Restrictions for n-Point Vertices in Higher-Spin Theories

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Abstract: We give a simple classification of the independent n-point interaction vertices for bosonic higher-spin gauge fields in d-dimensional Minkowski space-times. We first give a characterisation of such vertices for large dimensions, $d \geqslant 2n-1$, where one does not have to consider Schouten identities due to over-antisymmetrisation of space-time indices. When the dimension is lowered, such identities have to be considered, but their appearance only leads to equivalences of large-d vertices and does not lead to new types of vertices. We consider the case of low dimensions, $d < n$, in detail, where the large number of Schouten identities leads to strong restrictions on independent vertices. We also comment on the generalisation of our results to the intermediate case $n \leqslant d \leqslant 2n-2$. In all cases, the independent vertices are expressed in terms of elementary manifestly gauge-invariant quantities, suggesting that no deformations of the gauge transformations are induced.

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1 Introduction

In this paper, we investigate a Lagrangian formulation of higher-spin (HS) theories in arbitrary dimensions. The aim of this work is, in particular, to obtain restrictions for all possible independent interaction vertices of order $n \geqslant 4$ for massless higher-spin fields, extending the three-dimensional results of [1]. Together with the earlier results on the cubic vertices [2–9] (see also [10–15]), this work intends to complete the classification of *all independent interacting deformations* of free massless HS Lagrangians [16, 17] to the lowest order in the deformation parameters (coupling constants) in Minkowski space-time of arbitrary dimensions $d \geqslant 3$.

HS Gravities [18–20] (see, e.g., [21, 22] for reviews) are generalisations of Einstein's General Relativity which involve higher-spin gauge fields. These are symmetric tensor (Fronsdal) fields[1] $\phi_{\mu_1 \dots \mu_s}$, described by the Fronsdal action [16] at free level, describing massless particles of spin s upon quantisation[2]. A set of free HS fields can be described by a Lagrangian, which is a sum of Fronsdal Lagrangians for spin s fields. However, a full non-linear Lagrangian of *interacting* Fronsdal fields is not available to date.

Such theories are strongly constrained by gauge invariance, necessary for consistency. These gauge transformations extend those of General Relativity — space-time reparametrisations, or diffeomorphisms — to larger symmetries, involving gauge parameters that are Lorentz tensors of rank $(s-1)$ for each massless spin s field. This extension of symmetries can potentially resolve some problems of General Relativity (singularities, quantisation problem, etc), making HS Gravity an attractive field of investigation.

The corresponding gauge transformation for free fields reads [3]

$$\delta^{(0)} \phi_{\mu_1 \dots \mu_s} = s\, \partial_{(\mu_1} \epsilon_{\mu_2 \dots \mu_s)}\,, \tag{1.1}$$

which generalises the well known expressions for massless vector fields ($s = 1$) in gauge theory and the Graviton ($s = 2$) in linearised gravity theory.

The naive intuition from lower-spin model building suggests that one can pick an arbitrary collection of fields, including massless HS fields, and the gauge symmetries will partly constrain the interactions, leaving room for a large parameter space of theories. It turns out, that the severe constraints from HS gauge invariance rule out theories with an arbitrary choice of the particle content. Therefore, one is easily led to negative results if one chooses an arbitrary starting setup for constructing a theory with massless HS spectrum. This striking difference from textbook examples makes it tempting to conclude after some attempts that such theories cannot exist.

The problem can be traced to the global symmetries of the theory (see, e.g., [23]). Building such theories, therefore, can be addressed constructively by looking for suitable global symmetry algebras, which have to satisfy the so-called admissibility condition [24]. This condition rules out infinitely many potential candidate algebras (see, e.g., [25]) and

[1] The index s is the spin of the field and $\mu_i = 0, \dots, d-1$ in d dimensions.
[2] In this paper, we will restrict ourselves to integer-spin (bosonic) fields for simplicity.
[3] The round (square) brackets denote (anti-)symmetrisation with weight one.

was crucial in deriving the list of admissible HS algebras [26, 27] and constructing full non-linear HS equations [18–20] in the frame formulation, proving the existence of a theory with massless HS fields. This theory, however, has unusual properties: there is an infinite tower of massless higher-spin fields with $s = 0, 1, 2, \dots$ and a necessarily non-zero cosmological constant [28]. The need for a non-zero cosmological constant is related to diffeomorphism transformations, as explained in [29, 30], essential for the Fradkin-Vasiliev solution to the Aragone-Deser problem [31]. This argument, together with the holographic conjectures (see, e.g., [32, 33]) motivated the intense studies of HS interactions, especially in $(A)dS_d$ background [34–44].

The frame-like formulation of HS gravities that led to successful developments (including Vasiliev's non-linear equations [18–20]) registered less progress so far in understanding the corresponding Lagrangian formulation. On the other hand, the metric-like formulation [16, 17, 43] is a simple suitable setup for classifying interaction vertices and deriving restrictions on interacting Lagrangians. Here, we work in the framework of the Noether-Fronsdal program (see [45–58] for related literature and [9] for a recent summary of the status of the problem) to classify independent vertices of order $n \geqslant 4$ in arbitrary dimensions $d \geqslant 3$, generalising the $d = 3$ results obtained earlier in [1].

The situation is different only in three dimensions, where the interacting HS theories can admit arbitrary Einstein backgrounds (including Minkowski) as well as a finite spectrum of massless HS fields (see, e.g., [59–63]). However, such massless HS fields do not correspond to propagating particles in $d = 3$, while the inclusion of matter leads to a situation similar to the higher-dimensional story in many ways.

The Noether-Fronsdal program is a systematic approach to perturbatively construct a Lagrangian $\mathcal{L}$ for an arbitrary interacting HS theory order by order. In this procedure, $\mathcal{L}$ is expanded in powers of small parameters g_n,

$$\mathcal{L} = \mathcal{L}_2 + \sum_{n \geqslant 3} g_n \mathcal{L}_n + O(g_n^2)\,. \tag{1.2}$$

Here, $\mathcal{L}_2$ denotes the free Fronsdal Lagrangian and another sum over the different kinds of n-point vertices $\mathcal{L}_n$ is suppressed.

The action must be gauge-invariant, hence, $\delta\mathcal{L}$ equals a total derivative, where δ is obtained by a deformation of the free gauge transformation $\delta^{(0)}$,

$$\delta = \delta^{(0)} + \sum_{k \geqslant 1} \delta^{(k)}\,.$$

Here, the deformation $\delta^{(k)}$ is of k-th order in the fields. Since our aim is to find constraints for the *independent* vertex structures (i.e. linear in the coupling constants[4]), the n-point vertex must satisfy

$$\delta^{(0)}\mathcal{L}_n + \delta^{(n-2)}\mathcal{L}_2 = 0 \quad \text{up to total derivatives}\,. \tag{1.3}$$

[4]Gauge invariance provides constraints to fix the terms proportional to higher powers of coupling constants. We are interested here in the structures that parametrise the non-trivial deformations at the lowest order in the coupling constants.

In this paper, we find restrictions for all independent n-point vertices $\mathcal{L}_n$ for massless HS fields in arbitrary dimension $d \geqslant 3$, such that they satisfy Eq. (1.3) [5]. From that, we deduce a simple classification of vertices. For a summary of the explicit results, see the beginning of Section 7.

The paper is organised as follows: In Section 2, we set up notations and provide the mathematical framework for our analysis. There, we discuss that we have to analyse three different cases separately: Large dimensions $d \geqslant 2n-1$ (see Section 3), low dimensions $d < n$ (see Section 5) and the intermediate case (see comments in Section 7). We mostly consider parity-even vertices, but give a generalisation to parity-odd vertices in Section 6. We finally conclude in Section 7.

2 Preliminaries

We want to constrain the n-point independent vertices $\mathcal{L}_n$ that may constitute the lowest order deformations of the free Lagrangian for massless HS fields. For this purpose, it is sufficient to restrict ourselves to the traceless and transverse (TT) sector of the Lagrangian as in [1]. Hence, we can assume that the tensors $\phi_{\mu_1\ldots\mu_s}$ that describe the gauge fields, are traceless, divergence-free and the corresponding free equation of motion is given by the (massless) Klein-Gordon equation, hence

$$g^{\mu_1\mu_2}\phi_{\mu_1\ldots\mu_s} = 0\,, \qquad \partial^{\mu_1}\phi_{\mu_1\cdots\mu_s} = 0\,, \qquad \partial^\nu\partial_\nu\,\phi_{\mu_1\ldots\mu_s}\big|_{\text{free e.o.m.}} = 0\,. \tag{2.1}$$

The relaxation of these conditions will allow to reconstruct the full off-shell counterpart of the TT vertices as in [6, 43].

2.1 Vertex Generating Operators

It is very convenient to contract the indices of the fields each with an auxiliary vector variable a^μ,

$$\phi^{(s)}(x,a) = \frac{1}{s!}\phi_{\mu_1\ldots\mu_s}(x)a^{\mu_1}\cdots a^{\mu_s}\,. \tag{2.2}$$

This has several advantages: First, we do not have to tackle expressions with too many indices and secondly, the tensor $\phi_{\mu_1\cdots\mu_s}$ is by construction symmetric. We will also note later on, that the complexity of index contractions will be reduced a lot. For example, using the short-hand notation $P^\mu = \partial_{x^\mu}$ and $A^\mu = \partial_{a^\mu}$, the relations in Eq. (2.1) simplify to

$$A^2\,\phi^{(s)} = 0\,, \qquad A\cdot P\,\phi^{(s)} = 0\,, \qquad P^2\,\phi^{(s)}\big|_{\text{free e.o.m.}} = 0\,. \tag{2.3}$$

We call these relations collectively Fierz equations [64].

Now, each n-point vertex $\mathcal{L}_n$ in Eq. (1.2) is a product of n massless bosonic fields (and possibly derivatives thereof). But it has to be a Lorentz scalar, hence, all indices of the fields (and of the derivatives) must be fully contracted. For now, let us concentrate on

[5] We consider a flat Minkowski space-time but comment also on $(A)dS$ backgrounds in the Section 7.

parity-even vertices — we consider parity-odd vertices in Section 6. Then, we can write $\mathcal{L}_n$ in the following, very convenient way:

$$\mathcal{L}_n(x) = \mathcal{V}\left(\prod_{i=1}^{n} \phi_i(x_i, a_i)\right)\bigg|_{\substack{x_i=x\\ a_i=0}}. \tag{2.4}$$

This needs some explanation:

- We use the notation set up in Eq. (2.2) and dropped the spin labels of the fields: ϕ_i is a spin s_i field, $\phi_i = \phi^{(s_i)}$.

- The term in brackets represents a function of the spacetime coordinates x_i and the auxiliary vector variables a_i. The *vertex generating operator* $\mathcal{V}$ performs the index contractions between the fields ϕ_i as follows: Let $P_i^\mu = \partial_{x_i^\mu}$ and $A_i^\mu = \partial_{a_i^\mu}$ as in Eq. (2.3). Then, $\mathcal{V}$ must be a polynomial in the following commuting variables:

$$z_{ij} = A_i \cdot A_j\big|_{1\leqslant i\leqslant j\leqslant n}, \qquad y_{ij} = A_i \cdot P_j\big|_{1\leqslant i,j\leqslant n}, \qquad s_{ij} = P_i \cdot P_j\big|_{1\leqslant i\leqslant j\leqslant n}. \tag{2.5}$$

  The operator z_{ij} induces a single contraction of indices between the fields ϕ_i and ϕ_j, whereas y_{ij} will take one index of the field ϕ_i and contract it with a derivative which acts on the field ϕ_j. Finally, the operators s_{ij} will introduce extra derivatives (a derivative of ϕ_i is contracted with a derivative of ϕ_j). These are called Mandelstam variables.

- Since all of the indices in $\mathcal{L}_n$ have to be contracted, we discard all terms that still contain at least one of the auxiliary variables, when $\mathcal{V}$ acted on the terms in brackets. Thus, we set $a_i = 0$ in the end, which ensures that $\mathcal{L}_n$ is Lorentz invariant. Finally, we also set $x_i = x$. The splitting of the coordinates is useful to keep track of the derivatives acting on different fields, and has no physical consequences.

All in all, we translated the problem of 'what is the most general form of the parity-even n-point vertex $\mathcal{L}_n$' to the question 'what is the most general form of the vertex generating operator $\mathcal{V}$ in the polynomial ring $\mathbb{R}[y_{ij}, z_{ij}|_{i\leqslant j}, s_{ij}|_{i\leqslant j}]$'. The connection between Lagrangian $\mathcal{L}_n$ and operator $\mathcal{V}$ is given by Eq. (2.4). We also ensured that $\mathcal{L}_n$ is Lorentz invariant.

There are two questions arising now: First of all, there are equivalence relations for Lagrangians: e.g., two Lagrangians that differ by a total derivative lead to the same action. We call them equivalent in this case. What does this imply for the corresponding vertex generating operators? Secondly, how do we have to constrain $\mathcal{V}$, such that $\mathcal{L}_n$ is gauge invariant? We present a general answer to these questions in the next two sections and give more details in Sections 3 and 5.

2.2 Equivalence Relations for Vertex Generating Operators

We must take into account that different Lagrangians may describe the same theory. We say that they are *equivalent* in this case and evidently, we are only interested in $\mathcal{L}_n$ *up to* *equivalence.* When we encode the Lagrangians via vertex generating operators, we need to introduce a notion of equivalence for operators: *vertex operators $\mathcal{V}$ and $\mathcal{V}'$ are equivalent, $\mathcal{V} \approx \mathcal{V}'$, iff the two Lagrangians $\mathcal{L}_n$ and $\mathcal{L}'_n$, constructed from them via Eq. (2.4) are also equivalent.* We are hence only interested in $\mathcal{V}$ *up to equivalence* and summarise the different kinds of equivalence relations in the following.

The first kind of equivalence relations arises from field redefinitions $\phi_i \mapsto \phi_i + \delta\phi_i$, where $\delta\phi_i$ is non-linear in the fields. These do not change the theory, but affect the Lagrangian. For example, terms in $\mathcal{L}_2$ may contribute to $\mathcal{L}_n$ when the fields are redefined non-linearly. But in this way, the n-point vertices only change by terms that vanish when the free equations of motion are imposed. We say that two Lagrangians are equivalent, when they are related by such field redefinitions and deduce from Eq. (2.4) that we can choose $\mathcal{V}$ to be independent of s_{ii}. Furthermore, we assume that $\mathcal{V}$ does not depend on z_{ii} and y_{ii}, because the fields are traceless and divergence free.

