THE ENTROPIC REGULARIZATION OF THE MONGE PROBLEM ON THE REAL LINE

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Abstract. We study the entropic regularization of the optimal transport problem in dimension 1 when the cost function is the distance $c(x,y) = |y-x|$. The selected plan at the limit is, among those which are optimal for the non-penalized problem, the most "diffuse" one on the zones where it may have a density.

1. Introduction

In this paper, we are concerned with the following approximation of the optimal transportation problem: given two probability measures μ, ν on $\mathbb{R}^d$ and $\varepsilon > 0$, find the minimizer of

$$J_\varepsilon : \gamma \mapsto \iint |y-x| \,\mathrm{d}\gamma(x,y) + \varepsilon \iint \log \frac{\mathrm{d}\gamma}{\mathrm{d}(\mu\otimes\nu)} \,\mathrm{d}\gamma \tag{1}$$

among all the measures γ on $\mathbb{R}^d \times \mathbb{R}^d$ which have a density with respect to $\mu \otimes \nu$ and whose first and second marginals are equal to μ and ν, respectively. At the limit $\varepsilon \to 0$, we expect the minimizer γ_ε to converge to an optimal measure for the energy $\int |y-x| \,\mathrm{d}\gamma$ with prescribed marginals, and our goal is to understand which one is selected.

The corresponding problem with $\varepsilon = 0$ is the "distance cost" version of the classical optimal transport problem, whose original formulation, due to Monge in the 18th century [26], consists in looking for the map $T : \mathbb{R}^d \to \mathbb{R}^d$ which minimizes

$$\int_{\mathbb{R}^d} c(x, T(x)) \,\mathrm{d}\mu(x)$$

where c is a given positive function and T must satisfy the constraint

$$\text{for any Borel set } B \subset \mathbb{R}^d,\ \mu(T^{-1}(B)) = \nu(B). \tag{2}$$

Due to the difficulty of this image-measure constraint, this problem remained quite difficult to solve during many years. A suitable relaxation, which corresponds to our problem (1) with no penalization, was introduced by Kantorovich in the 40s [19, 20], namely the minimization problem:

$$\inf \left\{ \iint_{\mathbb{R}^d\times\mathbb{R}^d} c(x,y) \,\mathrm{d}\gamma(x,y) \ :\ \gamma \in \Pi(\mu,\nu) \right\} \tag{3}$$

where $\Pi(\mu,\nu)$ is the set of probability measures γ on $\mathbb{R}^d \times \mathbb{R}^d$ having μ and ν as marginals, that is,

$$\text{for any Borel set } B \subset \mathbb{R}^d, \quad \gamma(B \times \mathbb{R}^d) = \mu(B) \text{ and } \quad \gamma(\mathbb{R}^d \times B) = \nu(B). \tag{4}$$

The problem (3) is a generalization of the Monge's problem as, from any map T satisfying the above measure constraint, it is easy to build a measure $\gamma \in \Pi(\mu,\nu)$ which is concentrated on the graph of T and has same total energy; moreover, due to the compactness properties of $\Pi(\mu,\nu)$ for the weak topology of measures, proving the existence of solutions of (3) is easy by the direct methods of the calculus of variations. Thanks to a suitable convex duality argument,

it is possible to prove [11, 12, 18], under suitable assumptions on the data and the cost function c (which include the case $c = |y - x|^p$ for $1 < p < +\infty$), that the problem (3) admits a unique solution which is induced by a map T, yielding optimality of this map for the original Monge problem. In the case $c = |y - x|$, the existence results are more recent and the uniqueness is not guaranteed anymore, *cf.* [1]. We refer to the monographs [3, 31, 32, 28] for a complete overview of the optimal transportation theory.

Although it was already present in much earlier works, the penalization (1) has been recently reintroduced for numerical reasons. Indeed, computing numerically the optimal transport map T remained for a long time a very challenging problem; a first major achievement appeared in the beginning of the 2000s, when Benamou and Brenier [5] introduced the so-called *dynamical formulation* (based on the minimization of the kinetic energy among the curves of measures and velocity fields satisfying a mass conservation equation), which can be solved by an augmented Lagragian method after a convex change of variables. Let us also mention the algorithms due to Angenent, Hacker and Tannenbaum [4], which also rely on fluid-mechanics formulations, and more recently the approaches by discretization of the Monge-Ampère equation [7, 8] or via semi-discrete optimal transport [25, 23].

The numerical interest of the approaches similar to (1) has been show in the last few years. The general idea consists in perturbing the Kantorovich problem by the so-called *entropy functional* defined as

$$\mathrm{Ent}(\gamma|\rho) = \begin{cases} \displaystyle\int \log\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\rho}\right)\mathrm{d}\gamma & \text{if } \gamma \ll \rho, \\ +\infty & \text{otherwise.} \end{cases}$$

and in focusing, given a suitable small parameter $\varepsilon > 0$, on the problem

$$\inf\left\{\iint c(x,y)\,\mathrm{d}\gamma(x,y) + \varepsilon\,\mathrm{Ent}(\gamma|\mu\otimes\nu) \,:\, \gamma \in \Pi(\mu,\nu)\right\}. \tag{5}$$

Among other properties, the function $\mathrm{Ent}(\cdot|\mu\otimes\nu)$ enforces γ to be an absolutely continuous measure with respect to the tensor product $\mu\otimes\nu$ and favors such transport plans which are *as diffuse as possible* (on the set $\Pi(\mu,\nu)$, its unique minimizer is $\mu\otimes\nu$). The idea of the entropic penalization goes actually back from Schrödinger's works [29]; moreover, the algebraic properties of the entropy functional and of the dual problem of (5) make much easier the numerical computation of its solution, thanks to the so-called *Sinkhorn's algorithm* involving alternated projections. We refer to the papers [15, 6] and the PhD. thesis [27] for more details on the theoretical and numerical properties of this class of algorithms.

From a theoretical point of view, the convergence of the solution of (5) as $\varepsilon \to 0$ has been recently proven in [14] (we also mention [21, 22] in which similar problems are studied in a much more abstract framework): therein, the authors showed that the family of functionals

$$\gamma \in \Pi(\mu,\nu) \mapsto \int c\,\mathrm{d}\gamma + \varepsilon\,\mathrm{Ent}(\gamma|\mu\otimes\nu) \tag{6}$$

is Γ-converging, as $\varepsilon \to 0$, to the transport energy $\gamma \mapsto \int c\,\mathrm{d}\gamma$. Therefore, when c is one of the costs for which there exists a unique optimal plan γ for the Kantorovich problem (3), the family $(\gamma_\varepsilon)_\varepsilon$ of minimizers of (6) converges to γ.

In this paper, we are interested in the following theoretical question: *what's happening if there are several minimal plans γ for the cost function c?* As usual in minimization problems of penalized functionals, it is natural to guess that, when the set $\mathcal{O}_c$ of optimal plans for cost function c has at least two elements, the family $(\gamma_\varepsilon)_\varepsilon$ of minimizers of J_ε will converge to the "most diffuse" element of $\mathcal{O}_c$, namely the one which minimizes the entropy functional $E(\cdot|\mu\otimes\nu)$.

However, the existence of an optimal plan with *finite entropy* is far to be clear, and is generally false depending on the cost function c and the data.

We here focus on the case where $c(x, y) = |y - x|$, which was the original cost function proposed by Monge. This cost function is certainly the most studied one for which it is known that the uniqueness of optimal plan and optimal map fails, and approximating this transport energy through more "regular" ones is quite common: this is classically done through strictly convex costs which brings to the *monotone transport*, see for instance [13, 17, 30] for existence of optimal maps and [24] for very partial regularity results (we also mention the quite different approximation proposed in [16], where the "regularizing" term enforces the transport plan to be induced by a regular map). In the present paper, we concentrate on the entropic regularization of this problem where the measures are supported on the real line. In higher dimension, let us just notice that the structure of optimal Monge's plans is much more complicated, involving the geometric notion of *transport rays* (see for instance [1] or [28, Chapter 3]), and although one may guess that the "most diffuse transport plans on each rays" will be selected, attacking this regularization problem on the Euclidean space would probably require very different techniques from ours.

In the one-dimensional case, the contributions and results of this paper are the following:

- First of all, we need a complete description of the set of one-dimensional optimal transport plans for the distance cost $c(x, y) = |y - x|$; although this result is natural and its proof is not complicated, it was not present in the literature at that time for the best of our knowledge. The structure of optimal plans is described in Proposition 3.1, and can be summarized as follows: denoting by T the monotone rearrangement (that is, the unique non-decreasing transport map) between μ to ν, the optimal plans γ are those such that, if (x, y) is in the support of γ, then
  - if $T(x) = x$, then y must be exactly equal to x;
  - if x belongs to some interval where $T - \mathrm{id} > 0$, then y must belong to the same interval and be larger than x; and the analogous constraint holds where $T - \mathrm{id} < 0$.

  In particular, any optimal transport plan can be decomposed into a "bad plan" which is concentrated on the graph on the identity map (and therefore is singular with respect to the Lebesgue measure), and a "possibly good plan" which must only respect the sign of $T - \mathrm{id}$, and is allowed to have a density.
- Once stated this structural result, our main theorem studies the Γ-convergence of the penalized functional. Actually, we need to consider the "rescaled functional"

$$F_\varepsilon : \gamma \mapsto \frac{1}{\varepsilon}\left(\int |y - x| \,\mathrm{d}\gamma - W_1(\mu, \nu)\right) + \mathrm{Ent}(\gamma|\mu \otimes \nu) - \mu(A)|\log(2\varepsilon)|,$$

  where A is the set $\{x \in \mathbb{R} : T(x) = x\}$ and $W_1(\mu, \nu)$ the minimal transport energy for the distance cost. For fixed ε, the functional F_ε admits exactly the same minimizers as J_ε. In Theorem 4.1 we prove, under technical assumptions on the data, that its Γ-limit is $+\infty$ outside of the set of optimal plans, and that, in this set, it is equal to

$$F : \gamma \mapsto \mathrm{Ent}(\gamma \llcorner (\mathbb{R} \setminus A)^2 | \mu \otimes \nu) + C$$

  where the constant C only depends on the data. In particular, the unique minimizer of F is the plan which is optimal for distance cost and whose "good part" is "as diffuse as possible", as we could naturally expect. Moreover, this result implies the asymptotic expansion

$$\min J_\varepsilon = W_1(\mu, \nu) + \mu(A)\varepsilon|\log(2\varepsilon)| + \varepsilon \min F + \mathrm{o}(\varepsilon);$$

notice that the excess of order $\varepsilon|\log\varepsilon|$ is a common phenomenon with the penalization of the Monge problem proposed in [16].

- Finally, Section 5 studies the explicit form of the optimal plan which is selected at the limit, *i.e.* the minimizer of F; the precise result is stated in Theorem 5.1, which also gives a necessary and sufficient condition for F to be not identically $+\infty$. This section relies on entropy-minimization problems of "good part of optimal plans", which can be re-written
$$\min\Big\{\operatorname{Ent}\big(\gamma\,|\,(\mu(x)\otimes\nu(y))\llcorner(I\times I)\cdot\mathbb{1}_{y\geq x}\big)\Big\}$$
on maximal positivity intervals I of $T-\mathrm{id}$ (and as analogous minimization problems on their negative counterparts). Similar problems were already deeply analyzed in [9, 10], and we here provide new self-contained proofs more suitable for our needs.

2. Notations and preliminary results

In this section, we collect the notations and well-known facts on measure theory and optimal transportation which will be used throughout the paper.

