

General trilinear interaction for arbitrary even higher spin gauge fields

Ruben Manvelyan^{†‡}, Karapet Mkrtchyan^{†‡}
and Werner Rühl[†]

[†]*Department of Physics
Erwin Schrödinger Straße*

*Technical University of Kaiserslautern, Postfach 3049
67653 Kaiserslautern, Germany*

[‡]*Yerevan Physics Institute
Alikhanian Br. Str. 2, 0036 Yerevan, Armenia*

manvel,ruehl@physik.uni-kl.de; karapet@yerphi.am

ABSTRACT

Using Noether's procedure we present a complete solution for the trilinear interactions of arbitrary spins s_1, s_2, s_3 in a flat background, and discuss the possibility to enlarge this construction to higher order interactions in the gauge field. Some classification theorems of the cubic (self)interaction with different numbers of derivatives and depending on relations between the spins are presented. Finally the expansion of a general spin s gauge transformation into powers of the field and the related closure of the gauge algebra in the general case are discussed.

1 Introduction and notations

The motivation for the investigation of higher spin gauge field (HSF) interactions can be summarized as the following list of the essential three points:

1. The construction of interacting higher spin theories has been considered as an interesting task for itself and was always in the center of attention during the last thirty years [1]-[17].
2. Particular attention arose during the last decade after discovering the holographic duality between the $O(N)$ sigma model in $d = 3$ space and HSF gauge theory living in the space AdS_4 [18]. This case of holography is especially important by the existence of two conformal points of the boundary theory and the possibility to describe them by the same HSF gauge theory with the help of *spontaneous breaking of higher spin gauge symmetry and mass generation by a corresponding Higgs mechanism* [21]-[23].
3. Another still open point is verifying the holographic correspondence on the level of loop diagrams in the general case, and the possibility to use this correspondence for real constructions of unknown local interacting theories on the bulk from more or less well known conformal field theories on the boundary.

These complicated physical tasks necessitate quantum loop calculations for the HSF field theory [24], [25],[26] and therefore information about the manifest, off-shell and Lagrangian formulation of possible interactions for HSF. On the other hand one loop calculations are mainly interesting in the framework of their ultra-violet behaviour, when the difference between an AdS and a flat space background can be neglected at least in the leading order. These motivations caused us during last years to spend some effort on the construction of possible couplings which we started in series of articles that involved couplings among different higher spin fields [2, 3, 4, 19] . In our previous article [1] we directly construct a complete cubic selfinteraction for the case of spin $s = 4$ in a flat background, and discuss the cubic selfinteraction for general spin s with s derivatives in the same background. The leading term of the latter interaction together with the gauge transformation of first order in the field was presented and investigated.

Here we turn to the trilinear interaction of Fronsdal's [20] general spin s_1, s_2, s_3 gauge fields in a flat background (Section 2) and present the full solution of the corresponding Noether's equation (Section 3). Then we discuss a general classification theorem for (self)interactions based on our construction and the relation with other known couplings involving (Weyl) tensors integrated into the interaction Lagrangians (Section 4). The last section 5 is devoted to a discussion of gauge transformations that are nonlinear in the gauge field, as they were invented in the classical paper [7], and the possibility to form Lie algebras of

gauge transformations for even spin abelian gauge fields and in this context the construction of the fourth and higher order selfinteractions.

To handle these sections we should introduce here briefly our standard notations coming from our previous papers about HSF [1, 2, 3, 4, 19]. As usual we utilize instead of symmetric tensors such as $h_{\mu_1\mu_2\dots\mu_s}^{(s)}(z)$ the homogeneous polynomials in the vector a^μ of degree s at the base point z

$$h^{(s)}(z; a) = \sum_{\mu_i} \left(\prod_{i=1}^s a^{\mu_i} \right) h_{\mu_1\mu_2\dots\mu_s}^{(s)}(z). \quad (1.1)$$

Then we can write the symmetrized gradient, trace and divergence *

$$Grad : h^{(s)}(z; a) \Rightarrow Grad h^{(s+1)}(z; a) = (a\nabla)h^{(s)}(z; a), \quad (1.2)$$

$$Tr : h^{(s)}(z; a) \Rightarrow Tr h^{(s-2)}(z; a) = \frac{1}{s(s-1)} \square_a h^{(s)}(z; a), \quad (1.3)$$

$$Div : h^{(s)}(z; a) \Rightarrow Div h^{(s-1)}(z; a) = \frac{1}{s} (\nabla\partial_a)h^{(s)}(z; a). \quad (1.4)$$

Moreover we introduce the notation $*_a, *_b, \dots$ for a contraction in the symmetric spaces of indices a or b

$$*_a = \frac{1}{(s!)^2} \prod_{i=1}^s \overleftarrow{\partial}_a^{\mu_i} \overrightarrow{\partial}_{\mu_i}^a. \quad (1.5)$$

Then we see that the operators $(a\partial_b), a^2, b^2$ are dual (or adjoint) to $(b\partial_a), \square_a, \square_b$ with respect to the "star" product of tensors with two sets of symmetrized indices (1.5)

$$\frac{1}{n} (a\partial_b) f^{(m-1,n)}(a, b) *_a *_b g^{(m,n-1)}(a, b) = f^{(m-1,n)}(a, b) *_a *_b \frac{1}{m} (b\partial_a) g^{(m,n-1)}(a, b), \quad (1.6)$$

$$a^2 f^{(m-2,n)}(a, b) *_a *_b g^{(m,n)}(a, b) = f^{(m-2,n)}(a, b) *_a *_b \frac{1}{m(m-1)} \square_a g^{(m,n)}(a, b). \quad (1.7)$$

In the same fashion gradients and divergences are dual with respect to the full scalar product in the space (z, a, b)

$$(a\nabla) f^{(m-1,n)}(z; a, b) *_a *_b g^{(m,n)}(z; a, b) = -f^{(m-1,n)}(z; a, b) *_a *_b \frac{1}{m} (\nabla\partial_a) g^{(m,n)}(z; a, b). \quad (1.8)$$

Analogous equations can be formulated for the operators b^2 or $b\nabla$.

*To distinguish easily between "a" and "z" spaces we introduce the notation ∇_μ for space-time derivatives $\frac{\partial}{\partial z^\mu}$.

Here we will only present Fronsdal's Lagrangian in terms of these conventions[†]:

$$\mathcal{L}_0(h^{(s)}(a)) = -\frac{1}{2}h^{(s)}(a) *_a \mathcal{F}^{(s)}(a) + \frac{1}{8s(s-1)}\square_a h^{(s)}(a) *_a \square_a \mathcal{F}^{(s)}(a), \quad (1.9)$$

where $\mathcal{F}^{(s)}(z; a)$ is the Fronsdal tensor

$$\mathcal{F}^{(s)}(z; a) = \square h^{(s)}(z; a) - s(a\nabla)D^{(s-1)}(z; a), \quad (1.10)$$

and $D^{(s-1)}(z; a)$ is the deDonder tensor or traceless divergence of the higher spin gauge field

$$D^{(s-1)}(z; a) = Div h^{(s-1)}(z; a) - \frac{s-1}{2}(a\nabla)Tr h^{(s-2)}(z; a), \quad (1.11)$$

$$\square_a D^{(s-1)}(z; a) = 0. \quad (1.12)$$

The initial gauge variation of order zeroth in the spin s field is

$$\delta_{(0)}h^{(s)}(z; a) = s(a\nabla)\epsilon^{(s-1)}(z; a), \quad (1.13)$$

with the traceless gauge parameter for the double traceless gauge field

$$\square_a \epsilon^{(s-1)}(z; a) = 0, \quad (1.14)$$

$$\square_a^2 h^{(s)}(z; a) = 0. \quad (1.15)$$

Therefore at this point we can see from (1.13) and (1.14) that the de Donder gauge condition

$$D^{(s-1)}(z; a) = 0, \quad (1.16)$$

is a correct generalization of the Lorentz gauge condition in the case of spin $s > 2$. Finally we note that in deDonder gauge (1.16) $\mathcal{F}^{(s)}(z; a) = \square h^{(s)}(z; a)$ and the field $h^{(s)}$ decouples from its trace in Fronsdal's Lagrangian (1.9).

2 Noether's theorem in leading order: Trinomial coefficients

We consider three potentials $h^{(s_1)}(z_1; a)$, $h^{(s_2)}(z_2; b)$, $h^{(s_3)}(z_3; c)$ whose spins s_i are assumed to be ordered

$$s_1 \geq s_2 \geq s_3. \quad (2.1)$$

[†]From now on we will presuppose integration everywhere where it is necessary (we work with a Lagrangian as with an action) and therefore we will neglect all d dimensional space-time total derivatives when making a partial integration.

