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Spectral analysis of a two body problem with zero-range perturbation

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Abstract

We consider a class of singular, zero-range perturbations of the Hamiltonian of a quantum system composed by a test particle and a harmonic oscillator in dimension one, two and three and we study its spectrum. In fact we give a detailed characterization of point spectrum and its asymptotic behavior with respect to the parameters entering the Hamiltonian. We also partially describe the positive spectrum and scattering properties of the Hamiltonian.

Keywords: Solvable models in quantum mechanics; Zero-range interactions

1. Introduction

We consider in $\mathbb{R}^d$, $d = 1, 2, 3$, a system composed of a test particle and a harmonic oscillator interacting through a zero-range force.

The Hamiltonian is formally written as

$$H_\alpha^\omega \equiv H_0^\omega + \text{“}\alpha\delta(\vec{x} - \vec{y})\text{”}, \qquad H_0^\omega \equiv -\frac{1}{2}\Delta_{\vec{x}} - \frac{1}{2}\Delta_{\vec{y}} + \frac{\omega^2 y^2}{2} - \frac{\omega d}{2}. \tag{1.1}$$

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The precise meaning of the formal expression "$\alpha\delta(\vec{x}-\vec{y})$" will be given shortly.

Hamiltonians with formal zero-range forces have been introduced in physics since the early 1930s (see, e.g., [10,15,16,18]), in particular for the study of scattering of low energy atoms and electrons from a target.

The operators often considered in literature are approximations of (1.1), which correspond, very roughly speaking, to interactions with "very massive" nuclei through a potential of "very short range" and the nuclei are supposed so massive that they can be regarded as fixed scattering centers.

Here we consider a more general model in which the nuclei of the target are regarded as quantum particles harmonically bound to their equilibrium positions. This kind of model is widely used in physics to reconstruct the structure of the target (e.g., the distribution of the equilibrium positions) from scattering data.

In this paper we treat only the case of one harmonic oscillator, in which case α is a real parameter. We will come back in a forthcoming paper to the case in which several oscillators are present.

From the point of view of mathematics, zero-range interactions are interesting non-trivial models (see, e.g., [1,3,8]), for which it is possible to find simple explicit solutions to the Schrödinger equation and to compute physically relevant quantities. These models can be indexed by a small number of parameters which codify the "strength" and the position of the interactions.

The cases $d = 1, 2, 3$ will be treated in Sections 2, 3 and 4 respectively, and in each section we shall recall some results about the rigorous definition of zero range interactions in the corresponding dimension, often referring to [7] for proofs and further details.

Here we only remark that the definition of zero range interaction is much easier in dimension one when it can be given in terms of boundary conditions at $x = y$.

On the contrary, in dimensions two and three the definition requires a more sophisticated analysis in terms of self-adjoint extensions of the restriction of the operator H_0^ω to smooth functions which vanish in some neighborhood of the hyperplane $\vec{x} = \vec{y}$. This extension may be obtained at a purely formal level, considering a suitable interaction of range ϵ and letting $\epsilon \to 0$, after having applied a suitable renormalization prescription. An equivalent rigorous definition is obtained through the theory of quadratic forms. This is the definition we shall use in our analysis, indeed we shall show that in dimensions 1, 2 and 3 the operators H_α^ω are determined by simple quadratic forms, closed and bounded below.

It is worth remarking that zero-range interactions, as we have defined them, do not exist when $d \geqslant 4$ (the restriction mentioned above defines in this case an operator which is essentially self-adjoint) and that we consider only part of the self-adjoint extensions, i.e., those commonly called "δ"-type.

This is by no means the only way to define an interaction supported by a manifold of codimension 1 (in our case the manifold $\vec{x} = \vec{y}$). The extension we choose is known in the physical literature as "single layer potential." Other extensions are possible, among them the ones corresponding, roughly speaking, to double layer potential (dipole layers) and to various forms of the Robin conditions. A detailed mathematical study of the general case for pseudodifferential operators in a bounded domain with regular boundaries can be found, e.g., in [4,11,17].

We remark that in two and three dimensions the interaction is supported on immersed manifolds of codimensions 2 and 3 respectively. In this case one can use a higher order Poisson kernel or equivalently define the Schrödinger Hamiltonian by boundary triple theory (see, e.g., [5,19]). In this approach the auxiliary Hilbert space is a Sobolev space built over the Laplace–Beltrami of the manifold and the auxiliary maps are roughly speaking the trace maps on the regular and the singular part of the Poisson kernel at the manifold. In the two-dimensional and three-dimensional cases this construction leads to all self-adjoint extensions of the Laplacian restricted to function which vanish in a neighborhood of the manifold (one can take in place of the Laplacian any strictly elliptic operator with smooth coefficients with a possible addition of a potential and a regular magnetic field). While the method can be used for any immersed manifold, in our case one can, as we have done, give rather explicit analytic formulae and estimates. Notice that when the codimension is greater than three there is only one such extension, the Laplacian defined on the Sobolev space $\mathcal{H}^2$.

In the following sections, we shall prove that the essential spectrum of H_α^ω is the half-line $[0, +\infty)$, for all values of the parameters and $d = 1, 2, 3$, and the wave operators $\Omega_\pm(H_\alpha^\omega, H_0^\omega)$ exist and are complete.

We shall also fully characterize the negative part of the spectrum and give estimates of the number of eigenvalues.

We plan to come back to the scattering problem in a forthcoming paper and give a complete description of the multi-channel scattering associated with the pair H_α^ω, H_0^ω.

1.1. Notation

We introduce in this section some notation and basic facts which will be used in the rest of the paper.

Vectors in $\mathbb{R}^d$ will be denoted by $\vec{x}$, the modulus of $\vec{x}$ by x and $\langle x\rangle$ stands for $(1 + x^2)^{1/2}$.

Unless stated otherwise $\|\cdot\|$ will denote both the norm of functions in $L^2(\mathbb{R}^d)$ and the norm of bounded endomorphism of $L^2(\mathbb{R}^d)$.

Given any function $f \in L^2(\mathbb{R}^d)$, its Fourier transform, denoted by $\hat{f}$, will be defined by

$$\hat{f}(\vec{k}) \equiv \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} d\vec{x}\, e^{-i\vec{k}\cdot\vec{x}} f(\vec{x}). \tag{1.2}$$

We shall denote the Sobolev space of order m by $\mathcal{H}^m(\mathbb{R}^d)$, i.e.,

$$\mathcal{H}^m(\mathbb{R}^d) \equiv \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; \langle k\rangle^m \hat{f} \in L^2(\mathbb{R}^d)\right\},$$
$$\|f\|_{\mathcal{H}^m} = \left\|\langle k\rangle^m \hat{f}\right\|,$$

and the logarithmic Sobolev space by

$$\mathcal{H}^{\log}(\mathbb{R}^d) \equiv \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; \log(1 + \langle k\rangle)\hat{f} \in L^2(\mathbb{R}^d)\right\},$$
$$\|f\|_{\mathcal{H}^{\log}} = \left\|\log(1 + \langle k\rangle)\hat{f}\right\|.$$

We introduce the Hamiltonian of the harmonic oscillator by $H_{\mathrm{osc}}^\omega = \frac{1}{2}(p^2 + \omega^2 x^2)$, which will be used as a reference Hamiltonian in some technical estimates, and denote by $\Psi_{\vec{n}}^{(\omega)}(\vec{x})$, $n \in \mathbb{N}^d$, its normalized eigenvectors. The integral kernel of the semigroup (Mehler kernel, see [2]) is given by

$$e^{-H_{\mathrm{osc}}^\omega t}(\vec{x}; \vec{x}') \equiv \frac{e^{-\frac{\omega d t}{2}}}{\pi^{\frac{d}{2}}(1 - e^{-2\omega t})^{\frac{d}{2}}} \exp\left\{ -\frac{\omega(x^2 + x'^2)}{2\tanh \omega t} + \frac{\omega \vec{x}\cdot\vec{x}'}{\sinh \omega t}\right\}. \tag{1.3}$$

Let us recall for the reader's convenience some facts about compact operators. Let $\mathscr{H}$ and $\mathscr{H}'$ be two Hilbert spaces, we shall denote the space of compact operators from $\mathscr{H}$ to $\mathscr{H}'$ by $\mathscr{B}_0(\mathscr{H},\mathscr{H}')$; if $A \in \mathscr{B}_0(\mathscr{H},\mathscr{H}')$, we denote by $\mu_n(A)$ its singular values with decreasing ordering $\mu_0(A) \geqslant \mu_1(A) \geqslant \cdots \geqslant 0$. For $1 \leqslant p \leqslant \infty$, we shall denote the Schatten ideals[1] by

$$\mathscr{B}_p(\mathscr{H},\mathscr{H}') = \left\{ A \in \mathscr{B}_0(\mathscr{H},\mathscr{H}') \;\Big|\; \sum_{n=1}^{\infty} \big(\mu_n(A)\big)^p < +\infty \right\},$$

$$\|A\|_p = \left(\sum_{n=0}^{\infty} \big(\mu_n(A)\big)^p \right)^{\frac{1}{p}},$$

and, for $p = \infty$, we simply have $\mathscr{B}_\infty(\mathscr{H},\mathscr{H}') = \mathscr{B}(\mathscr{H},\mathscr{H}')$ and $\|A\|_\infty = \|A\|$. Let us recall also that, for $A \in \mathscr{B}_p(\mathscr{H},\mathscr{H}')$, we have $\mu_n(A) = \mu_n(A^*)$, where $*$ denotes the adjoint, and $A^* \in \mathscr{B}_p(\mathscr{H}',\mathscr{H})$ (see, e.g., [14]).

We shall denote the spectrum of an operator A by $\sigma(A)$, the pure point spectrum by $\sigma_{\mathrm{pp}}(A)$ and the essential spectrum by $\sigma_{\mathrm{ess}}(A)$.

The resolvent of the operator

$$H_0^\omega \equiv -\frac{1}{2}\Delta_{\vec{x}} - \frac{1}{2}\Delta_{\vec{y}} + \frac{\omega^2 y^2}{2} - \frac{\omega d}{2}$$

is given by the following integral kernel,[2]

$$\begin{aligned} G_\omega^\lambda(\vec{x},\vec{y};\vec{x}',\vec{y}') &\equiv \big(H_0^\omega + \lambda\big)^{-1}(\vec{x},\vec{y};\vec{x}',\vec{y}') \\ &= \frac{\omega^{d-1}}{2^{\frac{d}{2}}\pi^d} \int_0^1 d\nu \, \frac{\nu^{\frac{\lambda}{\omega}-1}}{(1-\nu^2)^{\frac{d}{2}}(\ln\frac{1}{\nu})^{\frac{d}{2}}} \\ &\quad \times \exp\left\{ -\frac{\omega}{2}\frac{1-\nu}{1+\nu}\big(y^2+y'^2\big) - \frac{\omega}{2\ln\frac{1}{\nu}}(\vec{x}-\vec{x}')^2 - \frac{\omega\nu}{1-\nu^2}(\vec{y}-\vec{y}')^2 \right\}, \end{aligned} \tag{1.4}$$

where $\lambda > 0$ and $\vec{x},\vec{y},\vec{x}',\vec{y}' \in \mathbb{R}^d$. The above expression has been obtained in [7]. Note that by separation of variables the kernel (1.4) can be expressed as well as[3]

$$G_\omega^\lambda(\vec{x},\vec{y};\vec{x}',\vec{y}') = 2\sum_{\vec{n}\in\mathbb{N}^d} \Psi_{\vec{n}}^{(\omega)}(\vec{y})\Psi_{\vec{n}}^{(\omega)}(\vec{y}')\big[-\Delta_{\vec{x}} + 2\omega n + 2\lambda\big]^{-1}(\vec{x};\vec{x}'), \tag{1.5}$$

[1] The norm $\|A\|_1$ will also be denoted by $\mathrm{Tr}(|A|)$, the usual trace class norm.

[2] In the following we shall often omit the suffix ω and set $G^\lambda \equiv G_1^\lambda$. Similarly we denote by H_α and H_0 the operators H_α^1 and H_0^1 respectively.

[3] For any $\vec{n} \in \mathbb{N}^d$, we set $n \equiv \sum_{i=1}^d n_i$.

which in the one-dimensional case becomes

$$G_\omega^\lambda(x,y;x',y') = \sum_{n=0}^{+\infty} \frac{\Psi_n^{(\omega)}(y)\Psi_n^{(\omega)}(y')}{\sqrt{2(\omega n+\lambda)}} \exp\bigl\{-\sqrt{2(\omega n+\lambda)}|x-x'|\bigr\}. \tag{1.6}$$

The symbol Π will stand for the collision plane

$$\Pi \equiv \bigl\{(\vec{x},\vec{y}) \in \mathbb{R}^{2d} \mid \vec{x}=\vec{y}\bigr\}, \tag{1.7}$$

and $\mathcal{G}_\omega^\lambda f(\vec{x},\vec{y})$ for the potential associated with $f \in L^2(\mathbb{R}^d)$, i.e.,

$$\mathcal{G}_\omega^\lambda f(\vec{x},\vec{y}) \equiv \int_{\mathbb{R}^d} d\vec{x}'\, G_\omega^\lambda(\vec{x},\vec{y};\vec{x}',\vec{x}') f(\vec{x}'). \tag{1.8}$$

If we denote by $\mathcal{P}:\mathcal{H}^m(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^d)$, $m > d/2$, the restriction to the plane Π, we trivially have $\mathcal{G}^\lambda = G_\omega^\lambda \mathcal{P}^*$.

