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A generating function for the cubic interactions of higher spin fields

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Abstract

We present off-shell generating function for all cubic interactions of Higher Spin gauge fields constructed in [1]. It is a generalization of on-shell generating function proposed in [2]. It's written in a very compact way and turns out to have remarkable structure.

1 Introduction and notations

Higher Spin gauge field theory is one of the most important and puzzling problems in the modern quantum field theory. It is a subject of many articles and always stay in the center of attention during last thirty years. Despite the fact that consistent equations of motion for Higher Spin gauge fields are known over twenty years [3], the question of existence of Lagrangian for interacting Higher Spin gauge fields is still open. The subject of special interest is a minimal selfinteraction of even spin gauge fields, where one can naively expect existence of Einstein-Hilbert type nonlinear action for any single even spin gauge field. Although there are known restrictions on Higher Spin theories in flat space-time, recent development [1] has shown that *there is a local higher derivative cubic lagrangian for gauge fields with any higher spins*. This shifts the no-go theorems to the quartic level on fields in interaction Lagrangian, where one can expect the final battle for existence of local (or nonlocal) Lagrangian for interacting HS gauge field theory in flat space.

The free Lagrangian for Higher Spin gauge fields both in flat space and in constantly curved backgrounds (dS and AdS) are known over thirty years [4]. In contrast to free theory, attempts to construct Lagrangian for interacting theory haven't been successful yet beyond the cubic vertexes. In this letter we are going to discuss only trilinear interactions of Higher Spin gauge fields.

Our recent results [1], [5]-[8] on Higher Spin gauge field cubic interactions in flat space, which certainly reproduce the flat limit of the Fradkin-Vasiliev vertex for higher spin coupling to gravity [9], show that all interactions of higher spin gauge fields with any spins s_1, s_2, s_3 both in flat space and in dS or AdS are unique*. This was already proven for some low spin cases of both Fradkin-Vasiliev vertex $2, s, s$ and nonabelian vertex $1, s, s$ in [10].

The first important step towards cubic interactions in Higher Spin gauge field theory was done in 1984 by Berends, Burgers and van Dam [11]. They constructed cubic selfinteraction Lagrangian for spin three gauge fields and proved impossibility of extension to higher orders. Their arguments are based on gauge algebra, which does not close for single spin three nonabelian field. The authors give an optimistic hope that it will be possible to extend this Lagrangian to higher orders if one take into account corrections from interactions with gauge fields with spins higher than three. Recent discussion on this subject was done by Bekaert, Boulanger and Leclerq [12]. They show impossibility to close this (spin 3) algebra taking into account corrections from interactions of other fields with spins higher (or lower) than three. It is not yet known whether there is or there is no nonlinear selfinteraction for spin three gauge field in the background space with nonzero cosmological constant (pure spin three theory).

Another important step was done by Fradkin and Vasiliev in [9] where coupling of Higher Spin gauge fields to linearized gravity was constructed in the constantly curved background. The interesting property of this Lagrangian is it's non-

*The cubic interaction Lagrangian is unique due to partial integration and field redefinition.

analyticity in cosmological constant, therefore not admitting flat space limit. However it was shown already in [10] that after rescaling of Higher Spin gauge fields one can observe flat limit for Fradkin-Vasiliev interactions. In our approach spin s gauge field has scaling dimension $[lenght]^{s-2}$, therefore Fradkin-Vasiliev vertex has flat limit with $2s-2$ derivatives (minimal possible number) in the $2-s-s$ interaction which has the same scaling dimension as Einstein-Hilbert Lagrangian terms. As it was shown by Metsaev in [13] using light cone gauge approach, there are three different couplings to linearized gravity with different numbers of derivatives for any higher spin s field, and in general $min\{s_1, s_2, s_3\}+1$ different possibilities with different numbers of derivatives for $s_1 - s_2 - s_3$ interaction. All these interactions were derived in a covariant off-shell formulation in [1].

In the recent paper [2] by Sagnotti and Taronna the authors proposed on-shell generating function for the general interaction presented in [1] from string theory consideration. In this article we are going to present off-shell extension of that generating function which surprisingly enhanced with beautiful structure of Grassman variables, the string origin of which is not clear yet.