Let X and Y be two Polish spaces (in this paper, we will only focus on the case $X=Y=\mathbb{R}$), and μ, ν be two positive measures on X, Y whose total masses are finite and equal. We denote $\Pi(\mu,\nu)$ the set of *transport plans* from μ to ν, that is, the set of positive measures on $X\times Y$ satisfying (4); recall that this constraint can be reformulated as
$$\text{for any } (\varphi,\psi)\in C_b(X)\times C_b(Y),\ \iint\varphi(x)\,\mathrm{d}\gamma(x,y)=\int\varphi\,\mathrm{d}\mu \text{ and } \iint\psi(y)\,\mathrm{d}\gamma(x,y)=\int\psi\,\mathrm{d}\nu.$$
When $T:X\to Y$ is a map, we call it *transport map* from μ to ν if it satisfies (2). This is equivalent to say that the measure γ_T defined as
$$\text{for any } f\in C_b(X\times Y),\ \iint f\,\mathrm{d}\gamma_T=\int f(x,T(x))\,\mathrm{d}\mu(x)$$
belongs to $\Pi(\mu,\nu)$. Equivalently, this means that the equality $\int\varphi\circ T\,\mathrm{d}\mu=\int\varphi\,\mathrm{d}\nu$ for any $\varphi\in C_b(X)$. A useful property of $\Pi(\mu,\nu)$ is the following:

Proposition 2.1. *Let X,Y be Polish spaces, and μ,ν positive measures on X,Y with finite mass. Then the set $\Pi(\mu,\nu)$ is compact with respect to the narrow topology of measures.*

We now introduce the definition of *relative entropy*:

Definition 2.1. *Let ρ be a fixed positive measure on $\mathbb{R}^d$, $d\geq 1$. For any positive measure γ on $\mathbb{R}^d$, we set*
$$\operatorname{Ent}(\gamma|\rho):=\begin{cases}\displaystyle\int\log\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\rho}\right)\mathrm{d}\gamma & \text{if } \gamma\ll\rho,\\ +\infty & \text{otherwise.}\end{cases}$$
and the functional $\operatorname{Ent}(\cdot|\rho)$ *is called* relative entropy with respect to ρ.

The following properties of the functional Ent are classical (see for instance [2, Theorem 2.34]):

Proposition 2.2. *Let ρ be a fixed positive measure on $\mathbb{R}^d$, $d\geq 1$. Then the functional* $\operatorname{Ent}(\cdot|\rho)$ *is strictly convex, and it is lower semi-continuous with respect to the narrow convergence of measures.*

In this paper, we are interested in the optimal transport problem when c is given by the distance $c(x,y)=|y-x|$. In that case, the so-called *duality formula of the optimal transport problem* takes the following form:

Theorem 2.1. *Given $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, we have the equality*

$$\inf\left\{\int |y-x|\,\mathrm{d}\gamma(x,y) \,:\, \gamma \in \Pi(\mu,\nu)\right\} = \sup\left\{\int u\,\mathrm{d}\nu - \int u\,\mathrm{d}\mu \,:\, u \in \mathrm{Lip}_1(\mathbb{R}^d)\right\} \tag{7}$$

where $\mathrm{Lip}_1(\mathbb{R}^d)$ stands for the set of Lipschitz functions on $\mathbb{R}^d$ having Lipschitz constant at most 1. The common optimal value will be denoted by $W_1(\mu,\nu)$ and the set of optimal transport plans by $\mathcal{O}_1(\mu,\nu)$; a maximizer of the dual problem will be called Kantorovich potential.

As a consequence of (7), given a function $u \in \mathrm{Lip}_1(\mathbb{R}^d)$ and a transport plan $\gamma \in \Pi(\mu,\nu)$, the following properties are equivalent:

(i) u is a Kantorovich potential and $\gamma \in \mathcal{O}_1(\mu,\nu)$;
(ii) for γ-a.e. (x,y), $|y-x| = u(y) - u(x)$.

The last theoretical notion we will need in this paper is the notion of Γ-convergence:

Definition 2.2. *Let X be a complete metric space and $(F_n)_n, F$ be functionals $X \to \mathbb{R}\cup\{+\infty\}$. We say that $(F_n)_n$ Γ-converges to F if, for any $x \in X$, the two following inequalities are satisfied:*

- *for any sequence $(x_n)_n$ of X converging to x, $\liminf_n F_n(x_n) \geq F(x)$ (Γ-liminf inequality);*
- *there exists a sequence $(x_n)_n$ of X converging to x and such that $\limsup_n F_n(x_n) \leq F(x)$ (Γ-limsup inequality).*

The main interests of this notion are its implications in terms of convergence of minima and minimizers:

Theorem 2.2. *Let $(F_n)_n$ be a sequence of functions $X \to \mathbb{R}\cup\{\pm\infty\}$ and assume that $F_n \xrightarrow[n]{\Gamma} F$. Assume moreover that there exists a compact and non-empty subset K of X such that*

$$\forall n \in N,\ \inf_X F_n = \inf_K F_n$$

(we say that $(F_n)_n$ is equi-mildly coercive on X). Then F admits a minimum on X and the sequence $(\inf_X F_n)_n$ converges to $\min F$. Moreover, if $(x_n)_n$ is a sequence of X such that

$$\lim_n F_n(x_n) = \lim_n (\inf_X F_n)$$

and if $(x_{\varphi(n)})_n$ is a subsequence of $(x_n)_n$ having a limit x, then $F(x) = \inf_X F$.

In our settings, all the functionals that we consider will be defined on the space $\Pi(\mu,\nu)$. Thanks to its compactness property, the equi-coercivity assumption of Theorem 2.2 will always be satisfied in this paper: therefore, in order to conclude that the minimal values and the minimizers of a given family of functionals are converging to those of a given "target" functional, checking the upper and lower limit conditions of Def. 2.2 will be enough.

We recall now the already known Γ-convergence result in entropic regularization, see [14, Theorem 2.7] (where the proof is given for $c(x,y) = |y-x|^2$, but can easily be adapted for a much larger class of cost functions):

Theorem 2.3 (Zeroth-order Γ-convergence)**.** *Let $c : \mathbb{R}^d \to \mathbb{R}^d$ be continuous and μ, ν be two probability measures on $\mathbb{R}^d$ having compact support and being absolutely continuous with respect to the Lebesgue measure. Assume that*

$$\int \log \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^d}\,\mathrm{d}\mu,\ \int \log \frac{\mathrm{d}\nu}{\mathrm{d}\mathcal{L}^d}\,\mathrm{d}\nu < +\infty.$$

Then the family of functionals

$$J_\varepsilon : \gamma \in \Pi(\mu,\nu) \mapsto \int c\,\mathrm{d}\gamma + \varepsilon\,\mathrm{Ent}(\gamma|\mu\otimes\nu)$$

is Γ*-converging, as* $\varepsilon \to 0$*, to* $\gamma \mapsto \displaystyle\int c \, \mathrm{d}\gamma$.

We end this section by the "first-order Γ-convergence" result, which completes the previous one:

Proposition 2.3 (First-order Γ-convergence)**.** *Under the same assumptions as Theorem 2.3, denoting by* $W_c(\mu,\nu)$ *the minimal transport energy for cost* c *and by* $\mathcal{O}_c(\mu,\nu)$ *the set of corresponding optimal plans, the family of functionals*

$$H_\varepsilon : \gamma \in \Pi(\mu,\nu) \mapsto \frac{1}{\varepsilon}\left(\int c \, \mathrm{d}\gamma - W_c(\mu,\nu)\right) + \mathrm{Ent}(\gamma|\mu\otimes\nu)$$

is Γ*-converging to*

$$H : \gamma \in \Pi(\mu,\nu) \mapsto \begin{cases} \mathrm{Ent}(\gamma|\mu\otimes\nu) & \text{if } \gamma \in \mathcal{O}_c(\mu,\nu), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. The Γ-limsup inequality is trivial since, given $\gamma \in \Pi(\mu,\nu)$:

- if $H(\gamma) = +\infty$, there is nothing to prove;
- if $H(\gamma) < +\infty$, this means that $E(\gamma|\mu\otimes\nu)$ is finite and $\int c \, \mathrm{d}\gamma = W_c(\mu,\nu)$. Taking for $(\gamma_\varepsilon)_\varepsilon$ the constant family equal to γ, we have immediately that $H_\varepsilon(\gamma_\varepsilon) = H(\gamma)$ for any ε, such that the upper limit condition is satisfied.

As for the Γ-liminf, given $\gamma_\varepsilon \to \gamma$ in $\Pi(\mu,\nu)$, then:

- if $\gamma \notin \mathcal{O}_c(\mu,\nu)$, since the entropy is positive on $\Pi(\mu,\nu)$, and $\int c \, d\gamma_\varepsilon \to \int c \, d\gamma$, we have that for ε small enough
$$H_\varepsilon(\gamma_\varepsilon) \geq \frac{1}{\varepsilon}\left(\int c \, \mathrm{d}\gamma_\varepsilon - W_c(\mu,\nu)\right) \geq \frac{1}{2\varepsilon}\left(\int c \, \mathrm{d}\gamma - W_c(\mu,\nu)\right) \to +\infty;$$
- otherwise, it is enough to observe, for any $\varepsilon > 0$, the inequality $H_\varepsilon(\gamma_\varepsilon) \geq \mathrm{Ent}(\gamma_\varepsilon|\mu\otimes\nu)$, and to conclude by semi-continuity of the entropy. □

The consequences of Theorem 2.3 and Prop. 2.3 are the following:

- the minimal value and the minimizers of J_ε are converging (up to subsequences) to the minimal transport energy and to an optimal transport plan for cost c, respectively. In particular, if there is only one optimal transport plan, then this plan is selected at the limit by the entropic regularization;
- if the set $\mathcal{O}_c(\mu,\nu)$ contains at least one plan with finite entropy (which is equivalent to say that H is not identically $+\infty$), then the plan which is selected at the limit is the one having minimal entropy in the set $\mathcal{O}_c(\mu,\nu)$, and we have the asymptotic expansion
$$\min J_\varepsilon = W_c(\mu,\nu) + \varepsilon \min H + \mathrm{o}(\varepsilon) \qquad \text{as } \varepsilon \to 0;$$
- in the converse case, and if there is no uniqueness of the optimal plan for cost c, then the plan which is selected at the limit is unknown, and the only information on the behavior of the minimal value is $(\inf J_\varepsilon - W_c(\mu,\nu))/\varepsilon \to +\infty$.

As we will see later on in the paper, this last case may occur when $c(x,y) = |y-x|$, depending on the data μ and ν; precisely, the next section gives a general description of the optimal plans for distance cost in dimension one.

3. Structure of one-dimensional optimal plans

3.1. **Additional notations and the monotone transport map.** First of all, let us recall the precise definition of the monotone transport map that we will use, which is essentially taken from [28, Chapter 2]. Let μ, ν be two probability measures on $\mathbb{R}$ compactly supported and with no atom, and let us denote by F_μ, F_ν the cumulative distribution functions of μ, ν respectively; we notice that since neither μ or ν have any atom, both F_μ and F_ν are continuous on $\mathbb{R}$. We define

$$T(x) := \inf\Big\{y \in \mathbb{R} \,:\, F_\nu(y) \geq F_\mu(x)\Big\}.$$

We can then prove the following properties, thanks to the boundedness of the supports of μ, ν and to the continuity of F_μ, F_ν:

- the map T is finitely valued on $(\inf(\operatorname{supp}\mu), +\infty)$;
- for any x with $\inf(\operatorname{supp}\mu) < x$, the infimum in the definition of $T(x)$ is attained; therefore,

$$\begin{cases} F_\nu(T(x)) = F_\mu(x) \\ \text{and, for any } y < T(x),\ F_\nu(y) < F_\mu(x)\,; \end{cases} \tag{8}$$

The following theorem then holds:

Theorem 3.1. *The map T is non-decreasing and satisfies $T_\#\mu = \nu$. Moreover, any other such map coincides with T except on a μ-negligible set. Finally, for any cost function c having form $c(x, y) = h(y - x)$ with h convex, the map T is optimal for the transport problem with cost c.*

The following lemma is useful in order to precise the links between F_μ, F_ν and the sign of $T - \mathrm{id}$.

Lemma 3.1. *For μ-a.e. every point of $\mathbb{R}$, the following equivalences are true:*

(a) *$T(x) = x$ if and only if $F_\mu(x) = F_\nu(x)$;*
(b) *$T(x) < x$ if and only if $F_\mu(x) < F_\nu(x)$;*
(c) *$T(x) > x$ if and only if $F_\mu(x) > F_\nu(x)$.*

Proof. Let us start by the equivalence (b). First of all, let x be a point of the support of μ, different of its bounds, and such that $T(x) < x$; in particular, the monotonicity of F_ν implies that $F_\nu(T(x)) \leq F_\nu(x)$. Assume moreover that $F_\mu(x) \geq F_\nu(x)$: keeping in mind that $F_\nu \circ T$ and F_μ coincide, this enforces $F_\nu(x) = F_\nu(T(x))$. We deduce that the interval $(T(x), x)$ does not meet the support of ν: therefore, $T(x)$, which belongs to $\operatorname{supp}\nu$, is actually a boundary point of one of the connected components of $\mathbb{R} \setminus \operatorname{supp}\nu$. These boundary points being countably many, they form a ν-negligible set, and the pre-image of this set by T is therefore μ-negligible, proving that the direct implication in (b) is true for μ-a.e. x.