For the interaction we make the cyclic ansatz

$$\begin{aligned} \mathcal{L}_I^{(0,0)}(h^{(s_1)}(a), h^{(s_2)}(b), h^{(s_3)}(c)) &= \sum_{n_i} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \int dz_1 dz_2 dz_3 \delta(z_3 - z_1) \delta(z_2 - z_1) \\ \hat{T}(Q_{12}, Q_{23}, Q_{31} | n_1, n_2, n_3) & h^{(s_1)}(z_1; a) h^{(s_2)}(z_2; b) h^{(s_3)}(z_3; c), \end{aligned} \quad (2.2)$$

where

$$\hat{T}(Q_{12}, Q_{23}, Q_{31} | n_1, n_2, n_3) = (\partial_a \partial_b)^{Q_{12}} (\partial_b \partial_c)^{Q_{23}} (\partial_c \partial_a)^{Q_{31}} (\partial_a \nabla_2)^{n_1} (\partial_b \nabla_3)^{n_2} (\partial_c \nabla_1)^{n_3}, \quad (2.3)$$

and the notation $(0, 0)$ as a superscript means that it is an ansatz for terms without $Div h^{(s_i-1)} = (\nabla_i \partial_{a_i}) h^{(s_i)}(a_i)$ and $Tr h^{(s_i-2)} = \frac{1}{s_i(s_i-1)} \square_{a_i} h^{(s_i)}(a_i)$. Denoting the number of derivatives by Δ we have

$$n_1 + n_2 + n_3 = \Delta. \quad (2.4)$$

We shall later determine and then use the minimal possible Δ . As balance equations we have

$$\begin{aligned} n_1 + Q_{12} + Q_{31} &= s_1, \\ n_2 + Q_{23} + Q_{12} &= s_2, \\ n_3 + Q_{31} + Q_{23} &= s_3. \end{aligned} \quad (2.5)$$

These equations are solved by

$$\begin{aligned} Q_{12} &= n_3 - \nu_3, \\ Q_{23} &= n_1 - \nu_1, \\ Q_{31} &= n_2 - \nu_2. \end{aligned} \quad (2.6)$$

Since the l.h.s. cannot be negative, we have

$$n_i \geq \nu_i. \quad (2.7)$$

The ν_i are determined to be

$$\nu_i = 1/2(\Delta + s_i - s_j - s_k), \quad i, j, k \text{ are all different.} \quad (2.8)$$

These ν_i must also be nonnegative, since otherwise the natural limitation of the n_i to nonnegative values would imply a boundary value problem which has only a trivial solution (see below). It follows that the minimally possible Δ is expressed by Metsaev's (see [11] equ. (5.11)-(5.13)) formula (using the ordering of the s_i).

$$\Delta_{min} = \max[s_i + s_j - s_k] = s_1 + s_2 - s_3. \quad (2.9)$$

From now on we choose mainly this Δ_{min} for Δ remembering that our ansatz for the interaction describes all other cases with more derivatives also. For example

$$\Delta = 6 \quad \text{for} \quad s_1 = s_2 = 4, s_3 = 2. \quad (2.10)$$

This value and the ordering of the s_i implies for the ν_i

$$\begin{aligned} \nu_1 &= s_1 - s_3, \\ \nu_2 &= s_2 - s_3, \\ \nu_3 &= 0. \end{aligned} \quad (2.11)$$

From this result and the experience with the cubic selfinteraction for $s = 4$ we can guess that the coefficient C in the ansatz is a *trinomial*[‡]

$$C_{n_1, n_2, n_3}^{s_1, s_2, s_3} = \text{const} \binom{s_3}{n_1 - s_1 + s_3, n_2 - s_2 + s_3, n_3}, \quad (2.12)$$

which entails

$$\begin{aligned} \sum_{ij} Q_{ij} &= \Delta - \sum_i \nu_i, \\ \sum_i \nu_i &= 3/2\Delta - 1/2 \sum_i s_i, \end{aligned} \quad (2.13)$$

and the expression (2.9) for Δ_{min} .

For the proof of this equation (2.12) we use Noether's theorem to derive recursion relations which are then solved. By variation w.r.t. $h^{(s_i)}$ we obtain three currents whose divergences must vanish on shell. We need only do the explicit variation once:

$$\begin{aligned} J^{(3)}(z_3; c) &= \sum C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \int dz_1 dz_2 \delta(z_3 - z_1) \delta(z_3 - z_2) \\ &\quad (\partial_a \partial_b)^{Q_{12}} (\partial_b c)^{Q_{23}} (c \partial)^{Q_{31}} (\partial_a \nabla_2)^{n_1} (\partial_b \nabla_3)^{n_2} (c \nabla_1)^{n_3} \\ &\quad h^{(s_1)}(z_1; a) h^{(s_2)}(z_2; b), \end{aligned} \quad (2.14)$$

having the divergence

$$\begin{aligned} (\partial_c \nabla_3) J^{(3)}(z_3; c) &= \sum C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \\ &\quad \{ n_3 (\nabla_1 \nabla_3) (\partial_a \partial_b)^{Q_{12}} (\partial_b c)^{Q_{23}} (c \partial_a)^{Q_{31}} (\partial_a \nabla_2)^{n_1} (\partial_b \nabla_3)^{n_2} (c \nabla_1)^{n_3 - 1} \\ &\quad + Q_{23} (\partial_a \partial_b)^{Q_{12}} (\partial_b c)^{Q_{23} - 1} (c \partial_a)^{Q_{31}} (\partial_a \nabla_2)^{n_1} (\partial_b \nabla_3)^{n_2 + 1} (c \nabla_1)^{n_3} \\ &\quad + Q_{31} (\partial_a \partial_b)^{Q_{12}} (\partial_b c)^{Q_{23}} (c \partial_a)^{Q_{31} - 1} (\partial_a \nabla_2)^{n_1} (\partial_a \nabla_3) (\partial_b \nabla_3)^{n_2} (c \nabla_1)^{n_3} \} \\ &\quad h^{(s_1)}(z_1; a) h^{(s_2)}(z_2; b) \Big|_{z_1 = z_2 = z_3}. \end{aligned} \quad (2.15)$$

[‡]We use the standard definition

$$\binom{s}{\alpha, \beta, \gamma} = \frac{s!}{\alpha! \beta! \gamma!}, \quad \alpha + \beta + \gamma = s.$$

This divergence (and the corresponding divergences of the currents $J^{(1,2)}$) must vanish on shell.

We shall develop now a recursive algorithm. First we study the terms not containing any deDonder expression $D^{(s_i-1)}$, $i = 1, 2, 3$:

$$D^{(s_i-1)} = \frac{1}{s_i} [(\partial_{a_i} \nabla_i) - 1/2(a_i \nabla_i) \square_{a_i}] h^{(s_i)}(z_i; a_i), \quad a_i = a, b, c. \quad (2.16)$$

We use that

$$(\nabla_1 \nabla_3) = 1/2[\square_2 - \square_1 - \square_3], \quad (2.17)$$

and

$$\square_i h^{(s_i)}(z_i; a_i) = \mathcal{F}^{(s_i)}(z_i; a_i) + s_i D^{(s_i-1)}, \quad (2.18)$$

$$\square_i \epsilon^{(s_i-1)}(z_i; a_i) = \delta_i^{(0)} D^{(s_i-1)}, \quad (2.19)$$

where $\mathcal{F}^{(s_i)}(z_i; a_i)$ is Fronsdal's gauge invariant equation of motion and can be dropped on shell. So the n_3 -term of (2.15) does not contribute to the leading order terms. On the other hand the Q_{23} -term is purely leading order. The Q_{31} -term contains

$$(\partial_a \nabla_3) = -(\partial_a \nabla_2) - (\partial_a \nabla_1). \quad (2.20)$$

Only the first term yields a leading order contribution, the next one is a divergence term. A possibility to classify the higher order terms is to count the divergence and the deDonder operators separately, say by numbers m_1, m_2 respectively.