For some important results on higher spin cubic interactions see [14]-[18] and references therein. For recent reviews see [19].

In this paper we consider a Higher Spin gauge field theory in Fronsdal formulation. The spin s field is rank s symmetric, double traceless tensor and we consider here only one copy of any spin, so these interactions are for even spin gauge field (self)interactions only.

To continue with this subject we introduce here briefly our standard notations coming from our previous papers about HSF (see for example [20]). As usual we utilize instead of symmetric tensors such as $h^{(s)}_{\mu_1\mu_2\dots\mu_s}(z)$ the homogeneous polynomials in the vector a^μ of degree s at the base point z

$$h^{(s)}(z;a) = \sum_{\mu_i}(\prod_{i=1}^{s} a^{\mu_i})h^{(s)}_{\mu_1\mu_2\dots\mu_s}(z). \tag{1.1}$$

Then we can write the symmetrized gradient, trace and divergence †

$$Grad : h^{(s)}(z;a) \Rightarrow Gradh^{(s+1)}(z;a) = (a\nabla)h^{(s)}(z;a), \tag{1.2}$$

$$Tr : h^{(s)}(z;a) \Rightarrow Trh^{(s-2)}(z;a) = \frac{1}{s(s-1)}\Box_a h^{(s)}(z;a), \tag{1.3}$$

$$Div : h^{(s)}(z;a) \Rightarrow Divh^{(s-1)}(z;a) = \frac{1}{s}(\nabla\partial_a)h^{(s)}(z;a). \tag{1.4}$$

†To distinguish easily between "a" and "z" spaces we introduce the notation ∇_μ for space-time derivatives $\frac{\partial}{\partial z^\mu}$.

Here we only present Fronsdal's Lagrangian in terms of these conventions[‡]:

$$\mathcal{L}_0(h^{(s)}(a)) = -\frac{1}{2}h^{(s)}(a) *_a \mathcal{F}^{(s)}(a) + \frac{1}{8s(s-1)}\Box_a h^{(s)}(a) *_a \Box_a \mathcal{F}^{(s)}(a), \tag{1.5}$$

where $\mathcal{F}^{(s)}(z;a)$ is the Fronsdal tensor

$$\mathcal{F}^{(s)}(z;a) = \Box h^{(s)}(z;a) - s(a\nabla)D^{(s-1)}(z;a), \tag{1.6}$$

and $D^{(s-1)}(z;a)$ is the deDonder tensor or traceless divergence of the higher spin gauge field

$$D^{(s-1)}(z;a) = Divh^{(s-1)}(z;a) - \frac{s-1}{2}(a\nabla)Trh^{(s-2)}(z;a), \tag{1.7}$$

$$\Box_a D^{(s-1)}(z;a) = 0. \tag{1.8}$$

The initial gauge variation of order zeroth in the spin s field is

$$\delta_{(0)}h^{(s)}(z;a) = s(a\nabla)\epsilon^{(s-1)}(z;a), \tag{1.9}$$

with the traceless gauge parameter for the double traceless gauge field

$$\Box_a \epsilon^{(s-1)}(z;a) = 0, \tag{1.10}$$

$$\Box_a^2 h^{(s)}(z;a) = 0. \tag{1.11}$$

2 Free lagrangian for all higher spin gauge fields

We introduce generating function for HS gauge fields by

$$\Phi(z;a) = \sum_{s=0}^{\infty}\frac{1}{s!}h^{(s)}(z;a) \tag{2.12}$$

Where we assume that spin s field has scaling dimension $s-2$, the a_i vectors have dimension -1, therefore all terms in the generating function for higher spin gauge fields (2.12) have the same dimension -2.