Conversely, let x be such that $F_\mu(x) < F_\nu(x)$ and assume that $T(x) \geq x$: therefore, since F_ν is nondecreasing,

$$F_\nu(T(x)) \geq F_\nu(x) > F_\mu(x)$$

which is impossible since $F_\nu(T(x)) = F_\mu(x)$.

The equivalence (c) can be proved by pretty similar arguments, and (a) is an obvious consequence of (b) and (c). □

In what follows in the paper, we will often use the following sets:

$$A := \Big\{x \in \mathbb{R} \,:\, F_\mu(x) = F_\nu(x)\Big\}$$

$$\text{and} \qquad A^+ := \Big\{x \in \mathbb{R} \,:\, F_\mu(x) > F_\nu(x)\Big\}, \qquad A^- := \Big\{x \in \mathbb{R} \,:\, F_\mu(x) < F_\nu(x)\Big\}.$$

We notice A^+ and A^- are both open, and that A, which is the complementary set of their union, is closed. Moreover, thanks to the Lemma 3.1, these three sets coincide, up to μ-negligible sets,

with the sets where $T - \mathrm{id}$ is respectively zero, positive and negative. The following remark, though obvious, will be also sometimes useful:

Lemma 3.2. *Let I be a maximal interval fully included in A^- or in A^+. Then $\mu(I) = \nu(I)$.*

Proof. Such an I is necessarily an open and bounded interval of $\mathbb{R}$; calling it (a, b) and using its maximality and the continuity of F_μ, F_ν, we can easily see that $F_\mu(a) = F_\nu(a)$ and $F_\mu(b) = F_\nu(b)$, so that $\mu(I) = \nu(I)$. □

We can now state the precise result on the structure of one-dimensional optimal plans:

Proposition 3.1. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ be atomless and compactly supported. Then the optimal plans for the Monge problem are exactly the transport plans from μ to ν such that, for γ-a.e. $(x, y) \in \mathbb{R}^2$:*

(a) *if $x \in A$, then $y = x$;*
(b) *if $x \in A^+$, then $y \geq x$, and y belongs to the same connected component of A^+ as x;*
(c) *similarly, if $x \in A^-$, then $y \leq x$, and y belongs to the same connected component of A^- as x.*

The key point of the proof of Prop. 3.1 consists in building a suitable Kantorovich potential, which is the topic of the next paragraph.

3.2. Construction of a Kantorovich potential.

Lemma 3.3. *For $x \in \mathbb{R}$, define*

$$u(x) = \int_0^x \big(\mathbb{1}_{A^+} - \mathbb{1}_{A^-}\big)(t)\,\mathrm{d}t.$$

Then u is a Kantorovich potential from μ to ν, that is, a maximizer of the problem (7).

Proof of Lemma 3.3. The fact that u is 1-Lipschitz being obvious, the only non-trivial point of the proof is checking that

$$u(T(x)) - u(x) = |T(x) - x| \quad \text{for } \mu\text{-a.e. } x\,, \tag{9}$$

which will ensure that u is optimal for the dual problem. Let then x be a point of the support of μ such that $T(x) \neq x$ (otherwise the equality (9) is obviously satisfied); assume without loss of generality that $T(x) > x$. By contradiction, suppose that $u(T(x)) - u(x) < |T(x) - x|$ (the converse inequality cannot hold since u is a 1-Lipschitz function): from the definition of u, this implies that $\mathcal{L}([x, T(x)] \setminus A^+) \neq 0$ and in particular there exists a point y such that $x < y < T(x)$ and $y \notin A^+$, that is, $F_\mu(y) \leq F_\nu(y)$. By the property (8), we deduce $y \geq T(y)$: summarizing,

$$\begin{cases} x < y; \\ T(x) > y \geq T(y). \end{cases}$$

Since T is a non-decreasing function, this is impossible. The arguments are identical in the case $T(x) < x$. □

The two following paragraphs are then devoted to the rigorous proof of Prop. 3.1.

3.3. Necessary condition to be optimal.

In this paragraph, we prove that all the optimal plans for the Monge problem must satisfy properties (a)-(c) of Prop. 3.1. We then fix $\gamma \in \mathcal{O}_1(\mu, \nu)$.

Step I: for any $\overline{x} \in A$, we have $\gamma((-\infty, \overline{x}] \times [\overline{x}, +\infty)) = \gamma([\overline{x}, +\infty) \times (-\infty, \overline{x}]) = 0$. Let $\overline{x} \in A$, and assume by contradiction that

$$\gamma((-\infty, \overline{x}] \times [\overline{x}, +\infty)) > 0. \tag{10}$$

Notice that (10) implies that $0 < F_\mu(\overline{x}) = F_\nu(\overline{x}) < 1$. We first show that

$$\gamma([\overline{x}, +\infty) \times (-\infty, \overline{x}]) > 0. \tag{11}$$

Indeed, if (11) was false, then we would have

$$\nu([\overline{x}, +\infty)) = \mu([x, +\infty)) = \gamma([\overline{x}, +\infty) \times \mathbb{R}) = \gamma([\overline{x}, +\infty)^2)$$

and consequently

$$\gamma((-\infty, \overline{x}] \times [\overline{x}, +\infty)) = \nu([\overline{x}, +\infty)) - \gamma([\overline{x}, +\infty) \times [\overline{x}, +\infty)) = 0,$$

which contradicts (10)

From (10) and (11), we deduce that there exists (x_1, y_1) and (x_2, y_2) in the support of γ with

$$x_1 < \overline{x} < y_1 \quad \text{and} \quad y_2 < \overline{x} < x_2.$$

Theorem 2.1 and the continuity of u imply that $u(y_1) - u(x_1) = y_1 - x_1$ and $u(y_2) - u(x_2) = -(y_2 - x_2)$: since u is 1-Lipschitz, this enforces u to be an affine function with slope 1 and -1 on the whole intervals $[x_1, y_1]$ and$[y_2, x_2]$, respectively. But these intervals have non-trivial intersection, which leads to a contradiction.

Step II: γ satisfies the property (a) *of Prop. 3.1.* Let $x \in A$, let y such that (x, y) belongs to the support of γ, and assume by contradiction that $y > x$. Without loss of generality, we may assume

$$(x, y) \cap A \neq \emptyset.$$

Indeed, if this was not the case, x would be a boundary point of one of the connected components of $\mathbb{R} \setminus A$: since this set is open, such points are countably many, forming then a μ-negligible set. Let then $x' \in A$ such that $x < x' < y$. From the result of Step I, it holds

$$\gamma((-\infty, x'] \times [x', +\infty)) = 0. \tag{12}$$

In particular, selecting a positive $\varepsilon < min(x' - x, y - x')$, we deduce from (12) that $\gamma([x \pm \varepsilon] \times [y \pm \varepsilon]) = 0$ so that (x, y) does not belong to the support of γ, a contradiction. Analogously, we prove that the inequality $y < x$ can not hold except for at most countably points x.

Step III: γ satisfies the properties (b) *and* (c) *of Prop. 3.1.* Let $x \in A^+$ and $y \in \mathbb{R}$ be such that (x, y) belongs to the support of γ and $u(y) - u(x) = |y - x|$. First, assume that $y < x$, which implies

$$-\int_y^x (\mathbb{1}_{A^+} - \mathbb{1}_{A^-}) = x - y;$$

in particular, since A^+ is an open set and does not meet A^-, we must have $A^+ \cap (y, x) = \emptyset$; but since x belongs to A^+ which is open, this is impossible.

We then have $y \geq x$; let us now prove that y belongs to the same connected component of A^+ as x. Since A^+ is an open set it suffices to prove that $(x, y) \subset A^+$. As just above, we deduce from the equality $u(y) - u(x) = |y - x|$ that $A^- \cap (x, y) = \emptyset$; therefore it suffices to prove that (x, y) does not meet A. Assume then that there exists $z \in A$ with $x < z < y$; the result of Step I gives then

$$\gamma((-\infty, z] \times [z, +\infty)) = 0.$$

By selecting a positive $\varepsilon < \min(z - x, y - z)$ we obtain again $\gamma([x - \varepsilon, x + \varepsilon] \times [y - \varepsilon, y + \varepsilon]) = 0$, a contradiction since (x, y) belongs to the support of γ. As before, the arguments in the case $x \in A^-$ are identical.

3.4. **Sufficient condition to be optimal.** We now focus on the converse implication of Prop. 3.1 and denote by $(I_j^-)_j$, $(I_k^+)_k$ the connected components of A^-, A^+ respectively (which are at most countably many). We also set for each j,k:

$$\mu_j^- = \mu \llcorner I_j^-,\ \mu_k^+ = \mu \llcorner I_k^+,$$
$$\text{and} \quad \nu_j^- = \nu \llcorner I_j^-,\ \nu_k^+ = \nu \llcorner I_k^+.$$

Lemma 3.2 implies that μ_j^-, ν_j^- (resp. μ_k^+, ν_k^+) have same mass. Moreover, thanks to the previous paragraph, the points (a)-(c) of Prop. 3.1 apply for the plan which is induced by the map T, which enforces, for any j,k,

$$\nu_j^- = T_\# \mu_j^- \qquad \text{and} \qquad \nu_k^+ = T_\# \mu_k^+.$$

Using also Lemma 3.1, we deduce

$$\begin{aligned} W_1(\mu,\nu) &= \int |T(x)-x| \,\mathrm{d}\mu(x) \\ &= \sum_j \int_{I_j^-} (T(x)-x)\,\mathrm{d}\mu(x) + \sum_k \int_{I_k^+} (x-T(x))\,\mathrm{d}\mu(x) \\ &= \sum_j \left(\int y \,\mathrm{d}\nu_j^-(y) - \int x \,\mathrm{d}\mu_j^-(x) \right) + \sum_k \left(\int x \,\mathrm{d}\mu_k^+(x) - \int y \,\mathrm{d}\nu_k^+(y) \right). \end{aligned} \tag{13}$$

Let now γ be a transport plan from μ to ν satisfying conditions (a)-(c) of Prop. 3.1. Then we have

$$\int |y-x| \,\mathrm{d}\gamma(x,y) = \int_{A\times\mathbb{R}} |y-x| \,\mathrm{d}\gamma(x,y) + \sum_j \int_{I_j^-\times\mathbb{R}} |y-x| \,\mathrm{d}\gamma(x,y) + \sum_k \int_{I_k^+\times\mathbb{R}} |y-x| \,\mathrm{d}\gamma(x,y).$$

From the property (a), we immediately deduce that the first integral in the sum above is zero; and from (b) and (c), we infer that, for any j,k,

$$\int_{I_j^-\times\mathbb{R}} |y-x| \,\mathrm{d}\gamma(x,y) = \int_{I_j^-\times I_j^-} (x-y)\,\mathrm{d}\gamma(x,y) = \int x \,\mathrm{d}\mu_j^- - \int y \,\mathrm{d}\nu_j^-$$
$$\text{and} \qquad \int_{I_k^+\times\mathbb{R}} |y-x| \,\mathrm{d}\gamma(x,y) = \int_{I_k^+\times I_k^+} (y-x)\,\mathrm{d}\gamma(x,y) = \int y \,\mathrm{d}\nu_k^+ - \int x \,\mathrm{d}\mu_k^+.$$

Adding on all the indexes j,k and using (13) leads to $\int |y-x| \,\mathrm{d}\gamma(x,y) = W_1(\mu,\nu)$, so that γ is optimal, as required.

4. Γ-convergence result

4.1. **Statement of the main theorem.** Having at hand the notations of the previous section and of Prop. 3.1, we can now state the main result of this paper.

Theorem 4.1. *Let $\mu,\nu \in \mathcal{P}(\mathbb{R})$ be two probability measures with bounded support such that $\mathrm{Ent}(\mu|\mathcal{L}^1), \mathrm{Ent}(\nu|\mathcal{L}^1) < \infty$; in particular they are absolutely continuous with respect to the Lebesgue measure. Assume that the following assumptions are true.*

(H1) *The set $A \cap \operatorname{supp}\mu$ has Lebesgue-negligible boundary, and its interior has form*

$$\operatorname{int}(A \cap \operatorname{supp}\mu) = \bigcup_i (a_i, b_i)$$

where the intervals (a_i, b_i) are pairwise disjoint and satisfy

$$-\sum_i \int_{a_i}^{b_i} \log\big(\min\{x-a_i, b_i-x\}\big)\,\mathrm{d}\mu(x) < +\infty. \tag{14}$$

This assumption is in particular satisfied in the two following cases:

(H1a) *μ has bounded density on A and* $\sum_i |a_i - b_i| \log |a_i - b_i| < +\infty$, *or*

(H1b) *we have* $\sum_i |a_i - b_i|^{1-\delta} < +\infty$ *for some* $\delta > 0$.