Any positive constant will be denoted by c, whose value may change from line to line.

2. The one-dimensional case

2.1. Preliminary results

The easiest way to give the expression (1.1) a rigorous meaning is to consider the (formal) quadratic form associated with such an operator: for any $\alpha \in \mathbb{R}$, at least formally, we have

$$\langle u|H_\alpha^\omega|u\rangle = \langle u|H_0^\omega|u\rangle + \alpha \int_{\mathbb{R}} dx\, \bigl|u(x,x)\bigr|^2.$$

This formal expression identifies a closed quadratic form bounded below (see [7] for the proof).

Definition 2.1 *(Quadratic form F_α^ω). The quadratic form $(F_\alpha^\omega, \mathcal{D}(F_\alpha^\omega))$ is defined as follows,*

$$\begin{aligned} F_\alpha^\omega[u] &\equiv F_0^\omega[u] + \alpha F_{\mathrm{int}}[u] \\ &\equiv \int_{\mathbb{R}^2} dx\, dy \left\{ \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{2}\left|\frac{\partial u}{\partial y}\right|^2 + \frac{\omega^2 y^2}{2}|u|^2 - \frac{\omega}{2}|u|^2 \right\} + \alpha \int_{\mathbb{R}} dx\, \bigl|u(x,x)\bigr|^2, \end{aligned} \tag{2.1}$$

$$\mathcal{D}\bigl(F_\alpha^\omega\bigr) = \bigl\{u \in L^2\bigl(\mathbb{R}^2\bigr) \bigm| F_\alpha^\omega[u] < +\infty\bigr\}. \tag{2.2}$$

The main properties of F_α^ω are summarized in the following theorem.

Theorem 2.2 *(Closure of the form F_α^ω). The quadratic form $(F_\alpha^\omega, \mathcal{D}(F_\alpha^\omega))$ is closed and bounded below on*

$$\mathcal{D}\bigl(F_\alpha^\omega\bigr) = \mathcal{D}\bigl(F_0^\omega\bigr) = \bigl\{u \in L^2\bigl(\mathbb{R}^2\bigr) \bigm| u \in \mathcal{H}^1\bigl(\mathbb{R}^2\bigr), yu \in L^2\bigl(\mathbb{R}^2\bigr)\bigr\}.$$

We denote by $(H_\alpha^\omega, \mathcal{D}(H_\alpha^\omega))$ the bounded from below self-adjoint operator on $L^2(\mathbb{R}^2)$ defined by (2.1) and (2.2). Concerning the resolvent of H_α^ω we have the following result.

Theorem 2.3 *(Operator H_α^ω). The domain and the action of H_α^ω are the following,*

$$\mathcal{D}(H_\alpha^\omega) = \{u \in L^2(\mathbb{R}^2) \mid u = \varphi^\lambda + \mathcal{G}_\omega^\lambda q, \varphi^\lambda \in \mathcal{D}(H_0^\omega), q + \alpha \mathcal{P} u = 0\}, \tag{2.3}$$

$$(H_\alpha^\omega + \lambda) u = (H_0^\omega + \lambda)\varphi^\lambda. \tag{2.4}$$

Moreover there exists $\lambda_0 > 0$ such that, for $\lambda > \lambda_0$ and for any $f \in L^2(\mathbb{R}^2)$, one has

$$(H_\alpha^\omega + \lambda)^{-1} f = (H_0^\omega + \lambda)^{-1} f + \mathcal{G}_\omega^\lambda q_f, \tag{2.5}$$

where the charge q_f is a solution to the following equation,

$$q_f + \alpha\{K_\omega^\lambda q_f + \mathcal{P} G_\omega^\lambda f\} = 0 \tag{2.6}$$

and $K_\omega^\lambda \equiv \mathcal{P} G_\omega^\lambda \mathcal{P}^*$ has integral kernel $K_\omega^\lambda(x; x') \equiv G_\omega^\lambda(x, x; x', x')$.

It is straightforward to see that H_α^ω is an extension of $\tilde{H}_0$ defined by

$$\mathcal{D}(\tilde{H}_0) = \{u \in C_0^\infty(\mathbb{R}^2 \setminus \Pi)\},$$
$$\tilde{H}_0 u = \left[-\frac{1}{2}\Delta_x - \frac{1}{2}\Delta_y + \frac{\omega^2 y^2}{2} - \frac{\omega}{2}\right] u.$$

Then, by definition, H_α^ω is a perturbation of H_0^ω supported by the null set Π, i.e., a rigorous counterpart of (1.1). It follows from a general argument (see [1, Lemma C.2]) that H_α^ω is a local operator, i.e., if $u = 0$ in an open set Ω, then $H_\alpha^\omega u = 0$ in Ω.

The effect of the interaction is equivalent to the boundary condition $q + \alpha \mathcal{P} u = 0$ satisfied by $u \in \mathcal{D}(H_\alpha^\omega)$ (see (2.3)) which fixes a unique self-adjoint extension of $\tilde{H}_0$. Such a boundary condition is manifestly local, i.e., the value of q at a given point $x \in \mathbb{R}$ is proportional to the value of u at the point (x, x). In this sense the constructed Hamiltonian H_α^ω defines a local zero-range interaction.

Note that the quadratic form (2.1) is not the most general zero-range perturbation of F_0^ω: it is clear, for instance, that, if we take a real function $\alpha(x)$ such that $\alpha \in L^\infty$, then

$$F'_\alpha[u] \equiv \int_{\mathbb{R}^2} dx\, dy \left\{\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{2}\left|\frac{\partial u}{\partial y}\right|^2 + \frac{\omega^2 y^2}{2}|u|^2 - \frac{\omega}{2}|u|^2\right\} + \int_{\mathbb{R}} dx\, \alpha(x)\left|u(x, x)\right|^2 \tag{2.7}$$

define another zero-range perturbation of F_0^ω.

The boundary condition corresponding to F'_α is $q(x) + \alpha(x)(\mathcal{P}u)(x) = 0$. The perturbation we use is distinguished by its invariance under translations along the coincidence manifold Π. In fact, the quadratic form (2.1) gives the simplest "δ"-like zero-range perturbation of F_0^ω which correspond to a local boundary condition. Similar remarks hold for the two- and three-dimensional cases too.

2.2. Spectral analysis

In this section we shall study the spectrum of (1.1). We analyze first the properties of the operator K^{λ}_{ω}, for fixed $\omega = 1$.

Proposition 2.4 *(Spectral analysis of K^{λ}_{1}). The operator $K^{\lambda}_{1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is compact, positive definite and self-adjoint. Let $\mu_n(\lambda)$, $n \in \mathbb{N}$, be its eigenvalues arranged in a decreasing order. Then $\mu_n(\lambda)$ is a decreasing function of λ, for any $n \in \mathbb{N}$, and $\lim_{\lambda\to 0} \mu_0(\lambda) = +\infty$, whereas $\lim_{\lambda\to 0} \mu_n(\lambda) < +\infty$, for $n > 0$.*

Furthermore the following estimate

$$\left|\mu_n(\lambda) - \frac{1}{\sqrt{2(n+\lambda)}}\right| \leqslant \frac{3}{5}\sqrt{\frac{2}{\pi}} \tag{2.8}$$

holds true for any $n \in \mathbb{N}$.

Proof. In order to simplify the notation, we shall denote by K^{λ} the operator K^{λ}_{1}, i.e., K^{λ}_{ω} for fixed $\omega = 1$.

Using the boundedness criterium for integral operators, see [12], it is straightforward to see that

$$\left\|K^{\lambda}\right\| \leqslant c \int_0^1 d\nu \, \frac{\nu^{\lambda-1}}{\sqrt{1-\nu^2+\ln\frac{1}{\nu}}}. \tag{2.9}$$

Estimate (2.9) implies that $\|K^{\lambda}\| \leqslant c\lambda^{-\frac{1}{2}}$ for $\lambda \to 0$ and for $\lambda \to +\infty$. The operator K^{λ} is manifestly self-adjoint.

We introduce the following decomposition:

$$K^{\lambda}(x;x') = \int_0^1 d\nu \, m^{\lambda}(\nu) k_{\nu}(x;x'), \tag{2.10}$$

$$m^{\lambda}(\nu) \equiv \frac{\nu^{\lambda-1}}{\sqrt{2}\pi\sqrt{1-\nu^2}\sqrt{\ln\frac{1}{\nu}}}, \tag{2.11}$$

$$k_{\nu}(x;x') \equiv \exp\left\{-\frac{1}{2}\frac{1-\nu}{1+\nu}\left(x^2+x'^2\right) - \frac{(x-x')^2}{2\ln\frac{1}{\nu}} - \frac{\nu(x-x')^2}{1-\nu^2}\right\}. \tag{2.12}$$

Since k_{ν} is a positive operator-valued function and $m^{\lambda}(\nu)$ is a positive function for $\nu \in (0,1)$, the operator K^{λ} is positive and has empty kernel.

Let us prove now that K^{λ} is a compact operator. Using lemma in [21, p. 65], it is straightforward to prove that

$$\int_0^{1-\delta} d\nu \, m^{\lambda}(\nu) k_{\nu}(x;x')$$

is the kernel of a trace class operator, for any $0 < \delta < 1$. By Halmos criterium, it converges to K^{λ} in the uniform topology, therefore K^{λ} is compact and positive.

By (2.10), the inequality $K^{\lambda_1} > K^{\lambda_2}$ holds for $\lambda_1 < \lambda_2$, therefore the eigenvalues are decreasing functions of λ by Min-Max theorem (see, e.g., [22, Theorem XIII.1]).

In order to study the behavior of the eigenvalues of K^{λ} for $\lambda \to 0$, it suffices to notice that, if f is orthogonal to $\exp\{-\frac{y^2}{2}\}$, then $\lim_{\lambda\to 0}\langle f|K^{\lambda}|f\rangle < +\infty$; the statement follows by Min-Max theorem and estimate (2.9).

We prove now the eigenvalue upper bound. Note first that (see, e.g., [25])

$$\left(H^1_{\mathrm{osc}} + \lambda - 1/2\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} dt\, \frac{e^{-(\lambda-\frac{1}{2})t} e^{-H^1_{\mathrm{osc}} t}}{\sqrt{t}},$$

which together with (1.3) yields

$$\begin{aligned}
&\left(H^1_{\mathrm{osc}} + \lambda - 1/2\right)^{-\frac{1}{2}}(x;x') \\
&\quad = \frac{1}{\pi}\int_0^1 d\nu\, \frac{\nu^{\lambda-1}}{\sqrt{2\pi}\sqrt{1-\nu^2}\sqrt{\ln\frac{1}{\nu}}} \exp\left\{-\frac{1}{2}\frac{1-\nu}{1+\nu}\left(x^2+x'^2\right) - \frac{\nu}{1-\nu^2}(x-x')^2\right\}.
\end{aligned}$$

In order to apply Schur test (see [12]) to the operator $(H^1_{\mathrm{osc}} + \lambda - 1/2)^{-\frac{1}{2}} - \sqrt{2}K^{\lambda}$, we estimate

$$\begin{aligned}
&\int_{\mathbb{R}} dx' \left\{\left(H^1_{\mathrm{osc}} + \lambda - 1/2\right)^{-\frac{1}{2}}(x;x') - \sqrt{2}K^{\lambda}(x;x')\right\} \\
&\quad = \frac{1}{\sqrt{\pi}}\int_0^1 d\nu\, \frac{\nu^{\lambda-1}}{\sqrt{1+\nu^2}\sqrt{\ln\frac{1}{\nu}}} \exp\left[-\frac{(1-\nu^2)x^2}{2(1+\nu^2)}\right] \\
&\qquad \times \left\{1 - \sqrt{\frac{(1+\nu^2)\ln\frac{1}{\nu}}{1-\nu^2+(1+\nu^2)\ln\frac{1}{\nu}}} \exp\left[-\frac{(1-\nu)^3 x^2}{2(1+\nu^2)[1-\nu^2+(1+\nu^2)\ln\frac{1}{\nu}]}\right]\right\} \\
&\quad \leqslant \frac{1}{2\sqrt{\pi}}\int_0^1 d\nu\, \frac{\nu^{\lambda-1}(1-\nu)^3}{\sqrt{\ln\frac{1}{\nu}}(1+\nu^2)^{\frac{3}{2}}[1-\nu^2+(1+\nu^2)\ln\frac{1}{\nu}]},
\end{aligned}$$

where we have used the inequality

$$\exp\left\{-a_1x^2\right\} - b\exp\left\{-a_2x^2\right\} \leqslant \frac{b(a_2-a_1)}{a_1}\exp\left\{-a_2x^2\right\} \leqslant \frac{a_2-a_1}{a_1},$$

which holds true for any $0 < a_1 < a_2$, $0 < b < 1$ and $ba_2 > a_1$. The last integral can be easily estimated by

$$\int_0^1 d\nu \, \frac{\nu^{\lambda-1}(1-\nu)^3}{\sqrt{\ln\frac{1}{\nu}}(1+\nu^2)^{\frac{3}{2}}[1-\nu^2+(1+\nu^2)\ln\frac{1}{\nu}]}$$
$$\leqslant \int_0^\infty dt \, \frac{(1-e^{-t})^3}{t^{\frac{3}{2}}} \leqslant \int_0^1 dt \, t^{\frac{3}{2}} + \int_1^\infty dt \, t^{-\frac{3}{2}} \leqslant \frac{12}{5},$$

so that, using the kernel symmetry, one has

$$\left\| \left(H^1_{\mathrm{osc}} + \lambda - 1/2\right)^{-\frac{1}{2}} - \sqrt{2} K^\lambda \right\| \leqslant \frac{6}{5\sqrt{\pi}}. \tag{2.13}$$

The result is thus a simple consequence of Min-Max theorem. □

We are now able to study the point spectrum of H^ω_α.

