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Fourier estimation method applied to forward interest rates

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Abstract

Principal component analysis (PCA) is a general method to analyse the factors of the term structure of interest rates. There are usually two or three factors. However, it is shown by Liu that when we apply PCA to forward rates, not spot rates, we need more factors to explain 95% of variability. In order to verify the robustness of this result, we introduce another method based on Fourier series, which is proposed by Malliavin and Mancino. The results reconfirm the observation of Liu with different data sets. In particular, the Fourier series method gives us similar results to PCA.

Keywords term structure of interest rates, principal component analysis, Fourier series method

Research Activity Group Mathematical Finance

1. Introduction

The use of principal component analysis (PCA) on the term structure of interest rates is a common method to reduce the dimensionality of the vector space of the original variables. It is a well known result that three factors are sufficient to explain most of the spot rate variability (see e.g. [1]). Nevertheless, the empirical results of [2] show that the number of factors for the forward rates is much greater than generally believed. Briefly speaking, for the 40 maturities of real market data in [2], we require more than 20 factors in order to explain at least 95% of variability.

In order to verify the validity of PCA applied to the term structure of interest rates, we introduce another method which has been proposed by Malliavin and Mancino [3] and has been developed by Barucci and Rèno [4], Malliavin and Thalmaier [5], and Malliavin and Mancino [6]. They have presented a method to compute the volatility based on Fourier series. This method is nonparametric and can be applied to high-frequency financial data. Thus, we apply this method to the term structure of forward rates.

In this paper, we will compute the volatility matrix of the forward rates by using the Fourier estimation methodology, and then compare this with the results of applying PCA. The organization of the present paper is as follows:

First, we summarize the Fourier series methodology presented by Malliavin and Mancino [3] in Section 2. Then, the method of estimating the volatility by using the Fourier series method will be given in Section 3. Next we perform a numerical study and give the results in Section 4. Finally, we summarize our findings in Section 5.

2. Fourier series method

We briefly recall the Fourier series method introduced by Malliavin and Mancino [3].

Let X be a d-dimensional stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ given by

$$dX_i(t) = \mu_i(t)dt + B_{i,j}(t)dW^j(t), \quad 0 \le t \le T,$$

where W is a d_1-dimensional standard Brownian motion, μ_i is a d-dimensional drift process and $B_{i,j}$ is a $\mathbf{R}^{d\times d_1}$-valued càdlàg volatility process, both of which are adapted to $(\mathcal{F}_t)$.

The volatility matrix $\Sigma = (\Sigma_{i,j})_{1\le i,j\le d}$ of process X is an adapted process defined by

$$\Sigma_{i,j}(t) = \sum_{k=1}^{d_1} B_{i,k}(t)B_{j,k}(t), \quad 0 \le t \le T.$$

Suppose that $T = 2\pi$. Now we shall show how the Fourier series method reconstructs $\Sigma(t)$ for all $t \in (0, 2\pi)$. Let us denote the (random) Fourier coefficients of "dX_j", $j = 1, \dots, d$ by

$$a_k(dX_j) := \frac{1}{\pi}\int_{(0,2\pi)} \cos(kt)dX_j(t),$$

$$b_k(dX_j) := \frac{1}{\pi}\int_{(0,2\pi)} \sin(kt)dX_j(t).$$

The Fourier coefficients of each cross volatility $\Sigma_{i,j}$, $1 \le i, j \le d$, are defined by

$$a_k(\Sigma_{i,j}) = \frac{1}{\pi}\int_{(0,2\pi)} \cos(kt)\Sigma_{i,j}(t)dt,$$

$$b_k(\Sigma_{i,j}) = \frac{1}{\pi}\int_{(0,2\pi)} \sin(kt)\Sigma_{i,j}(t)dt.$$

It follows from the Fourier-Féjer inversion formula that we can reconstruct Σ from its Fourier coefficients by

$$\Sigma_{i,j}(t) = \lim_{N\to\infty} \sum_{k=0}^{N} \left(1 - \frac{k}{N}\right) \big(a_k(\Sigma_{i,j})\cos(kt) + b_k(\Sigma_{i,j})\sin(kt)\big).$$

