Conformal invariant interaction of a scalar field with the higher spin field in AdS_D

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Abstract

The explicit form of linearized gauge invariant interactions of scalar and general higher even spin fields in the AdS_D space is obtained. In the case of general spin ℓ a generalized 'Weyl' transformation is proposed and the corresponding 'Weyl' invariant action is constructed. In both cases the invariant actions of the interacting higher even spin gauge field and the scalar field include the whole tower of invariant actions for couplings of the same scalar with all gauge fields of smaller even spin. For the particular value of $\ell = 4$ all results are in exact agreement with [1]

1 Introduction

After discovering the AdS_4/CFT_3 correspondence of the critical $O(N)$ sigma model [2] interest in the interacting theory of an arbitrary even high spin field drastically increased. So in the center of our attention is a theory of Fradkin-Vasiliev type [3] in the Fronsdal's metric formulation [4]. This case of AdS_D/CFT_{D-1} correspondence is also of great interest because supersymmetry and BPS arguments are absent and because both conformal points of the boundary theory (i.e. unstable free field theory and critical interacting point, in the large N limit) correspond to the same higher spin theory and are connected on the boundary by a Legendre transformation which corresponds to different boundary conditions (regular dimension one or shadow dimension two) in the quantization of the bulk scalar field [5]. Existence of this scalar field in higher spin gauge theory is also an interesting and important phenomenon and supports the spontaneous symmetry breaking mechanism and mass creation for initially massless gauge fields due to corresponding possible interactions (see for example [6],[7]). From this point of view any construction of a reasonable even linearized interaction is an interesting and important task in this reconstruction of the higher spin gauge theory from the holographic dual CFT and can be controlled by corresponding information about the anomalous dimensions of the dual global symmetry currents that fulfill the conservation conditions in the large N limit. Therefore we see that construction of the conformal coupling of the scalar with a general even higher spin gauge field appears as an interesting example of an interaction which is applicable for many different quantum one-loop calculations such as the trace anomaly of the scalar in the external higher spin gauge field and so on [8].

In this article we construct a generalization of the well known action for the conformally coupled scalar field in D dimensions in external gravity

$$S = \frac{1}{2}\int d^D z\sqrt{-G}\left[G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{(D-2)}{4(D-1)}R(G)\phi^2\right]. \tag{1}$$

to the coupling with the linearized external higher spin ℓ gauge field. Actually in this article we succeeded to generalize the result of [1] for spin four, obtained four years ago, to the general spin ℓ case. We show that the gauge and 'Weyl' invariant interaction of the scalar with the spin ℓ Fronsdal gauge field can be constructed only if we add the same type of interaction with all lower even spin gauge fields. In other words we can construct a self-consistent interaction of a gauge field with the conformally coupled scalar only with the whole finite tower of gauge fields with spins in the range $2 \leq s \leq \ell$. In the next section we fix the notation and conventions and briefly review the results of [1]. In section 3 we explicitly construct a linearized interaction *Lagrangian* of the conformal scalar field with the spin ℓ gauge field using Noether's procedure for higher spin *gauge* invariance. In section 4 we extend our investigation including Noether's procedure for *generalized Weyl invariance* and obtain a unique interacting action after nontrivial and tedious calculations summarized in several appendices. Note also that some consideration of nonlinear gauge invariant couplings of the scalar field on the level of the equation of motion can be found in [9] and on the level of the BRST formalism for higher spin fields in [10]. Finalizing introduction we can say that this is a linearized interaction with the scalar for *conformal higher spin theory* of the type discussed in [11],[12].

2 The cases of spin two and spin four

We work in Euclidian AdS_D with the following metric, curvature and covariant derivatives:

$$
\begin{aligned}
& ds^2 = g_{\mu\nu}(z)dz^\mu dz^\nu = \frac{L^2}{(z^0)^2}\delta_{\mu\nu}dz^\mu dz^\nu, \quad \sqrt{-g} = \frac{L^D}{(z^0)^D}\ , \\
& [\nabla_\mu,\, \nabla_\nu]\, V^\rho_\lambda = R_{\mu\nu\lambda}{}^\sigma V^\rho_\sigma - R_{\mu\nu\sigma}{}^\rho V^\sigma_\lambda\ , \\
& R_{\mu\nu\lambda}{}^\rho = -\frac{1}{(z^0)^2}\left(\delta_{\mu\lambda}\delta^\rho_\nu - \delta_{\nu\lambda}\delta^\rho_\mu\right) = -\frac{1}{L^2}\left(g_{\mu\lambda}(z)\delta^\rho_\nu - g_{\nu\lambda}(z)\delta^\rho_\mu\right)\ , \\
& R_{\mu\nu} = -\frac{D-1}{(z^0)^2}\delta_{\mu\nu} = -\frac{D-1}{L^2}g_{\mu\nu}(z) \quad , \quad R = -\frac{(D-1)D}{L^2}\ .
\end{aligned}
$$

In [1] the authors constructed gauge and generalized Weyl invariant actions for spin two and four gauge fields interacting with a scalar field. Here we review these results in the form suitable for a generalization to arbitrary higher even spin fields*. For the case $l = 2$ one can see [1] that a Weyl invariant action is

$$
S^{WI}(\phi, h^{(2)}) = S_0(\phi) + S_1^{\Psi^{(2)}}(\phi, h^{(2)}) + S_1^{r^{(2)}}(\phi, h^{(2)}). \tag{2}
$$

where

$$
\begin{aligned}
S_0(\phi) &= \frac{1}{2}\int d^D z\sqrt{-g}[\nabla_\mu\phi\nabla^\mu\phi + \frac{D(D-2)}{4L^2}\phi^2], && (3) \\
S_1^{\Psi^{(2)}}(\phi, h^{(2)}) &= \frac{1}{2}\int d^D z\sqrt{-g}h^{(2)\mu\nu}\Psi^{(2)}_{\mu\nu}(\phi) && (4) \\
\Psi^{(2)}_{\mu\nu}(\phi) &= -\nabla_\mu\phi\nabla_\nu\phi + \frac{g_{\mu\nu}}{2}(\nabla_\mu\phi\nabla^\mu\phi + \frac{D(D-2)}{4L^2}\phi^2), && (5) \\
S_1^{r^{(2)}}(\phi, h^{(2)}) &= \frac{1}{8}\frac{D-2}{D-1}\int d^D z\sqrt{-g}r^{(2)}(h^{(2)})\phi^2, && (6) \\
r^{(2)}(h^{(2)}) &= \nabla_\mu\nabla_\nu h^{(2)\mu\nu} - \Box h^{(2)\mu}_\mu - \frac{D-1}{L^2}h^{(2)\mu}_\mu && (7)
\end{aligned}
$$

