Spinor-Helicity Three-Point Amplitudes from Local Cubic Interactions

Eduardo CONDE **Euihun JOUNG** **Karapet MKRTCHYAN**

School of Physics & Astronomy and Center for Theoretical Physics
Seoul National University, Seoul 08826 KOREA
Gauge, Gravity & Strings, Center for Theoretical Physics of the Universe
Institute for Basic Sciences, Daejeon 34047 KOREA

E-mail: econdepe@snu.ac.kr, euihun.joung@snu.ac.kr, karapet@snu.ac.kr

Abstract: We make an explicit link between the cubic interactions of off-shell fields and the on-shell three-point amplitudes in four dimensions. Both the cubic interactions and the on-shell three-point amplitudes had been independently classified in the literature, but their relation has not been made explicit. The aim of this note is to provide such a relation and discuss similarities and differences of their constructions. For the completeness of our analysis, we also derive the covariant form of all parity-odd massless vertices.

Contents

1 Introduction

From the dawning of Quantum Field Theory, the relation between local fields and unitary irreducible representations (UIRs) of Poincaré algebra has provided important guidelines in constructing field theory Lagrangians as well as in understanding various physical consequences of them. For instance, the relation between free fields and the corresponding UIRs is very well understood by now: one can either begin with the field equations to show that their general solutions are in one-to-one correspondence with the UIRs (and their quantization leads to the Fock multi-particle space which corresponds to the tensor product representation of the UIR), or reciprocally start from the UIRs and move to the Fock space, then finally construct the quantum fields as the operators transforming in a covariant fashion under Poincaré symmetries. Concerning the familiar lower spin cases, one can find the detailed account *e.g.* in [1].

For these lower spins, the relation between free fields and UIRs can be further extended to the interacting level in the sense that all consistent cubic interaction vertices allowed in a local field theory correspond to tri-linear invariant forms of the UIRs. Since the UIRs of Poincaré algebra — typically labeled by the mass and spin (m, s) and the spatial momenta and helicity state $(\vec{p}, h)$ — can be directly interpreted as the physical states, the tri-linear invariant forms admit the interpretation of three-point amplitudes. From the representation point of view, the Clebsch-Gordan coefficients must enjoy the same tri-linear invariance condition that the three-point amplitudes satisfy. Hence these two objects actually coincide up to overall constants depending on (m, s)'s. The latter constants remain arbitrary for the amplitudes unless the underlying theory is fixed (the non-vanishing ones correspond, up to linear combinations, to the coupling constants of the theory), but are fixed for the Clebsch-Gordan coefficients by the completeness condition.

This understanding in lower spins was soon extended towards more general UIRs. The local free field theories for massive and massless higher-spin representations were constructed in [2, 3] and [4, 5]. About the interactions, on the one hand there have been extensive studies about the Clebsch-Gordan coefficients for the generic UIRs of Poincaré algebra (see *e.g.* [6]). On the other hand, from the field-theory point of view, all consistent local cubic interactions of massless fields in four dimensions have been derived in the light-cone gauge [7, 8] then generalized to higher dimensions [9]. See *e.g.* [10] for general discussions of this program. However, due to various no-go results on the flat-space massless interactions [11–13] (see also [14, 15]) and the success of higher-spin theories in AdS background [16, 17], this direction of research lost its dynamics for a while. A renewed interest on this issue came from, at least, two different directions.

The first direction is the AdS/CFT duality, which generically involves higher-spin fields on the AdS side (even massless ones in certain cases, typically when the corresponding CFT becomes free). Vasiliev's equations [17, 18] describe the dynamics of massless fields interacting with each other within the framework of the so-called unfolded formulation. Even if the latter provides a fully consistent picture, it is still interesting and illuminating to understand the duality from a more mundane field-theoretical point of view. Therefore, the interest on the nature of the flat-space cubic interactions was revived: the light-cone vertices for massive and fermionic fields were obtained in [19, 20], and the covariant form of the light-cone vertices was identified first for certain examples, *e.g.* [21–25], then generalized in [26–30] to arbitrary spins. For an overview of this line of investigations and an exhaustive list of references, the reader may consult the review [31].

The second direction is the ongoing progress in calculating scattering amplitudes using various on-shell methods (see *e.g.* [32, 33] and references therein). An important ingredient in exposing the simplicity of certain four-dimensional scattering amplitudes (especially those involving massless particles) is the use of spinor-helicity variables, as typically exemplified by the Parke-Taylor n-gluon amplitude [34, 35]. Although the helicity-spinor formalism was developed in the 80's mainly for phenomenological purposes, it possesses certain theoretical advantages like making covariant properties manifest; and sparked by Witten's twistor string [36], many theorists have adopted it in their works. As a byproduct of this wave of activity, all possible structures of three-point amplitudes of massless particles were identified in [37] using spinor-helicity variables. Recently, the authors in [38] have extended the classification to the case involving massive particles.

The methods typically used in the higher-spin and amplitude communities do not share the same philosophy: whereas Local Field Theory plays a prominent role in the former, the latter tries to escape from it as much as possible. It is therefore interesting to compare both methodologies and see what are the points of agreement and disagreement, if any at all. In this paper, we aim to make an explicit link between the developments in the two fields.

We consider both massless and massive particles, and show how the local Lagrangian vertices of Field Theory give rise to the known three-point amplitudes. In doing so, instead of using the original light-cone form of the vertices, we use their covariant version together with several generalized Kronecker-delta identities, valid only in four dimensions, to select the non-trivial vertices. For the completeness of massless interactions, we also rederive all parity-odd vertices in this way. Regarding massive interactions, although our procedure is completely general, we focus just on two types of interactions: the first type involving only one massive particle and two massless ones and the other type involving one massless particle and two massive particles of equal mass.

Besides providing an explicit link between the two results, we hope that our analysis helps to understand better the interplay between local Field Theory and the corresponding representation theory. Moreover, this work may also be considered as a toy exercise of the AdS/CFT duality where the local fields in flat space mimic those in AdS while the three-point amplitudes play the role of the three-point correlation functions of the CFT. In fact, there have been several attempts to get the flat-space S-matrix from AdS/CFT by taking a proper flat limit (see for instance [39, 40] for a very general prescription).

The organization of the paper is as follows. In Section 2, we review the spinor realization of the massless and massive UIRs of the Poincaré algebra with some discussions on its generality. In Section 3, we consider the massless case: beginning with the local cubic vertices, we explicitly calculate the corresponding three-point amplitudes using the spinor-helicity variables. In particular, we show how the inclusion of both parity-even and -odd vertices exhaust all possible structures found from the amplitude side, up to some subtleties. In Section 4, we move to the massive case: after reviewing the massive cubic vertices of different types, we focus on two cases. The case of one massive and two massless fields is analyzed in Section 4.2, while Section 4.3 contains the analysis of the case with two equal-mass and one massless particle. In all these cases, we find a good agreement with the result obtained from the amplitude side. After presenting our conclusions in Section 5, we include in Appendix A some technical details regarding the massive UIRs of the Poincaré algebra that we use in the manuscript.

2 Spinor Realization of Massless and Massive UIRs of Poincaré Algebra

In this work, we consider two types of UIRs of the Poincaré algebra. The first one is the massless helicity representation ($P^2 = 0$) with little group $SO(d-2)$, and the other one is the massive representation ($P^2 < 0$ in the mostly positive signature metric) with little group $SO(d-1)$. These are the only UIRs with a finite number of degrees of freedom having positive-definite energies.

Typically, these representations are realized in a way that the Poincaré covariance is not manifest because a certain reference momentum $P_\mu = p_\mu$ must be used to fix the little group. However in $d = 4$ dimensions, we can realize these representations in a manifestly covariant fashion by making use of the spinor representation of the Lorentz algebra. The price to pay for the covariance is that the representation should be realized on a projective space. In the following, we shall review this realization, first for the massless UIRs, then for the massive ones.

Before moving to such details, let us first fix the basic conventions used in this paper. To construct the Weyl representation of the Lorentz algebra, we use the Pauli matrices σ^i to build the following combinations,

$$(\sigma^\mu)_{a\dot{b}} = (1, \vec{\sigma})_{a\dot{b}}\ , \qquad (\bar{\sigma}^\mu)^{\dot{a}b} = \epsilon^{\dot{a}\dot{d}}\,\epsilon^{bc}\,(\sigma^\mu)_{c\dot{d}} = (1, -\vec{\sigma})^{\dot{a}b}\ , \tag{2.1}$$

$$(\sigma^{\mu\nu})_a{}^b = \frac{1}{4}\,(\sigma^\mu\,\bar{\sigma}^\nu - \sigma^\nu\,\bar{\sigma}^\mu)_a{}^b\,, \qquad (\bar{\sigma}^{\mu\nu})^{\dot{a}}{}_{\dot{b}} = -\frac{1}{4}\,(\bar{\sigma}^\mu\,\sigma^\nu - \bar{\sigma}^\nu\,\sigma^\mu)^{\dot{a}}{}_{\dot{b}}\,, \tag{2.2}$$

Here, the spinor indices are lowered and raised as

$$\psi_a = \epsilon_{ab}\,\psi^b\ , \qquad \psi^a = \epsilon^{ab}\,\psi_b\,, \qquad \epsilon^{ac}\,\epsilon_{cb} = \delta^a_b\,, \tag{2.3}$$

and equivalently for dotted indices. The matrices σ^μ, $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ can be used to express the components of any vector v_μ and anti-symmetric tensor $w_{\mu\nu}$ in terms of spinorial ones as

$$v^\mu = -\frac{1}{2}\,(\bar{\sigma}^\mu)^{\dot{b}a}\ v_{a\dot{b}}\,, \qquad w^{\mu\nu} = \frac{1}{2}\,w_{ab}\,(\sigma^{\mu\nu})^{ab} + \frac{1}{2}\,\bar{w}_{\dot{a}\dot{b}}\,(\bar{\sigma}^{\mu\nu})^{\dot{a}\dot{b}}\ , \tag{2.4}$$

and vice-versa:

$$v_{a\dot{b}} = (\sigma^\mu)_{a\dot{b}}\ v_\mu\,, \qquad w_{ab} = (\sigma^{\mu\nu})_{ab}\ w_{\mu\nu}\,, \quad \bar{w}_{\dot{a}\dot{b}} = (\bar{\sigma}^{\mu\nu})_{\dot{a}\dot{b}}\,w_{\mu\nu}\,. \tag{2.5}$$

When v_μ and $w_{\mu\nu}$ are real, the spinorial components should satisfy ${v_{a\dot{b}}}^* = v_{b\dot{a}}$ and ${w_{ab}}^* = \bar{w}_{\dot{b}\dot{a}}$. Henceforth, we shall use only these spinorial components for the Poincaré generators.

2.1 Massless Representations

Let us begin the discussion with the massless representation. By making use of a Weyl spinor λ_a , one can realize the following representation of the Poincaré algebra $\mathfrak{iso}(3,1)$:

$$P_{a\dot{b}} = \lambda_a \, \tilde{\lambda}_{\dot{b}} \,, \qquad L_{ab} = \lambda_{(a} \frac{\partial}{\partial \lambda^{b)}} \,, \qquad \tilde{L}_{\dot{a}\dot{b}} = \tilde{\lambda}_{(\dot{a}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{b})}} \,, \tag{2.6}$$

where we should understand that $\tilde{\lambda}$ and λ are complex-conjugate of each other, as it corresponds to real momenta. Later on, we shall analytically extend them to complex values. From the vanishing of the quadratic Casimir:

$$P^2 = 0 \,, \tag{2.7}$$

one can see that (2.6) is a massless representation. One can notice that the $\mathfrak{sl}_2$'s generated by L_{ab} and $\tilde{L}_{\dot{a}\dot{b}}$ are in fact in the Schwinger representation, hence each of them commutes respectively with their *number operator*,

$$N = \lambda_a \frac{\partial}{\partial \lambda_a} \,, \qquad \tilde{N} = \tilde{\lambda}_{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}_{\dot{a}}} \,. \tag{2.8}$$

Given the form of the translation generator $P_{a\dot{b}}$, only the linear combination

$$H = N - \tilde{N} \,, \tag{2.9}$$

commutes with all the generators of the Poincaré algebra. Therefore, the representation space $V = Fun(\mathbb{C}^2)$ can be block-diagonalized with respect to H as

$$V = \bigoplus_{h \in \mathbb{Z}} V_h \,, \tag{2.10}$$

where the space V_h, isomorphic to $\mathbb{C}^2/U(1)$, is given by

$$V_h = \Big\{ \; f(\lambda, \tilde{\lambda}) \; \Big| \; \forall \, e^{i\theta} \in U(1), \;\; f(e^{i\theta} \, \lambda, e^{-i\theta} \, \tilde{\lambda}) = e^{-ih\theta} \, f(\lambda, \tilde{\lambda}) \; \Big\} \,. \tag{2.11}$$

Each space V_h should still carry a faithful representation of Poincaré algebra as H commutes with $\mathfrak{iso}(3,1)$. In fact, V_h carries the massless helicity h representation because H coincides with the helicity operator:

$$\frac{W_0}{P_0} = \frac{(\sigma_0)^{a\dot{b}} \, W_{a\dot{b}}}{(\sigma_0)^{a\dot{b}} \, P_{a\dot{b}}} = \frac{1}{2} \, H \,, \tag{2.12}$$

where $W_{a\dot{b}}$ is the Pauli-Lubanski vector,

$$W_{a\dot{b}} = P^{c}{}_{\dot{b}} \, L_{ac} - P_{a}{}^{\dot{c}} \, \tilde{L}_{\dot{b}\dot{c}} \,. \tag{2.13}$$

After analytic continuation, the $U(1)$ coset condition will be replaced by the one, $f(\Omega \, \lambda, \Omega^{-1} \, \tilde{\lambda}) = \Omega^{-h} \, f(\lambda, \tilde{\lambda})$ for an arbitrary element Ω in $\mathbb{C} \backslash \{0\}$.

