Rainbow Valley of Colored (Anti) de Sitter Gravity in Three Dimensions

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Abstract: We propose a theory of three-dimensional (anti) de Sitter gravity carrying Chan-Paton color charges. We define the theory by Chern-Simons formulation with the gauge algebra $(\mathfrak{gl}_2 \oplus \mathfrak{gl}_2) \otimes \mathfrak{u}(N)$, obtaining a color-decorated version of interacting spin-one and spin-two fields. We also describe the theory in metric formulation and show that, among N^2 massless spin-two fields, only the singlet one plays the role of metric graviton whereas the rest behave as *colored spinning matter* that strongly interacts at large N. Remarkably, these *colored spinning matter* generates a non-trivial potential of staircase shape. At each extremum labelled by $k = 0, \ldots, [\frac{N-1}{2}]$, the $\mathfrak{u}(N)$ color gauge symmetry is spontaneously broken down to $\mathfrak{u}(N-k) \oplus \mathfrak{u}(k)$ and provides different (A)dS backgrounds with the effective cosmological constants $\left(\frac{N}{N-2k}\right)^2 \Lambda$. When this gauge symmetry breaking takes place, the spin-two Goldstone modes combine with (or are eaten by) the spin-one gauge fields to become partially massless spin-two fields. We discuss various aspects of this theory and highlight physical implications.

Contents

"It's time to try. Defying gravity.
I think I'll try. Defying gravity.
And you can't pull me down!"
— Gregory Maguire
'Wicked: The Life and Times of the Wicked Witch of the West'

1 Introduction

The Einstein gravity is known to be rigid. Variety of modifications have been challenged with diverse motivations (for reviews see [1] and references therein), yet nearly all such attempts so far have not proven successful and led to a no-go theorem [2–7] against an interacting theory of multiple gravitons. Recently, two situations that defy this no-go theorem were actively studied. One is the massive modification of gravity [8–11] along with diverse variants in three dimensions [12, 13]. Another is higher derivative modifications of gravity [14, 15].

In this work, we critically reexamine the no-go theorem and study a route that deforms the three-dimensional (anti) de Sitter gravity to multi-gravitons: *the color decoration.* As we shall demonstrate below, we find that, in certain situations, we can construct consistent theories of colored gravity. We start with three-dimensional (anti)-de Sitter gravity, and then color-deform it. The deformation we study is not limited to Einstein gravity and can be applied to various extensions. In particular, higher-spin theories, which were conceived in [16–18] can be all color decorated as discussed in [19–23]. In a forthcoming paper [24], we will construct three-dimensional color-decorated higher-spin gravity theory, which extends naturally the three-dimensional higher-spin gravity [25, 26] and higher-spin supergravity [27].

The color decoration of gravity evokes various conceptual issues. Clearly, the colored gravity is analogous to Yang-Mills theory if the Einstein gravity is compared to Maxwell theory. Beside the presence of multiple gauge bosons in the system, the Yang-Mills theory as color-decorated Maxwell theory has far-reaching consequences that are not shared by the Maxwell theory.[1] Likewise, we anticipate that color-decorated gravity brings out surprising new features one could not simply guess on a first look. In this paper, we define and study a version of the color-decorated Einstein gravity, and uncover remarkable new features not shared by the Einstein gravity itself. Most remarkably, we will find that this color-decorated gravity admits a number of metastable (A)dS backgrounds with different effective cosmological constants as classical vacua.

In anaylzing the color-decorated gravity, we shall utilize both the Chern-Simons formulation and the metric-like formulation in parallel. Various features of the theory are more transparent in one formulation over another. For instance, the existence of multiple (A)dS vacua with different cosmological constant can be understood more intuitively in the metric-like formulation, whereas consistency of the theory can be understood more

[1] It is said that even Wolfgang Pauli dismissed to anticipate such consequences of the Yang-Mills theory.

manifest in the Chern-Simons formulation with the gauge algebra,

$$(\mathfrak{gl}_2 \oplus \mathfrak{gl}_2) \otimes \mathfrak{u}(N)\,. \tag{1.1}$$

The $\mathfrak{u}(N)$ part of the algebra corresponds to the color gauge symmetry of the theory, whereas $\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ corresponds to the gravitational dynamics when it is not color decorated. So, compared to the spin-one situation that the Maxwell theory turns into the Yang-Mills theory once color-decorated, here we have nonabelian Chern-Simons gravity that turns into Chern-Simons gauge theory with enlarged nonabelian gauge algebra once color-decorated. In other words, the gravitational counterpart of Maxwell theory in the Yang-Mills theory is already non-abelian.

We stress that, compared to the usual gravity with $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ gauge algebra, our color-decorated gravity theory has two additional identity generators from each of $\mathfrak{gl}_2$. They are indispensable for the consistency of color decoration and correspond to two additional Chern-Simons gauge fields on top of the graviton. Hence, when colored-decorated, we get a massless spin-two field and two Yang-Mills spin-one fields, both taking adjoint values of $\mathfrak{u}(N)$. In general, a classical vacuum may spontaneously break the $\mathfrak{u}(N)$ gauge equivalence. We find that there always are multiple vacua, each distinguished by the symmetry breaking pattern $\mathfrak{u}(N) \to \mathfrak{h}$. They are also distinguished by $\mathfrak{h}$-dependent effective cosmological constant. Re-expressing the theory in metric-like formulation makes it clear that, among N^2 massless spin-two fields, only the singlet one plays the role of genuine graviton whereas the rest rather behave as *colored spinning matter* fields with minimal covariant coupling to the gravity as well as to the $\mathfrak{u}(N)$ gauge fields. We explicitly determine the Lagrangian of these *colored spinning matter* fields and find that they have a strong self-coupling compared to the gravitational one by the factor of $\sqrt{N}$. Analyzing the potential of the Lagrangian, we identify all the extrema: there are $[\frac{N+1}{2}]$ number of them and they have different effective cosmological constants,

$$\left(\frac{N}{N-2k}\right)^2 \Lambda\,, \tag{1.2}$$

where $k = 0, \ldots, [\frac{N-1}{2}]$ is the label of the extrema and Λ is the cosmological constant of the trivial, color-unbroken vacuum. Hence, not only (A)dS but also any exact gravitational backgrounds such as BTZ blackholes [28] lie multiple times with different cosmological constants (1.2) in the vacua of the colored gravity. All other extrema than the trivial vacuum spontaneously breaks the color symmetry $U(N)$ down to $U(N-k) \times U(k)$. When this symmetry breaking takes place, the corresponding $2\,k\,(N-k)$ gravitational Goldstone modes are combined with the gauge fields to become partially massless [29–33] spin-two fields.

The organization of the paper is as follows. In Section 2, we recapitulate the no-go theorem of interacting theory of multiple massless spin-two fields. In Section 3, we define the color-decorated (A)dS$_3$ gravity in Chern-Simons formulation, and discuss how this theory evades the no-go theorem. In Section 4, we recast the Chern-Simons action into metric formulation by solving torsion condition and obtain the Lagrangian for the *colored massless* spin-two fields. In Section 5, we solve the equations of motion and find a class of classical vacua with varying degrees of color symmetry breaking. We show that these (A)dS vacua have different cosmological constants. In Section 6, we expand the theory around a color non-singlet vacuum and analyze the field spectrum contents. We demonstrate that the fields corresponding to the broken part of the color symmetry describe the spectrum of partially massless spin-two field. Finally, Section 7 contains various discussions of our results and outlooks.

2 No-Go Theorem on Multiple Spin-Two Theory

The Einstein gravity describes the dynamics of massless spin-two field on a chosen vacuum. Conversely, it can be also verified that the Einstein gravity is the only interacting theory of a massless spin-two field (see e.g. [34–36]). In this context, one may ask whether there exists a non-trivial theory of multiple massless spin two fields. This possibility has been examined in [4–7], and we shall begin our discussion by reviewing these results[2].

The no-go theorem asserts that there is no consistent theory of interacting multiple massless spin-two fields only. The first point to note in this consideration is that any gauge-invariant two-derivative cubic interactions among the spin-two fields is in fact equivalent to that of Einstein-Hilbert (EH) action, modulo *color-decorated* cubic coupling constants g_{IJK}:

$$g_{IJK}\left(h^{I}_{\mu\rho}\,\partial^{\rho}\,h^{J}_{\nu\lambda}\partial^{\lambda}h^{K\,\mu\nu}+\cdots\right). \tag{2.1}$$

Here, $h^I_{\mu\nu}$ are the massless spin-two fields with *color* index I, and the tensor structure inside of the bracket is that of the EH cubic vertex. For the consistency with the color indices, it is required that the coupling constants are fully symmetric: $g_{IJK}=g_{(IJK)}$. Moreover, the gauge invariance requires that these constants define a Lie algebra spanned by the colored isometry generators. For instance, in the Minkowski spacetime, the colored generators P^I_μ and $M^I_{\mu\nu}$ obey

$$[\,M^I_{\mu\nu}\,,\,P^J_\rho\,]=2\,g^{IJ}{}_K\,\eta_{\rho[\nu}\,P^K_{\mu]}\,,\qquad [\,M^I_{\mu\nu}\,,\,M^J_{\rho\lambda}\,]=4\,g^{IJ}{}_K\,\eta_{[\nu[\rho}\,M^K_{\lambda]\mu]}\,. \tag{2.2}$$

Relating these colored generators to the usual isometry ones as $P^I_\mu=P_\mu\otimes\boldsymbol{T}^I$ and $M^I_{\mu\nu}=M_{\mu\nu}\otimes\boldsymbol{T}^I$, one can straightforwardly conclude that the color algebra $\mathfrak{g}_c$ generated by $\boldsymbol{T}^I$ must be *commutative* and *associative* [4–6]. Moreover, one can even show that $\mathfrak{g}_c$ necessarily reduces to a direct sum of one-dimensional ideals [7]: $\boldsymbol{T}^I\,\boldsymbol{T}^J=0$ for $I\neq J$. Therefore, in this set-up, the only possibility is the simple sum of several copies of Einstein gravity which do not interact with each other.

On the other hand, the no-go theorem can be evaded with a slight generalization of the setup. Firstly, if the isometry algebra can be consistently extended from a Lie algebra to an associative one, then the commutativity condition on the color algebra $\mathfrak{g}_c$ can be relaxed. The associative extension of isometry algebra typically requires to include other spectra, such as spin-one and possibly higher spins [19, 22, 23]. Moreover, it is not necessary to require that the structure constants g_{IJK} of $\mathfrak{g}_c$ be totally symmetric, but sufficient to

[2] See also related discussion in [37, 38].

assume that the totally symmetric part is non-vanishing, $g_{(IJK)} \neq 0$, so that massless spin-two fields have non-trivial interactions among themselves.

