Topological results *on* definably compact groups *in* o-minimal structures

Ph.D. thesis

Candidate *Marcello Mamino*
Supervisor *Professor Alessandro Berarducci*

Scuola Normale Superiore – Pisa

Contents

Introduction

The investigation of o-minimality has been tightly linked, almost from its beginning, with the study of groups definable in o-minimal structures.

Since A. Pillay's work [Pil–88], where he proves that any definable group has a unique structure of definable group manifold, early results [Raz–91] [NPR–91] [MS–92] [Str–94a] [Str–94b] [PPS–00a] [PPS–02] showed interesting connections between definable groups and classic mathematical objects as real Lie groups and Nash groups. Nevertheless striking differences [PS–99] also came to light. An important milestone has been reached with the solution of the so called *Pillay's conjectures* [Pil–04] [BOPP–05] [HPP–08a], providing a strong link between definable groups and real Lie groups: namely there is a real Lie group G/G^{00} associated to each definable group G. Under suitable assumptions, G/G^{00} is known to embody many group-theoretic, topological, and model-theoretic properties of G. The aim of the present work is to proceed with the investigation of the links between a definable group and its associated Lie group, in particular from the topological point of view.

Early explorations in the field focused on low-dimensional cases ([Raz–91], [NPR–91], [MS–92], [Str–94b]) and on several algebraic aspects. The o-minimal Euler characteristic [Str–94a] provided an analogue in the definable context of the cardinality in the finite groups context. Using the o-minimal Euler characteristic, A. Strzebonski obtained analogues of the Sylow theorems for definable groups, and a finiteness result for the torsion on definable abelian groups. Also the concept of *definable compactness* introduced by Y. Peterzil and C. Steinhorn in [PS–99] was to play an important role in the future developments. In particular, they prove that each *non* definably compact definable group has a one-dimensional definable subgroup. However a definably compact (infinite) group may non even have a single definable one-dimensional subgroup.

Recall that an infinite definable group G is *definably semisimple* if it has no abelian infinite normal definable subgroup. A profound study of semisimple definable groups, carried out by M.Otero, Y. Peterzil, A. Pillay, and S. Starchenko [OPP–96] [PPS–00a] [PPS–02], brought to light a fundamental dichotomy between the semisimple and the abelian case. A fundamental fact to be mentioned, in this respect, is that every definable group G is either abelian-by-finite or it is an extension

$$1 \to Z(G) \to G \to G/Z(G) \to 1$$

of a definable semisimple group $G/Z(G)$ by a definable abelian group $Z(G)$ (see [PS–00, Corollary 5.4]).

The semisimple case is very well understood, since it reduces to the semialgebraic case because of the so called *very good reduction*: Suppose that N is an o-minimal expansion of an ordered group. If H is a definably connected semisimple definable group, then there exists a definable real closed field R such that H is definably isomorphic in N, over parameters, to an R-semialgebraic group H' defined without parameters[1]. Moreover, note that the real close field R does not depend on H [OPP–96, Theorem 1.1]. The quoted fact is the endpoint of a chain of earlier results (for example [PPS–00a, Theorem 2.37] [PPS–00b, Theorem 1.1] [PS–00, Corollary 5.1]). The starting point of it has been the centerless case [PPS–00a, Theorem 2.37], which is dealt with trough Lie algebræ (adapted to the definable context).

On the other hand, in the abelian case, the Lie algebra has no structure to work with. In this rather different situation, the most important results [EO–04] – regarding the classification of the torsion subgroups of a definably compact abelian group – have been obtained trough the methods of algebraic topology.

A great breakthrough came with the formulation of the Pillay's conjectures (Theorem 1.4.4, as stated in [Pil–04, Conjecture 1.1]). The conjectures, now proven, state that:

i. for each definable group G *in a saturated enough o-minimal structure* there is a smallest type-definable subgroup G^{00} of bounded index—that subgroup being of course normal;
ii. the quotient $G/_{G^{00}}$ equipped with the *logic topology* (i.e. a subset of $G/_{G^{00}}$ is closed if its preimage in G is type-definable) is a real Lie group;
iii. if G is definably compact, then the dimension of $G/_{G^{00}}$ as a Lie group coincides with the o-minimal dimension of G.

The proof of the existence of the *infinitesimal subgroup* G^{00} and that $G/_{G^{00}}$ is a Lie group is due to A. Berarducci, M. Otero, Y. Peterzil and A. Pillay [BOPP–05]. The equality of the dimension has been proven by E. Hrushovski, Y. Peterzil and A. Pillay in [HPP–08a] assuming that the underlying o-minimal structure expands a real closed field—this will be our standing assumption. Entirely new model-theoretic ingredients entered the scene with the solution of the conjectures: NIP theories and Keisler measures. These and the methods used have significance in the field of model theory going beyond the realm of o-minimality.

Focusing on the definably compact case, the connections between G and $G/_{G^{00}}$ are not limited to the identity of the dimensions. For example, the two groups have been proved to have the same cohomology [Ber–09] (using results in [HP–09]); when G is *definably connected* they are even elementarily equivalent (in the group language) [HPP–08a]. A. Berarducci also proves [Ber–07, Theorem 5.2] that the correspondence between G and $G/_{G^{00}}$ is indeed an exact functor. In this work, we will investigate further the connection between a definably compact group G definable in an o-minimal expansion of a real closed field and its Lie counterpart $G/_{G^{00}}$.

Our principal results are that the homotopy groups of $G/_{G^{00}}$ do coincide with the *definable homotopy* groups of G (Theorem 3.1.9), and that $G/_{G^{00}}$

[1] M. Edmundo, G. Jones, N. Peatfield, private communication.

determines the definable homotopy type of G (Theorem 4.8.7). And we show that the isomorphisms between the fundamental group of G and the definable fundamental group of G/G^{00} given by Theorem 3.1.9 (and already established in the abelian case – the hard one – in [EO–04]) comes from a natural map. Examples in [PS–99] show that given two definably compact groups G and H definable in the same o-minimal structure, the isomorphism of G/G^{00} and H/H^{00} does not imply the definable isomorphism of G and H—i.e. G/G^{00} does not determine G up to definable isomorphism. However, a sensible question is whether G/G^{00} determines the topology of G. Our Theorem 3.3.4 reduces the question to the abelian case.

I will now review in more detail the contents of this thesis. The original results contained in this work have been partly obtained by the author in collaboration with A. Berarducci and M. Otero, and have been published in [BMO–08], [BM–10], and [Mam–10]. In particular: Lemma 2.3.2, Section 2.4, and Section 3.1 contain results obtained in collaboration with A. Berarducci and M. Otero; and Chapter 4 contains results obtained in collaboration with A. Berarducci.

In chapter 2, we will introduce some technical machinery regarding an o-minimal version of the classical notions of fibre bundle and fibration. This part of the dissertation may be considered *pure* o-minimal geometry in the spirit of [vdD–98]. The only prerequisite to chapter 2 is, in fact, that very book. The results presented can be regarded as generalizations of similar ones known for definable covering maps (see [EO–04] and [BO–09]). In particular we prove the homotopy lifting property for definable maps and definable *bundle maps*, and we show that a definable map is a fibration if and only if it is *locally* a fibration (hence, in particular, a definable fibre bundle is a definable fibration). We will apply the theory developed here to the definable groups using Lemma 2.1.2 (that a quotient of definable groups is a definable bibre bundle). This will enable us to apply algebraic topologic tools (like the long exact sequence of homotpy groups) developed in [BO–09] for definable fibrations.

Although the results are akin to their classical topological counterparts, there are some remarkable differences. For example, we deemed appropriate *not* to equip definable fibre bundles with a definable bundle group, because many examples would have been lost otherwise—in fact, even a single definable homeomorphism of a definable set may not be an element of any definable group of homeomorphisms, like $x \to x^3$ in $(\mathbb{R}, +, \cdot)$. Hence we call *definable fibre bundle* basically a locally trivial continuous definable map having fibres definably homeomorphic to some fixed definable set, and as bundle map we accept any fibre-preserving continuous definable map. In order for this large structure group (all the definable homeomorphisms of the fibre) to behave as a topological group, we must however impose some restriction on the fibre, the correct condition being that the fibre must be locally definably compact. As a second example, observe that many of the usual topological arguments make use of devices as the compact-open topology on path-spaces or the Lebesgue number: none of these have meaningful counterparts in the o-minimal setting.

In chapter 3, we will prove several results regarding the topology of definably compact definable groups. The main propositions here are Theorem 3.1.9 stating that the definable homotopy groups of G are isomorphic to the homotopy groups of G/G^{00}, and Theorem 3.3.4, which says that every definably compact group is definably homeomorphic to the topological product of its derived subgroup – which is definable by [HPP–08b] – and an abelian group. We will also point out a perhaps unexpected discrepancy between the definably compact world and the compact Lie world: the short exact sequence

$$1 \to [G, G] \to G \to G/[G, G] \to 1$$

always splits for compact Lie groups, but the same may not happen for definably compact definable groups (Example 3.3.6).

Theorem 3.1.9 (that the definable homotopy groups of G are isomorphic to the homotopy groups of G/G^{00}) is established in a somewhat indirect way. For semisimple groups the result is an easy application of known theory since it reduces to the semialgebraic case via results in [PPS–00a] and [PPS–02] (see also the *very good reduction* for definable semisimple groups [EJP–07]), the semialgebraic case, in turn, is dealt with using results in [BO–09]. The hard case is for abelian groups, because of known examples which show that definably compact abelian groups do not factorize into products of one-dimensional subgroups (as opposed to compact abelian Lie groups). In the abelian case, we have been able to prove that the higher homotopy groups are divisible and finitely generated. This implies that the the higher homotopy groups are trivial. Our proof works again by transfer to the reals, however we can not transfer the full group structure, instead we work with the induced definable H-space structure, which proves flexible enough be transferred. Finally we join the two cases using the long exact sequence for definable homotopy groups [BO–09, Section 4].

The same difficulty with the factorization of definably compact abelian groups is met in the proof of Theorem 3.3.4. Here we introduce a homotopic invariant (Definition 3.2.2) to characterize trivial principal definable fibre bundle over a definably compact abelian group, and we use Theorem 3.1.6 (every definably compact abelian group is definably homotopy equivalen to a standard torus of the same dimension) to replace a possibly wild definable abelian group with a factorized torus. Theorem 3.1.6 is, in turn, a consequence of Theorem 3.1.9 and the o-minimal Whitehead Theorem [BO–09, Theorem 5.6].

Finally, in chapter 4, we intend to elaborate on the isomorphism between the definable fundamental group of G and the fundamental group of G/G^{00}. The fundamental group was computed in the abelian case, the hard one, in [EO–04]. We start from the consideration that the original proof (i.e. [EO–04]+[HPP–08a]) is extremely indirect and doesn't lead to a natural nor canonical isomorphism. Using results in [Ber–09] – which in turn use the *compact domination*, proven in [HP–09] – we can show that a natural isomorphism indeed exists. Moreover the fundamental group of each open subset of G/G^{00} turns out to be isomorphic to the definable fundamental group of its preimage in G (the definable fundamental group is well-defined even though the preimage is just $\bigvee$-definable).

In order to construct an isomorphism between the definable fundamental group of G and the fundamental group of G/G^{00}, we work, in fact, with the fundamental groupoids (the set of all paths modulo homotopies, with the concatenation operation). Basically we show that there is a finite open covering $\mathcal{U}$ of G/G^{00} with simply connected sets such that the preimages of the sets in $\mathcal{U}$ are open $\bigvee$-definable definably simply connected subsets of G. Using these isomorphic coverings we get an induced morphism on the fundamental groupoids, which restricts to an isomorphism of the fundamental groups. Finally we prove that the constructed isomorphism does not depend on the choice of the coverings, and it is indeed natural. As a consequence the finite connected covering maps of G/G^{00} are in 1-to-1 correspondence with the finite definably connected definable covering maps of G, and the quotient of the definable universal cover of G modulo G^{00} is isomorphic to the universal cover of G/G^{00}. From this, we can draw consequences on the definable homotopy type of definably compact groups.

The knowledgeable reader is advised to skip the introductory chapter 1 altogether.

Chapter 1

Facts & definitions

This short chapter contains a small compendium of established results, definitions, and notations. Moreover, we will establish hereunder our own notations and conventions.

Our aim, hence, is twofold. First, we would like to give the reader a concise summary of what we will tacitly assume in the following chapters. Second, for mere completeness, we would like to define somewhere several almost-standard terms like *definable homotopy* or *infinitesimal subgroup*; even though arguably our readers don't need that much care.

The default reference for o-minimality is [vdD–98]. However, on the specific topic of groups definable in o-minimal structures, as far as we know, there is no comprehensive treatise, and the vast body of mathematical work in this field is still scattered through a number of research papers. Nevertheless, a clean panorama of most of the established theory, with pointers to the specific works, can be found in [Ote–08].

Nothing in this chapter is original, except, perhaps, the mistakes.

1.1 O-minimality

Definition 1.1.1. An o-minimal structure $\mathcal{M} = (R, <, \dots)$ is a totally ordered first order structure in which all definable subsets of R are definable from the order alone.

In Definition 1.1.1, as well as in the rest of this work, definable means *definable with parameters*. When intending otherwise, we will stress it explicitly, or write 0-definable to mean *definable without parameters*. Notably, although not immediate from the definition itself, o-minimality is preserved through elementary equivalence. The first order theory of an o-minimal structure is, hence, called o-minimal theory: clearly each of its models is o-minimal. For the basics of o-minimality the reader is referred to [vdD–98]. Nevertheless, we will collect here those results which will be used more commonly in the rest of this work.

By the classical quantifier elimination result of Tarski, the theory of real closed fields with plus and times is o-minimal. From now on, until the end of the bibliography, we will assume $\mathcal{M} \stackrel{def}{=} (R, <, 0, 1, +, \cdot, \dots)$ to be an o-minimal expansion of a real closed field. Besides the reals themselves, further examples of such expansions are:

Example 1.1.2.

i. Other real closed fields, such as the real closure of $\mathbb{Q}$ or the Puiseux series
ii. $\mathbb{R}^{exp}$, the real field with the exponential function [Wil–96]
iii. $\mathbb{R}^{an}$, the real field with all the analytic functions $[0, 1] \to \mathbb{R}$, by [DvdD–88].

We will now briefly summarize those properties of $\mathcal{M}$ thoroughly explained in [vdD–98]. We do this for completeness, and to help us introduce our notation in a gradual, organic way. References to Van den Dries' classical book will be omitted.

Fact 1.1.3 (Cell decomposition theorem)**.** *Each definable subset of* R^n *(from now on simply definable set) is a finite union of cells, where a* cell *is either:*

i. *a point*
ii. *the graph of a definable function defined on a cell having* R *as codomain*
iii. *the region above or below such a graph, or between two such graphs.*

Clearly we have an associated notion of dimension where a point has dimension 0, a cell of type ii has the same dimension of the domain of the function, and a cell of type iii has dimension one plus that of the domain. The dimension of a definable set is the maximal dimension of its cells, it does not depend on the particular decomposition chosen, and it is invariant under definable bijections. Clearly, dimension 0 means finite.

A definable function is a function having definable graph. We put on R the order topology, on R^n we get a product topology, so it makes sense to speak about continuity of definable functions. A definable map is a definable continuous function. A definable 1-to-1 map having continuous inverse will often be named definable homeomorphism, and its domain and codomain will be definably homeomorphic. Definable functions are very regular: the set of the discontinuity points of a definable function has dimension smaller than the domain of the function, and the same holds for differentiability up to any order.

While we are at functions: let I denote the definable subset $[0,1]$ of R. We will call DEFINABLE HOMOTOPY just any definable map having the cartesian product of any definable set times I as domain. A definable homotopy $h\colon X \times I \to Y$ is *relative* $X' \subset X$ if for every $x \in X'$ the function $h(x,-)$ is constant. A DEFINABLE CURVE is a definable function having I as domain. A DEFINABLE PATH is a continuous definable curve. Of course, definable paths may be concatenated, i.e. given a definable set X and two definable curves $\alpha, \beta\colon I \to X$ we define

$$\begin{aligned} \alpha+\beta\colon I &\to X \\ t &\mapsto \begin{cases} \alpha(2t) & \textit{when } t < \frac{1}{2} \\ \beta(2t-1) & \textit{when } \frac{1}{2} \leqslant t \end{cases} \end{aligned}$$

however concatenation makes sense just when $\alpha(1) = \beta(0)$, in this case $\alpha+\beta$ is itself a definable path.

The inverse of a definable 1-to-1 function is clearly definable, however *an* inverse of a definable, not necessarily 1-to-1, function is, although less clearly, definable too. This property, known as *definable choice* or *definable Skolem functions* is true whenever $\mathcal{M}$ is an o-minimal expansion of an ordered group.

As well as the cell decomposition theorem, we have the following

Fact 1.1.4 (Triangulation theorem)**.** *Each definable set is definably homeomorphic to the realization of some finite simplicial complex.*

Where by FINITE SIMPLICIAL COMPLEX we mean a finite set K of open affine simplexes in R^n such that for each $\sigma_1, \sigma_2 \in K$ either $\overline{\sigma_1} \cap \overline{\sigma_1} = \emptyset$ or $\overline{\sigma_1} \cap \overline{\sigma_1} = \overline{\tau}$ for some common face τ of σ_1 and σ_2. The realization $|K|$ of a simplicial complex is the union of its simplexes. In particular a finite simplicial complex is definably homeomorphic to a 0-definable subset of some standard closed simplex

$$\overline{\Delta}_n \stackrel{def}{=} \{(x_0, \dots, x_n) \in R^n \textit{ s.t. } x_0 + \cdots + x_n = 1 \textit{ and } x_{0,\dots,n} \geqslant 0\}$$

1.2 Definable spaces, compactness & connectedness

A DEFINABLE SPACE is built by gluing finitely many definable sets. Fix a definable set X and finitely many injective definable functions $\{g_i\}_i$

$$g_i\colon U_i \to R^{n_i}$$

whose domains $\{U_i\}_i$ are definable subsets of X. Assume that for each i, j the subset $g_i(U_i \cap U_j)$ is relatively open in the image of U_i. And, finally, equip X with the unique topology that makes all the functions g_i into homeomorphisms.

A definable space X is a REGULAR DEFINABLE SPACE if for every definable closed subset of X and every point $p \notin X$, we can find disjoint definable open neighbourhoods of p and X. Regular definable spaces are little surprise, in fact they are just definable sets:

Theorem 1.2.1. *Every regular definable space is definably homeomorphic to some definable set (with the subspace topology).*

Clearly definable sets are regular definable spaces. For this reason we will always speak of *definable sets* instead of *regular definable spaces*. Our decision will fail to stress the fact that we are often interested just in *the space* (i.e. the set with its subset topology modulo definable homeomorphisms) and not in *the set* itself as a particular subset of something else (like R^n). On the positive side, however, we think that this improves readability, in particular once the reader has been warned—as you have been right now.

The notions of compactness and connectedness are not preserved by elementary extensions. For example $[0, 1] \subset \mathbb{R}$ is compact and connected, but the same interval isn't compact in any non-standard extension of $\mathbb{R}$, and it is even totally disconnected. To remedy this, suitable notions of *definable* compactness and connectedness are given, which work for definable sets.

Definition 1.2.2 (Definable compactness, see [PS–99])**.** A definable set X is DEFINABLY COMPACT if for every definable function $f\colon (a, b) \to X$ the left and right limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist in X.

Definition 1.2.3 (Definable connectedness)**.** A definable set X is DEFINABLY CONNECTED if it can not be split into the union of two relatively open nonempty disjoint definable subsets.

Definition 1.2.2 makes sense even when X is just a definable space, however, for definable sets, it is equivalent to the more immediate *closed and bounded* [PS–99, Theorem 2.1].

About Definition 1.2.3, we would like to remark that a set is definably connected if and only if it is definably *path*-connected (i.e. any two points can be connected through a definable path inside the set), and that definable sets split into finite union of DEFINABLY CONNECTED COMPONENTS; both are consequences of cell decomposition.

1.3 Algebraic topology in the definable context

Definable homotopies have been introduced to the reader in section 1.1. We say that two definable sets X and Y are DEFINABLY HOMOTOPY EQUIVALENT if there are two definable maps $f\colon X \to Y$ and $g\colon Y \to X$ which are homotopy inverse of each other, i.e. $f \circ g$ and $g \circ f$ are both definably homotopic to the identity on their respective domains. In this section we will review several definable homotopic invariants which will play a role in the present work.

Singular homology [Woe–96] and cohomology [EO–04] have been developed in the o-minimal context. Basically, given a definable set, to it is associated a chain complex of definable singular simplices, i.e. of definable maps form a standard simplex to the set. The definable homology is defined, following the classical case, as the homology of that chain complex; the cohomology is defined by duality. All the algebraic properties are the same as for classical (co-)homology. We will use the standard notations H_* and H^* for the o-minimal homology and co-homology, no confusion will ensue.

A more general approach has been adopted in [EJP–06], which works in any o-minimal structure (i.e. even not expanding a field). The idea is to work with the O-MINIMAL SPECTRUM of a definable set, where the spectrum of X is the set of all complete types of the elementary diagram of $\mathcal{M}$ containing a defining formula for X (topologized as follows: a basic open set is the set of those types which contain a defining formula for some definable relatively open subset of X; this topology does not coincide with the Stone topology). The cohomology of a definable set is, hence, defined trough the sheaf cohomology of its o-minimal spectrum. This, more abstract, approach proved useful in later works on the cohomology of definably compact groups [Ber–07], [Ber–09]. Nevertheless, under our assumptions, the cohomology defined as in [EJP–06] turns out to be equivalent to that investigated in [EO–04] (see [Ber–07, remark 7.3]).

O-minimal homotopy groups are defined as in the topological case: equivalence classes of definable maps $I^n \to X$ modulo definable homotopy relative to the boundary $\delta(I)$. The group operation is as in the classical case, and as in the classical case all homotopy groups except the fundamental group are abelian. We will denote the n-th definable homotopy group of (X, x_0) by $\pi_n^{def}(X, x_0)$; since, when X is definably connected, the base point is irrelevant, in this case we will just write $\pi^{def}(X)$. The definable fundamental group was first introduced by M. Edmundo as the set of all equivalence classes of definable paths modulo homotopy, it has been studied in [BO–02]; the definable higher homotopy groups have been introduced in [BO–09], to which we refer the reader for a complete discussion.

$H_n(-; G)$, $H^*(-; R)$, and π_n^{def} are respectively co- contra- and co-variant functors from the category of definable sets (*pointed* sets for π_n^{def}) with definable maps to respectively the categories of abelian groups, graded algebras, and groups.

In particular, there is a long exact sequence for definable homotopy groups, formally identical to the classical one (see [BO–09, section 4]). Moreover, we will need the following analogues of well-known classical results (see [BO–09, Theorem 5.3], and [BO–09, Theorem 5.6]):

Theorem 1.3.1 (The o-minimal Hurewicz theorems). *Let X be a definably connected definable set, suppose that $\pi_i^{def}(X) = 0$ for each $i < n$, then $H_n(X; \mathbb{Z})$ is isomorphic to the abelianization of $\pi_n^{def}(X)$.*

Theorem 1.3.2 (The o-minimal Whitehead theorem). *Let X and Y be two definably connected sets, and let $f \colon X \to Y$ be a definable map such that, for some $x_0 \in X$ and all $n \geqslant 1$ the group homomorphism*

$$f_* \colon \pi_n^{def}(X, x_0) \to \pi^{def}(Y, f(x_0))$$

is an isomorphism. Then f is a definable homotopy equivalence.