0 order gauge transformation for this field reads as

$$\delta^0_\Lambda \Phi(z;a) = (a\nabla)\Lambda(z;a), \tag{2.13}$$

$$\delta^0_\Lambda D_a \Phi(z;a) = \Box\Lambda(z;a), \tag{2.14}$$

$$\delta^0_\Lambda \Box_a \Phi(z;a) = 2(\nabla\partial_a)\Lambda(z;a). \tag{2.15}$$

[‡]From now on we will presuppose integration everywhere where it is necessary (we work with a Lagrangian as with an action) and therefore we will neglect all d dimensional space-time total derivatives when making a partial integration.

where

$$\Lambda(z;a) = \sum_{s=1}^{\infty} \frac{1}{(s-1)!}\epsilon^{(s-1)}(z;a), \tag{2.16}$$

is the generating function of gauge parameters and is dimensionless[§].

Fronsdal constraint for gauge parameter reads as

$$\Box_a \Lambda(z;a) = 0, \tag{2.17}$$

For spin s field gauge variation we get as expected

$$\delta^0_\epsilon h^{(s)}(z;a) = s(a\nabla)\epsilon^{(s-1)}(z;a), \tag{2.18}$$

The second Fronsdal constraint for gauge field reads in this notations

$$\Box^2_a \Phi(z;a) = 0, \tag{2.19}$$

We introduced the "de Donder" operator

$$D_{a_i} = (\partial_{a_i}\nabla_i) - \frac{1}{2}(a_i\nabla_i)\Box_{a_i} \tag{2.20}$$

This operator is "linear" in ∂_{a_i}.

Here we write the quadratic lagrangian for free higher spin gauge fields in general form using generating function for HS fields (2.12). First we introduce Fronsdal operator

$$\mathcal{F}_{a_i} = \Box_i - (a_i\nabla_i)(\nabla_i\partial_{a_i}) + \frac{1}{2}(a_i\nabla_i)^2\Box_{a_i}, \tag{2.21}$$

or with the help of (2.20)

$$\mathcal{F}_{a_i} = \Box_i - (a_i\nabla_i)D_{a_i}. \tag{2.22}$$

The operator of equation of motion can be written in the form

$$\mathcal{G}_{a_i} = \mathcal{F}_{a_i} - \frac{a_i^2}{4}\Box_{a_i}\mathcal{F}_{a_i} \tag{2.23}$$

Now we can write free lagrangian for all gauge fields of any spin in a symmetric elegant form

$$\begin{aligned}\mathcal{L}^{free}(z) &= \frac{\kappa}{2}exp[\lambda^2\partial_{a_1}\partial_{a_2}]\int_{z_1z_2}\delta(z_1-z)\delta(z_2-z)\\ &\{(\nabla_1\nabla_2) - \lambda^2 D_{a_1}D_{a_2} - \frac{\lambda^4}{4}(\nabla_1\nabla_2)\Box_{a_1}\Box_{a_2}\}\Phi(z_1;a_1)\Phi(z_2;a_2)\mid_{a_1=a_2=0}\end{aligned} \tag{2.24}$$

[§]The gauge parameter for spin s field $\epsilon^{(s-1)}$ has scaling dimension $s-1$, therefore after contraction with $s-1$ a-s becomes dimensionless.

Where λ has scaling dimension -1, therefore λ^2 compensates the dimension of the operator in the exponent. We will see that all relative coupling constants of HS interactions will be expressed as powers of λ. The κ is a constant which makes action dimensionless (analogous to Einstein's constant and simply connected with the latter). It has scaling dimension $6-d$, where d is space-time dimension. For Einstein's constant κ_E we get

$$\kappa_E^{-2} = \kappa\lambda^4 \tag{2.25}$$

It is now obvious that in free lagrangian there is no mixing between gauge fields of different spin. It can be also written in such forms

$$\begin{aligned}\mathcal{L}^{free}(z) &= -\frac{1}{2}exp[\lambda^2\partial_{a_1}\partial_{a_2}]\int_{z_1}\delta(z_1-z)(\mathcal{G}_{a_1})\Phi(z_1;a_1)\Phi(z;a_2)\mid_{a_1=a_2=0}\\ &= -\frac{1}{2}exp[\lambda^2\partial_{a_1}\partial_{a_2}]\int_{z_2}\delta(z_2-z)(\mathcal{G}_{a_2})\Phi(z;a_1)\Phi(z_2;a_2)\mid_{a_1=a_2=0}\end{aligned} \tag{2.26}$$

This expressions reproduce Fronsdal lagrangians for all gauge fields with any spin.