In the leading order (l. o.) terms we renumber the powers $n_1 \rightarrow n_1 + 1$ in the Q_{23} -term and $n_2 \rightarrow n_2 + 1$ in the l.o. Q_{31} term. We get

$$\begin{aligned} & [(n_1 + 1 - \nu_1) C_{n_1+1, n_2, n_3}^{s_1, s_2, s_3} - (n_2 + 1 - \nu_2) C_{n_1, n_2+1, n_3}^{s_1, s_2, s_3}] \quad (2.21) \\ & (\partial_a \partial_b)^{n_3 - \nu_3} (\partial_b c)^{n_1 - \nu_1} (c \partial_a)^{n_2 - \nu_2} (\partial_a \nabla_2)^{n_1+1} (\partial_b \nabla_3)^{n_2+1} (c \nabla_1)^{n_3} = 0. \end{aligned}$$

It follows that the factor in the square bracket must vanish. Two analogous relations follow from the two other currents. The solution of these three recursion relations is

$$C_{n_1, n_2, n_3}^{s_1, s_2, s_3} = const \left(\begin{array}{c} \sum n_i - \sum \nu_i \\ n_1 - \nu_1, n_2 - \nu_2, n_3 - \nu_3 \end{array} \right), \quad (2.22)$$

which is equivalent to (2.12) for $\Delta = \Delta_{min}$ and therefore $\nu_3 = 0$, and describes also all other $\Delta > \Delta_{min}$ cases. Comparison with (2.9), (2.13) proves that we can present the trinomial coefficient also as

$$C_{Q_{12}, Q_{23}, Q_{31}}^{s_1, s_2, s_3} = const \left(\begin{array}{c} s_{min} \\ Q_{12}, Q_{23}, Q_{31} \end{array} \right). \quad (2.23)$$

We see that the number of contractions between indices of our three fields Q_{12}, Q_{23}, Q_{31} define our interaction completely.

Finally we want to make a remark concerning the case where two or all three of these fields are equal. Then we get only two or one current whose divergences vanish on shell. But in this case we have a symmetry which restores the result (2.12), (2.21) and shows that this is correct in all cases.

3 Cubic interactions for arbitrary spins: Complete solution of the Noether's procedure

To derive the next terms of interaction containing one deDonder expression we turn to the Lagrangian formulation of the task and solve Noether's equation

$$\sum_{i=1}^3 \delta_i^{(1)} \mathcal{L}_i^0(h^{(s_i)}(a)), + \sum_{i=1}^3 \delta_i^{(0)} \mathcal{L}_I(h^{(s_1)}(a), h^{(s_2)}(b), h^{(s_3)}(c)) = 0. \quad (3.1)$$

where

$$\delta_i^{(0)} h^{(s_i)}(a_i) = s_i(a_i \nabla_i) \epsilon^{s_i-1}(z_i; a_i) \quad (3.2)$$

$$\mathcal{L}_i^0(h^{(s_i)}(a)) = -\frac{1}{2} h^{(s_i)}(a_i) *_{a_i} \mathcal{F}^{(s_i)}(a_i) + \frac{1}{8s_i(s_i-1)} \square_{a_i} h^{(s_i)}(a_i) *_{a_i} \square_{a_i} \mathcal{F}^{(s_i)}(a_i) \quad (3.3)$$

Shifting $\delta_i^{(1)}$ by a trace term in the same way as in [1] we obtain the following functional equation:

$$\sum_{i=1}^3 \delta_i^{(0)} \mathcal{L}_I(h^{(s_1)}(a), h^{(s_2)}(b), h^{(s_3)}(c)) = 0 + O(\mathcal{F}^{(s_i)}(a_i)). \quad (3.4)$$

We solve this equation starting from the ansatz (2.2), (2.3) and integrating level by level in means of its dependence on deDonder tensors and traces of higher spin gauge fields.

Actually we have to solve the following equation:

$$\begin{aligned} C_{\{n_i\}}^{\{s_i\}} \hat{T}(Q_{ij}|n_i) [(a \nabla_1) \epsilon^{(s_1-1)} h^{(s_2)} h^{(s_3)} + h^{(s_1)} (b \nabla_2) \epsilon^{(s_2-1)} h^{(s_3)} + h^{(s_1)} h^{(s_2)} (c \nabla_3) \epsilon^{(s_3-1)}] \\ = 0 + O(\mathcal{F}^{(s_i)}(a_i), D^{(s_i-1)}(a_i), \square_{a_i} h^{(s_i)}(a_i)). \end{aligned} \quad (3.5)$$

Taking into account that due to (2.5)

$$\hat{T}(Q_{ij}|n_i)(a_i \nabla_i) \epsilon^{(s_i-1)}(a_i) = [\hat{T}(Q_{ij}|n_i), (a_i \nabla_i)] \epsilon^{(s_i-1)}(a_i), \quad (3.6)$$

we see that all necessary information for the recursion can be found calculating these commutators

$$\begin{aligned} [\hat{T}(Q_{ij}|n_i), (a \nabla_1)] &= Q_{31} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1 | n_1, n_2, n_3 + 1) \\ &\quad - Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 + 1, n_3) \\ &\quad + n_1 \hat{T}(Q_{12}, Q_{23}, Q_{31} | n_1 - 1, n_2, n_3) (\nabla_1 \nabla_2) \\ &\quad - Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2, n_3) (\partial_b \nabla_2), \end{aligned} \quad (3.7)$$

$$\begin{aligned}
[\hat{T}(Q_{ij}|n_i), (b\nabla_2)] &= Q_{12}\hat{T}(Q_{12}-1, Q_{23}, Q_{31}|n_1+1, n_2, n_3) \\
&\quad - Q_{23}\hat{T}(Q_{12}, Q_{23}-1, Q_{31}|n_1, n_2, n_3+1) \\
&\quad + n_2\hat{T}(Q_{12}, Q_{23}, Q_{31}|n_1, n_2-1, n_3)(\nabla_2\nabla_3) \\
&\quad - Q_{23}\hat{T}(Q_{12}, Q_{23}-1, Q_{31}|n_1, n_2, n_3)(\partial_c\nabla_3), \tag{3.8}
\end{aligned}$$

$$\begin{aligned}
[\hat{T}(Q_{ij}|n_i), (c\nabla_3)] &= Q_{23}\hat{T}(Q_{12}, Q_{23}-1, Q_{31}|n_1, n_2+1, n_3) \\
&\quad - Q_{31}\hat{T}(Q_{12}, Q_{23}, Q_{31}-1|n_1+1, n_2, n_3) \\
&\quad + n_3\hat{T}(Q_{12}, Q_{23}, Q_{31}|n_1, n_2, n_3-1)(\nabla_3\nabla_1) \\
&\quad - Q_{31}\hat{T}(Q_{12}, Q_{23}, Q_{31}-1|n_1, n_2, n_3)(\partial_a\nabla_1), \tag{3.9}
\end{aligned}$$

where we used relations like (2.18) and (2.20). In these commutators we can use also the following identities

$$\begin{aligned}
\nabla_1\nabla_2 &= \frac{1}{2}(\square_3 - \square_2 - \square_1), \\
\nabla_2\nabla_3 &= \frac{1}{2}(\square_1 - \square_2 - \square_3), \\
\nabla_3\nabla_1 &= \frac{1}{2}(\square_2 - \square_3 - \square_1). \tag{3.10}
\end{aligned}$$

Now we see immediately from the first two lines of (3.7)-(3.9) that these contribute to (3.4) as leading order terms and yield the same equations for the $C_{n_i}^{s_i}$ coefficients as (2.21)

$$(Q_{31}+1)C_{n_1, n_2+1, n_3}^{s_1, s_2, s_3} - (Q_{12}+1)C_{n_1, n_2, n_3+1}^{s_1, s_2, s_3} = 0, \tag{3.11}$$

$$(Q_{12}+1)C_{n_1, n_2, n_3+1}^{s_1, s_2, s_3} - (Q_{23}+1)C_{n_1+1, n_2, n_3}^{s_1, s_2, s_3} = 0, \tag{3.12}$$

$$(Q_{23}+1)C_{n_1+1, n_2, n_3}^{s_1, s_2, s_3} - (Q_{31}+1)C_{n_1, n_2+1, n_3}^{s_1, s_2, s_3} = 0. \tag{3.13}$$

with the solution (2.22) or (2.23).

To find the full interaction we follow the same strategy as in the case $s=4$ [1] and introduce the following classification for the higher order interaction terms in D and $\bar{h} = Trh$:

$$\mathcal{L}_I = \sum_{\substack{i,j=0,1,2,3 \\ i+j \leq 3}} \mathcal{L}_I^{(i,j)}(h^{(s)}), \tag{3.14}$$

where

$$\mathcal{L}_I^{(i,j)}(h^{(s)}) \sim \nabla^{s-i}(D)^i(\bar{h}^{(s)})^j(h^{(s)})^{3-j-i}. \tag{3.15}$$

In this notation the leading term described in the second section is $\mathcal{L}_I^{(0,0)}(h^{(s)})$.