Mathematically speaking, we impose the equivalence relations

$$y_{ii} \approx 0, \qquad z_{ii} \approx 0, \qquad s_{ii} \approx 0 \tag{2.6}$$

and deduce that each operator in the ideal $\langle y_{ii}, z_{ii}, s_{ii}\rangle \subset \mathbb{R}[y_{ij}, z_{ij}|_{i\leqslant j}, s_{ij}|_{i\leqslant j}]$ is equivalent to 0. Hence, we can construct equivalence classes of vertex generating operators,

$$[\mathcal{V}] \in \frac{\mathbb{R}[y_{ij}, z_{ij}|_{i\leqslant j}, s_{ij}|_{i\leqslant j}]}{\langle y_{ii}, z_{ii}, s_{ii}\rangle}\,.$$

But the quotient ring is isomorphic to the subring $\mathcal{R} = \mathbb{R}\big[y_{ij}|_{i\neq j}, z_{ij}|_{i<j}, s_{ij}|_{i<j}\big]$,

$$\frac{\mathbb{R}[y_{ij}, z_{ij}|_{i\leqslant j}, s_{ij}|_{i\leqslant j}]}{\langle y_{ii}, z_{ii}, s_{ii}\rangle} \simeq \mathcal{R} \subset \mathbb{R}[y_{ij}, z_{ij}|_{i\leqslant j}, s_{ij}|_{i\leqslant j}]\,,$$

so we can choose the vertex generating operator as $\mathcal{V} \in \mathcal{R}$. In other words, we simply dropped the dependence of $\mathcal{V}$ on y_{ii}, z_{ii} and s_{ii}.

Secondly, acting with the operator $D^\mu = \sum_{j=1}^n P_j^\mu$ on the term in brackets in Eq. (2.4) gives a total derivative in the Lagrangian. This does not change the action and hence, does not affect the theory. Therefore, we impose the equivalence relations

$$A_i \cdot D = \sum_{j=1}^n y_{ij} \approx 0, \qquad P_i \cdot D = \sum_{j=1}^n s_{ij} \approx 0\,. \tag{2.7}$$

These together generate an ideal $\mathcal{I}_D \subset \mathcal{R}$ and in the following, we consider equivalence classes of vertex generating operators in the quotient ring

$$[\mathcal{V}] \in \frac{\mathcal{R}}{\mathcal{I}_D}\,.$$

As for the equivalence relations in Eq. (2.6), we could choose a convenient representative $\mathcal{V}$ in $\mathcal{R}$, but it turns out to be better to keep the quotient ring structure for now.

A last equivalence stems from 'Schouten identities', i.e. relations following from over-antisymmetrisation of spacetime indices. These spacetime dimension-dependent identities are exact relations at the Lagrangian level. In the polynomial ring $\mathcal{R}$, however, we forgot that we work in d dimensions. Therefore, we have to impose Schouten identities as equivalence relations for vertex generating operators [6], which form an ideal $\mathcal{I}_S \subset \mathcal{R}$ as follows: Let $b = (P_1, \ldots, P_n, A_1, \ldots A_n)$ be a vector of derivative operators and consider the symmetric $2n \times 2n$ matrix

$$\mathcal{B} = \left(b_K \cdot b_L\right)\big|_{K,L\in(1,\ldots,2n)} = \begin{pmatrix} \mathcal{S} & \mathcal{Y}^T \\ \mathcal{Y} & \mathcal{Z} \end{pmatrix} . \tag{2.8}$$

Here, $\mathcal{S} = (s_{ij})$, $\mathcal{Y} = (y_{ij})$, $\mathcal{Z} = (z_{ij})$ are symmetric $(n \times n)$-matrices with elements in $\mathcal{R}$. With the equivalence relations in Eq. (2.6), the diagonal elements of $\mathcal{S}$, $\mathcal{Y}$ and $\mathcal{Z}$ vanish equivalently. We also keep in mind that there are further equivalence relations from Eq. (2.7) which introduce a linear relation among the first n rows (and columns) of $\mathcal{B}$, but we do not apply them right now.

Then, the ideal $\mathcal{I}_S$ is generated by all $(d+1) \times (d+1)$ minors of $\mathcal{B}$. We show this in a moment, but note first that this implies that $\mathcal{I}_S$ is trivial for $d \geqslant 2n-1$. Indeed, in this case, there is only one such minor, namely when equality holds. This minor is $\det \mathcal{B}$, which is equivalent to zero due to the equivalence relations in $\mathcal{I}_D$ (the first n rows add up to a total derivative). Now we show that for $d < 2n-1$, the above statement is true. Indeed, remove $(2n-d-1)$ rows and columns from $\mathcal{B}$, such that only the rows $K_1, \ldots, K_{d+1} \in (1, \ldots, 2n)$ and the columns $L_1, \ldots, L_{d+1} \in (1, \ldots, 2n)$ remain and call the resulting $(d+1) \times (d+1)$-matrix M. Then,

$$\begin{aligned} \det M &= \delta^{\mu_1}_{\nu_1} \cdots \delta^{\mu_{d+1}}_{\nu_{d+1}} \, B^{\mu_1}_{[K_1} \cdots B^{\mu_{d+1}}_{K_{d+1}]} \, B^{\nu_1}_{L_1} \cdots B^{\nu_{d+1}}_{L_{d+1}} \\ &= \delta^{\mu_1 \cdots \mu_{d+1}}_{\nu_1 \cdots \nu_{d+1}} \, B^{\mu_1}_{K_1} \cdots B^{\mu_{d+1}}_{K_{d+1}} \, B^{\nu_1}_{L_1} \cdots B^{\nu_{d+1}}_{L_{d+1}} \end{aligned} \tag{2.9}$$

and acting with it on the term in brackets in Eq. (2.4) yields a term in the Lagrangian with over-antisymetrised indices. On the other hand, each term in the Lagrangian with over-antisymmetrised indices corresponds to a vertex generating operator $\mathcal{V}$ that contains a factor of the form on the rhs of Eq. (2.9) for a certain set of indices $K_i, L_i \in (1, \ldots, 2n)$. Hence, $\mathcal{V} \in \mathcal{I}_S$.

At this step, it is convenient to introduce the notion of the *level* of a Schouten identity. To this end, let us first define the *level* of the rows and columns of $\mathcal{B}$ as follows: The first n rows and columns of $\mathcal{B}$ are of level 0 and all others are of level 1. Furthermore, each $(d+1) \times (d+1)$-submatrix M of $\mathcal{B}$ that is obtained by removing rows and columns inherits those row and column levels from $\mathcal{B}$. Then, the sum of row and column levels of M equals the power of A_i^μ operators in $\iota_d(\det M)$. This is what we call the *level* of the Schouten

[6] Formally, let ι_d be the map

$$\begin{aligned} \iota_d : \qquad \mathcal{R} \qquad &\rightarrow \mathbb{R}[P_i^\mu, A_i^\mu] \\ \mathcal{V}(z_{ij}, y_{ij}, s_{ij}) &\mapsto \mathcal{V}(A_i \cdot A_j, A_i \cdot P_j, P_i \cdot P_j) \end{aligned}$$

that replaces the operators z_{ij}, y_{ij} and s_{ij} by their definitions in Eq. (2.5). ι_d therefore reintroduces the operators P_i and A_i and hence, spacetime indices in d dimensions in the vertex generating operator $\mathcal{V}$. The kernel $\iota_d^{-1}(0)$ of this map is what we call the ideal of Schouten identities in d dimensions.

identity $\det M = 0$. Denote by $I(k)$ the ideal generated by all Schouten identities of level k, then we have

$$\mathcal{I}_S = \sum_{k=0}^{2d+2} I(k)\,, \tag{2.10}$$

where again, d denotes the spacetime dimension.

Now, we consider three cases:

- For large dimensions, $d \geqslant 2n-1$, as discussed before, there are no non-trivial Schouten identities at all (the only possible Schouten identities arise in the case $d = n+1$, but they are zero up to total derivatives, so they are already contained in $\mathcal{I}_D$). This case is much simpler and we treat it separately in Section 3.

- For large values of n, $d < n$, only the subideal $I(0)$ might be trivial (namely for $d+1 = n$, where the level 0 Schouten identities vanish up to a total derivative and thus are already contained in $\mathcal{I}_D$). Thanks to the variety of Schouten identities available, we are able to perform a lot of simplifications. We treat this case in Section 5.

- In the intermediate case $2n-2 \geqslant d \geqslant n$ only the ideals of level $2d-2n+4, \ldots, 2n$ are non-trivial. We will not study this case in full detail here, but a general characterisation of the corresponding vertices is given in Section 7.

All in all, we have now considered all possible equivalences for parity-even Lagrangians. Because of the freedom of field redefinitions, we consider $\mathcal{V} \in \mathcal{R}$ and we divide out the ideals generated by total derivatives ($\mathcal{I}_D$) and Schouten identities ($\mathcal{I}_S$),

$$[\mathcal{V}] \in \frac{\mathcal{R}}{\mathcal{I}}\,, \qquad \mathcal{I} = \mathcal{I}_S + \mathcal{I}_D\,. \tag{2.11}$$

2.3 Imposing Gauge Invariance

Finally, we require that $\mathcal{L}$ is gauge invariant, i.e. it satifies Eq. (1.3). What does this imply for the corresponding vertex generating operator $\mathcal{V}$? Note first that the second term in Eq. (1.3) vanishes when the free equations of motions are imposed. In other words, the requirement of gauge invariance for the independent vertex structures reads

$$\delta_k^{(0)} \mathcal{L}_n \approx 0\,, \tag{2.12}$$

where $\delta_k^{(0)}$ is the free gauge transformation of the field ϕ_k (see Eq. (1.1)).

The latter can be simplified by contracting the tensor for the gauge parameter in Eq. (1.1) with auxiliary vector variables a^μ as well,

$$\epsilon^{(s-1)}(x,a) = \frac{1}{(s-1)!}\epsilon_{\mu_1\ldots\mu_{s-1}}(x) a^{\mu_1}\cdots a^{\mu_{s-1}}\,. \tag{2.13}$$

Again, we drop the spin index, $\epsilon_k = \epsilon^{(s_k-1)}$, and the linearised gauge transformation of the k-th field ϕ_k in Eq. (1.1) reads

$$\delta_k^{(0)} \phi_k(x_k, a_k) = a_k \cdot P_k\, \epsilon_k(x_k, a_k), \qquad \text{(no sum)}.$$

Note that this gauge transformation must be consistent with Eqs. (2.3). We therefore impose the Fierz equations also for the gauge parameter.

All in all, we can now impose the restrictions for the vertex generating operators $\mathcal{V}$ from gauge invariance, Eq. (2.12):

$$\delta_k^{(0)}\mathcal{L}_n = \mathcal{V}\, a_k \cdot P_k \left(\epsilon_k(x_k, a_k) \prod_{1\leqslant i\leqslant n}^{i\neq k} \phi_i(x_i, a_i)\right)\Bigg|_{\substack{x_i=x \\ a_i=0}} \approx 0\,.$$

Since all the auxiliary vector variables a_i are set to zero in the end, it immediately follows that $\mathcal{L}_n$ is gauge invariant if and only if the corresponding vertex generating operator $\mathcal{V} \in \mathcal{R}$ (via Eq. (2.4)) satisfies

$$\text{for all } k \in \{1, \dots, n\} \;:\quad [\mathcal{V}, a_k \cdot P_k] =: D_k\mathcal{V} \in \mathcal{I}_S + \mathcal{I}_D\,. \tag{2.14}$$

Here, we defined the operators D_k of gauge variations. These act as linear first-order differential operators on the vertex $\mathcal{V}$:

$$D_k = \sum_{j=1}^{n}\left(y_{jk}\frac{\partial}{\partial z_{kj}} + s_{kj}\frac{\partial}{\partial y_{kj}}\right). \tag{2.15}$$

3 The case $2n - 1 \leqslant d$

We start with the case of sufficiently high space-time dimensions where the classification of vertices is the simplest because there are no Schouten identities and we only have to take into account total derivatives, hence, $\mathcal{I} = \mathcal{I}_D$.

3.1 Gauge Invariants

To derive the n-th order independent vertices we first recall the constraints on the vertex generating operators $y_{ij}\,, z_{ij}\,, s_{ij}$ in Eqs. (2.6) and (2.7) and count the independent variables:

$$y_{ii} \approx 0\,,\quad \sum_{j=1}^{n} y_{ij} \approx 0\,,\quad n(n-2) \text{ variables } y_{ij}\,, \tag{3.1a}$$

$$z_{ij} = z_{ji}\,,\quad z_{ii} \approx 0\,,\quad \frac{n(n-1)}{2} \text{ variables } z_{ij}\,, \tag{3.1b}$$

$$s_{ij} = s_{ji}\,,\quad s_{ii} \approx 0\,,\quad \sum_{j=1}^{n} s_{ij} \approx 0\,,\quad \frac{n(n-3)}{2} \text{ variables } s_{ij}\,. \tag{3.1c}$$

The vertex depends altogether on $2n(n-2)$ variables, and is subject to n linear differential equations that stem from Eqs. (2.14) and (2.15) [7]. If these differential equations are linearly independent, the solution should depend on $2n(n-2) - n = n(2n-5)$ variables.

[7] Notice that the operators D_k are consistent with these constraints (3.1), which means that D_k acting on a constraint will lead to a constraint. Therefore we can leave the operators D_k in the general form stated in Eq. (2.15) and do not need to express them in terms of a set of independent variables.

For cubic vertices, $n = 3$, this would give three invariants, while we know that the solution depends on four invariants y_{12}, y_{23}, y_{31} and $G = y_{12}\, z_{23} + y_{23}\, z_{31} + y_{31}\, z_{12}$. The reason is that the three equations in that case are not linearly independent: $y_{12}\, D_1 + y_{23}\, D_2 + y_{31}\, D_3 = 0$. Due to this relation, we have, e.g., the Yang-Mills cubic vertex $V_3^{YM} = G$ and the Einstein-Hilbert cubic vertex $V_3^{EH} = G^2$.