(H2) *There exists an optimal plan γ for the Monge problem such that $\gamma \llcorner (\mathbb{R} \setminus A)^2$ has finite entropy with respect to $\mu \otimes \nu$.*

Then the family $(F_\varepsilon)_\varepsilon$ of functionals defined on the set $\Pi(\mu, \nu)$ by

$$F_\varepsilon(\gamma) = \frac{1}{\varepsilon}\left(\int |y - x| \, \mathrm{d}\gamma - W_1(\mu, \nu)\right) + \mathrm{Ent}(\gamma | \mu \otimes \nu) - |\log(2\varepsilon)| \mu(A)$$

is Γ-converging, as $\varepsilon \to 0$, to the functional F which is finite only on the set $\mathcal{O}_1(\mu, \nu)$ of optimal plans for the Monge problem, and is then equal to

$$F(\gamma) = \mathrm{Ent}(\gamma \llcorner (\mathbb{R} \setminus A)^2 | \mu \otimes \nu) - \int_A \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^1}\right) \mathrm{d}\mu.$$

The main qualitative consequence of this result is that the minimizer which is select by the entropic regularization is exactly "the most diffuse one on the zone where an optimal transport plan may have a density", if such a plan having also finite entropy exists (which is the sense of the assumption **(H2)**). Indeed, Prop. 3.1 implies that, if the set A has positive mass for μ, all the elements of $\mathcal{O}_1(\mu, \nu)$ have infinite entropy, since their restriction to $A \times A$ is concentrated on the graph on the one-dimensional function T. On the contrary, it is natural to guess that there should exist 2-dimensional densities on the squares $(I_j^-)^2$, $(I_k^+)^2$ which respect the sign of $T - \mathrm{id}$: selecting the most diffuse such density (that is, the one which minimizes the entropy with respect to $\mu \otimes \mu$ on these squares, provided that such a density exists), we obtain exactly the minimizer of the target functionals F. Since, for fixed $\varepsilon > 0$, the functional F_ε is only a rescaling version of the original functional J_ε, it admits exactly the same minimizers, which are therefore converging to the (only) minimizer of F.

As we said in the introduction, we may also notice that Theorem 4.1 allows to know precisely the asymptotic behavior of the minimal value of the penalized problem: precisely, as $\varepsilon \to 0$,

$$\min J_\varepsilon = W_1(\mu, \nu) + \varepsilon |\log(2\varepsilon)| \mu(A) + \varepsilon \min F + o(\varepsilon).$$

The excess of order $|\log \varepsilon|$ is a common phenomenon with another type of approximation of the Monge problem where the regularization impacts the transport map itself, see [16]. Moreover, this was also the order of convergence of the construction proposed by Carlier *et al.* in [14] ("block approximation") to prove the density, in the set of transport plans, of transport plans having finite entropy (see Prop. 2.14 therein).

We postpone to the last section an explicit representation formula for the minimizer of the entropy restricted to the squares $(I_j^-)^2$, $(I_k^+)^2$, and in particular a necessary and sufficient condition for the assumption **(H2)**. Concerning the assumption **(H1)**, it is essentially needed for technical reasons; we do not claim it to be necessary for the Γ-convergence result, but unfortunately we were not able to conclude the lower limit estimate without it (see below the proof in Paragraph 4.3).

4.2. **Γ-limsup inequality.** In this section, we prove the Γ-limsup inequality of Theorem 4.1, that is, for any fixed $\gamma \in \Pi(\mu, \nu)$, building a family $(\gamma_\varepsilon)_\varepsilon$ in $\Pi(\mu, \nu)$ having γ as limit and such that

$$\limsup_{\varepsilon \to 0} F_\varepsilon(\gamma_\varepsilon) \leq F(\gamma).$$

We then select $\gamma \in \Pi(\mu,\nu)$ such that $F(\gamma) < +\infty$, otherwise there is nothing to prove: therefore, γ satisfies the conditions (a)-(c) and the assumption **(H2)** of Prop. 3.1. Let $\varepsilon > 0$ be fixed. We will define separately two densities whose sum will be our approximating transport plan.

Our first transport plan is simply $\gamma^1_\varepsilon := \gamma \llcorner (\mathbb{R} \setminus A)^2$. Thanks to the result of Prop. 3.1, it is exactly the part of γ where there is some displacement, so that its Monge cost has value $W_1(\mu,\nu)$.

In order to get a transport plan γ_ε from μ to ν, we have to build a "complementary transport plan" $\gamma^2_\varepsilon \in \Pi(\mu \llcorner A, \mu \llcorner A)$, and to set $\gamma_\varepsilon = \gamma^1_\varepsilon + \gamma^2_\varepsilon$. Let us now see what are the requirement so that the family $(\gamma_\varepsilon)_\varepsilon$ that we will obtain satisfies the Γ-limsup condition. Assume then that $\gamma^2_\varepsilon \in \Pi(\mu \llcorner A, \mu \llcorner A)$ is given: then, γ_ε connects μ to ν and we have, on one hand

$$\int |y-x| \,\mathrm{d}\gamma_\varepsilon(x,y) = \int |y-x| \,\mathrm{d}\gamma^2_\varepsilon(x,y) + W_1(\mu,\nu) \tag{15}$$

and on the other one, since γ^1_ε and γ^2_ε have disjoint supports,

$$\mathrm{Ent}(\gamma_\varepsilon|\mu\otimes\nu) = \mathrm{Ent}(\gamma^2_\varepsilon|\mu\otimes\nu) + \mathrm{Ent}(\gamma^1_\varepsilon|\mu\otimes\nu). \tag{16}$$

Plugging back the equalities (15) and (16) into the expression of F_ε, we can see that the measure γ^2_ε that we want to build must be a transport plan from $\mu \llcorner A$ to itself which satisfies

$$\frac{1}{\varepsilon}\int |y-x| \,\mathrm{d}\gamma^2_\varepsilon(x,y) + \mathrm{Ent}(\gamma^2_\varepsilon|\mu\otimes\nu) \le \mu(A)|\log(2\varepsilon)| - \int_A \log\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^1}\,\mathrm{d}\mu + \mathrm{o}(1). \tag{17}$$

From now, we will denote by μ_A the density of the measure $\mu \llcorner A$; with a slight abuse of notations, we will also use, when required, the same symbols for positive measure on $\mathbb{R}^2$ which are absolutely continuous with respect to the Lebesgue measure and, in that case, their densities (notice that by "density", we mean "density with respect to $\mathcal{L}^2$", and not to $\mu\otimes\nu$).

Step I: the definition of γ^2_ε. The strategy would be easy if we knew exactly the expression of the transport plan which minimizes, for fixed $\varepsilon > 0$, the energy $\int |y-x| \,\mathrm{d}\gamma + \varepsilon\,\mathrm{Ent}(\gamma|\mu\otimes\nu)$ on the set $\Pi(\mu_A,\mu_A)$. From the dual formulation of this last problem, it is not hard to see that this plan must have form $a_\varepsilon(x)a_\varepsilon(y)\mathrm{e}^{-|y-x|/\varepsilon}$ for some positive function a_ε; unfortunately, the function a_ε seems difficult to compute explicitly. Instead, our strategy will be to define directly a first density $\overline{\gamma}_\varepsilon$ which "looks like $a_\varepsilon(x)a_\varepsilon(y)\mathrm{e}^{-|y-x|/\varepsilon}$", and whose marginals are almost $\mu \llcorner A$ and are *dominated* by $\mu \llcorner A$; then, we will build another "complement" $\widetilde{\gamma}_\varepsilon$ between the remaining part of the marginals.

Precisely, the first density that we define is given by

$$\overline{\gamma}_\varepsilon(x,y) := \min(\mu_A(x),\mu_A(y))\frac{\mathrm{e}^{-|y-x|/\varepsilon}}{2\varepsilon}.$$

We notice that, for μ-a.e. $x \in A$, we have

$$\int_{\mathbb{R}} \overline{\gamma}_\varepsilon(x,y)\,\mathrm{d}y \le \mu_A(x)\int_{\mathbb{R}} \frac{\mathrm{e}^{-|y-x|/\varepsilon}}{2\varepsilon}\,\mathrm{d}y = \mu_A(x). \tag{18}$$

Let us denote by $\overline{\mu}_\varepsilon$ the first marginal of $\overline{\gamma}_\varepsilon$, which coincides with the second one, and let us set $\widetilde{\mu}_\varepsilon := \mu_A - \overline{\mu}_\varepsilon$: therefore, (18) implies that $\widetilde{\mu}_\varepsilon$ is a *positive* measure on $\mathbb{R}$. It then remains to build a complementary transport plan $\widetilde{\gamma}_\varepsilon$ belonging to $\Pi(\widetilde{\mu}_\varepsilon,\widetilde{\mu}_\varepsilon)$; then, setting $\gamma^2_\varepsilon := \overline{\gamma}_\varepsilon + \widetilde{\gamma}_\varepsilon$ will be enough to get a transport plan from μ_A to itself.

Our construction of $\widetilde{\gamma}_\varepsilon$ is inspired by the "block approximation" in [14, Definition 2.9] and consists in building a density which is piecewise equal, on small squares, to the multiplication of $\widetilde{\mu}_\varepsilon \otimes \widetilde{\mu}_\varepsilon$ by appropriate constants, keeping in mind that $\widetilde{\gamma}_\varepsilon$ must satisfy a marginal condition. Precisely, we select a family $(Q_i)_i$ of segments recovering the support of $\mu \llcorner A$ (and so of $\widetilde{\mu}_\varepsilon$) and having all length at most ε; notice that thanks to the boundedness of A, the number N_ε of such

segments which are needed is bounded by C/ε for a universal constant C. Now we set, for any Borel set $B \subset \mathbb{R}^2$,

$$\widetilde{\gamma}_\varepsilon(B) = \sum_i \frac{\widetilde{\mu}_\varepsilon \otimes \widetilde{\mu}_\varepsilon(B \cap (Q_i \times Q_i))}{\widetilde{\mu}_\varepsilon(Q_i)}.$$

It is then easy to check that $\widetilde{\gamma}_\varepsilon$ is, as required, a transport plan from $\widetilde{\mu}_\varepsilon$ to itself. In the next steps, we prove that $\overline{\gamma}_\varepsilon + \widetilde{\gamma}_\varepsilon$ satisfies (17).

Step II: the total mass of $\widetilde{\mu}_\varepsilon$ vanishes as $\varepsilon \to 0$. The result of this step will be proved by a dominated convergence argument: we thus begin by showing that

$$\text{for } \mu_A\text{-a.e. } x,\ \widetilde{\mu}_\varepsilon(x) \to 0 \text{ as } \varepsilon \to 0.$$

From the definition of $\overline{\mu}_\varepsilon$ and $\widetilde{\mu}_\varepsilon$, it is clear that, for μ_A-a.e. x,

$$0 \le \widetilde{\mu}_\varepsilon(x) \le \int_{y\in\mathbb{R}} |\mu_A(y) - \mu_A(x)| \frac{\mathrm{e}^{-|y-x|/\varepsilon}}{2\varepsilon}\,\mathrm{d}y.$$

Denoting by $\alpha_\varepsilon(r) := \dfrac{r\mathrm{e}^{-r/\varepsilon}}{\varepsilon^2}$, it follows

$$\begin{aligned}\int_{y\in\mathbb{R}} |\mu_A(y) - \mu_A(x)| \frac{\mathrm{e}^{-|y-x|/\varepsilon}}{2\varepsilon}\,\mathrm{d}y &= \int_{y\in\mathbb{R}} \left(\int_{r=|y-x|}^{+\infty} \frac{\alpha_\varepsilon(r)}{2r}\,\mathrm{d}r\right) |\mu_A(y) - \mu_A(x)|\,\mathrm{d}y \\ &= \int_{r=0}^{+\infty} \left(\frac{1}{2r}\int_{x-r}^{x+r} |\mu_A(y) - \mu_A(x)|\,\mathrm{d}y\right) \alpha_\varepsilon(r)\,\mathrm{d}r.\end{aligned}$$

by the Fubini theorem. Now we notice that, as a measure, α_ε weakly converges to the Dirac mass δ_0; on the other hand, we have

$$\lim_{r\to 0} \left(\frac{1}{2r}\int_{x-r}^{x+r} |\mu_A(y) - \mu_A(x)|\,\mathrm{d}y\right) = 0$$

as soon as x is a Lebesgue point of μ_A. This proves that, for μ_A-a.e. $x \in \mathbb{R}$, the pointwise convergence of densities $\widetilde{\mu}_\varepsilon(x) \to 0$. The domination assumption can then easily be checked, concluding the proof of this step.