To integrate Noether's equation next to the leading term we have to insert in (3.4) the last two lines of (3.7)-(3.9) and use two important relations (2.18),

(2.19). Thus we arrive at the following $O(D)$ solution:

$$\begin{aligned} \mathcal{L}_I^{(1,0)} &= \sum_{n_i} \int dz dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_3 - z) \delta(z_2 - z) \\ &\left[+ \frac{s_1 n_1}{2} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \hat{T}(Q_{ij} | n_1 - 1, n_2, n_3) D^{(s_1-1)} h^{(s_2)} h^{(s_3)} \right. \\ &+ \frac{s_2 n_2}{2} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \hat{T}(Q_{ij} | n_1, n_2 - 1, n_3) h^{(s_1)} D^{(s_2-1)} h^{(s_3)} \\ &\left. + \frac{s_3 n_3}{2} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \hat{T}(Q_{ij} | n_1, n_2, n_3 - 1) h^{(s_1)} h^{(s_2)} D^{(s_3-1)} \right]. \end{aligned} \quad (3.16)$$

The detailed proof of this formula can be found in the Appendix where we describe also derivations of all other terms.

The next $O(D^2)$ and $O(D^3)$ level Lagrangians are

$$\begin{aligned} \mathcal{L}_I^{(2,0)} &= \sum_{n_i} \int dz dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_3 - z) \delta(z_2 - z) \\ &\left[+ \frac{s_3 n_3 s_1 n_1}{2} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \hat{T}(Q_{ij} | n_1 - 1, n_2, n_3 - 1) D^{(s_1-1)} h^{(s_2)} D^{(s_3-1)} \right. \\ &+ \frac{s_1 n_1 s_2 n_2}{2} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \hat{T}(Q_{ij} | n_1 - 1, n_2 - 1, n_3) D^{(s_1-1)} D^{(s_2-1)} h^{(s_3)} \\ &\left. + \frac{s_2 n_2 s_3 n_3}{2} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \hat{T}(Q_{ij} | n_1, n_2 - 1, n_3 - 1) h^{(s_1)} D^{(s_2-1)} D^{(s_3-1)} \right], \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} \mathcal{L}_I^{(3,0)} &= \sum_{n_i} \int dz dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_3 - z) \delta(z_2 - z) \\ &\left[+ \frac{s_3 n_3 s_2 n_2 s_1 n_1}{2} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \hat{T}(Q_{ij} | n_1 - 1, n_2 - 1, n_3 - 1) D^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)} \right]. \end{aligned} \quad (3.18)$$

The remaining terms in the Lagrangian contain at least one trace:

$$\mathcal{L}_I^{(0,1)} = \mathcal{L}_I^{(0,2)} = 0, \quad (3.19)$$

$$\begin{aligned} \mathcal{L}_I^{(0,3)} &= \sum_{n_i} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \frac{Q_{12} Q_{23} Q_{31}}{8} \int dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_2 - z) \delta(z_3 - z) \\ &\left[\hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} - 1 | n_1, n_2, n_3) \square_a h^{(s_1)} \square_b h^{(s_2)} \square_c h^{(s_3)} \right], \end{aligned} \quad (3.20)$$

$$\begin{aligned} \mathcal{L}_I^{(1,1)} &= \sum_{n_i} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \int dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_2 - z) \delta(z_3 - z) \\ &\left[+ \frac{s_1 Q_{12} n_2}{4} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3) D^{(s_1-1)} \square_b h^{(s_2)} h^{(s_3)} \right. \\ &+ \frac{s_2 Q_{23} n_3}{4} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31} | n_1, n_2, n_3 - 1) h^{(s_1)} D^{(s_2-1)} \square_c h^{(s_3)} \\ &\left. + \frac{s_3 Q_{31} n_1}{4} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1 | n_1 - 1, n_2, n_3) \square_a h^{(s_1)} h^{(s_2)} D^{(s_3-1)} \right], \end{aligned} \quad (3.21)$$

$$\begin{aligned}
\mathcal{L}_I^{(1,2)} &= \sum_{n_i} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \int dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_2 - z) \delta(z_3 - z) \\
&\left[+ \frac{s_1 Q_{12} Q_{23} n_3}{8} \hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} | n_1, n_2, n_3 - 1) D^{(s_1-1)} \square_b h^{(s_2)} \square_c h^{(s_3)} \right. \\
&+ \frac{s_2 Q_{23} Q_{31} n_1}{8} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31} - 1 | n_1 - 1, n_2, n_3) \square_a h^{(s_1)} D^{(s_2-1)} \square_c h^{(s_3)} \\
&\left. + \frac{s_3 Q_{31} Q_{12} n_2}{8} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} - 1 | n_1, n_2 - 1, n_3) \square_a h^{(s_1)} \square_b h^{(s_2)} D^{(s_3-1)} \right], \tag{3.22}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_I^{(2,1)} &= \sum_{n_i} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \int dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_2 - z) \delta(z_3 - z) \\
&\left[+ \frac{s_2 s_3 Q_{31} n_1 n_2}{4} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1 | n_1 - 1, n_2 - 1, n_3) \square_a h^{(s_1)} D^{(s_2-1)} D^{(s_3-1)} \right. \\
&+ \frac{s_1 s_2 Q_{23} n_3 n_1}{4} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31} | n_1 - 1, n_2, n_3 - 1) D^{(s_1-1)} D^{(s_2-1)} \square_c h^{(s_3)} \\
&\left. + \frac{s_3 s_1 Q_{12} n_2 n_3}{4} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3 - 1) D^{(s_1-1)} \square_b h^{(s_2)} D^{(s_3-1)} \right]. \tag{3.23}
\end{aligned}$$

So we integrated all cells of the following classification table corresponding to (3.14)

$\bar{h} \backslash D$	0	1	2	3
0	hhh	Dhh	DDh	DDD
1	000	$\bar{h}Dh$	$\bar{h}DD$	
2	000	$\bar{h}\bar{h}D$		
3	$\bar{h}\bar{h}\bar{h}$			

(3.24)

and prove that *after fixing the freedom of partial integration in the leading term (i.e. our cyclic ansatz) all other terms of interaction can be integrated in a unique way when we avoid additional partial integration during recursions.*

Summarizing we see that the interaction Lagrangian in deDonder gauge $D^{(s-1)}(z; a) =$

0 can be expressed as a sum

$$\mathcal{L}_I^{dD}(h^{(s)}) = \sum_{j=0}^3 \mathcal{L}_I^{(0,j)}(h^{(s)}). \quad (3.25)$$

and it is nothing else than the first column of this table. Therefore (3.19) means that *in deDonder gauge the traces of the HS fields decouple from the fields as they do in the free Lagrangian.*

4 Classification of gauge invariant cubic vertices

From the recent literature [1], [2], [15] we can first speculatively group the cubic gauge invariant vertices into four classes depending on the number of Weyl tensors they contain. In [15] vertices with (say even) spins s_1, s_2, s_3 were explicitly constructed e.g. for the case 4, 4, 2, and the result was presented in a form which was linear in the (linearized) Weyl tensor of the minimal spin. In the remaining bracket of terms a second Weyl tensor (say for spin 2) is apparently not hidden. In [2] the ensemble of vertices constructed contain those of the spin type $2s, s, s$, and these are quadratic in the (linearized) Weyl tensor of the spin s field. In all these cases the number of derivatives is minimal and equals $\Delta = s_3 + s_2 - s_1$, where s_1 is the minimal spin. In this section we order the spins such as $s_1 \leq s_2 \leq s_3$, because the smaller spins give rise to the Weyl tensors, and these are represented by differential operators acting from left to right. Abandoning the constraint of minimality on the number of derivatives, one can construct vertices from three Riemann tensors by contraction having $\Delta_{BI} = s_1 + s_2 + s_3$ derivatives [§] (Born-Infeld interaction). Finally a selfinteracting vertex of spin type s, s, s has been derived in [1] for $s = 4$, which apparently did not factorize into any Weyl tensor at all. If we consider only the leading order terms discussed in Section 2, the Weyl tensor reduces to the Riemann tensor. The Riemann tensor for spin s gauge field [27], [28] is best defined for our purpose by the differential expression

$$R^{(s)}(z; a, b) = [(a\nabla)(b\partial_c) - (a\partial_c)(b\nabla)]^s h^{(s)}(z; c). \quad (4.1)$$

With the cyclic ansatz for a vertex of an arbitrary spin type presented in the preceding section, we can derive results on the factorization of the l. o. terms into Riemann tensors. Denoting the powers of the Riemann tensors maximally appearing by n , and letting the number of derivatives be $\Delta* \geq \Delta$ we may define classes of vertices $\mathcal{V}_n(\Delta*)$ and characterize them in terms of the spin s_i . There remain two tasks. First we may consider all explicitly known vertices for minimal derivative number Δ and rewrite their leading orders in terms of the cyclic notation for the l. o. terms discussed in the preceding section. This was in all cases

[§]This case is also described by our general formula for the case $\Delta = \Delta_{BI}$ and the triangle inequality $s_1 + s_2 \geq s_3$

possible. On the other hand we can turn this issue around and derive from the cyclic ansatz of Section 1 with minimal derivative number the maximum amount of information possible on the factorization into Riemann tensors, which gives new general insights on which spin type belongs to which class $\mathcal{V}_n(\Delta)$.