*We work with double traceless higher spin fields in Fronsdal's formulation [4].

which is of course the linearized form of (1) and is invariant with respect to the gauge and Weyl transformations

$$\delta^1_\varepsilon \phi = \varepsilon^\mu(z)\nabla_\mu\phi, \quad \delta^0_\varepsilon h^{(2)}_{\mu\nu} = 2\nabla_{(\mu}\varepsilon_{\nu)}; \tag{8}$$

$$\delta^1_\sigma \phi(z) = \Delta\sigma(z)\phi(z), \quad \delta^0_\sigma h^{(2)}_{\mu\nu} = 2\sigma(z)g_{\mu\nu}. \tag{9}$$

Now we turn to the case $l = 4$. In [1] the authors started from the action (3) and applied Noether's procedure using the following higher spin 'reparametrization' of the scalar field with a traceless third rank symmetric tensor parameter

$$\delta^1_\epsilon \phi(z) = \epsilon^{\mu\nu\lambda}(z)\nabla_\mu\nabla_\nu\nabla_\lambda\phi(z), \quad \epsilon^\alpha_{\alpha\mu} = 0. \tag{10}$$

The variation of (3) is[†]

$$\begin{aligned}\delta^1_\epsilon S_0(\phi) = \int d^D z\sqrt{-g}\{&-\nabla^{(\alpha}\epsilon^{\mu\nu\lambda)}\nabla_\mu\nabla_\alpha\phi\nabla_\nu\nabla_\lambda\phi + \epsilon^{\mu\nu}_{(1)}[\frac{1}{2}\nabla_\mu\nabla_\alpha\phi\nabla_\nu\nabla^\alpha\phi \\ &+\frac{D(D+2)}{8L^2}\nabla_\mu\phi\nabla_\nu\phi] - \nabla^{(\mu}\epsilon^{\nu)}_{(2)}[-\nabla_\mu\phi\nabla_\nu\phi + \frac{g_{\mu\nu}}{2}(\nabla_\lambda\phi\nabla^\lambda\phi + \frac{D(D-2)}{4L^2}\phi^2)] \\ &+[\nabla^2\phi - \frac{D(D-2)}{4L^2}\phi]\nabla_\mu(\epsilon^{\mu\nu}_{(1)}\nabla_\nu\phi)\}.\end{aligned} \tag{12}$$

We see immediately that the first two lines of (12) produce interactions with the spin four and two currents. From the other hand the last line in (12) is proportional to the equation of motion following from $S_0(\phi)$ and therefore can be absorbed after gauging by the trace of the spin four gauge field ($2\epsilon^{\mu\nu}_{(1)} \to h^{(4)\alpha\mu\nu}_\alpha$) performing the following field redefinition of ϕ

$$\phi \to \phi + \frac{1}{2}\nabla_\mu(h^{(4)\alpha\mu\nu}_\alpha\nabla_\nu\phi) \tag{13}$$

Such a type of field redefinition is a standard correction of Noether's procedure and means that we always can drop from the cubic part of the action terms proportional to the equation of motion following from the quadratic part of the initial action.

So finally we see that the action

$$S^{GI}(\phi, h^{(2)}, h^{(4)}) = S_0(\phi) + S^{\Psi^{(2)}}_1(\phi, h^{(2)}) + S^{\Psi^{(4)}}_1(\phi, h^{(4)}), \tag{14}$$

where $S_0(\phi)$, $S^{\Psi^{(2)}}_1(\phi, h^{(2)})$ are defined in (3)-(5) and

$$S^{\Psi^{(4)}}_1(\phi, h^{(4)}) = \frac{1}{4}\int d^D z\sqrt{-g}h^{(4)\mu\nu\alpha\beta}\Psi^{(4)}_{\mu\nu\alpha\beta}(\phi) \tag{15}$$

$$\Psi^{(4)}_{\mu\nu\alpha\beta}(\phi) = \nabla_{(\mu}\nabla_\nu\phi\nabla_\alpha\nabla_{\beta)}\phi - g_{(\mu\nu}[\nabla_\alpha\nabla^\gamma\phi\nabla_{\beta)}\nabla_\gamma\phi + \frac{D(D+2)}{4L^2}\nabla_\alpha\phi\nabla_{\beta)}\phi], \tag{16}$$

[†]For compactness we introduce shortened notations for divergences of the tensor's symmetry parameters

$$\epsilon^{\mu\nu\dots}_{(1)} = \nabla_\lambda\epsilon^{\lambda\mu\nu\dots}, \quad \epsilon^{\mu\dots}_{(2)} = \nabla_\nu\nabla_\lambda\epsilon^{\nu\lambda\mu\dots}, \quad \dots \tag{11}$$

is invariant with respect to the gauge transformations of the spin four field with an additional spin two field gauge transformation inspired by the second divergence of the spin four gauge parameter‡

$$\delta^1_\epsilon \phi(z) = \epsilon^{\mu\nu\lambda}(z)\nabla_\mu\nabla_\nu\nabla_\lambda\phi(z), \tag{17}$$
$$\delta^0_\epsilon h^{(4)\mu\nu\alpha\beta} = 4\nabla^{(\mu}\epsilon^{\nu\alpha\beta)}, \quad \delta^0_\epsilon h^{(4)\alpha\mu\nu}_{\alpha} = 2\epsilon^{\mu\nu}_{(1)}, \tag{18}$$
$$\delta^0_\epsilon h^{(2)\mu\nu} = 2\nabla^{(\mu}\epsilon^{\nu)}_{(2)}. \tag{19}$$

Thus we introduced a gauge invariant interaction of the scalar with the spin four gauge field $h^{(4)}_{\mu\nu\alpha\beta}$ in the minimal way. The next step is the spin four Weyl invariant interaction.