Based on the observations of X at times $t_i = 2\pi i/n$, $i = 0, \ldots, n$, and by fixing some positive integer N, we can approximate Σ as follows:

(1) Fourier coefficients $a_k(dX_j)$, $b_k(dX_j)$, $k = 0, \ldots, 2N$, are approximated by

$$\hat{a}_k(dX_j) = \frac{1}{\pi} \sum_{i=1}^{n} \big(\cos(kt_{i-1}) - \cos(kt_i)\big) X_j(t_{i-1}) + \frac{1}{\pi}\big(X_j(t_n) - X_j(t_0)\big),$$

$$\hat{b}_k(dX_j) = \frac{1}{\pi} \sum_{i=1}^{n} \big(\sin(kt_{i-1}) - \sin(kt_i)\big) X_j(t_{i-1}).$$

(2) Fourier coefficients of each cross volatility $\Sigma_{i,j}$, $1 \le i, j \le d$, are approximated by

$$\hat{a}_0(\Sigma_{i,j}) = \frac{\pi}{2(N+1-n_0)} \sum_{s=n_0}^{N} \big(\hat{a}_s(dX_i)\hat{a}_s(dX_j) + \hat{b}_s(dX_i)\hat{b}_s(dX_j)\big),$$

$$\hat{a}_k(\Sigma_{i,j}) = \frac{\pi}{N+1-n_0} \sum_{s=n_0}^{N} \big(\hat{a}_s(dX_i)\hat{a}_{s+k}(dX_j) + \hat{a}_s(dX_j)\hat{a}_{s+k}(dX_i)\big),$$

for $k = 1, 2, \ldots, N$, and

$$\hat{b}_k(\Sigma_{i,j}) = \frac{\pi}{N+1-n_0} \sum_{s=n_0}^{N} \big(\hat{a}_s(dX_i)\hat{b}_{s+k}(dX_j) + \hat{a}_s(dX_j)\hat{b}_{s+k}(dX_i)\big),$$

for $k = 0, 1, \ldots, N$.

(3) The volatilities $\Sigma_{i,j}(t)$ are approximated by

$$\hat{\Sigma}^{N,n}_{i,j}(t) = \sum_{k=0}^{N} \left(1 - \frac{k}{N}\right) \big(\hat{a}_k(\Sigma_{i,j})\cos(kt) + \hat{b}_k(\Sigma_{i,j})\sin(kt)\big). \tag{1}$$

3. Method of estimating the volatility

Let $r_t(T)$ denote the spot rate during the period $[t, T]$; i.e. the rate of the spot borrowing until the maturity T. The forward rate at t during the period $[T_i, T_{i+1}]$ is given by

$$F_t(T_i, T_{i+1}) = \frac{(T_{i+1} - t) r_t(T_{i+1}) - (T_i - t) r_t(T_i)}{T_{i+1} - T_i} \tag{2}$$
$$=: F_i(t), \quad i = 1, \ldots, d.$$

We will write $\mathbf{F}(t) = (F_1(t), \ldots, F_d(t))$.

Suppose that sample data of the forward rate curve $\mathbf{F}(t)$, $t = t_0, t_1, \ldots, t_N$ are given. From the data, we can calculate $\Delta\mathbf{F}(1), \ldots, \Delta\mathbf{F}(N)$, where for each $l = 1, \ldots, N$, $\Delta\mathbf{F}(l) := \mathbf{F}(t_l) - \mathbf{F}(t_{l-1})$ is a d-dimensional vector $(\Delta F_1(l), \ldots, \Delta F_d(l))$.