We prove the following theorem: *The class $\mathcal{V}_0(\Delta)$ consists of those spin types with $s_3 < 2s_1$, the class $\mathcal{V}_1(\Delta)$ of the spin types $s_3 \geq 2s_1, s_2 > s_1$, and the class $\mathcal{V}_2(\Delta)$ of the types $s_3 \geq 2s_1 = 2s_2$. As a corollary we obtain that $\mathcal{V}_0(\Delta)$ contains all selfinteractions.*

For the proof we start from (2.4), (2.12). We obtain by replacing $n_1 = s_1 - p$ and summing over n_2, n_3

$$\begin{aligned} & \sum_{n_2, n_3} \binom{s_1}{n_1, n_2 - s_2 + s_1, n_3 - s_3 + s_1} (\partial_a \nabla_2)^{n_1} (\partial_b \nabla_3)^{n_2} (\partial_c \nabla_1)^{n_3} \\ & (\partial_a \partial_b)^{n_3 - s_3 + s_1} (\partial_b \partial_c)^{n_1} (\partial_c \partial_a)^{n_2 - s_2 + s_1} h^{(s_1)}(z_1; a) h^{(s_2)}(z_2; b) h^{(s_3)}(z_3; c) \\ & = \binom{s_1}{p} [(\partial_a \nabla_2)(\partial_b \partial_c)]^{s_1 - p} (\partial_c \nabla_1)^{s_3 - s_1} (\partial_b \nabla_3)^{s_2 - s_1} \\ & \{(\partial_c \nabla_1)(\partial_b \partial_a) + (\partial_c \partial_a)(\partial_b \nabla_3)\}^p h^{(s_1)}(z_1; a) h^{(s_2)}(z_2; b) h^{(s_3)}(z_3; c). \end{aligned} \quad (4.2)$$

Since we want to neglect all higher order terms (such as divergences, traces and contractions of gradients $(\nabla_i \nabla_j)^\natural$), use partial integrations implied by $\nabla_1 + \nabla_2 + \nabla_3 = 0$, and keep only the l.o. expressions, we can in the square bracket replace $(\partial_b \nabla_3)$ by $-(\partial_b \nabla_1)$. Introducing two arbitrary tangential vectors e, f and the star product we get

$$[(\partial_b \partial_a)(\partial_c \nabla_1) - (\partial_c \partial_a)(\partial_b \nabla_1)]^p = [(e \partial_a)(f \nabla_1) - (f \partial_a)(e \nabla_1)]^p *_e *_f [(e \partial_b)(f \partial_c)]^p, \quad (4.3)$$

where we obtained p differential operator factors acting on $h^{(s_1)}(z_1, a)$ of a Riemann tensor. The other $s_1 - p$ components are produced as follows.

The remaining factors in (4.2) can be reordered by

$$\begin{aligned} & [(\partial_a \nabla_2)(\partial_b \partial_c)]^{s_1 - p} (\partial_c \nabla_1)^{s_3 - s_1} (\partial_b \nabla_3)^{s_2 - s_1} \\ & = [(\partial_a \nabla_2)(\partial_c \nabla_1)]^{s_1 - p} (\partial_b \partial_c)^{s_1 - p} (\partial_c \nabla_1)^{s_3 - 2s_1 + p} (\partial_b \nabla_3)^{s_2 - s_1}, \end{aligned} \quad (4.4)$$

which is permitted if $s_3 - 2s_1 \geq 0$, since p runs from 0 to s_1 . Then the square bracket can be contracted in the same way with the tangential vectors e, f and neglecting a contraction $(\nabla_1 \nabla_2)$ as of higher order we get

$$[(e \partial_a)(f \nabla_1) - (f \partial_a)(e \nabla_1)]^{s_1 - p} *_e *_f [(e \nabla_2)(f \partial_c)]^{s_1 - p}. \quad (4.5)$$

Multiplication of the two products over the square bracket operators (4.3) and (4.5) yields the Riemann tensor

$$R^{s_1}(z_1; f, e) = [(e \partial_a)(f \nabla_1) - (f \partial_a)(e \nabla_1)]^{s_1} h^{(s_1)}(z_1; a). \quad (4.6)$$

[¶]Note that the product of derivatives $(\nabla_i \nabla_j)$ in this case leads also to $\square R^{(s)}(a; b; h^{(s)}(c; z))$ which is equal to $R^{(s)}(a; b; (e \nabla) D^{(s-1)}) = 0$ on shell due to gauge invariance of the linearized curvature

However, in the remaining factors there is still the term

$$(\partial_c \nabla_1)^{s_3 - 2s_1 + p}, \quad (4.7)$$

which acts on $h^{(s_1)}$ (i.e. the Riemann tensor), but by partial integration and neglect of divergences as higher order terms, we can let it act on $h^{(s_2)}$. Then we obtain the first Riemann tensor formula

$$\begin{aligned} R^{(s_1)}(z_1; f, e) *_{e} *_{f} \sum_{p=0}^{s_1} (-1)^p \binom{s_1}{p} (e \partial_b)^p (e \nabla_2)^{s_1 - p} (f \partial_c)^{s_1} \\ (\partial_b \partial_c)^{s_1 - p} (\partial_c \nabla_2)^{s_3 - 2s_1 + p} (\partial_b \nabla_3)^{s_2 - s_1} h^{(s_2)}(z_2; b) h^{(s_3)}(z_3; c). \end{aligned} \quad (4.8)$$

The next construction is also intended to be a Riemann tensor for the spin s_1 . For this purpose we construct a homogeneous function in b of degree s_1 . Using the appropriate factors from (4.8) we define

$$H(z_2, z_3; b, c, f) = (f \partial_c)^{s_1} (\partial_b \nabla_3)^{s_2 - s_1} (\partial_c \nabla_2)^{s_3 - 2s_1} h^{(s_2)}(z_2; b) h^{(s_3)}(z_3; c), \quad (4.9)$$

which is homogeneous both in b and c of degree s_1 . Then we can perform the sum in (4.8) and get

$$[(e \nabla_2)(\partial_b \partial_c) - (\partial_c \nabla_2)(e \partial_b)]^{s_1} H(z_2, z_3; b, c, f). \quad (4.10)$$

Concerning the differential operator in front of H this expression looks like a Riemann tensor $R^{(s_1)}(z_2)$. But a gauge transformation $\delta h^{(s_2)}(z_2, b)$ acts as

$$\begin{aligned} (\partial_b \nabla_3)^{s_2 - s_1} (b \nabla_2) \epsilon^{(s_2 - 1)}(z_2; b) = \\ (s_2 - s_1) (\nabla_2 \nabla_3) (\partial_b \nabla_3)^{s_2 - s_1 - 1} \epsilon^{(s_2 - 1)}(z_2; b) \\ + (b \nabla_2) (\partial_b \nabla_3)^{s_2 - s_1} \epsilon^{(s_2 - 1)}(z_2, b). \end{aligned} \quad (4.11)$$

We conclude that only for $s_2 = s_1$ the expression (4.10) can be considered as a proper Riemann tensor. This completes the proof of the theorem.

5 Discussion: Towards gauge transformations as open Lie algebras

If all spins in the cubic interaction are equal s , we can derive the first order gauge transformation of $h^{(s)}$ from the r. h. s. of Noether's equation (3.4) taken off shell

$$\begin{aligned}
& [O(\mathcal{F}) \text{ part of } \delta_{\epsilon^{(s_1-1)}}^0 \mathcal{L}_I] \\
&= \sum_{n_i} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \int dz_1 dz_2 dz_3 \delta(z_1 - z) \delta(z_2 - z) \delta(z_3 - z) \\
&\left[- \frac{s_1 n_1}{2} \hat{T}(Q_{ij} | n_1 - 1, n_2, n_3) \epsilon^{(s_1-1)} \mathcal{F}^{(s_2)} h^{(s_3)} \right. \\
&+ \frac{s_1 Q_{12} n_2}{4} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3) \epsilon^{(s_1-1)} \square_b \mathcal{F}^{(s_2)} h^{(s_3)} \\
&+ \frac{s_3 s_1 Q_{12} n_2 n_3}{4} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3 - 1) \epsilon^{(s_1-1)} \square_b \mathcal{F}^{(s_2)} D^{(s_3-1)} \\
&+ \frac{s_1 Q_{12} Q_{23} n_3}{8} \hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} | n_1, n_2, n_3 - 1) \epsilon^{(s_1-1)} \square_b \mathcal{F}^{(s_2)} \square_c h^{(s_3)} \\
&+ \frac{s_1 n_1}{2} \hat{T}(Q_{ij} | n_1 - 1, n_2, n_3) \epsilon^{(s_1-1)} h^{(s_2)} \mathcal{F}^{(s_3)} \\
&+ \frac{s_1 Q_{12} n_2}{4} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3) \epsilon^{(s_1-1)} \square_b h^{(s_2)} \mathcal{F}^{(s_3)} \\
&+ \frac{s_1 s_2 n_1 n_2}{4} \hat{T}(Q_{12}, Q_{23}, Q_{31} | n_1 - 1, n_2 - 1, n_3) \epsilon^{(s_1-1)} D^{(s_2-1)} \mathcal{F}^{(s_3)} \\
&\left. - \frac{s_1 Q_{12} Q_{23} n_3}{8} \hat{T}(Q_{12} - 1, Q_{23} - 1, Q_{31} | n_1, n_2, n_3 - 1) \epsilon^{(s_1-1)} \square_b h^{(s_2)} \square_c \mathcal{F}^{(s_3)} \right]. \tag{5.1}
\end{aligned}$$