Hence, an interacting theory of multiple massless spin-two fields might be viable once other fields are added and coupled to them. As the next consistency check, one can examine the fate of the general covariance in such a theory: if there exists a genuine metric field among these massless spin-two fields, the others should be subject to interact covariantly with gravity. Moreover, one can also examine whether the multiple massless spin two fields can be color decorated bona fide by carrying non-Abelian charges. In principle, a theory can be made to covariantly interact with gravity or non-Abelian gauge field by simply replacing all its derivatives by the covariant ones with respect to both the diffeomorphism transformation and the non-Abelian gauge transformation. However, as for the diffeomorphism-covariant interactions of higher-spin fields [40], such replacements spoil the gauge invariance of the original system [41]. The problematic term in the gauge variation is proportional to the curvatures, namely, Riemann tensor $R_{\mu\nu\rho\lambda}$ or non-abelian gauge field strength $F_{\mu\nu}$. In three-dimensions, fortuitously, this is not a problem as these curvatures are just proportional to the field equations of Eintein gravity or Chern-Simons gauge theory, respectively. In higher dimensions, these terms can be compensated by introducing a non-trivial cosmological constant, but at the price of adding higher-derivative interactions [42, 43].

All in all, to have a consistent interacting theory of color-decorated massless spin-two fields, we need an (A)dS isometry gauge algebra which can be extended to an associative one. An immediate candidate is higher spin algebra in any dimensions, since Vasiliev's higher-spin theory can be consistently color-decorated, as mentioned before. Other option is to take the isometry algebras of $(A)dS_3$ and $(A)dS_5$ which are isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{sl}_4$ and can be extended to associative ones, $\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ and $\mathfrak{gl}_4$ by simply adding unit elements corresponding to spin-one fields.

3 Color-Decorated (A)dS$_3$ Gravity: Chern-Simons Formulation

Let us now move to the explicit construction of a theory of colored gravity. In this paper, we focus on the case of three dimensional gravity.

3.1 Color-Decorated Chern-Simons Gravity

In the uncolored case, it is known that the three-dimensional gravity can be written as a Chern-Simons theory with the action

$$S[\mathcal{A}] = \frac{\kappa}{4\pi} \int \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\,\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big)\,, \tag{3.1}$$

for the gauge algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. The constant κ is the level of Chern-Simons action. We are interested in color-decorating this theory. Physically, this can be done by attaching Chan-Paton factors to the gravitons. Mathematically, this amounts to requiring the fields to take values in the tensor-product space $\mathfrak{g}_i \otimes \mathfrak{g}_c$, where the $\mathfrak{g}_i$ is the isometry part of the algebra including $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and the $\mathfrak{g}_c$ is a finite-dimensional Lie algebra of a matrix group

$\mathfrak{G}_c$. For generic Lie algebras $\mathfrak{g}_i$ and $\mathfrak{g}_c$, their tensor product do not form a Lie algebra, as is clear from the commutation relations:

$$[\, M_X \otimes \boldsymbol{T}_I, M_Y \otimes \boldsymbol{T}_J \,] = \frac{1}{2}\,[\, M_X, M_Y \,] \otimes \{\, \boldsymbol{T}_I, \boldsymbol{T}_J \,\} + \frac{1}{2}\,\{\, M_X, M_Y \,\} \otimes [\, \boldsymbol{T}_I, \boldsymbol{T}_J \,]\,. \tag{3.2}$$

The anticommutators $\{\boldsymbol{T}_I, \boldsymbol{T}_J\}$ and $\{M_X, M_Y\}$ do not make sense within the Lie algebras. Instead, if we start from associative algebras $\mathfrak{g}_i$ and $\mathfrak{g}_c$, their direct product $\mathfrak{g}_i \otimes \mathfrak{g}_c$ will form an associative algebra. Hence, we need to select associative algebras for $\mathfrak{g}_i$ and $\mathfrak{g}_c$. For the color algebra $\mathfrak{g}_c$, we will consider the matrix algebra $\mathfrak{u}(N)$, but any finite-dimensional associative algebra can be used as the color algebra. For the isometry algebra $\mathfrak{g}_i$, we shall take $\mathfrak{g}_i = \mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ (instead of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$). We also need for the fields to obey Hermicity conditions compatible with the real form of the complex associate algebra.

Note that if the isometry algebra $\mathfrak{g}_i$ is not associative — as is the case with Poincaré algebra discussed in [4–7] — then the requirement of the closure of the algebra is that the color algebra $\mathfrak{g}_c$ be associative (for the first term in (3.2) to be in the product algebra) and commutative (for the second term in (3.2) to vanish).

Therefore, our model of colored gravity is the Chern-Simons theory (3.1) where the one-form gauge field $\mathcal{A}$ takes value in

$$\mathfrak{g} = (\mathfrak{gl}_2 \oplus \mathfrak{gl}_2) \otimes \mathfrak{u}(N) \;\ominus\; \mathrm{id} \otimes \boldsymbol{I}\,. \tag{3.3}$$

Notice that we have subtracted the $\mathrm{id} \otimes \boldsymbol{I}$ — where id and $\boldsymbol{I}$ are the centers of $\mathfrak{gl}_M \oplus \mathfrak{gl}_M$ and $\mathfrak{u}(N)$, respectively — since it only adds an abelian Chern-Simons which do not interact with other fields. It would be worth to remark as well that the algebra $\mathfrak{g}$ necessarily contains elements in $\mathrm{id} \otimes \mathfrak{su}(N)$ which corresponds to the gauge symmetries of $\mathfrak{su}(N)$ Chern-Simons theory. In this sense, this $\mathfrak{su}(N)$ will be referred to as the color algebra.[3] The trace Tr of (3.1) should be defined also in the tensor product space and is given by the product of two traces as

$$\mathrm{Tr}(\mathfrak{g}_i \otimes \mathfrak{g}_c) = \mathrm{Tr}(\mathfrak{g}_i)\, \mathrm{Tr}(\mathfrak{g}_c)\,. \tag{3.4}$$

It turns out useful [4] to decompose the algebra $\mathfrak{g}$ (3.3) into two orthogonal parts as

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c} \qquad \text{such that} \qquad \mathrm{Tr}(\mathfrak{b}\,\mathfrak{c}) = 0\,, \tag{3.5}$$

where $\mathfrak{b}$ is the subalgebra:

$$[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}\,, \tag{3.6}$$

corresponding to the *gravity plus gauge* sector, whereas $\mathfrak{c}$ corresponds to the *matter* sector — including all colored spin-two fields — subject to the covariant transformation,

$$[\mathfrak{b}, \mathfrak{c}] \subset \mathfrak{c}\,. \tag{3.7}$$

[3] In the Introduction, we have explained our model without taking into account this subtraction for the simplicity sake.

[4] Later, we will take advantage of this decomposition in solving the torsionless condition and convert the Chern-Simons formulation to the metric formulation.

Corresponding to the decomposition (3.5), the one-form gauge field $\mathcal{A}$ can be written as the sum of two parts

$$\mathcal{A} = \mathcal{B} + \mathcal{C}\,, \tag{3.8}$$

where $\mathcal{B}$ and $\mathcal{C}$ takes value in $\mathfrak{b}$ and $\mathfrak{c}$, respectively. In terms of $\mathcal{B}$ and $\mathcal{C}$, the Chern-Simons action (3.1) is reduced to

$$S[\mathcal{B},\mathcal{C}] = \frac{\kappa}{4\pi}\int \mathrm{Tr}\left(\mathcal{B}\wedge d\,\mathcal{B} + \frac{2}{3}\,\mathcal{B}\wedge\mathcal{B}\wedge\mathcal{B} + \mathcal{C}\wedge D_{\mathcal{B}}\,\mathcal{C} + \frac{2}{3}\,\mathcal{C}\wedge\mathcal{C}\wedge\mathcal{C}\right), \tag{3.9}$$

where $D_{\mathcal{B}}$ is the the $\mathcal{B}$-covariant derivative:

$$D_{\mathcal{B}}\,\mathcal{C} = d\,\mathcal{C} + \mathcal{B}\wedge\mathcal{C} + \mathcal{C}\wedge\mathcal{B}\,. \tag{3.10}$$

This splitting turns out to be a useful guideline in keeping manifest covariance with respect to the diffeomorphism and the non-abelian gauge transformation.

3.2 Basis of Associative Algebra

For further detailed analysis, we set our conventions and notations of the associative algebra involved. The $\mathfrak{sl}_2$ has three generators J_1, J_2, J_3. Combining them with the center generator J , one obtains $\mathfrak{gl}_2 = \mathrm{Span}\{J, J_1, J_2, J_3\}$ with the product

$$J_a\,J_b = \eta_{ab}\,J + \epsilon_{abc}\,J^c \qquad\qquad (a,b,c = 1,2,3) \tag{3.11}$$

The η_{ab} is the flat metric of $\mathfrak{sl}_2$ with mostly positive signs and ϵ_{abc} is the Levi-civita tensor of $\mathfrak{sl}_2$ with sign convention $\epsilon_{012} = +1$. The generators of the other $\mathfrak{gl}_2$ will be denoted by $\tilde{J}_a$ and $\tilde{J}$. In the case of AdS$_3$ background, the real form of the isometry algebra corresponds to $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, which satisfy

$$(J_a, \tilde{J}_a)^\dagger = -(J_a, \tilde{J}_a)\,, \qquad (J, \tilde{J}\,)^\dagger = (J, \tilde{J}\,)\,. \tag{3.12}$$

In the case of dS$_3$ background, the real form of the isometry algebra corresponds to $\mathfrak{so}(1,3) \simeq \mathfrak{sl}(2,\mathbb{C})$, which satisfy

$$(J_a, \tilde{J}_a)^\dagger = -(\tilde{J}_a, J_a)\,, \qquad (J, \tilde{J}\,)^\dagger = (\tilde{J}, J\,)\,. \tag{3.13}$$

Defining the Lorentz generator M_{ab} and the translation generator P_a as

$$M_{ab} = \frac{1}{2}\,\epsilon_{ab}{}^{c}\left(J_c + \tilde{J}_c\right), \qquad P_a = \frac{1}{2\sqrt{\sigma}}\left(J_a - \tilde{J}_a\right), \tag{3.14}$$

where $\sigma = +1$ for AdS$_3$ and $\sigma = -1$ for dS$_3$, we recover the standard commutation relations

$$[\,M_{ab}, M_{cd}\,] = 2\left(\eta_{d[a}\,M_{b]c} + \eta_{c[b}\,M_{a]d}\right), \quad [\,M_{ab}, P_c\,] = 2\,\eta_{c[b}\,P_{a]}\,, \quad [\,P_a, P_b\,] = \sigma\,M_{ab}\,, \tag{3.15}$$

of $\mathfrak{so}(2,2)$ and $\mathfrak{so}(1,3)$ for $\sigma = +1$ and -1, respectively.

The color algebra $\mathfrak{su}(N)$ can be supplemented with the center $\boldsymbol{I}$ to form the associative algebra $\mathfrak{u}(N)$, with the product

$$\boldsymbol{T}_I\,\boldsymbol{T}_J = \frac{1}{N}\,\delta_{IJ}\boldsymbol{I} + \left(g_{IJ}{}^{K} + i\,f_{IJ}{}^{K}\right)\boldsymbol{T}_K\,. \qquad (I,J,K = 1,\cdots,N^2-1). \tag{3.16}$$

The totally symmetric and anti-symmetric structure constants g_{IJK} and f_{IJK} are both real-valued.