3 Cubic Interactions

We are going to present very beautiful and compact form of all HS gauge field interactions derived in [1]. First we rewrite leading term of general trilinear interaction of higher spin gauge fields with any spins s_1, s_2, s_3 ¶

$$\begin{aligned}&\mathcal{L}^{leading}_{(1)}(h^{(s_1)}(z), h^{(s_2)}(z), h^{(s_3)}(z))\\ &= \sum_{\alpha+\beta+\gamma=n}\frac{1}{\alpha!\beta!\gamma!}\int_{z_1,z_2,z_3}\delta(z-z_1)\delta(z-z_2)\delta(z-z_3)\\ &\left[(\nabla_1\partial_c)^{s_3-n+\gamma}(\nabla_2\partial_a)^{s_1-n+\alpha}(\nabla_3\partial_b)^{s_2-n+\beta}(\partial_a\partial_b)^{\gamma}(\partial_b\partial_c)^{\alpha}(\partial_c\partial_a)^{\beta}\right]\\ &h^{(s_1)}(a;z_1)h^{(s_2)}(b;z_2)h^{(s_3)}(c;z_3),\end{aligned} \tag{3.27}$$

where the number of derivatives is

$$\Delta = s_1+s_2+s_3-2n, \tag{3.28}$$

$$0 \leq n \leq min(s_1, s_2, s_3) \tag{3.29}$$

as we see the minimal and maximal possible numbers of derivatives are

$$\Delta_{min} = s_1+s_2+s_3-2min(s_1,s_2,s_3), \tag{3.30}$$

$$\Delta_{max} = s_1+s_2+s_3. \tag{3.31}$$

This interactions trivialize only if we have two equal spin values and the third value is odd. The example is odd spin self-interaction. This we call $\ell - s - s$ case, where

¶ $\nabla_2\partial_a = \frac{\partial}{\partial a^{\mu}}\nabla_2^{\mu}$ and so on.

ℓ is odd. In that case we should have multiplet of spin s field, with at least two charges to couple to spin ℓ field. In the case of $\ell - \ell - \ell$ odd spin self interaction the number of possible charges in the multiplet should be at least 3. The case of Δ_{min} is important also because only in that case interaction (3.27) has the same dimension as the lowest spin field free Lagrangian.

The same lagrangian can be written in a following way (due to constant normalization factor 2^{Δ})

$$\begin{aligned} &\mathcal{L}^{leading}_{(1)}(h^{(s_1)}(z), h^{(s_2)}(z), h^{(s_3)}(z)) \\ &= \sum_{\alpha+\beta+\gamma=n} \frac{1}{\alpha!\beta!\gamma!} \int_{z_1,z_2,z_3} \delta(z-z_1)\delta(z-z_2)\delta(z-z_3) \\ &\big[(\nabla_{12}\partial_c)^{s_3-n+\gamma}(\nabla_{23}\partial_a)^{s_1-n+\alpha}(\nabla_{31}\partial_b)^{s_2-n+\beta}(\partial_a\partial_b)^{\gamma}(\partial_b\partial_c)^{\alpha}(\partial_c\partial_a)^{\beta}\big] \\ &h^{(s_1)}(a;z_1)h^{(s_2)}(b;z_2)h^{(s_3)}(c;z_3), \end{aligned} \tag{3.32}$$

Where

$$\nabla_{12} = \nabla_1 - \nabla_2, \tag{3.33}$$
$$\nabla_{23} = \nabla_2 - \nabla_3, \tag{3.34}$$
$$\nabla_{31} = \nabla_3 - \nabla_1. \tag{3.35}$$

Now we can see that the following expression is a generating function for the leading term of all interactions of HS gauge fields.

