_E, which is given by

$$u_E(x) = \begin{cases} 1, & x \in E^1 \\ 0, & x \in E^0 \\ \frac{1}{2}, & x \in \partial^* E \end{cases}.$$

Since $\mathcal{H}^{N-1}(\mathbb{R}^N \setminus (E^1 \cup \partial^* E \cup E^0)) = 0$, the function u_E is well defined $\mathcal{H}^{N-1}$-a.e..

In order to prove Theorem 4.1, we need to use the classical coarea formula, for which we refer to [4, Section 3.4, Theorem 1].

Theorem 2.4. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz. Then, for any $\mathcal{L}^N$-measurable set A, we have*

$$\int_A |\nabla u|\, dx = \int_{\mathbb{R}} \mathcal{H}^{N-1}(A \cap u^{-1}(t))\, dt. \tag{2.4}$$

3. The approximation of E with respect to any $\mu << \mathcal{H}^{N-1}$

The one-sided approximation theorem allows to extend (1.1) to any Radon measure μ such that $\mu << \mathcal{H}^{N-1}$. More precisely, for any bounded set of finite perimeter E, there exist smooth sets $E_{k;i}$, $E_{k;e}$, such that

$$\mu(E_{k;i}) \to \mu(E^1), \quad P(E_{k;i}) \to P(E) \tag{3.1}$$

and

$$\mu(E_{k;e}) \to \mu(E), \quad P(E_{k;e}) \to P(E). \tag{3.2}$$

The convergence of the perimeters in (3.1) and (3.2) follows as in the standard proof of (1.1). However, the convergence with respect to μ is a consequence of the following result.

Theorem 3.1. *Let μ be a Radon measure such that $\mu << \mathcal{H}^{N-1}$ and E be a bounded set of finite perimeter in $\mathbb{R}^N$. Then:*

- **(a)** $\|\mu\| (E^1 \Delta A_{k;t}) \to 0$, *for* $\frac{1}{2} < t < 1$;
- **(b)** $\|\mu\| (E \Delta A_{k;t}) \to 0$, *for* $0 < t < \frac{1}{2}$.

Proof. We have

$$u_k(x) \to u_E(x) \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x. \tag{3.3}$$

Since $\{0 < |u_k - u_E| \leq 1\} \subset E_\delta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, E) \leq \delta\}$, for any k if $\delta > \max \varepsilon_k$, and E_δ is bounded, then we can apply the dominated convergence theorem with respect to the measure $\|\mu\|$, taking 1 as summable majorant since μ is a Radon measure. Hence, for any $\varepsilon > 0$, there exists k large enough such that, if $\frac{1}{2} < t < 1$, we have

$$\begin{aligned}\varepsilon &\geq \int_{\mathbb{R}^n} |u_k(x) - u_E(x)| d\,\|\mu\| \\ &\geq \int_{A_{k;t}\setminus E^1} |u_k(x) - u_E(x)| d\,\|\mu\| + \int_{E^1\setminus A_{k;t}} |u_E(x) - u_k(x)| d\,\|\mu\| \\ &\geq (t - \frac{1}{2}) \|\mu\| (A_{k;t} \setminus E^1) + (1 - t) \|\mu\| (E^1 \setminus A_{k;t}) \\ &\geq \min\left\{t - \frac{1}{2}, 1 - t\right\} \|\mu\| (A_{k;t} \Delta E^1).\end{aligned}$$

Thus, for k large enough and $\frac{1}{2} < t < 1$, we obtain

$$\|\mu\| (A_{k;t} \Delta E^1) \leq \frac{\varepsilon}{\min\left\{t - \frac{1}{2}, 1 - t\right\}},$$

which is (a). Analogously, for $0 < t < \frac{1}{2}$, we have

$$\begin{aligned}\varepsilon &\geq \int_{\mathbb{R}^n} |u_k(x) - u_E(x)| d\,\|\mu\| \\ &\geq \int_{A_{k;t}\setminus E} |u_k(x) - u_E(x)| d\,\|\mu\| + \int_{E\setminus A_{k;t}} |u_E(x) - u_k(x)| d\,\|\mu\| \\ &\geq t \|\mu\| (A_{k;t} \setminus E) + (\frac{1}{2} - t) \|\mu\| (E \setminus A_{k;t}) \\ &\geq \min\left\{t, \frac{1}{2} - t\right\} \|\mu\| (A_{k;t} \Delta E).\end{aligned}$$