We write the generalized Weyl transformation law for the spin four case as in the [1]

$$\delta^0_\sigma h^{(4)\mu\nu\alpha\beta}(z) = 12\sigma^{(\mu\nu}(z)g^{\alpha\beta)}, \quad \delta^1_\sigma\phi(z) = \Delta_4\sigma^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi, \tag{20}$$

where we introduced a "conformal" weight Δ_4 for the scalar field. Then following [1] one can make (14) Weyl invariant introducing the following terms

$$S^{r^{(4)}}_1 = \frac{1}{2}\xi^1_4\int d^Dz\sqrt{-g}r^{(4)\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}\xi^0_4\int d^Dz\sqrt{-g}\nabla_\mu\nabla_\nu r^{(4)\mu\nu}\phi^2, \tag{21}$$

where§

$$r^{(4)\mu\nu} = \nabla_\alpha\nabla_\beta h^{(4)\alpha\beta\mu\nu} - \Box h^{(4)\alpha\mu\nu}_{\alpha} - \nabla^{(\mu}\nabla_\beta h^{(4)\nu)\beta\alpha}_{\alpha} - \frac{3(D+1)}{L^2}h^{(4)\alpha\mu\nu}_{\alpha}, \tag{22}$$
$$\delta^1_\epsilon r^{(4)\mu\nu} = 0, \quad r^{(4)\mu}_{\mu} = 0, \tag{23}$$
$$\xi^1_4 = -\frac{1}{4}\frac{D}{D+3}, \quad \xi^0_4 = \frac{1}{32}\frac{D(D-2)}{(D+1)(D+3)}, \quad \Delta_4 = \Delta = 1 - \frac{D}{2}. \tag{24}$$

Thus the linearized action for a scalar field interacting with the spin two and four fields in a conformally invariant way is

$$S^{WI}(\phi, h^{(2)}, h^{(4)}) = S^{WI}(\phi, h^{(2)}) + S^{\Psi^{(4)}}_1(\phi, h^{(4)}) + S^{r^{(4)}}_1(\phi, h^{(4)}), \tag{25}$$

which is invariant with respect to gauge and generalized Weyl transformations

$$\delta^1\phi = \varepsilon^\mu\nabla_\mu\phi + \epsilon^{\mu\nu\lambda}\nabla_\mu\nabla_\nu\nabla_\lambda\phi + \Delta\sigma\phi + \Delta\sigma^{\mu\nu}\nabla_\mu\nabla_\nu\phi, \tag{26}$$
$$\delta^0 h^{(2)\mu\nu} = 2\nabla^{(\mu}\varepsilon^{\nu)} + 2\nabla^{(\mu}\epsilon^{\nu)}_{(2)} + 2(1-\Delta-4D\xi^1_4)\nabla^{(\mu}\sigma^{\nu)}_{(1)}$$
$$+2\sigma g^{\mu\nu} + 2\xi^1_4\sigma_{(2)}g^{\mu\nu} \tag{27}$$
$$\delta^0 h^{(4)\mu\nu\alpha\beta} = 4\nabla^{(\mu}\epsilon^{\nu\alpha\beta)} + 12\sigma^{(\mu\nu}g^{\alpha\beta)}. \tag{28}$$

‡Note that the spin two part of our action continues to be invariant in respect of usual linearized reparametrization (8)

§We have to mention that our Δ_4 here differs from $\widetilde{\Delta}$ in [1] because we threw out terms, proportional to the equation of motion following from the quadratic action and therefore $S^{\Psi^{(4)}}_1$ from [1] turned into (16), in the Weyl variation of which we use denoted Δ_4.

3 Gauge invariant interaction for the spin ℓ case

Here we generalize our construction to the general spin ℓ case. Again following [1] we apply the following gauge transformation

$$\delta^1_\epsilon \phi(z) = \epsilon^{\mu_1\mu_2...\mu_{l-1}}_\ell(z)\nabla_{\mu_1}\nabla_{\mu_2}...\nabla_{\mu_{l-1}}\phi(z), \quad (29)$$

$$\delta^0_\epsilon h^{(\ell)\mu_1...\mu_l} = l\nabla^{(\mu_l}\epsilon^{\mu_1\mu_2...\mu_{l-1})}_\ell, \quad \delta^0_\epsilon h^{(\ell)\alpha\mu_1...\mu_{l-2}}_\alpha = 2\epsilon^{\mu_1...\mu_{l-2}}_{\ell(1)}, \quad (30)$$

$$\epsilon^\alpha_{\ell\alpha\mu_3...\mu_{l-1}} = 0 \quad (31)$$

to the action (3) and obtain the following starting variation for Noether's procedure

$$\delta^1_\epsilon S_0(\phi) = \int d^Dz\sqrt{-g}\{\nabla^\alpha\epsilon^{\mu_1...\mu_{l-1}}_\ell\nabla_\alpha\phi\nabla_{\mu_1}...\nabla_{\mu_{l-1}}\phi +$$
$$\epsilon^{\mu_1...\mu_{l-1}}_\ell\nabla_\alpha\phi\nabla^\alpha\nabla_{\mu_1}...\nabla_{\mu_{l-1}}\phi + \frac{D(D-2)}{4L^2}\epsilon^{\mu_1...\mu_{l-1}}_\ell\phi\nabla_{\mu_1}...\nabla_{\mu_{l-1}}\phi\}. \quad (32)$$

Using the following notations

$$T(n,k) = \nabla^\alpha\epsilon^{\mu_1...\mu_{n-1}}_{\ell(l-n)}\nabla_{\mu_1}...\nabla_{\mu_{k-1}}\nabla_\alpha\phi\nabla_{\mu_k}...\nabla_{\mu_{n-1}}\phi, \quad (33)$$

$$M(n,k) = \epsilon^{\mu_1...\mu_n}_{\ell(l-n-1)}\nabla_{\mu_1}...\nabla_{\mu_k}\nabla_\alpha\phi\nabla_{\mu_{k+1}}...\nabla_{\mu_n}\nabla^\alpha\phi, \quad (34)$$

$$N(n,k) = \epsilon^{\mu_1...\mu_n}_{\ell(l-n-1)}\nabla_{\mu_1}...\nabla_{\mu_k}\phi\nabla_{\mu_{k+1}}...\nabla_{\mu_n}\phi. \quad (35)$$

and commutation relation (B.1) from Appendix B we rewrite (32) in the form

$$\delta^1_\epsilon S_0(\phi) = \int d^Dz\sqrt{-g}\{T(l,1) + M(l-1,0) +$$
$$+\frac{(l-1)(l-2)}{2L^2}N(l-1,1) + \frac{D(D-2)}{4L^2}N(l-1,0)\}. \quad (36)$$