Theorem 2.5 *(Negative spectrum of H^ω_α). For $\alpha \geqslant 0$, H^ω_α has no negative eigenvalues, while, for $\alpha < 0$, there is a finite number $N_\omega(\alpha)$ of negative eigenvalues $-E_0(\alpha,\omega) \leqslant -E_1(\alpha,\omega) \leqslant \cdots \leqslant 0$ satisfying the scaling property*

$$E_n(\alpha,\omega) = \omega E_n(\alpha/\sqrt{\omega}, 1). \tag{2.14}$$

The corresponding eigenvectors are given by $u_n = \mathcal{G}^{E_n}_\omega q_n$, where q_n is a solution[4] of the homogeneous equation $q_n + \alpha K^{E_n}_\omega q_n = 0$.

Furthermore there exists $\alpha_0 > 0$ such that, for $-\alpha_0 < \alpha < 0$, $N_\omega(\alpha) = 1$, whereas, for $|\alpha| \geqslant \alpha_0$, $N_\omega(\alpha) > 1$. For fixed $\omega > 0$, the ground state energy $E_0(\alpha,\omega)$ satisfies the asymptotics

$$E_0(\alpha,\omega) \sim \frac{\alpha^2}{2} \tag{2.15}$$

as $\alpha \to 0$.

Proof. First we derive an integral equation equivalent to the eigenvalue problem. Let u be a solution to $H^\omega_\alpha u = -Eu$, $E > 0$. Using (2.4), this proves to be equivalent to

$$\left(H^\omega_0 + E\right)\phi^\lambda = (\lambda - E)\mathcal{G}^\lambda_\omega q.$$

The first resolvent identity yields from (2.5),

$$\phi^\lambda = \mathcal{G}^E_\omega q - \mathcal{G}^\lambda_\omega q,$$

which implies

$$u = \mathcal{G}^E_\omega q.$$

[4] Such a solution is actually unique once the L^2-norm of q_n is fixed, as it is by the L^2-normalization of u_n.

On the other hand by using the boundary conditions in (2.3), we arrive at the following homogeneous equation for q and E:

$$q + \alpha K_{\omega}^{E} q = 0. \tag{2.16}$$

By scaling the above equation is equivalent to the following one:

$$\tilde{q} + \frac{\alpha}{\sqrt{\omega}} K_1^{E/\omega} \tilde{q} = 0 \tag{2.17}$$

i.e., q solves (2.16) if and only if

$$\tilde{q}(x) \equiv \omega^{-\frac{1}{4}} q(x/\sqrt{\omega})$$

solves (2.17), which implies (2.14).

The other properties of the negative eigenvalues follow then from Proposition 2.4: if $\alpha \geqslant 0$, (2.17) has no solution, since $K_1^{E/\omega}$ is a positive operator; if $\alpha < 0$, by projecting (2.17) onto the eigenvectors of $K_1^{E/\omega}$, one obtains the algebraic equation

$$1 + \frac{\alpha \mu_n(E/\omega)}{\sqrt{\omega}} = 0, \tag{2.18}$$

and the eigenvalue equation is equivalent to find some $n \in \mathbb{N}$ and $E > 0$ satisfying (2.18).

The monotonicity of μ_n together with their asymptotics as $E \to 0$ (see Proposition 2.4) imply that, for any $\alpha < 0$, (2.18) has only a finite number of solutions E_n. More precisely, for $|\alpha| < \alpha_0$, it has only one solution E_0, since $\lim_{E \to 0} \mu_0(E/\omega) = +\infty$ and $\lim_{E \to 0} \mu_n(E/\omega) < +\infty$, for $n > 0$.

For fixed ω, the estimate (2.13) yields

$$\mu_0(E/\omega, 1) = \sqrt{\frac{\omega}{2E}} + \mathcal{O}(\sqrt{E}) \tag{2.19}$$

as $E \to 0$ and Eq. (2.15) easily follows from (2.18). □

Note that the result contained in theorem above yields also the expected asymptotic behavior as $\omega \to 0$: the limiting system is given by two particles freely moving on the line with a mutual zero-range interaction. The spectrum of such an operator is absolutely continuous for any sign of α, because of the translation invariance associated with the motion of the center of mass and it is $[0, \infty)$ for $\alpha > 0$, $[-\alpha^2/2, \infty)$ for $\alpha < 0$. If $\alpha < 0$, the scaling property (2.14) implies that the eigenvalues accumulate at the bottom of the continuous spectrum as $\omega \to 0$ and the corresponding bound states eventually disappear.

More interesting is the opposite asymptotics, that is the limit $\omega \to \infty$: in this case the strength of the harmonic oscillator becomes so large that, roughly speaking, one of the two particles remains fixed at the origin. More precisely we expect that the reduced dynamics of the other particle is generated by an Hamiltonian formally given by

$$h_\alpha = -\frac{1}{2}\Delta_x + \text{“}\alpha\delta(x)\text{”}.$$

We define h_α as the self-adjoint operator corresponding to the closed and bounded from below quadratic form f_α given by

$$f_\alpha[u] = \frac{1}{2}\int_{\mathbb{R}} dx \left|\frac{du}{dx}\right|^2 + \alpha|u(0)|.$$

It is straightforward to compute the domain and the action of h_α (see, e.g., [1]):

$$h_\alpha = -\frac{1}{2}\frac{d^2}{dx^2}, \tag{2.20}$$

$$\mathcal{D}(h_\alpha) = \{u \in \mathcal{H}^1(\mathbb{R}) \cap \mathcal{H}^2(\mathbb{R}\setminus\{0\}) \mid u'(0^+) - u'(0^-) = 2\alpha u(0)\}, \tag{2.21}$$

and, for $\alpha < 0$, its spectrum contains only one negative eigenvalue $-\alpha^2/2$ with (normalized) eigenvector

$$\xi_\alpha(x) \equiv \sqrt{|\alpha|}e^{-|\alpha||x|}.$$

For fixed $\alpha < 0$ and ω sufficiently large (larger than $\alpha^2/2$), the operator H_α^ω has only one negative eigenvalue $-E_0(\alpha,\omega)$ with normalized eigenvector $u_{\alpha,\omega}(x,y)$. We denote by $\rho_{\alpha,\omega}$ the reduced density matrix associated with the ground state $u_{\alpha,\omega}(x,y)$, i.e., the trace class operator $\rho_{\alpha,\omega} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with integral kernel

$$\rho_{\alpha,\omega}(x;x') \equiv \int_{\mathbb{R}} dy\, u^*_{\alpha,\omega}(x,y)u_{\alpha,\omega}(x',y). \tag{2.22}$$

Proposition 2.6 *(Ground state asymptotics as $\omega \to \infty$). For any fixed $\alpha < 0$ and for $\omega \to \infty$,*

$$E_0(\alpha,\omega) = \frac{\alpha^2}{2} + \mathcal{O}(\omega^{-1}), \tag{2.23}$$

and the reduced density matrix $\rho_{\alpha,\omega}$ converges to the one-dimensional projector onto ξ_α, i.e.,

$$\rho_{\alpha,\omega} \underset{\omega\to\infty}{\longrightarrow} |\xi_\alpha\rangle\langle\xi_\alpha| \tag{2.24}$$

in the norm topology of $\mathcal{B}_1(L^2(\mathbb{R}))$.

Proof. We first notice that the bound $\|K^\lambda\| \leqslant c\lambda^{-\frac{1}{2}}$ (see (2.9)) together with the eigenvalue equation (2.18) imply the bound $E_0(\alpha,\omega) \leqslant c\alpha^2$, so that the ground state energy of H_α^ω is bounded (from above) uniformly in ω and $E_0/\omega \to 0$, as $\omega \to \infty$, for fixed α. Therefore the first part of the statement can be proved exactly as the asymptotics (2.15) (see, e.g., (2.19)).

We now consider the ground state wave function $u_{\alpha,\omega}$, which can be expressed as $u_{\alpha,\omega} = \mathcal{G}_\omega^{E_0} q_0$ (see Proposition 2.5), where q_0 is a solution to the homogeneous equation $q_0 + \alpha K_\omega^{E_0} q_0 = 0$. Note that the L^2-norm of q_0 is actually fixed by the normalization of $u_{\alpha,\omega}$. Let us decompose q_0 as $q_0 = Q_0\Psi_0^{(\omega)} + \xi$, with $\langle\Psi_0^{(\omega)}|\xi\rangle = 0$. We are going to prove that

$$\|u_{\alpha,\omega} - w_{\alpha,\omega}\|^2 \leqslant \frac{c\|q_0\|^2}{\omega^{\frac{3}{2}}}, \tag{2.25}$$

$$w_{\alpha,\omega}(x,y) \equiv \frac{Q_0}{\sqrt{2E_0}} \Psi_0^{(\omega)}(y) \exp\{-\sqrt{2E_0}|x|\}. \tag{2.26}$$

The proof is done in two steps. We first show that

$$\|u_{\alpha,\omega} - v_{\alpha,\omega}\|^2 \leqslant \frac{c\|q_0\|^2}{\omega^{\frac{3}{2}}},$$

$$v_{\alpha,\omega}(x,y) \equiv \frac{1}{\sqrt{2E_0}} \Psi_0^{(\omega)}(y) \int_{\mathbb{R}} dx'\, e^{-\sqrt{2E_0}|x-x'|} \Psi_0^{(\omega)}(x') q_0(x').$$

Indeed, by using the representation (1.6), we can easily estimate

$$\|u_{\alpha,\omega} - v_{\alpha,\omega}\|^2 \leqslant \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}(\omega n + E_0)^{\frac{3}{2}}} \left|\left\langle \left|\Psi_n^{(\omega)}\right| \,\middle|\, |q_0| \right\rangle\right|^2 \leqslant \frac{c\|q_0\|^2}{\omega^{\frac{3}{2}}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \leqslant \frac{c\|q_0\|^2}{\omega^{\frac{3}{2}}}.$$

On the other hand, setting $\tilde{q}_0(x) \equiv \omega^{-\frac{1}{4}} q_0(x/\sqrt{\omega})$, one has

$$\|w_{\alpha,\omega}\|^2 = \frac{|Q_0|^2}{(2E_0)^{\frac{3}{2}}}, \tag{2.27}$$

$$\|v_{\alpha,\omega}\|^2 = \frac{|Q_0|^2}{(2E_0)^{\frac{3}{2}}} + \frac{1}{2E_0\omega^{\frac{3}{2}}} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dx'\, |x-x'| \Psi_0^{(1)}(x) \Psi_0^{(1)}(x') \tilde{q}_0^*(x) \tilde{q}_0(x')$$
$$\leqslant \frac{|Q_0|^2}{(2E_0)^{\frac{3}{2}}} + \frac{c\|q_0\|^2}{\omega^{\frac{3}{2}}},$$

and

$$2\Re\langle v_{\alpha,\omega}|w_{\alpha,\omega}\rangle = \frac{2|Q_0|^2}{(2E_0)^{\frac{3}{2}}} + \frac{Q_0^*}{\sqrt{\omega}(2E_0)^{\frac{3}{2}}} \int_{\mathbb{R}} dx\, |x| \Psi_0^{(1)}(x) \tilde{q}_0(x) \exp\left\{-\sqrt{\frac{2E_0}{\omega}}|x|\right\}$$
$$\geqslant \frac{2|Q_0|^2}{(2E_0)^{\frac{3}{2}}} - \frac{c\|q_0\|^2}{\omega^{\frac{3}{2}}}$$

so that $\|v_{\alpha,\omega} - w_{\alpha,\omega}\|^2 \leqslant c\|q_0\|^2/\omega^{\frac{3}{2}}$ and (2.25) is proven.

Let us now consider the charge q_0. The asymptotic behavior of the operator $K_\omega^{E_0}$, as $\omega \to \infty$, is given by the following estimate

$$\left\| K_\omega^{E_0} - \frac{1}{\sqrt{2E_0}} \left|\Psi_0^{(\omega)}\right\rangle \left\langle \Psi_0^{(\omega)}\right| \right\| = \mathcal{O}(\omega^{-\frac{1}{2}}), \tag{2.28}$$

which easily follows from (2.13) and the simple inequality

$$\left\| \left(2H_{\mathrm{osc}}^{\omega} + 2E_0 - 1\right)^{-\frac{1}{2}} - \frac{1}{\sqrt{2E_0}} \left|\Psi_0^{(\omega)}\right\rangle\!\left\langle\Psi_0^{(\omega)}\right| \right\| = \mathcal{O}\left(\omega^{-\frac{1}{2}}\right).$$

Therefore by projecting the homogeneous equation (2.16) onto ξ, we get

$$\|\xi\|^2 - \frac{c|\alpha|\|\xi\|\|q_0\|}{\sqrt{\omega}} \leqslant 0$$

because of (2.28), which in turn implies $\|q_0\|^2 = |Q_0|^2 + \mathcal{O}(\omega^{-1})$ and $\|\xi\| = \mathcal{O}(\omega^{-1})$.