Thus, for large k and $0 < t < \frac{1}{2}$,

$$\|\mu\| (A_{k;t} \Delta E) \leq \frac{\varepsilon}{\min\left\{t, \frac{1}{2} - t\right\}},$$

which gives (b). □

Remark 3.2. The convergence in (3.1) follows easily from Theorem 3.1: we have

$$|\mu(E^1) - \mu(A_{k;t})| = |\mu(E^1 \setminus A_{k;t}) - \mu(A_{k;t} \setminus E^1)|$$

and it is clear that (a) implies

$$|\mu(E^1 \setminus A_{k;t})| \leq \|\mu\| (E^1 \setminus A_{k;t}) \to 0,$$
$$|\mu(A_{k;t} \setminus E^1)| \leq \|\mu\| (A_{k;t} \setminus E^1) \to 0.$$

One can show (3.2) in a similar way using (b).
We also notice that Theorem 3.1 has been proved for any $t \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. However, since the sets $A_{k;t}$ have smooth boundary only for almost every t, we shall consider only $t \notin Z$, where Z is the set of singular values defined in the preliminaries.

Remark 3.3. With $\mu = \mathcal{H}^{N-1} \llcorner \partial^* E$, we obtain from Theorem 3.1:
(a) $\mathcal{H}^{N-1}(\partial^* E \cap A_{k;t}) \to 0$ for $\frac{1}{2} < t < 1$;
(b) $\mathcal{H}^{N-1}(\partial^* E \cap (\mathbb{R}^N \setminus A_{k;t})) \to 0$ for $0 < t < \frac{1}{2}$.
Indeed, this is clear from the following identities

$$\partial^* E \cap (E^1 \Delta A_{k;t}) = \partial^* E \cap [(E^1 \setminus A_{k;t}) \cup (A_{k;t} \setminus E^1)] = \partial^* E \cap A_{k;t},$$

$$\partial^* E \cap (E \Delta A_{k;t}) = \partial^* E \cap [(E \setminus A_{k;t}) \cup (A_{k;t} \setminus E)] = \partial^* E \cap (\mathbb{R}^N \setminus A_{k;t}).$$

Remark 3.4. Using Remark 3.3 we can also show that we have:
(a) $\mathcal{H}^{N-1}(\partial^* E \cap u_k^{-1}(t)) \to 0$ for $\frac{1}{2} < t < 1$;
(b) $\mathcal{H}^{N-1}(\partial^* E \cap u_k^{-1}(t)) \to 0$ for $0 < t < \frac{1}{2}$.
Indeed, $u_k^{-1}(t) \subset A_{k;s}$ for $\frac{1}{2} < s < t < 1$ and $u_k^{-1}(t) \subset \mathbb{R}^N \setminus A_{k;s}$ for $0 < t \leq s < \frac{1}{2}$.

In addition, we observe that $\|\mu\|\,(u_k^{-1}(t)) = 0$ for $\mathcal{L}^1$-a.e. t, since μ is a Radon measure. It is in fact clear that $u_k^{-1}(t) = \partial A_{k;t}$, that $\overline{A_{k;t}} \subset A_{k;s} \subset A_{k;0}$ if $0 < s < t < 1$, with $A_{k;0}$ bounded, and that the sets $\partial A_{k;t}$ are pairwise disjoint. Hence, since $\|\mu\|$ is finite on bounded sets and additive, the set

$$\{t \in (0,1) : \|\mu\|\,(\partial A_{k;t}) > \varepsilon\}$$

is finite for any $\varepsilon > 0$. This implies that the set $\{t \in (0,1) : \|\mu\|\,(\partial A_{k;t}) > 0\}$ is at most countable (see also the observation at the end of Section 1.4 of [1]).

Then we obtain also:
(a) $\mathcal{H}^{N-1}(\partial^* E \cap u_k^{-1}(t)) = 0$ for a.e. $\frac{1}{2} < t < 1$;
(b) $\mathcal{H}^{N-1}(\partial^* E \cap u_k^{-1}(t)) = 0$ for a.e. $0 < t < \frac{1}{2}$.

4. The main approximation result

The following theorem, together with Theorem 3.1, shows that indeed we have an *interior* approximation of E for a.e. $t \in (\frac{1}{2}, 1)$.

Theorem 4.1. *Let E be a set of finite perimeter in $\mathbb{R}^N$. There exists a sequence ε_k converging to 0 such that, if $u_k := \chi_E * \rho_{\varepsilon_k}$, we have*

$$\lim_{k\to+\infty} \mathcal{H}^{N-1}(u_k^{-1}(t) \setminus E^1) = 0 \tag{4.1}$$

for a.e. $t \in (\frac{1}{2}, 1)$.