On the other hand, one can easily see from Eq. (2.15) that the operators D_k are linearly independent for $n \geqslant 4$. Hence, the general form of the vertices should depend on $n(2n-5)$ invariants composed of s_{ij}, y_{ij}, z_{ij}.

At this point, we introduce gauge invariant operators, which are more suitable as the building blocks of n-th order vertices. These are given through the following variables:

$$s_{ij} = s_{ji} \qquad \frac{n(n-3)}{2} \text{ variables}\,, \tag{3.2}$$

$$c_{ij} = y_{ij}\, y_{ji} - s_{ij}\, z_{ij} = c_{ji}\,, \qquad \frac{n(n-1)}{2} \text{ variables}\,, \tag{3.3}$$

$$c_{i,jk} = y_{ij}\, s_{ik} - y_{ik}\, s_{ij} = -c_{i,kj}\,, \qquad \frac{n(n-2)(n-3)}{2} \text{ variables}\,. \tag{3.4}$$

It is easy to show that these expressions are gauge invariant:

$$D_k\, s_{ij} = 0\,, \quad D_k\, c_{ij} = 0\,, \quad D_k\, c_{i,jl} = 0\,. \tag{3.5}$$

Counting the number of the variables s_{ij} and c_{ij} is straightforward. In order to count the number of $c_{i,jk}$ variables, we count separately the number of choices for i and the number of choices for the antisymmetric pair jk for a given i and multiply them. Naively, we choose i in n possible ways, and the antisymmetric pair jk takes values in $\{i+1\,, \ldots, i-2 \text{ (mod n)}\}$, therefore takes $\frac{(n-2)(n-3)}{2}$ values, hence the number of $c_{i,jk}$'s is given above. These variables $c_{i,jk}$ are not linearly independent though, satisfying the following relations:

$$3\, c_{i,[jk}\, s_{i|l]} \equiv c_{i,jk}\, s_{il} + c_{i,kl}\, s_{ij} + c_{i,lj}\, s_{ik} = 0\,,. \tag{3.6}$$

These naively are $\frac{n(n-2)(n-3)(n-4)}{6}$ many, given by multiplying the n possible choices of i and $\frac{(n-2)(n-3)(n-4)}{6}$ choices of the antisymmetric triple jkl. But again, this counting is redundant, due to linear relations between equations, involving different choices of jkl. These relations are also given by adding another s_{im} and antisymmetrising the four indices $jklm$. This chain of reducibility can be resummed to get all linearly independent variables of $c_{i,jk}$. This is done by finding the number of possible values of jk antisymmetrised pairs that correspond to the independent variables, by summing up with changing signs the numbers of components of antisymmetric tensors of $gl(n)$, starting from rank two:

$$\sum_{i=2}^{n-2} (-1)^i \binom{n-2}{i} = n-3\,. \tag{3.7}$$

This means that the number of independent variables $c_{i,jk}$ is $n(n-3)$. We see that the variables $c_{i,jk}$ are redundant and we choose the following set of independent variables:

$$Y_i^j := c_{i,i+j\, i+1}\,, \tag{3.8}$$

where now $j = 2, \ldots, n-2$, taking $n-3$ possible values (indices are always meant modulo n). Thus, the number of variables Y_i^j is altogether $n(n-3)$. It is elementary to show that any other variable $c_{i,jk}$ can be expressed through Y_i^j using Eq. (3.6):

$$c_{i,jk} = \frac{c_{i,j\,i+1}\, s_{ik} - c_{i,k\,i+1}\, s_{ij}}{s_{i\,i+1}} = \frac{Y_i^{j-i}\, s_{ik} - Y_i^{k-i}\, s_{ij}}{s_{i\,i+1}}\,. \tag{3.9}$$

Therefore altogether we have:

$$\frac{n(n-3)}{2} + \frac{n(n-1)}{2} + n(n-3) = n(2n-5) \text{ invariants.} \tag{3.10}$$

Given that the number of independent invariants s_{ij} , c_{ij} , Y_i^j is the same as the number of variables that should constitute the building blocks of n-th order independent vertices, it is already tempting to conclude that the most general solution is an arbitrary function of these variables. We will show this now, by allowing for dividing by Mandelstam variables and making the replacements

$$z_{ij} = \frac{1}{s_{ij}}(y_{ij}\, y_{ji} - c_{ij})\,, \tag{3.11}$$

and, consecutively,

$$y_{ii+j} = \frac{1}{s_{ii+1}}(y_{ii+1}\, s_{ii+j} - Y_i^j)\,, \qquad\qquad j = 2, \ldots, n-2 \mod n\,, \tag{3.12}$$

expressing the vertex operator in terms of the variables s_{ij}, c_{ij}, Y_i^j and y_{ii+1}. Correspondingly, the gauge variation in terms of these variables is generated by the operators

$$D_k = s_{kk+1}\frac{\partial}{\partial y_{kk+1}}\,, \tag{3.13}$$

which turn into a single derivative. Therefore, the new gauge invariance equations for the vertex operator give:

$$D_k \mathcal{V}(s_{ij}, c_{ij}, Y_i^j, y_{ii+1}) = s_{kk+1}\frac{\partial}{\partial y_{kk+1}}\mathcal{V}(s_{ij}, c_{ij}, Y_i^j, y_{ii+1}) \approx 0\,. \tag{3.14}$$

If we go to a set of independent variables, we can conclude that the y_{ii+1}-derivative is equal to zero, and the vertex can be solely written in terms of the gauge invariant combinations s_{ij}, c_{ij}, Y_i^j. A gauge invariant local vertex generating operator $\mathcal{V}$ in high enough dimension ($d \geqslant 2n-1$) is then in one-to-one correspondence to a polynomial in s_{ij}, c_{ij}, Y_i^j, allowing at most those inverse powers of Mandelstam variables, such that $\mathcal{V}$ becomes polynomial in the variables s_{ij}, y_{ij} and z_{ij}, when re-expressing the combinations c_{ij} and Y_i^j.

3.2 Building blocks of vertices

We have just shown that any gauge-invariant vertex $\mathcal{V}$ of order n for $d \geqslant 2n-1$ can be rewritten as a function of the invariants c_{ij}, Y_i^j and s_{ij}. This function is polynomial in c_{ij} and Y_i^j, but can contain inverse powers of the Mandelstam variables s_{ij}.

In this subsection we address the question: 'what is the most general form of this function if we assume that the vertex is local?' First of all it is clear that any polynomial of c_{ij}, Y_i^j and s_{ij} defines a local and gauge-invariant vertex. Now let us analyse the case that the vertex contains a single pole in one s_{ij} when written in terms of the invariants:

$$\mathcal{V} = \frac{1}{s_{ij}} Q(c_{ij}, Y_i^j, s_{kl})\,. \tag{3.15}$$

Here, we assume that the polynomial Q does not explicitly depend on this specific s_{ij}. For $\mathcal{V}$ to be local, the inverse of s_{ij} has to be compensated by a term proportional to s_{ij} that arises when the invariants are rewritten in terms of s_{ij}, y_{ij}, z_{ij}. One can show that in this case, $\mathcal{V}$ is a linear combination of

$$b_{ijk\ell} = \frac{1}{s_{ij}} \left(c_{ij}\, s_{ik}\, s_{j\ell} - c_{i,jk} c_{j,i\ell}\right) \quad \text{and} \quad \frac{1}{s_{ij}} \left(s_{ik} c_{i,j\ell} - s_{i\ell} c_{i,jk}\right) \tag{3.16}$$

multiplied by polynomials in c_{ij}, Y_i^j and the Mandelstam variables [8]. The second expression is simply equal to $c_{i,k\ell}$ (see Eq. (3.6)), so it is again a polynomial in s_{kl} and c variables. The first one can be rewritten as

$$b_{ijk\ell} = \det \begin{pmatrix} s_{ij} & s_{ik} & y_{ji} \\ s_{\ell j} & s_{\ell k} & y_{j\ell} \\ y_{ij} & y_{ik} & z_{ij} \end{pmatrix} + s_{k\ell} c_{ij}\,. \tag{3.17}$$

Hence, up to a shift by a polynomial in Mandelstam and c variables, the building block $b_{ijk\ell}$ can be written as a determinant of a 3×3-submatrix of the matrix $\mathcal{B}$ (see Eq. (2.8)). This nicely fits with the observation that also the c invariants are just minors of $\mathcal{B}$,

$$c_{ij} = -\det \begin{pmatrix} s_{ij} & y_{ji} \\ y_{ij} & z_{ij} \end{pmatrix} \quad , \quad c_{i,jk} = \det \begin{pmatrix} s_{ik} & s_{ij} \\ y_{ik} & y_{ij} \end{pmatrix}\,. \tag{3.18}$$

Notice that these minors as well as the (3×3)-example above have the property that each $(n+i)$-th row (column) of the second block is accompanied by the corresponding (i-th) row (column) of the first block. This ensures gauge invariance because the i-th gauge variation transforms the $(n+i)$-th row (column) into the i-th row (column) leading to a vanishing determinant. Translating such a building block to the fields, the resulting expression is a pure curvature term: a tensor index of a field i occurs in an antisymmetric combination with an index of a derivative acting on the field.

Of course all such minors can be written as polynomials in the c invariants with negative powers of Mandelstam variables allowed. This can be explicitly seen when in the determinant we add to the $(j+n)$-th column the j-th column multiplied by $-\frac{y_{jj+1}}{s_{jj+1}}$, and similarly we add to the $(i+n)$-th row the i-th row multiplied by $-\frac{y_{ii+1}}{s_{ii+1}}$. Then one arrives at

$$\det \begin{pmatrix} (s_{ij}) & (y_{ji}) \\ (y_{ij}) & (z_{ij}) \end{pmatrix} = \det \begin{pmatrix} (s_{ij}) & \left(\frac{1}{s_{jj+1}} c_{j,ij+1}\right) \\ \left(\frac{1}{s_{ii+1}} c_{i,ji+1}\right) & \left(\frac{1}{s_{ij}s_{ii+1}s_{jj+1}} (c_{j,j+1\,i} c_{i,i+1\,j} - s_{ii+1}s_{jj+1}c_{ij})\right) \end{pmatrix}. \tag{3.19}$$

[8]Note that $c_{i,jk}$ can be expressed as a polynomial in Y's and Mandelstam variables via Eq. (3.9).

Here, the labels i and j only run through the values that correspond to the rows and columns present in the minor that we are considering.

There is one additional possibility due to the linear dependencies in $\mathcal{B}$: we can take the determinant of the $(2n-1)\times(2n-1)$ submatrix that is obtained by deleting, e.g., the first row and column. This is still gauge invariant because the gauge transformation with respect to the variables of the first field transforms the first row of the second block into a linear combination of the $n-1$ rows of the first block, and the determinant still vanishes. Expressed in terms of fields, such a building block corresponds to a term of the form

$$\delta^{[\mu_2\cdots\mu_{2n}]}_{\nu_2\cdots\nu_{2n}}\phi^{(1)}_{\mu_{n+1}}{}^{\nu_{n+1}}\partial_{\mu_2}\partial^{\nu_2}\phi^{(2)}_{\mu_{n+2}}{}^{\nu_{n+2}}\cdots\partial_{\mu_n}\partial^{\nu_n}\phi^{(n)}_{\mu_{2n}}{}^{\nu_{2n}}\,, \tag{3.20}$$

which is gauge invariant up to total derivatives. This Lovelock-type vertex can be generalised in a way, where one computes the determinant of the minor of $\mathcal{B}$ containing $n-1$ rows and columns from the first block and arbitrary number m of rows and columns from the second block, but these do not introduce new building blocks [9].

Note that also the Mandelstam variables s_{ij} are 1×1-minors. It is tempting to speculate that all gauge invariant local vertices $\mathcal{V}$ can be written as polynomials in the types of minors of $\mathcal{B}$ mentioned above. If this speculation is correct, then for a spin configuration $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n$ the lowest number of derivatives in a local vertex is $s_1+s_2+\cdots+s_{n-1}$.

4 Lower dimension: dealing with Schouten identities

In the previous section we have discussed the gauge-invariant vertices when we do not have to consider Schouten identities. When we go to lower dimensions, the ideal of relations is enlarged from $\mathcal{I}_D$ to $\mathcal{I}_D+\mathcal{I}_S$. Gauge-invariant vertex generating operators for large dimensions still define gauge-invariant operators in lower dimensions, but a priori, enlarging the ideal could have two effects: First, inequivalent vertices become equivalent, and second, new vertices arise that are gauge-invariant only up to the now larger set of equivalence relations. We will show in the following that the latter possibility does not lead to new equivalence classes of vertices, but that for all gauge-invariant vertex generating operators there are equivalent operators[10] which are gauge-invariant already without the use of Schouten identities.

To show this, start with a vertex generating operator $\mathcal{V}$ as a polynomial in s_{ij}, y_{ij}, z_{ij} that in d dimensions is gauge invariant,

$$D_k\mathcal{V}\in\mathcal{I}_D+\mathcal{I}_S\,. \tag{4.1}$$

In $\mathcal{V}$ we now express the variables z_{ij} and y_{ij} in terms of c_{ij}, Y_i^j and y_{ii+1},

$$\mathcal{V}=Q_{\mathcal{V}}(c_{ij},Y_i^j,y_{ii+1})\,, \tag{4.2}$$

[9] By adding total derivatives they can be transformed to a expression of the type (3.19) where the $n-1$ rows (columns) of the first block contain the m rows (columns) corresponding to those of the second block.