2.2 Massive Representations

Now let us consider the tensor-product of n copies of the representation (2.6):

$$P_{a\dot{b}} = \lambda^I{}_a\,\tilde{\lambda}_{I\,\dot{b}}\,, \qquad L_{ab} = \lambda^I{}_{(a}\,\frac{\partial}{\partial\lambda^{I\,b)}}\,, \qquad \tilde{L}_{\dot{a}\dot{b}} = \tilde{\lambda}_{I(\dot{a}}\,\frac{\partial}{\partial\tilde{\lambda}_I{}^{\dot{b})}}\,. \tag{2.14}$$

where $I = 1,\ldots,n$. Since each copy is a Poincaré UIR, their tensor products also carry unitary representations under Poincaré group, but reducible ones. We can reduce this representation into smaller ones by imposing certain conditions compatible with the Poincaré action. In this way, we may end up with an irreducible representation. Appropriate conditions can be found using differential operators acting on the spinor space which commute with those in (2.14), which realize the Poincaré generators. We first note that the Lorentz generators commute with two copies of $\mathfrak{gl}_n$:

$$N^I{}_J = \lambda^I{}_a\,\frac{\partial}{\partial\lambda^J{}_a}\,, \qquad \tilde{N}_J{}^I = \tilde{\lambda}_{J\dot{a}}\,\frac{\partial}{\partial\tilde{\lambda}_{I\,\dot{a}}}\,, \tag{2.15}$$

which are in fact the centralizers of $\mathfrak{sl}_2$'s (generated by L_{ab} and $\tilde{L}_{\dot{a}\dot{b}}$) within $\mathfrak{gl}_{2n}$'s (generated by $\lambda^I_a\frac{\partial}{\partial\lambda^J_b}$ and $\tilde{\lambda}^I_{\dot{a}}\frac{\partial}{\partial\tilde{\lambda}^J_{\dot{b}}}$). If we extend $\mathfrak{gl}_{2n}$ — that is, any differential operators bilinear in λ and $\partial/\partial\lambda$ — to $\mathfrak{sp}_{4n}$ by including the operators of the type $\lambda\,\lambda$ and $\frac{\partial^2}{\partial\lambda\,\partial\lambda}$'s, then the centralizers are extended with the antisymmetric $n\times n$ tensors

$$M^{IJ} = \lambda^I{}_a\,\lambda^{Ja}\,, \qquad \tilde{M}_{IJ} = \tilde{\lambda}_{I\dot{a}}\,\tilde{\lambda}_J{}^{\dot{a}}\,, \tag{2.16}$$

and their $\lambda \leftrightarrow \partial/\partial\lambda$ conjugates. The latter commute with M^{IJ} and $\tilde{M}_{IJ}$, however they have non-trivial commutators with the translation generators. Moreover among $N^I{}_J$ and $\tilde{N}_J{}^I$ only the combination

$$K^I_J = N^I{}_J - \tilde{N}_J{}^I\,, \tag{2.17}$$

commutes with the whole Poincaré algebra. Therefore, we can decompose the spinor space $\mathbb{C}^{2n}$ in terms of the UIRs of the algebra, to which we shall refer to as $\mathfrak{A}_n$, generated by K^I_J, M^{IJ} and $\tilde{M}_{IJ}$: it is a semi-direct sum of $\mathfrak{u}(n)$ (generated by K^I_J) and the antisymmetric tensor product of fundamental (for M^{IJ}) and anti-fundamental (for $\tilde{M}_{IJ}$) representations. By choosing a particular UIR of $\mathfrak{A}_n$, we can reduce the tensor product representation of Poincaré algebra into a smaller one. As in the Poincaré case, the UIRs of $\mathfrak{A}_n$ can be classified according to the value of the quadratic Casimir,

$$C_2(\mathfrak{A}_n) = -\frac{1}{2}\,M^{IJ}\,\tilde{M}_{IJ}\,, \tag{2.18}$$

which in fact coincides with that of the Poincaré algebra:

$$C_2(\mathfrak{iso}(3,1)) = P^2 = -\frac{1}{2}\,P_{a\dot{b}}\,P^{a\dot{b}} = C_2(\mathfrak{A}_n)\,. \tag{2.19}$$

Depending on this value, we may classify the representations of $\mathfrak{A}_n$.

Case $n = 2$

For the massive representations, it will be sufficient to consider the $n = 2$ case, where we have only one component for each of $M^{IJ} = -\epsilon^{IJ} M$ and $\tilde{M}_{IJ} = \epsilon_{IJ} \tilde{M}$. The latter generators commute with the $\mathfrak{su}(2) \subset \mathfrak{u}(2)$ generated by $\mathcal{K}^I_J$:

$$\mathcal{K}^I_J = K^I_J - \tfrac{1}{2}\, \delta^I_J\, K^K_K = \mathcal{N}^I{}_J - \tilde{\mathcal{N}}_J{}^I\,, \tag{2.20}$$

$$\mathcal{N}^I{}_J = N^I{}_J - \tfrac{1}{2}\, \delta^I_J\, N^K{}_K\,, \qquad \tilde{\mathcal{N}}_I{}^J = N_I{}^J - \tfrac{1}{2}\, \delta^I_J\, \tilde{N}_K{}^K\,, \tag{2.21}$$

whereas the $\mathfrak{u}(1)$ part $K = K^I{}_I$ satisfies the commutation relations:

$$[K, M] = 2\, M\,, \qquad [K, \tilde{M}] = -2\, \tilde{M}\,. \tag{2.22}$$

We see that $K, M, \tilde{M}$ form an $\mathfrak{iso}(2)$ algebra. Hence, we see that $\mathfrak{A}_2 \simeq \mathfrak{su}(2) \oplus \mathfrak{iso}(2)$. It turns out that the determination of an irreducible representation under $\mathfrak{iso}(2)$ fixes the mass value of the Poincaré representation, while choosing an irreducible representation for $\mathfrak{su}(2)$ fixes the spin. We can therefore associate the massive little group $SO(3)$ of the Poincaré group with the $\mathfrak{su}(2)$ generated by $\mathcal{K}^I_J$. One of the simplest ways to see this is to compute the corresponding Casimir operators. The quadratic Casimir P^2 of Poincaré algebra is already fixed in this case by (2.19), and given by $C_2(\mathfrak{A}_2) = C_2(\mathfrak{iso}(2)) = -M\, \tilde{M}$. Similarly to P^2, the square of Pauli-Lubanski vector (2.13) commutes with all the other generators. After a simple manipulation, we get

$$W^2 = -\frac{1}{2}\, W_{a\dot{b}}\, W^{a\dot{b}} = -\frac{1}{4}\, P_{a\dot{b}}\, P^{a\dot{b}} \left(L_{cd}\, L^{cd} + \tilde{L}_{\dot{c}\dot{d}}\, \tilde{L}^{\dot{c}\dot{d}} \right) + P^{a\dot{c}}\, P^{b\dot{d}}\, L_{ab}\, \tilde{L}_{\dot{c}\dot{d}}\,, \tag{2.23}$$

which can be further simplified using the identities, valid only for $n = 2$,

$$L_{ab}\, L^{ab} = \mathcal{N}^I{}_J\, \mathcal{N}^J{}_I\,, \qquad \tilde{L}_{\dot{a}\dot{b}}\, \tilde{L}^{\dot{a}\dot{b}} = \tilde{\mathcal{N}}_I{}^J\, \tilde{\mathcal{N}}_J{}^I\,, \tag{2.24}$$

$$P^{a\dot{c}}\, P^{b\dot{d}}\, L_{ab}\, \tilde{L}_{\dot{c}\dot{d}} = M\, \tilde{M}\, \mathcal{N}^I{}_J\, \tilde{\mathcal{N}}_I{}^J\,. \tag{2.25}$$

Finally, combining the above formulas, we can show that

$$W^2 = -\frac{1}{2}\, M\, \tilde{M}\, \mathcal{K}^I_J\, \mathcal{K}^J_I\,. \tag{2.26}$$

This makes it clear that when $P^2 = -M\, \tilde{M} = -m^2 < 0$, by taking the 'spin s' representation of $\mathfrak{su}(2)$, the corresponding Poincaré representation becomes also that of massive spin s. To recapitulate, starting from $V = Fun(\mathbb{C}^4)$, we get

$$V = \bigoplus_{m \in \mathbb{R},\, s \in \mathbb{N}/2} V_{m,s}\,, \tag{2.27}$$

where $V_{m,s}$, isomorphic to $\mathbb{C}^4/\mathfrak{A}_2$, is given by

$$V_{m,s} = \left\{ f_{r,h}(\lambda, \tilde{\lambda}) \;\middle|\; \begin{array}{c} \forall\, g \in SU(2)\,,\; f_{r,h}(g\,\lambda, g^{-1}\,\tilde{\lambda}) = D^s_{hh'}(g)\, f_{n,h'}(\lambda, \tilde{\lambda}) \\ \forall\, e^{i\,\theta} \in U(1)\,,\; f_{r,h}(e^{i\,\theta}\,\lambda, e^{-i\,\theta}\tilde{\lambda}) = e^{2\,i\,r\,\theta}\, f_{r,h}(\lambda, \tilde{\lambda}) \\ \lambda_{I\,a}\, \lambda^{I\,a}\, f_{r,h}(\lambda, \tilde{\lambda}) = 2\,m\, f_{r+1,h}(\lambda, \tilde{\lambda}) \\ \tilde{\lambda}_{I\,\dot{a}}\, \tilde{\lambda}^{I\,\dot{a}}\, f_{r,h}(\lambda, \tilde{\lambda}) = 2\,m\, f_{r-1,h}(\lambda, \tilde{\lambda}) \end{array} \right\}, \tag{2.28}$$

where $D^s_{hh'}(g)$ is the Wigner D-matrix. Since this matrix does not depend on the label r, representations with different values of r can be considered physically equivalent. We will make use of this fact later on in Section 4. See Appendix A for detailed account of how (2.28) is related to the standard massive representation.

3 Massless Interactions

We will match now each local cubic interaction to one of the three-point amplitudes classified using the spinor-helicity method. We begin this analysis with the case where all three fields or representations are massless. We restrict to the case of bosonic fields for the sake of simplicity. Before making an explicit link between them, we review the classification of massless three-point amplitudes in spinor-helicity variables and the construction of gauge-invariant local cubic-interaction vertices.

3.1 Three-point Amplitudes

Let us denote the asymptotic states of the three massless particles by

$$|\lambda^I, \tilde{\lambda}_I; h_I\rangle \,, \quad I = 1, 2, 3 \,. \tag{3.1}$$

As we discussed in Section 2, each asymptotic state furnishes a representation of $\mathfrak{iso}(3,1)$. The three-particle amplitude carries information about this representation in the sense of being a function living in the spaces V_{h_I} defined in (2.11). A way to re-state this is by saying that the helicity operator in (2.9) acts on the amplitude as it acts on the one-particle states. This fact essentially determines the three-point amplitude up to a coupling constant [37]. Discarding delta-function contributions (apart from the momentum-conserving delta-function, that we omit in what follows), the solution to the differential equations that follow from applying the three helicity operators to the amplitude is

$$M_3^{h_1,h_2,h_3} = \langle 1,2\rangle^{h_3-h_1-h_2}\langle 3,1\rangle^{h_2-h_3-h_1}\langle 2,3\rangle^{h_1-h_2-h_3} f\left(\langle I,J\rangle[I,J]\right) \,, \tag{3.2}$$

with $\langle I,J\rangle = \lambda^I{}_a\,\lambda^{Ja}$ and $[I,J] = \tilde{\lambda}_{I\,\dot{a}}\,\tilde{\lambda}^{\dot{a}}_J$. Here, f is an unknown function that can be determined with three physical requirements. One is that the amplitude should not be singular. Another is momentum conservation, that implies that either $\langle I,J\rangle = 0$ or $[I,J] = 0$. The remaining one is the fact that whenever $h_1 + h_2 + h_3 \neq 0$, the amplitude must vanish on the real sheet[1]. This gives

$$M_3^{h_1,h_2,h_3} = \begin{cases} g_{\text{H}}\,\langle 1,2\rangle^{h_3-h_1-h_2}\,\langle 3,1\rangle^{h_2-h_3-h_1}\,\langle 2,3\rangle^{h_1-h_2-h_3} & \text{when } h_1+h_2+h_3<0 \\ g_{\text{A}}\,[1,2]^{h_1+h_2-h_3}\,[3,1]^{h_3+h_1-h_2}\,[2,3]^{h_2+h_3-h_1} & \text{when } h_1+h_2+h_3>0 \end{cases} . \tag{3.3}$$

The coupling constants g_{H}, g_{A} above[2] are unrelated in a theory with no well-defined parity. For parity-even or -odd theories, they are equal up to a sign . Let us remark here that if we drop the non-singular requirement, we could formally have singular amplitudes obeying (3.2) (recall that $\langle I,J\rangle[I,J] = 0$ by momentum conservation). This can make sense in certain contexts [41], and we will see at the end of this section another example of this happening.