Then using relations between $T(m,n)$, $M(m,n)$ and $N(m,n)$ from Appendix A and after some algebra we 'diagonalize' (36)

$$\delta^1_\epsilon S_0(\phi) = \sum_{m=1}^{\frac{l}{2}}(-1)^m\binom{\ell-m-1}{m-1}\int d^Dz\sqrt{-g}\{-T(2m,m) + \frac{1}{2}M(2m-2,m-1)$$
$$+\frac{(D+2m-2)(D+2m-4)}{8L^2}N(2m-2,m-1)$$
$$-\frac{m-1}{l-2m+1}\epsilon^{\mu_1...\mu_{2m-2}}_{\ell(l-2m+1)}(\nabla_{\mu_1}...\nabla_{\mu_{m-1}}[\nabla^2\phi - \frac{D(D-2)}{4L^2}\phi]\nabla_{\mu_m}...\nabla_{\mu_{2m-2}}\phi)\} \quad (37)$$

Then performing a final symmetrization in (37) we obtain the following elegant expression

$$\delta^1_\epsilon S_0(\phi) = \int d^D z\sqrt{-g}\Big\{\sum_{m=1}^{\frac{l}{2}}\begin{pmatrix}\ell-m-1\\ m-1\end{pmatrix}[-\nabla^{(\mu_{2m}}\epsilon^{\mu_1\dots\mu_{2m-1})}_{\ell(l-2m)}\Psi^{(2m)}_{\mu_1\dots\mu_{2m}}]$$
$$+[\nabla^2\phi-\frac{D(D-2)}{4L^2}\phi]\sum_{m=2}^{\frac{l}{2}}\begin{pmatrix}\ell-m-1\\ m-2\end{pmatrix}\nabla_{\mu_1}\dots\nabla_{\mu_{m-1}}(\epsilon^{\mu_1\dots\mu_{2m-2}}_{\ell(l-2m+1)}\nabla_{\mu_m}\dots\nabla_{\mu_{2m-2}}\phi)\Big\}, \tag{38}$$

where

$$\Psi^{(2m)}_{\mu_1\dots\mu_{2m}} = (-1)^m\{\nabla_{\mu_1}\dots\nabla_{\mu_m}\phi\nabla_{\mu_{m+1}}\dots\nabla_{\mu_{2m}}\phi$$
$$-\frac{m}{2}g_{\mu_{2m-1}\mu_{2m}}g^{\alpha\beta}\nabla_{(\mu_1}\dots\nabla_{\mu_{m-1}}\nabla_{\alpha)}\phi\nabla_{(\mu_m}\dots\nabla_{\mu_{2m-2}}\nabla_{\beta)}\phi$$
$$-\frac{m(D+2m-2)(D+2m-4)}{8L^2}g_{\mu_{2m-1}\mu_{2m}}\nabla_{\mu_1}\dots\nabla_{\mu_{m-1}}\phi\nabla_{\mu_m}\dots\nabla_{\mu_{2m-2}}\phi\}, \tag{39}$$

and we admitted symmetrization for the set $\mu_1,\dots\mu_{2m}$ of indices. So we see that miraculously the coefficients in (39) don't depend on l ! All ℓ- dependence is concentrated in the second line of (38) proportional to the equation of motion for the action (3). This part like in the spin four case can be removed by appropriate field redefinition (see (44), (45), (B.6))

$$\phi \to \phi + \sum_{m=2}^{\frac{l}{2}}\frac{m-1}{2(l-2m+1)}\nabla_{\mu_1}\dots\nabla_{\mu_{m-1}}(h^{(2m)\alpha\mu_1\dots\mu_{2m-2}}_{\alpha}\nabla_{\mu_m}\dots\nabla_{\mu_{2m-2}}\phi) \tag{40}$$

and we can drop these terms from our consideration. Thus we arrive at the following spin ℓ gauge invariant action

$$S^{GI}(\phi,h^{(2)},h^{(4)},\dots,h^{(\ell)}) = S_0(\phi)+\sum_{m=1}^{\frac{l}{2}}S_1^{\Psi^{(2m)}}(\phi,h^{(2m)}) \tag{41}$$

where

$$S_1^{\Psi^{(2m)}}(\phi,h^{(2m)}) = \frac{1}{2m}\int d^D z\sqrt{-g}h^{(2m)\mu_1\dots\mu_{2m}}\Psi^{(2m)}_{\mu_1\dots\mu_{2m}}$$
$$=\frac{(-1)^m}{2m}\int d^D z\sqrt{-g}\{h^{(2m)\mu_1\dots\mu_{2m}}\nabla_{\mu_1}\dots\nabla_{\mu_m}\phi\nabla_{\mu_{m+1}}\dots\nabla_{\mu_{2m}}\phi$$
$$-\frac{m}{2}h^{(2m)\alpha\mu_1\dots\mu_{m-1}}_{\alpha\mu_m\dots\mu_{2m-2}}\nabla_{(\mu_1}\dots\nabla_{\mu_{m-1}}\nabla_{\mu)}\phi\nabla^{(\mu_m}\dots\nabla^{\mu_{2m-2}}\nabla^{\mu)}\phi$$
$$-\frac{m(D+2m-2)(D+2m-4)}{8L^2}h^{(2m)\alpha\mu_1\dots\mu_{2m-2}}_{\alpha}\nabla_{\mu_1}\dots\nabla_{\mu_{m-1}}\phi\nabla_{\mu_m}\dots\nabla_{\mu_{2m-2}}\phi\}, \tag{42}$$

and the final form of the improved gauge transformations

$$\delta^1_\epsilon \phi(z) = \epsilon_\ell^{\mu_1\mu_2...\mu_{l-1}}(z)\nabla_{\mu_1}\nabla_{\mu_2}...\nabla_{\mu_{l-1}}\phi(z), \tag{43}$$

$$\delta^0_\epsilon h^{(2m)\mu_1...\mu_{2m}} = 2m\nabla^{(\mu_{2m}}\varepsilon_\ell^{(2m)\mu_1...\mu_{2m-1})}, \quad \delta^0_\epsilon h_\alpha^{(2m)\alpha\mu_1...\mu_{2m-2}} = 2\varepsilon_{\ell(1)}^{(2m)\mu_1...\mu_{2m-2}}, \tag{44}$$

$$\varepsilon_\ell^{(2m)\mu_1...\mu_{2m-1}} = \binom{\ell-m-1}{m-1}\epsilon_{\ell(l-2m)}^{\mu_1...\mu_{2m-1}}. \tag{45}$$