In order to derive from (2.25) the L^2-convergence of the ground state to $w_{\alpha,\omega}$, we need then to bound $\|q_0\|$ uniformly in ω; this can be done by exploiting the L^2-normalization of $u_{\alpha,\omega}$: (2.27), (2.25) and the estimate for $\|\xi\|$ yield

$$\frac{|Q_0|}{(2E_0)^{\frac{3}{4}}} = \|w_{\alpha,\omega}\| \leqslant \|u_{\alpha,\omega}\| + \frac{c\|q_0\|}{\omega^{\frac{3}{4}}} = 1 + \frac{c\|q_0\|}{\omega^{\frac{3}{4}}} \leqslant 1 + \frac{c|Q_0|}{\omega^{\frac{3}{4}}} + \mathcal{O}\left(\omega^{-\frac{5}{4}}\right)$$

which together with the reverse inequality imply that $|Q_0| = 1 + o(1)$ and $\|q_0\| = 1 + o(1)$.

Hence the integral operator with kernel

$$\rho_{\alpha,\omega}^{(w)}(x;x') \equiv \int_{\mathbb{R}} dy\, w_{\alpha,\omega}^*(x,y) w_{\alpha,\omega}(x',y) = \frac{|Q_0|^2}{2E_0} \exp\left\{-\sqrt{2E_0}\left(|x| + |x'|\right)\right\}$$

converges in trace class norm to $|\xi_\alpha\rangle\langle\xi_\alpha|$, since it is a projector onto a vector which converges in L^2-norm to ξ_α ($|Q_0| \to 1$ and $E_0 \to \alpha^2/2$, as $\omega \to \infty$). On the other hand estimate (2.25) gives $\|\rho_{\alpha,\omega} - \rho_{\alpha,\omega}^{(w)}\| = o(1)$ and the convergence of the operators in the norm topology implies weak convergence in $\mathscr{B}_1(L^2(\mathbb{R}))$, but, since $\mathrm{Tr}(\rho_{\alpha,\omega}) = \mathrm{Tr}(|\xi_\alpha\rangle\langle\xi_\alpha|) = 1$, the convergence is actually in trace class norm. □

Note that in one dimension point interactions share much of the properties of interactions through potentials, which may give the possibility of proving the above result via methods used in spectral theory, such as a variation, due to Grushin (see, e.g., [24]), of Schur complement method.

Now we give some partial results on the positive spectrum of (1.1).

Theorem 2.7 *(Positive spectrum of H_α^ω). The essential spectrum of H_α^ω is equal to* $[0, +\infty)$ *and the wave operators $\Omega_\pm(H_\alpha^\omega, H_0^\omega)$ exist and are complete.*

Proof. It is sufficient to prove that $(H_\alpha^\omega + \lambda)^{-1} - (H_0^\omega + \lambda)^{-1}$ is a trace class operator for some $\lambda > 0$ and $\omega = 1$, then the thesis follows from Weyl's theorem (see [22, Theorem XIII.14]) and Kuroda–Birman theorem (see [21, Theorem XI.9]).

Let us introduce the operator $Q^\lambda \equiv (I + \alpha K_1^\lambda)^{-1}$. For $\lambda > \lambda_0$, Q^λ is bounded and positive (see Proposition (2.4)); the resolvent equation (2.5) can be cast in the following form

$$(H_\alpha + \lambda)^{-1} - G^\lambda = \mathcal{G}^\lambda Q^\lambda \mathcal{G}^{\lambda *}. \tag{2.29}$$

It is immediate to notice that the right-hand side of (2.29) is a positive operator; using Cauchy–Schwarz inequality and the boundedness of Q^λ, one can prove that $\mathcal{G}^\lambda Q^\lambda \mathcal{G}^{\lambda*}(x, y; x', y')$ is a continuous bounded function and that

$$\left|\mathcal{G}^\lambda Q^\lambda \mathcal{G}^{\lambda*}(x, y; x', y')\right| \leqslant c\left[\int_0^1 d\nu\, \frac{\nu^{\frac{\lambda}{\omega}-1}}{\sqrt{1-\nu^2+\ln\frac{1}{\nu}}}\right]^2. \tag{2.30}$$

Then using the lemma in [21, p. 65], together again with Cauchy–Schwarz inequality and the boundedness of Q^λ, one has

$$\begin{aligned}
&\int_{\mathbb{R}^2} dx\, dy\, \mathcal{G}^\lambda Q^\lambda \mathcal{G}^{\lambda*}(x, y; x, y) \\
&\quad \leqslant c\int_{\mathbb{R}^2} dx\, dy \left[\int_{\mathbb{R}} dy' \left|\mathcal{G}^\lambda(x, y; y', y')\right|^2\right]^{1/2} \left[\int_{\mathbb{R}} dy' \left|\mathcal{G}^{\lambda*}(y', y'; x, y)\right|^2\right]^{1/2} \\
&\quad \leqslant c\int_{\mathbb{R}^2} dx\, dy\, dy' \left|\mathcal{G}^\lambda(x, y; y', y')\right|^2 \leqslant c. \qquad \square
\end{aligned}$$

3. The two-dimensional case

3.1. Preliminary results

In order to rigorously define the operator (1.1), we use again the theory of quadratic forms. We refer to [7] for proofs and the heuristic derivation of the quadratic form associated with (1.1).

Definition 3.1 *(Quadratic form F_α^ω). The quadratic form $(F_\alpha^\omega, \mathcal{D}(F_\alpha^\omega))$ is defined as follows*

$$\mathcal{D}\left(F_\alpha^\omega\right) = \left\{u \in L^2\left(\mathbb{R}^4\right) \,\middle|\, \exists q \in \mathcal{D}\left(\Phi_\alpha^{\lambda,\omega}\right), \varphi^\lambda \equiv u - \mathcal{G}_\omega^\lambda q \in \mathcal{D}\left(F_0^\omega\right)\right\}, \tag{3.1}$$

$$F_\alpha^\omega[u] \equiv \mathcal{F}^{\lambda,\omega}[u] + \Phi_{\alpha,\omega}^\lambda[q], \tag{3.2}$$

where $\lambda > 0$ is a positive parameter and

$$\mathcal{F}^{\lambda,\omega}[u] \equiv \int_{\mathbb{R}^4} d\vec{x}\, d\vec{y} \left\{\frac{1}{2}\left|\nabla_{\vec{x}}\varphi^\lambda\right|^2 + \frac{1}{2}\left|\nabla_{\vec{y}}\varphi^\lambda\right|^2 + \lambda\left|\varphi^\lambda\right|^2 - \lambda|u|^2 - \omega|u|^2 + \frac{\omega^2 y^2}{2}\left|\varphi^\lambda\right|^2\right\}, \tag{3.3}$$

$$\mathcal{D}\left(\Phi_{\alpha,\omega}^\lambda\right) = \left\{q \in L^2\left(\mathbb{R}^2\right) \,\middle|\, \Phi_\alpha^{\lambda,\omega}[q] < +\infty\right\},$$

$$\Phi_{\alpha,\omega}^\lambda[q] \equiv \int_{\mathbb{R}^2} d\vec{x}\,\left(\alpha + a_\omega^\lambda(x)\right)\left|q(\vec{x})\right|^2 + \frac{1}{2}\int_{\mathbb{R}^4} d\vec{x}\, d\vec{x}'\, G_\omega^\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}')\left|q(\vec{x}) - q(\vec{x}')\right|^2, \tag{3.4}$$

$$a_\omega^\lambda(x) \equiv \frac{1}{4\pi}\left\{C + \int_0^1 d\nu\, \frac{1}{(1-\nu)}\left[1 - \frac{4\nu^{\lambda/\omega-1}(1-\nu)}{(1+\nu^2)\ln\frac{1}{\nu} + 1 - \nu^2}\right.\right.$$
$$\left.\left. \times \exp\left(-\frac{(1-\nu^2)\ln\frac{1}{\nu} + 2(1-\nu)^2}{2[(1+\nu^2)\ln\frac{1}{\nu} + 1 - \nu^2]}\omega x^2\right)\right]\right\}, \tag{3.5}$$
$$C \equiv -\left(\int_0^1 d\nu\, \frac{e^{-\frac{1}{\nu}}}{\nu} + \int_1^\infty d\nu\, \frac{e^{-\frac{1}{\nu}}}{\nu^2}\ln\frac{1}{\nu}\right). \tag{3.6}$$

Note that the decomposition $u = \varphi^\lambda + \mathcal{G}_\omega^\lambda q$ is well defined and unique (for fixed λ), $\mathcal{G}_\omega^\lambda q \in L^2(\mathbb{R}^4)$ for any $q \in L^2(\mathbb{R}^2)$ and the quadratic form (3.2) is independent of the parameter λ (see [7]); as a matter of fact λ plays here the role of a free parameter and its value will be chosen later.

Theorem 3.2 *(Closure of the form F_α^ω). The quadratic form $(F_\alpha^\omega, \mathcal{D}(F_\alpha^\omega))$ is closed and bounded below on the domain* (3.1) *for any $\omega \geqslant 0$.*

Let us denote by Γ_ω^λ the positive self-adjoint operator on $L^2(\mathbb{R}^2)$ associated with the quadratic form $\Phi_{\alpha,\omega}^\lambda$, i.e.

$$\langle q|\Gamma_\omega^\lambda|q\rangle \equiv \Phi_{\alpha,\omega}^\lambda[q] - \alpha\|q\|^2, \tag{3.7}$$

and by H_α^ω the Hamiltonian defined by F_α^ω. Then we have

Theorem 3.3 *(Operator H_α^ω). The domain and the action of H_α^ω are the following*

$$\mathcal{D}(H_\alpha^\omega) = \left\{u \in L^2(\mathbb{R}^4) \;\middle|\; u = \varphi^\lambda + \mathcal{G}_\omega^\lambda q, \varphi^\lambda \in \mathcal{D}(H_0^\omega), q \in \mathcal{D}(\Gamma_\omega^\lambda), (\alpha + \Gamma_\omega^\lambda)q = \mathcal{P}\varphi^\lambda\right\}, \tag{3.8}$$
$$(H_\alpha^\omega + \lambda)u = (H_0^\omega + \lambda)\varphi^\lambda, \tag{3.9}$$

and the resolvent of H_α^ω can be represented as

$$(H_\alpha^\omega + \lambda)^{-1} f = G_\omega^\lambda f + \mathcal{G}_\omega^\lambda q_f, \tag{3.10}$$

where, for any $f \in L^2(\mathbb{R}^4)$, q_f is a solution to

$$(\alpha + \Gamma_\omega^\lambda)q_f = \mathcal{P}G_\omega^\lambda f. \tag{3.11}$$

As in the one-dimensional case, one can recognize in (3.9) a self-adjoint extension of the symmetric operator $\tilde{H}_0$ defined by

$$\mathcal{D}(\tilde{H}_0) = \left\{u \in C_0^\infty(\mathbb{R}^4 \setminus \Pi)\right\},$$
$$\tilde{H}_0 u = \left[-\frac{1}{2}\Delta_{\vec{x}} - \frac{1}{2}\Delta_{\vec{y}} + \frac{\omega y^2}{2} - \omega\right]u$$

so that H_α^ω is a singular perturbation of H_0^ω supported on Π.

Note that the unperturbed Hamiltonian, corresponding to the case of no interaction between the particle and the harmonic oscillator, belongs to the family H_α^ω and is given by $\alpha = +\infty$. This fact, which could seem surprising, if one considers the formal Hamiltonian (1.1), is due to the renormalization procedure required to give a rigorous meaning to such a formal expression. Therefore we stress that in the two- and three-dimensional cases α does not play the role of coupling constant of the system. Also in the two-dimensional case, the one-parameter family of extensions considered has local boundary conditions (see [7]).

3.2. Spectral analysis

In order to study the spectrum of H_α^ω, we first need to state some spectral properties of the operator Γ_ω^λ.

Proposition 3.4 *(Spectral analysis of Γ_ω^λ). For $\omega > 0$ the domain $\mathcal{D}(\Phi_{\alpha,\omega}^\lambda)$ can be characterized in the following way:*

$$\mathcal{D}\left(\Phi_{\alpha,\omega}^\lambda\right) = \left\{q \in L^2\left(\mathbb{R}^2\right) \mid q \in \mathcal{H}^{\log}\left(\mathbb{R}^2\right), \hat{q} \in \mathcal{H}^{\log}\left(\mathbb{R}^2\right)\right\}. \tag{3.12}$$

On this domain $\Phi_{\alpha,\omega}^\lambda$ is closed and defines a self-adjoint operator Γ_ω^λ. For any $\lambda > 0$ the spectrum, $\sigma(\Gamma_\omega^\lambda)$, is purely discrete, i.e., $\sigma(\Gamma_\omega^\lambda) = \sigma_{\mathrm{pp}}(\Gamma_\omega^\lambda)$.

Let $\gamma_n(\lambda)$, $n \in \mathbb{N}$, be the eigenvalues of Γ_1^λ arranged in an increasing order ($\lim_{n\to\infty} \gamma_n(\lambda) = +\infty$). *For every $n \in \mathbb{N}$, $\gamma_n(\lambda)$ is a non-decreasing function of λ. Furthermore* $\lim_{\lambda\to 0} \gamma_0(\lambda) = -\infty$ *and the other eigenvalues remain bounded below, i.e., for any $\lambda \geqslant 0$, there exists a finite constant c such that $\gamma_n(\lambda) \geqslant -c$, for any $n \in \mathbb{N}$, $n > 0$.*

Proof. For the sake of simplicity we fix $\omega = 1$ from the outset and omit the dependence on ω in the notation.