$$\begin{aligned} \mathcal{A}^{00} = \textstyle\int_{z_1,z_2,z_3} \delta(z-z_1)\delta(z-z_2)\delta(z-z_3) exp W \\ \Phi_1(z_1; a_1 + \tfrac{1}{2}\nabla_{23})\Phi_2(z_2; a_2 + \tfrac{1}{2}\nabla_{31})\Phi_3(z_3; a_3 + \tfrac{1}{2}\nabla_{12}) \mid_{a_1=a_2=a_3=0} \end{aligned} \tag{3.36}$$

with

$$W = \frac{\lambda^2}{2}[(\partial_{a_1}\partial_{a_2})(\partial_{a_3}\nabla_{12}) + (\partial_{a_2}\partial_{a_3})(\partial_{a_1}\nabla_{23}) + (\partial_{a_3}\partial_{a_1})(\partial_{a_2}\nabla_{31})] \tag{3.37}$$

Which can be written in another form

$$\mathcal{A}^{00}(\Phi(z)) = \int_{z_1,z_2,z_3} \delta(z - z_{1,2,3}) exp\hat{W} \times \Phi(z_1; a_1)\Phi(z_2; a_2)\Phi(z_3; a_3) \mid_{a_1=a_2=a_3=0} \tag{3.38}$$

where

$$\begin{aligned} \hat{W} = \tfrac{\lambda^2}{2}[(\partial_{a_1}\partial_{a_2})(\partial_{a_3}\nabla_{12}) + (\partial_{a_2}\partial_{a_3})(\partial_{a_1}\nabla_{23}) + (\partial_{a_3}\partial_{a_1})(\partial_{a_2}\nabla_{31})] \\ + \tfrac{1}{2}[(\partial_{a_3}\nabla_{12}) + (\partial_{a_1}\nabla_{23}) + (\partial_{a_2}\nabla_{31})], \end{aligned} \tag{3.39}$$
$$\textstyle\int_{z_1,z_2,z_3} \delta(z - z_{1,2,3}) = \int_{z_1,z_2,z_3} \delta(z-z_1)\delta(z-z_2)\delta(z-z_3) \tag{3.40}$$

for brevity. Furthermore we will always assume this integration with delta functions, without writing it explicitly. The operator in the second row of (3.39) is a dimensionless operator, therefore it does not need any dimensional constant multiplier.

Now we can derive all other terms in the lagrangian using following important relation

$$[exp\hat{W}, A] = exp\hat{W}[\hat{W}, A] + exp\hat{W}[\hat{W}, [\hat{W}, A]] + exp\hat{W}[\hat{W}, [\hat{W}, [\hat{W}, A]]] + ... \tag{3.41}$$

for any operator A. And therefore

$$[exp\hat{W}, (a_1\nabla_1)] = exp\hat{W}[\hat{W}, (a_1\nabla_1)], \tag{3.42}$$
$$[exp\hat{W}, (a_2\nabla_2)] = exp\hat{W}[\hat{W}, (a_2\nabla_2)], \tag{3.43}$$
$$[exp\hat{W}, (a_3\nabla_3)] = exp\hat{W}[\hat{W}, (a_3\nabla_3)]. \tag{3.44}$$

Following commutators will be used many times while deriving trace and divergence terms

$$[\hat{W}, (a_1\nabla_1)] = -\frac{\lambda^2}{4}[(\partial_{a_2}\nabla_2)(\partial_{a_3}\nabla_{12}) + (\partial_{a_3}\nabla_3)(\partial_{a_2}\nabla_{31})] + \frac{1}{2}[\lambda^2(\partial_{a_2}\partial_{a_3}) + 1]\nabla_1\nabla_{23}, \tag{3.45}$$
$$[\hat{W}, (a_2\nabla_2)] = -\frac{\lambda^2}{4}[(\partial_{a_3}\nabla_3)(\partial_{a_1}\nabla_{23}) + (\partial_{a_1}\nabla_1)(\partial_{a_3}\nabla_{12})] + \frac{1}{2}[\lambda^2(\partial_{a_3}\partial_{a_1}) + 1]\nabla_2\nabla_{31}, \tag{3.46}$$
$$[\hat{W}, (a_3\nabla_3)] = -\frac{\lambda^2}{4}[(\partial_{a_1}\nabla_1)(\partial_{a_2}\nabla_{31}) + (\partial_{a_2}\nabla_2)(\partial_{a_1}\nabla_{23})] + \frac{1}{2}[\lambda^2(\partial_{a_1}\partial_{a_2}) + 1]\nabla_3\nabla_{12}, \tag{3.47}$$