If we assume moreover that the gauge transformations form a Lie algebra of power series in some "coupling constant" g , we can following along the ideas of Berends, Burger and Van Dam in their classical paper [7] derive conclusions on the higher order interactions. We sum up simple results:

- (1) The arguments of these authors to show that such power series algebra does not exist for $s = 3$, cannot be generalized to even spins;
- (2) the quartic interaction Lagrangian is nonzero and contains $2s - 2$ derivatives.

For a given gauge function $\epsilon^{(s-1)}(z; a)$ the gauge transformation is a substitution (classically) with expansion

$$h \rightarrow h + \delta_\epsilon h = h + \nabla \epsilon + \sum_{n \geq 1} g^n \Theta_n(h, h, \dots; \epsilon), \tag{5.2}$$

with Θ_n depending on ϵ linearly and on h in the n 'th power. Moreover we assume that the commutator of two such transformations is given by

$$[\delta_\epsilon, \delta_\eta] h = \delta_{C(h; \epsilon, \eta)} h, \tag{5.3}$$

with the expansion

$$C(h; \epsilon, \eta) = g \sum_{n \geq 0} g^n C_n(h, h, \dots, h; \epsilon, \eta), \quad (5.4)$$

where each C_n depends on ϵ and η linearly and on h in the n 'th power. As substitutions gauge transformations are associative and their infinitesimals must satisfy the Jacobi identity. At order g^2 this is e.g.

$$\sum_{\eta, \epsilon, \zeta \text{ cyclic}} \{C_1(\nabla \zeta; \eta, \epsilon) + C_0(C_0(\eta, \epsilon), \zeta)\} = 0. \quad (5.5)$$

The commutator can also be expanded

$$\begin{aligned} [\delta_\eta, \delta_\epsilon] &= g(\Theta_1(\nabla \epsilon; \eta) - \Theta_1(\nabla \eta; \epsilon)) \\ &+ g^2 \{[\Theta_1(\Theta_1(h; \epsilon); \eta) - \Theta_1(\Theta_1(h; \eta); \epsilon)] \\ &+ [\Theta_2(\nabla \epsilon, h; \eta) - \Theta_2(\nabla \eta, h; \epsilon)] \\ &+ [\Theta_2(h, \nabla \epsilon; \eta) - \Theta_2(h, \nabla \eta; \epsilon)]\} + O(g^3). \end{aligned} \quad (5.6)$$

Inserting this expansion into the definition of the functions C_n we obtain

$$\begin{aligned} \nabla C_0(\eta, \epsilon) &= \Theta_1(\nabla \epsilon; \eta) - \Theta_1(\nabla \eta; \epsilon), \quad (5.7) \\ \nabla C_1(h; \eta, \epsilon) &= \Theta_1(\Theta_1(h; \epsilon); \eta) - \Theta_1(\Theta_1(h; \eta); \epsilon) - \Theta_1(h; C_0(\eta, \epsilon)) \\ &+ \Theta_2(\nabla \epsilon, h; \eta) - \Theta_2(\nabla \eta, h; \epsilon) + \Theta_2(h, \nabla \epsilon; \eta) - \Theta_2(h, \nabla \eta; \epsilon). \end{aligned} \quad (5.8)$$

Assume that $\Theta_1(h; \epsilon)$ has been extracted from (5.1) for the case of equal spins s . Then the order of derivations in Θ_1 is

$$\sharp \Theta_1(h; \epsilon) = s - 1. \quad (5.9)$$

Inserting this result into (5.5), (5.7) we obtain the number of derivations in C_0, C_1 as

$$\sharp C_0(\eta, \epsilon) = s - 2 \quad \text{and} \quad \sharp C_1(h; \eta, \epsilon) = 2s - 3. \quad (5.10)$$

This implies

$$\sharp \Theta_2(h, h; \epsilon) = 2s - 3. \quad (5.11)$$

Consequently the quartic interaction must contain $2s - 2$ derivatives. The argument can be continued to still higher interactions: For n 'th order interactions the number is $(s - 2)(n - 2) + 2$. This result is equivalent to introducing a scale L and dimensions in the following way

$$[h] = L^{s-2}, \quad [\nabla] = \frac{1}{L}, \quad (5.12)$$

with a dimensionless coupling constant g , so that each term in the power series has the same dimension. Note that in the case of $\Delta = \Delta_{min}$ we obtained in the previous sections and in [1] the same dimensions for cubic selfinteractions and a free Fronsdal's action.

In [7] the argument was presented that for spin $s = 3$ a Lie algebra of gauge transformations in the form of power series does not exist, the problem starting with the second power. The argument was based on the term

$$(\partial_a \nabla_2)^{s-1} \epsilon^{(s-1)}(z_1; a) h^{(s)}(z_2; b), \quad (5.13)$$

which exists in Θ_1 . Such term is present in fact for any spin, as can be inspected from (5.1). Namely, in the fifth term of the square bracket of (5.1) (this is the unique localization) we get such expression for $n_1 = s, n_2 = n_3 = 0$. In equation (5.8) in the first line we have thus $2s - 2$ derivations acting on the field h in either term. In no other terms of (5.8) such expression appears. Therefore they must cancel inside this line and they do cancel indeed for even spin only. There is in this case no obstruction of the power series algebra by these arguments. A deeper investigation of such algebras will follow in the future.

Acknowledgements

This work is supported in part by Alexander von Humboldt Foundation under 3.4-Fokoop-ARM/1059429. Work of K.M. was made with partial support of CRDF-NFSAT UCEP06/07 and CRDF-NFSAT-SCS MES RA ECSP 09_01.

References

- [1] R. Manvelyan, K. Mkrtchyan and W. Rühl, "Direct construction of a cubic selfinteraction for higher spin gauge fields," arXiv:1002.1358 [hep-th].
- [2] R. Manvelyan, K. Mkrtchyan and W. Rühl, "Off-shell construction of some trilinear higher spin gauge field interactions," Nucl. Phys. B **826** (2010) 1 [arXiv:0903.0243 [hep-th]].
- [3] R. Manvelyan and K. Mkrtchyan, "Conformal invariant interaction of a scalar field with the higher spin field in AdS_D ," [arXiv:0903.0058 [hep-th]].
- [4] R. Manvelyan and W. Rühl, "Conformal coupling of higher spin gauge fields to a scalar field in AdS(4) and generalized Weyl invariance," Phys. Lett. B **593** (2004) 253, [arXiv:hep-th/0403241].
- [5] F. A. Berends, G. J. H. Burgers and H. van Dam, "Explicit Construction Of Conserved Currents For Massless Fields Of Arbitrary Spin," Nucl. Phys. B **271** (1986) 429;