The self-adjointness of Γ^λ immediately follows from the properties of the quadratic form Φ_α^λ (see [7]). Note that Γ^λ can be written in the following way

$$\Gamma^\lambda = a^\lambda + \Gamma_0^\lambda,$$

where a^λ is the multiplication operator for the unbounded function (3.5) and Γ_0^λ is the self-adjoint operator associated with the positive quadratic form

$$\Phi_0^\lambda[q] \equiv \frac{1}{2} \int_{\mathbb{R}^2} d\vec{x}\, d\vec{x}'\, G^\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') \left|q(\vec{x}) - q(\vec{x}')\right|^2. \tag{3.13}$$

Since Φ_0^λ is positive and $a^\lambda(x)$ is bounded below but not above, Γ^λ is an unbounded operator. Notice that $a^\lambda(x)$ is a monotone increasing function of x and $a^\lambda(x) \simeq c \log x$ for $x \to \infty$; furthermore $a^\lambda(x) \geqslant a^\lambda(0)$, $a^\lambda(0)$ is a monotone increasing function of λ and $a^\lambda(0) \simeq c \log \lambda$ for $\lambda \to \infty$. Hence the following lower bound

$$\Phi_\alpha^\lambda[q] \geqslant \int_{\mathbb{R}^2} d\vec{x}\, \left(\alpha + a^\lambda(x)\right)\left|q(\vec{x})\right|^2 \geqslant \left(\alpha + a^\lambda(0)\right)\|q\|^2 \tag{3.14}$$

proves that for any $\alpha \in \mathbb{R}$ there exists λ_0 such that for $\lambda > \lambda_0$ the quadratic form Φ_α^λ is positive; for such λ the operator Γ^λ is invertible and its inverse is a bounded operator.

We shall prove now that $\sigma(\Gamma^\lambda)$ is purely discrete for any λ. By [22, Theorem XIII.64] it suffices to prove that

$$\mathcal{D}_\eta \equiv \left\{ q \in \mathcal{D}\left(\Phi_\alpha^\lambda\right) \;\middle|\; \Phi_\alpha^\lambda[q] \leqslant \eta \right\}$$

is a compact subset of $L^2(\mathbb{R}^2)$ for any positive η; this will be proved using the Rellich's criterion [22, Theorem XIII.65]. The positivity of Γ_0^λ implies that, if $\Phi_\alpha^\lambda[q] \leqslant \eta$, then

$$\int_{\mathbb{R}^2} d\vec{x}\, a^\lambda(x) \left| q(\vec{x}) \right|^2 \leqslant \eta.$$

Moreover, applying the Fourier transform, Φ_α^λ can be rewritten in the following equivalent form

$$\Phi_\alpha^\lambda[q] = \int_{\mathbb{R}^2} d\vec{k} \left(\alpha + \tilde{a}^\lambda(k) \right) \left| \hat{q}(\vec{k}) \right|^2 + \frac{1}{2} \int_{\mathbb{R}^4} d\vec{k}\, d\vec{k}'\, \tilde{G}^\lambda(\vec{k}; \vec{k}') \left| \hat{q}(\vec{k}) - \hat{q}(\vec{k}') \right|^2, \tag{3.15}$$

where

$$\tilde{a}^\lambda(k) \equiv \frac{1}{4\pi} \left\{ C + \int_0^1 d\nu\, \frac{1}{1-\nu} \left[1 - \frac{4\nu^{\lambda-1}(1-\nu)}{(1+\nu^2)\ln\frac{1}{\nu} + 1 - \nu^2} \times \exp\left(- \frac{(1-\nu^2)\ln\frac{1}{\nu}}{2[(1+\nu^2)\ln\frac{1}{\nu} + 1 - \nu^2]} k^2 \right) \right] \right\}, \tag{3.16}$$

$$\tilde{G}^\lambda(\vec{k}; \vec{k}') \equiv \frac{1}{2\pi^2} \int_0^1 d\nu\, \frac{\nu^{\lambda-1}}{(1-\nu^2)\ln\frac{1}{\nu} + 2(1-\nu)^2} \times \exp\left\{ - \frac{[(1+\nu^2)\ln\frac{1}{\nu} + 1 - \nu^2](k^2 + k'^2)}{2[(1-\nu^2)\ln\frac{1}{\nu} + 2(1-\nu)^2]} - \frac{[1 - \nu^2 + 2\nu \ln\frac{1}{\nu}]\vec{k}\cdot\vec{k}'}{(1-\nu^2)\ln\frac{1}{\nu} + 2(1-\nu)^2} \right\}. \tag{3.17}$$

In order to prove (3.15), it is convenient to introduce a regularized quadratic form $\Phi_\alpha^{\lambda,\delta}$ obtained by restricting the integration domain in ν to the set $[0, 1-\delta]$, for some $0 < \delta < 1$.

It is straightforward to notice that $\Phi_\alpha^{\lambda,\delta}$ is a bounded form and that for every $q \in L^2(\mathbb{R}^2)$, $\Phi_\alpha^{\lambda,\delta}[q]$ is a monotone function of δ; therefore for $q \in \mathcal{D}(\Phi_\alpha^\lambda)$ we have

$$\lim_{\delta \to 0} \Phi_\alpha^{\lambda,\delta}[q] = \Phi_\alpha^\lambda[q]. \tag{3.18}$$

On the other hand, due to the regularization, with straightforward calculations one can prove that

$$\Phi_\alpha^{\lambda,\delta}[q] \equiv \left\{\alpha + \frac{1}{4\pi}\left[C + \int_0^{1-\delta} \frac{d\nu}{1-\nu}\right]\right\}\|q\|^2 - \int_{\mathbb{R}^4} d\vec{x}\, d\vec{x}'\, G^{\lambda,\delta}(\vec{x},\vec{x};\vec{x}',\vec{x}')q^*(\vec{x})q(\vec{x}')$$
$$= \left\{\alpha + \frac{1}{4\pi}\left[C + \int_0^{1-\delta} \frac{d\nu}{1-\nu}\right]\right\}\|\hat{q}\|^2 - \int_{\mathbb{R}^4} d\vec{k}\, d\vec{k}'\, \tilde{G}^{\lambda,\delta}(\vec{k};\vec{k}')\hat{q}^*(\vec{k})\hat{q}(\vec{k}'), \tag{3.19}$$

which can be rewritten in the following way:

$$\Phi_\alpha^{\lambda,\delta}[q] = \int_{\mathbb{R}^2} d\vec{k}\, \big(\alpha + \tilde{a}^{\lambda,\delta}(k)\big)\big|\hat{q}(\vec{k})\big|^2 + \frac{1}{2}\int_{\mathbb{R}^4} d\vec{k}\, d\vec{k}'\, \tilde{G}^{\lambda,\delta}(\vec{k};\vec{k}')\big|\hat{q}(\vec{k}) - \hat{q}(\vec{k}')\big|^2, \tag{3.20}$$

where $\tilde{a}^{\lambda,\delta}$ and $\tilde{G}^{\lambda,\delta}$ are the regularization of (3.16) and (3.17). Notice that (3.19) shows how Φ_α^λ can be obtained by a renormalization of the formal quantity $\langle q|\mathcal{G}^\lambda|q\rangle$.

Due to the monotonicity in δ, we can take the limit $\delta \to 0$ of (3.20) and, by (3.18), we obtain (3.15). It is immediate to notice that (3.15) has the same structure as (3.4) and in particular, if $\Phi_\alpha^\lambda[q] \leqslant \eta$, then

$$\int_{\mathbb{R}^2} d\vec{k}\, \tilde{a}^\lambda(k)\big|\hat{q}(\vec{k})\big|^2 \leqslant \eta.$$

The function $\tilde{a}^\lambda(k)$ has the same properties of $a^\lambda(x)$, namely it is a monotone function of k, $\hat{a}^\lambda(k) \simeq c\log k$ for $k \to \infty$ and $\hat{a}^\lambda(0) \simeq c\log\lambda$ for $\lambda \to 0$. Hence Rellich's criterion guarantees that $\mathcal{D}_\eta$ is a compact subset of $L^2(\mathbb{R}^2)$ and therefore Γ^λ has only pure point spectrum.

Notice that also the following bound holds,

$$\Phi_0^\lambda[q] \leqslant c\|q\|^2_{\mathcal{H}^{\log}(\mathbb{R}^2)}. \tag{3.21}$$

Indeed using the following inequality

$$G^\lambda(\vec{x},\vec{x};\vec{x}',\vec{x}') \leqslant c\int_0^1 d\nu\, \frac{\nu^{\lambda-1}}{(1-\nu^2)\ln\frac{1}{\nu}} \exp\left\{-\frac{1}{2\ln\frac{1}{\nu}}(\vec{x}-\vec{x}')^2 - \frac{\nu}{1-\nu^2}(\vec{x}-\vec{x}')^2\right\}$$

in (3.13) and taking the Fourier transform, we have

$$\Phi_0^\lambda[q] \leqslant c\int_{\mathbb{R}^2} d\vec{k} \int_0^1 d\nu\, \frac{\nu^{\lambda-1}}{2\nu\ln\frac{1}{\nu} + 1 - \nu^2}$$
$$\times \left\{1 - \exp\left[-\frac{2(1-\nu^2)\ln\frac{1}{\nu}}{4(2\nu\ln\frac{1}{\nu} + 1 - \nu^2)}k^2\right]\right\}\big|\hat{q}(k)\big|^2, \tag{3.22}$$

and, since

$$0 \leqslant \int_0^1 d\nu \, \frac{\nu^{\lambda-1}}{2\nu \ln \frac{1}{\nu} + 1 - \nu^2} \left\{ 1 - \exp\left[-\frac{2(1-\nu^2)\ln\frac{1}{\nu}}{4(2\nu\ln\frac{1}{\nu} + 1 - \nu^2)} k^2 \right] \right\}$$
$$\leqslant c \log\bigl(1 + \langle k \rangle\bigr),$$

(3.21) is proven. Therefore, taking into account the behavior of a^λ, (3.21) implies that there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ we have

$$\Gamma^\lambda \leqslant c\bigl(\log\langle x \rangle + \log\langle p \rangle + \log \lambda\bigr). \tag{3.23}$$

We can prove also a similar lower bound for Γ^λ. Putting together the lower bound (3.14) and a corresponding lower bound for (3.15), we obtain:

$$\langle q|\Gamma^\lambda|q\rangle \geqslant \frac{1}{2}\int_{\mathbb{R}^2} d\vec{x}\, a^\lambda(x)\bigl|q(\vec{x})\bigr|^2 + \frac{1}{2}\int_{\mathbb{R}^2} d\vec{k}\, \hat{a}^\lambda(k)\bigl|\hat{q}(\vec{k})\bigr|^2$$
$$\geqslant \left[\frac{a^\lambda(0)}{2} + \frac{\hat{a}^\lambda(0)}{2}\right]\|q\|^2, \tag{3.24}$$

and in particular (3.12) holds true, due to (3.23) and (3.24).

The monotonicity in λ of the eigenvalues $\gamma_n(\lambda)$ follows from the monotonicity of $\Phi_\alpha^\lambda[q]$ with respect to λ. This can be easily seen by observing that the regularized expression (3.19) is a non-decreasing function of λ, as it must be its limit as $\delta \to 0$.

In order to analyze the asymptotics for $\lambda \to 0$ of $\gamma_0(\lambda)$, we shall show that there exists a function q belonging to $\mathcal{D}(\Gamma^\lambda)$, such that $\lim_{\lambda\to 0}\langle q|\Gamma^\lambda|q\rangle = -\infty$. The result is then a consequence of the Min-Max theorem. Indeed taking the ground state of the 2d harmonic oscillator $\Psi_0^{(1)}(\vec{x})$, one has

$$\lim_{\lambda\to 0}\bigl\langle\Psi_0^{(1)}\bigl|\Gamma^\lambda\bigr|\Psi_0^{(1)}\bigr\rangle = \lim_{\lambda\to 0}\frac{1}{4\pi}\left\{C + \int_0^1 d\nu\, \frac{1}{1-\nu}\left[1 - \frac{8\nu^{\lambda-1}(1-\nu)}{(3+\nu^2)\ln\frac{1}{\nu} + 4(1-\nu)}\right]\right\}$$
$$\leqslant c_1 - c_2 \lim_{\lambda\to 0}\int_0^{\frac{1}{2}} d\nu\, \frac{\nu^{\lambda-1}}{1 + \ln\frac{1}{\nu}} = -\infty. \tag{3.25}$$

The boundedness from below of the other eigenvalues can be proved by showing that the quadratic form remains bounded as $\lambda \to 0$, if q is orthogonal to the above function $\Psi_0^{(1)}(\vec{x})$. Let $q^\perp(\vec{x})$ be an L^2-normalized function[5] in $\mathcal{D}(\Phi_\alpha^\lambda)$ such that $\langle\Psi_0^{(1)}(\vec{x})|q^\perp\rangle = 0$. From the expression of the quadratic form (3.4), it is clear that we can restrict the integrations in ν in (3.5) and G^λ to the interval $[0, 1 - 1/e]$, because the remainder is uniformly bounded in λ, i.e., there exists a finite constant c independent of λ such that

[5] For instance one can take $q^\perp = \Psi_{\vec{n}}^{(1)}$, $n_i \neq 0$ for some i.