Note that

$$\nabla_1\nabla_{23} = \Box_3 - \Box_2, \tag{3.48}$$
$$\nabla_2\nabla_{31} = \Box_1 - \Box_3, \tag{3.49}$$
$$\nabla_3\nabla_{12} = \Box_2 - \Box_1, \tag{3.50}$$

which is obvious because‖

$$\nabla_1 + \nabla_2 + \nabla_3 = 0. \tag{3.51}$$

We are working with the same type of diagram as in [1].

$\Box_{a_i}$ \ D_{a_i}	0	1	2	3
0	$\mathcal{A}^{00}$	$\mathcal{A}^{10}$	$\mathcal{A}^{20}$	$\mathcal{A}^{30}$
1	$\mathcal{A}^{01}$	$\mathcal{A}^{11}$	$\mathcal{A}^{21}$	
2	$\mathcal{A}^{02}$	$\mathcal{A}^{12}$		
3	$\mathcal{A}^{03}$			

(3.52)

‖We always admit partial integration, working with lagrangian as with an action.

Now we take gauge variation of $\mathcal{A}^{00}$, and find generating functions for all other terms in cubic Lagrangian. Simple but elegant structure shows first row of the diagram.

$$\mathcal{A}^{10}(\Phi(z)) = \mathcal{A}^{30}(\Phi(\quad z \quad)) = 0, \tag{3.53}$$

$$\begin{aligned}\mathcal{A}^{20}(\Phi(z)) = \frac{1}{4}exp\hat{W}\{ \quad & +[\lambda^2(\partial_{a_1}\partial_{a_2}) + 1][\lambda^2(\partial_{a_2}\partial_{a_3}) + 1]D_{a_3}D_{a_1} \\ & +[\lambda^2(\partial_{a_2}\partial_{a_3}) + 1][\lambda^2(\partial_{a_3}\partial_{a_1}) + 1]D_{a_1}D_{a_2} \\ & +[\lambda^2(\partial_{a_3}\partial_{a_1}) + 1][\lambda^2(\partial_{a_1}\partial_{a_2}) + 1]D_{a_2}D_{a_3}\} \\ & \Phi(z_1; a_1)\Phi(z_2; a_2)\Phi(z_3; a_3) \mid_{a_1=a_2=a_3=0}\end{aligned} \tag{3.54}$$

Another terms are

$$\mathcal{A}^{01}(\Phi(z)) = 0, \tag{3.55}$$

$$\begin{aligned}\mathcal{A}^{11}(\Phi(z)) = \frac{\lambda^2}{16}exp\hat{W}\{ \quad & +[\lambda^2(\partial_{a_1}\partial_{a_2}) + 1](\partial_{a_1}\nabla_{23})\Box_{a_3}D_{a_2} \\ & -[\lambda^2(\partial_{a_1}\partial_{a_2}) + 1](\partial_{a_2}\nabla_{31})\Box_{a_3}D_{a_1} \\ & +[\lambda^2(\partial_{a_2}\partial_{a_3}) + 1](\partial_{a_2}\nabla_{31})\Box_{a_1}D_{a_3} \\ & -[\lambda^2(\partial_{a_2}\partial_{a_3}) + 1](\partial_{a_3}\nabla_{12})\Box_{a_1}D_{a_2} \\ & +[\lambda^2(\partial_{a_3}\partial_{a_1}) + 1](\partial_{a_3}\nabla_{12})\Box_{a_2}D_{a_1} \\ & -[\lambda^2(\partial_{a_3}\partial_{a_1}) + 1](\partial_{a_1}\nabla_{23})\Box_{a_2}D_{a_3}\} \\ & \Phi(z_1; a_1)\Phi(z_2; a_2)\Phi(z_3; a_3) \mid_{a_1=a_2=a_3=0}\end{aligned} \tag{3.56}$$

and so on.