Proof. We take $s > \frac{1}{2}$ and a sequence ε_k, with $\varepsilon_k \to 0$, and we consider the set $A_{k;s} := \{u_k > s\}$. By the coarea formula (2.4), we have

$$\begin{aligned} \int_{A_{k;s}\setminus E^1} |\nabla u_k|\,dx &= \int_0^1 \mathcal{H}^{N-1}(u_k^{-1}(t) \cap (A_{k;s} \setminus E^1))\,dt \\ &= \int_s^1 \mathcal{H}^{N-1}(u_k^{-1}(t) \setminus E^1)\,dt, \end{aligned} \tag{4.2}$$

since, for $t \leq s$, $u_k^{-1}(t) \cap (A_{k;s} \setminus E^1) = \emptyset$, while, for $t > s$, $u_k^{-1}(t) \cap (A_{k;s} \setminus E^1) = u_k^{-1}(t) \setminus E^1$.

We claim that

$$\|\nabla u_k\|_{L^1(A_{k;s}\setminus E^1)} \to 0. \tag{4.3}$$

In order to prove the claim, we observe that, for any $x \in \mathbb{R}^N$,

$$\begin{aligned} \nabla u_k(x) = \int_{\mathbb{R}^N} \chi_E(y) \nabla_x \rho_{\varepsilon_k}(x-y)\,dy &= -\int_{\mathbb{R}^N} \chi_E(y) \nabla_y \rho_{\varepsilon_k}(x-y)\,dy \\ &= \int_{\mathbb{R}^N} \rho_{\varepsilon_k}(x-y) \nu_E(y)\,d\,\|D\chi_E\|\,(y) = (\rho_{\varepsilon_k} * D\chi_E)(x). \end{aligned}$$

Hence, $\nabla u_k = (D\chi_E * \rho_{\varepsilon_k}) = (\|D\chi_E\|\,\nu_E * \rho_{\varepsilon_k})$, which implies

$$|\nabla u_k| \leq \|D\chi_E\| * \rho_{\varepsilon_k}. \tag{4.4}$$

Recalling from (2.1) that $\mathcal{L}^N(E\Delta E^1) = 0$, (4.4) leads to

$$\begin{aligned}\|\nabla u_k\|_{L^1(A_{k;s}\setminus E^1)} &= \int_{\mathbb{R}^N} |\nabla u_k|\, \chi_{A_{k;s}\setminus E}\, dx \\ &\le \int_{\mathbb{R}^N} (\|D\chi_E\| * \rho_{\varepsilon_k})\, \chi_{A_{k;s}\setminus E}\, dx = \int_{\mathbb{R}^N} (\rho_{\varepsilon_k} * \chi_{A_{k;s}\setminus E})\, d\,\|D\chi_E\| = \\ &= \int_{\partial^* E} (\rho_{\varepsilon_k} * \chi_{A_{k;s}\setminus E})\, d\mathcal{H}^{N-1}.\end{aligned}$$

Thus, we need to investigate, for any $x \in \partial^* E$, the behaviour of $(\rho_{\varepsilon_k} * \chi_{A_{k;s}\setminus E})(x)$ as $k \to +\infty$. We have

$$\begin{aligned}(\rho_{\varepsilon_k} * \chi_{A_{k;s}\setminus E})(x) &= \int_{\mathbb{R}^N} \varepsilon_k^{-N} \rho\left(\frac{x-y}{\varepsilon_k}\right) \chi_{A_{k;s}}(y)\, \chi_{(\mathbb{R}^N\setminus E)}(y)\, dy \\ &= [y = x + \varepsilon_k z] = \int_{B(0,1)} \rho(z)\, \chi_{A_{k;s}}(x+\varepsilon_k z)\, \chi_{(\mathbb{R}^N\setminus E)}(x+\varepsilon_k z)\, dz.\end{aligned}$$

We observe that $x + \varepsilon_k z \in \mathbb{R}^N \setminus E$ if and only if $z \in \frac{(\mathbb{R}^N\setminus E)-x}{\varepsilon_k}$, hence

$$\chi_{(\mathbb{R}^N\setminus E)}(x+\varepsilon_k \cdot) = \chi_{\frac{(\mathbb{R}^N\setminus E)-x}{\varepsilon_k}}(\cdot) \to \chi_{H^-_{\nu_E(x)}}(\cdot) \;\text{ in }\; L^1(B(0,1)) \;\text{ as }\; k \to +\infty.$$