$$\langle q^{\perp}|\Gamma^{\lambda}|q^{\perp}\rangle \geqslant \left[-c+\frac{1}{4\pi}\int_0^{\frac{1}{e}} d\nu\,\frac{1}{1-\nu}\right]\|q^{\perp}\|^2 - \int_{\mathbb{R}^4} d\vec{x}\,d\vec{x}'\,G^{\lambda}_{1/e}(\vec{x},\vec{x};\vec{x}',\vec{x}')\big(q^{\perp}(\vec{x})\big)^* q^{\perp}(\vec{x}')$$

where we have expanded the second term in the form as in (3.19). The first term on the right-hand side of the expression above is again bounded by a finite constant, whereas, as we are going to prove, the only unbounded contribution comes from $G^{\lambda}_{1/e}$, but it contains a projection to the subspace spanned by $\Psi_0^{(1)}$:

$$\int_{\mathbb{R}^4} d\vec{x}\,d\vec{x}'\,G^{\lambda}_{1/e}(\vec{x},\vec{x};\vec{x}',\vec{x}')\big(q^{\perp}(\vec{x})\big)^* q^{\perp}(\vec{x}') \leqslant \frac{1}{2\pi^2}\int_0^{\frac{1}{e}} d\nu\,\frac{\nu^{\lambda-1}}{1-\nu^2}\langle q^{\perp}|k_{\nu}|q^{\perp}\rangle \equiv \langle q^{\perp}|K^{\lambda}|q^{\perp}\rangle,$$

where k_ν is the integral operator whose kernel is the two-dimensional analogous of (2.12). Moreover

$$\langle q^{\perp}|K^{\lambda}|q^{\perp}\rangle \leqslant \frac{1}{2\pi}\langle q^{\perp}|\big(H^1_{\mathrm{osc}}+\lambda-1\big)^{-1}|q^{\perp}\rangle + \frac{1}{2\pi^2}\int_0^{\frac{1}{e}} d\nu\,\frac{\nu^{\lambda-1}}{1-\nu^2}\langle q^{\perp}|k_{\nu}-\bar{k}_{\nu}|q^{\perp}\rangle, \tag{3.26}$$

$\bar{k}_\nu$ denoting the integral operator with kernel

$$\bar{k}_{\nu}(x,x') \equiv \exp\left\{-\frac{1}{2}\frac{1-\nu}{1+\nu}\big(x^2+x'^2\big)-\frac{\nu(\vec{x}-\vec{x}')^2}{1-\nu^2}\right\}. \tag{3.27}$$

The last term in (3.26) can be estimated as follows:

$$\begin{aligned}
&\frac{1}{2\pi^2}\int_0^{\frac{1}{e}} d\nu\,\frac{\nu^{\lambda-1}}{1-\nu^2}\langle q^{\perp}|k_{\nu}-\bar{k}_{\nu}|q^{\perp}\rangle \\
&\quad\leqslant \frac{1}{4\pi^2}\int_0^{\frac{1}{e}} d\nu\,\frac{\nu^{\lambda-1}}{(1-\nu^2)\ln\frac{1}{\nu}}\int_{\mathbb{R}^4} d\vec{x}\,d\vec{x}'\,|\vec{x}-\vec{x}'|^2\bar{k}_{\nu}(\vec{x};\vec{x}')\big|q^{\perp}(\vec{x})\big|\big|q^{\perp}(\vec{x}')\big| \\
&\quad\leqslant \frac{1}{\pi^2}\int_0^{1} d\nu\,\frac{\nu^{\lambda-1}}{1-\nu^2}\int_{\mathbb{R}^4} d\vec{x}\,d\vec{x}'\,x^2\bar{k}_{\nu}(\vec{x};\vec{x}')\big|q^{\perp}(\vec{x})\big|\big|q^{\perp}(\vec{x}')\big| \\
&\quad\leqslant \frac{1}{\pi}\langle|q^{\perp}||H^1_{\mathrm{osc}}\big(H^1_{\mathrm{osc}}+\lambda-1\big)^{-1}||q^{\perp}|\rangle \leqslant \frac{\|q^{\perp}\|^2}{\pi}.
\end{aligned}$$

Since, for any $q^{\perp}$ orthogonal to the ground state of H^1_{osc},

$$\langle q^{\perp} \big| \big(H^1_{\mathrm{osc}} + \lambda - 1\big)^{-1} \big| q^{\perp} \rangle \leqslant \|q^{\perp}\|^2,$$

we thus obtain

$$\langle q^{\perp} \big| K^{\lambda} \big| q^{\perp} \rangle \leqslant \frac{3}{2\pi} \|q^{\perp}\|^2$$

and the boundedness from below of the operator Γ^{λ} on the subspace of functions orthogonal to $\Psi_0^{(1)}$. □

The spectral properties of the operator $\Gamma^{\lambda}_{\omega}$ allow us to give a complete characterization of the discrete spectrum of H^{ω}_{α}.

Theorem 3.5 *(Negative spectrum of H^{ω}_{α}). For any $\alpha \in \mathbb{R}$ and $\omega \in \mathbb{R}^+$ the discrete spectrum $\sigma_{\mathrm{pp}}(H^{\omega}_{\alpha})$ of H^{ω}_{α} is not empty and it contains a number $N_{\omega}(\alpha) \geqslant 1$ of negative eigenvalues $-E_0(\alpha,\omega) \leqslant -E_1(\alpha,\omega) \leqslant \cdots \leqslant 0$, satisfying the scaling*

$$E_n(\alpha,\omega) = \omega E_n(\alpha,1). \tag{3.28}$$

The corresponding eigenvectors are given by $u_n = \mathcal{G}^{E_n}_{\omega} q_n$, where q_n is a solution to the homogeneous equation $\alpha q_n + \Gamma^{E_n}_{\omega} q_n = 0$.

Moreover there exists $\alpha_0 \in \mathbb{R}$ such that, if $\alpha > \alpha_0$, $N_{\omega}(\alpha) = 1$ and, for fixed ω and $\alpha \to -\infty$, $\ln N_{\omega}(\alpha) \geqslant c|\alpha|$.

The ground state energy has the following asymptotic behavior for fixed ω: $E_0 \simeq -c\alpha^{-1}$ for $\alpha \to +\infty$ and $\ln E_0 \simeq c|\alpha|$ for $\alpha \to -\infty$.

Proof. Following the proof of Theorem 2.5, we get that u_E is an eigenfunction of H^{ω}_{α} relative to the eigenvalue $-E$, $E > 0$, only if

$$u_E = \mathcal{G}^E_{\omega} q \tag{3.29}$$

for some $q \in \mathcal{D}(\Phi^{\lambda}_{\alpha,\omega})$. On the other hand u_E belongs to the domain of H^{ω}_{α} and then it must satisfy the boundary condition on Π, which for a function of this form becomes $\alpha q + \Gamma^E_{\omega} q = 0$, or

$$\Phi^E_{\alpha,\omega}[q] = \alpha \|q\|^2 + \langle q | \Gamma^E_{\omega} | q \rangle = 0. \tag{3.30}$$

So that there is a one-to-one correspondence between the negative eigenvalues of H^{ω}_{α} and non-trivial solutions to the homogeneous equation above. In other words $-E$ is an eigenvalue of H_{α}, if and only if 0 is an eigenvalue of $\alpha + \Gamma^E_{\omega}$. Note that, by scaling, q solves (3.30), if and only if $\tilde{q}(\vec{x}) \equiv \omega^{-\frac{1}{2}} q(\vec{x}/\sqrt{\omega})$ is a solution to the homogeneous equation

$$\alpha \tilde{q} + \Gamma_1^{E/\omega} \tilde{q} = 0,$$

which implies (3.28).

The other results are simple consequences of Proposition 3.4. In particular in order to complete the asymptotic analysis for $\alpha \to +\infty$ it is sufficient to notice that in fact (3.14) and (3.25) imply that $-c_1\lambda^{-1} \leqslant \gamma_0(\lambda) \leqslant -c_2\lambda^{-1}$ as $\lambda \to 0$, due to the asymptotic behavior of $a_\omega^\lambda(0)$ and $\hat{a}_\omega^\lambda(0)$ in such a limit; this is sufficient to conclude that $E_0 = \mathcal{O}(\alpha^{-1})$ for $\alpha \to +\infty$.

The previous argument can be repeated for $\lambda \to -\infty$ and gives that $\gamma_0(\lambda) \geqslant c \ln\lambda$ for $\lambda \to +\infty$, which means $\ln E_0 = \mathcal{O}(|\alpha|)$ for $\alpha \to -\infty$.

In order to conclude the proof, it is sufficient to notice that, for fixed $\omega = 1$, $N_1(\alpha)$ is bounded below by the cardinality of $\{n \in \mathbb{N} \mid \gamma_n(0) \leqslant -\alpha\}$; therefore any upper bound on $\gamma_n(0)$ provides a lower bound on $N_1(\alpha)$. Due to the monotonicity in λ of $\gamma_n(\lambda)$, to (3.23) and to the straightforward estimate $(\log\langle x\rangle + \log\langle p\rangle) \leqslant c\log(H^1_{\mathrm{osc}} + 1)$, we can use the eigenvalue distribution of the logarithm of the harmonic oscillator to estimate $N_1(\alpha)$ which gives $\ln N_1(\alpha) \geqslant c|\alpha|$ for $\alpha \to \infty$. □

We underline that for $\omega > 0$ the interaction is attractive in the sense that there exists at least one bound state irrespective of the sign of α. This fact is essentially due to the renormalization procedure used to rigorously define the quadratic form in (3.2) and to the presence of the harmonic oscillator; this is a common phenomenon in the theory of point interactions (see, e.g., [1] for a similar effect).

Note also that the different scaling (3.28) in ω is due to the scaling properties of the Green function (1.4) (more precisely its restriction to the planes Π), i.e., in d dimensions,

$$G_\omega^\lambda(\vec{x}, \vec{y}; \vec{x}', \vec{y}') = \omega^{d-1} G_1^{\lambda/\omega}(\sqrt{\omega}\vec{x}, \sqrt{\omega}\vec{y}; \sqrt{\omega}\vec{x}', \sqrt{\omega}\vec{y}').$$

The asymptotics for $\omega \to 0$ can be easily derived from (3.28): the spacing between different eigenvalues goes to 0 and in the limit they form a continuum, so that no bound state survives in the limit. On the opposite all the eigenvalues corresponding to excited states diverge as $\omega \to \infty$. More detailed results on the eigenvalue asymptotics could be obtained by applying usual techniques in semiclassical analysis (see, e.g., [6,9,13]), but such an investigation goes beyond the aim of this paper.

Before giving a partial characterization of the positive spectrum, let us prove a technical lemma.

Lemma 3.6. *Define the operator* $T_\omega^k \equiv \mathcal{G}_\omega^{\lambda *}(G_\omega^\lambda)^k \mathcal{G}_\omega^\lambda : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, *for any* $k \in \mathbb{N}$. *Then, if* $\lambda > k$, $T_\omega^k \in \mathcal{B}_p(L^2(\mathbb{R}^2), L^2(\mathbb{R}^2))$, *for any* $p > 2(k+1)^{-1}$.

Proof. Setting $\omega = 1$ for the sake of clarity and omitting the ω-dependence in the notation, we have the identity

$$T^k = \left(-\frac{d}{d\lambda}\right)^{k+1} \mathcal{P} G^\lambda \mathcal{P}^*, \tag{3.31}$$

so that, using (1.4), we get the integral kernel of T^k, i.e.,

$$T^k(\vec{x}; \vec{x}') = c \int_0^1 d\nu \, \frac{\nu^{\lambda-1}(\ln\frac{1}{\nu})^k}{(1-\nu^2)} \times \exp\left\{-\frac{1}{2}\frac{1-\nu}{1+\nu}\left(x^2 + x'^2\right) - \frac{(\vec{x}-\vec{x}')^2}{2\ln\frac{1}{\nu}} - \frac{\nu(\vec{x}-\vec{x}')^2}{1-\nu^2}\right\}.$$

Notice that, using the same argument as in the proof of Proposition 2.4, we can view T^k as the integral over the parameter ν of positive operator-valued functions t_ν, i.e.,

$$T^k = c\int_0^1 d\nu\, m_k(\nu) t_\nu, \tag{3.32}$$

$$m_k(\nu) = \frac{\nu^{\lambda-1}(\ln\frac{1}{\nu})^k}{(1-\nu^2)}, \tag{3.33}$$

$$t_\nu(\vec{x};\vec{x}') = \exp\left\{-\frac{1}{2}\frac{1-\nu}{1+\nu}\left(x^2+x'^2\right) - \frac{(\vec{x}-\vec{x}')^2}{2\ln\frac{1}{\nu}} - \frac{\nu(\vec{x}-\vec{x}')^2}{1-\nu^2}\right\}. \tag{3.34}$$

Applying Schur test to the operator t_ν, one has $\|t_\nu\|_\infty = \|t_\nu\| \leqslant c(1-\nu)$, whereas a simple calculation yields $\|t_\nu\|_1 = \mathrm{Tr}(t_\nu) \leqslant c(1-\nu)^{-1}$. On the other hand Hölder inequality in Schatten ideals (see [23]) gives

$$\|t_\nu\|_p \leqslant \|t_\nu\|_1^{1/p}\|t_\nu\|^{1-1/p} \leqslant (1-\nu)^{1-2/p}. \tag{3.35}$$

It is then straightforward to check that

$$\left\|T^k\right\|_p \leqslant \int_0^1 d\nu\, m_k(\nu)\|t_\nu\|_p < +\infty \tag{3.36}$$

for any $p > 2(k+1)^{-1}$. □

Now we present some partial results on the continuous spectrum of (1.1) by means of a characterization of the mapping properties of the resolvent; in the following we shall fix $\lambda > 1$, such that $(\Gamma_\omega^\lambda + \alpha)^{-1}$ exists and is bounded. Therefore Eq. (3.10) can be cast in the following form

$$\left(H_\alpha^\omega + \lambda\right)^{-1} = G_\omega^\lambda - \mathcal{G}_\omega^\lambda\left(\Gamma_\omega^\lambda + \alpha\right)^{-1}\mathcal{G}_\omega^{\lambda *}. \tag{3.37}$$

Theorem 3.7 *(Positive spectrum of H_α^ω). The essential spectrum of H_α^ω is equal to* $[0,+\infty)$ *and the wave operators $\Omega_\pm(H_\alpha^\omega, H_0^\omega)$ exist and are complete.*

Proof. We shall drop the dependence on ω for brevity. It is sufficient to prove that $(H_\alpha+\lambda)^{-1} - (H_0+\lambda)^{-1}$ is a compact operator and that $(H_\alpha+\lambda)^{-3} - (H_0+\lambda)^{-3}$ is trace class for some $\lambda > 0$, then the thesis follows from Weyl's theorem (see [22, Theorem XIII.14]) and [21, Corollary 3 of Theorem XI.11]).