Now we can put in (39) $m = \frac{l}{2}$ and compare our general expression for $S_1^{\Psi^{(l)}}(\phi, h^{(l)})$ with already known cases of spin two (5) and four (16). We can easily see that for these cases $S_1^{\Psi^{(l=2,4)}}(\phi, h^{(l)})$ exactly reproduces (5) and (16) respectively. So we found the gauge invariant action for a general spin l gauge field coupled to a scalar and this action has the following property: *The gauge invariant action $S^{GI}(\phi, h^{(2)}, h^{(4)}, ..., h^{(\ell)})$ for a spin l gauge field coupled to a scalar includes gauge invariant actions of a tower of all smaller even spin gauge fields coupled to the same scalar in an analogous way.*

4 Weyl invariant action for a higher spin field coupled to a scalar

In this section we introduce generalized Weyl transformations for higher spin fields and derive a Weyl invariant action for a higher spin field coupled to a scalar field. Following [1] we write the generalized Weyl transformation for the even spin l field in the form

$$\delta^0_\sigma h^{(\ell)\mu_1...\mu_l} = l(l-1)\sigma_\ell^{(\mu_1...\mu_{l-2}}g^{\mu_{l-1}\mu_l)}, \tag{46}$$

$$\delta^0_\sigma h_\alpha^{(\ell)\alpha\mu_1...\mu_{l-2}} = 2(D+2l-4)\sigma_\ell^{\mu_1...\mu_{l-2}}, \tag{47}$$

$$\delta^1_\sigma \phi = \Delta_\ell \sigma_\ell^{\mu_1...\mu_{l-2}}\nabla_{\mu_1}...\nabla_{\mu_{l-2}}\phi. \tag{48}$$

Then we assume that the Weyl invariant action for a spin l field should be accompanied with similar Weyl invariant actions for smaller spin gauge fields and therefore can be constructed from (41) adding $\frac{l}{2}$ additional terms

$$S^{WI}(\phi, h^{(2)}, h^{(4)}, ..., h^{(\ell)}) = S^{GI}(\phi, h^{(2)}, ..., h^{(\ell)}) + \sum_{m=1}^{l/2} S_1^{r^{(2m)}}(\phi, h^{(2m)}), \tag{49}$$

where each $S_1^{r^{(2m)}}$ is gauge invariant itself. In the case of spin two we had only the linearized Ricci scalar (see (6)) and for the spin four case we had two terms constructed from the spin four generalization of the Ricci scalar (see (21)). Now we will see that the generalization of the Ricci scalar for a higher spin field namely the trace of Fronsdal's operator [4],[7]

$$r^{(\ell)\mu_1\ldots\mu_{l-2}} = -\frac{1}{2}Tr\mathcal{F}(h^\ell) = \nabla_\alpha\nabla_\beta h^{(\ell)\alpha\beta\mu_1\ldots\mu_{l-2}} - \Box h_\alpha^{(\ell)\alpha\mu_1\ldots\mu_{l-2}}$$
$$-\frac{l-2}{2}\nabla^{(\mu_1}\nabla_\alpha h_\beta^{(\ell)\mu_2\ldots\mu_{l-2})\alpha\beta} - \frac{(l-1)(D+l-3)}{L^2}h_\alpha^{(\ell)\alpha\mu_1\ldots\mu_{l-2}}. \quad (50)$$

is the only gauge invariant combination of two derivatives and a higher spin field which we need to construct the Weyl invariant action (49) starting from (41). We will use the following strategy for solving our problem: We apply transformation (46)-(48) to (41) and try to compensate it with the variation of

$$\sum_{m=1}^{l/2} S_1^{r^{(2m)}}(\phi, h^{(2m)}), \ \ where$$
$$S_1^{r^{(\ell)}}(\phi, h^{(2)}, ..., h^{(\ell)}) =$$
$$= \frac{1}{2}\sum_{m=0}^{\frac{l}{2}-1}\xi_\ell^m \int d^Dz\sqrt{-g}\nabla_{\mu_{2m+1}}\ldots\nabla_{\mu_{l-2}}r^{(\ell)\mu_1\ldots\mu_{l-2}}\nabla_{\mu_1}\ldots\nabla_{\mu_m}\phi\nabla_{\mu_{m+1}}\ldots\nabla_{\mu_{2m}}\phi \quad (51)$$

introducing necessarily gauge and Weyl transformations for lower spin gauge fields

$$\delta_{\sigma_\ell}h^{(2m)\mu_1\ldots\mu_{2m}} = 2m(2m-1)C_\ell^m\sigma_{\ell(l-2m)}^{(\mu_1\ldots\mu_{2m-2}}g^{\mu_{2m-1}\mu_{2m})}, \quad m = 1, ..., l/2, \quad (52)$$
$$C_\ell^{\ell/2} = 1. \quad (53)$$