[1] In the case $h_1 + h_2 + h_3 = 0$, the amplitude does not need to vanish a priori for real momenta, which can line up along a null direction. However, no consistent interactions are known of this type. We will comment on this point again at the end of this Section 3.

[2] The subindices H and A stand for holomorphic and anti-holomorphic, referring to their dependence on the λ and $\tilde{\lambda}$ spinors respectively.

Finally note that there is no restriction on the spin of the scattering particles, which can be arbitrarily large. For given spins s_1, s_2 and s_3, there are generically four types of amplitudes associated:

$$(h_1, h_2, h_3) \in \{(\pm s_1, \pm s_2, \pm s_3)\,, (\mp s_1, \pm s_2, \pm s_3)\,, (\pm s_1, \mp s_2, \pm s_3)\,, (\pm s_1, \pm s_2, \mp s_3)\}\,, \quad (3.4)$$

which we grouped in parity-conjugated pairs.

3.2 Cubic vertices

Let us now turn to the Lagrangian description of cubic interactions. Local and gauge-invariant cubic Lagrangians have been completely classified in any D-dimensional space-time ($D \geq 4$). We briefly review here the derivation of covariant cubic vertices in four dimensions.

As is common practice, to deal with arbitrary higher-spin fields we introduce auxiliary variables u^μ and define the generating functions:

$$\varphi(x,u) = \sum_{s=0}^{\infty} \frac{1}{s!}\, \varphi_{\mu_1 \cdots \mu_s}(x)\, u^{\mu_1} \cdots u^{\mu_s}\,. \qquad (3.5)$$

The massless system requires gauge symmetries in order to propagate the correct number of on-shell degrees of freedom, and the gauge transformation takes the following form:

$$\delta\varphi(x,u) = u \cdot \partial_x\, \varepsilon(x,u) + \cdots\,, \qquad (3.6)$$

where the dots contain terms of higher order in the number of fields (namely they contain the non-linear part of the gauge transformation), which are not needed at cubic level. Indeed, from the gauge invariance of the full action, it can be shown that the generic cubic vertex

$$S^{(3)} = \int d^4x\, C(\partial_{x_I}, \partial_{u_I})\, \varphi^1(x_1,u_1)\, \varphi^2(x_2,u_2)\, \varphi^3(x_3,u_3)\Big|_{\substack{x_I = x\\ u_I = 0}}\,, \qquad (3.7)$$

should satisfy

$$[C(\partial_{x_J}, \partial_{u_J}), u_I \cdot \partial_{x_I}] \approx 0\,, \quad I = 1,2,3\,, \qquad (3.8)$$

where $\approx$ means modulo the Frondsal equations of motion [4]. In order to solve equation (3.8), one has to analyze what are the variables that C can possibly depend on. Since we will only require the on-shell content of the vertex later on, we can just focus on its transverse and traceless (TT) part, that we denote by C^{TT}, and which also satisfies equation (3.8) where $\approx$ means now modulo the Fierz system:

$$\partial_{x_I}^2\, \varphi^I \approx 0\,, \qquad \partial_{x_I} \cdot \partial_{u_I}\, \varphi^I \approx 0\,, \qquad \partial_{u_I}^2 \varphi^I \approx 0\,. \qquad (3.9)$$

In order to continue the analysis, we need to distinguish the cases where the vertices involve a Levi-Civita epsilon tensor (hence parity-odd) or not (parity-even).

3.2.1 Parity-even vertices

Let us begin with the parity-even cases. Since the vertices do not involve any $\epsilon_{\mu\nu\rho\sigma}$ tensor, the vertex function $C^{\rm TT}$ can only depend on the six variables,

$$Y_I = \partial_{u_I} \cdot \partial_{x_{I+1}} \,, \qquad Z_I = \partial_{u_{I+1}} \cdot \partial_{u_{I-1}} \,. \tag{3.10}$$

$C^{\rm TT}$ is then easily determined using the commutators,

$$[Y_I, u_J \cdot \partial_{x_J}] = 0 \,, \quad [Z_I, u_I \cdot \partial_{x_I}] = 0 \,, \quad \big[Z_I, u_{I\pm 1} \cdot \partial_{x_{I\pm 1}}\big] = \mp Y_{I\mp 1} \,. \tag{3.11}$$

The solution to the equations (3.8) is

$$C^{\rm TT} = \sum_{n=0}^{s_1} \lambda_n^{(s_1,s_2,s_3)} \, G^n \, Y_1^{s_1-n} \, Y_2^{s_2-n} \, Y_3^{s_3-n} \,, \tag{3.12}$$

where the $\lambda_n^{(s_1,s_2,s_3)}$'s are independent coupling constants that ought to be fixed by the quest for consistency of higher order interactions, and we defined G as the combination,

$$G = Y_1 \, Z_1 + Y_2 \, Z_2 + Y_3 \, Z_3 \,. \tag{3.13}$$

The discussion up to here is actually valid in any space-time dimension. In four dimensions the variables Y_I and Z_I are not independent, as generalized Kronecker-delta identities in four dimensions[3] imply $Y_1 \, Y_2 \, Y_3 \, G \approx 0$. This makes the expression (3.12) collapse to just two possible parity-even vertices:

$$C^{\rm TT} = g_{\rm min} \, G^{s_1} \, Y_2^{s_2-s_1} \, Y_3^{s_3-s_1} + g_{\rm non} \, Y_1^{s_1} \, Y_2^{s_2} \, Y_3^{s_3} \,, \tag{3.14}$$

where we assumed $s_1 \leq s_2 \leq s_3$ without loss of generality. The first vertex is the minimal coupling and contains $s_3 + s_2 - s_1$ derivatives. The second one contains $s_1 + s_2 + s_3$ derivatives instead, and it is usually called non-minimal coupling. Notice that these two vertices coincide when $s_1 = 0$.

3.2.2 Parity-odd vertices

The analysis of parity-odd cubic vertices is analogous to the parity-even case, except that now

$$C^{\rm TT} = \sum_{I=1}^{3} V_I \, F_I^{(V)}(Y,Z) + W_I \, F_I^{(W)}(Y,Z) \,, \tag{3.15}$$

depends linearly on the variables

$$V_I = \epsilon^{\mu\nu\rho\sigma} \partial_{u_{I+1}^\mu} \partial_{x_{I+1}^\nu} \partial_{u_{I-1}^\rho} \partial_{x_{I-1}^\sigma} \,, \qquad W_I = \epsilon^{\mu\nu\rho\sigma} \partial_{u_1^\mu} \partial_{u_2^\nu} \partial_{u_3^\rho} \partial_{x_I^\sigma} \,. \tag{3.16}$$

It is important to note here that the expression (3.15) contains in general a redundancy because the six variables V_I's and W_I's are not independent — using Schouten identities and momentum conservation we get the following six relations:

$$\begin{aligned} W_I \, Y_I &\approx V_{I+1} Z_{I-1} + V_{I-1} Z_{I+1} \,, \qquad & W_1 + W_2 + W_3 &\approx 0 \,, \\ V_1 \, Y_1 &\approx V_2 \, Y_2 \approx V_3 \, Y_3 \,, & V_I \, Y_I \, G &\approx 0 \,. \end{aligned} \tag{3.17}$$

[3] More explicitly, we should consider the identity $\delta^{\nu_1\nu_2\nu_3\nu_4\nu_5}_{\mu_1\mu_2\mu_3\mu_4\mu_5} \partial_{u_1}^{\mu_1} \partial_{u_2}^{\mu_2} \partial_{u_3}^{\mu_3} \partial_{x_1}^{\mu_4} \partial_{x_2}^{\mu_5} \partial_{u_1^{\nu_1}} \partial_{u_2^{\nu_2}} \partial_{u_3^{\nu_3}} \partial_{x_1^{\nu_4}} \partial_{x_2^{\nu_5}} = 0$. Hereupon we will refer to these sort of identities as Schouten identities.

The redundancy can be removed by expressing V_2, V_3 and W_I's in terms of the other variables as

$$V_I \approx V_1 \frac{Y_1}{Y_I}\,, \quad W_I \approx V_1 \frac{Y_{I+1}Z_{I+1} + Y_{I-1}Z_{I-1}}{Y_2\,Y_3}\,, \quad I = 2,3 \quad ; \quad W_1 \approx -V_1\,\frac{G + Y_1 Z_1}{Y_2\,Y_3}\,. \tag{3.18}$$

Since these relations commute with the gauge variations, removing the redundancy before solving the gauge invariance condition (3.8) yields a simpler expression for the vertex, namely $C^{\rm TT} = V_1\,F(Y,Z)$. Notice however that since the replacements (3.18) involve negative powers of Y_2 and Y_3, the function F is allowed to have terms proportional to the negative powers Y_2^{-1}, Y_3^{-1} or $(Y_2 Y_3)^{-1}$. Negative powers of Y_I do not make sense as it would mean a negative number of contractions, but actually they might be just an artifact of our procedure, which is purposed to remove redundancies. As we will show below, it is possible that even when these negative powers show up, the vertex still admits a polynomial expression if the relations (3.18) can be inverted.

Given that $[V_1, u_I \cdot \partial_{x_I}] = 0$, we can immediately see that there are only two possible gauge-invariant parity-odd vertices:

$$C^{\rm TT} = g_{\rm min,PO}\,V_1\,G^{s_1}\,Y_2^{s_2-s_1-1}\,Y_3^{s_3-s_1-1} + g_{\rm non,PO}\,V_1\,Y_1^{s_1}\,Y_2^{s_2-1}\,Y_3^{s_3-1}\,. \tag{3.19}$$

They respectively have $s_2+s_3-s_1$ and $s_1+s_2+s_3$ derivatives, hence can be naturally paired with the parity-even minimal and non-minimal coupling vertices. For this reason, we also refer to these parity-odd vertices as minimal and non-minimal couplings. Let us note that, similarly to parity-even case, the two vertices coincide for $s_1 = 0$. Notice that the minimal-coupling vertex has negative powers of Y_I's when $s_1 = s_2$. However, for $s_2 < s_3$, it can be brought to a polynomial form using the identities (3.17) as

$$V_1\,G^{s_1}\,Y_2^{-1}\,Y_3^{s_3-s_1-1} \approx \frac{1}{2}\,[V_1\,Z_2 - V_2\,Z_1 + (W_2 - W_1)\,Y_3]\,G^{s_1-1}\,Y_3^{s_3-s_1-1}\,, \tag{3.20}$$

where we have chosen the form symmetric under exchange of 1 and 2 among various equivalent expressions. About the case with $s_1 = s_2 = s_3$, it is impossible to remove completely the negative powers of Y_2 and Y_3 using (3.17) from the minimal-coupling vertices. Therefore we conclude that, compared to the parity-even vertices, the parity-odd vertices miss the minimal ones with all equal spins.

There is another case of coincident spins: $s_1 < s_2 = s_3$, for which the equation (3.19) does not contain negative powers of Y_I's. In this case, the vertex operator has symmetry property with respect to exchange of second and third fields, up to a factor $(-1)^{s_1}$, which suggests, similarly to parity-even cases, to include Chan-Paton structures in the case of odd s_1. Instead in the case of $s_1 = s_2 < s_3$, the vertex (3.20) has the opposite property — Chan-Paton structures are needed for even s_3. This strange difference suggests intuitive understanding of the case $s_1 = s_2 = s_3$, which, belonging to both of the above classes of vertices, should have both symmetric and antisymmetric properties with respect to exchange of any two fields, which cannot be satisfied by any non-vanishing vertex operator.

3.3 Match

We will investigate here if the amplitudes (3.3) match the ones obtained from the cubic vertices (3.14) and (3.19). For that it suffices to translate the Lagrangian vertices into amplitudes.