We normalize the center generators of both algebras such that their traces are given by [5]

$$\mathrm{Tr}(J) = 2\sqrt{\sigma}, \qquad \mathrm{Tr}(\tilde{J}) = -2\sqrt{\sigma}, \qquad \mathrm{Tr}(\boldsymbol{I}) = N\,. \tag{3.17}$$

The trace of all other elements vanish. This also defines the trace convention in the Chern-Simons action (3.1). With the associative product defined in (3.11), these traces yields all the invariant multilinear forms. For instance, we get the bilinear forms,

$$\mathrm{Tr}(J_a\,J_b) = 2\,\sqrt{\sigma}\,\eta_{ab}\,, \qquad \mathrm{Tr}(\tilde{J}_a\,\tilde{J}_b) = -2\,\sqrt{\sigma}\,\eta_{ab}\,, \qquad \mathrm{Tr}(\,\boldsymbol{T}_I\,\boldsymbol{T}_J) = \delta_{IJ}\,, \tag{3.18}$$

which extract the quadratic part of action.

3.3 Classical Vacua

In the Chern-Simons formulation, the equation of motion is the zero curvature condition: $\mathcal{F} = 0$. In searching for solutions, we choose to decompose the subspaces $\mathfrak{b}$ and $\mathfrak{c}$ in (3.5) as

$$\mathfrak{b} = \mathfrak{b}_{\mathrm{GR}} \oplus \mathfrak{b}_{\mathrm{Gauge}}\,, \qquad \mathfrak{c} = \mathfrak{iso} \otimes \mathfrak{su}(N)\,. \tag{3.19}$$

Here, for the gravity plus gauge sector,

$$\mathfrak{b}_{\mathrm{GR}} = \mathfrak{iso} \otimes \boldsymbol{I}\,, \qquad \mathfrak{b}_{\mathrm{Gauge}} = \mathrm{id} \otimes \mathfrak{su}(N)\,, \tag{3.20}$$

in which $\mathfrak{iso}$ stands for the isometry algebra of the (A)dS$_3$ space:

$$\mathfrak{iso} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2\,. \tag{3.21}$$

The vacuum solution is the configuration for which the connection $\mathcal{A}$ is nonzero only for the color singlet component and is given by the (A)dS$_3$ space:

$$\mathcal{B} = \left(\frac{1}{2}\,\omega^{ab}\,M_{ab} + \frac{1}{\ell}\,e^a\,P_a\right)\boldsymbol{I}\,, \qquad \mathcal{C} = 0, \tag{3.22}$$

where ℓ is the curvature radius of (A)dS$_3$. The zero-curvature condition imposes to ω^{ab} and e^a the usual zero (A)dS curvature and zero torsion conditions:

$$d\,\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} + \frac{\sigma}{\ell^2}\,e^a \wedge e^b = 0\,, \tag{3.23}$$

$$d\,e^a + \omega^{ab} \wedge e_b = 0\,, \tag{3.24}$$

which define the (A)dS$_3$ vacuum with the cosmological constant $\Lambda = -(\sigma/\ell^2)$.

For a general solution, we again decompose $\mathcal{A} = \mathcal{B} + \mathcal{C}$ according to (3.19). The gravity plus gauge sector takes the form

$$\mathcal{B} = \left[\frac{1}{2}\left(\omega^{ab} + \frac{1}{\ell}\,\Omega^{ab}\right)M_{ab} + \frac{1}{\ell}\,e^a\,P_a\right]\boldsymbol{I} + \boldsymbol{A} + \tilde{\boldsymbol{A}}\,, \tag{3.25}$$

[5] We use the same notation Tr for the traces of both the isometry algebra and the matrix algebra. We trust this will generate maximal confusion to the readers.

where $\boldsymbol{A} = A^I\, J\, \boldsymbol{T}_I$ and $\tilde{\boldsymbol{A}} = \tilde{A}^I\, \tilde{J}\, \boldsymbol{T}_I$ are two copies of $\mathfrak{su}(N)$ gauge field with

$$(\boldsymbol{A}, \tilde{\boldsymbol{A}})^\dagger = -\begin{cases} (\boldsymbol{A}, \tilde{\boldsymbol{A}}) & [\sigma = +1] \\ (\tilde{\boldsymbol{A}}, \boldsymbol{A}) & [\sigma = -1] \end{cases}. \tag{3.26}$$

In (3.25), the splitting $\omega^{ab} + \frac{1}{\ell}\Omega^{ab}$ in the gravity part is arbitrary and is purely for later convenience. The matter sector is composed of

$$\mathcal{C} = \frac{1}{\ell}\left(\boldsymbol{\varphi}^a\, J_a + \tilde{\boldsymbol{\varphi}}^a\, \tilde{J}_a\right). \tag{3.27}$$

Here, the colored massless spin-two fields $\boldsymbol{\varphi}^a = \varphi^{a,I}\, \boldsymbol{T}_I$ and $\tilde{\boldsymbol{\varphi}}^a = \tilde{\varphi}^{a,I}\, \boldsymbol{T}_I$ take a value in $\mathfrak{su}(N)$ carrying the adjoint representation. They satisfy

$$(\boldsymbol{\varphi}^a, \tilde{\boldsymbol{\varphi}}^a)^\dagger = +\begin{cases} (\boldsymbol{\varphi}^a, \tilde{\boldsymbol{\varphi}}^a) & [\sigma = +1] \\ (\tilde{\boldsymbol{\varphi}}^a, \boldsymbol{\varphi}^a) & [\sigma = -1] \end{cases}. \tag{3.28}$$

Note that the above has a sign difference from (3.26).

We may find solutions by demanding that (3.25) and (3.27) solve for the zero curvature condition. While this procedure straightfowardly yields nontrivial solutions, for better physical interpretations, we shall first recast the Chern-Simons formulation to the metric formulation and then obtain these nontrivial solutions by solving the latter's field equations.

4 Color-Decorated (A)dS$_3$ Gravity: Metric-like Formulation

So far, we described the theory in terms of the gauge field $\mathcal{A}$, so the fact that we are dealing with color-decorated gravity is not tangible. For the sake of concreteness and the advantage of intuitiveness, we shall recast the theory in a metric-like formulation.

We first need to solve the torsionless conditions. This is technically a cumbersome step. Here, we take a short way out from this problem. The idea is that, instead of solving the torsionless conditions for all the colored fields, we shall do it only for the singlet graviton. This will still allow us to write the action in a metric-like form but, apart from the gravity, all other colored fields are still described by a first-order Lagrangian.

In three dimensions, any spectrum can be written as a first-order Lagrangian which describes only one helicity mode. If one solves the torsionless conditions for the remaining non-gravity fields, the two fields describing helicity positive and negative modes will combine to generate a single field with a standard second-order Lagrangian. However, this last step is not necessary and even impossible for certain spectra.

In the following, we will derive the full metric-like action for the first-order Lagrangian description. For the second-order Lagrangian description, we shall only identify the potential term, leaving aside the explicit form of kinetic terms.

4.1 Colored gravity action around the singlet vacuum

Starting from the Chern-Simons formulation, described in terms of e^a, $\omega^{ab} + \Omega^{ab}/\ell$, $(\boldsymbol{A}, \tilde{\boldsymbol{A}})$ and $(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}})$, we construct a metric-like formulation by solving the torsionless condition of the gravity sector. This condition is given by

$$de^a + \left(\omega^{ab} + \frac{1}{\ell}\,\Omega^{ab}\right) \wedge e_b + \frac{\sqrt{\sigma}}{N\,\ell}\,\epsilon^{abc}\,\mathrm{Tr}\,(\boldsymbol{\varphi}_b \wedge \boldsymbol{\varphi}_c - \tilde{\boldsymbol{\varphi}}_b \wedge \tilde{\boldsymbol{\varphi}}_c) = 0\,. \tag{4.1}$$

We shall first consider the colored gravity around the singlet vacuum, (A)dS$_3$, for which ω^{ab} is the spin connection obeying (3.24). This forces us to set $\Omega^{ab} = \epsilon^{abc}\,\Omega_c$ is subject to

$$\Omega^{[a} \wedge e^{b]} - \frac{\sqrt{\sigma}}{N}\,\mathrm{Tr}\left(\boldsymbol{\varphi}^a \wedge \boldsymbol{\varphi}^b - \tilde{\boldsymbol{\varphi}}^a \wedge \tilde{\boldsymbol{\varphi}}^b\right) = 0\,. \tag{4.2}$$

With the above solution (4.2) to the torsionless conditions, the action (3.1) can be recast to the sum of three parts:

$$S = S_{\text{Gravity}} + S_{\text{CS}} + S_{\text{Matter}}\,. \tag{4.3}$$

The first term S_{Gravity} is the action for the (A)dS$_3$ gravity, given by [6]

$$\begin{aligned} S_{\text{Gravity}} &= \frac{\kappa\,N}{4\pi\,\ell}\int \epsilon_{abc}\,e^a \wedge \left(d\omega^{bc} + \omega^{bd}\wedge\omega_d{}^c + \frac{\sigma}{3\,\ell^2}\,e^b \wedge e^c\right) \\ &= \frac{1}{16\pi\,G}\int d^3x\sqrt{|g|}\left(R + \frac{2\,\sigma}{\ell^2}\right), \end{aligned} \tag{4.4}$$

where the Chern-Simons level is related to the Newton's constant G, the (A)dS$_3$ curvature scale ℓ and the rank of the color algebra N by

$$\kappa = \frac{\ell}{4\,N\,G}\,. \tag{4.5}$$

The second term S_{CS} is the doubled Chern-Simons action for $\mathfrak{su}(N)\oplus\mathfrak{su}(N)$ gauge algebra:

$$S_{\text{CS}} = \frac{\kappa\sqrt{\sigma}}{2\pi}\int\left[\mathrm{Tr}\left(\boldsymbol{A}\wedge d\boldsymbol{A} + \frac{3}{2}\,\boldsymbol{A}\wedge\boldsymbol{A}\wedge\boldsymbol{A}\right) - \mathrm{Tr}\left(\tilde{\boldsymbol{A}}\wedge d\tilde{\boldsymbol{A}} + \frac{3}{2}\,\tilde{\boldsymbol{A}}\wedge\tilde{\boldsymbol{A}}\wedge\tilde{\boldsymbol{A}}\right)\right]. \tag{4.6}$$

In the uncolored Chern-Simons gravity, it is unclear whether the Chern-Simons level κ has to be quantized as the gauge group is not compact. However, in the case of colored Chern-Simons gravity, the level κ should take an integer value for the consistency of S_{CS} (4.6) under a large $SU(N)\times SU(N)$ gauge transformation.

The last term S_{Matter} is the action for the colored massless spin-two fields $\boldsymbol{\varphi}^a$ and $\tilde{\boldsymbol{\varphi}}^a$. To derive it, we use the decompositions (3.25) and (3.27), and simplify by using (4.2). We get

$$S_{\text{Matter}} = \frac{1}{16\,\pi\,G}\int\left[\frac{1}{N}\,L[\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}},\ell] - \frac{1}{\ell^2}\,\epsilon_{abc}\,e^a\wedge\Omega^b\wedge\Omega^c\right], \tag{4.7}$$

where the three-form Lagrangian $L[\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}};\ell]$ is given by

$$\begin{aligned} &L[\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}},\ell] = L_+[\boldsymbol{\varphi},\ell] - L_-[\tilde{\boldsymbol{\varphi}},\ell]\,, \\ &L_\pm[\boldsymbol{\varphi},\ell] = 2\sqrt{\sigma}\,\mathrm{Tr}\left[\frac{1}{\ell}\,\boldsymbol{\varphi}_a\wedge D\,\boldsymbol{\varphi}^a + \frac{1}{\ell^2}\,\epsilon_{abc}\left(\frac{\pm 1}{\sqrt{\sigma}}\,e^a\wedge\boldsymbol{\varphi}^b\wedge\boldsymbol{\varphi}^c + \frac{2}{3}\,\boldsymbol{\varphi}^a\wedge\boldsymbol{\varphi}^b\wedge\boldsymbol{\varphi}^c\right)\right]. \end{aligned} \tag{4.8}$$

[6] In our normalization, $d^3x\sqrt{|g|} = \frac{1}{6}\,\epsilon_{abc}\,e^a\wedge e^b\wedge e^c$.