Before passing to the next steps, let us decompose the energy we are interested in in several terms, namely

$$\begin{aligned}\frac{1}{\varepsilon}\int |y-x|\,\mathrm{d}\gamma_\varepsilon^2(x,y) + \mathrm{Ent}(\gamma_\varepsilon^2|\mu\otimes\nu) \\ &= -2\int \mu_A \log \mu_A\,\mathrm{d}x \\ &\quad + \frac{1}{\varepsilon}\int |y-x|\,\mathrm{d}\overline{\gamma}_\varepsilon(x,y) + \int \log(\overline{\gamma}_\varepsilon)\,\mathrm{d}\overline{\gamma}_\varepsilon \qquad \text{(I)} \\ &\quad + \frac{1}{\varepsilon}\int |y-x|\,\mathrm{d}\widetilde{\gamma}_\varepsilon(x,y) \qquad \text{(II)} \\ &\quad + \int \log\left(1 + \frac{\widetilde{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon}\right)\mathrm{d}\overline{\gamma}_\varepsilon \qquad \text{(III)} \\ &\quad + \int \log(\widetilde{\gamma}_\varepsilon + \overline{\gamma}_\varepsilon)\,\mathrm{d}\widetilde{\gamma}_\varepsilon \qquad \text{(IV)}.\end{aligned}$$

We will estimate successively the terms (I), (II), (III) and (IV).

Step III: estimates on the terms (I), (II) *and* (III). Starting from the definition of $\overline{\gamma}_\varepsilon(x,y)$, we have for any $(x,y)\in\mathbb{R}^2$:

$$\log\overline{\gamma}_\varepsilon(x,y)=\log\min(\mu_A(x),\mu_A(y))-\log(2\varepsilon)-\frac{1}{\varepsilon}|y-x|.$$

Integrating with respect to $\overline{\gamma}_\varepsilon$ and coming back on the expression of (I) leads to

$$\text{(I)}=-\log(2\varepsilon)\overline{\gamma}_\varepsilon(\mathbb{R}^2)+\int\log\min(\mu_A(x),\mu_A(y))\min(\mu_A(x),\mu_A(y))\frac{\mathrm{e}^{-|y-x|/\varepsilon}}{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}y. \tag{19}$$

Denoting by $f_A(x,y):=\min\{\mu_A(x),\mu_A(y)\}\log(\min\{\mu_A(x),\mu_A(y)\})$ and $g(x)=\mu_A(x)\log(\mu_A(x))$, we claim that

$$\iint f_A(x,y)\frac{\mathrm{e}^{-|y-x|/\varepsilon}}{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}y\xrightarrow[\varepsilon\to0]{}\int_{\mathbb{R}}g(x)\,\mathrm{d}x. \tag{20}$$

This can be proved exactly in the same way as the result of Step II; more precisely, we notice that we have $|f_A(x,y)-g(x)|\le|g(y)-g(x)|$, since $f_A(x,y)$ is either $g(x)$ or $g(y)$. In particular we have

$$\int_{y\in\mathbb{R}}|f_A(x,y)-g(x)|\frac{\mathrm{e}^{-|y-x|/\varepsilon}}{2\varepsilon}\,\mathrm{d}y\le\int_{r>0}\left(\frac{1}{2r}\int_{x-r}^{x+r}|g(y)-g(x)|\,\mathrm{d}y\right)\alpha_\varepsilon(r)\,\mathrm{d}r$$

with the same kernel α_ε as above. It can be checked that the mean value in r goes to zero for a.e. x thanks to the fact that $g\in L^1$, and since $(\alpha_\varepsilon)_\varepsilon$ has δ_0 as limit in the weak sense of measures, this proves (20).

By combining (19) and (20), we conclude

$$\text{(I)}=\overline{\gamma}_\varepsilon(\mathbb{R}^2)\cdot|\log(2\varepsilon)|+\int\log\mu_A\,\mathrm{d}\mu_A+\mathrm{o}(1). \tag{21}$$

We now pass to the terms (II) and (III), which are the two easiest terms. First, since $\widetilde{\gamma}_\varepsilon$ is supported in the union of the squares $Q_i\times Q_i$ which have length at most ε, we have $|y-x|\le\sqrt{2}\varepsilon$ for $\widetilde{\gamma}_\varepsilon$-a.e. (x,y), so that

$$0\le\text{(II)}\le\frac{1}{\varepsilon}\cdot\widetilde{\gamma}_\varepsilon(\mathbb{R}^2)\cdot\sqrt{2}\varepsilon=\sqrt{2}\widetilde{\mu}_\varepsilon(\mathbb{R})\to0$$

since, thanks to the result of Step II, we have $\widetilde{\mu}_\varepsilon(\mathbb{R})\to0$ as $\varepsilon\to0$.

On the other hand, it also holds by the concavity of the logarithm:

$$0\le\text{(III)}=\int\log\left(1+\frac{\widetilde{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon}\right)\mathrm{d}\overline{\gamma}_\varepsilon\le\int\frac{\widetilde{\gamma}_\varepsilon}{\overline{\gamma}_\varepsilon}\,\mathrm{d}\overline{\gamma}_\varepsilon=\widetilde{\gamma}_\varepsilon(\mathbb{R}^2)$$

which is again equal to $\widetilde{\mu}_\varepsilon(\mathbb{R})$ and therefore vanishes as $\varepsilon\to0$. The two terms (II) and (III) have therefore zero as limit as $\varepsilon\to0$.

Step IV: estimate on the term (IV). This one is the most difficult. Recall that, from the definition of $\overline{\gamma}_\varepsilon$ and $\widetilde{\gamma}_\varepsilon$, we have for any $(x,y)\in\mathbb{R}^2$:

$$\widetilde{\gamma}_\varepsilon(x,y)=\frac{\widetilde{\mu}_\varepsilon(x)\widetilde{\mu}_\varepsilon(y)}{\widetilde{\mu}_\varepsilon(Q_i)}\le\frac{\mu_A(x)\mu_A(y)}{\widetilde{\mu}_\varepsilon(Q_i)}$$

$$\text{and}\quad\overline{\gamma}_\varepsilon(x,y)=\min(\mu_A(x),\mu_A(y))\frac{\mathrm{e}^{-|y-x|/\varepsilon}}{2\varepsilon}\le\frac{\mu_A(x)}{2\varepsilon}.$$

It immediately follows

$$\text{(IV)}\le\sum_i\int_{Q_i^2}\log\left(\frac{\mu_A(x)\mu_A(y)}{\widetilde{\mu}_\varepsilon(Q_i)}+\frac{\mu_A(x)}{2\varepsilon}\right)\mathrm{d}\widetilde{\gamma}_\varepsilon(x,y).$$

We separate this sum into two terms, depending if $\widetilde{\mu}_\varepsilon(Q_i)$ is larger or smaller than 2ε; therefore, we denote by I_{small} the set of indexes i such that $\widetilde{\mu}_\varepsilon(Q_i) \le 2\varepsilon$, by I_{large} its complementary set and also set

$$A_{\text{small}} := \bigcup_{i\in I_{\text{small}}} Q_i \qquad \text{and} \qquad A_{\text{large}} := \bigcup_{i\in I_{\text{large}}} Q_i.$$

Denoting by (IV.a) the part of the sum above whose indexes belong to I_{small}, we have therefore

$$\begin{aligned}
\text{(IV.a)} &\le \sum_{i\in I_{\text{small}}} \int_{Q_i^2} \log\left(\frac{\mu_A(x)\mu_A(y)}{\widetilde{\mu}_\varepsilon(Q_i)} + \frac{\mu_A(x)}{\widetilde{\mu}_\varepsilon(Q_i)}\right) \mathrm{d}\widetilde{\gamma}_\varepsilon(x,y) \\
&\le \sum_{i\in I_{\text{small}}} \int_{Q_i^2} \big(\log(\mu_A(x)) + \log(\mu_A(y)+1)\big)\,\mathrm{d}\widetilde{\gamma}_\varepsilon - \widetilde{\mu}_\varepsilon(Q_i)\log\left(\widetilde{\mu}_\varepsilon(Q_i)\right) \\
&\le 2\int_{A_{\text{small}}} \log(\mu_A(x)+1)\,\mathrm{d}\widetilde{\mu}_\varepsilon - \sum_{i\in I_{\text{small}}} \widetilde{\mu}_\varepsilon(Q_i)\log\left(\widetilde{\mu}_\varepsilon(Q_i)\right).
\end{aligned}$$

Consequently, since $\log(\mu_A(x)+1)$ is μ_A integrable and $\mu_A \ge \widetilde{\mu}_\varepsilon \to 0$ thanks to results of Step I, by dominated convergence we get

$$\text{(IV.a)} \le \mathrm{o}(1) - \sum_{i\in I_{\text{small}}} \widetilde{\mu}_\varepsilon(Q_i)\log\widetilde{\mu}_\varepsilon(Q_i). \tag{22}$$

For the second term, denoting by N_ε the number of indexes in I_{small}, the convexity of $t \mapsto t\log t$ gives

$$\begin{aligned}
-\frac{1}{N_\varepsilon}\sum_{i\in I_{\text{small}}} \widetilde{\mu}_\varepsilon(Q_i)\log\widetilde{\mu}_\varepsilon(Q_i) &\le -\left(\frac{1}{N_\varepsilon}\sum_{i\in I_{\text{small}}} \widetilde{\mu}_\varepsilon(Q_i)\right)\log\left(\frac{1}{N_\varepsilon}\sum_{i\in I_{\text{small}}} \widetilde{\mu}_\varepsilon(Q_i)\right) \\
&\le \frac{1}{N_\varepsilon}\widetilde{\mu}_\varepsilon(A_{\text{small}})\left(\log N_\varepsilon - \log\widetilde{\mu}_\varepsilon(A_{\text{small}})\right).
\end{aligned}$$

Plugging this inequality into (22) provides

$$\text{(IV.a)} \le \widetilde{\mu}_\varepsilon(A_{\text{small}})\log N_\varepsilon + \mathrm{o}(1)$$

as, at ε goes to 0, the term $\widetilde{\mu}_\varepsilon(A_{\text{small}})\log(\widetilde{\mu}_\varepsilon(A_{\text{small}}))$ vanishes (thus gives a negligible term once we multiply it by ε). Keeping in mind that $N_\varepsilon \le C/\varepsilon$ for a constant C which does not depend on ε and using again the convergence $\widetilde{\mu}_\varepsilon(A_{\text{small}}) \to 0$, we infer

$$\text{(IV.a)} \le |\log\varepsilon|\widetilde{\mu}_\varepsilon(A_{\text{small}}) + \mathrm{o}(\varepsilon). \tag{23}$$

On the other hand, recall that we have (IV) $\le$ (IV.a) + (IV.b) with

$$\begin{aligned}
\text{(IV.b)} &:= \sum_{i\in I_{\text{large}}} \int_{Q_i} \log\left(\frac{\mu_A(x)\mu_A(y)}{2\varepsilon} + \frac{\mu_A(x)}{2\varepsilon}\right) \mathrm{d}\widetilde{\gamma}_\varepsilon(x,y) \\
&\le \sum_{i\in I_{\text{large}}} \int_{Q_i^2} \big(\log(\mu_A(x)) + \log(\mu_A(y)+1)\big)\,\mathrm{d}\widetilde{\gamma}_\varepsilon - \widetilde{\mu}_\varepsilon(Q_i)\log(2\varepsilon) \\
&\le \int_{A_{\text{large}}} \log(\mu_A(x)+1)\,\mathrm{d}\widetilde{\mu}_\varepsilon + |\log 2\varepsilon|\widetilde{\mu}_\varepsilon(A_{\text{large}}) \\
&= |\log 2\varepsilon|\widetilde{\mu}_\varepsilon(A_{\text{large}}) + \mathrm{o}(1)
\end{aligned}$$

using again that the $\mu_A \ge \widetilde{\mu}_\varepsilon \to 0$ and dominate convergence. Putting together this last estimate with (23), we get

$$\text{(IV)} \le |\log(2\varepsilon)|\widetilde{\mu}_\varepsilon(\mathbb{R}) + \mathrm{o}(1).$$

Finally, by adding this last estimate to (21) and using that both (II) and (III) have zero limit as $\varepsilon \to 0$, we get

$$\begin{aligned}\frac{1}{\varepsilon}\int |y-x|\,\mathrm{d}\gamma_\varepsilon^2(x,y) + \mathrm{Ent}(\gamma_\varepsilon^2|\mu\otimes\nu) &\le (\overline{\mu}_\varepsilon(\mathbb{R}) + \widetilde{\mu}_\varepsilon(\mathbb{R}))|\log(2\varepsilon)| - \int \log\mu_A\,\mathrm{d}\mu_A + \mathrm{o}(1)\\ &\le \mu(A)|\log(2\varepsilon)| - \int \mu_A\log\mu_A\,\mathrm{d}x + \mathrm{o}(1)\end{aligned}$$

so that (17) is proven.