Given the form of the vertex (3.7), Feynman rules instruct us to extract the coefficient of $\prod_{I=1}^{3}\prod_{k=1}^{s_I}\partial_{u_I^{\mu_k}}$ of $C(\partial_{x_I},\partial_{u_I})$, multiply it by $(-i)\,e^{i(p_1+p_2+p_3)\cdot x}$, and contract it with the polarization tensors of the three external particles. This is equivalent to writing

$$\widetilde{M}_3=\int d^4x\,C^{\rm TT}(G,Y_i,V_1)\,\varphi^1_{\text{O-S}}(x_1,u_1)\,\varphi^2_{\text{O-S}}(x_2,u_2)\,\varphi^3_{\text{O-S}}(x_3,u_3)\Big|_{\substack{x_I=x\\u_I=0}}\,, \tag{3.21}$$

where $\widetilde{M}_3$ is the formal sum of the position-space amplitudes for all helicity configurations and all spins, and the O-S subindex refers to the fact that the φ^I should satisfy the equations of motion. The equivalence holds because the on-shell evaluation of the massless higher-spin fields yields polarization tensors times plane-wave exponentials:

$$\varphi_{\text{O-S}}(x,u)=\sum_{s=0}^{\infty}\frac{1}{s!}\int d^4p\,\left(\varphi^-(p)\,\epsilon^-_{\mu_1\cdots\mu_s}(p)+\varphi^+(p)\,\epsilon^+_{\mu_1\cdots\mu_s}(p)\right)\,u^{\mu_1}\cdots u^{\mu_s}\,e^{ip\cdot x}\,. \tag{3.22}$$

Therefore, let us start by solving the on-shell conditions (3.9) for a generic $\varphi(x,u)$. We will do so in terms of spinor variables to connect with (3.3). For that purpose, it is quite convenient to use light-cone variables:

$$\begin{aligned}x^\pm&=x^0\pm x^3\,,\quad z=x^1+i\,x^2\,,\\ u^\pm&=u^0\pm u^3\,,\quad \omega=u^1+i\,u^2\,.\end{aligned} \tag{3.23}$$

We start by fixing the gauge. We impose the light-cone gauge condition,

$$\partial_{u^-}\,\varphi(x^+,x^-,z,\bar z\,;\,u^+,u^-,\omega,\bar\omega)=0\,, \tag{3.24}$$

which simply implies that φ cannot depend on u^-. In this gauge, the transverse condition reads

$$(-\partial_{u^+}\partial_{x^-}+\partial_z\,\partial_{\bar\omega}+\partial_{\bar z}\,\partial_\omega)\,\varphi(x^+,x^-,z,\bar z\,;\,u^-,\omega,\bar\omega)=0\,, \tag{3.25}$$

which can be easily solved by

$$\varphi(x^+,x^-,z,\bar z\,;\,u^-,\omega,\bar\omega)=\exp\left[\frac{u^+}{\partial_{x^-}}\left(\partial_z\,\partial_{\bar\omega}+\partial_{\bar z}\,\partial_\omega\right)\right]\varphi_{\text{l.c.}}(x^+,x^-,z,\bar z\,;\,\omega,\bar\omega)\,, \tag{3.26}$$

where the subindex l.c. refers to light-cone. The next condition to solve is the d'Alembertian equation, $\Box\,\varphi_{\text{l.c.}}=0$, whose solution is a simple superposition of plane waves:

$$\varphi_{\text{l.c.}}(x,\omega,\bar\omega)=\int\frac{d^2\lambda\,d^2\tilde\lambda}{\text{vol}(GL(1))}\,e^{i\left(\frac{i}{2}\,x^{a\dot a}\,\lambda^I{}_a\,\tilde\lambda_{I\dot a}\right)}\exp\left[\frac{\tilde\lambda_{\dot 2}}{\lambda_2}\,\omega\,\partial_\chi+\frac{\lambda_2}{\tilde\lambda_{\dot 2}}\,\bar\omega\,\partial_{\bar\chi}\right]\phi(\lambda,\tilde\lambda\,;\,\chi,\bar\chi)\Big|_{\chi=\bar\chi=0}\,, \tag{3.27}$$

where we have decided to introduce the second exponential so that the "wave-function" ϕ carries helicity representations:

$$\phi(\Omega\,\lambda,\Omega^{-1}\,\tilde\lambda\,;\,\Omega^{-2}\,\chi,\Omega^2\,\bar\chi)=\phi(\lambda,\tilde\lambda\,;\,\chi,\bar\chi)\qquad[\Omega\in\mathbb{C}]\,. \tag{3.28}$$

Note that the integrand above becomes singular when the second components of the spinors are vanishing. This just occurs because we assumed that $\partial_{x^-}\varphi \neq 0$ when imposing the light-cone condition (3.24). This is equivalent to assuming that $\lambda_2\tilde{\lambda}_{\dot{2}} \neq 0$.

Now, using the relations (3.26) and (3.27), the effect of derivatives and contractions on the wave-function φ are given by

$$
\begin{aligned}
(\,\partial_{x^+}, \partial_{x^-}, \partial_z, \partial_{\bar{z}}\,)\varphi \quad &\leftrightarrow \quad i\,(\,-\lambda_1\,\tilde{\lambda}_{\dot{1}}, -\lambda_2\,\tilde{\lambda}_{\dot{2}}, \lambda_1\,\tilde{\lambda}_{\dot{2}}, \lambda_2\,\tilde{\lambda}_{\dot{2}}\,)\phi\,,\\
(\,\partial_{u^+}, \partial_{u^-}, \partial_\omega, \partial_{\bar{\omega}}\,)\varphi \quad &\leftrightarrow \quad \left[(\,-\tilde{\lambda}_{\dot{1}}, 0, \tilde{\lambda}_{\dot{2}}, 0\,)\,\frac{\partial_\chi}{\lambda_2} + (\,-\lambda_1, 0, 0, \lambda_2\,)\,\frac{\partial_{\bar{\chi}}}{\tilde{\lambda}_{\dot{2}}}\right]\phi\,.
\end{aligned}
\tag{3.29}
$$

From here we can see that the traceless condition $\partial_u^2\,\varphi = 0$ also takes a very simple form:

$$
\partial_\chi\,\partial_{\bar{\chi}}\,\phi(\lambda, \tilde{\lambda}\,;\,\chi, \bar{\chi}) = 0 \quad \Rightarrow \quad \phi(\lambda, \tilde{\lambda}\,;\,\chi, \bar{\chi}) = \phi^+(\lambda, \tilde{\lambda}\,;\,\chi) + \phi^-(\lambda, \tilde{\lambda}\,;\,\bar{\chi})\,, \tag{3.30}
$$

forbidding mixed contractions. Notice that this gives rise to the decomposition (3.22). In order to evaluate (3.21), we can just plug (3.26) and (3.27) in there, then use (3.29) to compactly write the resulting expression. Indeed, the operators that can appear in the vertex cast nicely as

$$
\begin{aligned}
-4i\,G \leftrightarrow \;& \frac{[2,3]^3}{[1,2][3,1]}\,\partial_{\bar{\chi}_1}\,\partial_{\chi_2}\,\partial_{\chi_3} + \frac{[3,1]^3}{[2,3][1,2]}\,\partial_{\chi_1}\,\partial_{\bar{\chi}_2}\,\partial_{\chi_3} + \frac{[1,2]^3}{[3,1][2,3]}\,\partial_{\chi_1}\,\partial_{\chi_2}\,\partial_{\bar{\chi}_3}\\
&+ \frac{\langle 2,3\rangle^3}{\langle 1,2\rangle\langle 3,1\rangle}\,\partial_{\chi_1}\,\partial_{\bar{\chi}_2}\,\partial_{\bar{\chi}_3} + \frac{\langle 3,1\rangle^3}{\langle 2,3\rangle\langle 1,2\rangle}\,\partial_{\bar{\chi}_1}\,\partial_{\chi_2}\,\partial_{\bar{\chi}_3} + \frac{\langle 1,2\rangle^3}{\langle 3,1\rangle\langle 2,3\rangle}\,\partial_{\bar{\chi}_1}\,\partial_{\bar{\chi}_2}\,\partial_{\chi_3}\,, \qquad (3.31)\\
-2i\,Y_I \leftrightarrow \;& \frac{[I,J][I,K]}{[J,K]}\,\partial_{\chi_I} + \frac{\langle I,J\rangle\langle I,K\rangle}{\langle J,K\rangle}\,\partial_{\bar{\chi}_I}\,, \qquad (3.32)\\
-4i\,V_1 \leftrightarrow \;& [2,3]^2\partial_{\chi_2}\partial_{\chi_3} - \langle 2,3\rangle^2\partial_{\bar{\chi}_2}\partial_{\bar{\chi}_3}\,. \qquad (3.33)
\end{aligned}
$$

Notice that the expressions above contain singular terms of the form $\frac{0}{0}$ when momentum conservation is taken into account. The operators G, Y_I, V_1 are of course not singular. We have purposely introduced these singularities by substituting

$$
\frac{\lambda_2^J}{\lambda_2^I} = -\frac{[I,K]}{[J,K]}\,, \qquad \frac{\tilde{\lambda}_{J2}}{\tilde{\lambda}_{I2}} = -\frac{\langle I,K\rangle}{\langle J,K\rangle}\,, \tag{3.34}
$$

in order to make the final expressions more appealing, and also because this will allow us to make easier contact with (3.2), where momentum conservation is not explicitly imposed. Using the formulae (3.31)-(3.33), we can easily evaluate the cubic vertices.

Let us start with the non-minimal vertices with $(s_1 + s_2 + s_3)$ derivatives. Omitting factors of 2 and i, we have, schematically,

$$
\begin{gathered}
C^{\mathrm{TT}} = g_{\mathrm{non}}\,Y_1^{s_1}\,Y_2^{s_2}\,Y_3^{s_3} + g_{\mathrm{non,PO}}\,V_1\,Y_1^{s_1}\,Y_2^{s_2-1}\,Y_3^{s_3-1}\\
\updownarrow\\
(g_{\mathrm{non}} + g_{\mathrm{non,PO}})\,[1,2]^{s_1+s_2-s_3}\,[2,3]^{s_2+s_3-s_1}\,[3,1]^{s_3+s_1-s_2}\,\partial_{\chi_1}^{s_1}\,\partial_{\chi_2}^{s_2}\,\partial_{\chi_3}^{s_2}\\
+\,(g_{\mathrm{non}} - g_{\mathrm{non,PO}})\,\langle 1,2\rangle^{s_1+s_2-s_3}\,\langle 2,3\rangle^{s_2+s_3-s_1}\,\langle 3,1\rangle^{s_3+s_1-s_2}\,\partial_{\bar{\chi}_1}^{s_1}\,\partial_{\bar{\chi}_2}^{s_2}\,\partial_{\bar{\chi}_3}^{s_2}\,.
\end{gathered}
\tag{3.35}
$$

In view of equation (3.21), we see that this vertex corresponds to the on-shell amplitude (3.3) in the first helicity configuration of (3.4), identifying the coupling constants as

$$g_{\mathrm{H}} \sim g_{\mathrm{non}} - g_{\mathrm{non,PO}}\,, \qquad g_{\mathrm{A}} \sim g_{\mathrm{non}} + g_{\mathrm{non,PO}}\,. \tag{3.36}$$

When one of the vertices with well-defined parity is not present, we see that g_{H} and g_{A} are indeed related.

For the minimal-coupling vertices with $s_1 < s_2 \leq s_3$, we get

$$\begin{gathered}
C^{\mathrm{TT}} = g_{\mathrm{min}}\, G^{s_1}\, Y_2^{s_2}\, Y_3^{s_3} + g_{\mathrm{min,PO}}\, V_1\, G^{s_1}\, Y_2^{s_2-1}\, Y_3^{s_3-1} \\
\updownarrow \\
g_{\mathrm{A}} \frac{[2,3]^{s_1+s_2+s_3}}{[1,2]^{s_1+s_3-s_2}[3,1]^{s_1+s_2-s_3}}\, \partial_{\bar\chi 1}^{s_1}\, \partial_{\chi 2}^{s_2}\, \partial_{\chi 3}^{s_3} + g_{\mathrm{H}} \frac{\langle 2,3\rangle^{s_1+s_2+s_3}}{\langle 1,2\rangle^{s_1+s_3-s_2}\langle 3,1\rangle^{s_1+s_2-s_3}}\, \partial_{\chi 1}^{s_1}\, \partial_{\bar\chi 2}^{s_2}\, \partial_{\bar\chi 3}^{s_3} \\
+ g_{\mathrm{A}} f_2 \frac{[3,1]^{s_1+s_2+s_3}}{[1,2]^{s_2+s_3-s_1}[2,3]^{s_1+s_2-s_3}}\, \partial_{\chi 1}^{s_1}\, \partial_{\bar\chi 2}^{s_2}\, \partial_{\chi 3}^{s_3} + g_{\mathrm{H}} f_2 \frac{\langle 3,1\rangle^{s_1+s_2+s_3}}{\langle 1,2\rangle^{s_2+s_3-s_1}\langle 2,3\rangle^{s_1+s_2-s_3}}\, \partial_{\bar\chi 1}^{s_1}\, \partial_{\chi 2}^{s_2}\, \partial_{\bar\chi 3}^{s_3} \\
+ g_{\mathrm{A}} f_3 \frac{[1,2]^{s_1+s_2+s_3}}{[2,3]^{s_1+s_3-s_2}[3,1]^{s_2+s_3-s_1}}\, \partial_{\chi 1}^{s_1}\, \partial_{\chi 2}^{s_2}\, \partial_{\bar\chi 3}^{s_3} + g_{\mathrm{H}} f_3 \frac{\langle 1,2\rangle^{s_1+s_2+s_3}}{\langle 2,3\rangle^{s_1+s_3-s_2}\langle 3,1\rangle^{s_2+s_3-s_1}}\, \partial_{\bar\chi 1}^{s_1}\, \partial_{\bar\chi 2}^{s_2}\, \partial_{\chi 3}^{s_3}\,.
\end{gathered} \tag{3.37}$$

where we have made the identifications:

$$g_{\mathrm{H}} \sim g_{\mathrm{min}} - g_{\mathrm{min,PO}}\,, \qquad g_{\mathrm{A}} \sim g_{\mathrm{min}} + g_{\mathrm{min,PO}}\,, \tag{3.38}$$

and have denoted

$$f_2 = \left(\frac{\langle 1,2\rangle[1,2]\langle 2,3\rangle[2,3]}{\langle 3,1\rangle[3,1]}\right)^{s_2-s_1}\,, \quad f_3 = \left(\frac{\langle 3,1\rangle[3,1]\langle 2,3\rangle[2,3]}{\langle 1,2\rangle[1,2]}\right)^{s_3-s_1}\,. \tag{3.39}$$