We first analyze $\mathcal{G}^\lambda(\Gamma^\lambda+\alpha)^{-1}\mathcal{G}^{\lambda *}$ and prove that it is a compact operator. Due to Lemma 3.6 we have $\mathcal{G}^\lambda \in \mathscr{B}_p(L^2(\mathbb{R}^2), L^2(\mathbb{R}^4))$ with $p > 4$: by taking $k = 0$, one obtains $\mathcal{G}^{\lambda *}\mathcal{G}^\lambda \in \mathscr{B}_p(L^2(\mathbb{R}^2), L^2(\mathbb{R}^2))$ for $p > 2$, i.e., denoting by g_n^2, $n \in \mathbb{N}$, its singular values, $\{g_n\} \in \ell_p$ for $p > 4$. By a standard argument (see, e.g., [20, the proof of Theorem VI.17]), one can show that $\{g_n\}$ are the singular values of $\mathcal{G}^\lambda$ and the result easily follows. This also implies that $\mathcal{G}^{\lambda *} \in \mathscr{B}_p(L^2(\mathbb{R}^4), L^2(\mathbb{R}^2))$, $p > 4$, and both operators are compact. Moreover $(\Gamma^\lambda+\alpha)^{-1}$ is a bounded operator and then $\mathcal{G}^\lambda(\Gamma^\lambda+\alpha)^{-1}\mathcal{G}^{\lambda*}\in\mathscr{B}_p(L^2(\mathbb{R}^4),L^2(\mathbb{R}^4))$ with $p>2$, by Hölder inequality, and in particular is a compact operator.

In order to prove the existence of wave operators and asymptotic completeness, let us expand the difference of the resolvent to third power:

$$\begin{aligned}
&(H_\alpha+\lambda)^{-3}-(H_0+\lambda)^{-3}\\
&\quad=\mathcal{G}^\lambda\big(\Gamma^\lambda+\alpha\big)^{-1}\mathcal{G}^{\lambda*}\big(G^\lambda\big)^2\\
&\qquad+G^\lambda\mathcal{G}^\lambda\big(\Gamma^\lambda+\alpha\big)^{-1}\mathcal{G}^{\lambda*}G^\lambda+\big(G^\lambda\big)^2\mathcal{G}^\lambda\big(\Gamma^\lambda+\alpha\big)^{-1}\mathcal{G}^{\lambda*}\\
&\qquad+\big(\mathcal{G}^\lambda\big(\Gamma^\lambda+\alpha\big)^{-1}\mathcal{G}^{\lambda*}\big)^2G^\lambda+\mathcal{G}^\lambda\big(\Gamma^\lambda+\alpha\big)^{-1}\mathcal{G}^{\lambda*}G^\lambda\mathcal{G}^\lambda\big(\Gamma^\lambda+\alpha\big)^{-1}\mathcal{G}^{\lambda*}\\
&\qquad+G^\lambda\big(\mathcal{G}^\lambda\big(\Gamma^\lambda+\alpha\big)^{-1}\mathcal{G}^{\lambda*}\big)^2+\big(\mathcal{G}^\lambda\big(\Gamma^\lambda+\alpha\big)^{-1}\mathcal{G}^{\lambda*}\big)^3.
\end{aligned}\tag{3.38}$$

All the terms on the right-hand side of (3.38) are trace class operators. Indeed it is sufficient to use Lemma 3.6, with $k=2$, and Hölder inequality, as done when studying $\mathcal{G}^\lambda(\Gamma^\lambda+\alpha)^{-1}\mathcal{G}^{\lambda*}$. As an example let us consider the first term in the above expression: by Lemma 3.6, $\mathcal{G}^{\lambda*}(G^\lambda)^2\in\mathscr{B}_p(L^2(\mathbb{R}^4),L^2(\mathbb{R}^2))$ for $p>4/5$ (by the same argument applied to $\mathcal{G}^\lambda$) and thus it is a trace class operator. The claim then follows from boundedness of $\mathcal{G}^\lambda$ and $(\Gamma^\lambda+\alpha)^{-1}$ and Hölder inequality. □

4. The three-dimensional case

4.1. Preliminary results

As in the two-dimensional case, operator (1.1) can be rigorously defined by means of the theory of quadratic forms (see [7]).

Definition 4.1 *(Quadratic form F_α^ω). The quadratic form $(F_\alpha^\omega,\mathscr{D}(F_\alpha^\omega))$ is defined as follows*

$$\mathscr{D}\big(F_\alpha^\omega\big)=\big\{u\in L^2\big(\mathbb{R}^6\big)\;\big|\;\exists q\in\mathscr{D}\big(\Phi_{\alpha,\omega}^\lambda\big),\,\varphi^\lambda\equiv u-\mathcal{G}_\omega^\lambda q\in\mathscr{D}\big(F_0^\omega\big)\big\},\tag{4.1}$$

$$F_\alpha^\omega[u]\equiv\mathcal{F}^{\lambda,\omega}[u]+\Phi_{\alpha,\omega}^\lambda[u],\tag{4.2}$$

where $\lambda>0$ is a positive parameter and

$$\begin{aligned}\mathcal{F}^{\lambda,\omega}[u]\equiv\int_{\mathbb{R}^6}d\vec{x}\,d\vec{y}\,\bigg\{&\frac{1}{2}\big|\nabla_{\vec{x}}\varphi^\lambda\big|^2+\frac{1}{2}\big|\nabla_{\vec{y}}\varphi^\lambda\big|^2\\
&+\lambda\big|\varphi^\lambda\big|^2-\lambda|u|^2-\frac{3\omega}{2}|u|^2+\frac{\omega^2y^2}{2}\big|\varphi^\lambda\big|^2\bigg\},\end{aligned}\tag{4.3}$$

$$\mathscr{D}\big(\Phi_{\alpha,\omega}^\lambda\big)=\big\{q\;\big|\;q\in L^2\big(\mathbb{R}^3\big),\,\Phi_{\alpha,\omega}^\lambda[q]<+\infty\big\},$$

$$\Phi^{\lambda}_{\alpha,\omega}[q] \equiv \int_{\mathbb{R}^3} d\vec{x}\,\big(\alpha + a^{\lambda}_{\omega}(x)\big)\big|q(\vec{x})\big|^2 + \frac{1}{2}\int_{\mathbb{R}^6} d\vec{x}\,d\vec{x}'\,G^{\lambda}_{\omega}(\vec{x},\vec{x};\vec{x}',\vec{x}')\big|q(\vec{x}) - q(\vec{x}')\big|^2, \tag{4.4}$$

$$a^{\lambda}_{\omega}(x) \equiv \frac{\sqrt{\omega}}{(4\pi)^{\frac{3}{2}}}\left\{\frac{1}{2} + \int_0^1 d\nu\,\frac{1}{(1-\nu)^{\frac{3}{2}}}\left[1 - \frac{8\nu^{\lambda/\omega-1}(1-\nu)^{\frac{3}{2}}}{[(1+\nu^2)\ln\frac{1}{\nu} + 1 - \nu^2]^{\frac{3}{2}}} \times \exp\left(-\frac{(1-\nu^2)\ln\frac{1}{\nu} + 2(1-\nu)^2}{2[(1+\nu^2)\ln\frac{1}{\nu} + 1 - \nu^2]}\omega x^2\right)\right]\right\}. \tag{4.5}$$

The well-posedness of the definition above can be shown exactly as in the two-dimensional case. Moreover in the same way one can prove that the form is actually closed and bounded below (see [7] for the proofs).

Theorem 4.2 *(Closure of the form F^{ω}_{α}). The quadratic form $(F^{\omega}_{\alpha}, \mathcal{D}(F^{\omega}_{\alpha}))$ is closed and bounded below on the domain* (4.1).

Concerning the self-adjoint operators H^{ω}_{α} and $\Gamma^{\lambda}_{\omega}$ associated with the quadratic forms F^{ω}_{α} and $\Phi^{\lambda}_{\alpha,\omega}$ respectively, i.e.,

$$\langle q|\Gamma^{\lambda}_{\omega}|q\rangle \equiv \Phi^{\lambda}_{\alpha,\omega}[q] - \alpha\|q\|^2, \tag{4.6}$$

we have the following theorem.

Theorem 4.3 *(Operator H^{ω}_{α}). The domain and the action of H^{ω}_{α} are the following*

$$\mathcal{D}\big(H^{\omega}_{\alpha}\big) = \big\{u \in L^2\big(\mathbb{R}^6\big) \;\big|\; u = \varphi^{\lambda} + \mathcal{G}^{\lambda}_{\omega}q,\; \varphi^{\lambda} \in \mathcal{D}\big(H^{\omega}_0\big), q \in \mathcal{D}\big(\Gamma^{\lambda}_{\omega}\big), \big(\alpha + \Gamma^{\lambda}_{\omega}\big)q = \mathcal{P}\varphi\big\}, \tag{4.7}$$

$$\big(H^{\omega}_{\alpha} + \lambda\big)u = \big(H^{\omega}_0 + \lambda\big)\varphi^{\lambda}, \tag{4.8}$$

and the resolvent of H_{α} can be represented as

$$\big(H^{\omega}_{\alpha} + \lambda\big)^{-1} f = G^{\lambda}_{\omega} f + \mathcal{G}^{\lambda}_{\omega} q_f, \tag{4.9}$$

where, for any $f \in L^2(\mathbb{R}^6)$, q_f is a solution to

$$\big(\alpha + \Gamma^{\lambda}_{\omega}\big)q_f = \mathcal{P}G^{\lambda}_{\omega} f. \tag{4.10}$$

The operators (4.8) give rise to a one-parameter family of self-adjoint operators, which actually coincides with a family of self-adjoint extensions of the three-dimensional analogous of the operator $\tilde{H}_0$ introduced in the previous section. Note that the free Hamiltonian H^{ω}_0 belongs to the family and it is given by (4.8) for $\alpha = +\infty$, exactly as in the two-dimensional case.

4.2. Spectral analysis

Most of the results proved in the two-dimensional case apply also to the three-dimensional one and there are only minor differences in the proofs. Hence we shall often omit the details and refer to the two-dimensional case.

The spectral properties of H_α^ω are strictly related to spectral properties of the operator Γ_ω^λ, so we shall start by studying the latter.

Proposition 4.4 *(Spectral analysis of Γ_ω^λ). The domain $\mathcal{D}(\Phi_{\alpha,\omega}^\lambda)$ can be characterized in the following way*:

$$\mathcal{D}\big(\Phi_{\alpha,\omega}^\lambda\big) = \big\{q \in L^2\big(\mathbb{R}^3\big) \mid q \in \mathcal{H}^{1/2}\big(\mathbb{R}^3\big), \hat{q} \in \mathcal{H}^{1/2}\big(\mathbb{R}^3\big)\big\}. \tag{4.11}$$

On this domain $\Phi_{\alpha,\omega}^\lambda$ is closed and defines a self-adjoint operator Γ_ω^λ. For any $\lambda > 0$ the spectrum $\sigma(\Gamma_\omega^\lambda)$ is purely discrete, i.e., $\sigma(\Gamma_\omega^\lambda) = \sigma_{\mathrm{pp}}(\Gamma_\omega^\lambda)$.

Let $\gamma_n(\lambda)$, $n \in \mathbb{N}$, be the eigenvalues of Γ_1^λ arranged in an increasing order ($\lim_{n\to\infty} \gamma_n(\lambda) = +\infty$). *For every $n \in \mathbb{N}$, $\gamma_n(\lambda)$ is a non-decreasing function of λ. Furthermore* $\lim_{\lambda\to 0} \gamma_0(\lambda) = -\infty$ *and the other eigenvalues remain bounded below, i.e., for any $\lambda \geqslant 0$, there exists a finite constant c such that $\gamma_n(\lambda) \geqslant -c$, for any $n \in \mathbb{N}$, $n > 0$.*

Proof. Let us set again $\omega = 1$ and denote by Γ^λ the operator Γ_1^λ.