This all expressions can be written in very elegant form. First we introduce Grassman variables

$$\eta_{a_1}, \bar{\eta}_{a_1}, \eta_{a_2}, \bar{\eta}_{a_2}, \eta_{a_3}, \bar{\eta}_{a_3}. \tag{3.57}$$

Now we change expressions in the formula (3.64) in a following way

$$(\partial_{a_1}\partial_{a_2}) \to (\partial_{a_1}\partial_{a_2}) + \frac{1}{4}\eta_{a_1}\bar{\eta}_{a_2}\Box_{a_2} + \frac{1}{4}\eta_{a_2}\bar{\eta}_{a_1}\Box_{a_1}, \tag{3.58}$$

$$(\partial_{a_2}\partial_{a_3}) \to (\partial_{a_2}\partial_{a_3}) + \frac{1}{4}\eta_{a_2}\bar{\eta}_{a_3}\Box_{a_3} + \frac{1}{4}\eta_{a_3}\bar{\eta}_{a_2}\Box_{a_2}, \tag{3.59}$$

$$(\partial_{a_3}\partial_{a_1}) \to (\partial_{a_3}\partial_{a_1}) + \frac{1}{4}\eta_{a_3}\bar{\eta}_{a_1}\Box_{a_1} + \frac{1}{4}\eta_{a_1}\bar{\eta}_{a_3}\Box_{a_3}, \tag{3.60}$$

$$(\partial_{a_1}\nabla_{23}) \to (\partial_{a_1}\nabla_{23}) + \eta_{a_1}\bar{\eta}_{a_2}D_{a_2} - \eta_{a_1}\bar{\eta}_{a_3}D_{a_3} \tag{3.61}$$

$$(\partial_{a_2}\nabla_{31}) \to (\partial_{a_2}\nabla_{31}) + \eta_{a_2}\bar{\eta}_{a_3}D_{a_3} - \eta_{a_2}\bar{\eta}_{a_1}D_{a_1} \tag{3.62}$$

$$(\partial_{a_3}\nabla_{12}) \to (\partial_{a_3}\nabla_{12}) + \eta_{a_3}\bar{\eta}_{a_1}D_{a_1} - \eta_{a_3}\bar{\eta}_{a_2}D_{a_2}. \tag{3.63}$$

Now we can write

$$\mathcal{A}(\Phi(z)) = \int d\eta_{a_1} d\bar{\eta}_{a_1} d\eta_{a_2} d\bar{\eta}_{a_2} d\eta_{a_3} d\bar{\eta}_{a_3} (1+\eta_{a_1}\bar{\eta}_{a_1})(1+\eta_{a_2}\bar{\eta}_{a_2})(1+\eta_{a_3}\bar{\eta}_{a_3})$$
$$exp\hat{W}\Phi(z_1;a_1)\Phi(z_2;a_2)\Phi(z_3;a_3)\,|_{a_1=a_2=a_3=0} \quad (3.64)$$

Where

$$\begin{aligned}\hat{W} &= \frac{1}{2}[1+\lambda^2(\partial_{a_1}\partial_{a_2} + \frac{1}{4}\eta_{a_1}\bar{\eta}_{a_2}\Box_{a_2} + \frac{1}{4}\eta_{a_2}\bar{\eta}_{a_1}\Box_{a_1})][\partial_{a_3}\nabla_{12} + \eta_{a_3}\bar{\eta}_{a_1}D_{a_1} - \eta_{a_3}\bar{\eta}_{a_2}D_{a_2}] \\ &+ \frac{1}{2}[1+\lambda^2(\partial_{a_2}\partial_{a_3} + \frac{1}{4}\eta_{a_2}\bar{\eta}_{a_3}\Box_{a_3} + \frac{1}{4}\eta_{a_3}\bar{\eta}_{a_2}\Box_{a_2})][\partial_{a_1}\nabla_{23} + \eta_{a_1}\bar{\eta}_{a_2}D_{a_2} - \eta_{a_1}\bar{\eta}_{a_3}D_{a_3}] \\ &+ \frac{1}{2}[1+\lambda^2(\partial_{a_3}\partial_{a_1} + \frac{1}{4}\eta_{a_3}\bar{\eta}_{a_1}\Box_{a_1} + \frac{1}{4}\eta_{a_1}\bar{\eta}_{a_3}\Box_{a_3})][\partial_{a_2}\nabla_{31} + \eta_{a_2}\bar{\eta}_{a_3}D_{a_3} - \eta_{a_2}\bar{\eta}_{a_1}D_{a_1}] \quad (3.65)\end{aligned}$$