We see then that the minimal-coupling vertices potentially give the scattering amplitudes (3.3) in the last three helicity configurations of (3.4), completing the match among cubic vertices and on-shell amplitudes. However, the helicity configurations $(\pm s_1, \mp s_2, \pm s_3)$ and $(\pm s_1, \pm s_2, \mp s_3)$ are problematic as the corresponding amplitudes that come from cubic vertices are singular as they are dressed with the singular factors f_2 or f_3. In order to get finite amplitudes in these helicity configurations, a certain form of non-locality should be allowed in the cubic vertex that would absorb the singular factor. However, such non-locality may well violate other physical requirements, and we shall not pursue this line here. Notice though that we obtain in this way an indirect explanation of why the theories with $h_1 + h_2 + h_3 = 0$ are sick, as these can only be produced in these problematic helicity configurations (recall we assumed $s_1 \leq s_2 \leq s_3$).

In the special case of $s_1 = s_2 < s_3$, the factor f_2 becomes one, hence no more singular. In fact, this case is where the expression (3.19) involving V_1 is no longer valid and it should be replaced by (3.20). A new calculation shows a small deviation: the holomorphic and anti-holomorphic coupling constants of the second and third line of (3.37) are now related to g_{non} and $g_{\mathrm{non,PO}}$ as

$$g_{\mathrm{H}} \sim g_{\mathrm{min}} + g_{\mathrm{min,PO}}\,, \qquad g_{\mathrm{A}} \sim g_{\mathrm{min}} - g_{\mathrm{min,PO}}\,, \tag{3.40}$$

and the factor f_3 differs from that of (3.39) but anyway vanishes for conserved momenta. The last case left out is when all spins are equal $s_1 = s_2 = s_3$. In this case, there is no parity-odd minimal-coupling vertex whereas the amplitude has the two independent holomorphic and anti-holomorphic pieces.

The origin of this disparity may be again related to that of the problematic amplitudes for the helicity configurations $(\pm s_1, \mp s_2, \pm s_3)$ and $(\pm s_1, \pm s_2, \mp s_3)$, where the cubic vertices do provide the same amplitude structures, but they come with a factor which vanishes upon imposing momentum conservation. Actually, momentum conservation is what makes a total derivative vanish from the cubic interaction point of view. In fact, formally, we can construct many cubic interactions which are boundary terms. If we calculate their corresponding amplitudes, we would get formulas dressed again with singular factors like f_2 and f_3. In the case of a parity-odd vertex s–s–s with s derivatives, one can find vertices which are boundary terms and that presumably can reproduce the missing amplitudes with a singular factor.

As a final remark, we comment on the fact that while (3.3) is a non-perturbative result, the amplitudes produced from cubic vertices are presumably tree level. Actually, it looks reasonable that the arguments used to constrain the form of the cubic vertex (3.7) can be applied to the quantum effective action, as they are based just on gauge and Lorentz invariance. In such a case, the same non-perturbative conclusion is reached via cubic vertices.

4 Massive Interactions

In this section we analyze the match between three-point amplitudes and cubic vertices in the case where some of the particles/fields have a mass. There are three main cases to be distinguished, namely when only one, two, or the three particles are massive. These cases are divided into subcases depending on the relation among the masses if there are several of them. We will start by discussing the general way to proceed, and then illustrate it with the two simplest examples: when one particle is massive, and when two particles of the same mass interact with a massless one.

4.1 Generalities

We first quickly review some known facts about three-point amplitudes and cubic vertices involving massive fields, then show what will be the general strategy to match them.

Amplitudes

On the amplitude side, the classification of three-point amplitudes with the representation discussed in 2.2 was explicitly considered in [38]. We briefly summarize here the most salient features of the classification.

When only one particle is massive, the functional form of the amplitude is completely fixed up to a coupling constant, pretty much as it happens in the massless case. Nonetheless, there is a restriction on the helicities of the massless particles, say particles 1 and 2, depending on the spin s_3 of the massive particle. Namely, we must have $|h_1 - h_2| \leq s_3$.

This constraint can be physically understood as arising from the conservation of momentum and angular momentum, since the process is allowed for real kinematics.

If two particles, say 1 and 2, are massive we must separate the cases of equal and different masses. The equal-mass case is similar to the massless one in the sense that the process is kinematically forbidden for real momenta. The three-point amplitude contains $2\min(s_1, s_2) + 1$ "coupling constants", each accompanying a different functional structure. If the masses are different, then the number of structures depends on the precise relation between h_3 and s_1, s_2. This case is kinematically allowed for real momenta, and as in the case of the one massive leg, one gets a restriction $|h_3| \leq s_1 + s_2$ following from the conservation laws of momentum and angular momentum.

When the three particles are massive the functional form of the amplitude is much less constrained, and the number of possible kinematic structures grows quite large, being bounded by $(2s_i + 1)(2s_j + 1)$ if s_k is the biggest spin ($\{i, j, k\} = \{1, 2, 3\}$).

Vertices

The classification of parity-even cubic vertices with massive fields is done exactly as in Section 3.2, with just a few differences (see *e.g.* [42] for the details). One is that the gauge condition (3.8) needs only be imposed when particle I is massless. Another is that the presence of masses modifies the $[Y_I, u_I \cdot \partial_{x_I}]$ commutator in (3.11) as

$$[Y_I, u_I \cdot \partial_{x_I}] = \frac{m_I^2 + m_{I+1}^2 - m_{I-1}^2}{2} \,. \tag{4.1}$$

The last difference is that the massless Schouten identity $Y_1\, Y_2\, Y_3\, G \approx 0$ is modified to

$$Y_1\, Y_2\, Y_3\, G + \frac{1}{2}\left(\mu_1\, G_1^2 + \mu_2\, G_2^2 + \mu_3\, G_3^2\right) + \left(\mu_1\, \mu_2 + \mu_2\, \mu_3 + \mu_3\, \mu_1\right) Z_1\, Z_2\, Z_3 \approx 0\,, \tag{4.2}$$

where we are denoting $G_I = G - Y_I Z_I$ and $2\mu_I = m_{I+1}^2 + m_{I-1}^2 - m_I^2$. With all these ingredients it is simple to work out the form of the vertices for each of the cases specified above. For the sake of simplicity, we shall not complete the analysis with the parity-odd vertices for massive fields. Hence, we will not be able to check if opposite-helicity amplitudes can be independently produced from the local vertices.

Match

As done in Section 3.3, we want to extract the three-point amplitudes from the cubic vertices. For that, we just need to use formula (3.21), which involves the on-shell form of the fields. For a massless field, this was given in (3.26)-(3.27). Let us derive here the analogue for an on-shell massive field. We have to impose the two conditions

$$\left(\Box - m^2\right) \phi_{\text{O-S}}(x, u) = 0\,, \qquad \partial_u^2\, \phi_{\text{O-S}}(x, u) = 0\,. \tag{4.3}$$

The solution to the equation of motion is simply given by

$$\begin{aligned}\phi_{\text{O-S}}(x, u) = \int \frac{d^4\lambda\, d^4\tilde{\lambda}}{\text{vol}(U(2))}\, \delta\Big(\det(\lambda^I\, \tilde{\lambda}_I) + m^2\Big) \exp\left(\frac{i}{2}\, x^{a\dot{a}}\, \lambda^I{}_a\, \tilde{\lambda}_{I\dot{a}}\right) \times \\ \times \exp\left(u_{a\dot{a}}\, \frac{\partial^2}{\partial \chi_a\, \partial \bar{\chi}_{\dot{a}}}\right) \tilde{\phi}_{\text{O-S}}(\lambda^I, \tilde{\lambda}_I; \chi, \bar{\chi})\Big|_{\chi=0=\bar{\chi}}\,,\end{aligned} \tag{4.4}$$

where the division by the volume of $U(2)$ means that we quotient by the action of $U(2)$ on the space of $\tilde{\phi}_{\text{O-S}}$. This requires to fix an irrep of $U(2) = SU(2)\times U(1)$ that the $\tilde{\phi}_{\text{O-S}}$ should carry. Since the massive particles of the three-point amplitudes considered in [38] were in their lowest-weight state of $SU(2)$, we also assume here a lowest-weight representation:

$$\mathcal{K}_-\,\tilde{\phi}_{\text{O-S}} = \left(\lambda^1{}_a\,\frac{\partial}{\partial\lambda^2{}_a} - \tilde{\lambda}_{2\dot{a}}\,\frac{\partial}{\partial\tilde{\lambda}_{1\dot{a}}}\right)\tilde{\phi}_{\text{O-S}} = 0\,, \tag{4.5}$$

$$\mathcal{K}_0\,\tilde{\phi}_{\text{O-S}} = -\frac{1}{2}\left(\lambda^1{}_a\,\frac{\partial}{\partial\lambda^1{}_a} - \lambda^2{}_a\,\frac{\partial}{\partial\lambda^2{}_a} - \tilde{\lambda}_{1\dot{a}}\,\frac{\partial}{\partial\tilde{\lambda}_{1\dot{a}}} + \tilde{\lambda}_{2\dot{a}}\,\frac{\partial}{\partial\tilde{\lambda}_{2\dot{a}}}\right)\tilde{\phi}_{\text{O-S}} = s\,\tilde{\phi}_{\text{O-S}}\,, \tag{4.6}$$

where we are combining the $SU(2)$ generators defined in (2.20) as $\mathcal{K}_- = -\mathcal{K}^1_2$ and $\mathcal{K}_0 = \frac{1}{2}(\mathcal{K}^2_2 - \mathcal{K}^1_1)$. By imposing these conditions, the field, carrying a massive spin s representation, has the spin angular momentum $-s$ along the space-like direction:

$$Q_{a\dot{b}} = \lambda^1_a\,\tilde{\lambda}^1_{\dot{b}} - \lambda^2_a\,\tilde{\lambda}^2_{\dot{b}}\,. \tag{4.7}$$

See Appendix A for the details. Finally, by imposing the $U(1)$ condition,[4]

$$K\,\tilde{\phi}_{\text{O-S}} = \left(\lambda^I{}_a\,\frac{\partial}{\partial\lambda^I{}_a} - \tilde{\lambda}_{Ia}\,\frac{\partial}{\partial\tilde{\lambda}_{Ia}}\right)\tilde{\phi}_{\text{O-S}} = 0\,, \tag{4.8}$$

the transverse condition,

$$\lambda^I{}_a\,\tilde{\lambda}_{I\dot{a}}\,\frac{\partial^2}{\partial\chi_a\,\bar{\chi}_{\dot{a}}}\,\tilde{\phi}_{\text{O-S}}(\lambda^I,\tilde{\lambda}_I;\chi,\bar{\chi}) = 0\,, \tag{4.9}$$

can be solved by

$$\tilde{\phi}_{\text{O-S}}(\lambda^I,\tilde{\lambda}_I;\chi,\bar{\chi}) = \frac{1}{s!}\left(\chi_a\,\lambda^{1\,a}\,\bar{\chi}_{\dot{a}}\,\tilde{\lambda}_2{}^{\dot{a}}\right)^s\,\tilde{\phi}^{(s,-s)}_{\text{O-S}}(\lambda^I,\tilde{\lambda}_I)\,. \tag{4.10}$$

The wave-function $\tilde{\phi}^{(s,-s)}_{\text{O-S}}$ carries now the trivial representation under $U(2)$. With all the ingredients above, let us now proceed to perform the explicit match between cubic vertices and massive three-point amplitudes.