[10] as long as we can divide by Mandelstam variables

where $Q_{\mathcal{V}}$ is a polynomial in the given variables. We suppressed the dependence on Mandelstam variables which can also occur with negative powers. In these variables, the gauge variation D_k is written as a derivative with respect to y_{kk+1} as in Eq. (3.13), so we have

$$D_k \mathcal{V} = s_{kk+1} \frac{\partial}{\partial y_{kk+1}} Q_{\mathcal{V}}(c_{ij}, Y_i^j, y_{ii+1}) \in \mathcal{I}_D + \mathcal{I}_S \,. \tag{4.3}$$

When we expand Q in powers of y_{12},

$$Q_{\mathcal{V}}(c_{ij}, Y_i^j, y_{ii+1}) = \sum_{n=0}^{N} q_n(c_{ij}, Y_i^j, y_{23}, \ldots, y_{n1})(y_{12})^n \,, \tag{4.4}$$

we apply $(D_1)^N$ to the expression and obtain

$$N!(s_{12})^N q_N \in \mathcal{I}_D + \mathcal{I}_S \,. \tag{4.5}$$

When we allow us to divide by Mandelstam variables, we conclude that

$$q_N \in \frac{1}{(s_{12})^N} \left(\mathcal{I}_D + \mathcal{I}_S\right) \,. \tag{4.6}$$

Similar relations can be found for all other terms in the expansion in y_{12} and also in the other variables y_{ii+1}. Hence, we find that

$$\mathcal{V} - Q_{\mathcal{V}}(c_{ij}, Y_i^j, y_{ii+1})\big|_{y_{ii+1}=0} \in \frac{1}{\Delta} \left(\mathcal{I}_D + \mathcal{I}_S\right) \,, \tag{4.7}$$

where Δ is a product of powers of Mandelstam variables. Therefore, $\mathcal{V}$ is equivalent to an operator depending only on c_{ij} and Y_i^j which already defines a gauge invariant vertex operator without the need of Schouten identities.

We conclude that in all dimensions, vertex generating operators can be expressed in terms of the operators identified for large dimensions. The main task for lower dimension is therefore to work out explicitly the equivalences between such operators that are induced by Schouten identities. Here, the case of low dimensions, $d < n$, is special because many Schouten identities arise that reduce the independent equivalence classes considerably. This will be discussed in detail in the subsequent section. The identifications in the intermediate case will be stated in the discussion in Section 7.

In the remainder of this section we give a heuristic geometric argument why generically one does not expect new vertices to appear when we lower the dimension. In the sense of algebraic geometry, the ideal $\mathcal{I} = \mathcal{I}_S + \mathcal{I}_D$ defines a variety $V(\mathcal{I})$ as the zero-set of the polynomials contained in $\mathcal{I}$. If $\mathcal{I}$ was a prime ideal, we could think of the ring $\mathcal{R}/\mathcal{I}$ as the ring of polynomial functions on this variety. The gauge variations D_k define n vector fields on this variety, and we are looking for functions on $V(\mathcal{I})$ that are constant along the vector fields. When we enlarge the ideal to $\mathcal{I}' \supset \mathcal{I}$ by going from higher to lower dimensions where new Schouten identities occur, we concentrate on a subvariety $V(\mathcal{I}')$ of $V(\mathcal{I})$. Generically, if the vector fields do not degenerate on this subvariety, functions that are constant along D_k on $V(\mathcal{I}')$ can be lifted to constant functions on $V(\mathcal{I})$.

The above argument only gives a very rough picture, because apart from the possible degeneration of the vector fields, there are two subtleties: First, as it was said, the argument only applies to prime ideals, but the ideals that occur are usually not prime; secondly, there could be constant polynomials on $V(\mathcal{I}')$ whose lifts to $V(\mathcal{I})$ are not polynomial. Therefore, this picture can only be seen as a heuristic explanation why generically we do not expect new gauge invariant vertices to appear when we lower the dimension.

5 The case $n > d$

In this chapter, we find general restrictions for gauge invariant n-point vertices with $n > d$. Our result is a simple characterisation of equivalence classes $[\mathcal{V}] \in \mathcal{R}/\mathcal{I}$ for vertex generating operators. The results are summarised in Section 5.4.

As discussed in Section 2.2, we have the full set of Schouten identities at hand in order to find a simple representative $\mathcal{V}$ for a given vertex. This has the advantage that a lot of simplifications are possible. On the other hand, the structure of the set of Schouten identities is very complicated, and the number of independent Schouten identities in the polynomial ring is large. This problem was solved in [1] for $d = 3$ by observing that many Schouten identities become dependent when multiplied with an appropriate product Δ of Mandelstam variables. By multiplying a given vertex $\mathcal{V}$ with Δ, the remaining independent Schouten identities can be used to deduce strong constraints for the vertex $\mathcal{V}$ itself. Essentially, one can treat the Mandelstam variables in the manipulations like numbers and also divide by them. This concept can be also employed in higher dimensions.

Formally, to be able to divide by certain combinations of Mandelstam variables, we introduce the ring of fractions, $M^{-1}\mathcal{R}$. Here, M is a multiplicatively closed set containing all (finite) products of *non-zero minors of the submatrix $\mathcal{S}$ of $\mathcal{B}$* (see Eq. (2.8)): these are the expressions we want to divide by. More explicitly, let $\mathrm{Mi}(\mathcal{S})$ be the set of non-zero minors of $\mathcal{S}$ [11] and let $M = \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})]$ be the set of monomials in these minors. Then, the ring of fractions consists of formal quotients,

$$M^{-1}\mathcal{R} = \left\{ \frac{r}{\Delta} \,|\, \Delta \in M, \; r \in \mathcal{R} \right\}, \tag{5.1}$$

with the obvious rules for addition and multiplication. As also $1 \in M$, we can identify $\mathcal{R}$ via $r \mapsto \frac{r}{1}$ as subring of $M^{-1}\mathcal{R}$. The ideal $\mathcal{I} = \mathcal{I}_S + \mathcal{I}_D \subset \mathcal{R}$ can then be seen as a subset of $M^{-1}\mathcal{R}$ which generates an ideal $\mathcal{I}_M$ in $M^{-1}\mathcal{R}$. Using the embedding of $\mathcal{R}$ into $M^{-1}\mathcal{R}$, we have an induced map of the quotient rings,

$$i_M : \frac{\mathcal{R}}{\mathcal{I}} \to \frac{M^{-1}\mathcal{R}}{\mathcal{I}_M}\,. \tag{5.2}$$

As we will argue below, this map is injective, and therefore we can characterise equivalence classes of vertices uniquely by equivalence classes in the ring of fractions. The crucial

[11] First, non-zero minors of order one are just the Mandelstam variables s_{ij} with $i \neq j$. Secondly, all minors of order $2, 3, \ldots, d$ are generically non-zero — even when the equivalence relations in Eq. (2.7) are applied. Finally, all minors of order greater than d do vanish due to Schouten identities. Hence, $\mathrm{Mi}(\mathcal{S})$ consists of all (2×2), (3×3), $\ldots (d \times d)$ subdeterminants of $\mathcal{S}$ as well as the Mandelstam variables s_{ij} with $i \neq j$.

observation is now that in $M^{-1}\mathcal{R}$ many of the generators of the ideal become dependent, so that $\mathcal{I}_M$ has a simple set of generators. This section is structured as follows. In Section 5.1 we find a simple set of generators for $\mathcal{I}_M$. This enables us to find a convenient representative of $[\mathcal{V}]$ in the quotient of the ring of fractions in Section 5.2. We then impose gauge invariance in Section 5.3, which leads to strong restrictions on the vertex $\mathcal{V}$. In $d = 3$, these restrictions completely rule out independent vertices (as reported in [1]), in higher dimensions the restrictions are less strict, and we discuss them in Section 5.4. In order to make the structure of this paper better accessible, we collect some proofs in Section 5.5.

Before we proceed, we want to show that i_M is indeed injective. If $i_M([\mathcal{V}]) = [0]$, this means that $\mathcal{V} \in \mathcal{R} \cap \mathcal{I}_M$. Then, there is some $\Delta \in M$ such that $\Delta\mathcal{V} \in \mathcal{I}$. But if $\Delta\mathcal{V}$ defines a trivial vertex, then also $\mathcal{V}$ defines a trivial vertex, which can be seen in Fourier space, where the operators s_{ij} are numbers. In particular, the polynomial Δ is non-zero on the subvariety defined by $k_i^2 = 0$ and $\sum k_i = 0$. Now, if $\Delta\mathcal{V}$ defines a trivial vertex, then

$$\Delta\mathcal{V} \prod_i \widehat{\phi}_i(k_i, a_i)\Big|_{a_i=0} \tag{5.3}$$

vanishes on this subvariety. But Δ is non-vanishing almost everywhere. Hence, since $\mathcal{V}$ only depends polynomially on k_i^μ, $\mathcal{V}$ applied on the fields $\widehat{\phi}_i$ must vanish. So we conclude that $\mathcal{V} \approx 0$, hence $[\mathcal{V}] = [0]$.

5.1 A Minimal Generating Set of Schouten Identities

In this section, we find a simple set of generators for the ideal $\mathcal{I}_M$ in two steps. First, any Schouten identitiy multiplied with a certain $\Delta \in M = \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})]$ is an element in the ideal generated by the equivalence relations in Eq. (2.7) and all Schouten identities up to level 2 [12] (recall the notion of level introduced in the paragraph before Eq. (2.10)). In other words,

$$\text{there exists } \Delta \in \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})] : \quad \Delta \cdot \mathcal{I}_S \subset \sum_{k=0}^{2} I(k) + \mathcal{I}_D \,. \tag{5.4}$$

We show this in Section 5.5.1. This observation implies that in the ring of fractions where we are allowed to divide by Δ, we need far less generators for the Schouten identities.

In order to perform the second step, we introduce some more notations: First,

$$N_{ij} = \begin{pmatrix} s_{ij} & \cdots & s_{ij+d-1} \\ \vdots & \ddots & \vdots \\ s_{i+d-1\,j} & \cdots & s_{i+d-1j+d-1} \end{pmatrix} \tag{5.5}$$

is a $d \times d$ submatrix of $\mathcal{S}$, hence, $\det N_{ij} \in \mathrm{Mi}(\mathcal{S})$ and N_{ij} has full rank. Secondly, let $B_1(i,j)$ with $i,j = 1,\ldots,n$ be the following $(d+1)\times(d+1)$ submatrix of $\mathcal{B}$: It contains the rows and columns $i, i+1, \ldots, i+d-1$ (modulo n) as well as another row j and the

[12] This proof relies on the fact that $n > d$.

column $i+n$. Hence,

$$\det B_1(i,j) = \det \begin{pmatrix} & & 0 \\ & & y_{ii+1} \\ & N_{ii} & \vdots \\ & & y_{ii+d-1} \\ s_{ji} & \cdots \; s_{ji+d-1} & y_{ij} \end{pmatrix} \in I(1)\,^{13}. \tag{5.6}$$

Finally, let $B_2(i,j)$ with $i,j=1,\dots,n$ be the $(d+1)\times(d+1)$ submatrix of $\mathcal{B}$ containing the rows $i,i+1,\dots,i+d-1$ (modulo n) and $i+n$, as well as the columns $j,j+1,\dots,j+d-1$ (modulo n) and $j+n$. Hence,

$$\det B_2(i,j) = \det \begin{pmatrix} & & y_{ij} \\ & N_{ij} & \vdots \\ & & y_{ij+d-1} \\ y_{ij} & \cdots \; y_{ij+d-1} & z_{ij} \end{pmatrix} \in I(2)\,.$$

With these notations, we show in Section 5.5.2 that there exists $\Delta \in M = \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})]$ such that

$$\Delta \cdot (\mathcal{I}_S + \mathcal{I}_D) \subset I(0) + \left\langle \sum_{k=1}^{n} s_{ik}\,,\, \det B_1(i,j)\,,\, \det B_2(i,j) \,\middle|\, i,j=1,\dots,n \right\rangle . \tag{5.7}$$

Denote the family of generators of $I(0)$ by $(\det B_0(A))$, where A labels the different equivalence relations. Then, we can conclude that $\mathcal{I}_M$ is generated as

$$\mathcal{I}_M = \left\langle \sum_{k=1}^{n} s_{ik}\,,\, (\det B_0(A))\,,\, \det B_1(i,j)\,,\, \det B_2(i,j) \,\middle|\, i,j=1,\dots,n \right\rangle . \tag{5.8}$$

5.2 The choice of Representative

Now, let us investigate the relevant ideal $\mathcal{I}_M$ in order to choose a convenient representative for $\mathcal{V}$ in its equivalence class $[\mathcal{V}] \in M^{-1}\mathcal{R}/\mathcal{I}_M$.

We start by considering the Schouten identities $\det B_2(i,j) \in I(2)$, with $i \neq j$. Using a Laplace expansion along the last column, they read

$$0 \approx \det B_2(i,j) = z_{ij} \det N_{ij} + \text{terms that do not contain any } z_{kl}\,. \tag{5.9}$$

Since $\det N_{ij} \in \mathrm{Mi}(\mathcal{S})$, we can divide by it in $M^{-1}\mathcal{R}$, and express z_{ij} by an expression independent of any z_{kl}. Hence, we may choose the representative of $[\mathcal{V}]$ to be independent of z_{ij}. In the same way, the Schouten identities $\det B_1(i,j) \in I(1)$ take the form

$$0 \approx \det B_1(i,j) = y_{ij} \det N_{ii} + p(s_{ij}, y_{ii+1}, \dots, y_{ii+d-1})\,.$$

Here, the polynomial p only depends on $y_{ii+1},\dots,y_{ii+d-1}$ [14] and the Mandelstam variables. Using these Schouten identities, we can replace all of the operators y_{ij} in $\mathcal{V}$ except for $y_{ii+1},\dots,y_{ii+d-1}$.

[13] Note that this is true for all $j=1,\dots,n$. If for example $j=i$, then $\det B_1(i,j)=0 \in I(1)$.
[14] the indices are considered modulo n.