Both are obtained by transfer from the semialgebraic setting, basically showing that the structure of definable homotopies reduces to that of semialgebraic homotopies. In particular, we will make use of the following result [BO–09, Corollary 3.6]:

Fact 1.3.3. *Let* X *and* Y *be two* 0*-definable sets, and let* $f\colon X \to Y$ *be a definable map. Then there is a* 0*-definable map* $g\colon X \to Y$ *definably homotopic to* f*. Moreover, if* X *and* Y *are semialgebraic, then* g *can also be taken semialgebraic.*

For definably compact sets, a further homotopic invariant is the o-minimal Euler characteristic, however we will make no direct use of it in this work. Perhaps surprisingly, it is preserved by any definable bijection, even non-continuous.

1.4 Definable groups

A definable group G is a definable set equipped with a definable group operation.

There are many natural examples in mathematics, based on the structures of Example 1.1.2. Finite groups, algebraic groups, and even all compact Lie groups (since they are definable in $\mathbb{R}^{an}$, and also since they are isomorphic to linear groups).

By Pillay's work [Pil–88] each definable group has a unique structure of *definable manifold* that makes it into a topological group. Here, a definable manifold is a definable space locally definably homeomorphic to R^d, for some dimension d. By Theorem 1.2.1, under our assumptions, G with its manifold topology is definably homeomorphic to a definable set with its subset topology: from now on we will identify G with such a set, and call the manifold topology Pillay's topology. In particular, follows from the uniqueness of Pillay's topology that any definable homomorphism of definable groups is continuous.

A major result in the study of definable groups has been the solution of a set of conjectures known as Pillay's conjectures, linking the realm of definable groups with that of Lie groups. All of them deal with the so-called *infinitesimal subgroup* G^{00} of a definable group G. As we will see, a Lie group arises in a natural way from the quotient $G/_{G^{00}}$ (even when the underlying o-minimal structure $\mathcal{M}$ is not based on the reals). This group shares many properties with the original definable group. In fact, most of the present work will be devoted to the investigation of the relations between the definable topology of G and the topology of $G/_{G^{00}}$ in the definably compact case. In order to make sense of this, we will summarize some model-theoretic common sense.

We recall that a type-definable set is a set that can be presented as an infinite intersection $\bigcap_{i\in I} X_i$ where each X_i is definable and I is a possibly infinite index set. Type-definable sets come equipped with a presentation, so it makes sense to interpret them in elementary extensions. Unlike what happens for definable sets, an equality $\bigcap_{i\in I} X_i = \bigcap_{j\in J} Y_j$ can hold in some model and fail in an elementary extension of it. To have a notion of equality not dependent on the model, we must restrict ourselves to models that are sufficiently saturated, i.e. assume the following.

Assumption 1.4.1. We assume the model $\mathcal{M}$ to be a saturated structure of some *large enough* cardinality κ. Moreover we will call SMALL any set of cardinality less than κ.

With these conventions one has for instance that infinite conjunctions commute with the existential quantifier, namely

$$\exists x \bigcap_{i \in I} (x \in X_i) \equiv \bigcap_{i \in I} \exists x (x \in X_i)$$

provided the family of definable sets $\{X_i\}_{i \in I}$ is downward directed (e.g. it is closed under finite intersections). It is common practice to say that such equalities hold *by saturation*. A $\bigvee$-DEFINABLE set is a set presented as a union $\bigcup_{i \in I} X_i$, where each X_i is definable and the index set I is small. By saturation if a type-definable set is included in a $\bigvee$-definable set, there is some definable set between them.

Honestly, assumption 1.4.1 is asking a lot, since in order to have properly *saturated* models we need, in general, the generalized continuum hypothesis; and the question of whether this hypothesis can be eliminated in a completely general situation has no definite answer. However, in our specific case, this so-called MONSTER MODEL is a mere figment, intended to relieve the burden of explicitly stating lower bounds on how much saturated and strongly homogeneous the structure should be. A posteriori (i.e. considering that the Pillay's conjectures have been solved positively), we could as well assume that small means *two to the cardinality of the language* and $\mathcal{M}$ is k-saturated and strongly k-homogeneous for some k greater than that.

Definition 1.4.2. Let G be a definable group. Then the INFINITESIMAL SUBGROUP G^{00} of G is the smallest type-definable subgroup of G of index less than κ.

If such a G^{00} exists, then the quotient G/G^{00} does not depend on k, as long as it is greater than $2^{\#language}$—see [Pil–04]. It follows that, in this case, there is a well defined group G/G^{00} associated to G. Hence, we will rightfully omit any mention of k.

Definition 1.4.3. On the group G/G^{00} we put the following LOGIC TOPOLOGY. We say that a subset X of G/G^{00} if its preimage in G is type-definable.

By now, all of Pillay's conjectures have been solved positively by works of A. Berarducci, E. Hrushovski, M. Otero, Y. Peterzil, A. Pillay, and others, virtually involving in the background all the known theory of groups definable in o-minimal structures. Here are the conjectures, as stated in [Pil–04, Conjecture 1.1]:

Theorem 1.4.4 (Pillay's conjectures)**.**

C-I. G^{00} *exists*

C-II. *the group* G/G^{00} *equipped with the logic topology is a compact Lie group*

C-iii. *if* G *is definably compact then the o-minimal dimension of* G *coincides with the dimension of* G/G^{00} *as a Lie group*

C-iv. *if* G *is definably compact and abelian, then* G^{00} *is divisible and torsion-free.*

Conjectures C-i and C-ii are proved in [BOPP–05]. The rest is proved, under our hypothesis that $\mathcal{M}$ expands a real closed field, in [HPP–08a].

I would like to stress, here, the high degree of indirectness of the methods by which C-i· · ·iv have been obtained. For example, C-i&ii are proven through a merely topological characterization of compact abelian Lie groups (due to Pontryagin), hence the real field does not arise from the proof in any natural way. Moreover, C-iii, in the abelian case, is the result of G and G/G^{00} having isomorphic torsion, which has been proven – on the G side – by algebraic-topological means. The latter example is emblematic: since the *interesting* property – the dimension – could not be transferred directly between the definable side and the Lie side, one had to trace it back (in both contexts) to some auxiliary property – the structure of the torsion subgroup – which can, by virtue of some model-theoretic result, be pushed to the other side of the barrier.

The map $F\colon G \to G/G^{00}$ is indeed an exact functor from the category of definably compact definable groups to the category of Lie groups, respectively with definable group isomorphisms and Lie isomorphisms [Ber–07, Theorem 5.2]. Moreover, C-iii says that the map $F\colon G \to G/G^{00}$ preserves the dimension (using the respective notion of dimension in each category). It is also known that F preserve the cohomology [Ber–09, Corollary 5.2]. In this work, we will prove that F preserves the homotopy groups as well (Theorem 3.1.9).

1.5 Semisimple & abelian definable groups

Recall that an infinite Lie group is said *semisimple* if it has no infinite connected abelian normal subgroup. A similar definition is given in the definable context:

Definition 1.5.1. A definable group G is semisimple if it is infinite and it has no infinite abelian normal definable subgroup.

The class of semisimple definable groups coincides, in fact, with that of semisimple *semialgebraic* groups. Namely we have:

Fact 1.5.2. *For any semisimple definable group* G*, there is a group* G'*, semialgebraic without parameters, definably isomorphic to it.*

Proofs of the above fact can be found either in [EJP–07, Theorem 3.1] or in [HPP–08b, Theorem 4.4 (ii)]. All reduce to the centerless case taking the quotient $G/Z(G)$. The centerless case, in turn, is proven in [PPS–00a, Theorem 4.1] by methods involving the development of Lie algebra machinery in the definable context. Follows from Fact 1.5.2 that nothing can happen among semisimple definable groups that doesn't already happen in the well-known context of semialgebraic linear groups over the reals.

On the other hand, Hrushovski Peterzil and Pillay prove in [HPP–08b] that the derived subgroup $[G, G]$ of a definable group G is a definable semisimple subgroup of G. Moreover, we will show that – in the definably compact case – the definable map $G \to G/[G, G]$ admits a continuous inverse, i.e. that any definably compact definably connected group is definably homeomorphic to the product of its derived subgroup and a definably compact abelian group (see Theorem 3.3.4).

Combining the observations above, we have a strong indication that most of the intricacies in the investigation of the topology of definably compact groups must reside in the abelian case. The reason – we may argue – is that, in this case, there is little algebraic structure to work with. For example, let us observe that abelian-by-finite definable groups are the only ones that do not define an infinite field (see [PS–00, Corollary 5.1]). Moreover, as opposed to the semisimple case, there are striking dissymmetries between definably compact abelian groups and compact abelian Lie groups. Most notably, examples are known of definably compact definably connected abelian groups which do not split into products of one-dimensional definable groups (and don't even *have* one-dimensional definable subgroups: see [PS–99, section 5]), hence – as opposed to the case of compact abelian Lie groups – the definable isomorphism class of a definably compact definably connected abelian group is not determined by the mere dimension.

Chapter 2

Definable fibre bundles & fibrations

In this chapter we will develop some basic theory of definable fibre bundles and fibrations. The results established here will be used later to study the topology of definably compact definable groups.

The notions of definable fibre bundle 2.1.1 and fibration 2.4.1 are modelled on the classical ones. Both can be seen as generalizations of the notion of DEFINABLE COVERING MAP used in [EO–04] to analyze the torsion of definably compact abelian groups. The reason we have to introduce our devices is, as expressed by Lemma 2.1.2, that quotients of definable groups are definable fibre bundles.

Most of the techniques presented here are analogous to classical topological arguments. Not everything that works in the topological setting, however, is viable in the o-minimal setting as well. Hence, we must choose our techniques with care. The principal difficulties come from the imperfect analogy between definable notions and their classical counterparts. For example, we do not have o-minimal analogues of devices as the path space with its compact-open topology, and collections of definable functions are usually not definable object. Also *definably* compact sets are not actually compact, hence no Lebesgue number, converging subsequences, &c. The difficulties are solved by *ad hoc* methods whenever they arise, nevertheless there are general patterns: for example Lemma 2.2.4 says that the *structure group* of our bundles, which we assume to be the set of all definable homeomorphisms of the fibre, although not a definable object, works as a topological group (i.e. the inverse is continuous); a second example is Lemma 2.3.2 which has been used everywhere the Lebesgue number would in a classical argument.

2.1 Definable fibre bundles

Definition 2.1.1. A DEFINABLE FIBRE BUNDLE $\mathscr{B} = (B, X, p, F)$ consists of a definable set F and a definable map $p\colon B \to X$ between definable sets B and X with the following properties:

I. p is onto,

II. there is a finite open definable covering $\mathcal{U} = \{U_i\}_i$ of X such that for each i there exists a definable homeomorphism $\phi_i\colon U_i \times F \to p^{-1}(U_i)$.

The set B is said to be the BUNDLE SPACE of $\mathscr{B}$, the set X is said to be its BASE, and F its FIBRE. Any covering as in II is said to be a TRIVIALIZATION COVERING and the maps ϕ_i are called TRIVIALIZATION MAPS. We will refer to the map p as the PROJECTION, moreover, fixed a trivialization covering $\mathcal{U} = \{U_i\}_i$ with the associated trivialization maps, we will call PROJECTION ON THE FIBRE each of the maps $p_i\colon p^{-1}(U_i) \to F$ sending $x \in p^{-1}(U_i)$ to the second component of $\phi_i^{-1}(x)$. Finally we will call DEFINABLE CROSS SECTION of $\mathscr{B}$ any definable map $s\colon X \to B$ such that $p \circ s$ is the identity on X.

Examples of definable fibre bundles are given by definable covering maps, as defined in [EO–04, Section 2] which are, in fact, definable fibre bundles having finite fibre, and by projections onto a component of a cartesian product of definable sets $\pi\colon X \times F \to X$. The latter will be referred to as TRIVIAL BUNDLE of base X and fibre F. In particular, we will be interested in the example given by the following lemma.

Lemma 2.1.2. *Let H be a definable subgroup of a definable group G. Then we have a (principal) definable fibre bundle $\mathscr{B}_{G/H} \stackrel{def}{=} (G, G/H, p, H)$, where we put on G/H the quotient topology and $p\colon G \to G/H$ is the projection.*

Before proving the lemma, observe that it actually makes sense, even though, *a priori*, the topological space (not *group*: H may not be normal) G/H is not a definable set. In fact, by [Ber–08, Theorem 4.3], G/H is definably homeomorphic to a definable manifold, which, in turn, is definably homeomorphic to a definable set—since $\mathcal{M}$ expands a real closed field. The quotient will be tacitly identified with such a set, from now on.

Proof. By definable choice, fix a (possibly discontinuous) definable function $s\colon G/H \to G$ such that $p \circ s$ is the identity on G/H. Let X be a subset of G/H *large* in G/H (large in the sense of [Pil–88]: $\square$ is LARGE in $\triangle$ means that that the dimension of $\triangle \setminus \square$ is less than the dimension of $\triangle$ itself) such that s is continuous on X. Without loss of generality, we may assume X to be open in G/H, in fact the interior of X is large in G/H, since $\dim(\delta(X)) < \dim(X)$. By [PPS–00a, Claim 2.12] finitely many left translates $U_1 \stackrel{def}{=} a_1X, \ldots, U_n \stackrel{def}{=} a_nX$ of X cover G/H (here $a_1, \ldots, a_n \in G$ and G acts on G/H by left multiplication). For each $i \leqslant n$ we have an induced section $s_i\colon U_i \to G$

$$s_i\colon x \mapsto a_i s(a_i^{-1}x)$$

that is continuous on U_i. Therefore, the map $\phi_i\colon U_i \times H \to p^{-1}(U_i)$ defined by $\phi_i(x, y) = s_i(x)y$ is a definable homeomorphism with its inverse sending

x to $(xH, (s_i(xH))^{-1}x)$. Finally, for each i, the map $p \circ \phi_i = p_1 : U_i \times H \to U_i$ is the projection onto the first coordinate. □

Definition 2.1.3. Let $\mathscr{B} = (B, X, p, F)$ and $\mathscr{B}' = (B', X', p', F)$ be definable fibre bundles having the same fibre. A DEFINABLE BUNDLE MAP $f \colon \mathscr{B} \to \mathscr{B}'$ is a definable map $f \colon B \to B'$ such that:

I. there is a definable map $\bar{f} \colon X \to X'$ such that the following diagram commutes

$$\begin{array}{ccc} B & \xrightarrow{f} & B' \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} \\ X & \xrightarrow{\bar{f}} & X' \end{array}$$

II. for each $x \in X$ the map $f\restriction_{p^{-1}(x)}$ is an homeomorphism onto $p'^{-1} \circ \bar{f}(x)$.
The map $\bar{f}$ is said to be induced by f on the base spaces.

2.2 Equivalence of definable fibre bundles

Definition 2.2.1. A definable bundle map $f \colon \mathscr{B} \to \mathscr{B}'$ between definable fibre bundles $\mathscr{B}$ and $\mathscr{B}'$ is said to be a DEFINABLE BUNDLE ISOMORPHISM whenever it is a homeomorphism (of the bundle spaces of $\mathscr{B}$ and $\mathscr{B}'$). In this case $\mathscr{B}$ and $\mathscr{B}'$ are said to be *isomorphic*.

The following lemma gives us a useful criterion to tell when a definable bundle map is an isomorphism, under the fairly general hypothesis that the fibre is LOCALLY DEFINABLY COMPACT—i.e. that each point of it has a definably compact definable neighbourhood.

Lemma 2.2.2. *Let $f \colon \mathscr{B} \to \mathscr{B}'$ be a definable bundle map between the definable fibre bundles $\mathscr{B} = (B, X, p, F)$ and $\mathscr{B}' = (B', X', p', F)$, with the common fibre F locally definably compact. Suppose that the map $\bar{f} \colon X \to X'$ induced by f on the base spaces is an homeomorphism. Then f is a definable bundle isomorphism.*

We will postpone the proof of Lemma 2.2.2 until some additional machinery has been developed.

Definition 2.2.3. Let X, Y, and Z be definable sets. We will call the family $\mathcal{F} = \{f_x\}_{x \in X}$ of functions from Y to Z a DEFINABLE FAMILY if the function

$$\begin{aligned} f \colon X \times Y &\to Z \\ (x, y) &\mapsto f_x(y) \end{aligned}$$

is definable. We call $\mathcal{F}$ a CONTINUOUS FAMILY whenever f is continuous.

Lemma 2.2.4. *Let X and Y be definable sets, with Y locally definably compact. Consider a definable continuous family $\mathcal{F} = \{f_x\}_{x \in X}$* of homeomorphisms *of Y. We claim that the definable family $\mathcal{F}^{-1} \overset{\text{def}}{=} \{f_x^{-1}\}_{x \in X}$ of homeomorphisms of Y, which we will call the* INVERSE *of $\mathcal{F}$, is itself continuous.*

Proof. Clearly $\mathcal{F}^{-1}$ is definable: we want to prove that the function

$$\begin{aligned} g\colon X \times Y &\to Y \\ (x, y) &\mapsto f_x^{-1}(y) \end{aligned}$$

is continuous. By contradiction, suppose g to be discontinuous at some point $(a, b') \in X \times Y$ with $b' = f_a(b)$ for some $b \in Y$. Hence, there are definable paths $\gamma_1 \colon I \to X$ and $\gamma_2 \colon I \to Y$ such that $\gamma_1(0) = a$ and $\gamma_2(0) = b'$ but the curve $x \mapsto f^{-1}_{\gamma_1(x)} \circ \gamma_2(x)$ does not converge to b for x going to 0. Now, for each $x \in I$ define the curve

$$\begin{aligned} \gamma'_x \colon I &\to Y \\ t &\mapsto \begin{cases} \gamma_2\left((1-2t)x\right) & \text{for } t < 1/2 \\ f_{\gamma_1((2t-1)x)}(b) & \text{for } 1/2 \leqslant t \end{cases} \end{aligned}$$

which is continuous, definable, and joins $\gamma'_x(0) = \gamma_2(x)$ to $\gamma'_x(1) = f_{\gamma_1(x)}(b)$. Additionally, observe that, for x going to 0, the paths γ'_x converge uniformly to the constant path at b'. Fix a definably compact definable neighbourhood $\overline{V} \subset Y$ of b, and let $t \colon I \to I$ be the function mapping x to the least t such that $f^{-1}_{\gamma_1(x)} \circ \gamma'_x(t)$ is in $\overline{V}$, which is definable, and is well defined observing that

$$f^{-1}_{\gamma_1(x)} \circ \gamma'_x(1) = b \in \overline{V}$$

Define the curve, not necessarily continuous

$$\begin{aligned} \gamma_3 \colon I &\to \overline{V} \\ x &\mapsto f^{-1}_{\gamma_1(x)} \circ \gamma'_x \circ t(x) \end{aligned}$$

Clearly, for each $x \in I$, either $t(x) = 0$ or $\gamma_3(x)$ lies on the boundary of $\overline{V}$, hence exists $b'' \overset{def}{=} \lim_{x \to 0} \gamma_3(x)$ and $b'' \neq b$. However

$$f_a(b'') = \lim_{x \to 0} f_{\gamma_1(x)} \circ \gamma_3(x) = \lim_{x \to 0} \gamma'_x \circ t(x) = b' = f_a(b)$$

which contradicts $f_a \colon Y \to Y$ being an homeomorphism, and, in particular, 1 to 1. □

The following example shows that in the statement of Lemma 2.2.4 we can not drop the assumption on Y to be locally definably compact. Similarly, we must assume the fibre to be locally definably compact in Lemma 2.2.2.

Example 2.2.5. Working in the o-minimal structure $\mathbb{R}^{alg}$, let Q be the definable set $\mathbb{R}^{>0} \times \mathbb{R}^{>0} \cup \{(0, 0)\} \subset \mathbb{R}^2$. We will show a continuous definable family $\mathcal{F}$ of homeomorphisms of Q whose inverse $\mathcal{F}^{-1}$ is not continuous.

For $t \in \mathbb{R}^{\geqslant 0}$, let $\alpha_t \colon \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ be the map

$$\alpha_t(x) = \begin{cases} \min\left(1, t + \left(\frac{x-t}{t}\right)^2\right) & \text{if } t > 0 \\ 1 & \text{if } t = 0 \end{cases}$$

and define the family $\mathcal{F} = \{f_t\}_{t\in\mathbb{R}^{\geqslant 0}}$ as

$$f_t(x,y) = \begin{cases} (0,0) & \text{if } (x,y) = (0,0) \\ \alpha_t\left(\frac{y}{x}\right) \cdot (x,y) & \text{otherwise} \end{cases}$$

which is clearly continuous (at $(0,0)$ because $\alpha_-(-)$ is bounded by 1). However, $\mathcal{F}^{-1}$ is not continuous, in fact, considering the path

$$\begin{aligned} \gamma\colon I &\to Q \\ t &\mapsto (t,t^2) \end{aligned}$$

we have

$$f_0^{-1} \circ \gamma(0) = (0,0) \neq (1,0) = \lim_{t\to 0} f_t^{-1} \circ \gamma(t)$$

Now, consider the trivial bundle $\mathscr{B} = \mathbb{R}^{\geqslant 0} \times Q$, with base $\mathbb{R}^{\geqslant 0}$ and fibre Q. The map

$$\begin{aligned} \mathscr{B} &\to \mathscr{B} \\ (t,(x,y)) &\mapsto (t, f_t(x,y)) \end{aligned}$$

is a bundle map inducing the identity on the base space $\mathbb{R}^{\geqslant 0}$. However its inverse is not continuous, hence it can not be a bundle isomorphism.

Proof of Lemma 2.2.2. Clearly f is 1 to 1, so suffices to show that f^{-1} is continuous. By contradiction, let $f(x)$ be a discontinuity point of f^{-1}. Fix definable trivialization coverings $\mathcal{U} = \{U_i\}_i$ and $\mathcal{U}' = \{U'_j\}_j$ for $\mathscr{B}$ and $\mathscr{B}'$ respectively, and fix i and j such that $p(x) \in U_i$ and $p' \circ f(x) \in U'_j$. Observe that the set $U \overset{def}{=} U_i \cap \bar{f}^{-1}(U'_j)$ is an open neighbourhood of $p(x)$, so $\bar{f}(U)$ is an open neighbourhood of $p' \circ f(x) = \bar{f} \circ p(x)$, and $p'^{-1} \circ \bar{f}(U)$ is an open neighbourhood of $f(x)$. Hence suffices to show that $f^{-1}\restriction_{p'^{-1}\circ\bar{f}(U)}$ is continuous. For, let ϕ_i and ϕ'_j be trivialization maps associated to U_i and U'_j respectively. Since ϕ'_j sends $\bar{f}(U) \times F$ isomorphically onto $p'^{-1} \circ \bar{f}(U)$, we can reduce to prove that $\phi_i^{-1} \circ f^{-1} \circ \phi_j\restriction_{\bar{f}(U)\times F}$ is continuous. However, this is equivalent to the family

$$\mathcal{F}^{-1} \overset{def}{=} \left\{p_i \circ f^{-1} \circ \phi_j(x,-)\right\}_{x\in\bar{f}(U)}$$

of homeomorphisms of the fibre being continuous, which we have using Lemma 2.2.4, since it is the inverse of

$$\mathcal{F} \overset{def}{=} \left\{p'_j \circ f \circ \phi_i\left(\bar{f}^{-1}(x),-\right)\right\}_{x\in\bar{f}(U)}$$

where p_i and p'_j are the projections on the fibre corresponding to ϕ_i and ϕ'_j. □

Definition 2.2.6. Let $\mathscr{B} = (B,X,p,F)$ be a definable fibre bundle, and let $f\colon Y \to X$ be a definable map form the definable set Y to the base space of $\mathscr{B}$. We define the INDUCED BUNDLE $f^{-1}(\mathscr{B}) \overset{def}{=} (A,Y,p',F)$ where

(I) $A = \{(x,y) \in B \times Y \; s.t. \; p(x) = f(y)\}$

(ii) $p' \colon A \ni (x, y) \mapsto y \in Y$

Moreover we will refer to the map $\check{f} \colon A \ni (x, y) \mapsto x \in B$ as induced by f.