In this expression, the covariant derivative D is with respect to both the Lorentz transformation and the $\mathfrak{su}(N)$ gauge transformation:

$$\begin{aligned} D\,\boldsymbol{\varphi}^a &= d\,\boldsymbol{\varphi}^a + \omega^{ab}\wedge\boldsymbol{\varphi}_b + \boldsymbol{A}\wedge\boldsymbol{\varphi}^a + \boldsymbol{\varphi}^a\wedge\boldsymbol{A}\,, \\ D\,\tilde{\boldsymbol{\varphi}}^a &= d\,\tilde{\boldsymbol{\varphi}}^a + \omega^{ab}\wedge\tilde{\boldsymbol{\varphi}}_b + \tilde{\boldsymbol{A}}\wedge\tilde{\boldsymbol{\varphi}}^a + \tilde{\boldsymbol{\varphi}}^a\wedge\tilde{\boldsymbol{A}}\,. \end{aligned} \tag{4.9}$$

The last term in (4.7) is an implicit function of $\boldsymbol{\varphi}^a$ and $\tilde{\boldsymbol{\varphi}}^a$. It is proportional to

$$\epsilon_{abc}\,e^a\wedge\Omega^b\wedge\Omega^c = \frac{1}{3}\,\epsilon_{abc}\,e^a\wedge e^b\wedge e^c\,\Omega_{[d}{}^{,d}\,\Omega_{e]}{}^{,e}\,, \tag{4.10}$$

where $\Omega^a = \Omega_b{}^a\,e^b$. From (4.2), they are determined to be

$$\Omega_a{}^b = \frac{1}{N}\,W^a_b(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}})\,, \tag{4.11}$$

where $W^a_b(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}})$ is given by

$$\begin{aligned} W^a_b(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}}) &= W^a_b(\boldsymbol{\varphi}) - W^a_b(\tilde{\boldsymbol{\varphi}})\,, \\ W^a_b(\boldsymbol{\varphi}) &= 4\,\sqrt{\sigma}\,\mathrm{Tr}\left(\boldsymbol{\varphi}_{[a}{}^b\,\boldsymbol{\varphi}_{c]}{}^c - \frac{1}{4}\,\delta^b_a\,\boldsymbol{\varphi}_{[c}{}^c\,\boldsymbol{\varphi}_{d]}{}^d\right). \end{aligned} \tag{4.12}$$

Here, $\boldsymbol{\varphi}_b{}^a$ are the components of $\boldsymbol{\varphi}^a$: $\boldsymbol{\varphi}^a = \boldsymbol{\varphi}_b{}^a\,e^b$. Notice that only the term (4.10) — which is quartic in $\boldsymbol{\varphi}^a$ and $\tilde{\boldsymbol{\varphi}}^a$ — gives the cross couplings between $\boldsymbol{\varphi}$'s and $\tilde{\boldsymbol{\varphi}}$'s.

4.2 first-order description

Gathering all above results and replacing the dreibein e^a in terms of the metric $g_{\mu\nu}$, the colored gravity action reads

$$S = S_{\rm CS} + \frac{1}{16\pi\,G}\int \mathrm{d}^3x\sqrt{|g|}\left[R - V(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}}) + \frac{2\sqrt{\sigma}}{N\,\ell}\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\boldsymbol{\varphi}_\mu{}^\lambda D_\nu\boldsymbol{\varphi}_{\rho\lambda} - \tilde{\boldsymbol{\varphi}}_\mu{}^\lambda D_\nu\tilde{\boldsymbol{\varphi}}_{\rho\lambda}\right)\right], \tag{4.13}$$

where the covariant derivative is given by

$$D_\mu\boldsymbol{\varphi}_{\nu\rho} = \nabla_\mu\boldsymbol{\varphi}_{\nu\rho} + [\boldsymbol{A}_\mu,\boldsymbol{\varphi}_{\nu\rho}] \tag{4.14}$$

and the potential function is given by

$$\begin{aligned} & V(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}}) \\ = \;& -\frac{1}{N\,\ell^2}\,\mathrm{Tr}\left[2\,\sigma\,\boldsymbol{I} + 4\left(\boldsymbol{\varphi}_{[\mu}{}^\mu\,\boldsymbol{\varphi}_{\nu]}{}^\nu + \tilde{\boldsymbol{\varphi}}_{[\mu}{}^\mu\,\tilde{\boldsymbol{\varphi}}_{\nu]}{}^\nu\right) + 8\sqrt{\sigma}\left(\boldsymbol{\varphi}_{[\mu}{}^\mu\,\boldsymbol{\varphi}_\nu{}^\nu\,\boldsymbol{\varphi}_{\rho]}{}^\rho - \tilde{\boldsymbol{\varphi}}_{[\mu}{}^\mu\,\tilde{\boldsymbol{\varphi}}_\nu{}^\nu\,\tilde{\boldsymbol{\varphi}}_{\rho]}{}^\rho\right)\right] \\ & -\frac{16\,\sigma}{N^2\,\ell^2}\,\mathrm{Tr}\Big(\boldsymbol{\varphi}_{[\mu}{}^\nu\,\boldsymbol{\varphi}_{\rho]}{}^\rho - \tilde{\boldsymbol{\varphi}}_{[\mu}{}^\nu\,\tilde{\boldsymbol{\varphi}}_{\rho]}{}^\rho\Big)\,\mathrm{Tr}\Big(\boldsymbol{\varphi}_{[\nu}{}^\mu\,\boldsymbol{\varphi}_{\lambda]}{}^\lambda - \tilde{\boldsymbol{\varphi}}_{[\nu}{}^\mu\,\tilde{\boldsymbol{\varphi}}_{\lambda]}{}^\lambda\Big) \\ & +\frac{6\,\sigma}{N^2\,\ell^2}\left[\mathrm{Tr}\left(\boldsymbol{\varphi}_{[\mu}{}^\mu\,\boldsymbol{\varphi}_{\nu]}{}^\nu - \tilde{\boldsymbol{\varphi}}_{[\mu}{}^\mu\,\tilde{\boldsymbol{\varphi}}_{\nu]}{}^\nu\right)\right]^2. \end{aligned} \tag{4.15}$$

The potential function consists of single-trace and double-trace parts. The single-trace part originates from the matter action, while the double-trace part originates from solving the torsionless conditions. For a general configuration, all terms in the potential function contributes the same as the other terms in (4.13).

Already at this stage, the contents of the colored gravity is clearly demonstrated: it is a theory of colored massless left-moving and right-moving spin-two fields. They interact covariantly with the color singlet gravity and also with the Chern-Simons color gauge fields. Moreover, they interact with each others through the potential function $V(\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}})$. The self-interaction is governed by the constant $1/N$. The single-trace cubic interaction is stronger than the gravitational cubic interaction by the factor of $\sqrt{N}$. Therefore, at large N and fixed Newton's constant , the colored massless spin-two fields will be strongly coupled to each other.

4.3 second-order description

In principle, we could also solve the torsionless condition for the colored spin-two fields and obtain a second-order Lagrangian (although this spoils the minimal interactions to the $\mathfrak{su}(N)$ gauge fields $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$). It amounts to taking linear combinations

$$\boldsymbol{\chi}_{\mu\nu} = \sqrt{\sigma}\,(\boldsymbol{\varphi}_{\mu\nu} - \tilde{\boldsymbol{\varphi}}_{\mu\nu})\,, \qquad \boldsymbol{\tau}_{\mu\nu} = \boldsymbol{\varphi}_{\mu\nu} + \tilde{\boldsymbol{\varphi}}_{\mu\nu}\,, \tag{4.16}$$

and integrating out the torsion part $\boldsymbol{\tau}_{\mu\nu}$, while keeping $\boldsymbol{\chi}_{\mu\nu}$. The resulting action is given by

$$S = S_{\rm CS} + \frac{1}{16\pi\, G}\int d^3x\sqrt{|g|}\,\big[\,R - V(\boldsymbol{\chi}) + \mathcal{L}_{\rm CM}(\boldsymbol{\chi},\nabla\boldsymbol{\chi},\boldsymbol{A},\tilde{\boldsymbol{A}})\,\big]\,. \tag{4.17}$$

In (4.17), the Lagrangian $\mathcal{L}_{\rm CM}$ reads

$$\mathcal{L}_{\rm CM}(\boldsymbol{\chi},\nabla\boldsymbol{\chi},\boldsymbol{A},\tilde{\boldsymbol{A}}) = \frac{1}{N}\,{\rm Tr}\left(2\,\boldsymbol{\chi}_{\mu\nu}\,\nabla^2\,\boldsymbol{\chi}^{\mu\nu} + \cdots\right), \tag{4.18}$$

where the ellipses include other tensor contractions together with higher-order terms of the form, $\boldsymbol{\chi}^n\,(\nabla\,\boldsymbol{\chi})^2$ with $n \geq 1$ as well as couplings to the gauge fields $\boldsymbol{A}$ and $\tilde{\mathbf{A}}$. We do not attempt to obtain the complete structure of these terms.