Finally, we perform a change of variables in $\mathcal{V}$. Similarly to Eq. (3.8) we introduce the combinations

$$Y_i^j = s_{ii+1}y_{ii+j} - s_{ii+j}y_{ii+1} \qquad \text{for } j = 2,\dots,d-1\,, \tag{5.10}$$

and replace all $y_{ii+2},\dots,y_{ii+d-1}$ in terms of these variables and y_{ii+1}. This can be done, because $s_{ii+1} \in M$ and we can divide by it in $M^{-1}\mathcal{R}$. We arrive at

$$\mathcal{V} \approx Q_{\mathcal{V}}(y_{ii+1}, Y_i^j, s_{ij})\,, \tag{5.11}$$

where $Q_{\mathcal{V}}$ is a polynomial in y_{ii+1}, Y_i^j and the Mandelstam variables (with coefficients that can contain inverse powers of elements in $\mathrm{Mi}(\mathcal{S})$). More explicitly, we can see $[\mathcal{V}]$ as an element in the quotient

$$[\mathcal{V}] \in \frac{M^{-1}\mathbb{R}\left[y_{ii+1}, Y_i^j, s_{ij}\right]}{\left\langle (\det B_0(A))\,,\, \sum_{j=1}^n s_{ij}\,,\, \det B_2(i,i) \,\middle|\, i = 1,\dots,n\right\rangle}\,. \tag{5.12}$$

There are several reasons to introduce the Y_i^j variables. First, they are the gauge invariant combinations of the y_{ij} variables — we have discussed this already in Section 3 and it will become important in Section 5.3. Secondly, the remaining level-2 Schouten identities $\det B_2(i,i)$ can be written solely in terms of the Y_i^j's and the Mandelstam variables, and they do not depend explicitly on y_{ii+1}. We show this in the rest of this section: For this purpose, consider

$$s_{ii+1}^2 \det B_2(i,i) = \det \begin{pmatrix} & & & & 0 \\ & & & & s_{ii+1}y_{ii+1} \\ & & N_{ii} & & \vdots \\ & & & & s_{ii+1}y_{ii+d-1} \\ 0 & s_{ii+1}y_{ii+1} & \cdots & s_{ii+1}y_{ii+d-1} & 0 \end{pmatrix}.$$

The determinant of the matrix does not change when y_{ii+1} times the first row is subtracted from the last one and y_{ii+1} times the first column is subtracted from the last one. Hence, using the definition of Y_i^j in Eq. (5.10), we find

$$s_{ii+1}^2 \det B_2(i,i) = \det \begin{pmatrix} & & & & & 0 \\ & & & & & 0 \\ & & N_{ii} & & & Y_i^2 \\ & & & & & \vdots \\ & & & & & Y_i^{d-1} \\ 0 & 0 & Y_i^2 & \cdots & Y_i^{d-1} & 0 \end{pmatrix} = -\sum_{j,k=2}^{d-1} Y_i^j\,(\mathrm{adj}N_{ii})_{jk}\,Y_i^k =: q_2^i(Y_i^j, s_{jk})\,.$$

Here, we used a Laplace expansion along the last row and column. The resulting polynomials q_2^i are quadratic in the Y_i^j variables with coefficients that still depend on the Mandelstam variables. However, the q_2^i's are independent of y_{ii+1}. We comment on their structure in Section 5.4. All in all, we can replace the generators $\det B_2(i)$ by q_2^i because we are allowed to divide by Mandelstam variables. Hence, we have the following result:

$$[\mathcal{V}] \in \frac{M^{-1}\mathbb{R}\left[y_{ii+1}, Y_i^j, s_{ij}\right]}{\left\langle (\det B_0(A))\,,\, \sum_{j=1}^n s_{ij}\,,\, q_2^i \,\middle|\, i=1,\ldots,n\right\rangle}\,.$$

5.3 General Restrictions from Gauge Invariance

With the results of the previous sections, we now show that the polynomial $Q_\mathcal{V}$ introduced in Eq. (5.11) can be chosen to be independent of y_{ii+1} if the operator $\mathcal{V}$ corresponds to a gauge invariant Lagrangian $\mathcal{L}_n$. From now on, we will always consider $\mathcal{V}$ as an element in the bigger ring of fractions.

Starting from Eq. (2.14) and using that the operators $a_k \cdot P_k$ commute with all Mandelstam variables, we find that a gauge invariant vertex $\mathcal{L}_n$ requires

$$\text{for all } k \in \{1,\ldots n\} \;:\quad [\mathcal{V}, a_k \cdot P_k] \in \mathcal{I}_M\,,$$

where $\mathcal{L}_n$ and $\mathcal{V}$ are related via Eq. (2.4). Now, the ideal $\mathcal{I}_M$ is gauge invariant, hence, it commutes with the operators $a_k \cdot P_k$. We deduce that the polynomial in Eq. (5.11) satisfies

$$[Q_\mathcal{V}, a_k \cdot P_k] \in \left\langle (\det B_0(A))\,,\, \sum_{j=1}^n s_{ij}\,,\, q_2^i(Y_i^j) \,\middle|\, i=1,\ldots,n\right\rangle\,.$$

With

$$[y_{ii+1}, a_k \cdot P_k] = \delta_{ik} s_{ii+1} \qquad \Rightarrow \qquad [Y_i^j, a_k \cdot P_k] = \delta_{ik}\left(s_{ii+j}s_{ii+1} - s_{ii+1}s_{ii+j}\right) = 0\,,$$

it follows immediately that

$$\text{for all } k=1,\ldots,n \;:\quad s_{kk+1}\partial_{y_{kk+1}} Q_\mathcal{V} \in \left\langle (\det B_0(A))\,,\, \sum_{j=1}^n s_{ij}\,,\, q_2^i(Y_i^j) \,\middle|\, i=1,\ldots,n\right\rangle\,.$$

But the generators of the ideal on the rhs do not depend on y_{ii+1}. We conclude that $Q_\mathcal{V}$ can be chosen to be independent of y_{ii+1}. More explicitly,

$$\mathcal{V} \approx Q_\mathcal{V}(Y_i^j\,,\, s_{ij})\,, \qquad [\mathcal{V}] \in \frac{M^{-1}\mathbb{R}\left[Y_i^k, s_{ij}\right]}{\left\langle (\det B_0(A))\,,\, \sum_{j=1}^n s_{ij}\,,\, q_2^i(Y_i^j) \mid i=1,\ldots,n\right\rangle}\,. \tag{5.13}$$

5.4 Restrictions for $\mathcal{V}$

Let us summarise our results. Eq. (5.13) states that each gauge invariant vertex $\mathcal{V}$ is equivalent to a vertex $Q_\mathcal{V}$, which does only depend on Mandelstam variables and Y_i^j. In particular, translating back to the vertex in terms of P_i^μ and A_i^μ operators, we have the following relation:

$$\iota_d(Y_i^j) = 2P_{i\mu}A_{i\nu}P^\mu_{[i+1}A^\nu_{i+j]} = 2P_{i[\mu}A_{i\nu]}P^\mu_{i+1}A^\nu_{i+j}\,.$$

Now, in the vertex generated by $Q_{\mathcal{V}}$, an index of the ith field is only generated by A_i^μ via a corresponding Y_i^j. Hence, the ith field enters the Lagrangian via a curvature term (each index of the field is antisymmetrised with an index of a partial derivative acting on it). We deduce that Q_V generates a Lagrangian, which can be written solely in terms of curvature terms.

The drawback of this analysis is that we loose locality on the way to this result. $Q_{\mathcal{V}}$ might not have a local form, since it can have inverse powers of Mandelstam variables. We can only say that for each gauge invariant vertex (generated by $\mathcal{V}$), there is a $\Delta \in M$ such that $\Delta\mathcal{V}$ can be written only in terms of curvatures.

Much stricter conditions can be found in three dimensions [1]. In that case, there is only one Y_i^j and the corresponding Schouten identity is $q_2^i = -s_{ii+1}^2(Y_i^2)^2$. Hence, $\det B_2(i,i) = (Y_i^2)^2 \approx 0$ and $Q_{\mathcal{V}}$ is only linear in Y_i. One can then deduce that $\mathcal{V}$ itself is at most linear in each of the operators A_i^μ, which means that the corresponding vertex $\mathcal{L}_n$ contains no higher-spin fields at all. This argument can also be obtained from the observation that in $d = 3$, there are simply no curvature terms for higher-spin fields.

5.5 Proofs

5.5.1 Proof of Eq. (5.4)

Let $\det M = 0$ be a Schouten identity that stems from a $(d+1) \times (d+1)$-submatrix M of $\mathcal{B}$ such that $\det M \notin \mathcal{I}_D$. Let r (s) be the number of level-0 rows (columns) of M. Furthermore, let Let $\bar{r}$ ($\bar{s}$) be the number of level-1 rows (columns) of M. Hence, $r + \bar{r} = s + \bar{s} = d + 1$. Without loss of generality, we assume $r \geqslant s$ [15]. Furthermore, let $\bar{s} \geqslant 2$, hence, the level of the Schouten identity $\det M = 0$ is $\bar{r} + \bar{s} \geqslant 2$. In particular, equality holds if and only if $\bar{s} = 2$ and $\bar{r} = 0$.

With the submatrix M given, we construct a $(d+2) \times (d+2)$-submatrix $\widetilde{M}$ of $\mathcal{B}$ as follows:

- Removing $(2n - d - 2)$ rows and columns from $\mathcal{B}$ results in $\widetilde{M}$.
- There is a level-0 row (which we call *Row*) and a level-0 column (called *Col*) in $\widetilde{M}$, such that removing *Row* and *Col* in $\widetilde{M}$ yields M. Hence, $\widetilde{M}$ contains $(r+1)$ level-0 rows and $(s+1)$ level-0 columns.
- The construction of $\widetilde{M}$ might not be unique, but is always possible. This can be seen as follows: In order to construct $\widetilde{M}$, there must be at least one level-0 row of $\mathcal{B}$ that is not part of M (otherwise, M would contain all level-0 rows of $\mathcal{B}$ which means that $\det M \in \mathcal{I}_D$ which contradicts our assumption). Furthermore, there are at least two level-0 columns of $\mathcal{B}$ that are not part of M, because $\bar{s} \geqslant 2$ and hence, $s \leqslant d - 1$ [16]. In particular, we can always choose $\widetilde{M}$ such that the intersection of *Row* and *Col* contains a non-zero Mandelstam variable.

The construction of the matrix $\widetilde{M}$ is visualised in Figure 1.

[15] If $r < s$, we choose M^T instead of M, which yields the same Schouten identity $\det M^T = \det M$. M^T is a submatrix of $\mathcal{B}$ as well because $\mathcal{B}$ is symmetric.

[16] $\mathcal{B}$ has more than d level-0 columns, since $n > d$.

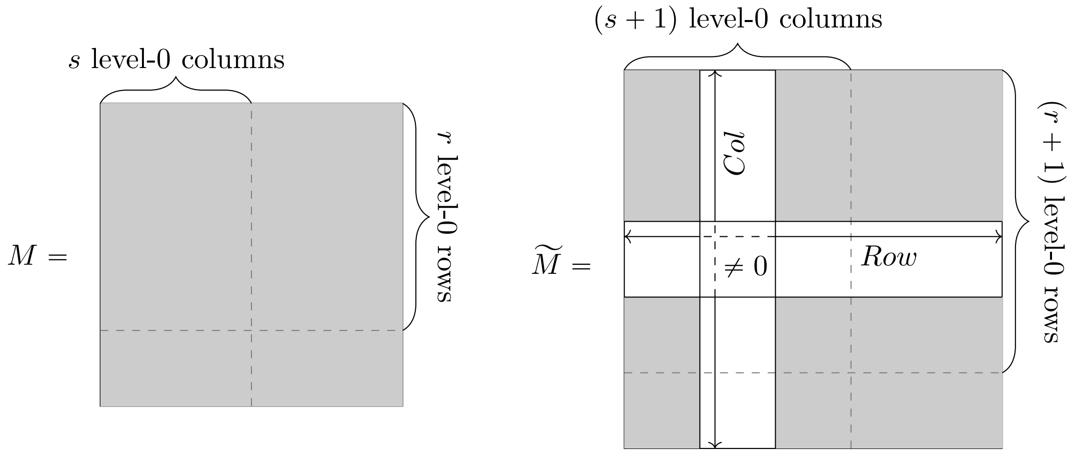

Figure 1. Visualisation of the matrices M and $\widetilde{M}$

For $\widetilde{M}$, Cramers rule states that

$$\mathbb{I}_{(d+2)\times(d+2)} \det \widetilde{M} - \widetilde{M} \cdot C^T = 0, \tag{5.14}$$

where $C = (c_{ij})$ denotes the cofactor matrix of $\widetilde{M} = (\widetilde{m}_{ij})$. In particluar, c_{ij} is (up to a factor of ± 1) equal to the determinant of the $(d+1)\times(d+1)$-submatrix obtained by deleting the i-th row and the j-th column from $\widetilde{M}$. In other words, c_{ij} is a $(d+1) \times (d+1)$-minor of $\mathcal{B}$, hence $c_{ij} \in \mathcal{I}_S$. In the following, we consider only part of Eq. (5.14):

$$\delta_{ji} \det \widetilde{M} - \sum_{k=1}^{s+1} \widetilde{m}_{jk} c_{ik} - \sum_{k=s+2}^{d+2} \widetilde{m}_{jk} c_{ik} = 0 \qquad i = 1, \ldots, s+1,\ j \in J\,. \tag{5.15}$$

Here, J is a (non-unique) subset of $s+1$ level-0 rows that contains *Row*. In other words,

$$J \subset \{1, \ldots, r+1\}\,, \qquad |J| = s+1\,, \qquad Row \in J\,.$$

Performing a Laplace expansion of $\det \widetilde{M}$ along the last column of $\widetilde{M}$ (which is of level 1 because of $\bar{s} \geqslant 2$), we deduce that $\det \widetilde{M}$ is a linear combination of Schouten identities of level $\bar{r} + \bar{s} - 1$ and $\bar{r} + \bar{s} - 2$. Hence,

$$\det \widetilde{M} \in I(\bar{r} + \bar{s} - 1) + I(\bar{r} + \bar{s} - 2)\,.$$

Furthermore, in the third term of Eq. (5.15), the Schouten identities c_{ik} with $k > s+1$ are of level $(\bar{r} + \bar{s} - 1)$. We therefore conclude that the middle term is an element in the following ideal:

$$\text{for all } i = 1, \ldots, s+1,\ j \in J\ :\quad \left(\sum_{k=1}^{s+1} \widetilde{m}_{jk} c_{ik}\right) \in I(\bar{r} + \bar{s} - 1) + I(\bar{r} + \bar{s} - 2)\,. \tag{5.16}$$

Now, denote by $N = (\widetilde{m}_{jk})$ (with $j \in J$ and $k \in \{1, \ldots, s+1\}$) the $(s+1) \times (s+1)$-submatrix of $\widetilde{M}$ that occurs in Eq. (5.16). It is also a submatrix of $\mathcal{S}$ as it only consists of level-0 rows and columns. Since $s+1 \leqslant d$, we deduce that $\det N \in \mathrm{Mi}(\mathcal{S})$ [17]. In particular, $\det N \neq 0$ and by inverting N in Eq. (5.16) using Cramers rule, we find

$$\text{for all } j \in J,\, k \in \{1,\ldots,s+1\}\ :\quad \det N \cdot c_{jk} \in I(\bar r+\bar s-1)+I(\bar r+\bar s-2)\,.$$

Finally, setting $j = \mathit{Row}$ and $k = \mathit{Col}$, we have $c_{jk} = \det M$ — which corresponds to the Schouten identity of level $(\bar r+\bar s)$ we started with. It follows directly that for all $\det M \in I(\bar r+\bar s)$, either $\det M \in \mathcal{I}_D$ or

$$\text{there exists } \det N \in \mathrm{Mi}(\mathcal{S})\ :\quad \det N \cdot \det M \in I(\bar r+\bar s-1)+I(\bar r+\bar s-2)\,. \tag{5.17}$$

In other words,

$$\text{there exists } \Delta \in \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})]\ :\quad \Delta\cdot I(\bar r+\bar s) \subset I(\bar r+\bar s-1)+I(\bar r+\bar s-2)+\mathcal{I}_D$$

and a recursion over $\bar r$ and $\bar s$ proves the general statement in Eq. (5.4).