Lemma 2.2.7. *Using the same notations of Definition 2.2.6 we have that $f^{-1}(\mathscr{B})$ is actually a definable fibre bundle, and $\check{f}$ is a definable bundle map $f^{-1}(\mathscr{B}) \to \mathscr{B}$.*

Proof. Straightforward: p' is clearly onto, so suffices to fix a trivialization covering $\mathcal{U} = \{U_i\}_i$ for $\mathscr{B}$ and check that $\mathcal{V} \overset{\text{def}}{=} \{f^{-1}(U_i)\}_i$ is a trivialization covering of $f^{-1}(\mathscr{B})$. For, let ϕ_i be a trivialization map for $\mathscr{B}$ associated to U_i, and observe that the following

$$\begin{aligned} \phi'_i \colon f^{-1}(U_i) \times F &\to p'^{-1} \circ f^{-1}(U_i) \\ (x, y) &\mapsto (\phi_i(f(x), y), x) \end{aligned}$$

is an homeomorphism, hence we may take it as the trivialization map for $f^{-1}(\mathscr{B})$ associated to $f^{-1}(U_i)$. That $\check{f}$ is a bundle map follows immediately by inspection of the definition. □

Observation 2.2.8. Consider three definable sets X_1,X_2, and X_3 with two definable maps $f_1 \colon X_1 \to X_2$ and $f_2 \colon X_2 \to X_3$ between them. Take a definable fibre bundle $\mathscr{B} = (B, X_3, p, F)$, and define $\mathscr{B}' = f_1^{-1}(f_2^{-1}(\mathscr{B}))$ and $\mathscr{B}'' = g^{-1}(\mathscr{B})$, where $g = f_2 \circ f_1$. Is easy to see from Definition 2.2.6 – and also, when F is locally definably compact, follows from Theorem 2.2.9 below – that $\mathscr{B}'$ and $\mathscr{B}''$ are isomorphic. In addition, there is a definable bundle isomorphism $h \colon \mathscr{B}' \to \mathscr{B}''$ such that $\check{g} \circ h = \check{f}_2 \circ \check{f}_1$.

Theorem 2.2.9. *Let $\mathscr{B} = (B, X, p, F)$ and $\mathscr{B}' = (B', X', p', F)$ be definable fibre bundles having the same fibre F, and let $f \colon \mathscr{B} \to \mathscr{B}'$ be a definable bundle map between them. Suppose that F is a locally definably compact definable set. Then the the bundle $\mathscr{B}'' \overset{\text{def}}{=} \bar{f}^{-1}(\mathscr{B}')$, where $\bar{f}$ is the map induced by f on the base spaces, is isomorphic to $\mathscr{B}$. Moreover there is a definable bundle isomorphism $g \colon \mathscr{B} \to \mathscr{B}''$ such that $\check{\bar{f}} \circ g = f$.*

Proof. By definition, $\mathscr{B}'' = (B'', X, p'', F)$ where

$$\begin{aligned} B'' &= \left\{(x, y) \in B' \times X \textit{ s.t. } p'(x) = \bar{f}(y)\right\} \\ p'' &\colon (x, y) \mapsto y \end{aligned}$$

We claim that

$$\begin{aligned} g \colon B &\to B'' \\ x &\mapsto (f(x), p(x)) \end{aligned}$$

works. In fact, g is clearly a definable bundle map, moreover it induces the identity on the common base space X of $\mathscr{B}$ and $\mathscr{B}''$, hence, by Lemma 2.2.2, it is a definable bundle isomorphism. The identity $\check{\bar{f}} \circ g = f$ is immediate. □

Observe that the hypothesis of local compactness can not be dropped from Theorem 2.2.9, in fact any definable bundle map for which the thesis of Lemma 2.2.2 fails would make Theorem 2.2.9 fail as well.

2.3 Homotopies of definable bundle maps

Definition 2.3.1. Let $\mathscr{B} = (B, X, p, F)$ and $\mathscr{B}' = (B', X', p', F)$ be definable fibre bundles having the same fibre, and let f and g be definable bundle maps $\mathscr{B} \to \mathscr{B}'$. We call HOMOTOPY OF DEFINABLE BUNDLE MAPS between f and g a definable map $h\colon B \times I \to B'$ having the following properties:

I. $h_0 \overset{def}{=} h(-, 0) = f$

II. $h_1 \overset{def}{=} h(-, 1) = g$

III. For each $t \in I$ the map $h_t \overset{def}{=} h(-, t)$ is a definable bundle map.

In this situation, the function $\overline{h}\colon X \times I \ni (x, t) \mapsto \overline{h_t}(x) \in X'$ is a definable homotopy between $\overline{f}$ and $\overline{g}$, to which we will refer as the HOMOTOPY INDUCED ON THE BASE SPACES by h.

Moreover, we will say that h is STATIONARY WITH THE INDUCED HOMOTOPY $\overline{h}$ if for any subinterval $[t_1, t_2]$ of I and any $x \in B$ the following happens: $h\restriction_{\{x\}\times[t_1,t_2]}$ is constant if and only if $\overline{h}\restriction_{\{p(x)\}\times[t_1,t_2]}$ is constant.

Lemma 2.3.2. *Let $\mathcal{U}$ be a finite definable open covering of $X \times I$. Then, there are continuous definable functions $0 \equiv g_1 \leqslant \ldots \leqslant g_{k+1} \equiv 1\colon X \to I$ such that:*
($\star$) for each $x \in X$ and each $i \in \{1, \ldots, k\}$, the set $\{x\} \times [g_i(x), g_{i+1}(x)]]$ is entirely contained in some element of $\mathcal{U}$.

Proof. Caveat: Observe that if ($\star$) holds for an *ordered* set of definable functions $\mathcal{G}$ then it also holds for any superset of $\mathcal{G}$.
The proof is by induction on the dimension of X. If $\dim(X) = 0$ then X is finite and the statement is trivial.

Suppose that the statement is true for $\dim(X) \leqslant n$, then we will prove it for $\dim(X) = n + 1$. Observe that for each $x \in X$ there are

$$0 = l_1 < \ldots < l_{h(x)+1} = 1$$

such that for each $i \in \{1, \ldots, h(x)\}$ the set $\{x\} \times [l_i, l_{i+1}]$ is entirely contained in some element of $\mathcal{U}$. Without loss of generality we may suppose $h(x)$ to be minimal for each x, then $h = \sup_{x\in X} h(x) < \infty$ by uniform finiteness. By definable choice (and by the caveat) we have definable functions

$$0 \equiv l_1 \leqslant \ldots \leqslant l_{h+1} \equiv 1\colon X \to I$$

not necessarily continuous, such that ($\star$) holds for them. Let $\mathcal{X}$ be a finite partition of X into definable sets such that each of the functions l_{-} is continuous on each of the sets of $\mathcal{X}$. Now consider the set $\mathcal{X}'$ of the interiors of the sets in $\mathcal{X}$, and let $Y = X \setminus \bigcup \mathcal{X}'$: clearly $\dim(Y) \leqslant n$. By the induction hypothesis we get definable continuous functions $0 \equiv g_1 \leqslant \ldots \leqslant g_{k+1} \equiv 1\colon Y \to I$ such that for each $x \in Y$ and each $i \in \{1, \ldots, k\}$ the set $\{x\} \times [g_i(x), g_{i+1}(x)]$ is entirely contained in some element of $\mathcal{U}$. Consider a definable open neighbourhood Z of Y in X such that there is a retraction $r\colon Z \to Y$. We claim that, by continuity of the functions $g_i \circ r$, we can find in Z a new definable open neighbourhood Z' of Y such that for each $x \in Z'$ and each $i \in \{1, \ldots, k\}$ the set $\{x\} \times [g_i \circ r(x), g_{i+1} \circ r(x)]$ is entirely contained in some element of $\mathcal{U}$.

In fact, let $\mathcal{U} = \{U_j\}_j$: working in Z, we observe that, since I is closed and bounded, the projection $p_1 : Z \times I \to Z$ on the first component is a closed function, hence, for each i and j, the set

$$\begin{aligned} Z_{i,j} &= \left\{x \in Z \textit{ s.t. } \{x\} \times [g_i \circ r(x), g_{i+1} \circ r(x)] \subset U_j\right\} \\ &= Z \setminus p_1\left(\{(x,t) \in Z \times I \textit{ s.t. } t \in [g_i \circ r(x), g_{i+1} \circ r(x)]\} \setminus U_j\right) \end{aligned}$$

is open, and so is $Z' = \bigcup_{i,j} Z_{i,j}$, which contains Y by hypothesis.

Finally, we have a definable open covering $\mathcal{X}'' = \mathcal{X}' \cup \{Z'\}$ of X such that the statement of the lemma holds for each element of $\mathcal{X}''$: we will show that this implies the statement for X. For, let $\{V_1, \dots, V_m\} = \mathcal{X}''$ and suppose that for each i the statement holds for V_i and definable functions $g_{i,j} : V_i \to I$. Consider a shrinking $\{V_1', \dots, V_m'\}$ of $\mathcal{X}''$ such that $\overline{V_i'} \subset V_i$ for each i. Since I is definably contractible, for each i and j we have a continuous definable extension $g_{i,j}' : X \to I$ of $g_{i,j}\restriction_{\overline{V_i'}}$. Hence, the continuous definable functions

$$g_r(x) = \textit{ the } r\textit{-th element of } \{g_{i,j}'(x)\}_{i,j} \textit{ in ascending order}$$

satisfy ($\star$), by the caveat, since a subset of them does on each of the sets $\overline{V_i'}$, which cover X. $\square$

Theorem 2.3.3. *Let $\mathscr{B} = (B, X, p, F)$ and $\mathscr{B}' = (B', X', p', F)$ be definable fibre bundles having the same fibre. Let $f\colon \mathscr{B} \to \mathscr{B}'$ be a definable bundle map and $\tilde{h}\colon X \times I \to X'$ be a definable homotopy of $\overline{f}$, i.e. $\tilde{h}_0 = \overline{f}$. Then $\tilde{h}$ can be lifted to an homotopy h of definable bundle maps such that $h_0 = f$ and $\overline{h} = \tilde{h}$. Moreover, h can be chosen so that it is stationary with $\overline{h}$.*

Proof. Let $\mathcal{U} = \{U_i\}_{1 \leqslant i \leqslant n}$ be a trivialization covering for $\mathscr{B}'$, and take a definable shrinking $\mathcal{U}' = \{U_i'\}_{1 \leqslant i \leqslant n}$ of $\mathcal{U}$; i.e. $\mathcal{U}'$ covers X' and for each n holds $\overline{U_n'} \subset U_n$. By Fact 2.3.2 we have finitely many definable maps

$$0 \equiv g_1 \leqslant \cdots \leqslant g_{k+1} \equiv 1 \colon X \to I$$

such that for each $x \in X$ and for each $j \in \{1, \dots, k\}$ the set $\tilde{h}(x, [g_j(x), g_{j+1}(x)])$ is entirely contained in some element of $\mathcal{U}'$. Not to clutter the argument with quantifications, from now on, we will assume an implicit *for each* $i \in \{1, \dots, n\}$ *and each* $j \in \{1, \dots, k\}$, unless stated otherwise.

Let

$$V_{j,i} = \left\{x \textit{ s.t. } \tilde{h}\left(x, [g_j(x), g_{j+1}(x)]\right) \subset U_i'\right\}$$

Clearly, for each j the finite family $\{V_{j,i}\}_i$ of open definable sets is a covering of X. Choose definable open subsets $W_{j,i}$ of X so that $\overline{W_{j,i}} \subset V_{j,i}$ and so that for each j the family $\{W_{j,i}\}_i$ covers X. Fix definable continuous functions $u_{j,i} : X \to I$ such that $u_{j,i}\restriction_{W_{j,i}} \equiv 1$ and $u_{j,i}\restriction_{X \setminus V_{j,i}} \equiv 0$, what is possible by definable partition of unity. Finally define

$$\begin{aligned} \sigma_{j,i} &: X \to I \\ &\quad x \mapsto \max\left(0, u_{j,1}(x), \dots, u_{j,i}(x)\right) \\ \tau_{j,i} &: X \to I \\ &\quad x \mapsto g_j(x) + \sigma_{j,i}(x)\left(g_{j+1}(x) - g_j(x)\right) \end{aligned}$$

and write τ_a for $\tau_{j,i}$ where (j,i) is the a-th pair of indices in lexicographical order (j is more important, both are increasing).

Stipulating that τ_0 denotes the constant 0, we have a finite family $\{\tau_a\}_{a \leqslant nk}$ of continuous definable functions with the following property: for each a let

$$X_a = \{(x,t) \in X \times I \textit{ s.t. } 0 \leqslant t \leqslant \tau_a(x)\}$$

then $X_a \subset X_{a+1}$, and for each a there is a $b_a \in \{1,\ldots,n\}$ such that

$$\tilde{h}(X_{a+1} \setminus X_a) \subset U'_{b_a}$$

in fact, b_a is i when (j,i) is the a-th pair. For each a let

$$Y_a = \{(x,t) \in B \times I \textit{ s.t. } (p(x),t) \in X_a\}$$

The homotopy h is given on $Y_0 = B \times \{0\}$: we will extend it inductively on the sets Y_- up to $Y_{nk} = B \times I$. Fix an a and suppose to have already defined h on Y_a. For $(x,t) \in Y_{a+1} \setminus Y_a$ let

$$h(x,t) = \phi'^{-1}_{b_a}\left(\tilde{h}\left(p(x),t\right), p'_{b_a} \circ h\left(x, \tau_a \circ p(x)\right)\right)$$

where the formula makes sense since

$$h(x, \tau_a \circ p(x)) \in \mathrm{Dom}(p'_{b_a}) = p'^{-1}(U_{b_a})$$

which we have observing that $\{p(x)\} \times (\tau_a \circ p(x), \tau_{a+1} \circ p(x)]$ is a non-empty subset of $X_{a+1} \setminus X_a$, hence $\tilde{h}(p(x), \tau_a \circ p(x)) \in \overline{U'_{b_a}} \subset U_{b_a}$.

Our function is clearly continuous and fibre preserving, hence a definable bundle map. Stationarity is immediate by inspection of the formula above. □

Corollary 2.3.4. *Any definable fibre bundle on a definably contractible base having locally definably compact fibre is trivial.*

Proof. Immediate from Theorem 2.3.3 and Theorem 2.2.9. □

Now we give a definition of homotopy equivalence for definable fibre bundles. Notice that two definable fibre bundles which are homotopy equivalent have definably homotopy equivalent bundle spaces, definably homotopy equivalent bases, and the same fibre.

Definition 2.3.5. Two definable fibre bundles $\mathscr{B}$ and $\mathscr{B}'$ are said to be HOMOTOPY EQUIVALENT DEFINABLE FIBRE BUNDLES if there are two definable bundle maps $f\colon \mathscr{B} \to \mathscr{B}'$ and $g\colon \mathscr{B}' \to \mathscr{B}$ such that $f \circ g$ and $g \circ f$ are both homotopic to the identity on their respective domains—i.e. there are homotopies *of definable bundle maps* between each of them and the identity. In this situation, f and g are said to be HOMOTOPY INVERSE of each other.

Observation 2.3.6. Fix an $f\colon \square \to \triangle$ and let $g'\colon \triangle \to \square$ be a LEFT HOMOTOPY INVERSE of f, which means that $g' \circ f$ is homotopic to the identity. Let $g''\colon \triangle \to \square$ be a RIGHT HOMOTOPY INVERSE of f, i.e. $f \circ g'' \sim \mathrm{Id}$. Then f has an homotopy inverse which is $g' \circ f \circ g''$.

Theorem 2.3.7. *Let X and X' be definable sets, and let $f\colon X \to X'$ be a definable homotopy equivalence—i.e. there is a definable homotopy inverse $g\colon X' \to X$ of f. Consider a definable fibre bundle $\mathscr{B}' = (B', X', p, F)$ having X' as its base space and locally definably compact fibre F. Then $\mathscr{B} \stackrel{\text{def}}{=} f^{-1}(\mathscr{B}')$ is homotopy equivalent to $\mathscr{B}'$.*

Proof. Take a definable homotopy $\overline{h}\colon X' \times I \to X'$ with $\overline{h}_0 = \mathrm{Id}$ and $\overline{h}_1 = f \circ g$. By Theorem 2.3.3 we have an homotopy of bundle maps $h\colon \mathscr{B}' \times I \to \mathscr{B}'$ such that $h_0 = \mathrm{Id}$ and the homotopy induced by h on the base spaces coincides with $\overline{h}$—what justifies our abuse of the notation $\overline{h}$. By Theorem 2.2.9 we know that $\mathscr{B}'$ is isomorphic to $\overline{h}_1^{-1}(\mathscr{B}')$ which, in turn, is isomorphic to $g^{-1}(\mathscr{B})$ by observation 2.2.8, and we have as well a definable bundle isomorphism $\psi\colon \mathscr{B}' \to g^{-1}(\mathscr{B})$ such that $\check{f} \circ \check{g} \circ \psi = h_1$. Hence $\check{f}$ has a right homotopy inverse, which is $\check{g} \circ \psi$.

By the very same argument $\check{g}$ – hence $\check{g} \circ \psi$ – has a right homotopy inverse. So $\check{g} \circ \psi$, having both right and left – which is $\check{f}$ – homotopy inverses, is a definable homotopy equivalence between $\mathscr{B}'$ and $\mathscr{B}$. □

2.4 Definable fibrations

Definition 2.4.1. We will say that a definable map $p\colon E \to B$ is a DEFINABLE FIBRATION if for every definable set X, every definable homotopy $f\colon X \times I \to B$, and every definable map $g\colon X \to E$ such that $p \circ g = f(-, 0)$ there is a definable homotopy $h\colon X \times I \to E$ such that $p \circ h = f$ (h is a LIFTING of f) and holds $h(-, 0) = g(-)$; i.e. p has the HOMOTOPY LIFTING PROPERTY with respect to all definable sets.

Examples of definable fibrations are definable covering maps (see [BO–09, Proposition 4.10]) and trivial fibre bundles. All of them generalize to the following o-minimal parallel of the classical result that each fibre bundle is a fibration.

Theorem 2.4.2. *Every definable fibre bundle is a definable fibration.*

Proof. Along the lines of Theorem 2.3.3. Let $\mathscr{B} = (E, B, p, F)$ be a definable fibre bundle, and let $f\colon X \times I \to B$ and $g\colon X \to E$ be as in Definition 2.4.1. Take a trivialization covering $\mathcal{U} = \{U_i\}_{1 \leqslant i \leqslant n}$ of $\mathscr{B}$ and a definable shrinking $\mathcal{U}' = \{U'_i\}_{1 \leqslant i \leqslant n}$ of $\mathcal{U}$. By Lemma 2.3.2 we have finitely many definable maps $0 \equiv g_1 \leqslant \cdots \leqslant g_{k+1} \equiv 1\colon X \to I$ such that for each $x \in B$ and for each $j \in \{1, \ldots, k\}$ the set $f(x, [g_j(x), g_{j+1}(x)])$ is entirely contained in some element of $\mathcal{U}'$.

Let

$$V_{j,i} = \left\{x \textit{ s.t. } f\left(x, [g_j(x), g_{j+1}(x)]\right) \subset U'_i\right\}$$

then define the sets $\{W_{j,i}\}_{j,i}$, and the functions $\{\sigma_{j,i}\}_{j,i}$ and $\{\tau_{j,i}\}_{j,i}$ from X to I, as in the proof of Theorem 2.3.3. Using again the notation τ_a to denote $\tau_{j,i}$ where (j, i) is the a-th pair of indices in lexicographical order, define

$$X_a = \{(x, t) \in X \times I \textit{ s.t. } 0 \leqslant t \leqslant \tau_a(x)\}$$

then $X_a \subset X_{a+1}$, and for each a there is a $b_a \in \{1, \ldots, n\}$ such that

$$\tilde{h}(X_{a+1} \setminus X_a) \subset U'_{b_a}$$

A lifting h of f is given on $X_0 = X \times \{0\}$ by g: we will extend it inductively on the sets X_- up to $X_{nk} = X \times I$. Fix an a and suppose to have already defined h on X_a. For $(x, t) \in X_{a+1} \setminus X_a$ let

$$h(x,t) = \phi'^{-1}_{b_a}\left(f\left(p(x),t\right), p'_{b_a} \circ h\left(x, \tau_a \circ p(x)\right)\right)$$

This definition works because of the same reasons as in the proof of Theorem 2.3.3. □

In the rest of this section, we will prove the following stronger result (stating that if a definable map is *locally* a definable fibration, then it is a definable fibration).

Theorem 2.4.3. *Given a definable map* $p \colon E \to B$*, if there is a finite definable open covering* $\mathcal{U}$ *of* B*, such that for each* $U \in \mathcal{U}$ *the map* $p\restriction_{p^{-1}(U)} \colon p^{-1}(U) \to U$ *is a definable fibration, then* p *is a definable fibration.*

In the rest of this section, $p \colon E \to B$ *is a fixed continuous definable map between definable sets* E *and* B*.*

Now, we are going to consider a definable subset X of the path-space of B. More precisely, we will consider a *uniform family* of paths parametrized by some definable set X, which we will represent using a continuous definable function from $X \times I$ to B. The reader may think of any $x \in X$ as a path itself, and, by abuse of notation, of $f(x, -)$ as $x(-)$. From this point of view f is a continuous map from X to the path-space of B with the compact-open topology. For each such object, we will call *lifting function* any continuous definable map sending each pair (e, x), where x is an element of X and e is a point in $p^{-1}(x(0))$, to a lifting of x starting at e. It is easily shown (Lemma 2.4.5) that the existence of a lifting function for any X is equivalent to p being a definable fibration. Moreover, in Lemma 2.4.8, we will show how, given lifting functions for each element of a finite definable open covering of X, we can get a lifting function for X (basically blending the pieces via a definable partition of unity). Finally, through lemmata 2.3.2 and 2.4.9, we will use the local triviality of p to get such a covering for any subset X of the pathspace of B.

Definition 2.4.4. For any definable map $f \colon X \times I \to B$, let $\bar{B}_f$ be the definable set

$$\bar{B}_f = \{(e, x) \in E \times X \textit{ s.t. } p(e) = f(x, 0)\}$$

We will call a definable map $\lambda \colon \bar{B}_f \times I \to E$ a LIFTING FUNCTION for f if for all $(e, x) \in \bar{B}_f$,

$$p \circ \lambda\left((e,x), -\right) = f(x, -) \quad \text{and} \quad \lambda\left((e,x), 0\right) = e$$

Lemma 2.4.5. *The map* p *is a definable fibration if and only if for any definable set* X *and for any definable map* $f \colon X \times I \to B$ *there is a lifting function for* f*.*

Proof. To prove the *if* part, consider functions $f\colon X \times I \to B$ and $g\colon X \to E$ such that $f(-,0) = p \circ g(-)$. Let λ be a lifting function for f. Then

$$h \overset{def}{=} \lambda((g \times \mathrm{Id})(-), -)\colon X \times I \to E$$

is a lifting of f and $h(-,0) = g(-)$.

To prove the *only if* part, consider the homotopy $h\colon \bar{B}_f \times I \to B$ defined by $h((e,x),t) = f(x,t)$. Since $h(-,0) = p \circ p_1$, where $p_1\colon \bar{B}_f \to E$ is the projection on the first component, we may lift h to a map $\lambda\colon \bar{B}_f \times I \to E$ such that $\lambda(-,0) = p_1$, which is a lifting function for f. □

Definition 2.4.6. For any definable map $f\colon X \times I \to B$ and for any definable subset $W \subset X$, let $\widetilde{W}_f$ be the definable set

$$\widetilde{W}_f = \{(e,x,s) \in E \times W \times I \textit{ s.t. } p(e) = f(x,s)\}.$$

We will call a definable map $\Lambda\colon \widetilde{W}_f \times I \to E$ an extended lifting function for f over W if for all $(e,x,s) \in \widetilde{W}_f$,

$$p \circ \Lambda((e,x,s),-) = f(x,-) \quad \text{and} \quad \Lambda((e,x,s),s) = e.$$

The idea of an extended lifting function is that it maps any triple (e,x,s), where x is a path, s is an element of I, and e is a point in $p^{-1}(x(s))$, to a lifting of x passing through e at time s. As we will see in the next lemma, an extended lifting function is nothing but a lifting function in disguise (basically because, using appropriate lifting functions, we can lift separately the two halves of any path which are before and after s); in fact, extended lifting functions were introduced for merely technical reasons.