The potential function $V(\boldsymbol{\chi})$ of the colored dreibein field $\boldsymbol{\chi}_{\mu\nu}$ corresponds to the extremum of

$$\begin{aligned} V(\boldsymbol{\chi},\boldsymbol{\tau}) = & -\frac{2\,\sigma}{N\,\ell^2}\,{\rm Tr}\Big(\boldsymbol{I} + \boldsymbol{\chi}_{[\mu}{}^{\mu}\,\boldsymbol{\chi}_{\nu]}{}^{\nu} + \sigma\,\boldsymbol{\tau}_{[\mu}{}^{\mu}\,\boldsymbol{\tau}_{\nu]}{}^{\nu} + \boldsymbol{\chi}_{[\mu}{}^{\mu}\,\boldsymbol{\chi}_{\nu}{}^{\nu}\,\boldsymbol{\chi}_{\rho]}{}^{\rho} + 3\,\sigma\,\boldsymbol{\chi}_{[\mu}{}^{\mu}\,\boldsymbol{\tau}_{\nu}{}^{\nu}\,\boldsymbol{\tau}_{\rho]}{}^{\rho}\Big) \\ & - \frac{4}{N^2\,\ell^2}\,{\rm Tr}\Big(\boldsymbol{\chi}_{[\mu}{}^{\nu}\,\boldsymbol{\tau}_{\rho]}{}^{\rho} + \boldsymbol{\tau}_{[\mu}{}^{\nu}\,\boldsymbol{\chi}_{\rho]}{}^{\rho}\Big)\,{\rm Tr}\Big(\boldsymbol{\chi}_{[\nu}{}^{\mu}\,\boldsymbol{\tau}_{\lambda]}{}^{\lambda} + \boldsymbol{\tau}_{[\nu}{}^{\mu}\,\boldsymbol{\chi}_{\lambda]}{}^{\lambda}\Big) \\ & + \frac{6}{N^2\,\ell^2}\,\left[{\rm Tr}\left(\boldsymbol{\chi}_{[\mu}{}^{\mu}\,\boldsymbol{\tau}_{\nu]}{}^{\nu}\right)\right]^2\,, \end{aligned} \tag{4.19}$$

along the $\boldsymbol{\tau}_{\mu\nu}$ direction. As the extremum equation for $\boldsymbol{\tau}_{\mu\nu}$ is linear in $\boldsymbol{\tau}_{\mu\nu}$,

$$\mathcal{M}(\boldsymbol{\chi})\cdot\boldsymbol{\tau} = 0\,, \tag{4.20}$$

it must be that the unique solution is $\boldsymbol{\tau}_{\mu\nu} = 0$ for a generic configuration of $\boldsymbol{\chi}_{\mu\nu}$ [7]. Proceeding with this situation, we end up with the cubic potential for the colored dreibein field $\boldsymbol{\chi}_{\mu\nu}$:

$$V(\boldsymbol{\chi}) = -\frac{2\,\sigma}{N\,\ell^2}\,{\rm Tr}\Big(\boldsymbol{I} + \boldsymbol{\chi}_{[\mu}{}^{\mu}\,\boldsymbol{\chi}_{\nu]}{}^{\nu} + \boldsymbol{\chi}_{[\mu}{}^{\mu}\,\boldsymbol{\chi}_{\nu}{}^{\nu}\,\boldsymbol{\chi}_{\rho]}{}^{\rho}\Big)\,. \tag{4.21}$$

This potential has a noticeably simple form, but also has rich implications as we shall now discuss in the next sections.

[7] There can also exist nontrivial $\boldsymbol{\tau}_{\mu\nu}$ solutions at special values of $\boldsymbol{\chi}_{\mu\nu}$, corresponding to kernel of $\mathcal{M}$ in (4.20). We relegate complete classification of these null solutions in a separate paper [24].

5 Classical Vacuua of Colored Gravity

Having constructed the action in metric-like formulation, we now search for classical vacua that solves the field equations of motion

$$-\frac{\delta\mathcal{L}_{\rm CM}}{\delta g_{\mu\nu}} = G_{\mu\nu} - \frac{1}{2}V(\boldsymbol{\chi})\,g_{\mu\nu}\,, \qquad \frac{\delta\mathcal{L}_{\rm CM}}{\delta\boldsymbol{\chi}_{\mu\nu}} = \frac{\partial V(\boldsymbol{\chi})}{\partial\boldsymbol{\chi}_{\mu\nu}}\,, \tag{5.1}$$

$$-\frac{N}{2\sqrt{\sigma}\,\ell}\frac{\delta\mathcal{L}_{\rm CM}}{\delta\boldsymbol{A}_\mu} = \epsilon^{\mu\nu\rho}\,\boldsymbol{F}_{\nu\rho}\,, \qquad -\frac{N}{2\sqrt{\sigma}\,\ell}\frac{\delta\mathcal{L}_{\rm CM}}{\delta\tilde{\boldsymbol{A}}_\mu} = \epsilon^{\mu\nu\rho}\,\tilde{\boldsymbol{F}}_{\nu\rho}\,. \tag{5.2}$$

For the identification of solutions, we assume that the colored massless spin-two fields are covariantly constant with the trivial $\mathfrak{su}(N)$ gauge connection,

$$\boldsymbol{A} = 0\,, \qquad \tilde{\boldsymbol{A}} = 0\,, \qquad \nabla_\rho\,\boldsymbol{\chi}_{\mu\nu} = 0 \tag{5.3}$$

This immediately implies that

$$\boldsymbol{\chi}_{\mu\nu} = g_{\mu\nu}\,\boldsymbol{X} \qquad \text{for} \qquad \boldsymbol{X} = \text{constant} \in \mathfrak{su}(N). \tag{5.4}$$

The equations in the second line (5.2) trivialize and the rest reduce to

$$G_{\mu\nu} - \frac{1}{2}V(\boldsymbol{X})\,g_{\mu\nu} = 0 \qquad \text{and} \qquad \frac{\partial V(\boldsymbol{X})}{\partial\boldsymbol{X}} = 0\,, \tag{5.5}$$

where $V(\boldsymbol{X}) = V(\boldsymbol{\chi}_{\mu\nu} = g_{\mu\nu}\,\boldsymbol{X})$ is given by

$$V(\boldsymbol{X}) = -\frac{2\,\sigma}{N\,\ell^2}\,\text{Tr}\left(\boldsymbol{I} + 3\,\boldsymbol{X}^2 + \boldsymbol{X}^3\right). \tag{5.6}$$

From (5.5), the extremum of the potential defines the effective cosmological constant of the extremum:

$$\Lambda = \frac{1}{2}\,V(\boldsymbol{X})\,. \tag{5.7}$$

Although cubic, being a matrix-valued function, the potential $V(\boldsymbol{X})$ may admit a large number of nontrivial extrema that depends on the color algebra $\mathfrak{su}(N)$. If exists, each of such extrema will define a distinct vacuum with a different cosmological constant (5.7). As an illustration of this potential, consider the function $f(\boldsymbol{X}) = \frac{1}{N}\,\text{Tr}\left(\boldsymbol{I} + 3\,\boldsymbol{X}^2 + \boldsymbol{X}^3\right)$ for the $\boldsymbol{X}$ belonging to $\mathfrak{su}(3)$. The 3×3 matrix $\boldsymbol{X}$ can be diagonalized by a $SU(3)$ rotation to

$$\mathbf{X} = a\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{bmatrix} + b\begin{bmatrix}-2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{5.8}$$

We plot the function $f(a,b)$ in Fig.1. It clearly exhibits four extremum points: $(0,0)$, $(2,0)$, $(0,2)$ and $(-2,-2)$. The first point at the origin gives $f=1$, whereas the other three points all give $f=9$. In fact, these three points are connected by $SU(3)$ rotation and hence connected in the eight dimensional space of $\mathfrak{su}(3)$.

We now explicitly identify the extrema of potential function (5.6) for arbitrary value of N. The extremum points are defined by the equation:

$$\delta V(\boldsymbol{X}) = -\frac{6\,\sigma}{N\,\ell^2}\,\text{Tr}\left[(2\,\boldsymbol{X} + \boldsymbol{X}^2)\,\delta\boldsymbol{X}\right] = 0\,. \tag{5.9}$$

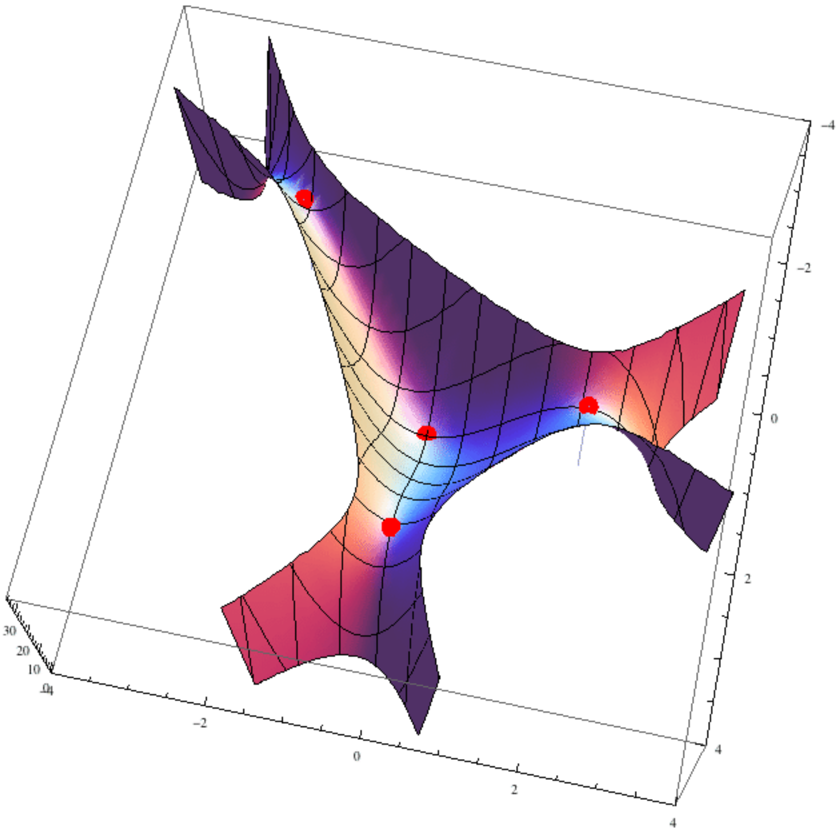

Figure 1. *The shape of the potential function for* $\mathfrak{su}(3)$.

Since $\boldsymbol{X}$ is traceless, it also follows that $\delta\boldsymbol{X}$ is also traceless. Thus, for finite N, it must be that

$$2\,\boldsymbol{X} + \boldsymbol{X}^2 = \frac{1}{N}\,\mathrm{Tr}\left(2\,\boldsymbol{X} + \boldsymbol{X}^2\right)\boldsymbol{I}\,. \tag{5.10}$$

Since $\mathrm{Tr}\,(\boldsymbol{I}+\boldsymbol{X})^2 \neq 0$ — otherwise it would follow from (5.10) that the matrix $\boldsymbol{I}+\boldsymbol{X}$ is nilpotent while having a non-trivial trace — one can redefine the matrix $\boldsymbol{X}$ in terms of $\boldsymbol{Z}$:

$$\boldsymbol{Z} = \sqrt{\frac{N}{\mathrm{Tr}(\boldsymbol{I}+\boldsymbol{X})^2}}\,(\boldsymbol{I}+\boldsymbol{X})\,, \quad \text{equivalently,} \quad \boldsymbol{X} = \frac{N}{\mathrm{Tr}(\boldsymbol{Z})}\,\boldsymbol{Z} - \boldsymbol{I} \in \mathfrak{su}(N), \tag{5.11}$$

and simplify the equation (5.10) as

$$\boldsymbol{Z}^2 = \boldsymbol{I}\,. \tag{5.12}$$

Complete solutions of this equation, up to $SU(N)$ rotations, are given by

$$\boldsymbol{Z}_k = \begin{bmatrix} \boldsymbol{I}_{(N-k)\times(N-k)} & 0 \\ 0 & -\boldsymbol{I}_{k\times k} \end{bmatrix}\,, \qquad k = 0, 1, \ldots, \left[\tfrac{N-1}{2}\right]. \tag{5.13}$$

where the upper bound of k is fixed by $[\frac{N-1}{2}]$ due to the property that $\boldsymbol{X}_{N-k}$ is a $SU(N)$ rotation of $\boldsymbol{X}_k$. Notice also that, when N is even, $k = \frac{N}{2}$ is excluded since it leads to

$\mathrm{Tr}(\boldsymbol{Z}) = 0$ for which $\boldsymbol{X}$ is ill-defined. Plugging the solutions (5.13) to the potential, we can identify the values of the potential at the extrema as

$$V(\boldsymbol{X}_k) = -\frac{2\,\sigma}{\ell^2}\left(\frac{N}{\mathrm{Tr}\,(\boldsymbol{Z}_k)}\right)^2 = -\frac{2\,\sigma}{\ell^2}\left(\frac{N}{N-2k}\right)^2. \tag{5.14}$$

These values play the role of the effective cosmological constant at k-th extremum, according to (5.7).