4.2 Two Massless and One Massive

The generic form of the amplitude for the interaction of two massless particles, with helicities h_1 and h_2, with a third massive particle with mass m and spin angular momentum $-s_3$ along the Q direction (4.7), is given by

$$M_3 = f_1(m,\langle 3,4\rangle)\,\langle 1,2\rangle^{-s_3-h_1-h_2}\langle 2,3\rangle^{h_1-h_2+s_3}\langle 3,1\rangle^{h_2-h_1+s_3}\,, \tag{4.11}$$

where we have parametrized the momenta as $p_1 = \lambda^1\tilde{\lambda}_1$, $p_2 = \lambda^2\tilde{\lambda}_2$ and $P_3 = \lambda^3\tilde{\lambda}_3 + \lambda^4\tilde{\lambda}_4$. In [38], the function f_1 was fixed to be constant. However, because of the condition (4.8),

[4] As we discussed below the equation (2.28), this is just a free choice we have in selecting a certain value for the representation label r. We notice here a small deviation with respect to [38], where instead of (4.8) the condition $K\,\tilde{\phi}_{\text{O-S}} = \mathcal{K}_0\,\tilde{\phi}_{\text{O-S}}$ is imposed as it was more natural for the reduction of massive amplitudes to massless ones. In this paper, we make the choice (4.8), which leads to a simpler expression for the on-shell fields in terms of the spinor-helicity variables.

$\sigma = s_3 - s_1 - s_2$	$\ldots, -4, -2$	$\ldots, -3, -1$	0	1	$2, 3, \ldots$
n	0	1	$0,\ 2+\sigma$	$1,\ 2+\sigma$	σ

Table 1. Selection among (4.13) of the possible cubic vertices for the interaction of one massive and two massless particles in four dimensions. In some cases, two vertices are possible.

here f_1 will be instead a homogeneous function of $\langle 3,4\rangle$ with weight $-s_3$. This amplitude is only allowed if

$$|h_1 - h_2| \leq s_3 \,. \tag{4.12}$$

Let us recover (4.11) and (4.12) from the cubic vertices.

Assuming without loss of generality that $s_1 \leq s_2$, the most general cubic vertex in this case takes the following form (see [42] for the derivation):

$$C^{\mathrm{TT}} = \sum_{n=\max\{0, s_3-s_2-s_1\}}^{s_3-s_2+s_1} \lambda_n^{(s_1,s_2,s_3)}\, H_1^{\frac{s_3+s_2-s_1-n}{2}}\, H_2^{\frac{s_3+s_1-s_2-n}{2}}\, H_3^{\frac{s_1+s_2-s_3+n}{2}}\, Y_3^n \,, \tag{4.13}$$

where we have introduced the following combinations

$$H_1 = Y_2\, Y_3 - \frac{m^2}{2}\, Z_1 \,, \quad H_2 = Y_3\, Y_1 - \frac{m^2}{2}\, Z_2 \,, \quad H_3 = Y_1\, Y_2 + \frac{m^2}{2}\, Z_3 \,, \tag{4.14}$$

which are useful because the Schouten identity (4.2) can be recast in this case as

$$H_1\, H_2\, H_3 \approx H_3^2\, Y_3^2 \,. \tag{4.15}$$

Using this identity, the number of possible vertices for given spins gets reduced to two or one, as shown in Table 1. These are the vertices that must be now used in (3.21). The action of the operators H_I and Y_3 on the on-shell fields casts as

$$\begin{aligned} -4H_1 &\leftrightarrow -\frac{m^4\langle 3,1\rangle^2}{\langle 1,2\rangle^2\langle 3,4\rangle}\partial_{\chi_2} + \frac{m^2\langle 2,3\rangle^2}{\langle 3,4\rangle}\partial_{\bar\chi_2} \,, \quad & -4H_2 &\leftrightarrow -\frac{m^4\langle 2,3\rangle^2}{\langle 1,2\rangle^2\langle 3,4\rangle}\partial_{\chi_1} + \frac{m^2\langle 3,1\rangle^2}{\langle 3,4\rangle}\partial_{\bar\chi_1} \,, \\ -4H_3 &\leftrightarrow \frac{m^2}{\langle 1,2\rangle^2}\partial_{\chi_1}\partial_{\chi_2} + \langle 1,2\rangle^2\partial_{\bar\chi_1}\partial_{\bar\chi_2} \,, & -2iY_3 &\leftrightarrow m^2\frac{\langle 3,1\rangle\langle 2,3\rangle}{\langle 1,2\rangle\langle 3,4\rangle} \,. \end{aligned} \tag{4.16}$$

The factors of m come from the use of momentum conservation, which tells us that a possible set of kinematically independent spinor products is $\{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 1,4\rangle, \langle 3,4\rangle\}$. The fact that $\langle 1,4\rangle$ does not appear in H_I, Y_3 parallels the fact that it neither appears in the amplitude (4.11). This is ultimately a consequence of the condition (4.5).

With expressions (4.16) at hand, it is immediate to reproduce the amplitude (4.11) from the vertices in Table 1. Omitting coupling constants and factors of 2 and i, the vertices with $\sigma \leq 1$ give

$$\begin{aligned} &\frac{m^{2s_1+2s_2+2s_3}}{\langle 3,4\rangle^{s_3}}\langle 1,2\rangle^{-s_1-s_2-s_3}\,\langle 2,3\rangle^{s_2+s_3-s_1}\,\langle 3,1\rangle^{s_3+s_1-s_2}\,\partial_{\chi_1}^{s_1}\,\partial_{\chi_2}^{s_2} \\ &+(-1)^{s_2-s_1}\,\frac{m^{2s_3}}{\langle 3,4\rangle^{s_3}}\langle 1,2\rangle^{s_1+s_2-s_3}\,\langle 2,3\rangle^{s_1+s_3-s_2}\,\langle 3,1\rangle^{s_3+s_2-s_1}\,\partial_{\bar\chi_1}^{s_1}\,\partial_{\bar\chi_2}^{s_2} \,, \end{aligned} \tag{4.17}$$

which correspond to the helicity configurations $(h_1, h_2) = (\pm s_1, \pm s_2)$. Notice that the two different off-shell vertices with $\sigma = 0, 1$ yield the same three-point on-shell amplitudes. Considering now the vertices with $\sigma \geq 2$, we get

$$\begin{aligned}
&\frac{m^{2s_1+2s_2+2s_3}}{\langle 3,4\rangle^{s_3}} \langle 1,2\rangle^{-s_1-s_2-s_3} \langle 2,3\rangle^{s_2+s_3-s_1} \langle 3,1\rangle^{s_3+s_1-s_2} \partial_{\chi 1}^{s_1} \partial_{\chi 2}^{s_2} \\
&+ (-1)^{s_2} \frac{m^{2s_1+2s_3}}{\langle 3,4\rangle^{s_3}} \langle 1,2\rangle^{s_2-s_1-s_3} \langle 2,3\rangle^{s_3-s_2-s_1} \langle 3,1\rangle^{s_1+s_2+s_3} \partial_{\chi 1}^{s_1} \partial_{\bar{\chi} 2}^{s_2} \\
&+ (-1)^{s_1} \frac{m^{2s_2+2s_3}}{\langle 3,4\rangle^{s_3}} \langle 1,2\rangle^{s_1-s_2-s_3} \langle 2,3\rangle^{s_1+s_2+s_3} \langle 3,1\rangle^{s_3-s_1-s_2} \partial_{\chi 1}^{s_1} \partial_{\bar{\chi} 2}^{s_2} \\
&+ (-1)^{s_2-s_1} \frac{m^{2s_3}}{\langle 3,4\rangle^{s_3}} \langle 1,2\rangle^{s_1+s_2-s_3} \langle 2,3\rangle^{s_1+s_3-s_2} \langle 3,1\rangle^{s_3+s_2-s_1} \partial_{\bar{\chi} 1}^{s_1} \partial_{\bar{\chi} 2}^{s_2} \,,
\end{aligned} \tag{4.18}$$

which also contains the helicity configurations $(h_1, h_2) = (\pm s_1, \mp s_2)$. Hence, interestingly, these configurations only occur when $s_3 \geq s_1 + s_2 + 2$. This is a restriction not captured by the amplitude, which allows these configurations also when $s_1 + s_2 = s_3 - 1, s_3$. While it would seem natural that the extra vertices in Table 1 that appear when $\sigma = 0, 1$ would match these configurations, this does not seem to be the case. We can only conjecture that, analogously to the massless case, the imposition of momentum conservation is preventing us from seeing these amplitudes from the cubic vertices.

To finish this subsection, let us also derive the condition (4.12) from the cubic-vertex analysis. One just needs to consider two cases. As we saw, the case where $(h_1, h_2) = (\pm s_1, \mp s_2)$ only happens when $s_3 \geq s_1 + s_2 + 2$, automatically implying that $s_3 > s_1 + s_2 = |h_1 - h_2|$. In the case where $(h_1, h_2) = (\pm s_1, \pm s_2)$, we have to check that $s_3 \geq s_2 - s_1$, which directly follows from the fact that the exponent of H_2 in (4.13) should be non-negative.

4.3 One Massless and Two Equal Massive

Let us start again by stating the result for the three-point amplitude obtained in [38]. We take particles 1 and 2 to have mass m, while particle 3 is massless. The spin angular momenta along the Q directions (4.7) of the massive particles are fixed to be $-s_1$ and $-s_2$, while the helicity of the third particle, h_3, is free. Without loss of generality, we assume that the spin of the first massive field is not larger than the second one: $s_1 \leq s_2$. Denoting the momenta as $P_1 = \lambda^1 \tilde{\lambda}_1 + \lambda^4 \tilde{\lambda}_4$, $P_2 = \lambda^2 \tilde{\lambda}_2 + \lambda^5 \tilde{\lambda}_5$ and $p_3 = \lambda^3 \tilde{\lambda}_3$, the amplitude takes the following form:

$$M_3 = f_2\Big(m, \langle 1,4\rangle, \langle 2,5\rangle, \frac{[4,5]}{\langle 1,2\rangle}\Big) \langle 1,2\rangle^{s_1+s_2+h_3} \langle 2,3\rangle^{s_2-s_1-h_3} \langle 3,1\rangle^{s_1-s_2-h_3} \,, \tag{4.19}$$

with the function f_2 equal to[5]

$$f_2 = \sum_{k=0}^{2s_1} c_k \Big(\frac{\langle 1,4\rangle}{m}, \frac{\langle 2,5\rangle}{m}\Big) \Big(1 - \frac{\langle 1,4\rangle\langle 2,5\rangle}{m^2} \frac{[4,5]}{\langle 1,2\rangle}\Big)^{s_1+s_2+h_3-k} \,, \tag{4.20}$$

[5] In [38] the combination appearing in f_2 was $\Big(1 + \frac{\langle 1,5\rangle\langle 2,4\rangle}{m^2} \frac{[4,5]}{\langle 1,2\rangle}\Big)$. The minus sign here is due to the definition of the angular bracket adopted there, $\langle I, J\rangle = \lambda^{I\,a} \lambda^{J}{}_{a}$, as opposed to $\langle I, J\rangle = \lambda^{I}{}_{a} \lambda^{J a}$ here.

where the free functions $c_k(x,y)$ were fixed to be constants in [38], but here the condition (4.8) will give homogeneous functions of weight $-s_1$ and $-s_2$ in $\langle 1,4\rangle$ and $\langle 2,5\rangle$ respectively. Notice we have $2s_1+1$ independent kinematic structures in (4.19), a conclusion we reproduce below via the analysis of cubic vertices.

In this case, the consistent cubic interaction is given in general dimensions by

$$C^{\rm TT} = \sum_{n,m} \lambda^{(s_1,s_2,s_3)}_{n,m}\, G^n\, Y_1^{s_1-n-m}\, Y_2^{s_2-n-m}\, Y_3^{s_3-n}\, Z_3^m\,. \tag{4.21}$$

Involving two parameters n and m, this type of cubic interactions contains many vertices with different number of derivatives. However, in four dimensions, again thanks to the Schouten identity (4.2), many of them trivialize. We can see that for two equal-mass massive particles the identity reduces to

$$Y_1\, Y_2\, Y_3\, G \approx -\frac{m^2}{2}\,(G - Y_3\, Z_3)^2\,. \tag{4.22}$$

Notice that the tensor structures involved in this identity have different number of derivatives. In order to remove the ambiguities due to (4.22), we take the representative vertex having the least derivatives. In other words, if a certain vertex contains the four-derivative structure $Y_1\,Y_2\,Y_3\,G$, then we replace it by the RHS of (4.22), having just two derivatives. Implementing this rule into the general form of interactions (4.21), we are left with three possibilities:

1. The first case is where the vertices do not involve any G:

$$C^{\rm TT} = \sum_m \lambda^{s_1\text{--}s_2\text{--}s_3}_{0,m}\, Y_1^{s_1-m}\, Y_2^{s_2-m}\, Y_3^{s_3}\, Z_3^m\,. \tag{4.23}$$

   In this case, the number of vertices are given by the possible values of m: there are s_1+1 vertices corresponding to $m=0,1,\ldots,s_1$.

2. The second case is where the vertices do not involve any Y_1:

$$C^{\rm TT} = \sum_n \lambda^{s_1\text{--}s_2\text{--}s_3}_{n,s_1-n}\, G^n\, Y_2^{s_2-s_1}\, Y_3^{s_3-n}\, Z_3^{s_1-n}\,. \tag{4.24}$$

   The possible vertices are parameterized by n: there are $\min\{s_1,s_3\}$ vertices corresponding to $n=1,2,\ldots,\min\{s_1,s_3\}$. We drop the possibility of $n=0$ as it overlaps with the case 1.