Lemma 2.4.7. *The following statements are equivalent:*

i. *for any definable set X and for any definable map $f\colon X \times I \to B$ there is lifting function for f;*
ii. *for any definable set X and for any definable map $f\colon X \times I \to B$ there is an extended lifting function for f over X.*

Proof. The direction ii $\implies$ i is trivial. For i $\implies$ ii, let $Y = X \times I$ and let $g', g''\colon Y \times I \to B$ be defined by

$$g'((x,s),t) = \begin{cases} f(x,s-t) & \text{for } t \leqslant s \\ f(x,0) & \text{for } t > s \end{cases}$$

$$g''((x,s),t) = \begin{cases} f(x,s+t) & \text{for } t \leqslant 1-s \\ f(x,1) & \text{for } t > 1-s \end{cases}$$

By i there are lifting functions λ' and λ'' for g' and g'' respectively. Now, an extended lifting function $\Lambda\colon \widetilde{X}_f \times I \to E$ for f over X is given by

$$\Lambda((e,x,s),t) = \begin{cases} \lambda'((e,(x,s)),s-t) & \text{for } t \leqslant s \\ \lambda''((e,(x,s)),t-s) & \text{for } t > s \end{cases}$$

which is easily proved continuous, moreover for $t \leqslant s$ we have

$$p \circ \Lambda((e,x,s),t) = p \circ \lambda'((e,(x,s)),s-t) = g'((x,s),s-t) = f(x,t)$$

and similarly for $s < t$; and finally

$$\Lambda((e,x,s),s) = \lambda'((e,(x,s)),0) = e$$

□

Lemma 2.4.8. *Let* X *be a definable set, and let* $f\colon X \times I \to B$ *be a definable map. Suppose that there is a finite open covering* $\mathcal{W}$ *of* X *such that for each element* W *of* $\mathcal{W}$ *there is an extended lifting function for* f*. Then there is a lifting function for* f*.*

Proof. Let $\mathcal{W} = \{W_j\}_{j \in J}$ and consider a definable partition of unity $\{g_j\}_{j \in J}$ on X (i.e. for each j, g_j is a definable map from X to $\mathcal{M}^{\geqslant 0}$ and $\sum_j g_j \equiv 1$) such that $\overline{W'_j} \subset W_j$ for each j, where $W'_j \overset{def}{=} \{x \in X \text{ s.t. } g_j(x) > 0\}$. For each subset κ of J define

$$W'_\kappa = \bigcup_{j \in \kappa} W'_j \qquad f_\kappa = f\restriction_{W'_\kappa \times I} \qquad \bar{B}_\kappa = \bar{B}_{f_\kappa} \qquad g_\kappa = \sum_{j \in \kappa} g_j$$

Consider a maximal subset α of J such that there is a lifting function λ_α for f_α. For the sake of contradiction, suppose there is a $j_0 \in J \setminus \alpha$. We shall construct a lifting function for f_β, where $\beta \overset{def}{=} \alpha \cup \{j_0\}$. Let $h = \frac{g_\alpha}{g_\beta}\colon W_\beta \to I$. Clearly h is the constant 1 on $W'_\alpha \setminus W'_{j_0}$, the constant 0 on $W'_{j_0} \setminus W'_\alpha$, and takes values in $(0,1)$ on $W'_\alpha \cap W'_{j_0}$. Define $\mu\colon \bar{B}_{\{j_0\}} \to E$ by:

$$\mu(e,x) = \begin{cases} e & \text{for } 0 \leqslant h(x) < 1/2 \\ \lambda_\alpha((e,x), 2h(x)-1) & \text{for } 1/2 \leqslant h(x) < 1 \end{cases}$$

Consider an extended lifting function Λ for f over W_{j_0}, then we claim that $\lambda_\beta\colon \bar{B}_\beta \times I \to E$ defined by:

$$\lambda_\beta((e,x),t) =$$
$$= \begin{cases} \Lambda((e,x,0),t) & \text{for } 0 \leqslant h(x) < 1/2 \\ \lambda_\alpha((e,x),t) & \text{for } 1/2 \leqslant h(x) \leqslant 1 \text{ and } 0 \leqslant t \leqslant 2h(x)-1 \\ \Lambda((\mu(e,x),x,2h(x)-1),t) & \text{for } 1/2 \leqslant h(x) \leqslant 1 \text{ and } 2h(x)-1 < t \leqslant 1 \end{cases}$$

is a lifting function for f_β. In fact, to prove the continuity of λ_β suffices to observe that:

$$\begin{aligned} \Lambda((e,x,0),t) &= \lambda_\alpha((e,x),t) && \text{for } h(x) = 1/2 \text{ and } t = 0 \\ \Lambda((e,x,0),t) &= \Lambda((\mu(e,x),x,2h(x)-1),t) && \text{for } h(x) = 1/2 \text{ and } t \geqslant 0 \\ \lambda_\alpha((e,x),t) &= \Lambda((\mu(e,x),x,2h(x)-1),t) && \text{for } 1/2 \leqslant h(x) \text{ and } t = 2h(x)-1 \end{aligned}$$

for $x \in \overline{W'_{j_0}} \subset W_{j_0}$. Moreover, equations

$$p \circ \lambda_\beta((e,x),t) = f(x,t) \quad \text{and} \quad \lambda_\beta((e,x),0) = e$$

which are required by the definition of lifting function, hold because Λ and λ_α are (extended) lifting functions. □

Lemma 2.4.9. *Let* $U_1, \dots, U_k$ *be definable subsets of* B. *Suppose that, for any definable set* X, *for any definable map* $f\colon X \times I \to B$, *and for any* i, *there is an extended lifting function for* f *over* $\{x \in X \textit{ s.t. } f(x, I) \subset U_i\}$. *Then, for any definable set* X, *for any definable continuous function* $f\colon X \times I \to B$, *and for any definable maps* $0 \equiv g_1 \leqslant \dots \leqslant g_{k+1} \equiv 1\colon X \to I$, *there is an extended lifting function for* f *over* W, *where*

$$W = \{x \in X \textit{ s.t. } f(x, [g_i(x), g_{i+1}(x)]) \subset U_i \textit{ for all } i = 1, \dots, k\}$$

Proof. The idea behind the following convoluted formulæ is that, for any i, using the extended lifting function for U_i (which we have by hypothesis), we can lift any path $x \in W$ restricted to the interval $[g_i(x), g_{i+1}(x)]$. Hence, all we have to do is to lift such paths interval by interval, taking care that they fit together properly.

Let $Y = \{(x, a, b) \in X \times I^2 \textit{ s.t. } a < b\}$, and let $h\colon Y \times I \to B$ be defined by

$$h((x,a,b),t) = \begin{cases} f(x,a) & \text{for } t \in [0,a) \\ f(x,t) & \text{for } t \in [a,b] \\ f(x,b) & \text{for } t \in (b,1] \end{cases}$$

By hypothesis, for each i, we have an extended lifting function Λ_i for h over $Y_i \overset{def}{=} \{y \in Y \textit{ s.t. } h(y, I) \subset U_i\}$. Let

$$L_{i,j} = \{((e,x,s),t) \in \widetilde{W}_f \times I \textit{ s.t. } s \in [g_i(x), g_{i+1}(x)] \text{ and } t \in [g_j(x), g_{j+1}(x)]\}$$

and define $\Lambda'_{i,j}\colon L_{i,j} \to E$ by

$$\Lambda'_{i,j}((e,x,s),t) =$$
$$= \begin{cases} \Lambda_i((e,(x,g_i(x),g_{i+1}(x)),s),t) & \text{for } i = j \\ \Lambda_j((\Lambda'_{i,j-1}((e,x,s),g_j(x)),(x,g_j(x),g_{j+1}(x)),g_j(x)),t) & \text{for } j > i \\ \Lambda_j((\Lambda'_{i,j+1}((e,x,s),g_{j+1}(x)),(x,g_j(x),g_{j+1}(x)),g_{j+1}(x)),t) & \text{for } j < i \end{cases}$$

It is routine to prove simultaneously by induction on $|j - i|$ that these are well-defined continuous functions and

$$\forall ((e,x,s),t) \in L_{i,j} \quad f(x,t) = p \circ \Lambda'_{i,j}((e,x,s),t) \tag{$\star$}$$

Now, let

$$l\colon X \times I \to \{1, \dots, k\}$$
$$l(x,s) = i < k \text{ iff } g_i(x) < g_{i+1}(x) \text{ and } s \in [g_i(x), g_{i+1}(x))$$
$$l(x,s) = k \text{ iff } s \in [g_k(x), g_{k+1}(x)]$$

and define

$$\Lambda'\colon \widetilde{W}_f \times I \to E$$
$$\Lambda'((e,x,s),t) = \Lambda'_{l(x,s),l(x,t)}((e,x,s),t)$$

We have the continuity of Λ' since the definable maps $\Lambda'_{i,j}$ coincide on the intersections of their domains, which are closed. That, in turn, may be proved easily observing that

$$\Lambda'_{i,i}((e,x,s),s) = e$$

by definition,

$$\Lambda'_{i,j}((e,x,g_i(x)),t) = \Lambda'_{i-1,j}((e,x,g_i(x)),t)$$

by induction on $|j-i|$, and

$$\Lambda'_{i,j}((e,x,s),g_j(x)) = \Lambda'_{i,j-1}((e,x,s),g_j(x))$$

by definition. Since $\Lambda'((e,x,s),s) = e$, by ($\star$) we can conclude that Λ' is an extended lifting function for f over W. □

Finally, we have all the ingredients to prove Theorem 2.4.3.

proof of Theorem 2.4.3. We apply Lemma 2.4.5. Fix a definable set X and a definable map $f\colon X \times I \to B$, we have to show that there is a lifting function for f over X. By Lemma 2.4.8, it suffices to find a definable finite open covering $\mathcal{W}$ of X and for each $W \in \mathcal{W}$ an extended lifting function for f over W.

On the other hand – observing that $f^{-1}(\mathcal{U})$ is a finite definable open covering of $X \times I$ – by Lemma 2.3.2, there are definable maps

$$0 \equiv g_1 \leqslant \ldots \leqslant g_{k+1} \equiv 1\colon X \to I$$

such that for each $x \in X$ and each $1 \leqslant i \leqslant k$ the set $f(x,[g_i(x),g_{i+1}(x)])$ is entirely contained in some element of $\mathcal{U}$. Let $\mathcal{U} = \{U_j\}_{j=1}^{h}$, and for each function $\sigma\colon \{1,\ldots,k\} \to \{1,\ldots,h\}$ let

$$W_\sigma = \{x \in X \textit{ s.t. } f(x,[g_i(x),g_{i+1}(x)]) \subset U_{\sigma(i)}, \text{ for all } i = 1,\ldots,k\}$$

The sets $\{W_\sigma\}_\sigma$ are a definable finite open covering of X. Therefore, it suffices to prove that for each σ there is an extended lifting function for f over W_σ. Fix such a σ and denote $U_{\sigma(i)}$ by U_i^σ, for each $i = 1,\ldots,k$. By hypothesis $p\restriction_{p^{-1}(U_i^\sigma)}\colon p^{-1}(U_i^\sigma) \to U_i^\sigma$ is a definable fibration, for each $i = 1,\ldots,k$. Hence by lemmata 2.4.5 and 2.4.7, for each i, each definable Y and each definable continuous $h\colon Y \times I \to U_i^\sigma$ there is an extended lifting function for h over Y. Therefore, for every i for every definable Z and every definable continuous $g\colon Z \times I \to B$ there is an extended lifting function for g over $\{z \in Z \textit{ s.t. } g(z,I) \subset U_i^\sigma\}$. That is, we are under the hypothesis of Lemma 2.4.9, and consequently there is an extended lifting function for f over W_σ, as required. □

Chapter 3

On the topology of definably compact groups

In this chapter, we will deal with the topology of definably compact definable groups. Our main results are Theorem 3.1.9 and Theorem 3.3.4.

The first theorem states that for any definably compact definably connected group G and for each n

$$\pi_n(G) = \pi_n^{def}(G/G^{00})$$

which is a natural addition to the set of known correlations between the topological invariants of G and G/G^{00} (see [EO–04] and [Ber–09]).

Theorem 3.3.4 states that each definably compact definably connected group G is definably homeomorphic to the product of its derived subgroup $[G, G]$ – recall that, by results in [HPP–08b], $[G, G]$ is a definable subgroup of G – and a definable abelian group. This group happens to be a finite quotient of the center of G. Our result can be seen as a definable analogue of [Bor–61, Proposition 3.1] which states that every compact connected Lie group is homeomorphic to the topological product of its derived subgroup and a torus. Perhaps surprisingly, in the Lie case a stronger result is known: the derived subgroup of a compact Lie group is indeed a semidirect factor of the group (Borel-Scheerer-Hofmann splitting theorem: see [HM–98, Theorem 9.39]). We will provide an example showing that the same statement fails in the definable case.

The relations of mutual dependence of Theorem 3.1.9 and Theorem 3.3.4 are somewhat complicated. To prove the latter we need the abelian case of the former and Fact 3.3.1. On the other and, given Theorem 3.3.4 we could obtain Theorem 3.1.9 from the semisimple and the abelian case separately. A second way to get Theorem 3.1.9 from the semisimple and the abelian case is to combine [BO–09, Corollary 4.11] and Fact 3.3.1. Nevertheless, our proof, resorting to algebraic topology devices, makes no use of Fact 3.3.1, which was not known at the time the proof was first conceived.

3.1 Homotopy groups of definable groups

The aim of this section is to prove Theorem 3.1.9. The statement is proven by combining the abelian and the semisimple case trough the long exact sequence for definable homotopy (see [BO–09]). The semisimple case is an easy consequence of results in [PPS–00a] (Peterzil, Pillay, and Starchenko prove there that every definably connected centerless semisimple definable group is definably isomorphic to a linear group).

For the abelian case, the result is known on the fundamental group [EO–04], so all that remains to be shown is that $\pi_n^{def}(G)$ is trivial for every $n > 1$. This apparently simple statement turns out to be definitely non-trivial. In fact, the direct route of the classical proof (factorizing G into a product of one-dimensional tori) is barred by examples as in [PS–99, Section 5]. Hence, we will take an indirect way: we will prove that $\pi_n^{def}(G)$ is divisible and finitely generated. This implies the theorem. Divisibility is not hard to prove using the group structure of G. To prove that $\pi_n^{def}(G)$ is finitely generated, however, we can not simply invoke the triangulation theorem, since the homotopy groups of a finite simplicial complex may not be finitely generated. Instead, we will triangulate G and then show that the finite simplicial complex thus obtained has a semialgebraic H-space structure. This will allow us to transfer the problem to the real case and apply the classical result that an H-space with finitely generated homology groups has finitely generated higher homotopy groups.

Definition 3.1.1. We will denote by T the definably compact group $[0, 1)$ with the sum modulo 1 as group operation. We will write $T^d(R)$ when there may be confusion on what structure the interval $[0, 1)$ is in.

Definition 3.1.2. A definable H-space is a definable pointed space (X, x_0) equipped with a definable continuous map $\mu\colon X \times X \to X$ such that both $\mu(-, x_0)$ and $\mu(x_0, -)$ are definably homotopic to the identity.

It is clear that a definable H-space definable over $\mathbb{R}$ is, in particular, an H-space (in the classical sense).

Theorem 3.1.3. *Let G be a definable group. Then $\pi_n^{def}(G)$ is finitely generated for each $n > 0$.*

Proof. We can assume, without loss of generality, that G is definably connected and bounded. By the triangulation theorem we can identify G with the realization $|K'|(\mathcal{M})$ in $\mathcal{M}$ of a finite simplicial complex K' with rational vertexes, one of which is the group identity e. Now, let K be a closed subcomplex of K' such that $K(\mathcal{M})$ is a semialgebraic deformation retract of $|K'|(\mathcal{M})$ containing e. The multiplication on $|K'|(\mathcal{M})$ induces a map $\nu\colon |K|(\mathcal{M}) \times |K|(\mathcal{M}) \to |K|(\mathcal{M})$ which gives to $(|K|(\mathcal{M}), e)$ the structure of a definable H-space. Moreover, by [BO–09, Corollary 3.6], the map ν is definably homotopic to a semialgebraic continuous map

$$\mu\colon |K|(\mathcal{M}) \times |K|(\mathcal{M}) \to |K|(\mathcal{M})$$

definable without parameters. Of course, μ induces again a definable H-space structure over $(|K|(\mathcal{M}), e)$. Moreover, again by [BO–09], the semialgebraic maps defined without parameters $\mu(-, e)$ and $\mathrm{Id}_{|K|(\mathcal{M})}$, which are *definably* homotopic, are also *semialgebraically* homotopic with an homotopy defined without parameters; similarly for $\mu(e, -)$ and $\mathrm{Id}_{|K|(\mathcal{M})}$.

Claim 3.1.4. Let $|K|(\mathbb{R})$ denote the realization of K in $\mathbb{R}$, then $\pi_n(|K|(\mathbb{R}))$ is finitely generated for each $n \geqslant 1$.

Proof. $(|K|(\mathbb{R}), e)$ is endowed with a definable H-space structure by μ, or more precisely by the function defined by the same formula as μ interpreted in $\mathbb{R}$. Therefore $(|K|(\mathbb{R}), e)$ is an H-space. Notice that every path-connected H-space is a SIMPLE SPACE (i.e. its fundamental group acts trivially on all homotopy groups, see [Spa–66, Chapter 7 Theorem 3.9]), and for simple spaces the homotopy groups are all finitely generated if the homology groups are so ([Whi–78, Chapter XIII Corollary 7.14]), hence we have the claim since $|K|(\mathbb{R})$ is a finite simplicial complex. □

Now, by [BO–09, Corollary 4.4], $\pi_n^{def}(|K|(\mathcal{M}))$ is isomorphic to $\pi_n(|K|(\mathbb{R}))$, hence it is a finitely generated abelian group. Since $\pi_n^{def}(|K|(\mathcal{M})) \cong \pi_n^{def}(G)$, we get the result. □

Corollary 3.1.5. *Let* G *be a definable abelian group. Then* $\pi_n^{def}(G) = 0$*, for each* $n \geqslant 2$.

Proof. By Theorem 3.1.3, $\pi_n(G)$ is a finitely generated group; moreover, we know that it is abelian for each $n \geqslant 2$. Since a finitely generated abelian group is divisible if and only if it is trivial, it suffices to show that $\pi_n(G)$ is divisible for each $n > 1$. We may assume that G is definably connected. By [EO–04, Corollary 2.12] the maps

$$\begin{aligned} p_k \colon G &\to G \\ x &\mapsto kx \end{aligned}$$

with $k \in \mathbb{N}^{>0}$ are definable covering maps, and hence, by [BO–09, Corollary 4.11] they induce isomorphisms on the higher homotopy groups. As a consequence, for each $[\alpha] \in \pi_n(G)$ and for each $k \in \mathbb{N}^{>0}$ we have some $[\beta] \in \pi_n(G)$ such that $[p_k \circ \beta] = [\alpha]$, and we can conclude that $\pi_n(G)$ is divisible observing that $[p_k \circ \beta] = k[\beta]$. □

Theorem 3.1.6. *Let* G *be a definably connected definably compact* d*-dimensional abelian group. Then* G *is definably homotopy equivalent to* T^d*, i.e. there are definable continuous maps* $g \colon G \to T^d$ *and* $f \colon T^d \to G$*, such that* $f \circ g$ *and* $g \circ f$ *are definably homotopic to the identity.*

Proof. Consider the following map, originally defined in the proof of [OP–09, Lemma 4.3]:

$$\begin{aligned} f \colon T^d &\to G \\ (t_1, \dots, t_d) &\mapsto \gamma_1(t_1) + \dots + \gamma_d(t_d) \end{aligned}$$

where $[\gamma_1], \ldots, [\gamma_d]$ are free generators of the definable fundamental group of G. Then, clearly, f induces an isomorphism between $\pi_1^{def}(T^d)$ and $\pi_1^{def}(G)$. To see the latter, consider for each $i = 1, \ldots, d$, the loop $\delta_i \colon [0,1] \to T^d$ defined by

$$\delta_i(t) = \begin{cases} (0, \ldots, \overset{i}{t}, \ldots, 0) & \text{for } t \in [0,1) \\ (0, 0, \ldots, 0) & \text{for } t = 1 \end{cases}$$

Hence, $\pi_1(T^d) = \langle [\delta_1], \ldots, [\delta_d] \rangle$ and so the map f induces on the fundamental groups is $f_*([\delta_i]) = [\gamma_i]$, for each $i = 1, \ldots, d$, which is an isomorphism.

Since by Theorem 3.1.5 all the higher homotopy groups of both G and T^d are trivial, f induces an isomorphism on them as well (in a trivial way), hence, by the o-minimal version of Whitehead theorem [BO–09, Theorem 5.6], f is an homotopy equivalence. □

By [EO–04, Theorem 1.1], and applying duality, we have $H_i(G; \mathbb{Q}) \cong \mathbb{Q}^{\binom{d}{i}}$ for each $i \geqslant 0$. Here we improve that result by proving the following.

Corollary 3.1.7. *Let G be a definably connected definably compact d-dimensional abelian group. Then the o-minimal homology group $H_i(G; \mathbb{Z}) \cong \mathbb{Z}^{\binom{d}{i}}$, for each $i \geqslant 0$.*

Proof. By Theorem 3.1.6, for each $i \geqslant 0$ we have that $H_i(G; \mathbb{Z}) \cong H_i(T^d(\mathcal{M}); \mathbb{Z})$, and by [BO–02, Proposition 3.2] $H_i(T^d(\mathcal{M}); \mathbb{Z})$ is isomorphic to $H_i(T^d(\mathbb{R}); \mathbb{Z})$, which in turn is isomorphic to $\mathbb{Z}^{\binom{d}{i}}$. □

Next we proceed to study the o-minimal homotopy groups of definably compact (noncommutative) groups. Towards this aim we first prove some results concerning homogeneous spaces of the type G/H, where G is a definable group (not necessary definably compact) and H is a definable subgroup of G. Notice that by [PPS–00a, Corollary 2.14] such G/H can be equipped with a definable manifold topology so that the canonical action of G on G/H is continuous. Moreover, by [Ber–08, Theorem 4.3], this topology coincides with the quotient topology (inherited by that of G), which in turn coincides with the definable group topology – provided H is normal – of G/H. Notice also that H being closed in G, the coset space G/H is a regular definable space, and hence we can consider its definable manifold topology induced by that of the ambient space.

Fact 3.1.8. *Let F be the functor from the category of definably compact groups to the category of compact Lie groups which sends G to G/G^{00}. Then:*

i. *F preserves dimension and connectedness (each concept in its category),*
ii. *F is exact i.e. transforms exact sequences in exact sequences.*

Proof. By [HPP–08a, Theorem 8.1] and [Ber–07, Theorem 5.2] respectively. □

Our next result says that the functor F also preserves the homotopy groups.