6 Rainbow Vacua of Colored Gravity

We learned that there are $[\frac{N+1}{2}]$ many distinct vacua having different effective cosmological constants. In this section, we study the colored gravity around each of these vacua and analyze the particle spectrum contents at each vacuum.

6.1 Decomposition of Associative Algebra

For an efficient treatment of the colored gravity at each distinct vacuum in the Chern-Simons formulation, it is important to identify the proper decomposition of the algebra (3.5). For that, we revisit the isometry and the color algebra decompositions. The isometry algebra can be divided into the rotation part $\mathcal{M}$ and the translation part $\mathcal{P}$ as

$$\mathfrak{iso} = \mathcal{M} \oplus \mathcal{P}\,, \tag{6.1}$$

the same as the trivial vacuum. For the color algebra, each vacuum spontaneously breaks the Chan-Paton $\mathfrak{su}(N)$ gauge symmetry down to $\mathfrak{su}(k) \oplus \mathfrak{su}(N-k) \oplus \mathfrak{u}(1)$, and hence the original algebra admits the decomposition:

$$\mathfrak{su}(N) \simeq \mathfrak{su}(N-k) \oplus \mathfrak{su}(k) \oplus \mathfrak{u}(1) \oplus \mathfrak{bs}\,. \tag{6.2}$$

Here, $\mathfrak{bs}$ is the vector space corresponding to the *broken symmetry*, spanned by $2k(N-k)$ generators. It is important to note that each part commutes or anti-commutes with the background matrix $\boldsymbol{Z}_k$ (5.13) as

$$\big[\,\boldsymbol{Z}_k\,,\, \mathfrak{su}(N-k) \oplus \mathfrak{su}(k) \oplus \mathfrak{u}(1)\,\big] = 0\,, \qquad \big\{\,\boldsymbol{Z}_k\,,\, \mathfrak{bs}\,\big\} = 0\,. \tag{6.3}$$

We now decompose the entire algebra (3.3) according to (3.5) in terms of the gravity plus gauge sector $\mathfrak{b}$ and the matter sector $\mathfrak{c}$. The former has again two parts similarly to the singlet vacuum case as $\mathfrak{b} = \mathfrak{b}_{\mathrm{GR}} \oplus \mathfrak{b}_{\mathrm{Gauge}}$, but the algebras to which the gravity and the gauge sectors correspond differ from (3.20) by

$$\mathfrak{b}_{\mathrm{GR}} = \big(\mathcal{M} \otimes \boldsymbol{I}\big) \oplus \big(\mathcal{P} \otimes \boldsymbol{Z}_k\big)\,, \qquad \mathfrak{b}_{\mathrm{Gauge}} = \mathrm{id} \otimes \Big(\mathfrak{su}(N-k) \oplus \mathfrak{su}(k) \oplus \mathfrak{u}(1)\Big)\,. \tag{6.4}$$

Compared to the trivial vacuum $k = 0$, the associative algebra of $k \neq 0$ vacua differ in both sectors. The gauge sector is concerned only with the unbroken part of the color algebra. The algebra of the gravity sector is deformed by $\boldsymbol{Z}_k$, but still satisfies the same commutation relations with the generators:

$$\boldsymbol{M}_{ab} = M_{ab}\,\boldsymbol{I}\,, \qquad \boldsymbol{P}_a = P_a\,\boldsymbol{Z}_k\,. \tag{6.5}$$

The one-form gauge connection fields associated with these sectors are given correspondingly by

$$\mathcal{B}_{\rm GR} = \frac{1}{2}\left(\omega^{ab}+\Omega^{ab}\right)\boldsymbol{M}_{ab}+\frac{1}{\ell_k}\,e^a\,\boldsymbol{P}_a\,,$$
$$\mathcal{B}_{\rm Gauge} = \boldsymbol{A}_+ + \boldsymbol{A}_- + \tilde{\boldsymbol{A}}_+ + \tilde{\boldsymbol{A}}_- + \left(A+\tilde{A}\right)\boldsymbol{Y}_k\,, \tag{6.6}$$

where the k-vacuum radius ℓ_k is related to the singlet one as

$$\ell_k = \left(\frac{N-2k}{N}\right)\ell\,, \tag{6.7}$$

and $\boldsymbol{Y}_k$ is the traceless matrix:

$$\boldsymbol{Y}_k = \frac{k\,\boldsymbol{I}_+ - (N-k)\,\boldsymbol{I}_-}{N}\,. \tag{6.8}$$

Here again, the spin connection ω^{ab} is the standard one satisfying (3.24), whereas Ω^{ab} will be determined in terms of other fields from the torsionless conditions. The gauge connection fields $\boldsymbol{A}_\pm$ and $\tilde{\boldsymbol{A}}_\pm$ take values in $\mathfrak{su}(N-k)$ for the subscript $+$ and $\mathfrak{su}(k)$ for the subscript $-$, whereas A and $\tilde{A}$ are Abelian gauge fields taking values in $\mathfrak{u}(1)$.

In the case of non-singlet vacua, the matter sector space has three parts:

$$\mathfrak{c} = \mathfrak{c}_{\rm CM}\oplus\mathfrak{c}_{\rm NM}\oplus\mathfrak{c}_{\rm BS}\,. \tag{6.9}$$

For the introduction of each elements, let us first define the generators of $\mathfrak{gl}_2\oplus\mathfrak{gl}_2$ deformed by $\boldsymbol{Z}_k$ as

$$\boldsymbol{J}_a = J_a\,\boldsymbol{I}_+ + \tilde{J}_a\,\boldsymbol{I}_-\,,\qquad \boldsymbol{J} = J\,\boldsymbol{I}_+ + \tilde{J}\,\boldsymbol{I}_-\,,$$
$$\tilde{\boldsymbol{J}}_a = J_a\,\boldsymbol{I}_- + \tilde{J}_a\,\boldsymbol{I}_+\,,\qquad \tilde{\boldsymbol{J}} = J\,\boldsymbol{I}_- + \tilde{J}\,\boldsymbol{I}_+\,, \tag{6.10}$$

where $\boldsymbol{I}_\pm$ are the identities associated with $\mathfrak{u}(N-k)$ and $\mathfrak{u}(k)$, respectively:

$$\boldsymbol{I}_\pm = \frac{1}{2}\left(\boldsymbol{I}\pm\boldsymbol{Z}_k\right). \tag{6.11}$$

These deformed $\mathfrak{gl}_2\oplus\mathfrak{gl}_2$ generators satisfy also the same relation as (3.11), and they are related to $\boldsymbol{M}_{ab}$ and $\boldsymbol{P}_a$ (6.5) analogously to (3.14) by

$$\boldsymbol{M}_{ab} = \frac{1}{2}\,\epsilon_{ab}{}^{c}\left(\boldsymbol{J}_c+\tilde{\boldsymbol{J}}_c\right)\qquad\text{and}\qquad \boldsymbol{P}_a = \frac{1}{2\sqrt{\sigma}}\left(\boldsymbol{J}_a-\tilde{\boldsymbol{J}}_a\right). \tag{6.12}$$

Therefore, if we define the matter fields using $\boldsymbol{J}_a$ and $\tilde{\boldsymbol{J}}_a$, then they will have the standard interactions with the gravity.

We now introduce each elements of (6.9). The first one $\mathfrak{c}_{\rm CM}$ is the residual color symmetry:

$$\mathfrak{c}_{\rm CM} = \mathfrak{iso}\otimes\left(\mathfrak{su}(N-k)\oplus\mathfrak{su}(k)\right), \tag{6.13}$$

describing colored spin-two fields associated with the one form

$$\mathcal{C}_{\rm CM} = \frac{1}{\ell_k}\left[\left(\boldsymbol{\varphi}_+^a+\boldsymbol{\varphi}_-^a\right)\boldsymbol{J}_a + \left(\tilde{\boldsymbol{\varphi}}_+^a+\tilde{\boldsymbol{\varphi}}_-^a\right)\tilde{\boldsymbol{J}}_a\right]. \tag{6.14}$$

The fields $\boldsymbol{\varphi}_+^a$ and $\tilde{\boldsymbol{\varphi}}_+^a$ take values in $\mathfrak{su}(N-k)$, whereas $\boldsymbol{\varphi}_-^a$ and $\tilde{\boldsymbol{\varphi}}_-^a$ in $\mathfrak{su}(k)$, both transforming in the adjoint representations.

The second element $\mathfrak{c}_{\rm NM}$ corresponds to the other color singlet:

$$\mathfrak{c}_{\rm NM} = \mathfrak{iso} \otimes \mathfrak{u}(1)\,, \tag{6.15}$$

which is linearly independent of the gravity sector. The associated one form reads

$$\mathcal{C}_{\rm NM} = \frac{1}{\ell_k}\left(\psi^a\,\boldsymbol{J}_a + \tilde{\psi}^a\,\tilde{\boldsymbol{J}}_a\right)\boldsymbol{Y}_k\,\boldsymbol{Z}_k\,, \tag{6.16}$$

where the last matrix factor $\boldsymbol{Y}_k\,\boldsymbol{Z}_k$ is inserted to ensure ${\rm Tr}(\mathfrak{b}_{\rm GR}\,\mathfrak{c}_{\rm NM}) = 0$, equivalently,

$${\rm Tr}\big(\boldsymbol{J}\,\boldsymbol{Y}_k\,\boldsymbol{Z}_k\big) = 0 = {\rm Tr}\big(\tilde{\boldsymbol{J}}\,\boldsymbol{Y}_k\,\boldsymbol{Z}_k\big)\,. \tag{6.17}$$

This part describes a color neutral, massless spin-two field.

The last element $\mathfrak{c}_{\rm BS}$ is what corresponds to the broken part of the color symmetries:

$$\mathfrak{c}_{\rm BS} = ({\rm id} \oplus \mathfrak{iso}) \otimes \mathfrak{bs}\,. \tag{6.18}$$

Unlike the above parts, this part does not describe massless spin-two fields. Rather, it describes so-called partially-massless [29–32, 32, 33] spin-two fields, as we shall demonstrate in the following. The corresponding one form is given by

$$\mathcal{C}_{\rm BS} = \frac{1}{\ell_k}\left(\boldsymbol{\phi}\,\boldsymbol{J} + \boldsymbol{\phi}^a\,\boldsymbol{J}_a + \tilde{\boldsymbol{\phi}}\,\tilde{\boldsymbol{J}} + \tilde{\boldsymbol{\phi}}^a\,\tilde{\boldsymbol{J}}_a\right), \tag{6.19}$$

where the fields $\boldsymbol{\phi}^a$, $\boldsymbol{\phi}$, $\tilde{\boldsymbol{\phi}}^a$ and $\tilde{\boldsymbol{\phi}}$ take values in $\mathfrak{bs}$, carrying the bi-fundamental representations of $\mathfrak{su}(N-k)$ and $\mathfrak{su}(k)$. Because these fields anti-commute with $\boldsymbol{Z}_k$, they also intertwine the left-moving and the right-moving $\mathfrak{gl}_2$'s. For instance,

$$\boldsymbol{\phi}^a\,\boldsymbol{J}_b = \tilde{\boldsymbol{J}}_b\,\boldsymbol{\phi}^a\,. \tag{6.20}$$

As a consequence, they transform differently under Hermitian conjugate:

$$(\boldsymbol{\phi}, \boldsymbol{\phi}^a, \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}^a)^\dagger = \begin{cases} (-\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}^a, -\boldsymbol{\phi}, \boldsymbol{\phi}^a) & [\sigma = +1] \\ (-\boldsymbol{\phi}, \boldsymbol{\phi}^a, -\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}^a) & [\sigma = -1] \end{cases}, \tag{6.21}$$

compared to the massless ones (3.28).