- [6] F. A. Berends, G. J. H. Burgers and H. Van Dam, “On Spin Three Selfinteractions,” *Z. Phys. C* **24** (1984) 247;
- [7] F. A. Berends, G. J. H. Burgers and H. van Dam, “On The Theoretical Problems In Constructing Interactions Involving Higher Spin Massless Particles,” *Nucl. Phys. B* **260** (1985) 295.
- [8] E. S. Fradkin and M. A. Vasiliev, “On The Gravitational Interaction Of Massless Higher Spin Fields,” *Phys. Lett. B* **189** (1987) 89.
- [9] E. S. Fradkin and M. A. Vasiliev, “Cubic Interaction In Extended Theories Of Massless Higher Spin Fields,” *Nucl. Phys. B* **291** (1987) 141. M. A. Vasiliev, “Cubic Interactions of Bosonic Higher Spin Gauge Fields in AdS_5 ”, [arXiv:hep-th/0106200]. M. A. Vasiliev, “ $N = 1$ Supersymmetric Theory of Higher Spin Gauge Fields in AdS_5 at the Cubic Level”, [arXiv:hep-th/0206068]
- [10] T. Damour and S. Deser, “Geometry of spin 3 gauge theories,” *Annales Poincare Phys. Theor.* **47**, 277 (1987); T. Damour and S. Deser, “Higher derivative interactions of higher spin gauge fields,” *Class. Quant. Grav.* **4**, L95 (1987).
- [11] R. R. Metsaev, “Cubic interaction vertices for massive and massless higher spin fields,” *Nucl. Phys. B* **759** (2006) 147 [arXiv:hep-th/0512342];
- [12] R. R. Metsaev, “Cubic interaction vertices for fermionic and bosonic arbitrary spin fields,” arXiv:0712.3526 [hep-th].
- [13] A. Sagnotti, “Higher Spins and Current Exchanges,” arXiv:1002.3388 [hep-th]; D. Francia, J. Mourad and A. Sagnotti, “Current exchanges and unconstrained higher spins,” *Nucl. Phys. B* **773** (2007) 203 [arXiv:hep-th/0701163]; D. Francia and A. Sagnotti, “Higher-spin geometry and string theory,” *J. Phys. Conf. Ser.* **33** (2006) 57 [arXiv:hep-th/0601199]; A. Sagnotti and M. Tsulaia, “On higher spins and the tensionless limit of string theory,” *Nucl. Phys. B* **682** (2004) 83 [arXiv:hep-th/0311257]; D. Francia and A. Sagnotti, “On the geometry of higher-spin gauge fields,” *Class. Quant. Grav.* **20** (2003) S473 [arXiv:hep-th/0212185].
- [14] I. G. Koh, S. Ouvry, “Interacting gauge fields of any spin and symmetry,” *Phys. Lett. B* **179** (1986) 115; Erratum-ibid. **183** B (1987) 434.
- [15] Nicolas Boulanger, Serge Leclercq, Per Sundell, “On The Uniqueness of Minimal Coupling in Higher-Spin Gauge Theory,” *JHEP* 0808:056,2008; [arXiv:0805.2764 [hep-th]]. Xavier Bekaert, Nicolas Boulanger, Sandrine Cnockaert, Serge Leclercq, “On Killing tensors and cubic vertices in higher-spin gauge theories,” *Fortsch. Phys.* **54** (2006) 282-290; [arXiv:hep-th/0602092].

- [16] A. Fotopoulos, N. Irges, A. C. Petkou and M. Tsulaia, “Higher-Spin Gauge Fields Interacting with Scalars: The Lagrangian Cubic Vertex,” *JHEP* **0710** (2007) 021; [arXiv:0708.1399 [hep-th]]. I. L. Buchbinder, A. Fotopoulos, A. C. Petkou and M. Tsulaia, “Constructing the cubic interaction vertex of higher spin gauge fields,” *Phys. Rev. D* **74** (2006) 105018; [arXiv:hep-th/0609082]. *Grav.* **21** (2004) S1457;
- [17] M. A. Vasiliev, “Higher Spin Gauge Theories in Various Dimensions”, *Fortsch. Phys.* **52**, 702 (2004) [arXiv:hep-th/0401177]. X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, “Nonlinear higher spin theories in various dimensions”, [arXiv:hep-th/0503128]. D. Sorokin, “Introduction to the Classical Theory of Higher Spins” *AIP Conf. Proc.* **767**, 172 (2005); [arXiv:hep-th/0405069]. N. Bouatta, G. Compere and A. Sagnotti, “An Introduction to Free Higher-Spin Fields”; [arXiv:hep-th/0409068].
- [18] I. R. Klebanov and A. M. Polyakov, “AdS dual of the critical O(N) vector model,” *Phys. Lett. B* **550** (2002) 213; [arXiv:hep-th/0210114].
- [19] R. Manvelyan, K. Mkrtchyan and W. Rühl, “Ultraviolet behaviour of higher spin gauge field propagators and one loop mass renormalization,” *Nucl. Phys. B* **803** (2008) 405 [arXiv:0804.1211 [hep-th]].
- [20] C. Fronsdal, “Singletons And Massless, Integral Spin Fields On De Sitter Space (Elementary Particles In A Curved Space Vii),” *Phys. Rev. D* **20**, (1979) 848; “Massless Fields With Integer Spin,” *Phys. Rev. D* **18** (1978) 3624.
- [21] W. Rühl, “The masses of gauge fields in higher spin field theory on AdS(4),” *Phys. Lett. B* **605** (2005) 413; [arXiv:hep-th/0409252]; the results presented here are based on extensive calculations performed by K. Lang and W. Rühl, *Nucl. Phys. B* **400** (1993) 597.
- [22] R. Manvelyan and W. Rühl, “The off-shell behaviour of propagators and the Goldstone field in higher spin gauge theory on AdS(d+1) space,” *Nucl. Phys. B* **717** (2005) 3; [arXiv:hep-th/0502123].
- [23] R. Manvelyan and W. Rühl, “The masses of gauge fields in higher spin field theory on the bulk of AdS(4),” *Phys. Lett. B* **613** (2005) 197; [arXiv:hep-th/0412252].
- [24] R. Manvelyan and W. Rühl, “The structure of the trace anomaly of higher spin conformal currents in the bulk of AdS(4),” *Nucl. Phys. B* **751**, (2006) 285; [arXiv:hep-th/0602067].

- [25] R. Manvelyan and W. Rühl, “The quantum one loop trace anomaly of the higher spin conformal conserved currents in the bulk of AdS(4),” Nucl. Phys. B **733** (2006) 104; [arXiv:hep-th/0506185].
- [26] R. Manvelyan and W. Rühl, “Generalized Curvature and Ricci Tensors for a Higher Spin Potential and the Trace Anomaly in External Higher Spin Fields in AdS_4 Space,” Nucl. Phys. B **796** (2008) 457; [arXiv:0710.0952 [hep-th]].
- [27] B. deWit and D.Z. Freedman, “Systematics of higher spin gauge fields,” Phys. Review D **21** (1980), 358-367.
- [28] R. Manvelyan and W. Rühl, “The Generalized Curvature and Christoffel Symbols for a Higher Spin Potential in AdS_{d+1} Space,” Nucl. Phys. B **797**, 371 (2008) [arXiv:0705.3528 [hep-th]].

Appendix

Proof of $\mathcal{L}_I^{(i,j)}$

The expression for $\mathcal{L}_I^{(1,0)}$ is right if the following remaining group of terms vanishes:

$$+ \frac{s_1 n_1 s_3}{2} C_{(n_i)}^{(s_i)} [\hat{T}(Q_{ij} | n_1 - 1, n_2, n_3), (c\nabla_3)] (\epsilon^{(s_1-1)} h^{(s_2)} D^{(s_3-1)} + D^{(s_1-1)} h^{(s_2)} \epsilon^{(s_3-1)}) \quad (\text{A.1})$$

$$+ \frac{s_1 n_1 s_2}{2} C_{(n_i)}^{(s_i)} [\hat{T}(Q_{ij} | n_1 - 1, n_2, n_3), (b\nabla_2)] (D^{(s_1-1)} \epsilon^{(s_2-1)} h^{(s_3)} - \epsilon^{(s_1-1)} D^{(s_2-1)} h^{(s_3)}) \quad (\text{A.2})$$

$$+ \frac{s_2 n_2 s_1}{2} C_{(n_i)}^{(s_i)} [\hat{T}(Q_{ij} | n_1, n_2 - 1, n_3), (a\nabla_1)] (D^{(s_1-1)} \epsilon^{(s_2-1)} h^{(s_3)} + \epsilon^{(s_1-1)} D^{(s_2-1)} h^{(s_3)}) \quad (\text{A.3})$$

$$+ \frac{s_2 n_2 s_3}{2} C_{(n_i)}^{(s_i)} [\hat{T}(Q_{ij} | n_1, n_2 - 1, n_3), (c\nabla_3)] (h^{(s_1)} D^{(s_2-1)} \epsilon^{(s_3-1)} - h^{(s_1)} \epsilon^{(s_2-1)} D^{(s_3-1)}) \quad (\text{A.4})$$

$$+ \frac{s_3 n_3 s_2}{2} C_{(n_i)}^{(s_i)} [\hat{T}(Q_{ij} | n_1, n_2, n_3 - 1), (b\nabla_2)] (h^{(s_1)} D^{(s_2-1)} \epsilon^{(s_3-1)} + h^{(s_1)} \epsilon^{(s_2-1)} D^{(s_3-1)}) \quad (\text{A.5})$$

$$+ \frac{s_3 n_3 s_1}{2} C_{(n_i)}^{(s_i)} [\hat{T}(Q_{ij} | n_1, n_2, n_3 - 1), (a\nabla_1)] (\epsilon^{(s_1-1)} h^{(s_2)} D^{(s_3-1)} - D^{(s_1-1)} h^{(s_2)} \epsilon^{(s_3-1)}) \quad (\text{A.6})$$

$$- s_1 Q_{12} s_2 C_{(n_i)}^{(s_i)} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_i) \epsilon^{(s_1-1)} D^{(s_2-1)} h^{(s_3)} \quad (\text{A.7})$$

$$- s_2 Q_{23} s_3 C_{(n_i)}^{(s_i)} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31} | n_i) h^{(s_1)} \epsilon^{(s_2-1)} D^{(s_3-1)} \quad (\text{A.8})$$

$$- s_3 Q_{31} s_1 C_{(n_i)}^{(s_i)} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1 | n_i) D^{(s_1-1)} h^{(s_2)} \epsilon^{(s_3-1)}. \quad (\text{A.9})$$