Theorem 3.1.9. *Let G be a definably connected definably compact group. Then $\pi_n^{def}(G) \cong \pi_n(G/G^{00})$ for all n.*

Proof.
Case G *abelian.* Let d be the dimension of G. By [EO–04, Theorem 1.1] and Corollary 3.1.5 above, we have $\pi_1^{def}(G) \cong \mathbb{Z}^d$ and $\pi_n^{def}(G) = 0$ for each $n > 1$, respectively. On the other hand, since G/G^{00} is an abelian compact Lie group of dimension d, we also have $\pi_1(G/G^{00}) \cong \mathbb{Z}^d$ and, for $n > 1$, we have $\pi_n(G/G^{00}) = 0$.
Case $n = 1$. First note that by the o-minimal Poincaré-Hurewicz theorem ([EO–04, Theorem 5.1]), and the corresponding classical result, it suffices to prove that $H_1(G) = H_1(G/G^{00})$. On the other hand, by [Ber–09, Corollary 5.2] and [Ber–07, Remark 7.3], the singular cohomology groups $H^1(G; L)$ and $H^1(G/G^{00}; L)$ are isomorphic for any coefficient group L. Hence, by the universal coefficient theorem (UCT) for cohomology, we are done in this case. The details are as follows: The UCT says that for any chain complex $\mathcal{C}$, for any abelian group L and for any $n > 0$, if $H_{n-1}(\mathcal{C})$ is free then $H^n(\mathcal{C}; L) \cong \hom(H_n(\mathcal{C}); L)$. Applying the UCT for $n = 1$ to the (honest) chain complex associated to the definable singular simplexes on our definable group G we get $H^1(G; L) \cong \hom(H_1(G); L)$. Hence, by the corresponding result for the singular chain complex of G/G^{00}, we have

$$\hom(H_1(G); L) \cong \hom\left(H_1\left(G/G^{00}\right); L\right)$$

for any abelian group L. It follows that $H_1(G) \cong H_1(G/G^{00})$, since both groups $H_1(G)$ and $H_1(G/G^{00})$ are abelian and finitely generated.

Before we prove the result in the remaining cases we have the following.

Claim 3.1.10. Let G and H be two definably connected definably compact groups. Suppose G is a finite extension of H, Then, for every $n > 1$, if $\pi_n(H) \cong \pi_n(H/H^{00})$ then $\pi_n(G) \cong \pi_n(G/G^{00})$.

Proof. By [EO–04, Proposition 2.11], the onto homomorphism $G \to H$ with finite kernel is a definable covering map. Then, $\pi_n(G) \cong \pi_n(H)$ for any $n > 1$ by [BO–09, Corollary 4.11]. By preservation of exactness, the induced map from G/G^{00} to H/H^{00} is also an onto homomorphism with finite kernel, hence a covering map and so $\pi_n(G/G^{00}) \cong \pi_n(H/H^{00})$, for any $n > 1$. □

Case G *definably semisimple and* $n > 1$. Since the center of G is finite, G is a finite extension of the definably semisimple centerless group $G/Z(G)$. By Claim 3.1.10, we may consider G as centerless. On the other hand, we may assume by results in [PPS–00a, Theorem 4.1] and [PPS–02, proof of Theorem 5.1(3)] (see [Ote–08, theorems 5.3, 4.2]) that $G = G(\mathcal{M})$ is a semialgebraic group over the real algebraic numbers. By [BO–09, Corollary 4.4], $\pi_n(G(\mathcal{M})) \cong \pi_n(G(\mathbb{R}))$, for every n. But $G(\mathbb{R})$ is G/G^{00} by the proof of [Pil–04, Proposition 3.6].
General case. Let Z be the center of G. Then, by [PS–00, Corollary 5.4], the group G/Z is definably semisimple, hence the result holds for G/Z. By Lemma 2.1.2 and Theorem 2.4.2 the projection map $G \to G/Z$ is a definable fibration. Therefore, by [BO–09, Theorem 4.9], for each $n \geqslant 2$, the o-minimal homotopy groups $\pi_n(G, Z)$ and $\pi_n(G/Z)$ are isomorphic; and, hence, the

o-minimal homotopy sequence of the pair (G, Z) is the following long exact sequence (see [BO–09, Section 4])

$$\cdots \to \pi_{n+1}(G/Z) \to \pi_n(Z) \to \pi_n(G) \to \pi_n(G/Z) \to \pi_{n-1}(Z) \to \cdots$$

On the other hand, using the exactness of the functor to the Lie category we have that $F(Z)$ is a closed normal subgroup of $F(G)$, so the projection map $F(G) \to F(G)/F(Z)$ ($\cong F(G/Z)$) is a fibration and hence we have the exact sequence

$$\cdots \to \pi_{n+1}\left(F(G/Z)\right) \to \pi_n\left(F(Z)\right) \to \pi_n\left(F(G)\right) \to \pi_n\left(F(G/Z)\right) \to \cdots$$

Since Z and $F(Z)$ are abelian, we have $\pi_n(Z) = 0$ and $\pi_n(F(Z)) = 0$ for all $n \geqslant 2$. So for $n \geqslant 3$, $\pi_n(G) \cong \pi_n(G/Z)$ and $\pi_n(F(G)) \cong \pi_n(F(G/Z))$. By the previous case $\pi_n(F(G/Z)) \cong \pi_n(G/Z)$. For $n = 2$, recall that the second homotopy group of a compact Lie group is trivial, so $\pi_2(F(G/Z)) = 0$, and therefore also $\pi_2(G/Z) = 0$ (by the semisimple case). Since $\pi_2(Z) = 0$, it follows that also $\pi_2(G) = 0$. □

Observation 3.1.11. Let k be a characteristic zero field. Then

$$H_n(G; k) \cong H_n(G/G^{00}; k)$$

for each n.

Proof. By a similar argument as the one done in the above proof with the UCT. Indeed, we can use the following facts: (i) the singular cohomology groups $H^1(G; L)$ and $H^1(G/G^{00}; L)$ are isomorphic for any coefficient group L ([Ber–09, Corollary 5.2]); (ii) the UCT; (iii) the fact that $H_n(G; k)$ are free, and (iv) $H_n(G; k)$ are finite-dimensional k-vector spaces. □

3.2 Bundles on the torus

In this section we will study the special case of definable fibre bundles whose base is a STANDARD TORUS. Intending by standard torus T^n for some n, where T denotes the definable group $[0, 1)$ with the sum modulo 1 and the Pillay's topology. T can be definably identified with a definable set with its subset topology. Alternatively, we could as well have defined T as SO_2—i.e. the set $\{(x, y) \in R^2$ *s.t.* $x^2 + y^2 = 1\}$ with the usual operation *and the subset topology*, or as any other one-dimensional definably compact group (notice, however, that for example SO_2 and $[0, 1)$ with the sum modulo 1 are in general not definably isomorphic).

Using the results of Chapter 2 we will give a necessary and sufficient condition for the triviality of *principal* definable fibre bundles on a torus.

Observation 3.2.1. Let X be a definably connected definable set such that $\pi_n^{def}(X)$ is trivial—e.g. T^m for any m and for $n > 1$. Let $f, g \colon I^n \to X$ be two definable maps, and suppose that $f\restriction_{\delta(I^n)} = g\restriction_{\delta(I^n)}$. Then f and g are definably homotopic via an homotopy $h \colon I^n \times I \to X$ such that $h_t\restriction_{\delta(I^n)} = f\restriction_{\delta(I^n)}$ for any $t \in I$.

Proof. This observation follows immediately from the definition, as in the the classical case. □

Definition 3.2.2. We say that a definable set X has property $\mathscr{P}$ if X is definably connected and, for all $n \in \mathbb{N}$, any definable map $f\colon \delta(I^n) \to X$ such that

$$\begin{aligned} f(0, x_2, x_3, \dots, x_n) &= f(1, x_2, x_3, \dots, x_n) \\ f(x_1, 0, x_3, \dots, x_n) &= f(x_1, 1, x_3, \dots, x_n) \\ &\vdots \\ f(x_1, x_2, x_3, \dots, 0) &= f(x_1, x_2, x_3, \dots, 1) \end{aligned} \tag{$\star$}$$

is definably homotopic to a constant.

Observation 3.2.3. Equivalently, a definably connected definable set X has property $\mathscr{P}$ if any definable map from $\delta(I^n)$ to X satisfying ($\star$) extends to a map from I^n to X.

We are interested in property $\mathscr{P}$ because of the following lemma (which will be used in the following section).

Lemma 3.2.4. *Let* G *be a definably connected definable group. Then* G *has property* $\mathscr{P}$.

Proof. Let $f\colon \delta(I^n) \to G$ be a definable map satisfying ($\star$). It suffices to prove that f is definably homotopic to a constant function. For each $i \in \{1, \dots, n\}$ define the projection $p_i\colon I^n \to \delta(I^n)$ by

$$(p_i(x_1, \dots, x_n))_j = \begin{cases} x_j & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

Now let $f_0 = f$ and, for each $i \in \{1, \dots, n\}$, define

$$\begin{aligned} f_i\colon \delta(I^n) &\to G \\ x &\mapsto f_{i-1}(x) \cdot (f_{i-1} \circ p_i(x))^{-1} \end{aligned}$$

Each of these maps satisfies ($\star$)—since f_0 does and f_i is defined inductively from f_{i-1} by means of operations that preserve ($\star$). Moreover, for each i, we have $f_i{\restriction}_{S_i} \equiv e_G$ where

$$S_i \overset{def}{=} \left\{(x_1, \dots, x_n) \in \delta(I^n) \textit{ s.t. } \exists j \leqslant i\ x_j \in \{0, 1\}\right\}$$

what is easy to prove by induction. Since $S_n = \delta(I^n)$, we have that $f_n \equiv e_G$, hence it suffices to prove that $f_{i-1} \sim f_i$ for each i. We claim that $f_{i-1} \circ p_i$ is definably homotopic to a constant, which is enough—in fact, by the definable connectedness of G, we may assume that constant to be e_G, hence

$$f_{i-1} = f_{i-1} \cdot e_G^{-1} \sim f_{i-1} \cdot (f_{i-1} \circ p_i)^{-1} = f_i$$

Here is the required homotopy

$$\begin{aligned} \delta(I^n) \times I &\to G \\ (x, t) &\mapsto f_{i-1} \circ p_i(tx) \end{aligned}$$

□

Lemma 3.2.5. *Property $\mathscr{P}$ is a definable homotopy invariant—i.e. given two definably homotopy equivalent definable sets X and X' one has property $\mathscr{P}$ if and only if the other has.*

Proof. Immediate from the definition. Take $g\colon X \to X'$ and $g'\colon X' \to X$ definable maps definably homotopy inverse of each other. Suppose that X' has property $\mathscr{P}$, and let $f\colon \delta(I^n) \to X$ satisfy $(\star)$. By construction f is homotopic to $g' \circ g \circ f$, but $g \circ f\colon \delta(I^n) \to X'$ satisfies $(\star)$, hence $g \circ f \sim \mathit{constant}$ by hypothesis. As a consequence $f \sim g' \circ g \circ f \sim g'(\mathit{constant}) = \mathit{constant}$. □

Lemma 3.2.6. *Let $\mathscr{B} = (B, T^n, p, F)$ be definable fibre bundle having the n-dimensional torus T^n as the base space and a locally definably compact definably connected definable set F as fibre. Suppose that the bundle space B of $\mathscr{B}$ has property $\mathscr{P}$. Then $\mathscr{B}$ admits a definable cross section.*

Proof. Let $\tau_n\colon I^n \to T^n$ be the canonical map

$$\tau_n\colon (x_1, \dots, x_n) \mapsto (x_1 \bmod 1, \dots, x_n \bmod 1)$$

By induction on the dimension n, suffices to show that any definable *partial* cross section s defined on $\tau_n(\delta(I^n))$ extends to a *global* cross section.
Case $n = 1$. Follows immediately from the definable connectedness of the fibre, using the fact that $\tau_1^{-1}(\mathscr{B})$ is isomorphic to a product bundle by Corollary 2.3.4. Observe, however, that the same argument doesn't work for $n > 1$, since we would need the fibre to be $(n-1)$-connected.
Case $n > 1$. Since B has property $\mathscr{P}$, by observation 3.2.3, the map $s \circ \tau_n$ extends to a map $f\colon I^n \to B$. By observation 3.2.1, $p \circ f$ is definably homotopic to τ_n. Since $f\restriction_{\delta(I^n)} = s \circ \tau_n\restriction_{\delta(I^n)}$ there is a well defined definable map $f'\colon T^n \to B$ such that $f = f' \circ \tau_n$, moreover $p \circ f'$ is definably homotopic to the identity on T^n. Hence, by Theorem 2.3.3, the identity on $\mathscr{B}$ is homotopic to a definable bundle map $g\colon \mathscr{B} \to \mathscr{B}$ which induces $p \circ f'$ on the base space T^n of $\mathscr{B}$. As a consequence, by Theorem 2.2.9, $\mathscr{B}' \stackrel{\text{def}}{=} (p \circ f')^{-1}(\mathscr{B})$ is isomorphic to $\mathscr{B}$. On the other hand, $\mathscr{B}'$, which has base space

$$B' \stackrel{\text{def}}{=} \left\{(x, y) \in B \times T^n \textit{ s.t. } p(x) = p \circ f'(y)\right\}$$

admits a definable cross section, which is $s'\colon y \mapsto (f'(y), y)$. Is thus proven that $\mathscr{B}$ has a definable cross section; some extra care must be taken in order to ensure that this cross section coincides with s on $\tau_n(\delta(I^n))$.

Using observation 3.2.1, we get a definable homotopy between $p \circ f'$ and the identity which is stationary on $\tau_n(\delta(I^n))$. Hence the bundle map g given by Theorem 2.3.3 restricts to the identity on $p^{-1} \circ \tau_n(\delta(I^n))$. Now, the bundle isomorphism from $\mathscr{B}$ to $\overline{g}^{-1}(\mathscr{B})$ given by the proof of Theorem 2.2.9 identifies $x \in \mathscr{B}$ with $(g(x), p(x))$, which is in the graph of s' if and only if $f' \circ p(x) = g(x)$. However, on $p^{-1} \circ \tau_n(\delta(I^n))$, we know that g is the identity, so $f' \circ p(x) = g(x)$ is equivalent to $f' \circ p(x) = x$, which holds if and only if x is in the graph of s. □

Lemma 3.2.6 gives us a necessary and sufficient condition for the triviality of a definable principal fibre bundle over a torus.

Observation 3.2.7. Fix any definable fibre bundle $\mathscr{B} = (B, X, p, F)$ having a definably connected base X. Then the definable fundamental group of X acts as a group of permutations of the definably connected components of the fibre $p^{-1}(x_0)$ over the base point x_0 of X—precisely $[\gamma] \in \pi_1^{def}(X)$ sends a definably connected component C of $p^{-1}(x_0)$ to the connected component containing $\gamma'(1)$ where γ' is any lifting of γ at a point in C.

The action is well defined, in fact consider two liftings γ' and δ' of γ and δ with $[\gamma] = [\delta]$. Then the concatenation of $\gamma(1 - -)$ and δ is null-homotopic, hence, by the homotopy lifting property, the concatenation of $\gamma'(1 - -)$ and δ' is homotopic to a curve contained in the fibre $p^{-1}(x_0)$ and joining $\delta'(1)$ with $\gamma'(1)$, i.e. $\delta'(1)$ and $\gamma'(1)$ are in the same definably connected component of the fibre.

Definition 3.2.8. We say that a definable fibre bundle $\mathscr{B} = (B, X, p, G)$ is a PRINCIPAL DEFINABLE FIBRE BUNDLE if the following holds:

I. the fibre G of $\mathscr{B}$ has a definable group structure that makes it into a topological group,

II. the fibre G acts on $\mathscr{B}$ as a group of definable bundle maps inducing the identity on the base space,

III. the action of G is definable and continuous, and, *on each fibre,* it is free and transitive.

Principal definable fibre bundles having a section, as their topological colleagues, are trivial.

Observation 3.2.9. Let $\mathscr{B} = (B, X, p, G)$ be a principal definable fibre bundle. Suppose that $\mathscr{B}$ admits a section $s \colon X \to B$. Then $\mathscr{B}$ is trivial.

Proof. Let us denote $g \cdot x$ with $g \in G$ and $x \in X$ the action of G. Suffices to check that

$$\begin{aligned} f \colon G \times B &\to X \\ (g, b) &\mapsto g \cdot s(b) \end{aligned}$$

is an homeomorphism. Clearly f is continuous and injective. Recall that, by [Pil–88], G is a definable manifold. Hence, by the o-minimal invariance of domain [Joh–01] (i.e. that a continuous definable injection $R^n \to R^n$ is open), it is an homeomorphism. □

We have the following lemma—which, as we will see, generalizes to the case where instead of T^n we have any definably compact abelian group.

Lemma 3.2.10. *A principal definable fibre bundle* $\mathscr{B} = (B, T^n, p, G)$ *having a torus* T^n *as the base space is trivial if and only if its bundle space* B *has property* $\mathscr{P}$ *and* $\pi_1^{def}(T^n)$ *acts trivially on the connected components of* G.

Proof (sketch). The *only if* part will become evident after Lemma 3.2.4 is proven in the next section.

For the *if* part, fix a definably connected component B' of B: let p' be $p\restriction_{B'}$, we claim that $\mathscr{B}' \stackrel{def}{=} (B', T^n, p', G^0)$ – where G^0 denotes the definably connected component of the identity of G – is a principal definable fibre bundle. In fact, $p'^{-1}(x_0)$ is definably connected: take two points in $p'^{-1}(x_0)$, then, by definable connectedness of B', take a definable path joining them, and observe that the triviality of the action of the fundamental group implies that the two end-points of this path are in the same connected component. Being homeomorphic to a definably connected component of G, the inverse image $p'^{-1}(x_0)$ must be homeomorphic to G^0. The local triviality of B, now, implies the local triviality of B', hence $\mathscr{B}'$ is a definable fibre bundle. Clearly, the action of $G^0 < G$ on the fibres of B can not permute their definably connected components, hence $\mathscr{B}'$ is principal (with the same action as $\mathscr{B}$ restricted to G^0).

By Lemma 3.2.6 and observation 3.2.9, $\mathscr{B}'$ is trivial. The statement follows working on each definably connected component separately. □

3.3 Failing to split definably compact groups

The aim of this section is to prove Theorem 3.3.4. A corresponding classical result [Bor–61, Proposition 3.1] can be proven rather straightforwardly by induction on the dimension:

> Let G be our Lie group and let $Z^0(G)$ denote the connected component of the identity of the center of G. Take a connected subgroup Z of $Z^0(G)$ having codimension 1. By induction, we can assume the statement on $Z \cdot [G, G]$, where $[G, G]$ denotes the derived subgroup. The base space of the principal fibre bundle induced by the quotient $G/_{Z \cdot [G,G]}$ is homeomorphic to a circle, hence, using [Ste–51, Corollary 18.6], the bundle is trivial.

The same approach fails in the o-minimal setting because of the already observed existence of definably compact abelian definable groups which don't factor into products of one-dimensional groups.

As usual we must take a detour. By observation 3.2.9, it suffices to prove the existence of a section of the definable fibre bundle $G \to G/_{[G,G]}$, however we can not assume a factorization of the definable abelian group $G/_{[G,G]}$. Nevertheless, it is a known fact that the existence of sections is, in fact, an homotopic – more precisely co-homologic – property (by a collection of techniques known as *obstruction theory*). Hence we can expect the statement to be reducible to the case of a factorized torus using Theorem 3.1.6. That is, indeed, what we are going to do, having identified property $\mathscr{P}$ (see Definition 3.2.2, and Lemma 3.2.6) as the correct homotopic invariant for our purpose. (A second approach would have been to transfer the result form the reals working in the spirit of [BO–09]: this involves constructing enough of obstruction theory on the o-minimal side, and showing that, whatever the co-homologic objects required, they reduce to the semialgebraic case. Although this was his original plan, the author didn't investigate this approach.)

After proving Theorem 3.3.4, we will show that the derived subgroup of a definably compact definable group may not be a semidirect factor of the group. The example is slightly surprising since the same statement is indeed true for compact Lie groups (see [HM–98, Theorem 9.39]).

Fact 3.3.1 ([HPP–08b, Corollary 6.4]). *Let* G *be a definably compact definably connected group, then the derived subgroup* $[G,G]$ *of* G *is a definably connected semisimple definable group. Let* $Z^0(G)$ *denote the definably connected component of the identity of the center of* G*. Then* $G = [G,G]\cdot Z^0(G)$ *and* $\Gamma_G \overset{def}{=} [G,G]\cap Z^0(G)$ *is finite.*

Clearly, from Fact 3.3.1, we have that the quotient $G/[G,G]$ is definably isomorphic to $Z^0(G)/\Gamma_G$. We will call $\mathscr{B}_G \overset{def}{=} (G, Z^0(G)/\Gamma_G, p, [G,G])$ the bundle obtained from $\mathscr{B}_{G/[G,G]}$ through this (group) isomorphism.

Observation 3.3.2. Let $\mathscr{B}' = (B', X', p', F)$ be a definable fibre bundle, and let f be a definable map from a definable set X to the base X' of $\mathscr{B}'$. Consider the bundle $\mathscr{B} \overset{def}{=} f^{-1}(\mathscr{B}')$. If $\mathscr{B}'$ admits a definable cross section $s'\colon X' \to B'$, then $\mathscr{B}$ has a definable cross section too. In fact, by definition,

$$B = \left\{(x,y) \in B' \times X \textit{ s.t. } p'(x) = f(y)\right\}$$

and

$$\begin{aligned} s\colon X &\to B \\ y &\mapsto \left(s' \circ f(y), y\right) \end{aligned}$$

is a cross section of $\mathscr{B}$. In particular, using Theorem 2.2.9, we have that given two homotopy equivalent definable fibre bundles having locally definably compact fibre, one has a definable cross section if and only if the other has.

Lemma 3.3.3. *Let* G *be a definably compact definably connect group. Then the definable fibre bundle* $\mathscr{B}_G$ *admits a definable cross section.*

Proof. First of all, observe that the fibre of $\mathscr{B}_G$ is definably connected and (locally) definably compact. By Theorem 3.1.6 we know that $Z^0(G)/\Gamma_G$ is definably homotopy equivalent to T^d, for some d. Let $f\colon T^d \to Z^0(G)/\Gamma_G$ be a definable homotopy equivalence, and consider the definable fibre bundle $\mathscr{B}' \overset{def}{=} f^{-1}(\mathscr{B}_G)$. By 2.3.7 we have that $\mathscr{B}'$ is homotopy equivalent to $\mathscr{B}_G$, hence the respective bundle spaces are definably homotopy equivalent. Since, by Lemma 3.2.4, the bundle space of $\mathscr{B}_G$ – which is G – has property $\mathscr{P}$, the same holds for the bundle space of $\mathscr{B}'$. As a consequence, by Lemma 3.2.6, $\mathscr{B}'$ has a definable cross section. The statement is thus proven using observation 3.3.2. □

Theorem 3.3.4. *Let* G *be a definably compact definably connected group. Then* G *is definably homeomorphic to the Cartesian product* $[G,G] \times Z^0(G)/\Gamma_G$ *of the derived subgroup of* G *and an abelian definable group.*

Proof. Immediate by Lemma 3.3.3: in fact the homeomorphism is

$$\begin{aligned} [G,G] \times Z^0(G)/\Gamma_G &\to G \\ (x,y) &\mapsto x \cdot s(y) \end{aligned}$$

where $\cdot$ denotes the group operation in G and s is a definable cross section of $\mathscr{B}_G$. □

The same argument proves the following generalization of Lemma 3.2.10.

Theorem 3.3.5. *Any principal definable fibre bundle* $\mathscr{B} = (B, A, p, G)$ *having a definably compact definably connected abelian group* A *as the base space is trivial if and only if its bundle space* B *has property* $\mathscr{P}$ *and* $\pi_1^{def}(A)$ *acts trivially on the connected components of* G.

Proof. Suffices to observe that the condition on the triviality of the action of $\pi_1^{def}(A)$ is invariant under homotopy equivalence *of definable fibre bundles*. Then follow the same argument in the proof of Lemma 3.3.3 and apply Lemma 3.2.10. □

The following example shows that a definably compact group may not be (definably isomorphic to) a definable semidirect product of its derived subgroup with some definable group.