In other words we solve the equation

$$\delta_{\sigma_\ell}S^{WI}(\phi, h^{(2)}, ..., h^{(\ell)}) = \delta^1_{\sigma_\ell}S_0 + \sum_{s=1}^{l/2}\delta^0_{\sigma_\ell}S_1^{\Psi^{(2s)}} + \sum_{s=1}^{l/2}\delta^0_{\sigma_\ell}S_1^{r^{(2s)}} = 0 \quad (54)$$

which consists of a system of $l+1$ equations for $(l/2+1)(l/2+2)/2$ variables ¶

$$\triangle_\ell, \quad (55)$$
$$C_\ell^m, \quad m = 1, 2, \ldots, l/2, \quad (56)$$
$$\xi_{2s}^n, \quad n = 0, 1, \ldots s-1; \ s = 1, ..., l/2. \quad (57)$$

but when we find $\xi_\ell^{\ell/2-k}$ we also find ξ_{2s}^{s-k} for any s. In other words we find a whole diagonal of this triangle matrix

$$\begin{pmatrix} C_\ell^1 & C_\ell^2 & . & . & . & C_\ell^{\ell/2-1} & C_\ell^{\ell/2} & \Delta_\ell \\ \xi_\ell^0 & \xi_\ell^1 & . & . & . & \xi_\ell^{\ell/2-2} & \xi_\ell^{\ell/2-1} & \\ \xi_{\ell-2}^0 & \xi_{\ell-2}^1 & . & . & . & \xi_{\ell-2}^{\ell/2-2} & & \\ . & . & . & . & . & & & \\ . & . & . & . & & & & \\ \xi_4^0 & \xi_4^1 & & & & & & \\ \xi_2^0 & & & & & & & \end{pmatrix} \quad (58)$$

¶This system includes also (53) as an equation for $C_\ell^{\ell/2}$.

which helps us to solve the whole system. We have two equations for any vertical line of this matrix besides the last, for which we have one equation for Δ. We start from the last vertical line and go to the left. When we take any line and two equations for that line of variables, we have only two variables to find if we already solved all lines to the right of that one. That means that our system has a unique solution. Placing in the Appendix C all complicated Weyl variations of (54) we present here the resulting system of equations for the unknown variables (55)-(57):

$$\Delta_\ell = 1 - \frac{D}{2} \tag{59}$$

$$\frac{(-1)^{l/2}}{2}(\Delta_\ell - \frac{l-2}{2}) - (D+2l-5)\xi_\ell^{\ell/2-1} = 0 \tag{60}$$

$$(-1)^m C_\ell^m + \sum_{s=m+1}^{l/2} m C_\ell^s \xi_{2s}^m = 0, \quad (m = 1, ..., l/2-1) \tag{61}$$

$$\begin{aligned}&\frac{(-1)^{m-1}}{2}(m-1)C_\ell^m - C_\ell^m (D+4m-5)\xi_{2m}^{m-1}\\ &+\frac{1}{2}\sum_{s=m+1}^{l/2} C_\ell^s[-m(m-1)\xi_{2s}^m - (2s-2m+2)(D+2s+2m-5)\xi_{2s}^{m-1}] = 0\\ &(m = 1, ..., l/2-1)\end{aligned} \tag{62}$$

The solution of this system is universal $\Delta_\ell = \Delta = 1 - \frac{D}{2}$ and

$$\xi_\ell^m = \frac{(-1)^m}{2^{\ell-2m}(\ell/2)}\binom{\ell/2}{m}\frac{(\frac{D}{2}+m-1)_{\ell/2-m}}{(\frac{D+\ell-1}{2}+m-1)_{\ell/2-m}} \tag{63}$$

$$C_\ell^m = \frac{(-1)^{\ell/2-m}}{2^{\ell-2m}}\binom{\ell/2-1}{m-1}\frac{(\frac{D}{2}+m-1)_{\ell/2-m}}{(\frac{D-1}{2}+2m)_{\ell/2-m}}. \tag{64}$$

These completely fix (51) and therefore the full Weyl invariant action (49) and also determine the transformation law for the whole tower of higher spin gauge fields (52).

Conclusion

We constructed a gauge and generalized Weyl invariant interacting Lagrangian for a linearized higher even spin gauge field and a conformally coupled scalar field in AdS_D space. The resulting Lagrangian for the spin ℓ field includes all lower even spin gauge fields also with the same type of interaction with the scalar. These results can be used for constructions of a more complicated interaction between different higher spin gauge fields in AdS space.

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Appendix A

Here we present the basic relations between different T-s, M-s and N-s which we use in section 3.

$$T(n,k) = (-1)^m \sum_{i=0}^{m} \binom{m}{i} T(n-i, k+m-i), \tag{A.1}$$

$$T(n,k) = (-1)^m \sum_{i=0}^{m} \binom{m}{i} T(n-i, k-m), \tag{A.2}$$

and the same for M and N. There is another important relation

$$T(n,k) = -M(n-1,k) - \frac{k(k-1)}{2L^2}N(n-1,k-1)$$
$$-[\frac{(n-k-1)(2D+n-k-4)}{2L^2} + \frac{D(D-2)}{4L^2}]N(n-1,k)$$
$$-\epsilon_\ell^{(l-n)\mu_1\ldots\mu_{n-1}}\nabla_{\mu_1}\ldots\nabla_{\mu_k}\phi\nabla_{\mu_{k+1}}\ldots\nabla_{\mu_{n-1}}(\Box - \frac{D(D-2)}{4L^2})\phi, \qquad \text{(A.3)}$$

and the 'symmetrization' relations

$$M(2k+1,k) = M(2k+1,k+1) = -\frac{1}{2}M(2k,k), \qquad \text{(A.4)}$$

$$N(2k+1,k) = N(2k+1,k+1) = -\frac{1}{2}N(2k,k), \qquad \text{(A.5)}$$

$$T(2m,m) = \nabla^{(\alpha}\epsilon_{\ell(l-2m)}^{\mu_1\ldots\mu_{2m-1})}\nabla_{\mu_1}\ldots\nabla_{\mu_{m-1}}\nabla_\alpha\phi\nabla_{\mu_m}\ldots\nabla_{\mu_{2m-1}}\phi$$
$$+\frac{(m-1)(m-2)}{6L^2}N(2m-2,m-1) - \frac{(m-1)(m-2)}{12L^2}N(2m-4,m-2), \qquad \text{(A.6)}$$

$$M(2m-2,m-1) = \epsilon_{\ell(l-2m+1)\mu_m\ldots\mu_{2m-2}}^{\mu_1\ldots\mu_{m-1}}\nabla_{(\mu_1}\ldots\nabla_{\mu_{m-1}}\nabla_{\alpha)}\phi\nabla^{(\mu_m}\ldots\nabla^{\mu_{2m-2}}\nabla^{\alpha)}\phi$$
$$+\frac{(m-1)(m-2)}{3L^2}N(2m-2,m-1,m-1) - \frac{(m-1)(m-2)}{6L^2}N(2m-4,m-2) \qquad \text{(A.7)}$$

Here we have to mention that these relations are satisfied due to full derivatives and therefore admit integration.