Remark 1 *Assume that the forward rates process $\mathbf{F}(t) = (F_1(t), \ldots, F_d(t))$ are Brownian semi-martingales given by*

$$dF_i(t) = \sum_{j=1}^{r} \mu_i(t)dt + \sigma_{i,j}(t) dW^j(t), \quad i = 1, \ldots, d.$$

where $W = (W^1, \ldots, W^r)$ is an r-dimensional Brownian motion and $\sigma_{i,j}$ and μ_i are adapted processes. Then we define the time-dependent volatility matrix by

$$\Sigma_{i,j}(t) = \sum_{l=1}^{r} \sigma_{i,l}(t)\sigma_{j,l}(t).$$

According to [6, Theorem 3.4], under a suitable condition, the following convergence holds in probability

$$\lim_{n,N\to\infty} \sup_t \left|\hat{\Sigma}^{N,n}_{i,j}(t) - \Sigma_{i,j}(t)\right| = 0.$$

4. Numerical study

We use time series of American zero rates and Japanese zero rates from May 2005 to May 2008 and from June 2005 to June 2008 respectively, where for each, we have a total of 777 and 723 daily observations. The maturities of the American zero rates are

$$T_1 = 2009/5/15,$$
$$T_2 = 2010/5/15,$$
$$T_3 = 2011/5/15,$$
$$\vdots$$
$$T_{13} = 2021/5/15,$$

and the maturities of the Japanese zero rates are

$$T_1 = 2009/6/20,$$
$$T_2 = 2010/6/20,$$
$$T_3 = 2011/6/20,$$
$$\vdots$$
$$T_{13} = 2021/6/20.$$

We use this data to calculate the forward rates by formula (2). Here we have $H_A = 777$ observations for American data and $H_J = 723$ observations for Japanese data. We follow the steps introduced previously to approximate the volatility matrix using the Fourier series method. We calculate the Fourier coefficients by $N_1 = H/2$ points and estimate the Fourier coefficients of cross volatility by $N_2 = H/4$. We smooth the Féjer kernel in (1) by replacing $(1 - k/N)$ with $\sin^2(\delta k)/(\delta k)^2$ for some appropriate parameter $\delta > 0$. In this study, we use $\delta_A = 2\pi/259$ and $\delta_J = 2\pi/241$ for American data and Japanese data, respectively. We use both the Fourier series method and PCA to analyze the interest rates and the results are as follows:

4.1 Analysis of spot rates

In order to compare our results with general beliefs, in this section we perform empirical studies on both meth-

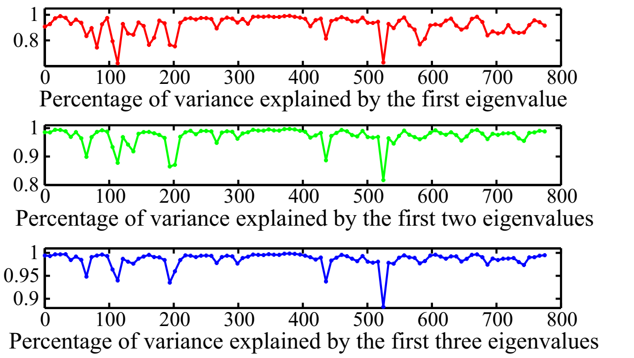

Fig. 1. Percentage of variance explained by the first three eigenvalues as a function of time for American spot rate.

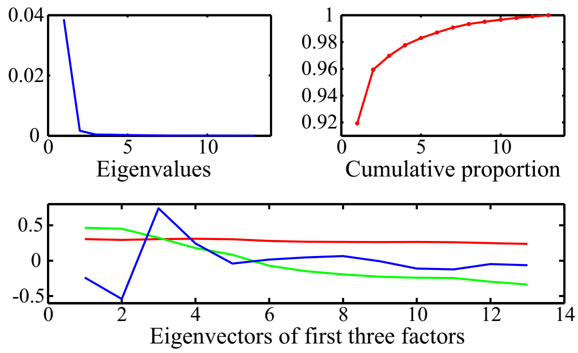

Fig. 2. Estimated eigenvalues using PCA, the proportion of contributions of principle component and eigenvectors of first three factors for American spot rate.

ods to analyse the spot rate.