In particular, this means that the L^1 limit of $\chi_{(\mathbb{R}^N\setminus E)}(x+\varepsilon_k z)$ is not $\mathcal{L}^N$-a.e. zero only if $z \cdot \nu_E(x) \le 0$, so we can restrict the integration domain to $B(0,1) \cap H^-_{\nu_E(x)}$. On the other hand, $x + \varepsilon_k z \in A_{k;s} = \{u_k > s\}$ if and only if $u_k(x+\varepsilon_k z) > s$. We see that

$$\begin{aligned}u_k(x+\varepsilon_k z) &= \int_{\mathbb{R}^N} \rho_{\varepsilon_k}(x+\varepsilon_k z - y)\, \chi_E(y)\, dy \\ &= [y = x + \varepsilon_k z + \varepsilon_k u] = \int_{B(0,1)} \rho(u)\, \chi_E(x + \varepsilon_k(u+z))\, du.\end{aligned}$$

Arguing as before, we obtain $\chi_E(x+\varepsilon_k(z+\cdot)) \to \chi_{H^+_{\nu_E(x)}}(z+\cdot)$ in $L^1(B(0,1))$ as $k \to +\infty$, for any $x \in \partial^* E$ and $z \in B(0,1)$. Now, we recall that $z\cdot\nu_E(x) \le 0$, and since we have $\chi_{H^+_{\nu_E(x)}}(z+u) = 1$ if and only if $0 \le (z+u)\cdot\nu_E(x)$, we conclude that $0 \le -z\cdot\nu_E(x) \le u\cdot\nu_E(x) \le 1$; that is, u belongs to the half ball $B(0,1) \cap H^+_{\nu_E}(x)$. This implies that, for any $x \in \partial^* E$ and $z \in B(0,1) \cap H^-_{\nu_E}(x)$,

$$\lim_{k\to+\infty} u_k(x+\varepsilon_k z) := v(x,z) = \int_{B(0,1)} \rho(u)\, \chi_{H^+_{\nu_E(x)}}(z+u)\, du \le \frac{1}{2}. \tag{4.5}$$

Therefore, since $0 \le \chi_{A_{k;s}}(x+\varepsilon_k z) \le 1$ and $0 \le \chi_{(\mathbb{R}^N\setminus E)}(x+\varepsilon_k z) \le 1$, these calculations yield

$$\begin{aligned}(\rho_{\varepsilon_k} * \chi_{A_{k;s}\setminus E})(x) &= \int_{B(0,1)} \rho(z)\, \chi_{A_{k;s}}(x+\varepsilon_k z)\, \chi_{(\mathbb{R}^N\setminus E)}(x+\varepsilon_k z)\, dz \\ &\to \int_{B(0,1)} \rho(z)\, \chi_{\{v(x,z)>s\}}(z)\, \chi_{H^-_{\nu_E(x)}}(z)\, dz,\end{aligned} \tag{4.6}$$

for any $x \in \partial^* E$.

Equation (4.5) shows then that the limit in (4.6) is identically zero, since

$$\left\{ z \in \mathbb{R}^N : v(x,z) > s > \frac{1}{2} \right\} \cap B(0,1) \cap H^-_{\nu_E}(x) = \emptyset,$$

for any $x \in \partial^* E$.

We can now apply the Lebesgue dominated convergence theorem with respect to the measure $\mathcal{H}^{N-1} \llcorner \partial^* E$ and the sequence of functions $\rho_{\varepsilon_k} * \chi_{A_{k;s}\setminus E}$ (since the constant 1 is clearly a summable majorant), thus obtaining (4.3).

Finally, up to passing to another subsequence (which we shall keep calling ε_k with a little abuse of notation), (4.2) and (4.3) yield (4.1), for a.e. $t > s$. Since $s > \frac{1}{2}$ is fixed arbitrarily, we can conclude that (4.1) is valid for a.e. $t > \frac{1}{2}$.

□

4.1. **Remark.** *An analogous result holds for the measure theoretic exterior; namely, there exists a sequence ε_k converging to 0 such that, if $u_k := \chi_E * \rho_{\varepsilon_k}$, we have*

$$\lim_{k \to +\infty} \mathcal{H}^{N-1}(u_k^{-1}(t) \setminus E^0) = 0 \tag{4.7}$$

for a.e. $t \in (0, \frac{1}{2})$.

4.2. **Remark.** *It is not difficult to see that Theorem 3.1 and Theorem 4.1 are also valid if we work in an open set Ω and we consider a set of finite perimeter $E \subset\subset \Omega$ (which is the framework of* [2]*). Indeed, the extension to 0 in $\mathbb{R}^N \setminus \Omega$ of the function χ_E is a function of bounded variations in $\mathbb{R}^N$; that is, E can be seen as a bounded set of finite perimeter in $\mathbb{R}^N$. In this way, the previous results follow.*

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