5.5.2 Proof of Eq. (5.7)

We prove Eq. (5.7) in three steps. It directly follows from Eq. (5.4), as well as Eqs. (5.18, 5.22 and 5.26).

Part 1: First of all, we show that

$$\text{there exists } \Delta \in \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})]\ :\quad \Delta I(2) \subset I(0)+I(1)+\big\langle \det B_2(i,j)\,\big|\, i,j=1,\ldots,n\big\rangle\,. \tag{5.18}$$

Therefore, consider an arbitrary level-2 Schouten identity $\det M \approx 0$, with M being a $(d+1)\times(d+1)$-submatrix of $\mathcal{B}$. As explained in the previous section, we may assume that M has either one ($\bar r = 1$, $\bar s = 1$) or two level-1 columns ($\bar r = 0$, $\bar s = 2$). In the latter case, the proof of the previous section goes through and Eq. (5.17) is satisfied. Hence, we only need to consider the other case ($\bar r = \bar s = 1$). In particular, we show that if M contains the level-1 row $i+n$ and the level-1 column $j+n$ of $\mathcal{B}$, then:

$$\text{there exists } \Delta \in \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})]\ :\quad \Delta \det M \in I(1)+\langle \det B_2(i,j)\rangle\,, \tag{5.19}$$

which implies Eq. (5.18).

We prove Eq. (5.19) by induction. Therefore, fix i and j. Let $I_2(K,L)$ be the ideal generated by all level-2 Schouten identities $\det M \approx 0$, such that the $(d+1)\times(d+1)$-submatrix M of $\mathcal{B}$ has the following properties:

i) M contains the level-1 row $i+n$ and the level-1 column $j+n$ of $\mathcal{B}$.

ii) K rows (L columns) of M stem from the rows $i,\ldots,i+d-1$ (columns $j,\ldots,j+d-1$) (modulo n) of $\mathcal{B}$.

[17] In the case that $s=0$, N is just a Mandelstam variable. But within the construction of $\widetilde M$, we chose *Row* and *Col* such that its intersection (which is N in that case) is non-zero. Therefore, $\det N = N \neq 0$.

Hence, $I_2(d,d) = \langle \det B_2(i,j) \rangle$ and Eq. (5.19) is a recursive consequence of the following two propositions:

$$\text{if } K < d\,, \quad \text{there exists } \Delta \in \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})] \;:\quad \Delta \cdot I_2(K,L) \subset I(1) + I_2(K+1,L)\,, \tag{5.20}$$
$$\text{if } L < d\,, \quad \text{there exists } \Delta \in \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})] \;:\quad \Delta \cdot I_2(K,L) \subset I(1) + I_2(K,L+1)\,. \tag{5.21}$$

Here, we present the proof of Eq. (5.21). Eq. (5.20) can be shown in the same way, except that the roles of rows and columns are interchanged. Therefore, we start with a $(d+1) \times (d+1)$-submatrix M of $\mathcal{B}$, such that $\det M \in I_2(K,L)$ with $L < d$. For M given, we construct a $(d+2) \times (d+2)$-matrix $\widetilde{M}$ as follows:

- Removing the first row from $\widetilde{M}$ yields a $(d+1) \times (d+2)$-submatrix $\widehat{M}$ of $\mathcal{B}$.
- There is a unique $k_0 \in \{1, \ldots, d+1\}$, such that removing the k_0-th column from $\widehat{M}$ yields M. We construct $\widetilde{M}$ such that the k_0-th column stems from one of the columns $j, \ldots, j+d-1$ (modulo n) of $\mathcal{B}$, which is possible because $L < d$.
- Note that $\widehat{M}$ has $d+1$ level-0 columns. Hence, at least one of those cannot stem from one of the columns $j, \ldots, j+d-1$ (modulo n) of $\mathcal{B}$. Let us agree that at least the l_0-th column has this property. Obviously, $l_0 \neq k_0$.
- The first two rows of $\widetilde{M}$ coincide, hence, $\det \widetilde{M} = 0$.

Now, Cramers rule states that $\widetilde{M} C^T = 0$, where $C = (c_{kl})$ is the cofactor matrix of $\widetilde{M} = (\widetilde{m}_{kl})$. The first column of this matrix equation reads

$$\sum_{l=1}^{d+2} \widetilde{m}_{kl} c_{1l} = 0\,.$$

Note that up to a factor of ± 1, c_{1l} is the determinant of the $(d+1) \times (d+1)$-matrix obtained by removing the first row and the lth column from $\widetilde{M}$. Hence, $c_{1l} \in I(2)$ for $l \leqslant d+1$ and $c_{1\,d+2} \in I(1)$. In particular, $c_{1k_0} \propto \det M \in I_2(K,L)$ and $c_{1l_0} \propto \det M_{l_0 \to k_0} \in I_2(K, L+1)$, where the matrix $M_{l_0 \to k_0}$ differs from M by only one column. Indeed, it contains the k_0-th column of $\widetilde{M}$ instead of the l_0-th. We deduce that

$$\sum_{1 \leqslant l \leqslant d+1}^{l \neq l_0} \widetilde{m}_{kl} c_{1l} \in I(1) + I_2(K, L+1)\,.$$

Now, consider only the rows $2, \ldots, d+1$ of that relation. The matrix $N = (\widetilde{m}_{kl})$ with $k \in \{2, \ldots, d+1\}$ and $l \in \{1, \ldots, d+1\} \backslash \{l_0\}$ is a $(d \times d)$-submatrix of $\mathcal{S}$ and can hence, be inverted using Cramers rule. We find that

$$\det N \cdot c_{1l} \in I(1) + I_2(K, L+1)$$

and setting $l = k_0$ finally proves Eq. (5.21).

Part 2: In a second step, we show that

$$\text{there exists } \Delta \in \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})] \quad : \quad \Delta I(1) \subset I(0) + \langle \det B_1(i,j) \,|\, i,j = 1,\ldots,n \rangle \,. \tag{5.22}$$

The proof is similar to the previous one. Let $\det M \approx 0$ be an arbitrary level-1 Schouten identity, where M is a $(d+1)\times(d+1)$-submatrix of $\mathcal{B}$. As explained in the previous section, we may assume that M has exactly one level-1 column ($\bar{r} = 0, \bar{s} = 1$). In particular, we show that if M contains the level-1 column $i+n$ of $\mathcal{B}$, then:

$$\text{there exists } \Delta \in \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})] : \quad \Delta \det M \in I(0) + \langle \det B_1(i,j) \,|\, j = 1,\ldots,n \rangle \,, \tag{5.23}$$

which implies Eq. (5.22).

Again, we prove Eq. (5.23) by induction. Therefore, fix i. Let $I_1(K,L)$ be the ideal generated by all level-1 Schouten identities $\det M \approx 0$, such that the $(d+1)\times(d+1)$-submatrix M of $\mathcal{B}$ has the following properties:

i) M contains the level-1 column $i+n$ of $\mathcal{B}$.

ii) K rows (L columns) of M stem from the rows (columns) $i,\ldots,i+d-1$ (modulo n) of $\mathcal{B}$.

Hence, $M(d,d) = \langle \det B_1(i,j) \,|\, j = 1,\ldots,n \rangle$ and Eq. (5.23) is a recursive consequence of the following two propositions:

$$\text{if } K < d\,, \quad \text{there exists } \Delta \in \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})] : \quad \Delta \cdot I_1(K,L) \subset I(0) + I_1(K+1,L)\,, \tag{5.24}$$

$$\text{if } L < d\,, \quad \text{there exists } \Delta \in \mathrm{Mon}[\mathrm{Mi}(\mathcal{S})] : \quad \Delta \cdot I_1(K,L) \subset I(0) + I_1(K,L+1)\,. \tag{5.25}$$

Here, we give the proof of Eq. (5.25), Eq. (5.24) follows analogously. For a given $(d+1)\times(d+1)$-submatrix M of $\mathcal{B}$, such that $\det M \in I_1(K,L)$ with $L < d$, we construct a $(d+2)\times(d+2)$-matrix $\widetilde{M}$ as follows:

- Removing the first row from $\widetilde{M}$ yields a $(d+1)\times(d+2)$-submatrix $\widehat{M}$ of $\mathcal{B}$.
- There is a unique $k_0 \in \{1,\ldots,d+1\}$, such that removing the k_0-th column from $\widehat{M}$ yields M. Again, we construct $\widetilde{M}$ such that the k_0-th column stems from one of the columns $i,\ldots,i+d-1$ (modulo n) of $\mathcal{B}$, which is possible because $L < d$.
- Note that $\widehat{M}$ has $d+1$ level-0 columns. Hence, at least one of those (say, the l_0-th) cannot stem from one of the columns $i,\ldots,i+d-1$ (modulo n) of $\mathcal{B}$. Again, $l_0 \neq k_0$.
- The first two rows of $\widetilde{M}$ coincide, hence, $\det \widetilde{M} = 0$.

Now, Cramers rule states that $\widetilde{M}C^T = 0$, where $C = (c_{kl})$ is the cofactor matrix of $\widetilde{M} = (\widetilde{m}_{kl})$. Considering the first column of this matrix equation, we have

$$\sum_{l=1}^{d+2} \widetilde{m}_{kl} c_{1l} = 0\,.$$

Up to a factor of ± 1, c_{1l} is the determinant of the $(d+1) \times (d+1)$-matrix obtained by removing the first row and the lth column from $\widetilde{M}$. Hence, $c_{1l} \in I(1)$ for $l \leqslant d+1$ and $c_{1\,d+2} \in I(0)$. In particular, $c_{1k_0} \propto \det M \in I_1(K, L)$ and $c_{1l_0} \propto \det M_{l_0 \to k_0} \in I_1(K, L+1)$, where the matrix $M_{l_0 \to k_0}$ differs from M by only one column (it contains the k_0-th column of $\widetilde{M}$ instead of the l_0-th). We deduce that

$$\sum_{1 \leqslant l \leqslant d+1}^{l \neq l_0} \widetilde{m}_{kl} c_{1l} \in I(0) + I_1(K, L+1)\,.$$

Again, we only consider the rows $2, \ldots, d+1$ of that relation. The matrix $N = (\widetilde{m}_{kl})$ with $k \in \{2, \ldots, d+1\}$ and $l \in \{1, \ldots, d+1\} \backslash \{l_0\}$ is a $(d \times d)$-submatrix of $\mathcal{S}$ and can be inverted using Cramers rule. Finally,

$$\det N \cdot c_{1l} \in I(0) + I_1(K, L+1)$$

and setting $l = k_0$ proves Eq. (5.25).

Part 3: Finally, we prove that for any $i \in \{1, \ldots, n\}$,

$$\det N_{ii} \sum_{j=1}^{n} y_{ij} \in \left\langle \sum_{j=1}^{n} s_{ij} \right\rangle + \langle \det B_1(i,j) \rangle\,, \tag{5.26}$$

where $N_{ii} \in \mathrm{Mi}(\mathcal{S})$ is defined in Eq. (5.5).

Fix $i \in \{1, \ldots, n\}$. Then, for any $j \in \{1, \ldots, n\}$, let $N_{ii}(k \to j)$ be the matrix N_{ii}, where the $(k+1)$st row is replaced by $\begin{pmatrix} s_{ji} & s_{ji+1} & \cdots & s_{ji+d-1} \end{pmatrix}$. In particular,

$$\sum_{j=1}^{n} \det N_{ii}(k \to j) \in \left\langle \sum_{j=1}^{n} s_{ij} \right\rangle\,, \tag{5.27}$$

because the determinant of $N_{ii}(k \to j)$ is linear (especially in the $(k+1)$st row).

Now, a Laplace expansion of Eq. (5.6) with respect to the last column results in

$$\det B_1(i,j) = y_{ij} \det N_{ii} - \sum_{k=0}^{d-1} y_{ii+1} \det N_{ii}(k \to j)\,,$$

which holds for all $j \in \{1, \ldots, n\}$. In particular,

$$\det N_{ii} \sum_{j=1}^{n} y_{ij} = \sum_{j=1}^{n} \det B_1(i,j) + \sum_{k=0}^{d-1} y_{ii+1} \sum_{j=1}^{n} \det N_{ii}(k \to j)\,,$$

which, taking Eq. (5.27) into account, proves Eq. (5.26).

6 Parity-Odd Vertices

So far, we only discussed parity-even vertices, i.e. terms in the Lagrangian which do not involve the epsilon tensor $\epsilon_{\mu_1 \cdots \mu_d}$. However, the discussion of the previous sections can simply be generalised also for parity-odd vertices.

First of all, the most general form of a parity-odd vertex is given by Eq. (2.4) but with $\mathcal{V}$ replaced by

$$\tilde{\mathcal{V}} = \sum Q_{I_1\cdots I_d}\tilde{\mathcal{V}}^{I_1\cdots I_d}\,, \tag{6.1}$$

where $\tilde{\mathcal{V}}^{I_1\cdots I_d} \in \mathbb{R}[y_{ij}, z_{ij}|_{i\leqslant j}, s_{ij}|_{i\leqslant j}]$ contains the parity-even contractions [18] and

$$Q_{I_1\cdots I_d} = \epsilon_{\mu_1\cdots\mu_d} b_{I_1}^{\mu_1}\cdots b_{I_d}^{\mu_d}$$

is totally antisymmetric in its indices ($I_k = 1,\ldots,2n$). The derivative operators b_I were introduced in Section 2.2, right before Eq. (2.8). Note that for $i = 1,\ldots,n$, we have $b_i = P_i$ and $b_{i+n} = A_i$. The structure of the gauge-invariant parity-odd vertices depends on the dimension:

- For $d \geqslant 2n$, there are no parity-odd $n-$point vertex operators, because $Q_{I_1\cdots I_d} = 0$ (the vector b has only $2n-1$ independent entries up to total derivatives).