4.1.1 American interest rate data

We analyse the Fourier series method and PCA each, to see what will happen to the American spot rates. Fig. 1 shows the result of the Fourier series method, that is, the percentage described by the first three eigenvalues as a function of time during the trading days. Fig. 2 shows the result of PCA. In the Fourier series method, the first eigenvalue can almost describe spectrum more than 98% except for a few points. It is similar to the result of PCA, where one eigenvalue can describe 92% and two eigenvalues can describe over 95% of variability.

4.1.2 Japanese interest rate data

The results for Japanese spot rates are organized similarly. Fig. 3 shows the result of the Fourier series method and Fig. 4 shows the result of PCA. Both of the results are similar to those in the previous section. The first eigenvalue excluding a few points can describe spectrum more than 98% in the Fourier series method, and one eigenvalue can describe around 94% of the variability in PCA.

4.2 Analysis of forward rates

Now we are going to explain the empirical studies of both methods to analyse the forward rate.

4.2.1 American interest rate data

First we analyse the Fourier series method. Fig. 5 shows the percentage described by the first three eigenvalues. As you see, the first eigenvalue can only describe spectrum of the volatility matrix from 30% to 60%. Even

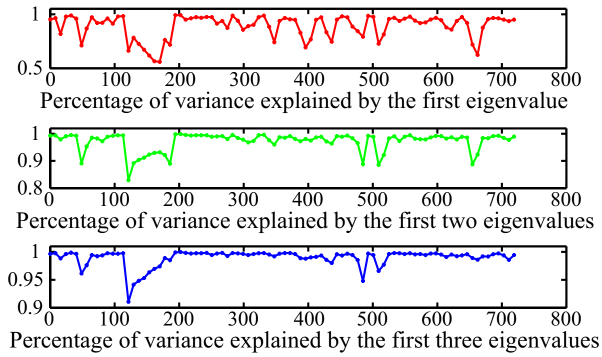

Fig. 3. Percentage of variance explained by the first three eigenvalues as a function of time for Japanese spot rate.

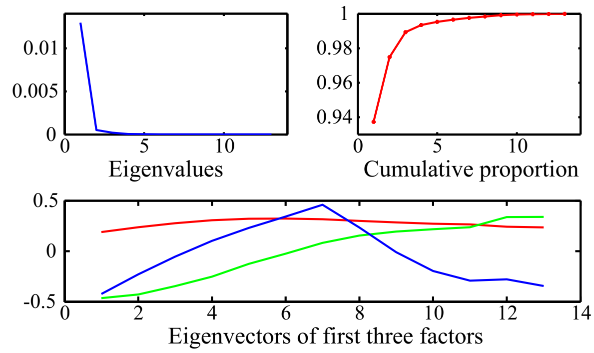

Fig. 4. Estimated eigenvalues using PCA, the proportion of contributions of principle component and eigenvectors of first three factors for Japanese spot rate.

if we consider three eigenvalues, it is still not significant as three eigenvalues only describe from 70% to 90%. This result is similar to [2]. We need more factors to explain most of the forward rate variability. We note that if we want to describe spectrum more than 95%, we need at least the first six eigenvalues. We also apply PCA to the term structure of American forward rates and the result is shown in Fig. 6. Three eigenvalues can only describe 50%.

4.2.2 Japanese interest rate data

We use the same arrangement as in the previous section. Fig. 7 shows the percentage described by the first three eigenvalues. The first eigenvalue can only describe spectrum of the volatility matrix from 30% to 80%. Furthermore, three eigenvalues only describe from 70% to 90%. If we want to describe spectrum of more than 95%, we need at least the first six eigenvalues. The result of applying PCA to the term structure of Japanese forward rates is shown in Fig. 8 and three eigenvalues describe 70%.