Example 3.3.6. We will construct a definably compact group G definable in the field $\mathbb{R}^{alg}$ of the algebraic real numbers such that the derived subgroup of G has no definable semidirect complement in G. Consider any definably compact group $(G, \cdot)$ such that its derived subgroup $[G, G]$ has a definable semidirect complement $H < G$. Let $\sigma\colon G \to [G, G]$ and $\tau\colon G \to H$ be defined by the equation $x = \sigma(x) \cdot \tau(x)$ for each $x \in G$. Observe that $\sigma\restriction_{Z(G)}$ is a group homomorphism, moreover it is the identity on $Z(G) \cap [G, G]$, hence, if $Z^0(G) \cap [G, G]$ is non-trivial, we have a non-trivial definable group homomorphism from $Z^0(G)$ to $[G, G]$. Now let T denote $[0, 1)$ with the sum modulo 1, and let $SU_2 = SU_2(\mathbb{R}^{alg})$ denote the group

$$SU_2 = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}\}_{\substack{a^2+b^2+c^2+d^2=1 \\ a,b,c,d \in \mathbb{R}^{alg}}}$$

with the usual quaternion multiplication. Consider $G \stackrel{def}{=} (T \times SU_2)/\Gamma$ where Γ is the normal subgroup $\{(0, 1), (1/2, -1)\}$. It is easy to check that $T \times \{1, -1\} \cong T$ is the center of G, and $\{0, 1/2\} \times SU_2 \cong SU_2$ is the derived subgroup of G, hence $(0, -1) \cdot \Gamma$ is an element of $Z^0(G) \cap [G, G]$, which therefore is non-trivial. Follows that, were $[G, G]$ a semidirect factor of G, we would have a definable non-trivial homomorphism $h\colon T \to SU_2$. Being a definably connected abelian definable subgroup of SU_2 of dimension 1, the image of h would be a 0-Sylow in the sense of [Str–94a]. Follows that $\mathrm{Img}(h)$ would be conjugated to the subgroup $SO_2 \stackrel{def}{=} \{a + b\mathbf{i}\} < SU_2$; however no non-trivial homomorphism $T \to SO_2$ can be definable in $\mathbb{R}^{alg}$ (if not we would be able to find a 0-definable homomorphism, and using it to define π without parameters).

Chapter 4

Definable fundamental groupoid & finite coverings

The definable fundamental group has played an important role in the investigation of definable groups, and in particular in the solution of Pillay's conjectures 1.4.4—where it provided the link between the o-minimal dimension of a definably compact abelian group and the dimension of the corresponding Lie group (see [EO–04] and [HPP–08a]). As of today, no alternative route has been devised for this point. Nevertheless, the argument to compute the fundamental group of a definably compact abelian group is very indirect (remember that the group can not be factorized), and, for instance, it doesn't give any natural isomorphism between the fundamental group of G and that of G/G^{00}. The obvious difficulty being that paths in G are parametrized by elements of the o-minimal structure, while paths in G/G^{00} are parametrized by real numbers—using the standard part map clearly won't work.

In this chapter, we try to analyze the situation using the *compact domination conjecture* proven in [HP–09]. We will prove a *local* version of the isomorphism between the fundamental group of G and that of G/G^{00}, namely that for any open subset of G its fundamental group is isomorphic to the definable fundamental group of its preimage in G/G^{00}. Using this, we will show that the said isomorphism between the fundamental groups of definably compact and Lie groups is indeed natural (Section 4.3). A second consequence is that the finite extensions of G and G/G^{00} respectively – as subcategories of the objects over G and G/G^{00} in their respective categories – are isomorphic categories (Theorem 4.5.3). Furthermore, we will be able to draw consequences on the definable universal cover (Theorem 4.5.2) and the homotopy type (Theorem 4.8.7) of definably compact groups.

4.1 Topological consequences of compact domination

In this section we will explore some topological consequences of the compact domination conjecture proven in [HP–09] (and extended in [HPP–08b] to the non-abelian case). One of its equivalent formulations says that given a definably compact group G, the image in G/G^{00} of a nowhere dense definable subset of G has Haar measure zero. We will only use the following consequence of compact domination, proven in [Ber–08]: the infinitesimal subgroup G^{00} is a decreasing intersection $\bigcap_{i \in \mathbb{N}} C_i$ of definably simply connected (actually even definably contractible) definable subsets C_i of G. We will explore this situation in the following more general setting:

Assumption 4.1.1. Let X be a $\bigvee$-definable set, let Y be a locally simply connected second countable locally compact space, and let $f \colon X \to Y$ be a surjective function with the following properties:

I. The preimage of every open subset of Y is $\bigvee$-definable (so in particular X is $\bigvee$-definable).

II. The preimage of every compact subset of Y is type-definable.

III. For all $y \in Y$ the type-definable set $f^{-1}(y)$ is a decreasing intersection $\bigcap_{i \in \mathbb{N}} C_i$ of definably simply connected definable open subsets $\{C_i\}_i$ of X.

Note that the second countability assumption ensures that $|Y| \leqslant 2^{\aleph_0}$, so in particular Y is small.

Example 4.1.2. The natural projection map $p \colon G \to G/G^{00}$ satisfies assumption 4.1.1 (where G is a definably compact definable group). More generally let O be an open subset of G/G^{00}, then $f{\restriction}_{p^{-1}(O)}$ satisfies assumption 4.1.1.

Definition 4.1.3 (Definable fundamental groupoid)**.** Given a $\bigvee$-definable set X, and a subset Γ of X, let $P^{def}(X, \Gamma)$ be the set of definable paths in X with endpoints in Γ. Let $\pi^{def}(X, \Gamma)$ be the quotient of $P^{def}(X, \Gamma)$ modulo definable homotopy of paths (relative to the endpoints). We define an operation $+$ on $\pi^{def}(X, \Gamma)$ by $[\alpha] + [\beta] = [\alpha + \beta]$ where $\alpha + \beta$ is the concatenation of the paths α and β. Clearly, this is defined only when the final point of α coincides with the starting point of β. With this operation $\pi^{def}(X, \Gamma)$ is a groupoid, namely a category in which every morphism is an isomorphism (the objects are the elements of Γ, the morphisms are the homotopy classes of paths, and the composition is the concatenation). In particular when $\Gamma = X$ we obtain the DEFINABLE FUNDAMENTAL GROUPOID $\pi^{def}(X, X)$ of X. When Γ is a singleton we obtain the definable fundamental group $\pi^{def}(X, x_0) \stackrel{def}{=} \pi^{def}(X, \{x_0\})$, which will also be written as $\pi_1^{def}(X)$ when the base point is clear from the context or irrelevant. Dropping *def* one obtains the corresponding classical notions.

In this section we will prove the following theorem.

Theorem 4.1.4. *Let* $f \colon X \to Y$ *be as in assumption 4.1.1. Then there is a unique morphism of groupoids* $f_* \colon \pi^{def}(X, X) \to \pi(Y, Y)$ *with the following properties:*

1. $f_* = f$ *on the object part of the groupoids, i.e.* f_* *maps the definable homotopy class of a path with endpoints* x *and* y*, to the homotopy class of a path with endpoints* $f(x)$ *and* $f(y)$ *respectively.*

2. *For any open* $O \subseteq Y$*, and for any* $[a] \in \pi^{def}(X, X)$ *such that* $\operatorname{Im} a \subseteq f^{-1}(O)$*, there is a path* b *in* Y *such that* $\operatorname{Im} b \subseteq O$ *and* $f_*([a]) = [b]$.
Moreover for this unique morphism f_* *we have:*
3. *If* Γ *is a (possibly infinite) subset of* X *such that* $f{\restriction}_\Gamma$ *is injective, then the restriction of* f_* *to* $\pi^{def}(X, \Gamma)$ *is an isomorphism onto* $\pi(Y, f(\Gamma))$*. In particular if* Γ *is a singleton, we obtain an isomorphism of the respective fundamental groups.*
4. *If* $f\colon X \to Y$ *satisfies assumption 4.1.1 and* O *is an open subset of* Y*, then the restriction* $f^O\colon f^{-1}(O) \to O$ *of* f *continues to satisfy the assumption (we have restricted also the codomain to make the map surjective). In this situation we have a commutative diagram*

$$\begin{array}{ccc} \pi^{def}(f^{-1}(O), f^{-1}(O)) & \xrightarrow{f^O_*} & \pi(O, O) \\ \downarrow & & \downarrow \\ \pi^{def}(X, X) & \xrightarrow{f_*} & \pi(Y, Y) \end{array}$$

where the vertical arrows are the homomorphisms induced by the inclusion maps.

Corollary 4.1.5. *Let* G *be a definably compact definably connected definable group. Then the projection* $p\colon G \to G/G^{00}$ *induces an isomorphism*

$$p_*\colon \pi_1^{def}(G) \xrightarrow{\cong} \pi_1(G/G^{00})$$

Moreover if O *is an open simply connected subset of* G/G^{00}*, then* $p^{-1}(O)$ *is an* $\bigvee$*-definable definably simply connected subset of* G

Proof. Apply 3 with Γ a single point in $p^{-1}(O)$. □

Proof of Theorem 4.1.4. The proof is split into a number of claims and definitions, and will be completed at the end of this section. The idea is the following. Given an open cover $\mathcal{U}$ of a topological space, the nerve of $\mathcal{U}$ is the simplicial complex whose n-simplexes are the n-tuples of open sets from $\mathcal{U}$ which have a non-empty intersection. We will show that, if $\mathcal{U}$ satisfies suitable assumptions (each element of $\mathcal{U}$ is path connected and the intersection of two elements of $\mathcal{U}$ is contained in a simply connected set), then the fundamental groupoid of the space is determined by the nerve of $\mathcal{U}$. We will apply this result to the topological space Y. A corresponding result holds also in the o-minimal category (considering definably path-connected and definably simply connected sets), so we can apply it to X. We will show (Claim 4) that X and Y admit two open covers $\mathcal{V}$ and $\mathcal{U}$ which satisfy the required assumptions in the respective categories and have isomorphic nerves (in fact $\mathcal{V}$ will consist of the f-preimages of the sets in $\mathcal{U}$: the difficulty here is in choosing $\mathcal{U}$ so that all the conditions are met on both sides *simultaneously*). This will essentially prove the result. Guided by the above idea, we will actually develop the proof without explicit mention of the nerves. Let us now come to the formal details.

Claim 1. f *is continuous.*

Proof of claim. By our assumptions, for every $y \in Y$, the preimage $f^{-1}(y)$ is a small directed intersections of definable open sets. So it suffices to show that (in a saturated model) the intersection $\bigcap_{i \in I} O_i$ of a small directed family of definable open sets O_i is open. So take $x \in \bigcap_{i \in I} O_i$ and let us show that x is in the interior. To this aim let $B_t(x) \subset X$ be the ball of center x and positive radius $t \in M$. Since O_i is open in X, there is $t_i > 0$ such that for all $0 < t < t_i$ we have $B_t(x) \cap X \subset O_i$. By saturation there is a single positive $t^* \in M$ which for each i is smaller than t_i. But then $B_{t^*}(x) \cap X \subset \bigcap_{i \in I} O_i$ so x is in the interior. □

Claim 2. *Let* D *be a definable subset of* X*. Then* $f(D)$ *is a compact subset of* Y*. (It follows that the same conclusion holds if* D *is only assumed to be type-definable.)*

Proof of claim. Let $(O_i \mid i \in I)$ be an open cover of $f(D)$. We must find a finite subcover. The definable set D is included in $\bigcup_{i \in J} f^{-1}(O_i)$ and each $f^{-1}(O_i)$ is $\bigvee$-definable. By saturation there is a finite $J \subset I$ such that $D \subset \bigcup_{i \in J} f^{-1}(O_i)$. It then follows that $f(D) \subset \bigcup_{i \in J} O_i$. □

Claim 3.

I. *Let* Z *be a compact connected subset of* Y*. Then the type-definable set* $f^{-1}(Z)$ *is definably connected.*

II. *Let* U *be an open connected subset of* Y*. Then the* $\bigvee$*-definable set* $f^{-1}(U)$ *is definably path-connected.*

Proof of claim.

I. By [BOPP–05, Lemma 2.2] if a type-definable set is the intersection of a filtered family of definably connected sets, then it is itself definably connected. It then follows from assumption 4.1.1 that for each $y \in Y$, the type-definable set $f^{-1}(y)$ is definably connected. Now let Z be a compact connected subset of Y, and suppose for a contradiction that $f^{-1}(Z)$ is the union of two relatively definable disjoint non-empty open sets A and B. Being relatively definable in a type-definable set, A and B are in fact type-definable. So their images $f(A)$ and $f(B)$ are compact by Claim 2. Since $Z = f(A) \cup f(B)$ and Z is connected, $f(A)$ and $f(B)$ have a non-empty intersection. Take $y \in f(A) \cap f(B)$. Then $f^{-1}(y)$ meets both A and B, contradicting the fact that $f^{-1}(y)$ is definably connected.

II. Let $x, y \in f^{-1}(U)$. Since Y is locally simply connected, it is in particular locally path connected, so its connected open subsets are path-connected. We can thus choose a path a in U connecting $f(x)$ to $f(y)$. Its image $Z \stackrel{def}{=} \mathrm{Im}(a)$ is a compact connected subset of U. So by I the type-definable set $f^{-1}(Z)$ is definably connected. Since this set is contained in the $\bigvee$-definable set $f^{-1}(U)$, by saturation there is a definable set D with $f^{-1}(Z) \subset D \subset f^{-1}(U)$. The definably connected component D' of D containing x must contain also y (since it contains $f^{-1}(Z)$). Now it suffices to recall that a definable set is definably connected if and only if it is definably path connected. □

Definition 4.1.6. An open cover $\mathcal{P}$ of a topological space is a STAR REFINEMENT of a cover $\mathcal{Q}$, if for every $P \in \mathcal{P}$, there is a $Q \in \mathcal{Q}$ such that if $P' \in \mathcal{P}$ has a non-empty intersection with P then $P' \subset Q$. In a metric space, and more generally in a uniform space, every open cover has a star refinement. Every Tychonoff space admits a compatible uniform structure, so the existence of star refinements applies to Tychonoff spaces. In particular it applies to any open subset of a locally compact Hausdorff space.

Claim 4. *There are open covers $\mathcal{U}$ of Y and $\mathcal{V}$ of X having the following properties.*

I. $\mathcal{V} \stackrel{def}{=} \{f^{-1}(U)\}_{U \in \mathcal{U}}$ *(so $\mathcal{V}$ is determined by $\mathcal{U}$).*

II. *Each element of $\mathcal{U}$ is path connected, and whenever two elements of $\mathcal{U}$ have a non-empty intersection, their union is contained in some simply connected subset of Y.*

III. *Each element of $\mathcal{V}$ is definably path connected (and $\bigvee$-definable), and whenever two elements of $\mathcal{V}$ have a non-empty intersection, their union is contained in some definably simply connected subset of X.*

Note that it is easy to find a cover $\mathcal{U}$ satisfying II (take a star-refinement of a cover by simply connected sets). Moreover it will turn out that if $\mathcal{U}$ satisfies II and we define $\mathcal{V}$ as in I, then $\mathcal{V}$ satisfies III. However for the moment we cannot assume this fact, so we need to do some more work to find the appropriate $\mathcal{U}$.

Proof of Claim 4. By assumption 4.1.1 we can choose, for each $y \in Y$, a definably simply connected open definable set $C_y \subset X$ containing $f^{-1}(y)$. Since Y is locally simply connected and second countable, we can find a decreasing sequence $\{O_n\}_{n \in \mathbb{N}}$ of simply connected open neighborhoods of y with $\bigcap_n O_n = \{y\}$. Moreover since Y is locally compact, we can arrange so that for each n the set $\overline{O_n}$ is compact and $\overline{O_{n+1}} \subset O_n$. By our assumptions $f^{-1}(\overline{O_{n+1}})$ is type-definable. Since $\bigcap_{n \in \mathbb{N}} f^{-1}(\overline{O_n}) = f^{-1}(y) \subset C_y$, by saturation, there is some n such that $f^{-1}(\overline{O_n}) \subset C_y$. Fix such an n and let $Z_y = O_n$. So $f^{-1}(Z_y) \subset C_y$. We have thus found a cover $\mathcal{Z} \stackrel{def}{=} \{Z_y\}_{y \in Y}$ of Y, such that the preimage of any element of the cover is contained in a definably simply connected set. Now let $\mathcal{U}$ be a star-refinement of $\mathcal{Z}$. We can assume that each element of $\mathcal{U}$ is path-connected, as otherwise we could replace it by its connected components. So $\mathcal{U}$ satisfies II. Finally let $\mathcal{V} = \{f^{-1}(U)\}_{U \in \mathcal{U}}$. Then $\mathcal{V}$ is a star-refinement of $\{f^{-1}(U)\}_{U \in \mathcal{Z}}$. We must prove that $\mathcal{V}$ satisfies III. By Claim 3 each member of $\mathcal{V}$ is definably path-connected. By construction whenever two members of $\mathcal{V}$ intersect, their union is contained in a set of the form C_y, which is a definably simply connected set. ☐

Claim 5. *Any $\bigvee$-definable open subset V of X is a small union of definable open sets.*

Proof of claim. We can write V as a small directed union $\bigcup_{i \in I} D_i$ of definable sets $\{D_i\}_i$. We claim that V is the union $\bigcup_{i \in I} \mathrm{Int}(D_i)$ of the interiors of the sets $\{D_i\}_i$. To this aim, let $x \in V$. Since V is open there is a definable open neighbourhood U of x contained in $\bigcup_{i \in I} D_i$. Hence by saturation there is some $i \in I$ such that $U \subset D_i$. But then $x \in \mathrm{Int}(D_i)$. ☐

Definition 4.1.7. Let $\mathcal{P}$ be a family of subsets of a given set. A set is $\mathcal{P}$-SMALL if it is contained in some member of $\mathcal{P}$. A function is $\mathcal{P}$-small if its image is $\mathcal{P}$-small.

Claim 6.

I. *Given a definable subset D of X, there are finitely many sets in $\mathcal{V}$ whose union covers D. Moreover there is a finite partition of D into definable sets whose closures are $\mathcal{V}$-small.*

II. *Given a definable path a in X there is a subdivision $a = a_1 + \cdots + a_n$ of a such that each a_i is $\mathcal{V}$-small.*

Proof of claim.

I. Since $f(D)$ is a compact subset of Y, it can be covered by finitely many members $U_1, \ldots, U_n$ of $\mathcal{U}$. Then, it follows that $D \subset V_1 \cup \cdots \cup V_n$ where we let $V_i \stackrel{def}{=} f^{-1}(U_i) \in \mathcal{V}$. Each V_i is $\bigvee$-definable and open, so by saturation there are definable open subsets O_i of V_i with $D \subset O_1 \cup \cdots \cup O_n$. By shrinking each O_i if necessary, we may assume that the closure of O_i is included in V_i for all i. It is now sufficient to take a cell decomposition of D compatible with $O_1, \ldots, O_n$.

II. Take a finite partition $\mathcal{P}$ of $\mathrm{Im}(a)$ into definable sets whose closures are $\mathcal{V}$-small. Then take a cell decomposition of $I = \mathrm{dom}(a)$ compatible with $f^{-1}(D)$ for all $D \in \mathcal{P}$. The endpoints of the decomposition yield the desired subdivision of a. □

Definition 4.1.8.

I. Given two definable paths a and b in X with the same endpoints, we say that they are $\mathcal{V}$-CONTIGUOUS (written $a \sim_{\mathcal{V}} b$) if there are definable paths u, v, a', b' in X such that $a = u + a' + v$ and $b = u + b' + v$ and $\mathrm{Im}(a') \cup \mathrm{Im}(b') \subset V$ for some $V \in \mathcal{V}$. Let $\sim_{\mathcal{V}}^{\star}$ be the transitive closure of $\sim_{\mathcal{V}}$. Two paths in the $\sim_{\mathcal{V}}^{\star}$ relation are said to be $\mathcal{V}$-EQUIVALENT.

II. Similarly, dropping all definability conditions, one defines the relation of $\mathcal{U}$-contiguity and $\mathcal{U}$-equivalence between paths in Y.

Claim 7.

I. *Two paths in Y are $\mathcal{U}$-equivalent if and only if they are homotopic (all the homotopies we consider are relative to the endpoints).*

II. *Two definable paths in X are $\mathcal{V}$-equivalent if and only if they are definably homotopic.*

Proof of claim. One direction is trivial (equivalent implies homotopic). We prove the other direction.

I. Implicit in the proof of the van Kampen theorem in [Bro–68]. One argues as follows. Given an homotopy $F\colon I \times I \to Y$ from a to b, we can subdivide the homotopy square $I \times I$ into small squares so that the image of each of them under F is $\mathcal{U}$-small. If n is the number of squares in the subdivision of $I \times I$, it is easy to see that a is $\mathcal{U}$-equivalent to b by a sequence of n contiguity moves. The converse is trivial, since two $\mathcal{U}$-contiguous paths differ for a subpath contained in a simply connected set.

II. Let $F\colon I \times I \to X$ be a definable homotopy between definable paths $a = F(0, -)$ and $b = F(1, -)$ in X. By Claim 6 the image of F can be partitioned into finitely many definable sets $D_1, \ldots, D_n$ whose closures are $\mathcal{V}$-small. Consider a cell decomposition of the homotopy square $I \times I$ compatible with $F^{-1}(D_i)$ for all i. We can then reason as above with the role of the small squares replaced by the cells of the decomposition. For the details see the proof of the o-minimal van Kampen theorem in [BO–02]. □

We are now ready to define $f_*\colon \pi^{def}(X, X) \to \pi(Y, Y)$.

Definition 4.1.9. On the object part of the groupoids we set $f_* = f$. Given a definable path a in X and a path b in Y we say that a corresponds to b if there is a subdivision $a = a_1 + \cdots + a_n$ into $\mathcal{V}$-small definable paths a_i, and a subdivision $b = b_1 + \cdots + b_n$ into $\mathcal{U}$-small paths such that, for all $i \leqslant n$, the endpoints of a_i are mapped by $f \colon X \to Y$ to the respective endpoints of b_i. If a corresponds to b, then we define $f_*([a]) = [b]$.

The proof of the following claim depends only on claims 4 and 7, so it is rather symmetric in X and Y. This observation will be needed later.