This operator generates all terms in cubic interaction of any three HS fields with any possible number of derivatives Δ in the range $\Delta_{min} \leq \Delta \leq \Delta_{max}$. Another possible form of $\hat{W}$ operator is

$$\begin{aligned}\hat{W} &= [1+\lambda^2(\partial_{a_1}\partial_{a_2} + \frac{1}{2}\eta_{a_1}\bar{\eta}_{a_2}\Box_{a_2})][(\partial_{a_3}\nabla_1) + \frac{1}{2}\eta_{a_3}\bar{\eta}_{a_1}D_{a_1} - \frac{1}{2}\eta_{a_3}\bar{\eta}_{a_2}D_{a_2} + \frac{1}{2}\eta_{a_3}\bar{\eta}_{a_3}D_{a_3}] \\ &+ [1+\lambda^2(\partial_{a_2}\partial_{a_3} + \frac{1}{2}\eta_{a_2}\bar{\eta}_{a_3}\Box_{a_3})][(\partial_{a_1}\nabla_2) + \frac{1}{2}\eta_{a_1}\bar{\eta}_{a_2}D_{a_2} - \frac{1}{2}\eta_{a_1}\bar{\eta}_{a_3}D_{a_3} + \frac{1}{2}\eta_{a_1}\bar{\eta}_{a_1}D_{a_1}] \\ &+ [1+\lambda^2(\partial_{a_3}\partial_{a_1} + \frac{1}{2}\eta_{a_3}\bar{\eta}_{a_1}\Box_{a_1})][(\partial_{a_2}\nabla_3) + \frac{1}{2}\eta_{a_2}\bar{\eta}_{a_3}D_{a_3} - \frac{1}{2}\eta_{a_2}\bar{\eta}_{a_1}D_{a_1} + \frac{1}{2}\eta_{a_2}\bar{\eta}_{a_2}D_{a_2}] \quad (3.66)\end{aligned}$$

This case generates the Lagrangian derived in [1]. The leading term of that Lagrangian is (3.27). These two operators (3.65) and (3.66) generate two lagrangians that differ from each other just by partial integration and field redefinition. All interactions of HS gauge fields with any number of derivatives are unique and are generated by both operators (3.65) and (3.66).

In the case of (3.65) we have

$\Box_{a_i}$ \ D_{a_i}	0	1	2	3
0	$\mathcal{A}^{00}$	0	$\mathcal{A}^{20}$	0
1	0	$\mathcal{A}^{11}$	$\mathcal{A}^{21}$	
2	$\mathcal{A}^{02}$	$\mathcal{A}^{12}$		
3	$\mathcal{A}^{03}$			

(3.67)

In the case of (3.66) we have

$\Box_{a_i}$ \ D_{a_i}	0	1	2	3
0	$\mathcal{A}^{00}$	$\mathcal{A}^{10}$	$\mathcal{A}^{20}$	$\mathcal{A}^{30}$
1	0	$\mathcal{A}^{11}$	$\mathcal{A}^{21}$	
2	0	$\mathcal{A}^{12}$		
3	$\mathcal{A}^{03}$			

(3.68)

Both forms of the same cubic Lagrangian are very useful for further investigations.

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Note added. When the present work was on its final stage for submission, the paper [21] appeared in the archive which includes an analysis of off-shell cubic interactions in HS field theories from String theory point of view, using BRST technique. The two results are closely related.

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