Remark 2 *There is no significant difference between the Fourier series method and PCA applied to the Japanese forward rate, while the two methods do not give a very close result for the American forward rate.*

5. Conclusions

In this paper we applied two methods to the term structure of interest rates. The numerical studies show that the results of [2] are reconfirmed with different data sets and different methods. In short, if we want to ex-

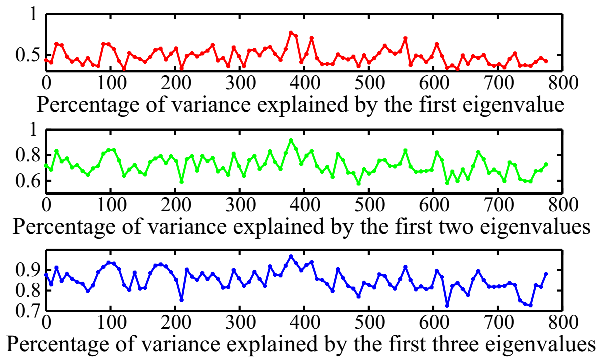

Fig. 5. Percentage of variance explained by the first three eigenvalues as a function of time for American forward rate.

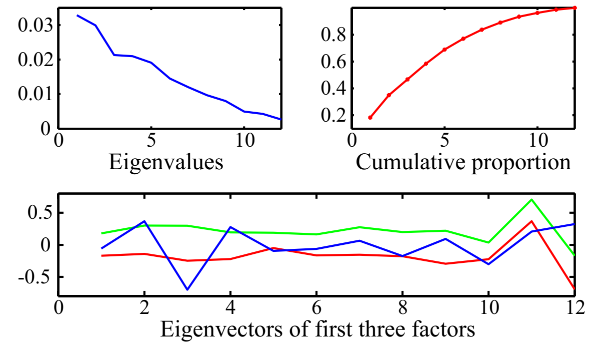

Fig. 6. Estimated eigenvalues using PCA, the proportion of contributions of principle component and eigenvectors of first three factors for American forward rate.

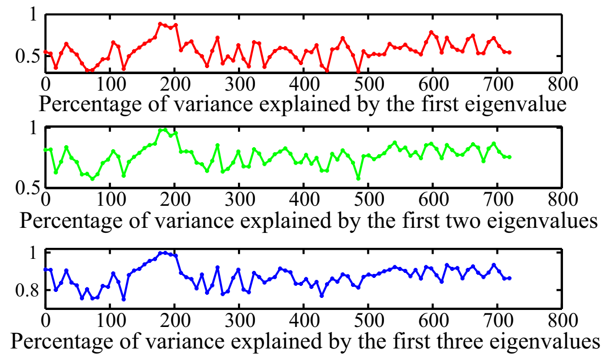

Fig. 7. Percentage of variance explained by the first three eigenvalues as a function of time for Japanese forward rate.

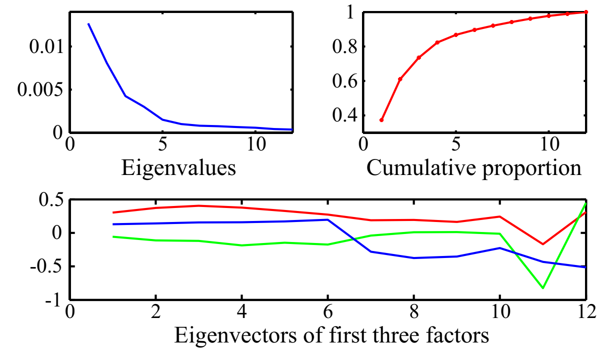

Fig. 8. Estimated eigenvalues using PCA, the proportion of contributions of principle component and eigenvectors of first three factors for Japanese forward rate.

plain up to 95% of forward rate variability, both methods seems to require strictly more than three eigenvalues, while a few eigenvalues are sufficient for spot rates.

In future work, we plan to generate a different estimator by using the Fourier series method to verify if this result is robust or not.

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