Claim 8. f_* *is well defined.*

Proof of claim.
Step 1. First note that two $\mathcal{V}$-small definable paths a and a' in X with the same endpoints are definably homotopic. In fact if $V \in \mathcal{V}$ and $V' \in \mathcal{V}$ contain the images of a and a' respectively, then by Claim 4 we have that $V \cup V'$ is contained in a definably simply connected set, and therefore a is definably homotopic to a'. Similarly two $\mathcal{U}$-small paths b and b' in Y with the same endpoints are homotopic. So $f_*([a]) = [b]$ is well defined at least when a and b are $\mathcal{V}$-small.
Step 2. We next show that if a corresponds to b, then the homotopy class of b is determined by a. So suppose that $a = a_1 + \ldots + a_n$ is a subdivision of a into $\mathcal{V}$-small definable paths, let a_i correspond to b_i (a path in Y), and let $b = b_1 + \ldots + b_n$. By step 1, the homotopy class of each b_j is determined by the corresponding a_j, but we must prove that the homotopy class of b does not depend on the chosen subdivision of a. Since any two subdivisions have a common refinement, it suffices to consider the case in which one of the a_i is further subdivided into $\mathcal{V}$-small paths. So without loss of generality suppose $i = 1$ and let $a_1 = a_1^1 + \cdots + a_1^k$ be a subdivision of a_1 into $\mathcal{V}$-small paths. We must show that b_1 is homotopic to $b_1^1 + \cdots + b_1^k$ where each b_1^j is such that a_1^j corresponds to b_1^j. To this aim let $U \in \mathcal{U}$ be such that $\mathrm{Im}(a_1) \subset f^{-1}(U)$. Then in particular the endpoints of a_1 and of each a_1^j are in $f^{-1}(U)$. Therefore the endpoints of b_1 and of each b_1^j are in U. Reasoning as in step 1 we can then assume that the image of b_1 and of each b_1^j is entirely contained in U since we can reduce to this case replacing each of these paths by a homotopic path. After these reductions, b_1 and $b_1^1 + \cdots + b_1^k$ are two paths in U with the same endpoints, and therefore they are homotopic (since U is contained in a simply connected set).
Step 3. Finally suppose that a is definably homotopic to a', and let us show that b is homotopic to b', where a corresponds to b and a' to b'. By Claim 7 we can assume that a' is $\mathcal{V}$-contiguous to a. So we can write $a = u + z + v$ and $a' = u + z' + v$ with $\mathrm{Im}(z) \cup \mathrm{Im}(z') \subset V$ for some $V \in \mathcal{V}$. Choose subdivisions of a and a' such that z and z' are segments of the chosen subdivisions. By step 2 we may assume that b and b' are obtained from a and a' using these subdivisions. So we can assume that b and b' are $\mathcal{U}$-contiguous. Therefore, by Claim 7, they are homotopic. □

Claim 9. *The definition of* f_* *does not depend on the particular choice of the cover* $\mathcal{U}$ *in Claim 4.*

Proof of claim. Suppose $\mathcal{U}'$ is a refinement of $\mathcal{U}$ still satisfying the conditions in Claim 4. Then clearly if we define $f_*([a])$ using $\mathcal{U}'$ instead of $\mathcal{U}$ we get the same function (since a subdivision of a compatible with the preimages of the sets in $\mathcal{U}'$ is also compatible with the preimages of the sets in $\mathcal{U}$). Now it it suffices to observe that for any two coverings $\mathcal{U}$ and $\mathcal{U}'$ satisfying the conditions in Claim 4 there is a common refinement which still satisfies the conditions (take the connected components of the pairwise intersections of an element of $\mathcal{U}$ and an element of $\mathcal{U}'$). □

Claim 10. f_* *is a morphism and satisfies points 1 and 2 in Theorem 4.1.4.*

Proof of claim. Note that in Definition 4.1.9, if $a = a_0 + a_1$ and a_0, a_1 correspond to b_0 and b_1 respectively, then a corresponds to $b \stackrel{def}{=} b_0 + b_1$. Hence f_* is a morphism. By construction it satisfies 1. We prove 2. Because of Claim 9, by enlarging $\mathcal{U}$ we can suppose it to be a base of the topology of Y. Given an open subset O of Y, we can then express O as the union of a subfamily $\mathcal{U}'$ of $\mathcal{U}$. Consider $\mathcal{V}' \stackrel{def}{=} \{f^{-1}(U)\}_{U \in \mathcal{U}'}$. Given a path a in $f^{-1}(O)$, the construction of $[b] = f_*([a])$ can be carried out subdividing a into $\mathcal{V}'$-small paths and associating to each of them a $\mathcal{U}'$-small path in Y. The image of b is clearly a subset of O. □

Claim 11. f^* *is the unique morphism satisfying 1 and 2 in Theorem 4.1.4.*

Proof of claim. Let O be a simply connected open subset of Y. If a is a definable path in X with $\operatorname{Im} a \subseteq f^{-1}(O)$, condition 2 in Theorem 4.1.4 forces $f_*([a])$ to be of the form $[b]$ for some path b with $\operatorname{Im} b \subseteq O$. Since O is simply connected, and the endpoints of b are the images of the endpoints of a, then $[b]$ is completely determined. So f_* is determined on the paths satisfying $\operatorname{Im}(a) \subseteq f^{-1}(O)$ for some open simply connected subset $O \subset Y$. By Claim 6 we can reduce to this situation by subdividing the paths. □

Claim 12. *Let* $\Gamma \subset X$ *be such that* $f{\restriction}_\Gamma \colon \Gamma \to Y$ *is injective. Then the restriction of* f_* *to* $\pi^{def}(X, \Gamma)$ *is an isomorphism onto* $\pi(Y, f(\Gamma))$.

Proof of claim. Choose a right inverse $\psi \colon Y \to X$ of $f \colon X \to Y$ extending $(f{\restriction}_\Gamma)^{-1}$. We define an inverse $\psi_* \colon \pi(Y, f(\Gamma)) \to \pi^{def}(X, \Gamma)$ of f_* as follows. Given $[b] \in \pi(Y, f(\Gamma))$, consider a subdivision $b = b_1 + \cdots + b_m$ of b into $\mathcal{U}$-small paths. For each i, let y_{i-1} and y_i be the endpoints of b_i, and let a_i be a $\mathcal{V}$-small path in X from $x_{i-1} \stackrel{def}{=} \psi(y_{i-1})$ to $x_i \stackrel{def}{=} \psi(y_i)$. Then $f_*([a_i]) = [b_i]$. Finally define $\psi_*([b]) = [a_1 + \cdots + a_m]$. Note that ψ_* is well defined by the same argument that proves that f_* is well defined. Indeed in that proof we only used claims 4 and 7, so we can repeat the argument with the roles of X and Y interchanged. We also claim that $\psi_* = (f_*{\restriction}_{\pi^{def}(X,\Gamma)})^{-1}$. In fact, by inspection of the definitions, the same pair of subdivisions $a = a_1 + \cdots + a_m$ and $b = b_1 + \cdots + b_m$ witnesses both $f_*([a]) = [b]$ and $\psi_*([b]) = [a]$ simultaneously. □

The proof of Theorem 4.1.4 is now complete, except for point 4, which is clear from the construction and left to the reader. □

4.2 Equivariance

We now specialize the results of the previous section to the case of spaces that carry a group structure. So let G be a definably compact group. Let $p^G \colon G \to G/G^{00}$ be the projection map, and let

$$p^G_* \colon \pi^{def}(G, G) \to \pi(G/G^{00}, G/G^{00})$$

be the induced morphism of groupoids as in Theorem 4.1.4.

Definition 4.2.1. The group G acts on $\pi^{def}(G, G)$ by $x \cdot [a] = [x \cdot a]$, where $x \in G$ and a is a definable path in G. Similarly we have an action of G/G^{00} on $\pi(G/G^{00}, G/G^{00})$ given by $y \cdot [b] = [y \cdot b]$. Finally we also have an action of G on $\pi(G/G^{00}, G/G^{00})$ sending $(x, [b])$ to $p^G(x) \cdot [b]$, where $x \in G$ and b is a path in G/G^{00}.

Theorem 4.2.2. *The map $p^G_* \colon \pi^{def}(G, G) \to \pi(G/G^{00}, G/G^{00})$ is equivariant under the action of G, namely for each $x \in G$ and $[a] \in \pi^{def}(G, G)$, we have*

$$p^G_*(x \cdot [a]) = p^G(x) \cdot p^G_*([a])$$

Proof. Let $x \in G$ and consider the map

$$\begin{aligned} p_x \colon \pi^{def}(G, G) &\to \pi(G/G^{00}, G/G^{00}) \\ [a] &\mapsto p^G(x^{-1}) \cdot p^G_*(x \cdot [a]) \end{aligned}$$

It is easy to check that p_x is a groupoid morphism satisfying the conditions 1 and 2 of Theorem 4.1.4. Hence, by uniqueness, $p_x = p^G_*$ for all $x \in G$. It follows that p^G_* is equivariant. □

4.3 Functors and natural transformations

In this section we establish the functoriality properties of the morphisms p^G_*. We can regard the correspondence $G \mapsto \pi^{def}(G, G)$ as the object part of a functor π^{def} from the category of definably compact groups (and definable group homomorphisms) to the category of groupoids (and groupoid homomorphisms). Similarly we have a functor $\pi_1^{def} \colon G \mapsto \pi_1^{def}(G)$ from definably compact groups to groups. Finally we have a functor $F \colon G \to G/G^{00}$ from definably compact groups to compact Lie groups (and Lie homomorphisms). By Theorem 4.1.4, for G a definably compact group, the projection $p^G \colon G \to G/G^{00}$ induces a morphism

$$p^G_* \colon \pi^{def}(G, G) \to \pi(G/G^{00}, G/G^{00})$$

Theorem 4.3.1. *Let G be a definably compact group. The family $p_* = (p^G_*)_G$ is a natural transformation of the functor*

$$\pi^{def} \colon G \mapsto \pi^{def}(G, G)$$

to the functor

$$\pi \circ F \colon G \mapsto \pi(G/G^{00}, G/G^{00})$$

In other words, given a definable morphism $f\colon G \to G'$ *we have a commutative diagram in the category of groupoids:*

$$\begin{array}{ccc} \pi^{def}(G,G) & \xrightarrow{\pi^{def}(f)} & \pi^{def}(G',G') \\ \Big\downarrow p_*^G & & \Big\downarrow p_*^{G'} \\ \pi(G/G^{00}, G/G^{00}) & \xrightarrow{\pi(F(f))} & \pi(G'/G'^{00}, G'/G'^{00}) \end{array}$$

where $F(f)\colon G/G^{00} \to G'/G'^{00}$ *is the induced Lie homomorphism.*

Proof. Consider an open cover $\mathcal{U}'$ of G'/G'^{00} by simply connected sets. Now consider an open cover $\mathcal{U}$ of G/G^{00} by simply connected sets which refines $\{F(f)^{-1}(U')\}_{U' \in \mathcal{U}'}$. Using these covers in Definition 4.1.9, the commutativity of the diagram follows immediately. □

Remark 4.3.2. Let G be a definably compact group. By [HPP–08a] the subgroup G^{00} is torsion free (see [Ber–07] for the non-abelian case), so if Γ is a finite subgroup of G, then p^G maps Γ isomorphically onto its image in G/G^{00}.

Corollary 4.3.3. *If* Γ *is a finite subgroup of a definably compact group* G*, the restriction of* p_*^G *to* $\pi^{def}(G,\Gamma)$ *is an isomorphism onto* $\pi(G/G^{00}, p^G(\Gamma))$.

4.4 Locally definable groups

A locally definable group is a countable union of definable sets, equipped with a group operation whose restriction to each definable set is definable. This definition is equivalent to the one in [Edm–06], and it is slightly more restrictive than the notion of $\bigvee$-definable group in [PS–99] where only the cardinality of the set of parameters is assumed to be countable (the two notions coincide if the language is countable). The restriction to countable unions, is useful in Proposition 4.4.1 below. As usual we assume that the underlying o-minimal structure $\mathcal{M}$ is sufficiently saturated (ω_1-saturated will suffice here). A locally definable function is a function between $\bigvee$-definable sets whose restriction to each definable set is definable. A locally definable homomorphism between locally definable groups is a homomorphism which is locally definable.

Proposition 4.4.1 ([Edm–06, Theorem 4.2])**.** *Let* $f\colon B \to C$ *be a surjective locally definable homomorphism of locally definable groups. Then there is a locally definable section* $s\colon C \to B$*, namely a locally definable function such that* $f \circ s$ *is the identity on* C.

Following [Edm–06, Definition 3.1] we say that subgroup A of a locally definable group B is compatible if it intersects every definable subset X of B into a definable set. (In the quoted paper it is actually required that X be open in the topology induced by the group structure, but this yields an equivalent definition since every definable subset is contained in an open definable subset.)

Proposition 4.4.2 ([Edm–06, Lemma 3.3] and [Edm–06, Theorem 4.2]). *A subgroup* $A < B$ *of a locally definable group* B *is compatible if and only if it is the kernel of a locally definable surjective homomorphism* $f \colon B \to C$ *between locally definable groups.*

If $1 \to A \to B \to C \to 1$ is an exact sequence of locally definable groups and locally definable homomorphisms, we denote C by $B/_A$: our notation is unambiguous, since by Proposition 4.4.1 the quotient C is unique up to locally definable isomorphism. By [Edm–06, Remark 4.7], the third isomorphism Theorem $B/_A \cong (B/_L)/(A/_L)$ holds in the category of locally definable groups and locally definable homomorphisms. More precisely we have:

Proposition 4.4.3 ([Edm–06, Remark 4.7]). *Consider an exact sequence*

$$1 \to A \to B \to C \to 1$$

of locally definable groups and locally definable homomorphisms and suppose that the map $A \to B$ *is the inclusion. Let* L *be a normal subgroup of* B *contained in* A *as a compatible subgroup (it follows that* L *is compatible also in* B*). Then there is an induced exact sequence of locally definable groups*

$$1 \to A/_L \to B/_L \to C \to 1$$

The proposition is contained in the cited result of [Edm–06], however for the reader's convenience we sketch a proof. The homomorphism $A \to A/_L$ admits a locally definable section $s \colon A/_L \to A$. The homomorphism $A/_L \to B/_L$ can be obtained as the composition

$$A/_L \xrightarrow{s} A \to B \to B/_L$$

hence it is locally definable. Similarly we obtain the locally definable homomorphism $B/_L \to C$.

4.5 Universal cover

Let G be a definable group. The (o-minimal) universal cover $\widetilde{G}$ of G has been studied in [EE–07], where in particular it is shown that there is a locally definable surjective homomorphism $f \colon \widetilde{G} \to G$ of locally definable groups whose kernel is isomorphic to $\pi_1^{def}(G)$. We can injectively embed $\widetilde{G}$ into the fundamental groupoid $\pi^{def}(G, G)$ in the following way. Given a definable path α in G starting at the identify, let $\tilde{\alpha}$ be its unique lifting to $\widetilde{G}$ starting at the identity, and let $\tilde{\alpha}(1) \in \widetilde{G}$ be its endpoint. Any element x of $\widetilde{G}$ is of the form $\tilde{\alpha}(1)$, and since $\tilde{\alpha}(1)$ depends only on the definable homotopy class $[\alpha]$, we have an injective function $\iota \colon \widetilde{G} \to \pi^{def}(G, G)$ sending x to $[\alpha]$. So we have $\iota(\widetilde{G}) \subset \pi^{def}(G, G)$. Notationally we can suppose that ι is the inclusion, namely we can define

$$\widetilde{G} \subset \pi^{def}(G, G)$$

as the subset consisting of all the definable homotopy classes of paths starting at the identity e_G. This identification is only a matter of notational convenience, since literally $\widetilde{G}$ is a subset of some cartesian product of $\mathcal{M}$ (being a locally definable group), while $\iota(\widetilde{G}) \subset \pi^{def}(G,G)$ is not. With our identification the group operation of $\widetilde{G} \subset \pi^{def}(G,G)$ is defined by $[a]\cdot[b] = [a] + [a(1)\cdot b]$ where $a(1)$ is the endpoint of the definable path a and $+$ denotes the operation of the groupoid (induced by concatenation of paths). The covering homomorphism $\widetilde{G} \to G$ sends $[a]$ to $a(1)$. The topology on $\widetilde{G} \subset \pi^{def}(G,G)$ is defined as follows: if U is a definably simply connected open subset of G, then the set of all $[a] \in \widetilde{G}$ such that $\mathrm{Im}(a) \subset U$ is a basic open neighbourhood of $e_{\widetilde{G}}$ in $\widetilde{G}$. By left translation we obtain the basic open neighbourhoods around the other points.

With analogous definitions the universal cover of the real Lie group G/G^{00} can be identified with the subset

$$\widetilde{G/G^{00}} \subset \pi(G/G^{00}, G/G^{00})$$

consisting of all the homotopy classes of paths starting at $e_{G/G^{00}} \in G/G^{00}$.

The purpose of these identifications is to be able to define an homomorphism from the universal cover of G to the universal cover of G/G^{00}.

Definition 4.5.1. The morphism of groupoids

$$p^G_* : \pi^{def}(G,G) \to \pi(G/G^{00}, G/G^{00})$$

given by Theorem 4.1.4 induces a map

$$\widetilde{p}^G : \widetilde{G} \to \widetilde{G/G^{00}}$$

by restriction, namely $\widetilde{p}^G([a]) \stackrel{def}{=} p^G_*([a])$. Since p^G_* is equivariant (Theorem 4.2.2), the map $\widetilde{p}^G$ is a morphism of groups.

Theorem 4.5.2. *Given a definably compact definably connected definable group G, the kernel of $\widetilde{p}^G$ is isomorphic to G^{00} via the map $[a] \mapsto a(1)$. So we have a commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G^{00} & \longrightarrow & \widetilde{G} & \longrightarrow & \widetilde{G/G^{00}} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G^{00} & \longrightarrow & G & \longrightarrow & G/G^{00} & \longrightarrow & 1 \end{array}$$

Proof. Let $[a] \in \widetilde{G}$ be in the kernel (where a is a definable path in G starting at the identity). Then $p^G_*([a]) = [b]$ where b is a contractible loop at $e_{G/G^{00}}$. Since p^G must send the endpoints of a to the endpoints of b (namely to the identity of G/G^{00}), it follows that $a(1) \in G^{00}$. So we have a well defined function $[a] \mapsto a(1)$ from $\ker(\widetilde{p}^G)$ to G^{00} which moreover is a group homomorphism. We must prove that it is an isomorphism.

(Surjectivity) Given $x \in G^{00}$ we must find $[a] \in \ker(\widetilde{p}^G)$ with $a(1) = x$. To this aim let U be a a simply connected open neighbourhood of $e_{G/G^{00}}$ in G/G^{00}. By Remark 4.1.5, its preimage V in G is a ($\bigvee$-definable) definably simply connected subset of G. In particular V is definably path connected, so there is a definable path a in U from e_G to x. By Theorem 4.1.4 there is a path b in U with $p^G_*([a]) = [b]$. Since $x \in G^{00}$, the endpoint $p^G(x)$ of b is the identity of G/G^{00}, namely b is a loop. Moreover since the image of b is contained in the simply connected set U, we know that $[b]$ is the identity of G/G^{00}, and therefore $[a] \in \ker(\widetilde{p}^G)$.

(Injectivity) Let $[a] \in \ker(\widetilde{p}^G)$ and suppose $a(1) = e_G$ (namely a is a loop). By Theorem 4.3.2, the projection p^G_* sends $\pi_1^{def}(G)$ bijectively to $\pi_1(G/G^{00})$. Since under this map $[a]$ goes to the identity, it follows that a is definably contractible, namely $[a]$ is the identity of $\widetilde{G}$. □

We will obtain similar results for the finite extensions of G/G^{00}.

Theorem 4.5.3. *Let G be a definably compact definably connected definable group. Given an extension of connected Lie groups $f\colon \mathbf{H} \to G/G^{00}$ with a finite kernel, there is a definable group extension $\pi\colon H \to G$ of G such that $H/H^{00} \cong \mathbf{H}$ (as coverings of G/G^{00}). We thus obtain a commutative diagram:*

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & \mathbf{H} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f} \\ G & \xrightarrow{p} & G/G^{00} \end{array}$$

where $\varphi\colon H \to \mathbf{H}$ is the composition of the projection $H \to H/H^{00}$ with the isomorphism $H/H^{00} \cong \mathbf{H}$.

Proof. Let L be the image of the homomorphism $\pi_1(f)\colon \pi_1(\mathbf{H}) \to \pi_1(G/G^{00})$. Then by classical results (see [Hat–02, Proposition 1.36 and 1.37]) we have an exact sequence

$$1 \to \pi_1(G/G^{00})/L \to \mathbf{H} \to G/G^{00} \to 1$$

Let $L' < \pi_1^{def}(G)$ be the preimage of L under the isomorphism between $\pi_1^{def}(G)$ and $\pi_1(G/G^{00})$ of Theorem 4.3.1. By [EE–07] we have an exact sequence of locally definable groups

$$1 \to \pi_1^{def}(G) \to \widetilde{G} \to G \to 1$$

so we can identify $\pi_1^{def}(G)$ with a compatible subgroup of $\widetilde{G}$. Since $\pi_1^{def}(G)$ is discrete, the compatibility condition implies that $\pi_1^{def}(G)$ (hence also its subgroup L') intersects every definable subset of $\widetilde{G}$ into a finite set. Moreover since $L' < \pi_1^{def}(G)$ and $\pi_1^{def}(G)$ is contained in the center of $\widetilde{G}$ ([Edm–06, Corollary 3.16]), L' is normal in $\widetilde{G}$. By the third isomorphism theorem we obtain an exact sequence

$$1 \to \pi_1^{def}(G)/L' \to \widetilde{G}/L' \xrightarrow{p} G \to 1$$

of locally definable groups. Since the kernel $\Gamma' \cong \pi_1^{def}(G)/L' \cong \pi_1(G/G^{00})/L$ is finite and G is definable, the locally definable group $H \stackrel{def}{=} \widetilde{G}/L'$ is actually definable. Note that the image of $\pi_1(p)\colon \pi_1^{def}(H) \to \pi_1^{def}(G)$ is L'. We have an induced homomorphism $F(p)\colon H/H^{00} \to G/G^{00}$, and by Theorem 4.3.1 the image of $\pi_1(F(p))\colon \pi_1(H/H^{00}) \to \pi_1(G/G^{00})$ is L, namely it coincides with the image of $\pi_1(f)\colon \pi_1(\mathbf{H}) \to \pi_1(G/G^{00})$. Since the covering spaces are classified by the subgroups of the fundamental group ([Hat–02, Proposition 1.36 and 1.37]), the two coverings $f\colon \mathbf{H} \to G/G^{00}$ and $F(p)\colon H/H^{00} \to G/G^{00}$ are isomorphic. □

4.6 Definably compact semisimple groups

Let G be a definably compact definably connected semisimple group. In this section we show that the (Lie-)isomorphism type of G/G^{00} determines the definable isomorphism type of G.

Lemma 4.6.1. *Work in an o-minimal expansion of* $\mathbb{R}$. *Let* $f\colon X \to B$ *be a definable continuous map. Let* $p\colon E \to B$ *be a definable covering map. Let* $\widetilde{f}\colon X \to E$ *be a lifting of* f *(i.e. a continuous function, not necessarily definable, such that* $p \circ \widetilde{f} = f$*). Then* $\widetilde{f}$ *is definable.*

Proof. By definition of definable covering, we have a definable finite cover $\mathcal{U}$ of B by definably connected definable open sets such that, for any $U \in \mathcal{U}$, the preimage $p^{-1}(U)$ is a finite disjoint union of definably connected open sets on each of which p is an homeomorphism onto U. Fix $U \in \mathcal{U}$ and let $E_1, \cdots, E_m$ be the definably connected components of $p^{-1}(U)$ and $X_1, \cdots, X_n$ be the definably connected components of $f^{-1}(U)$. Note that all these sets are definable. Fix an $i \in \{1, \cdots, n\}$. Since we are working over $\mathbb{R}$, a definably connected set is connected. So, by continuity of f, there is a $j \in \{1, \cdots, m\}$ such that $\widetilde{f}(X_i) \subset E_j$. Hence $(\widetilde{f}{\restriction}_{X_i})(x) = y$ if and only if $x \in X_i \wedge y \in E_j \wedge f(x) = p(y)$. This proves that $\widetilde{f}{\restriction}_{X_i}$ is definable, and the definability of $\widetilde{f}$ follows observing that the same hold for any $U \in \mathcal{U}$ and any i. □

Fact 4.6.2 ([EJP–07, Theorem 3.1] or [HPP–08b, Theorem 4.4(ii)])**.** *For any semisimple definable group* G_1*, there is a group* G_2*, semialgebraic without parameters, definably isomorphic to it.*

Lemma 4.6.3. *Let* G_1 *and* G_2 *be definably connected semialgebraic semisimple groups defined over* $\mathbb{R}$. *By [Pil–88],* $G_1(\mathbb{R})$ *and* $G_2(\mathbb{R})$ *have a natural Lie group structure. Suppose that* $f\colon G_1(\mathbb{R}) \to G_2(\mathbb{R})$ *is a Lie isomorphism. Then* f *is semialgebraic over* $\mathbb{R}$.