4.3. **Γ-liminf inequality.** Let $(\gamma_\varepsilon)_\varepsilon$ be a family of transport plans having a limit γ as $\varepsilon \to 0$ for the weak convergence of measures. First of all, we start by eliminating the case where γ is not an optimal transport plan for the Monge problem by noticing that, in that case,

$$F_\varepsilon(\gamma_\varepsilon) \ge \frac{M}{\varepsilon} - \mu(A)|\log(2\varepsilon)|$$

for some positive constant M, thus the Γ-liminf inequality is obviously satisfied. We now assume that $\gamma \in \mathcal{O}_1(\mu,\nu)$, so that it verifies the statement of Prop. 3.1; we moreover may assume without loss of generality that all the γ_ε, for $\varepsilon > 0$ small enough, have finite energy F_ε and consequently that all of them have a density with respect to the 2-dimensional Lebesgue measure. Again we will use generally the same notations for the measures γ_ε and their 2-dimensional densities with respect to the Lebesgue measure (and not to $\mu\otimes\nu$).

Let u be the Kantorovich potential given by Lemma 3.3. Using the 1-Lipschitz property of u, we may write

$$\begin{aligned}\frac{1}{\varepsilon}\left(\int |y-x|\,\mathrm{d}\gamma_\varepsilon(x,y) - W_1(\mu,\nu)\right) &= \frac{1}{\varepsilon}\int \Big(|y-x| - (u(y)-u(x))\Big)\,\mathrm{d}\gamma_\varepsilon(x,y)\\ &\ge \frac{1}{\varepsilon}\int_{A\times\mathbb{R}} \Big(|y-x| - (u(y)-u(x))\Big)\,\mathrm{d}\gamma_\varepsilon(x,y).\end{aligned}$$

Let us denote by $\gamma_\varepsilon^A = \gamma_\varepsilon \llcorner A\times\mathbb{R}$ and $\hat{\gamma}_\varepsilon = \gamma_\varepsilon - \gamma_\varepsilon^A$. Notice that the first marginal of γ_ε^A is precisely μ_A while the second marginal may be different and we call it ν_ε; notice that $\nu_\varepsilon \rightharpoonup \nu\llcorner A = \mu_A$ but, thanks to the fact that $\int \nu\log\nu < \infty$, we still have $\int \nu_\varepsilon\log\nu \to \int_A \mu_A\log\mu_A$.

Since $\hat{\gamma}_\varepsilon$ and γ_ε^A are concentrated on disjoint sets and have γ_ε as sum, we have

$$\mathrm{Ent}(\gamma_\varepsilon|\mu\otimes\nu) = \mathrm{Ent}(\hat{\gamma}_\varepsilon|\mu\otimes\nu) + \mathrm{Ent}(\gamma_\varepsilon^A|\mu\otimes\nu).$$

Moreover, it is clear that $\hat{\gamma}_\varepsilon$ weakly converges, as $\varepsilon \to 0$, to $\gamma\llcorner(\mathbb{R}\backslash A)^2$; by the lower-semicontinuity property of the entropy functional, it follows

$$\liminf_{\varepsilon\to 0} \mathrm{Ent}(\hat{\gamma}_\varepsilon|\mu\otimes\nu) \ge \mathrm{Ent}(\gamma\llcorner(\mathbb{R}\setminus A)^2|\mu\otimes\nu).$$

Putting together the above estimates, and using the lower semi-continuity of the entropy, it is clear that the Γ-liminf inequality we look for reduces to

$$\begin{aligned}&\iint \Big(|y-x| - (u(y)-u(x))\Big)\,\mathrm{d}\gamma_\varepsilon^A(x,y) + \varepsilon\,\mathrm{Ent}(\gamma_\varepsilon^A|\mu\otimes\nu)\\ &\qquad\qquad \ge \varepsilon\left(\mu(A)|\log(2\varepsilon)| - \int_A \log\Big(\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^1}\Big)\,\mathrm{d}\mu\right) + o(\varepsilon)\end{aligned}$$

which, by noticing that $\mathrm{Ent}(\gamma_\varepsilon^A|\mu\otimes\nu) = \iint \gamma_\varepsilon^A \log\gamma_\varepsilon^A - 2\int_A \mu_A \log\mu_A + o(1)$, is equivalent to

$$\iint_{A\times\mathbb{R}} \Big(|y-x| - (u(y)-u(x))\Big)\,\mathrm{d}\gamma_\varepsilon^A(x,y) + \varepsilon \iint_{A\times\mathbb{R}} \gamma_\varepsilon^A \log\gamma_\varepsilon^A \geq \varepsilon\left(\mu(A)|\log(2\varepsilon)| + \int_A \mu\log\mu\right) + \mathrm{o}(\varepsilon). \tag{24}$$

In order to prove (24), we will make use of the following lemma.

Lemma 4.1. *Let f, c be measurable positive functions, such that $cf \in L^1(\mathbb{R})$ and $f \in L\log L(\mathbb{R})$. Let $\varepsilon > 0$. Then the following inequality holds:*

$$\int \big(c(x)f(x) + \varepsilon f(x)\log(f(x))\big)\,\mathrm{d}x \geq \varepsilon\left(\left(\int f\right)\log\left(\int f\right) - \left(\int f\right)\log\left(\int \mathrm{e}^{-c/\varepsilon}\right)\right).$$

Proof of Lemma 4.1. We write the left-handside as

$$\varepsilon \times \left(\int \mathrm{e}^{-c/\varepsilon}\right) \times \frac{1}{\int \mathrm{e}^{-c/\varepsilon}} \int \frac{f(x)}{\mathrm{e}^{-c(x)/\varepsilon}} \log\left(\frac{f(x)}{\mathrm{e}^{-c(x)/\varepsilon}}\right) \mathrm{e}^{-c(x)/\varepsilon}\,\mathrm{d}x$$

and the claimed inequality is then a direct consequence of Jensen inequality, applied to the convex function $t \mapsto t\log t$ and the probability measure $\dfrac{\mathrm{e}^{-c(x)/\varepsilon}\,\mathrm{d}x}{\int \mathrm{e}^{-c/\varepsilon}}$. □

Let $\varepsilon > 0$ and $x \in A$ be fixed, and let $[a,b]$ be a bounded interval containing the supports of μ and ν. We apply Lemma 4.1 to the function $f_x : y \mapsto \gamma_\varepsilon^A(x,y)$ and the function c defined as

$$c_x(y) = (|y-x| - (u(y)-u(x)))\,\mathbb{1}_{a\leq y\leq b}.$$

We then obtain

$$\begin{aligned}
&\int_a^b \Big(|y-x| - (u(y)-u(x))\Big)\gamma_\varepsilon(x,y)\,\mathrm{d}y + \varepsilon\int_a^b \log\gamma_\varepsilon(x,y)\gamma_\varepsilon(x,y)\,\mathrm{d}y \\
&\qquad \geq \varepsilon\left(\int_a^b \gamma_\varepsilon(x,\cdot)\right)\log\left(\int_a^b \gamma_\varepsilon(x,\cdot)\right) - \varepsilon\left(\int_a^b \gamma_\varepsilon(x,\cdot)\,\mathrm{d}y\right)\log\left(\int_a^b \mathrm{e}^{-c_x(y)/\varepsilon}\,\mathrm{d}y\right) \\
&\qquad \geq \varepsilon\mu_A(x)\log\mu_A(x) - \varepsilon\mu(x)\log\left(\int_a^b \mathrm{e}^{-c_x(y)/\varepsilon}\,\mathrm{d}y\right).
\end{aligned}$$

After integrating with respect to x, it remains to prove the following lower estimate:

$$-\int_A \log\left(\int_a^b \mathrm{e}^{-c_x(y)/\varepsilon}\,\mathrm{d}y\right)\mathrm{d}\mu(x) \geq \mu(A)|\log(2\varepsilon)| + \mathrm{o}(1).$$

We notice that, from the definition of the Kantorovich potential u, it holds

$$\Big||y-x| - (u(y)-u(x))\Big| \leq \mathcal{L}^1(A\cap[x,y])$$

for any $x, y \in \mathbb{R}$ (and where the notation $[x,y]$ is used alternatively for $[x,y]$ and $[y,x]$, depending if $x \leq y$ or $y \leq x$). Consequently, it is enough to prove the following:

$$-\int_A \log\left(\int_a^b \mathrm{e}^{-\mathcal{L}^1([x,y]\cap A)/\varepsilon}\,\mathrm{d}y\right)\mathrm{d}\mu(x) \geq \mu(A)|\log(2\varepsilon)| + \mathrm{o}(1),$$

which can be written, after changing of variable $y = x + \varepsilon t$, as

$$\liminf_{\varepsilon\to 0} \int_A -\log\left(\frac{1}{2}\int_{I_\varepsilon(x)} \exp\left(-|t|\cdot\frac{\mathcal{L}^1([x,x+\varepsilon t]\cap A)}{\varepsilon|t|}\right)\mathrm{d}t\right)\mathrm{d}\mu(x) \geq 0\,, \tag{25}$$

where we denote by $I_\varepsilon(x)$ the interval $[-(x-a)/\varepsilon, (b-x)/\varepsilon]$.

We prove (25) by a dominated convergence argument. Denote by

$$\mathcal{I}_\varepsilon(x) := \frac{1}{2}\int_{I_\varepsilon(x)} \exp\left(-|t| \cdot \frac{\mathcal{L}^1([x, x+\varepsilon t] \cap A)}{\varepsilon |t|}\right) \mathrm{d}t.$$

Let x be a fixed Lebesgue point of A for μ and let $\alpha > 0$ be arbitrary small. Let $\eta > 0$ be fixed such that

$$1 - \alpha \leq \frac{\mathcal{L}^1([x, y] \cap A)}{|y - x|} \leq 1 + \alpha$$

for any $y \in [x - \eta, x + \eta]$. Setting $y = x + \varepsilon t$, we obtain after multiplying by t, composing with the function $\exp(-\cdot)$ and integrating:

$$\frac{1}{1+\alpha}\left(1 - \exp\left(-(1+\alpha)\frac{\eta}{\varepsilon}\right)\right) \leq \frac{1}{2}\int_{-\eta/\varepsilon}^{\eta/\varepsilon} \exp\left(-|t| \cdot \frac{\mathcal{L}^1([x, x+\varepsilon t] \cap A)}{\varepsilon |t|}\right) \mathrm{d}t \leq \frac{1}{1-\alpha}\left(1 - \exp\left(-(1-\alpha)\frac{\eta}{\varepsilon}\right)\right). \tag{26}$$

The other part of $\mathcal{I}_\varepsilon(x)$ can be estimated simply by noticing that, for any t such that $|t| \geq \eta/\varepsilon$, it holds

$$\mathcal{L}^1([x, x+\varepsilon t] \cap A) \geq \mathcal{L}^1([x, x+\eta] \cap A) > 0,$$

the last inequality coming from the fact that x is a Lebesgue point of A. Therefore,

$$\frac{1}{2}\int_{I_\varepsilon(x)\setminus[-\eta/\varepsilon, \eta/\varepsilon]} \mathrm{e}^{-\mathcal{L}^1([x, x+\varepsilon t]\cap A)} \mathrm{d}t \leq \frac{b-a}{2\varepsilon} \exp\left(-\frac{\mathcal{L}^1([x, x+\eta] \cap A)}{\varepsilon}\right). \tag{27}$$

Using (26) and (27) and sending ε to 0, we obtain

$$\frac{1}{1+\alpha} \leq \liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon(x) \leq \limsup_{\varepsilon \to 0} \mathcal{I}_\varepsilon(x) \leq \frac{1}{1-\alpha}$$

and since $\alpha > 0$ is arbitrary, we conclude that $-\log \mathcal{I}_\varepsilon(x) \to 0$, for μ-a.e. $x \in A$.