3. The final case is where the vertices do not involve any Y_3:

$$C^{\rm TT} = \sum_m \lambda^{s_1\text{--}s_2\text{--}s_3}_{s_3,m}\, G^{s_3}\, Y_1^{s_1-s_3-m}\, Y_2^{s_2-s_3-m}\, Z_3^m\,, \tag{4.25}$$

   which can occur only when $s_1\geq s_3$. There are s_1-s_3 possible vertices corresponding to $m=0,1,\ldots,s_1-s_3-1$. We drop the possibility of $m=s_1-s_3$ as it overlaps with the case 2.

Irrespective of what $\min\{s_1, s_3\}$ is, we see that there are always $2s_1 + 1$ possible vertices, coinciding with the number of different structures in the three-point amplitude. Moreover it is not difficult to see[6] that these vertices give rise to (4.19), with the c_k being linear combinations of the $\lambda^{(s_1,s_2,s_3)}_{s_3,m}$. It is sufficient to use the following expressions for the operators Y_I, Z_3, G,

$$\begin{aligned}
&-2iY_1 \leftrightarrow \frac{\langle 1,2\rangle\langle 3,1\rangle}{\langle 2,3\rangle}\frac{m^2}{\langle 1,4\rangle}\,\xi\,, \quad -2iY_3 \leftrightarrow -\frac{m^2\langle 1,2\rangle}{\langle 3,1\rangle\langle 2,3\rangle}\,\xi\,\partial_{\chi_3} + \frac{\langle 3,1\rangle\langle 2,3\rangle}{\langle 1,2\rangle}\,\xi^{-1}\,\partial_{\bar{\chi}_3}\,, \\
&-2iY_2 \leftrightarrow \frac{\langle 1,2\rangle\langle 2,3\rangle}{\langle 3,1\rangle}\frac{m^2}{\langle 2,5\rangle}\,\xi\,, \qquad 2Z_3 \leftrightarrow \langle 1,2\rangle[4,5]\,, \\
&\quad G \leftrightarrow -\frac{m^2\langle 1,2\rangle^3}{\langle 3,1\rangle\langle 2,3\rangle}\,\frac{m^2\,\xi}{\langle 1,4\rangle\langle 2,5\rangle}\partial_{\chi_3} + \langle 1,2\rangle\langle 3,1\rangle\langle 2,3\rangle\frac{\langle 1,4\rangle\langle 2,5\rangle}{m^2\,\xi}\left(\frac{[4,5]}{\langle 1,2\rangle}\right)^2\,\partial_{\bar{\chi}_3}\,,
\end{aligned} \tag{4.26}$$

where we have denoted

$$\xi = 1 - \frac{\langle 1,5\rangle\langle 2,4\rangle}{m^2}\,\frac{[4,5]}{\langle 1,2\rangle}\,. \tag{4.27}$$

The expressions in (4.26) are obtained by acting the operators Y_I, Z_3 and G on the on-shell fields (*cf.* equations (3.26), (3.27) for massless fields and (4.4), (4.10) for the massive ones), and using momentum conservation to express the twenty spinor products that one can build in terms of just eight of them. The independent set that we chose is the following: $\{\langle 1,2\rangle, \langle 3,1\rangle, \langle 2,3\rangle, \langle 1,4\rangle, \langle 1,5\rangle, \langle 2,4\rangle, \langle 2,5\rangle, [4,5]\}$. We can notice that, similarly to what happened in Section 4.2, the products $\langle 1,5\rangle, \langle 2,4\rangle$ do not appear in (4.26), consistently agreeing with the fact that the lowest-weight amplitude (4.19) does not depend on them either.

5 Conclusion

In this paper we have shown how the spinor-helicity three-point amplitudes can be produced from the local cubic-interaction vertices. Relating on-shell local fields to wave-functions in the spinor-helicity variables, we could derive the relations between the *building blocks* of the vertices such as Y_I's and G to simple rational functions of spinor contractions, $\langle I, J\rangle$ or $[I, J]$. These relations were then used to find the precise dictionary between the complete cubic vertices and three-point amplitudes.

Our result shows that most of the amplitude structures can be reproduced from the cubic interactions. In particular, the independence between the holomorphic and anti-holomorphic amplitudes could be obtained by including both parity-even and -odd vertices, as we have checked through the massless cases. Nonetheless, there remain several amplitudes which do not appear from the cubic interactions if we strictly impose the momentum conservation condition. However, when the latter condition is relaxed at intermediate levels, we could observe the missing amplitudes do appear but together with some factors which vanish when the momentum conservation condition is imposed. It seems that from the Poincaré covariance of the amplitude, the nature of locality is not transparent enough

[6] For the comparison, it is better to rewrite (4.20) as $f_2 = \xi^{-s_1+s_2+h_3}\sum_{k=0}^{2s_1} c_k\,\xi^k$, where what remains inside the sum is a polynomial in ξ, or alternatively in $\frac{[4,5]}{\langle 1,2\rangle}$, of order $2s_1+1$.

and there do exist more structures than what is allowed by the locality of the Lagrangian, although presumably boundary terms may produce these extra structures with singular factors.

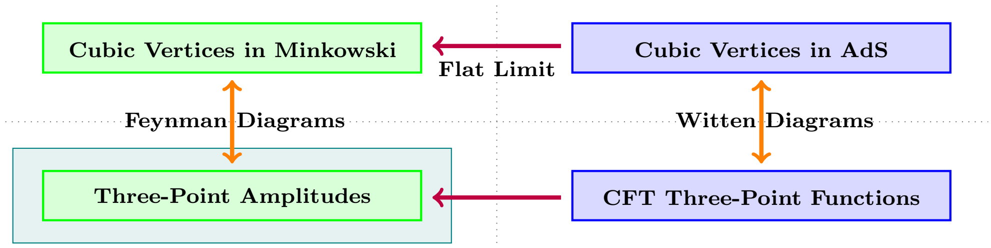

Figure 1. Schematic relation among flat/AdS Local Field Theories and amplitudes/correlators. The shaded region in the left-down corner corresponds to the amplitudes which satisfy the invariance condition but are not realized by Local Field Theory.

This small discrepancy is somewhat curious when viewing the matching procedure as the flat limit of the AdS/CFT duality. On the one hand, there are exactly the same number of local cubic interactions in AdS spacetime as in flat spacetime [43, 44] (while the classification of deforming and non-deforming vertices differs [45]). Actually, the AdS vertices can be obtained from the flat vertices by adding proper lower derivative terms required to compensate the non-commutativity of the AdS covariant derivatives. On the other hand, independent structures of CFT three-point functions can be identified by asking the invariance under the conformal group, which is isomorphic to the isometry group of AdS [46, 47]. In a sense, the CFT three-point functions are the AdS analog of the scattering amplitudes because the way they are determined is the same: by requiring invariance under the isometry group. It turns out that the number of AdS vertices and the number of CFT three-point functions exactly match, even though explicit links between them have not been made yet (see however [48] where massless AdS vertices have been determined from the three-point functions of free scalar CFT). Hence, it seems that there is no discrepancy in the AdS case like the one present for flat space. It is not clear what makes the Poincaré invariance — $\mathfrak{iso}(3,1)$ — differ from the AdS invariance — $\mathfrak{so}(3,2)$, but we presume that it is the algebraic nature: the latter algebra is simple but the former is not.

Acknowledgments

We thank Dario Francia and Andrea Marzolla for useful discussions and valuable feedback on the first draft. The work of EC and EJ was supported by the National Research Foundation of Korea through the grant NRF-2014R1A6A3A04056670. The work of EJ was supported by the Russian Science Foundation grant 14-42-00047 associated with Lebedev Institute. The work of KM was supported by the BK21 Plus Program funded by the Ministry of Education (MOE, Korea) and National Research Foundation of Korea (NRF). This research was also supported by the Munich Institute for Astro- and Particle Physics (MIAPP) of the DFG cluster of excellence "Origin and Structure of the Universe".

A Identifying massive states

In this section, we provide an explicit analysis of the $n=2$ spinor-helicity representation considered in Section 2.2. In particular, we will show how a wave-function,

$$\langle \lambda, \mu \,|\Psi\rangle = \langle \rho_a, \alpha_a, \beta_a, \gamma_a \,|\Psi\rangle \,, \tag{A.1}$$

is related to the usual wave-function of a massive particle. Here, λ_a and μ_a denote the two copies of spinors λ_a^I $(I=1,2)$, which are parameterized by eight variables $(\rho_a, \alpha_a, \beta_a, \gamma_a)$ as

$$\begin{aligned} \lambda_1 &= \rho_1 \cos\alpha_1 \, e^{i\theta_1} \,, \quad \lambda_2 = \rho_2 \cos\alpha_2 \, e^{i\theta_2} \,, \\ \mu_1 &= \rho_1 \sin\alpha_1 \, e^{i\phi_1} \,, \quad \mu_2 = \rho_2 \sin\alpha_2 \, e^{i\phi_2} \,, \end{aligned} \tag{A.2}$$

where we have redefined the angular variables as

$$\alpha_\pm = \alpha_1 \pm \alpha_2 \,, \quad \beta_\pm = \theta_1 - \phi_1 \pm (\theta_2 - \phi_2) \,, \quad \gamma_\pm = \theta_1 + \phi_1 \pm (\theta_2 + \phi_2) \,. \tag{A.3}$$

We take the state vector $|\Psi\rangle$ to be the eigenstate of momentum $P_{a\dot{b}}$ and K generators, $|\Psi\rangle = |\, p \otimes r \otimes \psi \,\rangle$:

$$P_{a\dot{b}} \,|\, p \otimes r \otimes \psi \,\rangle = p_{a\dot{b}} \,|\, p \otimes r \otimes \psi \,\rangle \,, \qquad K \,|\, p \otimes r \otimes \psi \,\rangle = r \,|\, p \otimes r \otimes \psi \,\rangle \,, \tag{A.4}$$

where $|\psi\rangle$ stands for the part of the state which is not yet determined by the $P_{a\dot{b}}$ and K conditions. For the sake of clarity and simplicity, we assume henceforth the momentum to be in the rest frame:

$$p_{a\dot{b}} = m\, \delta_{ab} \,. \tag{A.5}$$

This is translated into the following set of equations:

$$\rho_1 = \rho_2 = \sqrt{m} \,, \qquad \cos\alpha_+ = 0 \,, \qquad e^{i\,\beta_-} = -1 \,. \tag{A.6}$$

After fixing the kinematic condition (A.6), the functional dependence of the wave-function reduces to just three angular variables, $\alpha = \alpha_-$, $\beta = \beta_+$, $\gamma = \gamma_-$, as

$$\langle \rho_a, \alpha_a, \beta_a, \gamma_a \,|\, p \otimes r \otimes \psi \rangle = \delta(\rho_1 - \sqrt{m})\, \delta(\rho_2 - \sqrt{m})\, \delta(\cos\alpha_+)\, \delta(e^{i\,\beta_-} - 1)\, e^{i\,r\,\frac{\gamma_+}{4}} \,\langle \alpha, \beta, \gamma \,|\psi\rangle \,.$$

The actions of the generators K and $\mathcal{K}^I_J$ are realized by the differential operators $K = -4\,i\,\partial_{\gamma_+}$, and

$$\mathcal{K}_0 = \frac{1}{2}\left(\mathcal{K}^2_2 - \mathcal{K}^1_1\right) = 2\,i\,\partial_\beta \,, \qquad \mathcal{K}_- = -\mathcal{K}^1_2 = -i\, e^{\frac{i\beta}{2}} \left(\partial_\alpha - 2\,i\, \tan\alpha\, \partial_\beta - 2\,i\, \sec\alpha\, \partial_\gamma\right) \,. \tag{A.7}$$

The latter enjoy the commutation relations of $\mathfrak{su}(2)$:

$$[\mathcal{K}_+, \mathcal{K}_-] = 2\,\mathcal{K}_0 \,, \qquad [\mathcal{K}_0, \mathcal{K}_\pm] = \pm \mathcal{K}_\pm \,. \tag{A.8}$$

where with our conventions $\mathcal{K}_+ = -\mathcal{K}_-^\dagger$. Now we move to the little group $SO(3)$ operators, which leave (A.5) invariant. They are the combinations:

$$J_3 = L_{12} - \tilde{L}_{\dot{1}\dot{2}} = 2\,i\,\partial_\gamma \,, \qquad J_- = -L_{11} - \tilde{L}_{\dot{2}\dot{2}} = -i\, e^{\frac{i\gamma}{2}} \left(\partial_\alpha - 2\,i\, \tan\alpha\, \partial_\gamma - 2\,i\, \sec\alpha\, \partial_\beta\right) \,, \tag{A.9}$$

with the commutation relations (again with $J_+ = -J_-^\dagger$),

$$[J_+, J_-] = 2\,J_3\,, \qquad [J_3, J_\pm] = \pm\,J_\pm\,. \tag{A.10}$$

The quadratic Casimir of the above two algebras coincide:

$$\begin{aligned} C_2 &= {\mathcal{K}_0}^2 + \frac{1}{2}\{\mathcal{K}_+, \mathcal{K}_-\} = {J_3}^2 + \frac{1}{2}\{J_+, J_-\} \\ &= -\partial_\alpha^2 + \tan\alpha\,\partial_\alpha - 4\,\sec^2\alpha\left(\partial_\beta^2 + \partial_\gamma^2 + 2\,\sin\alpha\,\partial_\beta\,\partial_g\right)\,. \end{aligned} \tag{A.11}$$