Indeed calculating commutators in the leading order and using relation (3.12) we see that

$$\begin{aligned} & (\text{A.1}) + (\text{A.6}) \\ & = s_1 s_2 (Q_{23} + 1) C_{n_1+1, n_2, n_3}^{s_i} \hat{T}(Q_{12}, Q_{23}, Q_{31} | n_1, n_2 + 1, n_3) D^{(s_1-1)} h^{(s_2)} \epsilon^{(s_3-1)}, \quad (\text{A.10}) \end{aligned}$$

which exactly cancels (A.9) after a corresponding shift of n_2 and using relation (3.13). In a similar way we can prove cancellation of the other two sets of three lines.

To prove formulas for $\mathcal{L}_I^{(2,0)}$ and $\mathcal{L}_I^{(3,0)}$ we should manage the commutators of T operators with a,b,c, gradients in the following expression

$$\begin{aligned} & \frac{s_1 s_2 s_3}{2} C_{(n_i)}^{(s_i)} \left[\right. \\ & [n_1 n_3 \hat{T}(Q_{ij}|n_1 - 1, n_2, n_3 - 1), (b\nabla_2)] (D^{(s_1-1)} D^{(s_2-1)} \epsilon^{(s_3-1)} + D^{(s_1-1)} \epsilon^{(s_2-1)} D^{(s_3-1)}) \\ & [n_2 n_3 \hat{T}(Q_{ij}|n_1, n_2 - 1, n_3 - 1), (a\nabla_1)] (D^{(s_1-1)} \epsilon^{(s_2-1)} D^{(s_3-1)} + \epsilon^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)}) \\ & [n_1 n_2 \hat{T}(Q_{ij}|n_1 - 1, n_2 - 1, n_3), (c\nabla_3)] (\epsilon^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)} + D^{(s_1-1)} D^{(s_2-1)} \epsilon^{(s_3-1)}) \\ & - n_3 Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31}|n_1, n_2, n_3 - 1) (\epsilon^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)} - \epsilon^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)}) \\ & - n_2 Q_{31} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1|n_1, n_2 - 1, n_3) (D^{(s_1-1)} D^{(s_2-1)} \epsilon^{(s_3-1)} - D^{(s_1-1)} \epsilon^{(s_2-1)} D^{(s_3-1)}) \\ & \left. - n_1 Q_{23} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31}|n_1 - 1, n_2, n_3) (D^{(s_1-1)} \epsilon^{(s_2-1)} D^{(s_3-1)} - \epsilon^{(s_1-1)} D^{(s_2-1)} D^{(s_3-1)}) \right], \end{aligned} \quad (\text{A.11})$$

and use again (3.11)-(3.13) to show that (A.11) is zero.

The remaining terms are:

$$\begin{aligned} & \frac{1}{2} C_{(n_i)}^{(s_i)} \left[\right. \\ & -s_1 Q_{12} s_2 (s_2 - 1) [\hat{T}(Q_{12} - 1, Q_{23}, Q_{31}|n_i), (b\nabla_2)] \epsilon^{(s_1-1)} \bar{h}^{(s_2-2)} h^{(s_3)} \\ & -s_2 Q_{23} s_3 (s_3 - 1) [\hat{T}(Q_{12}, Q_{23} - 1, Q_{31}|n_i), (c\nabla_3)] h^{(s_1)} \epsilon^{(s_2-1)} \bar{h}^{(s_3-2)} \\ & \left. -s_3 Q_{31} s_1 (s_1 - 1) [\hat{T}(Q_{12}, Q_{23}, Q_{31} - 1|n_i), (a\nabla_1)] \bar{h}^{(s_1-2)} h^{(s_2)} \epsilon^{(s_3-1)} \right], \end{aligned} \quad (\text{A.12})$$

and

$$\begin{aligned} & \frac{s_1 s_2 s_3}{4} C_{(n_i)}^{(s_i)} \left[\right. \\ & -n_3 Q_{12} (s_2 - 1) [\hat{T}(Q_{12} - 1, Q_{23}, Q_{31}|n_1, n_2, n_3 - 1), (b\nabla_2)] \\ & (\epsilon^{(s_1-1)} \bar{h}^{(s_2-2)} D^{(s_3-1)} - D^{(s_1-1)} \bar{h}^{(s_2-2)} \epsilon^{(s_3-1)}) \\ & -n_1 Q_{23} (s_3 - 1) [\hat{T}(Q_{12}, Q_{23} - 1, Q_{31}|n_1 - 1, n_2, n_3), (c\nabla_3)] \\ & (D^{(s_1-1)} \epsilon^{(s_2-1)} \bar{h}^{(s_3-2)} - \epsilon^{(s_1-1)} D^{(s_2-1)} \bar{h}^{(s_3-2)}) \\ & -n_2 Q_{31} (s_1 - 1) [\hat{T}(Q_{12}, Q_{23}, Q_{31} - 1|n_1, n_2 - 1, n_3), (a\nabla_1)] \\ & (\bar{h}^{(s_1-2)} D^{(s_2-1)} \epsilon^{(s_3-1)} - \bar{h}^{(s_1-2)} \epsilon^{(s_2-1)} D^{(s_3-1)}) \left. \right], \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned}
& -\frac{s_3 s_1 (s_1 - 1) n_1}{4} C_{(n_i)}^{(s_i)} Q_{31} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1 | n_1 - 1, n_2, n_3) \\
& (\delta \bar{h}^{(s_1-2)} h^{(s_2)} D^{(s_3-1)} + 2(\nabla D)^{(s_1-2)} h^{(s_2)} \epsilon^{(s_3-1)}) \\
& -\frac{s_1 s_2 (s_2 - 1) n_2}{4} C_{(n_i)}^{(s_i)} Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3) \\
& (D^{(s_1-1)} \delta \bar{h}^{(s_2-2)} h^{(s_3)} + 2\epsilon^{(s_1-1)} (\nabla D)^{(s_2-2)} h^{(s_3)}) \\
& -\frac{s_2 s_3 (s_3 - 1) n_3}{4} C_{(n_i)}^{(s_i)} Q_{23} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31} | n_1, n_2, n_3 - 1) \\
& (h^{(s_1)} D^{(s_2-1)} \delta \bar{h}^{(s_3-2)} + 2h^{(s_1)} \epsilon^{(s_2-1)} (\nabla D)^{(s_3-2)}), \tag{A.14}
\end{aligned}$$

The last $DD\bar{h}$ terms coming from our calculation are:

$$\begin{aligned}
& \frac{s_1 s_2 s_3}{4} C_{(n_i)}^{(s_i)} \left[\right. \\
& -(s_3 - 1) n_1 n_3 Q_{23} \hat{T}(Q_{12}, Q_{23} - 1, Q_{31} | n_1 - 1, n_2, n_3 - 1) \\
& (D^{(s_1-1)} D^{(s_2-1)} \delta \bar{h}^{(s_3-2)} + 2D^{(s_1-1)} \epsilon^{(s_2-1)} (\nabla D)^{(s_3-2)}) \\
& -(s_2 - 1) n_2 n_3 Q_{12} \hat{T}(Q_{12} - 1, Q_{23}, Q_{31} | n_1, n_2 - 1, n_3 - 1) \\
& (D^{(s_1-1)} \delta \bar{h}^{(s_2-2)} D^{(s_3-1)} + 2\epsilon^{(s_1-1)} (\nabla D)^{(s_2-2)} D^{(s_3-1)}) \\
& -(s_1 - 1) n_1 n_2 Q_{31} \hat{T}(Q_{12}, Q_{23}, Q_{31} - 1 | n_1 - 1, n_2 - 1, n_3) \\
& \left. (\delta \bar{h}^{(s_1-2)} D^{(s_2-1)} D^{(s_3-1)} + 2(\nabla D)^{(s_3-2)} D^{(s_2-1)} \epsilon^{(s_3-1)}) \right]. \tag{A.15}
\end{aligned}$$

These terms can be used in the same fashion for proving the remaining part of $\mathcal{L}_I^{(i,j)}$ to contain traces.