To conclude the proof, we have to control $-\log \mathcal{I}_\varepsilon(x)$ by an μ-integrable function which does not depend on ε; here our assumption **(H1)** on the structure of the set A comes into play. Denote by

$$l(x) = \min\Big((x - a_i), (x - b_i)\Big) \quad \text{for } x \in \operatorname{int}(A \cap \operatorname{supp}\mu),\ a_i < x < b_i.$$

Since A has negligible boundary, the function l is well-defined μ-a.e. on A. Let now x be a point of the interior of A and i such that $a_i < x < b_i$. We claim that

$$\text{for any } y \in I,\ \mathcal{L}^1([x, y] \cap A) \geq \frac{|y - x|}{(b - a)} l(x). \tag{28}$$

Indeed, this inequality is clearly true if $y \in (a_i, b_i)$, since this implies $\mathcal{L}^1([x, y] \cap A) = |y - x|$ and since $l(x) < b - a$; on the other hand, if $y > b_i$ for instance, then $\mathcal{L}^1([x, y] \cap A) \geq (b_i - x) \geq l(x)$, and since y and x both belong to $[a, b]$, yielding $|y - x|/(b - a) \geq 1$.

Inequality (28) directly implies

$$\mathcal{I}_\varepsilon(x) \leq \frac{1}{2}\int_{I_\varepsilon(x)} \exp\left(-|t|\frac{l(x)}{b-a}\right) \mathrm{d}t \leq \frac{b-a}{l(x)}$$

and we also notice, by simply using the inequality $\mathcal{L}^1([x, x+\varepsilon t] \cap A) \leq \varepsilon t$ for any x, that $\mathcal{I}_\varepsilon(x)$ is also always at least equal to 1, for any x. Therefore,

$$0 \leq \log \mathcal{I}_\varepsilon(x) \leq \log\left(\frac{b-a}{l(x)}\right).$$

The domination assumption is therefore satisfied as soon as $\int_A |\log l(x)|\,\mathrm{d}\mu(x) < +\infty$, as it was stated in **(H1)**.

It remains to prove that **(H1)** holds as soon as (H1a) or (H1b) does. Let us write, for each i, $[a_i, b_i] = [a_i, \alpha_i] + [\alpha_i, \beta_i] + [\beta_i, b_i]$ where

$$\alpha_i = \min\left(a_i + 1, \frac{a_i + b_i}{2}\right) \quad \text{and} \quad \beta_i = \max\left(b_i - 1, \frac{a_i + b_i}{2}\right).$$

We then obtain, by controlling $l(x)$ with $b - a$ on each $[\alpha_, \beta_i]$:

$$\begin{aligned}
\int_A |\log l(x)|\,\mathrm{d}\mu(x) &= \sum_i \int_{a_i}^{b_i} |\log l(x)|\,\mathrm{d}\mu(x) \\
&\leq \sum_i \left(-\int_{a_i}^{\alpha_i} \log(x - a_i)\,\mathrm{d}\mu(x) - \int_{\beta_i}^{b_i} \log(b_i - x)\,\mathrm{d}\mu(x) \right) \\
&\quad + \sum_i |\log(b - a)|\mu([a_i, b_i]).
\end{aligned}$$

The last term in (29) is exactly $\mu(A)|\log(b-a)|$, so it remains to control the two first terms. In case assumption (H1a) is true, we simply notice

$$\begin{aligned}
0 &\leq -\int_{a_i}^{\alpha_i} \log(x - a_i)\,\mathrm{d}\mu(x) \\
&\leq -\|\mu\|\infty(\alpha_i - a_i)\Big(\log(\alpha_i - a_i) - 1\Big) \\
&\leq \|\mu\|\infty \frac{b_i - a_i}{2}\left(1 + \left|\log\frac{b_i - a_i}{2}\right|\right)
\end{aligned}$$

and the same bound holds for the term $\int_{\beta_i}^{b_i} |\log(b_i - x)|\,\mathrm{d}\mu(x)$ by exactly the same computation, leading to

$$\int_A |\log l(x)|\,\mathrm{d}\mu(x) \leq \|\mu\|\infty \mathcal{L}^1(A \cap \operatorname{supp}\mu) + |\log(b-a)|\mu(A) + \sum_i (b_i - a_i)|\log(b_i - a_i)|$$

which is finite by assumption (H1a).

Assume now instead that (H1b) holds. We then use the inequality $\delta t u \leq u \log u + e^{\delta t - 1}$ and get

$$\begin{aligned}
\delta \int_{a_i}^{\alpha_i} |\log(x - a_i)|\mu(x)\,\mathrm{d}x &\leq \int_{a_i}^{\alpha_i} e^{-\delta \log(x - a_i)}\,\mathrm{d}x + \int_{a_i}^{\alpha_i} \log(\mu(x))\mu(x)\,\mathrm{d}x \\
&= \int_{a_i}^{\alpha_i} \frac{1}{(x - a_i)^\delta}\,\mathrm{d}x + \int_{a_i}^{\alpha_i} \log(\mu(x))\mu(x)\,\mathrm{d}x \\
&= \frac{(\alpha_i - a_i)^{1-\delta}}{1 - \delta} + \int_{a_i}^{\alpha_i} \log(\mu(x))\mu(x)\,\mathrm{d}x.
\end{aligned}$$

Using a similar argument for the intervals β_i, b_i and then summing up we get

$$\delta \int_A |\log l(x)|\,\mathrm{d}\mu(x) \leq 2\int_A \log(\mu)\mu\,\mathrm{d}x + |\log(b-a)|\mu(A) + \frac{2}{1-\delta}\sum_i (b_i - a_i)^{1-\delta},$$

concluding the proof.

5. An explicit form of the minimizer

In this section we will try to compute explicitly what will be the plan we are selecting via the limit procedure, that is, the minimizer of F. Given the expression of F and the result of Prop. 3.1 on the structure of $\mathcal{O}_1(\mu,\nu)$, it is enough to know, on each maximal interval I where $F_\mu - F_\nu$ has constant sign, the minimizer of the entropy among all the plans γ such that $y-x$ has good sign for γ-a.e. pairs (x,y).

In particular, for each j (and using the notations of Paragraph 3.4), this plan will be the minimizer, on the set $\Pi(\mu\llcorner I, \nu\llcorner I)$, of the entropy with respect to the measure $k = \mu_j^- \otimes \nu_j^- \cdot \mathbb{1}_{y>x}$. This problem is well known in the literature (see for example [10]), and the shape of the minimizer is know to be of the form $a(x)\otimes b(y)\cdot k$. The proof of [10] relies on an abstract and general result of [9]: to be more precise the usual proof relies on the existence of maximizers in the dual problem if the entropy minimization has a finite value; then, using the duality relation, we can get that the minimizer has the special shape $a(x)\otimes b(y)\cdot k$.

We will provide a self-contained proof of this result, in a more suitable context for our needs: precisely, we prove the existence of a plan with this special shape, independently of the finiteness of the entropy (see Proposition 5.1 below), which we think it is a interesting result itself. It would anyway be interesting to know when there exists a plan of the form $a(x)\otimes b(y)\cdot k$ still when the entropy minimization problem is not finite, besides our case.

In Proposition 5.2 we then prove that the construction in Prop. 5.1 gives in fact the unique minimizer in the optimization problem, whenever its value is finite; here we in fact pass to the dual problem and we use the dual functions given by the particular shape found in Prop. 5.1.

Then, thanks to the explicit construction given in Prop. 5.1 we can state in Theorem. 5.1 a necessary and sufficient condition in order to satisfy assumption **(H2)**.

Proposition 5.1. *Let μ,ν be two probability measures on $\mathbb{R}$, compactly supported and with no atom, and denote by $\mathcal{F} = F_\mu - F_\nu$ (where F_μ, F_ν are the cumulative distribution functions of μ,ν respectively). Let $I=(a,b)$ be a maximal positivity interval for $\mathcal{F}$, that is $\mathcal{F}(a)=\mathcal{F}(b)=0$ and $\mathcal{F}>0$ on I. Then there exist two nonnegative measures ρ_1,ρ_2 on I such that, defining*

$$G(x) := \rho_2([x,b]) \quad \text{and} \quad F(y) := \rho_1([a,y])$$

for any $x,y\in I$, we have

$$\begin{cases} G\cdot\rho_1 = \mu \\ F\cdot\rho_2 = \nu \end{cases} \qquad \text{on } I.$$

Moreover G and F are continuous and positive on (a,b), and the positive measure

$$\gamma_0(x,y) := \rho_1(x)\otimes\rho_2(y)\cdot\mathbb{1}_{y>x}$$

is a transport plan between $\mu\llcorner I$ and $\nu\llcorner I$.

Similarly, if I is a maximal negativity interval of $\mathcal{F}$, there exist two positive measures ρ_1,ρ_2 with densities such that

$$\begin{cases} G\cdot\rho_1 = \mu \\ F\cdot\rho_2 = \nu \end{cases} \qquad \text{where } F(y)=\rho_1([y,b]),\ G(x)=\rho_2([a,x]),$$

the functions G and F being continuous and positive on (a,b), and

$$\gamma_0(x,y) := \rho_1(x)\otimes\rho_2(y)\cdot\mathbb{1}_{y<x}$$

belongs to $\Pi(\mu\llcorner I,\nu\llcorner I)$.

Proof. We treat only the case where I is a maximal positivity interval of $\mathcal{F}$ and claim that the proof is similar for their negative counterpart. Denote by

$$G_\mu : x \mapsto \mu([a,x]) = F_\mu(x) - F_\mu(a)$$

and define the function G_ν in the same way; in particular, the assumption on $\mathcal{F}, a, b$ implies that

$$G_\mu > G_\nu \text{ on } I, \quad \text{and } G_\mu(a) = G_\nu(a) = 0,\ G_\mu(b) = G_\nu(b) = \mu(I).$$

On the other hand, the function $\mathcal{F}$ is continuous and strictly positive on the whole I; we can thus define an integral of $\mu/\mathcal{F}$ on I and call it T. We then observe that $\mu/\mathcal{F} \geq \mu/G_\mu$ so that we have, for any $x_0, x \in I$ and using that $D \log G_\mu = \mu/G_\mu$:

$$T(x_0) - T(x) = \int_x^{x_0} \frac{1}{\mathcal{F}} \, \mathrm{d}\mu \geq \int_x^{x_0} \frac{1}{G_\mu} \, \mathrm{d}\mu = \log G_\mu(x_0) - \log G_\mu(x).$$

In particular, since $G_\mu(a) = 0$ we have $T(a) = -\infty$. Let us define

$$F : x \in I \mapsto e^{T(x)} \quad \text{and} \quad G : y \in I \mapsto \frac{\mathcal{F}(y)}{F(y)}.$$

The observations above imply $F(a) = 0$. Similarly, $G(b) = 0$ and $\log(G)$ is a primitive of $-\nu/\mathcal{F}$. Finally, we define $\rho_1 = DF$ and $\rho_2 = -DG$ as derivative of BV_{loc} functions (since they are increasing and locally bounded); notice by construction F and G are continuous, so ρ_1 and ρ_2 will have no atoms. Since T and F are BV_{loc} without jump part, the chain rule holds:

$$\rho_1 = DF = De^T = e^T \cdot DT = F \cdot \frac{\mu}{\mathcal{F}} = \frac{\mu}{G}.$$

In other words, we have $G \cdot \rho_1 = \mu$ and we can prove the equality $F \cdot \rho_2 = \nu$ in the same way. By construction F and G are continuous and nonnegative; moreover, $F \cdot G = \mathcal{F} > 0$ on I, so that neither F and G may vanish on I.

Define now γ_0 as in the statement of Prop. 5.1: it is then clear that, for any $\varphi \in C_b(\mathbb{R})$,

$$\int_{I \times I} \varphi(x) \, \mathrm{d}\gamma_0(x, y) = \int_I \varphi(x) G(x) \, \mathrm{d}\rho_1(x) = \int_I \varphi(x) \, \mathrm{d}\mu(x)$$

and that, similarly, the second marginal of γ_0 is ν. □

The next result proves that the transport plan γ_0 defined in Prop. 5.1 is actually the minimizer of the entropy among the transport plans on I^2 satisfying the corresponding sign constraint.