6.2 Colored Gravity around Non-Singlet Vacua

With the precise form of one form fields (6.6, 6.14, 6.16, 6.19), we now rewrite the Chern-Simons action into a metric-like form. It is given by the sum of three terms as in (4.3). Firstly, we have the standard gravity action

$$S_{\rm Gravity} = \frac{1}{16\pi\,G}\int d^3x\sqrt{|g|}\left(R + \frac{2\,\sigma}{\ell_k^2}\right), \tag{6.22}$$

with a k-dependent effective cosmological constant, set by (6.7). The Chern-Simons action $S_{\rm CS}$ for the gauge fields $\boldsymbol{A}_+$ for $\mathfrak{su}(N-k)$, $\boldsymbol{A}_-$ for $\mathfrak{su}(k)$ and A for $\mathfrak{u}(1)$ are given analogously to (4.6). Finally, the action for the matter sector takes the following form:

$$S_{\rm Matter} = \frac{1}{16\,\pi\,G}\int \frac{1}{N-2k}\Big(L[\boldsymbol{\varphi}_+,\tilde{\boldsymbol{\varphi}}_+,\ell_k] - L[\boldsymbol{\varphi}_-,\tilde{\boldsymbol{\varphi}}_-,\ell_k] + L_{\rm BS}[\boldsymbol{\phi},\tilde{\boldsymbol{\phi}},\ell_k] + L_{\rm cross}\Big) \\ - \frac{k(N-k)}{N^2}\,L[\psi,\tilde{\psi},\ell_k] - \frac{1}{\ell_k^2}\,\epsilon_{abc}\,e^a\wedge\Omega^b\wedge\Omega^c\,, \tag{6.23}$$

where L is the massless Lagrangian given in (4.9) whereas $L_{\rm BS}$ is given by

$$L_{\rm BS}[\boldsymbol{\phi},\tilde{\boldsymbol{\phi}},\ell] = \frac{4\sqrt{\sigma}}{\ell}\,{\rm Tr}\left[\left\{\tilde{\boldsymbol{\phi}}\wedge\left(D\,\boldsymbol{\phi} - \frac{1}{\sqrt{\sigma}\,\ell}\,e^a\wedge\boldsymbol{\phi}_a\right) - \tilde{\boldsymbol{\phi}}_a\wedge\left(D\,\boldsymbol{\phi}^a - \frac{1}{\sqrt{\sigma}\,\ell}\,e^a\wedge\boldsymbol{\phi}\right)\right\}\boldsymbol{Z}_k\right], \tag{6.24}$$

and the covariant derivatives $D\boldsymbol{\phi}^a$ and $D\boldsymbol{\phi}$ are given by

$$D\boldsymbol{\phi}^a = D_\omega\boldsymbol{\phi}^a + \left(\tilde{\boldsymbol{A}}_+ + \tilde{\boldsymbol{A}}_- + \tilde{A}\,\boldsymbol{Y}_k\right)\wedge\boldsymbol{\phi}^a - \boldsymbol{\phi}^a\wedge(\boldsymbol{A}_+ + \boldsymbol{A}_- + A\,\boldsymbol{Y}_k)\,, \\ D\boldsymbol{\phi} = {\rm d}\boldsymbol{\phi} + \left(\tilde{\boldsymbol{A}}_+ + \tilde{\boldsymbol{A}}_- + \tilde{A}\,\boldsymbol{Y}_k\right)\wedge\boldsymbol{\phi} - \boldsymbol{\phi}\wedge(\boldsymbol{A}_+ + \boldsymbol{A}_- + A\,\boldsymbol{Y}_k)\,, \tag{6.25}$$

as well as similarly for the tilde counter parts. The other terms in (6.23) give additional interactions: the last term gives quartic interaction through $\Omega_a{}^{,b}$:

$$\Omega_a{}^{,b} = \frac{1}{N-2k}\Big[W^a_b(\boldsymbol{\varphi}_+,\tilde{\boldsymbol{\varphi}}_+) + W^a_b(\boldsymbol{\varphi}_-,\tilde{\boldsymbol{\varphi}}_-) + W_{{\rm BS}}{}^a_b(\boldsymbol{\phi},\tilde{\boldsymbol{\phi}})\Big] - \frac{1}{k(N-k)}\,W^a_b(\psi,\tilde{\psi})\,. \tag{6.26}$$

where W^a_b is given by (4.12) and $W_{\rm BS}{}^a_b$ by

$$W_{\rm BS}{}^a_b(\boldsymbol{\phi},\tilde{\boldsymbol{\phi}}) = 8\,\sqrt{\sigma}\,{\rm Tr}\left[\left(\tilde{\boldsymbol{\phi}}_{[a}{}^{[b}\,\boldsymbol{\phi}_{c]}{}^{c]} - \frac{1}{4}\,\delta^b_a\,\tilde{\boldsymbol{\phi}}_{[c}{}^{c}\,\boldsymbol{\phi}_{d]}{}^{d}\right)\boldsymbol{Z}_k\right]. \tag{6.27}$$

The term $L_{\rm cross}$ given by

$$L_{\rm cross} = \frac{4\sqrt{\sigma}}{\ell_k^2}\,{\rm Tr}\Big[\,(\boldsymbol{\varphi}^a_+ - \boldsymbol{\varphi}^a_- + \psi^a\,\boldsymbol{Y}_k)\wedge\left(\tilde{\boldsymbol{\phi}}\wedge\boldsymbol{\phi}_a + \tilde{\boldsymbol{\phi}}_a\wedge\boldsymbol{\phi}\right) + \\ + \epsilon_{abc}\left(\boldsymbol{\varphi}^a_+\wedge\boldsymbol{\varphi}^b_+ + \boldsymbol{\varphi}^a_-\wedge\boldsymbol{\varphi}^b_-\right)\wedge\psi^c\,\boldsymbol{Y}_k\,\boldsymbol{Z}_k\Big] - \left([\ ]\leftrightarrow\widetilde{[\ ]}\right), \tag{6.28}$$

is the cross terms originating from the Chern-Simons cubic interactions.

In principle, we can further simplify the action as we did in the singlet vacuum case. However, already at this level, we can extrat a lot of physics. Firstly, we have a scalar potential as a function of four fields $\boldsymbol{\varphi}_\pm$, ψ, $\boldsymbol{\phi}$ (and their tilde counter parts) and the point where all fields vanish correspond to an extremum point whose potential value gives the cosmological constant $-\sigma/\ell_k^2$. This potential should be a shfit of the potential $V(\boldsymbol{\varphi},\tilde{\boldsymbol{\varphi}})$ (4.15) defined around the singlet vacuum, hence it will admit all other vacua as extrema. Secondly, the interaction strength of each fields can be easily read off from the action. The gravity and gauge interaction has the same strength controlled by G and κ as in the singlet vacuum case. The interaction of colored spin two fields $\boldsymbol{\varphi}_\pm$ is weakened from N to $N-2k$, and the same for the broken-symmetry field $\boldsymbol{\phi}$. Finally, the color-neutral massless spin two ψ has interaction strength controlled by $N^2/[k(N-k)]$. Therefore, when the color symmetry is maximally broken, that is $N-2k\sim 1$, the interaction between all these fields becomes as weak as the gravitational one.

6.3 Spectra of Broken Color Symmetry Parts

Around the non-singlet vacua, the fields $\boldsymbol{\varphi}_{\pm}$ and ψ both describe massless spin-two fields having the same quadratic Lagrangian given by (4.9). On the other hand, the fields $\boldsymbol{\phi}$ corresponding to the broken part of the color symmetries have different quadratic Lagrangian (6.24), hence describe different spectra. We have already mentioned that they correspond to partially massless particles [29–32, 32, 33]. In this section, we analyze the quadratic Lagrangian (6.24) to prove this statement. Here, we mainly concentrate on AdS$_3$. To get the dS$_3$ result, we just change the AdS radius to $\sqrt{-1}$ times dS radius.

Though the Lagrangian (6.24) has a rather non-standard form involving cross term between $\boldsymbol{\phi}$ and $\tilde{\boldsymbol{\phi}}$ together with an insertion of $\boldsymbol{Z}_k$, it can always be diagonalized with the help of the Hermicity property (6.21). Therefore, for the spectrum analysis, it will suffice to consider $S_{\rm BS}[\phi,\phi^a]$ taking the following expression:

$$S_{\rm BS}[\phi,\phi^a]=\int \phi\wedge\left(d\phi-\frac{1}{\ell}\,e_a\wedge\phi^a\right)-\phi_a\wedge\left(D\phi^a-\frac{1}{\ell}\,e^a\wedge\phi\right) \tag{6.29}$$

with the (A)dS dreibein and spin connection (e^a,ω^{ab}) . We first note that this action admits the gauge symmetries with parameters $(\varepsilon,\varepsilon^a)$,

$$\delta\,\phi=d\,\varepsilon-\frac{1}{\ell}\,e^a\,\varepsilon_a\,,\qquad \delta\,\phi^a=D\,\varepsilon^a-\frac{1}{\ell}\,e^a\,\varepsilon\,. \tag{6.30}$$

They are inherited from the Chern-Simons gauge symmetries.

For a closer look of this action involving three fields $h_{\mu\nu}=e^a{}_{(\mu}\,\phi_{\nu)a}$, $f_{\mu\nu}=e^a{}_{[\mu}\,\phi_{\nu]a}$ and $\phi_\mu=e^a{}_\mu\,\phi_a$, we consider two different but equivalent paths:

- We first derive the equation of motion for one-form fields ϕ^a and ϕ . They are given by

  $$D\phi^a-\frac{1}{\ell}e^a\wedge\phi=0\,,\qquad d\phi-\frac{1}{\ell}e^a\wedge\phi_a=0\,. \tag{6.31}$$

  The second equation implies that the antisymmetric field $f_{\mu\nu}$ is the field strength of ϕ_μ : $f_{\mu\nu}=\ell\,\partial_{[\mu}\phi_{\nu]}$. Then, by gauge fixing ϕ_μ to zero with the gauge parameter ε^a, the field $f_{\mu\nu}$ decouples from the first equation. We thus end up with only one field $h_{\mu\nu}$ satisfying the equation of motion,

  $$\nabla_{[\mu}h_{\nu]\rho}=0\,, \tag{6.32}$$

  and the gauge symmetry,

  $$\delta\,h_{\mu\nu}=\nabla_{(\mu}\,\partial_{\nu)}\,\varepsilon-\frac{1}{\ell^2}\,g_{\mu\nu}\,\varepsilon\,. \tag{6.33}$$

  This gauge symmetry is precisely the gauge symmetry of partially massless spin-two field [29–32, 32, 33].