- For $d = 2n-1$, there is a unique elementary parity-odd vertex operator

$$Q^{LL}_{1\cdots 2n-1} = \epsilon_{\mu_1\ldots\mu_{2n-1}} P_1^{\mu_1}\ldots P_{n-1}^{\mu_{n-1}}\, A_1^{\mu_n}\ldots A_n^{\mu_{2n-1}}\,, \tag{6.2}$$

which is gauge invariant up to total derivatives and squares to the Lovelock operator (3.20). This covers also the case of $n = 3$ and $d = 5$, consistent with [5].

- In the case $n > d$, we make use again of the fact that we consider $[\mathcal{V}]$ in the ring of fractions. The crucial point is that the general form of an elementary building block $Q_{I_1\cdots I_d}$ of parity-odd vertices can be highly simplified, when it is multiplied with the upper-left $d\times d$ submatrix of $\mathcal{S}$. Denote this matrix by S_d. Its determinant,

$$\det S_d = \frac{1}{d!}\epsilon_{\mu_1\cdots\mu_d}\epsilon_{\nu_1\cdots\nu_d} b_1^{\mu_1}\cdots b_d^{\mu_d} b_1^{\nu_1}\cdots b_d^{\nu_d}\,,$$

is a non-zero minor of $\mathcal{B}$, hence, $\det S_d \in \mathrm{Mi}(\mathcal{S})$ and we conclude that

$$\det S_d\cdot Q_{I_1\cdots I_d} = (\mathcal{B}_{1I_1}\cdots\mathcal{B}_{dI_d})\big|_{[I_1\cdots I_d]}\cdot Q_{1\cdots d}\,.$$

In other words, for any parity-odd vertex in the Lagrangian given by the vertex generating operator $\tilde{\mathcal{V}}$ in Eq. (6.1), we find

$$\det S_d\cdot\tilde{\mathcal{V}} = Q_{1\cdots d}\cdot\mathcal{V}\,, \tag{6.3}$$

where $\mathcal{V} \in \mathbb{R}[y_{ij}, z_{ij}|_{i\leqslant j}, s_{ij}|_{i\leqslant j}]$ as in the parity-even case.

Now, since we work in the ring of fractions, we can divide by $\det S_d \in \mathrm{Mi}(\mathcal{S})$. Furthermore, $Q_{1\cdots d}$ is gauge invariant:

$$[Q_{1\cdots d}, a_k\cdot P_k] = 0\,.$$

Hence, along the same lines as in Section 5, we find that

$$\tilde{\mathcal{V}} \approx Q_{1\cdots d}\cdot Q_{\mathcal{V}}(Y_i^j, s_{ij})\,. \tag{6.4}$$

[18] We discussed these in the previous sections, where they were called $\mathcal{V}$.

- For $n \leqslant d \leqslant 2n-2$, one has to be more careful, taking into account the Schouten identities, as was done for cubic vertices in $d=4$ [7] and $d=3$ [9]. The difference here is in the possibility to use negative powers of Mandelstam variables in the case of $n \geqslant 4$. The idea is to use Schouten identities to bring any parity-odd vertex structure to a form, where one has a gauge-invariant "square-root of a Horndeski-type operator":

$$Q^{(n-1)}_{I_1 \cdots I_d} = \epsilon_{\mu_1 \ldots \mu_d} P^{\mu_1}_{I_1} \ldots P^{\mu_{n-1}}_{I_{n-1}} A^{\mu_n}_{I_n} \ldots A^{\mu_d}_{I_d}\,, \qquad I_k \leqslant n\,. \tag{6.5}$$

This procedure can remove redundancies present due to Schouten identities, but the uniquely fixed form of the vertex may involve negative powers of the variables[19] s_{ij}. This scheme was instrumental in deriving parity-odd cubic vertices in four [7] and three dimensions [9]. We just need to show here, that any parity-odd vertex operator that involves less than $n-1$ derivatives, can be related to another operator with more derivatives by Schouten identities. We prove this in the rest of this section.

Let us take a generic operator of this type,

$$Q^{(k)}_{I_1 \ldots I_k\, J_1 \ldots J_{d-k}} = \epsilon_{\mu_1 \ldots \mu_d} P^{\mu_1}_{I_1} \ldots P^{\mu_k}_{I_k} A^{\mu_{k+1}}_{J_1} \ldots A^{\mu_d}_{J_{d-k}}\,, \tag{6.6}$$

with $k \leqslant n-2$, $1 \leqslant I_1 < I_2 < \cdots < I_k \leqslant n$ and $1 \leqslant J_1 < J_2 < \cdots < J_{d-k+1} \leqslant n$. We can form a Schouten identitiy

$$0 = \epsilon_{\mu_1 \ldots \mu_d} P^{\mu_1}_{I_1} \ldots P^{\mu_k}_{I_k} A^{[\mu_{k+1}}_{J_1} \ldots A^{\mu_d}_{J_{d-k}} P^{\nu_1}_{I_{k+1}} \ldots P^{\nu_{k+1}]}_{I_{2k+1}} P^{K_1}_{\nu_1} \ldots P^{K_{k+1}}_{\nu_{k+1}}\,, \tag{6.7}$$

or, schematically,

$$0 = Q^{(k)} \det S^{(k+1)} + O(Q^{(j>k)})\,, \tag{6.8}$$

where $\det S^{(k+1)} \in \mathrm{Mi}(\mathcal{S})$, while $O(Q^{(j>k)})$ refers to all the terms that contain parity-odd opeartors involving more than k derivatives P_I. Using the equation Eq. (6.8), one can replace the operator $Q^{(k)}$ with expressions that contain operators $Q^{(j)}$ with $j>k$, but also inverse powers of $\det S^{(k+1)}$ (which are non-zero). This is possible as long as $k \leqslant n-2$, therefore the procedure saturates when all the parity-odd operators are brought to the form (6.5).

7 Discussion

In this work, we complete the classification of independent vertices of arbitrary order $n \geqslant 3$ for massless bosonic fields with arbitrary spin in arbitrary space-time dimensions $d \geqslant 3$.

We briefly summarise the results:

- For dimensions $d \geqslant 2n-1$ there are no Schouten identities. After reducing to the independent Mandelstam variables, we find that all gauge invariant operators can be expressed as polynomials in the gauge-invariant combinations c_{ij} and Y_i^j,

$$\mathcal{V} \in M_1^{-1}\mathbb{R}[s_{ij}, c_{ij}, Y_i^j]\,, \tag{7.1}$$

[19] In the special case of cubic vertices in $d=4$ [7], one even gets negative powers of y_{ij}'s, which, however can be removed by inverting this procedure after solving for the vertex operator.

where M_1 is the set of all products of Mandelstam variables s_{ij} $(i \neq j)$. The invariant combinations Y_i^j are labelled by $i = 1, \ldots, n$ and $j = 2, \ldots, n-2$.

- For dimensions $d < n$ we have the full set of Schouten identities at our disposal. All gauge invariant operators are already generated by the Y_i^j where $i = 1, \ldots, n$ and $j = 2, \ldots, d-1$. All remaining relations are generated by level-0 Schouten identities and specific quadratic expressions q_2^i in the variables Y_i^j,

$$[\mathcal{V}] \in \frac{M^{-1}\mathbb{R}[s_{ij}, Y_i^j]}{\langle (\det B_0(A)), q_2^i \rangle}\,, \tag{7.2}$$

  where again we reduced to the independent Mandelstam variables.

- In the intermediate case $(2n-1 > d \geqslant n)$, we have Schouten identities, but because $d \geqslant n$ the non-trivial Schouten identities involve at least $(d-n)+2 \geqslant 2$ level-1 rows and columns. By an argument analogous to the one leading to Eq. (5.4) one can show that in the ring of fractions all Schouten identities are generated by those that contain $n-1$ level-0 rows and columns and $(d-n)+2$ rows and columns of level-1. Let us denote them by $\det B_{2(d-n)+4}(A)$, where A labels the possible choices of the level-1 rows and columns. These generators are all gauge-invariant (up to total derivatives), and hence we can express them in terms of the invariant combinations c_{ij} and Y_i^j as in Section 3. Then the gauge invariant vertices are classified by equivalence classes

$$[\mathcal{V}] \in \frac{M^{-1}\mathbb{R}[s_{ij}, c_{ij}, Y_i^j]}{\langle \det B_{2(d-n)+4}(A) \rangle}\,. \tag{7.3}$$

An interesting question is whether the higher order vertices can induce deformations of gauge transformations for the fields involved. Deformations arise when the gauge variation is non-trivial before imposing the equations of motion. Terms in the variation that contain the equations of motion have to be compensated by a non-trivial $\delta^{(n-2)}$ in Eq. (1.3). We have found that in all dimensions, as long as we are allowed to divide by Mandelstam variables, the independent gauge-invariant vertices can be expressed in terms of the combinations c_{ij} and $Y_i^j = c_{i,i+j i+1}$, but these — as defined in Eq. (3.3) and Eq. (3.4) — are manifestly gauge-invariant without need of equations of motion. This strongly suggests that the vertex does not induce a deformation. Strictly speaking we can only conclude that $\Delta\mathcal{V}$ for an appropriate product Δ of Mandelstam variables does not induce any deformation. However, in Fourier space Δ is simply a (generically non-zero) number and should not change the general structure of deformations, hence we do not expect that $\mathcal{V}$ itself can induce a deformation.

To recapitulate, as soon as we allow for dividing by Mandelstam variables (and hence, we loose manifest locality), the independent vertices of order $n \geqslant 4$ can be all written in terms of linearised curvatures of HS fields. Therefore they are manifestly gauge invariant with respect to linearised gauge transformations and do not introduce deformations for the latter. On the other hand, if such deformations of the gauge transformations, induced from cubic vertices, exist in the theory, then these vertices will be completed by further non-linear terms. This is similar to higher-curvature terms in Einstein Gravity, whose non-linear structure is gauge invariant with respect to full diffeomorphisms, induced from the Einstein-Hilbert cubic vertex. Such non-linear completions may make use of non-linear generalisation of de Wit-Freedman curvatures [65], which are not known in metric-like formulation (see, however, [66]). In the frame formulation, these vertices would correspond to structures that make use of Weyl tensors and their descendants (zero form sector of Vasiliev system). In the light of our findings here, the three dimensional results of [1] can be interpreted as particular case of the general dimensional results: all the independent vertices are given through linearised curvatures, which are on-shell trivial in $d = 3$.

Even though the classification is done for Minkowski spaces, we expect the vertices found here to deform smoothly to $(A)dS$ space-times as it happens for cubic vertices. Indeed, the existence of $(A)dS$ extensions for linearised de Wit-Freedman curvatures for HS fields [67] allows to straightforwardly lift vertices given through curvatures to $(A)dS_d$. Same is true for the operators (3.20) and (6.5), where one can simply replace derivatives with $(A)dS_d$ covariant ones.

Our results should have a direct analogue for correlation functions of conserved tensors in $d - 1$ dimensional conformal field theories, which can be classified with similar methods [68]. For $n = 3$ there is a precise match between independent vertices and three-point functions [5–7, 38, 43, 44, 68–70]. It would be interesting to compare our findings for $n \geqslant 4$ with the group theoretic results of [71].

Next, we would like to note that there is another interpretation of the equation Eq. (2.12) which we solved here. One can think of Eq. (2.12) as a Ward identity for an n-point amplitude computed in a theory of interacting HS fields. It is clear from our discussion, that the building blocks of the amplitudes are given through c_{ij}, $Y_i^j = c_{i,i+j i+1}$ and Mandelstam variables, including negative powers of the latter. They correspond to arbitrary tensor contractions of linearised curvatures [65] of HS gauge fields and their derivatives. These linear de Wit-Freedman curvatures (or their traceless part: the Weyl tensors) and their derivatives are the only on-shell non-zero gauge invariants with respect to the linearised gauge transformations. It is natural that the amplitudes for $n \geqslant 4$ should be given through gauge invariant quantities, as they are observable.

The amplitude interpretation might be less motivated in three dimensions, since there are no propagating HS massless particles in three dimensions. As proved in [1], there are no candidate invariants for amplitudes with such fields either for $d = 3$. There is one difference between amplitudes and vertices though — the latter are supposed to be local, while the former do not have to. Given that one can always multiply the candidate invariant vertices (amplitudes) by a non-vanishing function of Mandelstam variables, one can show that relaxing locality would not help to get non-zero amplitudes in $d = 3$. There is an interesting conclusion to be made here: since the amplitude is a sum of exchanges[20] and contact vertices, vanishing amplitudes imply that the exchanges and contact vertices

[20]The exchange is again a notion that is defined when there are particles to exchange, but this should not affect our argument, given that a propagator for massless HS fields can be formally defined in three dimensions. See, e.g., [72, 73]. We thank Shailesh Lal for a discussion about this point.

should cancel each other. This is only possible if the non-local parts of the exchanges sum up to zero, which should be specific to three dimensions and is presumably due to the special structure of vertices and Schouten identities present only in three dimensions. We plan to study the Lagrangian formulation of metric-like non-linear HS theories with(out) matter in the near future to expose these special properties of HS gravities in $d = 3$.

Note added We learned from Euihun Joung and Massimo Taronna about their preprint with related results [74], which will appear on arxiv simultaneously.

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References

[1] S. Fredenhagen, O. Krüger and K. Mkrtchyan, Phys. Rev. Lett. **123** (2019), 131601; `arXiv:1905.00093`.

[2] A. K. H. Bengtsson, I. Bengtsson and N. Linden, *"Interacting Higher Spin Gauge Fields on the Light Front,"* *Class. Quant. Grav.* **4** (1987) 1333.

[3] R. R. Metsaev, *"Poincare invariant dynamics of massless higher spins: Fourth order analysis on mass shell,"* *Mod. Phys. Lett. A* **6** (1991) 359.

[4] R. R. Metsaev, *"S matrix approach to massless higher spins theory. 2: The Case of internal symmetry,"* *Mod. Phys. Lett. A* **6** (1991) 2411.