We can decompose $\Gamma^\lambda = a^\lambda + \Gamma_0^\lambda$, where Γ_0^λ is the self-adjoint operator associated with the positive quadratic form

$$\Phi_0^\lambda[q] \equiv \frac{1}{2} \int_{\mathbb{R}^2} d\vec{x}\, d\vec{x}'\, G^\lambda(\vec{x}, \vec{x}; \vec{x}', \vec{x}') \big|q(\vec{x}) - q(\vec{x}')\big|^2. \tag{4.12}$$

Since Φ_0 is positive and $a^\lambda(x)$ is an unbounded function, which is however bounded below for any $\lambda > 0$, Γ^λ is an unbounded operator which is bounded below. Notice that $a^\lambda(x)$ is a monotone increasing function of x and $a^\lambda(x) \simeq cx$ for $x \to \infty$; furthermore $a^\lambda(x) \geqslant a^\lambda(0)$, $a^\lambda(0)$ is a monotone increasing function of λ and $a^\lambda(0) \simeq c\sqrt{\lambda}$ for $\lambda \to \infty$. Hence by the lower bound (3.14) we have that for any $\alpha \in \mathbb{R}$, there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ the quadratic form Φ_α^λ is positive and bounded below; for such λ the operator Γ^λ is invertible.

The claim on the spectrum of Γ^λ can be proved in the same way as the two-dimensional case, then it is sufficient to prove that Φ_α^λ can be written in the following way:

$$\Phi_\alpha^\lambda[q] = \int_{\mathbb{R}^2} d\vec{k}\, \big(\alpha + \tilde{a}^\lambda(k)\big) \big|\hat{q}(\vec{k})\big|^2 + \frac{1}{2} \int_{\mathbb{R}^4} d\vec{k}\, d\vec{k}'\, \tilde{G}^\lambda(\vec{k}; \vec{k}') \big|\hat{q}(\vec{k}) - \hat{q}(\vec{k}')\big|^2, \tag{4.13}$$

where

$$\tilde{a}^\lambda(k) \equiv \frac{1}{4\pi} \left\{ C + \int_0^1 d\nu\, \frac{1}{1-\nu} \left[1 - \frac{4\nu^{\lambda-1}(1-\nu)}{(1+\nu^2)\ln\frac{1}{\nu} + 1 - \nu^2} \right.\right.$$
$$\left.\left. \times \exp\left(-\frac{(1-\nu^2)\ln\frac{1}{\nu}}{2[(1+\nu^2)\ln\frac{1}{\nu} + 1 - \nu^2]} k^2 \right) \right] \right\}, \tag{4.14}$$

$$\tilde{G}^\lambda(\vec{k};\vec{k}') \equiv \frac{1}{2\pi^2}\int_0^1 d\nu\, \frac{\nu^{\lambda-1}}{(1-\nu^2)\ln\frac{1}{\nu}+2(1-\nu)^2} \times \exp\left\{-\frac{[(1+\nu^2)\ln\frac{1}{\nu}+1-\nu^2](k^2+k'^2)}{2[(1-\nu^2)\ln\frac{1}{\nu}+2(1-\nu)^2]} - \frac{[1-\nu^2+2\nu\ln\frac{1}{\nu}]\vec{k}\cdot\vec{k}'}{(1-\nu^2)\ln\frac{1}{\nu}+2(1-\nu)^2}\right\}. \tag{4.15}$$

The function $\tilde{a}^\lambda(k)$ has the same asymptotic behavior for $k\to\infty$ as $a^\lambda(x)$, namely $\hat{a}^\lambda(k)\simeq ck$ and by applying Rellich's criterion, the spectrum of Γ^λ is pure point.

Notice that the following bound holds:

$$\Phi_0^\lambda[q] \leqslant c\|q\|^2_{\mathcal{H}^{1/2}(\mathbb{R}^3)}. \tag{4.16}$$

Indeed, using the inequality in (3.13),

$$G^\lambda(\vec{x},\vec{x};\vec{x}',\vec{x}') \leqslant c\int_0^1 d\nu\, \frac{\nu^{\lambda-1}}{((1-\nu^2)\ln\frac{1}{\nu})^{3/2}}\exp\left\{-\frac{(\vec{x}-\vec{x}')^2}{2\ln\frac{1}{\nu}} - \frac{\nu(\vec{x}-\vec{x}')^2}{1-\nu^2}\right\},$$

and taking the Fourier transform, we have

$$\Phi_0^\lambda[q] \leqslant c\int_{\mathbb{R}} d\vec{k}\int_0^1 d\nu\, \frac{\nu^{\lambda-1}}{(2\nu\ln\frac{1}{\nu}+1-\nu^2)^{3/2}} \times \left\{1-\exp\left[-\frac{(1-\nu^2)\ln\frac{1}{\nu}}{2(1-\nu^2+2\nu\ln\frac{1}{\nu})}k^2\right]\right\}\left|\hat{q}(k)\right|^2. \tag{4.17}$$

Since

$$0 \leqslant \int_0^1 d\nu\, \frac{\nu^{\lambda-1}}{(2\nu\ln\frac{1}{\nu}+1-\nu^2)^{3/2}}\left\{1-\exp\left[-\frac{2(1-\nu^2)\ln\frac{1}{\nu}}{4(2\nu\ln\frac{1}{\nu}+1-\nu^2)}k^2\right]\right\} \leqslant c\langle k\rangle,$$

(4.16) is proved.

Therefore, taking into account the behavior of a^λ, (4.16) implies that there exists $\lambda_0>0$ such that

$$\Gamma^\lambda \leqslant c\big(\langle x\rangle^{1/2}+\langle p\rangle^{1/2}+\lambda^{1/2}\big) \tag{4.18}$$

holds for $\lambda>\lambda_0$. Also in the three-dimensional case the lower bound (3.24) holds as well, but let us stress that a^λ and $\tilde{a}^\lambda$ have a different behavior; in particular in the three-dimensional case (4.11) holds true, due to (4.18).

Monotonicity of the eigenvalues and unboundedness from below of $\gamma_0(\lambda)$, as $\lambda \to 0$, can be shown exactly as in the two-dimensional case. Note that one has to evaluate the form on the ground state of the three-dimensional harmonic oscillator.

Furthermore we have the lower bound,

$$\langle q^\perp | \Gamma^\lambda | q^\perp \rangle \geqslant \frac{1}{(4\pi)^{\frac{3}{2}}} \left[\frac{1}{2} + \int_0^{\frac{1}{e}} d\nu \, \frac{1}{(1-\nu)^{\frac{3}{2}}} \right] \| q^\perp \|^2 - \int_{\mathbb{R}^6} d\vec{x}\, d\vec{x}'\, G^\lambda_{1/e}(\vec{x}, \vec{x}; \vec{x}', \vec{x}') \big(q^\perp(\vec{x})\big)^* q^\perp(\vec{x}'),$$

but, acting as in the proof of (3.26), we get

$$\int_{\mathbb{R}^6} d\vec{x}\, d\vec{x}'\, G^\lambda_{1/e}(\vec{x}, \vec{x}; \vec{x}', \vec{x}') \big(q^\perp(\vec{x})\big)^* q^\perp(\vec{x}') \leqslant \frac{1}{(2\pi)^{\frac{3}{2}}} \big[\langle q^\perp | \big(H^1_{\mathrm{osc}} + \lambda - 3/2\big)^{-1} | q^\perp \rangle + 2 \| q^\perp \|^2 \big],$$

so that, if $q^\perp$ is a normalized function orthogonal to the ground state of the harmonic oscillator, $\langle q^\perp | \Gamma^\lambda | q^\perp \rangle \geqslant -c$, for some finite constant c. □

The discrete spectrum of H^ω_α can now be fully characterized.

Theorem 4.5 *(Negative spectrum of H^ω_α). For any $\alpha \in \mathbb{R}$, the discrete spectrum $\sigma_{\mathrm{pp}}(H^\omega_\alpha)$ of H^ω_α is not empty and it contains a number $N_\omega(\alpha)$ of negative eigenvalues $-E_0(\alpha, \omega) \leqslant -E_1(\alpha, \omega) \leqslant \cdots \leqslant 0$ satisfying the scaling*

$$E_n(\alpha, \omega) = \omega E_n(\alpha/\sqrt{\omega}, 1). \tag{4.19}$$

The corresponding eigenvectors are given by $u_n = \mathcal{G}^{E_n}_\omega q_n$, where q_n is a solution to the homogeneous equation $\alpha q_n + \Gamma^{E_n}_\omega q_n = 0$.

Moreover there exists $\alpha_0 \in \mathbb{R}$ such that, if $\alpha > \alpha_0$, $N_\omega(\alpha) = 1$ and, for fixed ω and $\alpha \to -\infty$, $N_\omega(\alpha) \simeq c|\alpha|^6$.

The ground state energy has the following asymptotic behavior for fixed ω: $E_0 \simeq c\alpha^{-1}$ for $\alpha \to +\infty$ and $E_0 \simeq c\alpha^2$ for $\alpha \to -\infty$.

Proof. See the proof of Theorem 3.5; notice that in the argument used to estimate the asymptotics of $N_\omega(\alpha)$, the spectral distribution of the square root of the three-dimensional harmonic oscillator is involved. □

An interesting consequence of the above theorem is the existence of a bound state for any α and $0 < \omega < \infty$, in particular even if $\alpha > 0$ and there is no bound states for the "reduced" system (we shall come back to this question in the concluding comments).

The asymptotics for $\omega \to 0$ and $\alpha > 0$ is exactly as in the one- and two-dimensional case, whereas the behavior for $\omega \to 0$ and $\alpha < 0$ proves to be much more complicated, due to the ground state asymptotics (see theorem above). If $\omega \to \infty$ we expect that the asymptotics depend on a crucial way on the sign of α, since at least one bound state should survive if $\alpha < 0$, whereas, if $\alpha > 0$, all bound states should disappear in the limit.

Now we shall give a partial characterization of the positive spectrum of (1.1), but we first state a result analogous[6] to Lemma 3.6.

Lemma 4.6. *Define the operator $T^k_\omega \equiv \mathcal{G}^\lambda_\omega (G^\lambda_\omega)^k \mathcal{G}^{\lambda *}_\omega$, for any $k \in \mathbb{N}$. Then, if $\lambda > k$, $T^k_\omega \in \mathscr{B}_p(L^2(\mathbb{R}^3), L^2(\mathbb{R}^3))$, for any $p > 3(k + 1/2)^{-1}$.*

Theorem 4.7 *(Positive spectrum of H^ω_α). The essential spectrum of H^ω_α is equal to* $[0, +\infty)$ *and the wave operators $\Omega_\pm(H^\omega_\alpha, H^\omega_0)$ exist and are complete.*

Proof. We shall omit the dependence on ω for brevity. It is sufficient to prove that $(H_\alpha + \lambda)^{-1} - (H_0 + \lambda)^{-1}$ is a compact operator and that $[(H_\alpha + \lambda)^{-1}]^4 - [(H_0 + \lambda)^{-1}]^4$ is trace class for some $\lambda > 0$, then the thesis follows from Weyl's theorem (see [22, Theorem XIII.14] and [21, Corollary 3 of Theorem XI.11]).

We shall fix λ sufficiently large such that $(\Gamma^\lambda + \alpha)^{-1}$ exists. Boundedness of $(\Gamma^\lambda + \alpha)^{-1}$, Hölder inequality and the fact that $\mathcal{G}^{\lambda *}\mathcal{G}^\lambda \in \mathscr{B}_p(L^2(\mathbb{R}^6), L^2(\mathbb{R}^6))$, $p > 6$, because of Lemma 4.6, imply compactness of $(H_\alpha + \lambda)^{-1} - (H_0 + \lambda)^{-1}$, as in the two-dimensional case. Besides one can show that $\mathcal{G}^\lambda$ belongs to $\mathscr{B}_p(L^2(\mathbb{R}^6), L^2(\mathbb{R}^3))$, $p > 12$, and $\mathcal{G}^{\lambda *} \in \mathscr{B}_p(L^2(\mathbb{R}^3), L^2(\mathbb{R}^6))$ for the same p. Finally the tedious but straightforward calculation of $[(H_\alpha + \lambda)^{-1}]^4 - [(H_0 + \lambda)^{-1}]^4$ and the application of Lemma 4.6 with $k = 3$ to each term of the expansion give the result. □

5. Conclusions and perspectives

We have studied a quantum system composed of a test particle and a harmonic oscillator interacting through a zero-range force. We have given a rigorous meaning to the Hamiltonian H^ω_α of the system, described the properties of its spectrum and established asymptotic completeness for the scattering operators $\Omega_\pm(H^\omega_\alpha, H^\omega_0)$, where H^ω_0 is the Hamiltonian of the system without the zero-range force.

The negative part of the spectrum of H^ω_α for $\omega > 0$ is discrete and we have given estimates of the number of bound states. There is a peculiar feature of this part of the spectrum: in the three-dimensional setting, in the case of a fixed center, i.e., $\omega = \infty$, when the parameter α is negative, there is exactly one bound state, while, in the case $\alpha > 0$, the spectrum is absolutely continuous. In our case, if $\alpha > 0$, there is always a bound state and, if $\alpha < 0$, the number of bound states increases as the strength of the harmonic force goes to zero.

We might interpret this feature as due to the fact that bound states of the harmonic oscillator provide a mechanism through which the test particle is bound, even if the interaction due to the zero-range force is "repulsive."

We have privileged explicit expressions because we regard our analysis as preliminary to a detailed treatment of the case in which many oscillators are present. This model is widely used in the physical literature, for instance in kinetic theory, under the name of Rayleigh gas and in that

[6] The proof follows exactly the proof of Lemma 3.6 and is omitted for the sake of brevity.

context one considers as relevant the spectral and scattering properties of a particle interacting with a background of scatterers, in particular detailed estimates on the scattering cross sections. We plan to extend our analysis to this more general setting and obtain rather detailed information through a multichannel scattering approach, the channels being labeled by the bound states of the harmonic oscillators.

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