Proposition 5.2. *Let $\mu, \nu, \rho_1, \rho_2, I$ and γ_0 as in Prop. 5.1. Let Γ_0 be the set of optimal plans from $\mu \llcorner I$ to $\nu \llcorner I$ (that is, the set of $\gamma \in \Pi(\mu \llcorner I, \nu \llcorner I)$ such that $y - x > 0$ for γ-a.e. (x, y)). Then we have*

$$\min_{\gamma \in \Gamma_0} \mathrm{Ent}(\gamma | \mu \otimes \nu) = \mathrm{Ent}(\gamma_0 | \mu \otimes \nu), \tag{29}$$

this equality being also true if both sides are $+\infty$. Moreover, whenever $\mathrm{Ent}(\gamma_0 | \mu \otimes \nu) < \infty$, the only minimizer in (29) is γ_0.

In the proof of Prop. 5.2 we will make use of the following lemma:

Lemma 5.1. *Let a, b two Borel functions defined on an open set Ω and let us consider a finite measure μ on Ω. Then, we have that*

$$\int_\Omega (\varphi_n(a) + \varphi_n(b))_+ \, \mathrm{d}\mu \to \int_\Omega (a + b)_+ \, \mathrm{d}\mu \qquad \int_\Omega (\varphi_n(a) + \varphi_n(b))_- \, \mathrm{d}\mu \to \int_\Omega (a + b)_- \, \mathrm{d}\mu$$

where $\varphi_n(t) = \max\{-n, \min\{t, n\}\}$. In particular, if either $(a + b)_-$ or $(a + b)_+$ is in $L^1(\mu)$, we have also

$$\int_\Omega (\varphi_n(a) + \varphi_n(b)) \, \mathrm{d}\mu \to \int_\Omega (a + b) \, \mathrm{d}\mu.$$

Proof. First we claim that

$$(\varphi_n(a)+\varphi_n(b))_+ \leq (a+b)_+ \tag{30}$$

(and similarly $(\varphi_n(a)+\varphi_n(b))_- \leq (a+b)_-$).

If $a,b>0$ this is obvious, so let us suppose $a>0$ and $b<0$, with $a+b=k>0$ then $a=k-b$ and so

$$\varphi_n(a)=\varphi_n(k-b)\leq \varphi_n(k)+\varphi_n(-b)=\varphi_n(k)-\varphi_n(b)\leq k-\varphi_n(b),$$

where we used that φ_n is odd and it is sub additive on the positive numbers. In order to conclude the proof of Equation 30 we need to show that $\varphi_n(a)+\varphi_n(b)\leq 0$ whenever $a+b\leq 0$, but this is obvious from the fact that φ_n is increasing and odd and in particular if $a+b\leq 0$ we have $a\leq -b$ and so

$$\varphi_n(a)+\varphi_n(b)\leq \varphi_n(-b)+\varphi_n(b)=0.$$

In particular, using Equation 30 we immediately get

$$\limsup_{n\to\infty}\int_\Omega (\varphi_n(a)+\varphi_n(b))_+\,\mathrm{d}\mu \leq \int_\Omega (a+b)_+\,\mathrm{d}\mu,$$

while, since $\lim_{n\to\infty}(\varphi_n(a)+\varphi_n(b))_+=(a+b)_+$, by Fatou lemma we have

$$\liminf_{n\to\infty}\int_\Omega (\varphi_n(a)+\varphi_n(b))_+\,\mathrm{d}\mu \geq \int_\Omega (a+b)_+\,\mathrm{d}\mu,$$

thus giving us that $\int(\varphi_n(a)+\varphi_n(b))_+\to\int(a+b)_+$. We deal with the negative part in the same way. □

Proof or Prop. 5.2. Again we only will deal with an interval I where $\mathcal{F}$ is positive. Assume that $\mathrm{Ent}(\cdot|\mu\otimes\nu)$ is not identically $+\infty$ on Γ_0 and let $\gamma\in\Gamma_0$ having finite entropy; with a slight abuse of notation, we will also denote by $\gamma(x,y)$ the density of the plan γ with respect to $\mu\otimes\nu$. Let us denote $\Omega=I\times I\cap\{y>x\}$; we know that for every bounded measurable couple of functions $A,B:I\to\mathbb{R}$, we have

$$\mathrm{Ent}(\gamma|\mu\otimes\nu)=\int_\Omega\big(\gamma\log(\gamma)-(A(x)+B(y))\gamma-\gamma\big)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)+\int_\Omega (A+B+1)\,\mathrm{d}\gamma_0.$$

Now we use that $t\log(t)-ct\geq -e^{c-1}$ for any real numbers c,t; applying this with $c=A(x)+B(y)+1$ and $t=\gamma(x,y)$, we deduce

$$\mathrm{Ent}(\gamma|\mu\otimes\nu)\geq -\int_\Omega e^{A(x)+B(y)}\,\mathrm{d}\mu\,\mathrm{d}\nu+\int_\Omega (A(x)+B(y)+1)\,\mathrm{d}\gamma_0(x,y). \tag{31}$$

Let now φ_n be defined as in Lemma 5.1 and define

$$A_n(x):=\varphi_n(-\log(G(y)))\quad\text{and}\quad B_n(y)=\varphi_n(-\log(F(y))).$$

Equation (31) holds in particular with A_n and B_n instead of A and B, and we will estimate the two terms of the right-handside from below by $\mathrm{Ent}(\gamma_0|\mu\otimes\nu)$ as $n\to+\infty$. By Lemma 5.1 we know that $(A_n(x)+B_n(y))_+\leq(-\log G(x)-\log F(y))_+$ and so $e^{A_n(x)+B_n(y)}\leq\max\{1/G(x)F(y),1\}$. Thus, by dominated convergence, we get

$$\int_\Omega e^{A_n+B_n}\,\mathrm{d}\mu\,\mathrm{d}\nu\to\int_\Omega\frac{1}{GF}\,\mathrm{d}\mu\,\mathrm{d}\nu=\int_\Omega\mathrm{d}\gamma_0.$$

As for the second term of (31), it is clear that $(-\log G(x)-\log(F(y))_-\in L^1(\gamma_0)$ since

$$\int_\Omega(-\log(G)-\log(F))_-\,\mathrm{d}\gamma_0=\int_\Omega\log\Big(\frac{1}{GF}\Big)_-\frac{1}{GF}\,\mathrm{d}\mu\otimes\nu\leq\int_\Omega e^{-1}\,\mathrm{d}\mu\otimes\nu<\infty.$$

In particular using again Lemma 5.1 we get

$$\int_\Omega (A_n(x) + B_n(y) + 1)\,\mathrm{d}\gamma_0 \to \int_\Omega \left(\log\Big(\frac{1}{F(y)G(x)}\Big) + 1\right)\,\mathrm{d}\gamma_0.$$

So, putting all the estimates together and using $\dfrac{\mathrm{d}\gamma_0}{\mathrm{d}(\mu\otimes\nu)} = \dfrac{1}{G(x)F(y)}$, we obtain

$$\begin{aligned}\mathrm{Ent}(\gamma\llcorner(I\times I)|\mu\otimes\nu) &\geq -\int_\Omega e^{A_n(x)+B_n(y)}\,\mathrm{d}x\,\mathrm{d}y + \int (A_n(x)+B_n(y)+1)\,\mathrm{d}\gamma_0(x,y)\\ &\to -\int_\Omega \mathrm{d}\gamma_0 + \int_\Omega \left(\log\Big(\frac{\mathrm{d}\gamma_0}{\mathrm{d}\mu\otimes\nu}\Big)+1\right)\,\mathrm{d}\gamma_0 = \mathrm{Ent}(\gamma_0|\mu\otimes\nu),\end{aligned}$$

proving that γ_0 minimizes $\mathrm{Ent}(\cdot|\mu\otimes\nu)$ on Γ_0. As for the uniqueness it is sufficient to observe that Γ_0 is a linear space and $\mathrm{Ent}(\cdot|\mu\otimes\nu)$ is strictly convex. □

We can now prove the following result, which expresses exactly when the above assumption **(H2)** holds depending on the cumulative distribution functions F_μ, F_ν.

Theorem 5.1. *Let μ,ν be two probability measures on $\mathbb{R}$, compactly supported and absolutely continuous with respect to the Lebesgue measure. Let $\mathcal{F} = F_\mu - F_\nu$, where F_μ, F_ν are the cumulative distribution functions of μ,ν, and A^+, A^- as in Section 3. Then a necessary and sufficient condition for assumption* **(H2)** *is*

$$-\int_{A^+\cup A^-} \log|\mathcal{F}(x)|\,\mathrm{d}\mu(x) < +\infty$$

and in that case, the minimal entropy among the elements of $\mathcal{O}_1(\mu,\nu)$ is reached by the plan γ which coincides with the identity map on $A\times A$ and, on each square I^2 where I is a maximal positivity (resp. negativity) interval of $\mathcal{F}$, is equal to the γ_0 defined by Prop. 5.2. Moreover, for any such interval I, we have

$$\mathrm{Ent}(\gamma_0\llcorner I^2|\mu\otimes\nu) = -\int_I \log|\mathcal{F}(x)|\,\mathrm{d}\mu(x) - \mu(I). \tag{32}$$

Proof. Let I be a maximal positivity interval of $\mathcal{F}$ (the set where $\mathcal{F}$ is negative being treated in a similar way). We first notice that, thanks to Proposition 3.1(b) we have that $\gamma\llcorner_{I^2} = \gamma\llcorner_{I\times\mathbb{R}} = \gamma\llcorner_{\mathbb{R}\times I}$ and so it is an (optimal) plan between $\mu\llcorner_I$ and $\nu\llcorner_I$. Then we can deduce from Prop. 5.2 that the minimal entropy of $\gamma\llcorner_{I^2}$ among the elements of $\mathcal{O}_1(\mu,\nu)$ is reached by γ_0, even if it is $+\infty$; so it is sufficient to prove that equality (32) holds, even when one of the two sides is not finite. Let us assume, for simplicity, that $I = (0,1)$. Then by definition

$$\mathrm{Ent}(\gamma_0|\mu\otimes\nu) = -\int_0^1\int_x^1 \log(G(x)F(y))\,\mathrm{d}\rho_2(y)\,\mathrm{d}\rho_1(x).$$

We can split the logarithm in the inner integral since $G(x)$ is constant and use the definition of F,G and the results from Proposition 5.1 to get

$$\mathrm{Ent}(\gamma_0|\mu\otimes\nu) = \int_0^1 \left(-\log(G(x))G(x) + \int_x^1 \log(F(y))\,\mathrm{d}\rho_2(y)\right)\,\mathrm{d}\rho_1(x).$$

Since $-\rho_2 = DG$ we can use integration by parts in order to get

$$\int_x^1 \log(F(y))\,\mathrm{d}DG(y)\,\mathrm{d}y = \Big[G(y)\log(F(y)\Big]_x^1 - \int_x^1 \frac{G(y)}{F(y)}\,\mathrm{d}DF = -G(x)\log F(x) - \int_x^1 \frac{\mathrm{d}\mu(y)}{F(y)},$$

where we used that $G(y)\log F(y) \to 0$ for $y \to 1$. This is true since for y sufficiently close to 1 we have $\mathcal{F}(y) < 1$ and so in particular $F(1/2) \leq F(y) = \mathcal{F}(y)/G(y) \leq 1/G(y)$. In particular, using $G(y) \to 0$ as $y \to 1$ we get

$$0 = \lim_{y\to 1} G(y)\log(F(1/2)) \leq \lim_{y\to 1} G(y)\log(F(y)) \leq \lim_{y\to 1} -G(y)\log(G(y)) = 0.$$

Now for any nonnegative measure η in $(0,1)$ we have that $\int_0^1 f(x)\int_x^1 \frac{\mathrm{d}\eta(y)}{F(y)}\,\mathrm{d}x = \eta(0,1)$: it is true by integration by parts for any measure compactly supported in $(0,1)$, and then we reach every measure by an approximation argument. We use this to get:

$$\begin{aligned}
\mathrm{Ent}(\gamma_0|\mu\otimes\nu) &= \int_0^1 \left(-\log(G(x))G(x) - \log(F(x))G(x) - \int_x^1 \frac{1}{F(y)}\,\mathrm{d}\mu(y)\right)\mathrm{d}\rho_1(x)\\
&= -\int_0^1 \log(\mathcal{F})\,\mathrm{d}(G\cdot\rho_1) - \int_0^1 \mathrm{d}\mu\\
&= -\int_0^1 \log(\mathcal{F})\,\mathrm{d}\mu - \mu(I),
\end{aligned}$$

where we used $F(x)\cdot G(x) = \mathcal{F}(x)$ and in the last passage $G\cdot\rho_1 = \mu$, proving the desired equality (32). □

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