[5] R. R. Metsaev, *"Cubic interaction vertices of massive and massless higher spin fields,"* *Nucl. Phys. B* **759** (2006) 147; `hep-th/0512342`.

[6] R. Manvelyan, K. Mkrtchyan and W. Rühl, *"General trilinear interaction for arbitrary even higher spin gauge fields,"* *Nucl. Phys. B* **836** (2010) 204; `arXiv:1003.2877`.

[7] E. Conde, E. Joung and K. Mkrtchyan, *"Spinor-Helicity Three-Point Amplitudes from Local Cubic Interactions,"* *JHEP* **1608** (2016) 040; `arXiv:1605.07402`.

[8] K. Mkrtchyan, *"Cubic interactions of massless bosonic fields in three dimensions,"* *Phys. Rev. Lett.* **120** (2018), 221601; `arXiv:1712.10003`.

[9] P. Kessel and K. Mkrtchyan, *"Cubic interactions of massless bosonic fields in three dimensions II: Parity-odd and Chern-Simons vertices,"* *Phys. Rev. D* **97** (2018) no.10, 106021; `arXiv:1803.02737`.

[10] A. Sagnotti and M. Taronna, *"String Lessons for Higher-Spin Interactions,"* *Nucl. Phys. B* **842** (2011) 299; `arXiv:1006.5242`.

[11] A. Fotopoulos and M. Tsulaia, *"On the Tensionless Limit of String theory, Off - Shell Higher Spin Interaction Vertices and BCFW Recursion Relations,"* *JHEP* **1011** (2010) 086; arXiv:1009.0727.

[12] R. Manvelyan, K. Mkrtchyan and W. Rühl, *"A Generating function for the cubic interactions of higher spin fields,"* *Phys. Lett. B* **696** (2011) 410; arXiv:1009.1054.

[13] K. Mkrtchyan, *"On generating functions of Higher Spin cubic interactions,"* *Phys. Atom. Nucl.* **75** (2012) 1264; arXiv:1101.5643.

[14] A. Sagnotti, *"Notes on Strings and Higher Spins,"* *J. Phys. A* **46** (2013) 214006; arXiv:1112.4285.

[15] R. R. Metsaev, *"BRST-BV approach to cubic interaction vertices for massive and massless higher-spin fields,"* *Phys. Lett. B* **720** (2013) 237; arXiv:1205.3131.

[16] C. Fronsdal, *"Massless Fields with Integer Spin,"* *Phys. Rev. D* **18** (1978) 3624.

[17] A. Campoleoni and D. Francia, *"Maxwell-like Lagrangians for higher spins,"* *JHEP* **1303** (2013) 168; arXiv:1206.5877.

[18] M. A. Vasiliev, *"Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions,"* *Phys. Lett. B* **B243** (1990) 378.

[19] S. F. Prokushkin and M. A. Vasiliev, *"Higher spin gauge interactions for massive matter fields in 3-D AdS space-time,"* *Nucl. Phys. B* **545** (1999) 385; arXiv:hep-th/9806236.

[20] M. A. Vasiliev, *"Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)"*, *Phys. Lett. B* **567** (2003) 139; arXiv:hep-th/0304049.

[21] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, *"Nonlinear higher spin theories in various dimensions,"* hep-th/0503128 .

[22] V. E. Didenko and E. D. Skvortsov, *"Elements of Vasiliev theory,"* arXiv:1401.2975.

[23] E. Joung and K. Mkrtchyan, *"Notes on higher-spin algebras: minimal representations and structure constants,"* *JHEP* **1405** (2014) 103; arXiv:1401.7977.

[24] S. E. Konshtein and M. A. Vasiliev, *Massless Representations and Admissibility Condition for Higher Spin Superalgebras*, *Nucl. Phys.* **B312** (1989) 402–418.

[25] E. Joung, K. Mkrtchyan and G. Poghosyan, *"Looking for partially-massless gravity,"* *JHEP* **1907** (2019) 116; arXiv:1904.05915.

[26] S. E. Konstein and M. A. Vasiliev, *"Extended Higher Spin Superalgebras and Their Massless Representations,"* *Nucl. Phys. B* **331** (1990) 475.

[27] M. A. Vasiliev, *"Higher Spin Algebras and Quantization on the Sphere and Hyperboloid,"* *Int. J. Mod. Phys. A* **6** (1991) 1115.

[28] E. S. Fradkin and M. A. Vasiliev, *"Cubic Interaction in Extended Theories of Massless Higher Spin Fields,"* *Nucl. Phys. B* **291** (1987) 141.

[29] E. S. Fradkin and M. A. Vasiliev, *"On the Gravitational Interaction of Massless Higher Spin Fields,"* *Phys. Lett. B* **189** (1987) 89.

[30] E. Joung and M. Taronna, *"Cubic-interaction-induced deformations of higher-spin symmetries,"* *JHEP* **1403** (2014) 103; arXiv:1311.0242.

[31] C. Aragone and S. Deser, *"Consistency Problems of Hypergravity,"* *Phys. Lett.* **86B** (1979) 161.

[32] S. Giombi, *"Higher Spin — CFT Duality,"* *New Frontiers in Fields and Strings* (2017) 137; arXiv:1607.02967.

[33] M. R. Gaberdiel and R. Gopakumar, *"Minimal Model Holography,"* *J. Phys. A* **46** (2013) 214002; arXiv:1207.6697.

[34] R. Manvelyan and K. Mkrtchyan, *"Conformal invariant interaction of a scalar field with the higher spin field in AdS(D),"* *Mod. Phys. Lett. A* **25** (2010) 1333; arXiv:0903.0058.

[35] X. Bekaert, E. Joung and J. Mourad, *"On higher spin interactions with matter,"* *JHEP* **0905** (2009) 126; arXiv:0903.3338.

[36] K. Mkrtchyan, *"Linearized interactions of scalar and vector fields with the higher spin field in AdSD,"* *Armenian J. Phys.* **3** (2010) 98 [*Phys. Part. Nucl. Lett.* **8** (2011) 266].

[37] M. A. Vasiliev, *"Cubic Vertices for Symmetric Higher-Spin Gauge Fields in $(A)dS_d$,"* *Nucl. Phys. B* **862** (2012) 341; arXiv:1108.5921.

[38] E. Joung and M. Taronna, *"Cubic interactions of massless higher spins in (A)dS: metric-like approach,"* *Nucl. Phys. B* **861** (2012) 145; arXiv:1110.5918.

[39] E. Joung, L. Lopez and M. Taronna, *"Solving the Noether procedure for cubic interactions of higher spins in (A)dS,"* *J. Phys. A* **46** (2013) 214020; arXiv:1207.5520.

[40] N. Boulanger, D. Ponomarev and E. D. Skvortsov, *"Non-abelian cubic vertices for higher-spin fields in anti-de Sitter space,"* *JHEP* **1305** (2013) 008; arXiv:1211.6979.

[41] X. Bekaert, J. Erdmenger, D. Ponomarev and C. Sleight, *"Towards holographic higher-spin interactions: Four-point functions and higher-spin exchange,"* *JHEP* **1503** (2015) 170; arXiv:1412.0016; *"Quartic AdS Interactions in Higher-Spin Gravity from Conformal Field Theory,"* *JHEP* **1511** (2015) 149; arXiv:1508.04292.

[42] C. Sleight and M. Taronna, *"Higher Spin Interactions from Conformal Field Theory: The Complete Cubic Couplings,"* *Phys. Rev. Lett.* **116** (2016) no.18, 181602; arXiv:1603.00022; *"Higher spin gauge theories and bulk locality: a no-go result,"* arXiv:1704.07859; *"Feynman rules for higher-spin gauge fields on AdS_{d+1},"* *JHEP* **1801** (2018) 060; arXiv:1708.08668.

[43] D. Francia, G. L. Monaco and K. Mkrtchyan, *"Cubic interactions of Maxwell-like higher spins,"* *JHEP* **1704** (2017) 068; arXiv:1611.00292.

[44] C. Sleight and M. Taronna, *"Spinning Witten Diagrams,"* *JHEP* **1706** (2017) 100; arXiv:1702.08619.

[45] A. K. H. Bengtsson, I. Bengtsson and L. Brink, *"Cubic Interaction Terms for Arbitrary Spin,"* *Nucl. Phys. B* **227** (1983) 31.

[46] F. A. Berends, G. J. H. Burgers and H. van Dam, *"On the Theoretical Problems in Constructing Interactions Involving Higher Spin Massless Particles,"* *Nucl. Phys. B* **260** (1985) 295.

[47] X. Bekaert, N. Boulanger, S. Cnockaert and S. Leclercq, *"On killing tensors and cubic vertices in higher-spin gauge theories,"* *Fortsch. Phys.* **54** (2006) 282; hep-th/0602092; N. Boulanger and S. Leclercq, *"Consistent couplings between spin-2 and spin-3 massless fields,"* *JHEP* **0611** (2006) 034; hep-th/0609221.

[48] D. Francia, J. Mourad and A. Sagnotti, *"Current exchanges and unconstrained higher spins,"* *Nucl. Phys. B* **773** (2007) 203; arXiv:hep-th/0701163.

[49] A. Fotopoulos and M. Tsulaia, *"Gauge Invariant Lagrangians for Free and Interacting Higher Spin Fields. A Review of the BRST formulation,"* *Int. J. Mod. Phys. A* **24** (2009) 1; arXiv:0805.1346.

[50] Y. M. Zinoviev, *"On spin 3 interacting with gravity,"* *Class. Quant. Grav.* **26** (2009) 035022; arXiv:0805.2226.

[51] N. Boulanger, S. Leclercq and P. Sundell, *"On The Uniqueness of Minimal Coupling in Higher-Spin Gauge Theory,"* *JHEP* **0808** (2008) 056; arXiv:0805.2764.

[52] R. Manvelyan, K. Mkrtchyan and W. Rühl, *"Off-shell construction of some trilinear higher spin gauge field interactions,"* *Nucl. Phys. B* **826** (2010) 1; arXiv:0903.0243.

[53] R. Manvelyan, K. Mkrtchyan and W. Rühl, *"Direct Construction of A Cubic Selfinteraction for Higher Spin gauge Fields,"* *Nucl. Phys. B* **844** (2011) 348; arXiv:1002.1358.

[54] W. Rühl, *"Solving Noether's equations for gauge invariant local Lagrangians of N arbitrary higher even spin fields,"* arXiv:1108.0225.

[55] A. K. H. Bengtsson, *"Investigations into Light-front Quartic Interactions for Massless Fields (I): Non-constructibility of Higher Spin Quartic Amplitudes,"* *JHEP* **1612** (2016) 134; arXiv:1607.06659.

[56] M. Taronna, *"On the Non-Local Obstruction to Interacting Higher Spins in Flat Space,"* *JHEP* **1705** (2017) 026; arXiv:1701.05772.

[57] R. Roiban and A. A. Tseytlin, *"On four-point interactions in massless higher spin theory in flat space,"* *JHEP* **1704** (2017) 139; arXiv:1701.05773.

[58] D. Ponomarev, *"A Note on (Non)-Locality in Holographic Higher Spin Theories,"* *Universe* **4** (2018) no.1, 2; arXiv:1710.00403.

[59] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *"Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields,"* *JHEP* **1011** (2010) 007; arXiv:1008.4744.

[60] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *"Towards metric-like higher-spin gauge theories in three dimensions,"* *J. Phys. A* **46** (2013) 214017 arXiv:1208.1851.

[61] S. Fredenhagen and P. Kessel, *"Metric- and frame-like higher-spin gauge theories in three dimensions,"* *J. Phys. A* **48** (2015) no.3, 035402; arXiv:1408.2712.

[62] S. Gwak, E. Joung, K. Mkrtchyan and S. J. Rey, *"Rainbow Valley of Colored (Anti) de Sitter Gravity in Three Dimensions,"* *JHEP* **1604** *(2016) 055*; arXiv:1511.05220; S. Gwak, E. Joung, K. Mkrtchyan and S. J. Rey, *"Rainbow vacua of colored higher-spin (A)dS$_3$ gravity,"* *JHEP* **1605** (2016) 150; arXiv:1511.05975.

[63] A. Campoleoni, S. Fredenhagen and J. Raeymaekers, *"Quantizing higher-spin gravity in free-field variables,"* *JHEP* **1802** (2018) 126; arXiv:1712.08078.

[64] M. Fierz, *"Uber die relativistische Theorie kräftefreier Teilchen mit beliebigem Spin,"* Helv. Phys. Acta **12** (1939) 297.

[65] B. de Wit and D. Z. Freedman, *"Systematics of Higher Spin Gauge Fields,"* *Phys. Rev. D* **21** (1980) 358.

[66] R. Manvelyan, K. Mkrtchyan, W. Ruhl and M. Tovmasyan, *"On Nonlinear Higher Spin Curvature,"* *Phys. Lett. B* **699** (2011) 187; arXiv:1102.0306.

[67] R. Manvelyan and W. Ruhl, *"The Generalized curvature and Christoffel symbols for a higher spin potential in AdS(d+1) space,"* *Nucl. Phys. B* **797** (2008) 371; arXiv:0705.3528.

[68] M. S. Costa, J. Penedones, D. Poland and S. Rychkov, *"Spinning Conformal Correlators,"* *JHEP* **1111** (2011) 071; arXiv:1107.3554.

[69] S. Giombi, S. Prakash and X. Yin, *"A Note on CFT Correlators in Three Dimensions,"* *JHEP* **1307** (2013) 105; arXiv:1104.4317.

[70] S. Fredenhagen, O. Krüger and K. Mkrtchyan, *"Constraints for Three-Dimensional Higher-Spin Interactions and Conformal Correlators,"* *Phys. Rev. D* **100** (2019), 066019; arXiv:1812.10462.

[71] P. Kravchuk and D. Simmons-Duffin, *"Counting Conformal Correlators,"* *JHEP* **1802** (2018) 096; arXiv:1612.08987.

[72] J. R. David, M. R. Gaberdiel and R. Gopakumar, *"The Heat Kernel on AdS(3) and its Applications,"* *JHEP* **1004** (2010) 125; arXiv:0911.5085.

[73] S. Giombi and I. R. Klebanov, *"One Loop Tests of Higher Spin AdS/CFT,"* *JHEP* **1312** (2013) 068; arXiv:1308.2337.

[74] E. Joung and M. Taronna, *To be published*.