Now, we can fix the state vector $|\psi\rangle$ to carry a UIR of $\mathcal{K}_i$ and J_i. For instance, we can choose $|\psi\rangle = |s, h_{\rm K}, h_{\rm J}\rangle$ to be an eigenstate of $\mathcal{K}_0$ and J_0 generators:

$$C_2\,|s, h_{\rm K}, h_{\rm J}\rangle = s(s+1)\,|s, h_{\rm K}, h_{\rm J}\rangle\,, \tag{A.12}$$

$$\mathcal{K}_0\,|s, h_{\rm K}, h_{\rm J}\rangle = h_{\rm K}\,|s, h_{\rm K}, h_{\rm J}\rangle\,, \tag{A.13}$$

$$J_3\,|s, h_{\rm K}, h_{\rm J}\rangle = h_{\rm J}\,|s, h_{\rm K}, h_{\rm J}\rangle\,. \tag{A.14}$$

From the expressions (A.8) and (A.9), we can conclude that $\langle\alpha, \beta, \gamma\,|s, h_{\rm K}, h_{\rm J}\rangle$ coincides with the Wigner function,

$$\langle\alpha, \beta, \gamma\,|s, h_{\rm K}, h_{\rm J}\rangle = \langle s, h_{\rm K}|\,R(\alpha, \beta, \gamma)\,|s, h_{\rm J}\rangle\,, \tag{A.15}$$

where $|s, h_{\rm J}\rangle$ is the eigenstate of $\mathfrak{su}(2)$ and $R(\alpha, \beta, \gamma)$ is an element of $SU(2)$ with (α, β, γ) related to the Euler angles. The $\mathcal{K}_i$ and J_i actions are realized respectively by the left and right multiplications on the element $R(\alpha, \beta, \gamma)$.

In this paper, we have not diagonalized the state vector with respect to J_3, but only with respect to $\mathcal{K}_0$ and C_2, hence it remains as a generic linear combination:

$$|\psi^s_{h_{\rm K}}\rangle = \sum_{h_{\rm J}} c_{h_{\rm J}}\,|s, h_{\rm K}, h_{\rm J}\rangle\,. \tag{A.16}$$

Defining similarly $|\psi^s\rangle = \sum_{h_{\rm J}} c_{h_{\rm J}}\,|s, h_{\rm J}\rangle$, we get

$$\langle\alpha, \beta, \gamma\,|\psi^s_{h_{\rm K}}\rangle = \langle s, h_{\rm K}|\,R(\alpha, \beta, \gamma)\,|\psi^s\rangle\,. \tag{A.17}$$

Even though the state vector $|\psi^s_{h_{\rm K}}\rangle$ is an undetermined one, the wave-function $\langle\alpha, \beta, \gamma\,|\psi^s_{h_{\rm K}}\rangle$ admits an intuitive interpretation as the $\vec{J}\cdot\hat{u}$ eigenstate with eigenvalue $h_{\rm K}$:

$$\langle\alpha, \beta, \gamma\,|\,\vec{J}\cdot\hat{u}\,|\psi^s_{h_{\rm K}}\rangle = h_{\rm K}\,\langle\alpha, \beta, \gamma\,|\psi^s_{h_{\rm K}}\rangle\,, \tag{A.18}$$

where the unit vector $\hat{u}$ is the rotation of $\hat{e}_3$ by $R(\alpha, \beta, \gamma)$:

$$\vec{J}\cdot\hat{u} = R_J^{-1}(\alpha, \beta, \gamma)\,J_3\,R_J(\alpha, \beta, \gamma)\,. \tag{A.19}$$

Relaxing the rest frame condition (A.5), we can also find the 'covariant' form of the four-vector Q — which reduces to $(0, m\,\hat{u})$ in the rest frame — as

$$Q_{a\dot{b}} = \lambda_a\,\tilde{\lambda}_{\dot{b}} - \mu_a\,\tilde{\mu}_{\dot{b}}\,. \tag{A.20}$$

It is space-like and orthogonal to the momentum vector:

$$Q^2 = m^2 , \qquad P_{a\dot{b}}\, Q^{a\dot{b}} = 0 . \tag{A.21}$$

Differently from $P_{a\dot{b}}$, the vector $Q_{a\dot{b}}$ does not commute with $\mathcal{K}^I_J$, and most importantly it satisfies

$$Q^{a\dot{b}}\, W_{a\dot{b}} = -2m^2\, \mathcal{K}_0 , \tag{A.22}$$

whereas $P^{a\dot{b}}\, W_{a\dot{b}} = 0$.

References

[1] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, 2005.

[2] L. P. S. Singh and C. R. Hagen, *Lagrangian formulation for arbitrary spin. 1. The boson case*, Phys. Rev. **D9** (1974) 898–909.

[3] L. P. S. Singh and C. R. Hagen, *Lagrangian formulation for arbitrary spin. 2. The fermion case*, Phys. Rev. **D9** (1974) 910–920.

[4] C. Fronsdal, *Massless Fields with Integer Spin*, Phys. Rev. **D18** (1978) 3624.

[5] J. Fang and C. Fronsdal, *Massless Fields with Half Integral Spin*, Phys. Rev. **D18** (1978) 3630.

[6] J. A. Strathdee, J. F. Boyce, R. Delbourgo and A. Salam, *Partial wave analysis. I*, IC-67-9.

[7] A. K. H. Bengtsson, I. Bengtsson and L. Brink, *Cubic Interaction Terms for Arbitrary Spin*, Nucl. Phys. **B227** (1983) 31–40.

[8] A. K. H. Bengtsson, I. Bengtsson and N. Linden, *Interacting Higher Spin Gauge Fields on the Light Front*, Class. Quant. Grav. **4** (1987) 1333.

[9] E. S. Fradkin and R. R. Metsaev, *A Cubic interaction of totally symmetric massless representations of the Lorentz group in arbitrary dimensions*, Class. Quant. Grav. **8** (1991) L89–L94.

[10] F. A. Berends, G. J. H. Burgers and H. van Dam, *On the Theoretical Problems in Constructing Interactions Involving Higher Spin Massless Particles*, Nucl. Phys. **B260** (1985) 295–322.

[11] S. Weinberg, *Photons and Gravitons in s Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass*, Phys. Rev. **135** (1964) B1049–B1056.

[12] S. R. Coleman and J. Mandula, *All Possible Symmetries of the S Matrix*, Phys. Rev. **159** (1967) 1251–1256.

[13] C. Aragone and S. Deser, *Consistency Problems of Hypergravity*, Phys. Lett. **B86** (1979) 161–163.

[14] R. R. Metsaev, *Poincare invariant dynamics of massless higher spins: Fourth order analysis on mass shell*, Mod. Phys. Lett. **A6** (1991) 359–367.

[15] R. R. Metsaev, *S matrix approach to massless higher spins theory. 2: The Case of internal symmetry*, Mod. Phys. Lett. **A6** (1991) 2411–2421.

[16] E. S. Fradkin and M. A. Vasiliev, *Cubic Interaction in Extended Theories of Massless Higher Spin Fields*, *Nucl. Phys.* **B291** (1987) 141–171.

[17] M. A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions*, *Phys. Lett.* **B243** (1990) 378–382.

[18] M. A. Vasiliev, *Cubic interactions of bosonic higher spin gauge fields in AdS(5)*, *Nucl. Phys.* **B616** (2001) 106–162, [hep-th/0106200].

[19] R. R. Metsaev, *Cubic interaction vertices of massive and massless higher spin fields*, *Nucl. Phys.* **B759** (2006) 147–201, [hep-th/0512342].

[20] R. R. Metsaev, *Cubic interaction vertices for fermionic and bosonic arbitrary spin fields*, *Nucl. Phys.* **B859** (2012) 13–69, [0712.3526].

[21] X. Bekaert, N. Boulanger and S. Cnockaert, *Spin three gauge theory revisited*, *JHEP* **01** (2006) 052, [hep-th/0508048].

[22] N. Boulanger and S. Leclercq, *Consistent couplings between spin-2 and spin-3 massless fields*, *JHEP* **11** (2006) 034, [hep-th/0609221].

[23] Yu. M. Zinoviev, *On spin 3 interacting with gravity*, *Class. Quant. Grav.* **26** (2009) 035022, [0805.2226].

[24] R. Manvelyan, K. Mkrtchyan and W. Ruhl, *Off-shell construction of some trilinear higher spin gauge field interactions*, *Nucl. Phys.* **B826** (2010) 1–17, [0903.0243].

[25] R. Manvelyan, K. Mkrtchyan and W. Ruehl, *Direct Construction of A Cubic Selfinteraction for Higher Spin gauge Fields*, *Nucl. Phys.* **B844** (2011) 348–364, [1002.1358].

[26] R. Manvelyan, K. Mkrtchyan and W. Ruhl, *General trilinear interaction for arbitrary even higher spin gauge fields*, *Nucl. Phys.* **B836** (2010) 204–221, [1003.2877].

[27] A. Sagnotti and M. Taronna, *String Lessons for Higher-Spin Interactions*, *Nucl. Phys.* **B842** (2011) 299–361, [1006.5242].

[28] A. Fotopoulos and M. Tsulaia, *On the Tensionless Limit of String theory, Off - Shell Higher Spin Interaction Vertices and BCFW Recursion Relations*, *JHEP* **11** (2010) 086, [1009.0727].

[29] R. Manvelyan, K. Mkrtchyan and W. Ruehl, *A Generating function for the cubic interactions of higher spin fields*, *Phys. Lett.* **B696** (2011) 410–415, [1009.1054].

[30] R. R. Metsaev, *BRST-BV approach to cubic interaction vertices for massive and massless higher-spin fields*, *Phys. Lett.* **B720** (2013) 237–243, [1205.3131].

[31] X. Bekaert, N. Boulanger and P. Sundell, *How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples*, *Rev. Mod. Phys.* **84** (2012) 987–1009, [1007.0435].

[32] H. Elvang and Y.-t. Huang, *Scattering Amplitudes*, 1308.1697.

[33] L. J. Dixon, *A brief introduction to modern amplitude methods*, in *Proceedings, 2012 European School of High-Energy Physics (ESHEP 2012): La Pommeraye, Anjou, France, June 06-19, 2012*, pp. 31–67, 2014. 1310.5353. DOI.

[34] S. J. Parke and T. R. Taylor, *An Amplitude for n Gluon Scattering*, *Phys. Rev. Lett.* **56** (1986) 2459.

[35] F. A. Berends and W. T. Giele, *Recursive Calculations for Processes with n Gluons*,

Nucl. Phys. **B306** (1988) 759.

[36] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189, [0312171].

[37] P. Benincasa and F. Cachazo, *Consistency Conditions on the S-Matrix of Massless Particles*, 0705.4305.

[38] E. Conde and A. Marzolla, *Lorentz Constraints on Massive Three-Point Amplitudes*, 1601.08113.

[39] J. Penedones, *Writing CFT correlation functions as AdS scattering amplitudes*, *JHEP* **03** (2011) 025, [1011.1485].

[40] A. L. Fitzpatrick, J. Kaplan, J. Penedones, S. Raju and B. C. van Rees, *A Natural Language for AdS/CFT Correlators*, *JHEP* **11** (2011) 095, [1107.1499].

[41] Z. Bern, L. J. Dixon and D. A. Kosower, *On-shell recurrence relations for one-loop QCD amplitudes*, *Phys. Rev.* **D71** (2005) 105013, [hep-th/0501240].

[42] E. Joung, L. Lopez and M. Taronna, *On the cubic interactions of massive and partially-massless higher spins in (A)dS*, *JHEP* **07** (2012) 041, [1203.6578].

[43] M. A. Vasiliev, *Cubic Vertices for Symmetric Higher-Spin Gauge Fields in* $(A)dS_d$, *Nucl. Phys.* **B862** (2012) 341–408, [1108.5921].

[44] E. Joung and M. Taronna, *Cubic interactions of massless higher spins in (A)dS: metric-like approach*, *Nucl. Phys.* **B861** (2012) 145–174, [1110.5918].

[45] E. Joung and M. Taronna, *Cubic-interaction-induced deformations of higher-spin symmetries*, *JHEP* **03** (2014) 103, [1311.0242].

[46] M. S. Costa, J. Penedones, D. Poland and S. Rychkov, *Spinning Conformal Correlators*, *JHEP* **11** (2011) 071, [1107.3554].

[47] M. S. Costa, J. Penedones, D. Poland and S. Rychkov, *Spinning Conformal Blocks*, *JHEP* **11** (2011) 154, [1109.6321].

[48] C. Sleight and M. Taronna, *Higher-spin Interactions from CFT: The Complete Cubic Couplings*, 1603.00022.