- Instead of first deriving the equation and then gauge fixing to $\phi_\mu=0$, one can reverse the procedure. We first gauge fix and eliminate ϕ_μ field in the action and obtain

  $$S_{\rm PM}[\phi_{\mu\nu}]=\int d^3x\sqrt{|g|}\;\epsilon^{\mu\nu\rho}\,\phi^\lambda{}_\mu\,\nabla_\nu\,\phi_{\rho\lambda}\,, \tag{6.34}$$

modulo a boundary term. We note that the field $\phi_{\mu\nu}$ contains both of the symmetric part $h_{\mu\nu}$ and the antisymmetric part $f_{\mu\nu}$, while only $h_{\mu\nu}$ admits the gauge symmetries (6.33). The equation of motion is now given by

$$C_{\mu\nu,\rho} := \nabla_{[\mu} h_{\nu]\rho} + \nabla_{[\mu} f_{\nu]\rho} = 0\,. \tag{6.35}$$

The totally anti-symmetric part $C_{[\mu\nu,\rho]} = \partial_{[\mu} f_{\nu\rho]} = 0$ can be readily solved as $f_{\mu\nu} = \partial_{[\mu}\, a_{\nu]}$. With the field redefinition $h_{\mu\nu} \to h_{\mu\nu} - \nabla_{(\mu} a_{\nu)}$, the trace of the above equation, $C^{\rho}{}_{\mu,\rho} = 0$, gives

$$a_\mu = \frac{\ell^2}{2}\,(\nabla^\rho h_{\mu\rho} - \nabla_\mu h^\rho{}_\rho)\,, \tag{6.36}$$

Taking now a divergence of $C_{\mu\nu,\rho}$, we arrive at the standard second-order massive spin-two equation,

$$\nabla^\rho C_{\rho(\mu,\nu)} = G^{\rm lin}_{\mu\nu} + \frac{1}{\ell^2}\,(h_{\mu\nu} - g_{\mu\nu}\, h^\rho{}_\rho) = 0\,, \tag{6.37}$$

with the linearized Einstein tensor $G^{\rm lin}_{\mu\nu}$. One can also check that the mass of the above equation corresponds to that of a partially-massless field. Furthermore, using Bianchi identity, we deduce that the left-hand side of (6.36) vanishes, so does $f_{\mu\nu}$. Therefore, we end up with the same equation (6.32).[8]

7 Discussions

In this paper, we proposed a Chan-Paton color-decorated gravity in three dimensions and studies its properties. We have shown that the theory describes a gravitational system of colored massless spin-two matter fields coupled to $\mathfrak{su}(N)$ gauge fields. The matter fields have a non-trivial potential whose $[\frac{N+1}{2}]$ extrema have different values of effective cosmological constants. All the extremum points but the origin spontaneously break the $\mathfrak{su}(N)$ color symmetry down to $\mathfrak{su}(N-k) \oplus \mathfrak{su}(k)$. We found that the spin-two Goldstone modes corresponding to the broken part of the symmetries are combined with the gauge fields and become partially massless spin-two fields. In the vacua with large $k \sim N/2$, the interactions of the matter fields are as weak as the gravitational one. In the small k vacua, their interaction becomes strong by the factor of $\sqrt{N}$.

Let us discuss more on the potential (5.6). Firstly, the cubic form shows that the potential is not bounded from below or above. Secondly, the overall factor σ shows that the overall sign of the potential depends whether we consider AdS_3 or dS_3 background. Thirdly, to understand better the stability of the extrema we found, let us consider the second variation of the potential,

$$\delta^2 V(\boldsymbol{X}_k) = -\frac{12\,\sigma}{(N-2k)\,\ell^2}\,\mathrm{Tr}\left(\boldsymbol{Z}_k\,\delta \boldsymbol{X}^2\right). \tag{7.1}$$

[8] Note that the equation (6.37) alone is weaker than the first-order one (6.32). The former describes one propagating degrees of freedom, while the latter does not have any bulk mode and corresponds to the spectrum described by (6.29). To recapitulate, in three dimensions (not in higher dimensions), there are two kinds of partially-massless fields for the maximal depth, which includes the spin-two partially-massless spectra. We shall discuss more about this subtlety in the companion paper [24].

The Hessian is not positive-definite for an arbitrary $\delta\boldsymbol{X}$ except the singlet vacuum $k=0$. So, all $k\neq 0$ vacua are saddle points and the $k=0$ vacuum is the minimum/maximum in dS_3/AdS_3 space. Another evidence that the $k\neq 0$ vacua are saddle points is that the kinetic terms of the massless fields $\boldsymbol{\varphi}_-$ and ψ have a relative negative sign compared to the singlet sector.

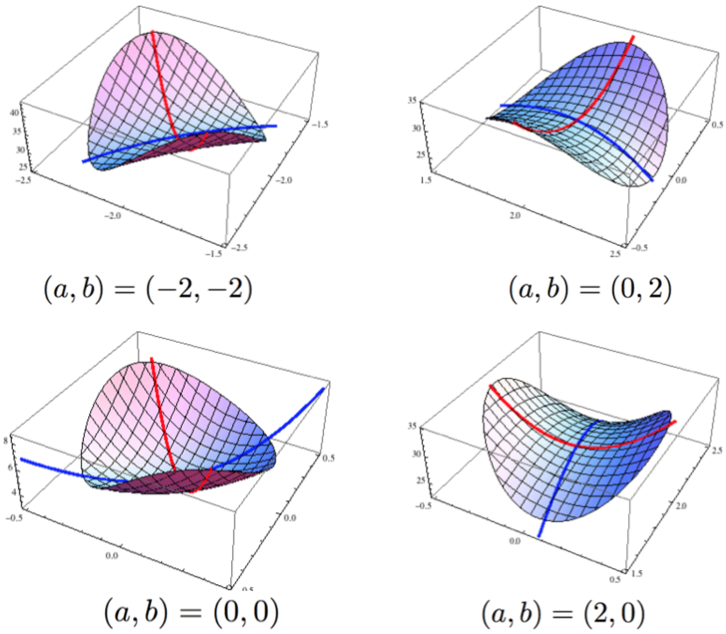

Figure 2. *Potentials around each rainbow vacua with $N=3$. The $(a,b)=(0,0)$ vacuum is the minimum/maximum in dS_3/AdS_3 space. The others, connected by $SU(3)$ transformation, are all saddle points.*

Considering the dS_3 branch, the potential takes a spiral stairwell shape (Fig.3) with $[\frac{N+1}{2}]$ many steps, having split cosmological constants that range from $\Lambda=1/\ell^2$ at the lowest step all the way up to $\sim N^2\,\Lambda$ at the highest step. The spacing gets dense in lower steps, while sparse in higher steps. If such features continue to hold in higher dimensions, the colored gravity with large N might be very relevant for the early universe cosmology in that the universe begins in an inflationary epoch with a large cosmological constant at a very high stairstep. The colored matter are weakly coupled there, and hence they are not confined. As the state of the universe decays towards lower stairsteps, the effective cosmological constant decreases sequentially and eventually exits the inflation. The colored matter fields start to interact stronger and eventually form heavy color-neutral composites. It is in this synopsis that the spin-two colored matter fields might play a novel role in the current paradigm of the inflationary cosmology.

We also speculate on a novel approach to the three-dimensional quantum colored gravity. At large N, the contribution of the $O(N/2)$ multiple vacua in the path integral might be captured by the $\mathfrak{su}(N)$ random matrix model given by

$$\mathcal{Z}_{\rm MM}=\int d\boldsymbol{X}\,\exp\left[i\,c\,V(\boldsymbol{X})\right], \tag{7.2}$$

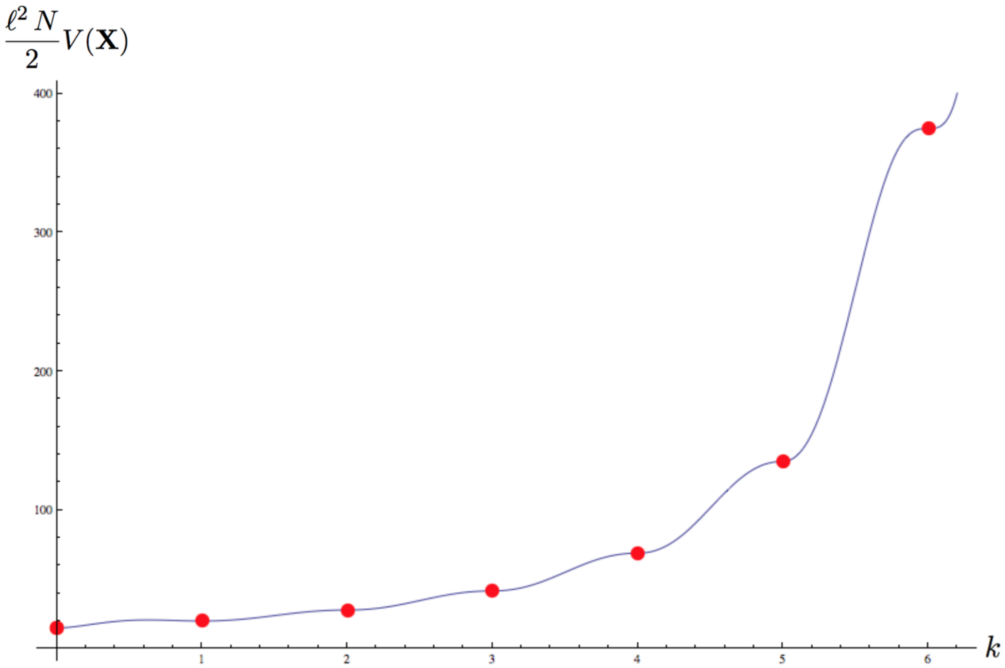

Figure 3. Potential of the colored gravity in dS ($N = 15$): k is the parameter of a curve in $\mathfrak{su}(15)$ that passes through the extremum points.

where $\boldsymbol{X}$ is a collection of tensor-valued constant $(N \times N)$ Hermitian matrix. It would be interesting to explore an initio definition of the three-dimensional quantum gravity starting from this class of matrix models.

This work brings in many open problem worth of further investigation. As already mentioned in Section 1, extensions to higher-spin (A)dS$_3$ gravity and supergravity is imminent. Further extensions to color-decoration of the known higher-spin gravity in three-dimensional Lifshitz spacetime [44]and flat spacetime [45] are also straightforward. Extension to higher-dimensional spacetime is also highly interesting. A version of such situation was already studied in the context of AdS/CFT correspondence [46]. Vasiliev equations for color-decorated higher-spin theories needs to be better understood, along with higher-dimensional counterpart of the stairstep potential we found in three dimensions. As the color dynamics is described by Chern-Simons gauge theory, one might anticipate to formulate colored gravity in any dimensions in terms of a version of Chern-Simons formulation, perhaps, along the lines of [47] and [48]. Quantum aspects of color-decorated gravity is an avenue to be explored, in particular, consequences and implications of strong color interactions among colored spin-two fields. Turning to the inflationary cosmology, it would be interesting how the color-decoration modifies the infrared dynamics of interacting massless spin-two fields at super-horizon scales. This brings one to investigate stochastic dynamics of these fields, as would be described by color-decorated version of the Langevin dynamics [49, 50].

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