Appendix B

We use the following commutation relations in AdS_D

$$\epsilon_\ell^{\mu_1\ldots\mu_{l-1}}[\nabla^\mu, \nabla_{\mu_1}\ldots\nabla_{\mu_k}]\phi = \frac{k(k-1)}{2L^2}\epsilon_\ell^{\mu\mu_2\ldots\mu_{l-1}}\nabla_{\mu_2}\ldots\nabla_{\mu_k}\phi, \qquad \text{(B.1)}$$

$$[\nabla_{\mu_1}\ldots\nabla_{\mu_k}, \nabla^\mu]\epsilon_\ell^{\mu_1\ldots\mu_{l-1}} = \frac{2k(D+l-2)-k(k+1)}{2L^2}\epsilon_{\ell(k-1)}^{\mu\mu_{k+1}\ldots\mu_{l-1}}, \qquad \text{(B.2)}$$

$$\epsilon_\ell^{\mu_1\ldots\mu_{l-1}}[\nabla_\mu, \nabla_{\mu_1}\ldots\nabla_{\mu_k}]\nabla^\mu\phi = \frac{k(2D+k-3)}{2L^2}\epsilon_\ell^{\mu_1\mu_2\ldots\mu_{l-1}}\nabla_{\mu_1}\ldots\nabla_{\mu_k}\phi, \qquad \text{(B.3)}$$

$$\epsilon_\ell^{\mu_1\ldots\mu_{l-1}}[\nabla^2, \nabla_{\mu_1}\ldots\nabla_{\mu_k}]\phi = \frac{k(D+k-2)}{L^2}\epsilon_\ell^{\mu_1\mu_2\ldots\mu_{l-1}}\nabla_{\mu_1}\ldots\nabla_{\mu_k}\phi, \qquad \text{(B.4)}$$

where $\epsilon_\ell^{\mu_1\ldots\mu_{l-1}}$ is the symmetric and traceless tensor. Finally we list all necessary binomial identities

$$\sum_{k=0}^{n-m}(-1)^k\binom{n}{k} = (-1)^{n-m}\binom{n-1}{m-1}, \quad \sum_{k=0}^{n-m}(-1)^k\binom{n}{m+k} = \binom{n-1}{m-1}, \qquad \text{(B.5)}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad \binom{\ell-m-1}{m-2} = \frac{m-1}{l-2m+1}\binom{\ell-m-1}{m-1} \qquad \text{(B.6)}$$

Appendix C

Here we present all Weyl variations necessary for the derivation of (59)-(62)

$$\delta^1_{\sigma_\ell} S_0 = \Delta_\ell \int d^D z\sqrt{-g}\{\sum_{m=1}^{\frac{\ell}{2}-1} \binom{\ell-m-2}{m-1} \nabla^{(\mu_{2m}}\sigma^{\mu_1...\mu_{2m-1})}_{\ell(l-2m-1)}\Psi^{(2m)}_{\mu_1...\mu_{2m}}$$
$$+\sum_{m=1}^{\frac{\ell}{2}-1} \frac{(-1)^{m-1}}{2}\binom{\ell-m-3}{m-1}\Box\sigma^{\mu_1...\mu_{2m}}_{\ell(l-2m-2)}\nabla_{\mu_1}...\nabla_{\mu_m}\phi\nabla_{\mu_{m+1}}...\nabla_{\mu_{2m}}\phi$$
$$+\sum_{m=1}^{\frac{\ell}{2}-1}(-1)^m\binom{\ell-m-3}{m-1}\sigma^{\mu_1...\mu_{2m}}_{\ell(l-2m-2)}\nabla_{\mu_1}...\nabla_{\mu_m}\nabla_\alpha\phi\nabla_{\mu_{m+1}}...\nabla_{\mu_{2m}}\nabla^\alpha\phi$$
$$+O(\frac{1}{L^2})\}. \tag{C.1}$$

We don't have to calculate $O(\frac{1}{L^2})$ terms because they can be fixed from flat space consideration and gauge invariance of Fronsdal's operator in AdS. The first term in (C.1) we can cancel by an additional gauge transformation of all gauge fields with spin less than ℓ. To cancel other terms we calculate the variation of $\sum_{m=1}^{\ell/2} S_1^{\Psi^{(2m)}}(\phi, h^{(2m)})$:

$$\delta^0_{\sigma_\ell} S_1^{\Psi^{(2m)}}(\phi, h^{(2)}, ..., h^{(2m)})$$
$$= C^m_\ell \int d^D z\sqrt{-g}\{-(m-1)[\nabla^{(\mu_{2m-2}}\sigma^{\mu_1...\mu_{2m-3})}_{\ell(l-2m+1)}\Psi^{(2m-2)}_{\mu_1...\mu_{2m-2}}$$
$$+\frac{(-1)^m}{2}\Box\sigma^{\mu_1...\mu_{2m-2}}_{\ell(l-2m)}\nabla_{\mu_1}...\nabla_{\mu_{m-1}}\phi\nabla_{\mu_m}...\nabla_{\mu_{2m-2}}\phi]$$
$$+(-1)^m(1-\frac{D}{2})\sigma^{\mu_1...\mu_{2m-2}}_{\ell(l-2m)}\nabla_{\mu_1}...\nabla_{\mu_{m-1}}\nabla_\alpha\phi\nabla_{\mu_m}...\nabla_{\mu_{2m-2}}\nabla^\alpha\phi\}. \tag{C.2}$$

and the variation of $\sum_{m=1}^{\ell/2} S_1^{r^{(2m)}}(\phi, h^{(2m)})$:

$$\delta^0_{\sigma_\ell} S_1^{r^{(\ell)}} = \frac{1}{2}\sum_{m=1}^{\frac{\ell}{2}-1}\int d^D z\sqrt{-g}\{[2m(2m-1)\xi_\ell^m - 2(m-1)(D+4m-8)\xi_\ell^{m-1}] \times$$
$$\times \nabla^{(\mu_{2m-2}}\sigma^{\mu_1\ldots\mu_{2m-3})}_{\ell(l-2m+1)}\Psi^{(2m-2)}_{\mu_1\ldots\mu_{2m-2}}\}$$
$$-\frac{1}{2}\int d^D z\sqrt{-g}\{(l-2)(D+2l-8)\xi_\ell^{\ell/2-1}\nabla^{(\mu_{l-2}}\sigma^{\mu_1\ldots\mu_{l-3})}_{\ell(1)}\Psi^{(\ell-2)}_{\mu_1\ldots\mu_{l-2}}\}$$
$$+\frac{1}{2}\sum_{m=1}^{\frac{\ell}{2}-1}\xi_\ell^m\int d^D z\sqrt{-g}\{2m(1-\frac{D}{2})\sigma^{\mu_1\ldots\mu_{2m-2}}_{\ell(l-2m)}\nabla_{\mu_1}\ldots\nabla_{\mu_{m-1}}\nabla_\alpha\phi\nabla_{\mu_m}\ldots\nabla_{\mu_{2m-2}}\nabla^\alpha\phi\}$$
$$+\frac{1}{2}\sum_{m=1}^{\frac{\ell}{2}-1}\int d^D z\sqrt{-g}\{[-m(m-1)\xi_\ell^m - (l-2m+2)(D+l+2m-5)\xi_\ell^{m-1}] \times$$
$$\times \Box\sigma^{\mu_1\ldots\mu_{2m-2}}_{\ell(l-2m)}\nabla_{\mu_1}\ldots\nabla_{\mu_{m-1}}\phi\nabla_{\mu_m}\ldots\nabla_{\mu_{2m-2}}\phi\}$$
$$-\frac{1}{2}\int d^D z\sqrt{-g}\{2(D+2l-5)\xi_\ell^{\ell/2-1}\Box\sigma_\ell^{\mu_1\ldots\mu_{l-2}}\nabla_{\mu_1}\ldots\nabla_{\mu_{l/2-1}}\phi\nabla_{\mu_{l/2}}\ldots\nabla_{\mu_{l-2}}\phi\}$$
$$+O(\frac{1}{L^2}) \tag{C.3}$$

Then finally we get

$$\delta_{\sigma_\ell} S^{WI}(\phi, h^{(2)}, ..., h^{(\ell)}) = \delta^1_{\sigma_\ell} S_0 + \sum_{s=1}^{\ell/2} \delta^0_{\sigma_\ell} S_1^{\Psi^{(2s)}} + \sum_{s=1}^{\ell/2} \delta^0_{\sigma_\ell} S_1^{r^{(\ell)}}$$

$$= \sum_{m=1}^{\ell/2-1} \int d^D z \sqrt{-g} \{ \binom{\ell-m-2}{m-1} \Delta_\ell - m C_\ell^{m+1} [1 + (D+4m-4)\xi^m_{2m+2}]$$

$$+ \frac{1}{2} \sum_{s=m+2}^{\ell/2} [(2m+2)(2m+1)\xi^{m+1}_{2s} - 2m(D+4m-4)\xi^m_{2s}] \} \times$$

$$\times \nabla^{(\mu_{2m}} \sigma_{\ell(l-2m-1)}^{\mu_1 ... \mu_{2m-1})} \Psi^{(2m)}_{\mu_1 ... \mu_{2m}}$$

$$+ \int d^D z \sqrt{-g} \{ \frac{(-1)^{l/2}}{2} (\Delta_\ell - \frac{l-2}{2}) - (D+2l-5)\xi_\ell^{\ell/2-1} \} \times$$

$$\times \Box \sigma_\ell^{\mu_1 ... \mu_{l-2}} \nabla_{\mu_1} ... \nabla_{\mu_{l/2-1}} \phi \nabla_{\mu_{l/2}} ... \nabla_{\mu_{l-2}} \phi$$

$$+ \sum_{m=1}^{\ell/2-1} \int d^D z \sqrt{-g} \{ \frac{(-1)^m}{2} [\binom{\ell-m-2}{m-2} \Delta_\ell - (m-1) C_\ell^m] - C_\ell^m (D+4m-5) \xi^{m-1}_{2m}$$

$$+ \frac{1}{2} \sum_{s=m+1}^{\ell/2} C_\ell^s [-m(m-1)\xi^m_{2s} - (2s-2m+2)(D+2s+2m-5)\xi^{m-1}_{2s}] \} \times$$

$$\times \Box \sigma_{\ell(l-2m)}^{\mu_1 ... \mu_{2m-2}} \nabla_{\mu_1} ... \nabla_{\mu_{m-1}} \phi \nabla_{\mu_m} ... \nabla_{\mu_{2m-2}} \phi$$

$$+ \int d^D z \sqrt{-g} \{ (-1)^{l/2} (1 - \frac{D}{2} - \Delta_\ell) \sigma_\ell^{\mu_1 ... \mu_{l-2}} \nabla_{\mu_1} ... \nabla_{\mu_{l/2-1}} \nabla_\alpha \phi \nabla_{\mu_{l/2}} ... \nabla_{\mu_{l-2}} \nabla^\alpha \phi \}$$

$$+ \sum_{m=1}^{\ell/2-1} \int d^D z \sqrt{-g} \{ (-1)^{m-1} \binom{\ell-m-2}{m-2} \Delta_\ell + (-1)^m (1 - \frac{D}{2}) C_\ell^m$$

$$+ (1 - \frac{D}{2}) \sum_{s=m+1}^{\ell/2} m C_\ell^s \xi^m_{2s} \} \sigma_{\ell(l-2m)}^{\mu_1 ... \mu_{2m-2}} \nabla_{\mu_1} ... \nabla_{\mu_{m-1}} \nabla_\alpha \phi \nabla_{\mu_m} ... \nabla_{\mu_{2m-2}} \nabla^\alpha \phi \qquad (C.4)$$

From this expression we can derive our system of equations (59)-(62).