Proof. We first prove the result under the additional assumption that G_1 and G_2 are centerless. The isomorphism $f\colon G_1(\mathbb{R}) \to G_2(\mathbb{R})$ induces an isomorphism $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ of the corresponding Lie algebras. Since we are in the centerless case, the adjoint representation $\mathrm{Ad}_{G_1}\colon G_1(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{g}_1)$ is an isomorphism onto $\mathrm{Aut}^0(\mathfrak{g}_1)$ and similarly for $G_2(\mathbb{R})$. Fixing a basis of the vector spaces $\mathfrak{g}_1$ and $\mathfrak{g}_2$, we can consider Ad_{G_1} and Ad_{G_2} as semialgebraic maps. Let $\widetilde{\phi}\colon \mathrm{Aut}^0(\mathfrak{g}_1) \to \mathrm{Aut}^0(\mathfrak{g}_2)$ be the isomorphism induced by ϕ. Then $f = \mathrm{Ad}_{G_1} \circ \widetilde{\phi} \circ {\mathrm{Ad}_{G_2}}^{-1}$ and therefore f is semialgebraic over $\mathbb{R}$.

To reduce the general case to the centerless case we use the fact that $G_1/Z(G_1)$ and $G_2/Z(G_2)$ are centerless. Clearly f induces an isomorphism $g\colon G_1/Z(G_1) \to G_2/Z(G_2)$. By the centerless case g is semialgebraic. By Lemma 4.6.1, we have that f is semialgebraic. □

Remark 4.6.4. In the above lemma we cannot ensure that f is semialgebraic over $\mathbb{R}^{alg}$ even assuming that G_1 and G_2 are semialgebraic over $\mathbb{R}^{alg}$. In fact let $G_1 = G_2 = SO_3$. The group of inner automorphisms of SO_3 is non-trivial and connected, so it has the cardinality of the continuum. Therefore there is some inner automorphism $f\colon SO_3 \to SO_3$ which is not definable over $\mathbb{R}^{alg}$.

Theorem 4.6.5. *Let G_1 and G_2 be definably compact definably connected semisimple definable groups. Suppose that there is a Lie isomorphism $\psi\colon G_1/G_1^{00} \to G_2/G_2^{00}$. Then there is a definable isomorphism $f\colon G_1 \to G_2$. If the o-minimal structure is sufficiently saturated we can consider the projections $p^{G_1}\colon G_1 \to G_1/G_1^{00}$ and $p^{G_2}\colon G_2 \to G_2/G_2^{00}$ and we can choose f so that $p^{G_2} \circ f = \psi \circ p^{G_1}$.*

Proof. By Fact 4.6.2 we may assume G_1 and G_2 to be semialgebraic without parameters. So it makes sense to consider the groups $G_1(\mathbb{R})$ and $G_2(\mathbb{R})$. If $\mathcal{M}$ is sufficiently saturated there is an elementary embedding of $\mathbb{R}$ into R (in the language of fields) and there is a surjective homomorphism $G_1(\mathcal{M}) \to G_1(\mathbb{R})$ (given by the *standard part map*) whose kernel is $G_1^{00} = G_1^{00}(M)$: see [Pil–04]. Similarly for G_2. So $G_1/G_1^{00} \cong G_1(\mathbb{R})$ and $G_2/G_2^{00} \cong G_2(\mathbb{R})$ (with the logic topology). Hence we have a Lie isomorphism $\psi'\colon G_1(\mathbb{R}) \to G_2(\mathbb{R})$ induced by ψ. By Lemma 4.6.3 ψ' is semialgebraic over $\mathbb{R}$. The same formula defines an isomorphism $f\colon G_1(M) \to G_2(M)$ with $p^{G_2} \circ f = \psi \circ p^{G_1}$. If $\mathcal{M}$ is not sufficiently saturated, then we can go to a saturated extension $\mathcal{M}'$ to get an $\mathcal{M}'$-definable isomorphism $f\colon G_1(\mathcal{M}') \to G_2(\mathcal{M}')$ as above, and therefore (quantifying over the parameters) also an $\mathcal{M}$-definable isomorphism from $G_1(M)$ to $G_2(M)$. □

4.7 Definably compact abelian groups

In this section we try to understand, in the abelian case, up to which extent G/G^{00} determines G. It is known that there are definably compact definably connected abelian groups G_1 and G_2 of the same dimension (hence with $G_1/G_1^{00} \cong G_2/G_2^{00}$) which are not definably isomorphic ([Str–94a, PS–99]). However by 3.1.6 any two definably compact definably connected abelian groups of the same dimension are definably homotopy equivalent. The same proof yields the following:

Lemma 4.7.1. *Let G_1 and G_2 be definably compact definably connected abelian groups of the same dimension n. (So $\pi_1(G_1) \cong \pi_1(G_2) \cong \mathbb{Z}^n$ by [EO–04].) Let $\theta\colon \pi_1(G_1) \to \pi_1(G_2)$ be an isomorphism. Then there is a definable continuous map $f\colon G_1 \to G_2$ with $\pi_1(f) = \theta$ and $f(e_{G_1}) = e_{G_2}$. Moreover, any such map f is a definable homotopy equivalence.*

Proof.
Special case. Suppose that G_1 is a direct product of 1-dimensional definable subgroups. Choose free generators $[a_1], \dots, [a_n]$ of $\pi_1^{def}(G_1)$ such that each $x \in G_1$ can be written uniquely in the form $x = a_1(t_1) + \dots + a_n(t_n)$ with $0 \leqslant t_i < 1$. Choose definable loops $b_1, \dots, b_n$ in G_2 such that $[b_1], \dots, [b_n] \in \pi_1^{def}(G_2)$ are the images of $[a_1], \dots, [a_n]$ under θ. Define $f(x) = b_1(t_1) + \ldots + b_n(t_n)$. Then clearly $\pi_1^{def}(f) = \theta$. Since the higher definable homotopy groups of G_1 and G_2 are zero by 3.1.5, f is a definable homotopy equivalence by the o-minimal version of Whitehead theorem in [BO–09].
General case. We reduce to the special case as follows. Remember that T is a definably compact definably connected one-dimensional abelian group, and T^n be the direct product of n copies of T, where $n = \dim G_1$. By [EO–04], we have $\pi_1^{def}(G_1) \cong \pi_1(T^n) \cong \mathbb{Z}^n$. Choose an isomorphism $\lambda\colon \pi_1^{def}(T^n) \to \pi_1^{def}(G_1)$.

Then $\theta \circ \lambda \colon \pi_1^{def}(T^n) \to \pi_1^{def}(G_2)$ is an isomorphism. By the special case we get definable homotopy equivalences g and h with $\pi_1(g) = \lambda$ and $\pi_1(h) = \theta \circ \lambda$. Let g' be a definable homotopy inverse of g. So $f \stackrel{def}{=} h \circ g'$ satisfies $\pi_1(f) = \theta$. □

To improve on the above result we need a definition.

Definition 4.7.2. Let G_1 and G_2 be definable groups. Given a subgroup $\Gamma_1 < G_1$ we say that a definable map $f \colon G_1 \to G_2$ is a Γ_1-equivariant definable homotopy equivalence if $f \restriction_{\Gamma_1}$ is an isomorphism onto its image Γ_2, and f admits a definable homotopy inverse f' such that the following holds:

I. $f(e_{G_1}) = e_{G_2}$ and $f(cx) = f(c)f(x)$ for any $c \in \Gamma_1$ and $x \in G_1$;
II. $f'(e_{G_2}) = e_{G_1}$ and $f'(c'x') = f'(c')f'(x')$ for any $c' \in \Gamma_2$ and $x' \in G_2$;
III. there is a definable homotopy $h \colon I \times G_1 \to G_1$ relative to Γ_1 between $f' \circ f$ and the identity map on G_1 such that $h_t(cx) = ch_t(x)$ for any $c \in \Gamma_1$, $x \in G_1$, and $t \in I$;
IV. there is a definable homotopy $h' \colon I \times G_2 \to G_2$ relative to Γ_2 between $f \circ f'$ and the identity map on G_2 such that $h'_t(c'x') = c'h'_t(x')$ for any $c' \in \Gamma_2$, $x' \in G_2$, and $t \in I$.

Note that $f' \restriction_{\Gamma_2}$ is the inverse of $f \restriction_{\Gamma_1}$.

To prove the existence of Γ-equivariant homotopy equivalences we need some preliminary results. The following lemma says that given a definable covering map $p \colon E \to B$ we can always lift a definable map $f \colon X \to B$ to a map $\widetilde{f} \colon X \to E$ provided there are no obstructions coming from the fundamental group.

Lemma 4.7.3. *Let $p \colon E \to B$ be a definable covering map, with B definably connected. And let $f \colon X \to B$ be a definable map from a definable definably connected set X to B. Fix base points $e_0 \in E$, $b_0 \in B$ and $x_0 \in X$ with $f(x_0) = p(e_0) = b_0$. Consider the homomorphisms $\pi_1(p)$ and $\pi_1(f)$ induced by p and f on the definable fundamental groups. If $\operatorname{Im} \pi_1^{def}(f) \subset \operatorname{Im} \pi_1^{def}(p)$ then there is a unique definable map $\widetilde{f} \colon X \to E$ lifting f (i.e. such that $p \circ \widetilde{f} = f$) with $\widetilde{f}(x_0) = e_0$.*

Proof. The proof of the corresponding classical result (see [Spa–66, Theorem 2.4.5]) can be adapted to the o-minimal category thanks to the definable homotopy lifting property 2.4.2. More precisely, for each $x \in X$ choose, uniformly in x, a definable path a_x from x_0 to x in X. Then $b_x \stackrel{def}{=} f \circ a_x$ is a definable path in B. Let $\widetilde{b_x}$ be its (unique) lifting to a definable path in E with starting point e_0. Define $\widetilde{f}(x)$ as the final point of $\widetilde{b_x}$. This is independent on the choice of the paths and works. □

Theorem 4.7.4. *Let G_1 and G_2 be definably compact definably connected abelian groups. Let*

$$\psi \colon G_1/G_1^{00} \to G_2/G_2^{00}$$

be an isomorphism of Lie groups. Let Γ_1 be a finite subgroup of G_1. Then there is a Γ_1-equivariant definable homotopy equivalence $f^{G_1} \colon G_1 \to G_2$ which agrees with ψ on Γ_1 (more precisely for each $c \in \Gamma_1$ we have $f^{G_1}(c)G_2^{00} = \psi(cG_1^{00})$).

Proof. To simplify notations let $\mathbf{G_1} = G_1/G_1^{00}$ and $\mathbf{G_2} = G_2/G_2^{00}$. Since Γ_1 is finite, the projection $G_1 \to \mathbf{G_1}$ maps Γ_1 isomorphically onto its image

$$\mathbf{\Gamma_1} \overset{def}{=} \Gamma_1 G_1^{00}/G_1^{00} < \mathbf{G_1}$$

Let $\mathbf{\Gamma_2} = \psi(\mathbf{\Gamma_1}) < \mathbf{G_2}$ and let Γ_2 be the unique finite subgroup of G_2 which is mapped to $\mathbf{\Gamma_2}$ under the projection $G_2 \to \mathbf{G_2}$.

Passing to the quotient, the isomorphism $\psi\colon \mathbf{G_1} \to \mathbf{G_2}$ induces an isomorphism $\phi\colon \mathbf{G_1}/\mathbf{\Gamma_1} \to \mathbf{G_2}/\mathbf{\Gamma_2}$ making the following diagram commute (where the vertical arrows are the projections):

$$\begin{array}{ccc} \mathbf{G_1} & \xrightarrow{\psi} & \mathbf{G_2} \\ \downarrow & & \downarrow \\ \mathbf{G_1}/\mathbf{\Gamma_1} & \xrightarrow{\phi} & \mathbf{G_2}/\mathbf{\Gamma_2} \end{array}$$

Since $\mathbf{\Gamma_1}$ and $\mathbf{\Gamma_2}$ are mapped to the identity of $\mathbf{G_1}/\mathbf{\Gamma_1}$ and $\mathbf{G_2}/\mathbf{\Gamma_2}$ respectively, we obtain an induced commutative diagram in the category of groupoids:

$$\begin{array}{ccc} \pi(\mathbf{G_1}, \mathbf{\Gamma_1}) & \xrightarrow{\pi(\psi)} & \pi(\mathbf{G_2}, \mathbf{\Gamma_2}) \\ \downarrow & & \downarrow \\ \pi(\mathbf{G_1}/\mathbf{\Gamma_1}, e) & \xrightarrow{\pi(\phi)} & \pi(\mathbf{G_2}/\mathbf{\Gamma_2}, e) \end{array} \tag{*}$$

where $\mathbf{G_1}/\mathbf{\Gamma_1}$ can be naturally identified with $(G_1/\Gamma_1)/(G_1/\Gamma_1)^{00}$ and similarly on the G_2 side. By Theorem 4.3.1 and Remark 4.3.2 we have commutative diagrams

$$\begin{array}{ccc} \pi^{def}(G_1, \Gamma_1) & \longrightarrow & \pi^{def}(G_1/\Gamma_1, e) \\ \downarrow p_*^{G_1} & & \downarrow p_*^{G_1/\Gamma_1} \\ \pi(\mathbf{G_1}, \mathbf{\Gamma_1}) & \longrightarrow & \pi(\mathbf{G_1}/\mathbf{\Gamma_1}, e) \end{array} \qquad \begin{array}{ccc} \pi^{def}(G_2, \Gamma_2) & \longrightarrow & \pi^{def}(G_2/\Gamma_2, e) \\ \downarrow p_*^{G_2} & & \downarrow p_*^{G_2/\Gamma_2} \\ \pi(\mathbf{G_2}, \mathbf{\Gamma_2}) & \longrightarrow & \pi(\mathbf{G_2}/\mathbf{\Gamma_2}, e) \end{array} \tag{**}$$

where the vertical arrows are isomorphisms and the horizontal arrows are induced by the quotient maps. By the composition of diagrams (*) and (**) we obtain a commutative diagram in the category of groupoids

$$\begin{array}{ccc} \pi^{def}(G_1, \Gamma_1) & \xrightarrow{\theta} & \pi^{def}(G_2, \Gamma_2) \\ \downarrow & & \downarrow \\ \pi^{def}(G_1/\Gamma_1, e) & \xrightarrow{\lambda} & \pi^{def}(G_2/\Gamma_2, e) \end{array}$$

where θ is uniquely determined by the equation

$$\pi(\psi) \circ p_*^{G_1} = p_*^{G_2} \circ \theta$$

For each $c \in \Gamma_1$ and $[a] \in \pi(G_1, \Gamma_1)$ we have $\theta(c \cdot [a]) = \overline{\psi}(c) \cdot \theta([a])$, where $\overline{\psi}\colon \Gamma_1 \to \Gamma_2$ is defined by $\overline{\psi}(c)G_1^{00} = \psi(cG_1^{00})$. In fact, using Theorem 4.2.2 we can verify that $\overline{\psi}(c) \cdot \theta([a])$ meets the definition of $\theta(c \cdot [a])$:

$$\begin{aligned}
\pi(\psi) \circ p_*^{G_1}(c \cdot [a]) &= \pi(\psi)(p^{G_1}(c) \cdot p_*^{G_1}([a])) \\
&= \psi \circ p^{G_1}(c) \cdot \pi(\psi) \circ p_*^{G_1}([a]) \\
&= \psi \circ p^{G_1}(c) \cdot p_*^{G_2} \circ \theta([a]) \\
&= p^{G_2} \circ \overline{\psi}(c) \cdot p_*^{G_2} \circ \theta([a]) = p_*^{G_2}(\overline{\psi}(c) \cdot \theta([a]))
\end{aligned}$$

Using Lemma 4.7.1 we can obtain a definable homotopy equivalence $f\colon G_1/\Gamma_1 \to G_2/\Gamma_2$ with $\pi_1(f) = \lambda$ and $f(e_{G_1/\Gamma_1}) = e_{G_2/\Gamma_2}$. So, by Lemma 4.7.3 there is a definable continuous map $f^{G_1}\colon G_1 \to G_2$ with $f^{G_1}(e_{G_1}) = e_{G_2}$ making the following diagram commute.

$$\begin{array}{ccc}
G_1 & \xrightarrow{f^{G_1}} & G_2 \\
\downarrow p & & \downarrow p' \\
G_1/\Gamma_1 & \xrightarrow{f} & G_2/\Gamma_2
\end{array}$$

It remains to be shown that f^{G_1} is a Γ_1-equivariant definable homotopy equivalence. The equation $f^{G_1}(c \cdot -) = f^{G_1}(c)f^{G_1}(-)$ for $c \in \Gamma_1$ holds because both maps coincide with the unique lifting of $f \circ p$ mapping e_{G_1} to $f^{G_1}(c) \in \Gamma_2$. Now let $f'\colon G_2/\Gamma_2 \to G_1/\Gamma_1$ be a homotopy inverse of f. Define $f^{G_1}{}'$ to be the unique lifting of $f' \circ p'$ at e_{G_1}. Then as above $f^{G_1}{}'(c'y) = f^{G_1}{}'(c')f^{G_1}{}'(y)$ for any $c' \in \Gamma_2$ and $y \in G_2$. Let $h\colon I \times G_1/\Gamma_1 \to G_1/\Gamma_1$ be a definable homotopy between the identity h_0 on G_1/Γ_1 and $f' \circ f$, then let $h'\colon I \times G_2/\Gamma_2 \to G_2/\Gamma_2$ be a definable homotopy between the identity h_0' on G_2/Γ_2 and $f \circ f' = h_1'$. We may assume $h_t(1) = e_{G_1}$ for all t (otherwise use $(t, x) \to (h_t(1))^{-1}h_t(x)$ instead of h) and the same for h'. Finally, define $\widetilde{h}\colon I \times G_1 \to G_1$ as the unique lifting of $h \circ (\mathrm{Id} \times p)\colon I \times G_1 \to G_1/\Gamma_1$ to G_1 and $\widetilde{h}'\colon I \times G_2 \to G_2$ as the unique lifting of $h' \circ (\mathrm{Id} \times p)\colon I \times G_2 \to G_2/\Gamma_2$ to G_2. By uniqueness of liftings, $\widetilde{h}$ is a definable homotopy between the identity and $f^{G_1} \circ f^{G_2}$. Similarly $\widetilde{h}'$ is a definable homotopy between the identity and $f^{G_2} \circ f^{G_1}$. Moreover $\widetilde{h}$ and $\widetilde{h}'$ are constant on $I \times \Gamma_1$ and $I \times \Gamma_2$ since h and h' are constant on $I \times \{e\}$. The equations $\widetilde{h}_t(cx) = c\widetilde{h}_t(x)$ and $\widetilde{h}_t'(c'x') = c'\widetilde{h}_t'(x')$, where $c \in \Gamma_1$ and $c' \in \Gamma_2$, follow by uniqueness of liftings. □

4.8 Almost direct products

Given a group G and two subgroups A and B of G, we recall that G is the almost direct product of A and B if $G = AB$ and the function $m\colon A \times B \to G$ sending (x, y) to xy is a surjective group homomorphism with a finite kernel. This implies that $ab = ba$ for all $a \in A$ and $b \in B$, and that $\Gamma \stackrel{def}{=} A \cap B$ is a finite central subgroup of G. In this situation we write $G = A \times_\Gamma B$. Note that the kernel of $m\colon A \times B \to A \times_\Gamma B$ is $\Gamma^\Delta \stackrel{def}{=} \{(c, c^{-1})\}_{c \in \Gamma}$.

Every definably compact definably connected group is an almost direct product of a definably connected abelian subgroup and a semisimple definable subgroup. More precisely we have:

Fact 4.8.1. *Let* G *be a definably compact definably connected group. Let* $Z^0(G)$ *be the definable identity component of the center* $Z(G)$ *of* G*. By [HPP–08b] the commutator subgroup* $[G,G]$ *is definable and semisimple, and* G *is an almost direct product of* $Z^0(G)$ *and* $[G,G]$*. The corresponding statement holds in the category of compact connected Lie groups.*

Lemma 4.8.2. *Consider almost direct products of definable groups* $G_1 = A_1 \times_{\Gamma_1} B_1$ *and* $G_2 = A_2 \times_{\Gamma_2} B_2$*. Suppose that there are:*

I. *an isomorphism* $f^{\Gamma_1}\colon \Gamma_1 \to \Gamma_2$*,*

II. *a* Γ_1*-equivariant definable homotopy equivalence* $f^{A_1}\colon A_1 \to A_2$*,*

III. *a* Γ_1*-equivariant definable homotopy equivalence* $f^{B_1}\colon B_1 \to B_2$*,*

satisfying $f^{A_1}\restriction_{\Gamma_1} = f^{B_1}\restriction_{\Gamma_1} = f^{\Gamma_1}$*. Then there is a* Γ_1*-equivariant definable homotopy equivalence* $f^{G_1}\colon G_1 \to G_2$ *such that* $f^{G_1}(ab) = f^{A_1}(a)f^{B_1}(b)$ *for all* $a \in A_1$ *and* $b \in B_1$*. In particular* $f^{G_1}\restriction_{A_1} = f^{A_1}$ *and* $f^{G_1}\restriction_{B_1} = f^{B_1}$*.*

Proof. By definition of almost direct product there is a (unique) well defined function $f^{G_1}\colon G_1 \to G_2$ satisfying $f^{G_1}(ab) = f^{A_1}(a)f^{B_1}(b)$ for $a \in A_1$ and $b \in B_1$. Moreover f^{G_1} is continuous since $m\colon A_1 \times B_1 \to G_1$ is a definable covering map (hence locally $f^{G_1} = (f^{A_1} \otimes f^{B_1}) \circ m^{-1}$). Let f'^{A_1} and f'^{B_1} be homotopy inverses for f^{A_1} and f^{B_1} satisfying the conditions of Definition 4.7.2 and let $f'^{G_1}\colon G_2 \to G_1$ be defined symmetrically. We claim that f^{G_1} is a definable homotopy equivalence with homotopy inverse f'^{G_1}. In fact, let $h^{A_1}\colon I \times A_1 \to A_1$ be a definable homotopy between $f'^{A_1} \circ f^{A_1}$ and the identity satisfying the conditions of Definition 4.7.2, and let $h^{B_1}\colon I \times B_1 \to B_1$ be the same for $f'^{B_1} \circ f^{B_1}$. Define

$$h_t^{G_1}(ab) = h_t^{A_1}(a)h_t^{B_1}(b) \tag{4.1}$$

for $a \in A_1$ and $b \in B_1$. The fact that $h_t^{G_1}$ is well defined follows by the conditions in Definition 4.7.2 and the definition of almost direct product. A definable homotopy between $f^{G_1} \circ f'^{G_1}$ and the identity can be defined symmetrically. The lemma is thus proven. □

Fact 4.8.3 ([Con–09])**.** *Let* A *and* B *be type-definable subgroups of a definable group* G*, with* A *normal in* G*. Then* AB *is a type-definable subgroup of* G *and* $(AB)^{00} = A^{00}B^{00}$*.*

Lemma 4.8.4. *Let* $G = A \times_\Gamma B$ *be an almost direct product of definable groups. Let* $p\colon G \to G/G^{00}$ *be the projection map. Then* $G/G^{00} = p(A) \times_{p\Gamma} p(B)$*.*

Proof. Consider the homomorphism

$$\begin{aligned} m\colon\ p(A) \times p(B) &\to G/G^{00} \\ (aG^{00}, bG^{00}) &\mapsto abG^{00} \end{aligned}$$

Since $G^{00} = A^{00}B^{00}$ (Fact 4.8.3), if abG^{00} is the identity of G/G^{00} we have $aa' = b^{-1}b'$ for some $a' \in A^{00}$ and $b' \in B^{00}$. But $A \cap B = \Gamma$, so there is $c \in \Gamma$ such that $aa' = b^{-1}b' = c$. It follows that $aG^{00} = cG^{00}$ and $bG^{00} = c^{-1}G^{00}$. We have thus proven that the kernel of m is

$$\ker m = p(\Gamma)^\Delta \overset{def}{=} \left\{(cG^{00}, c^{-1}G^{00})\right\}_{c\in\Gamma}$$

a finite subgroup. □

Remark 4.8.5. Let G be a definably compact group and let A be a definable subgroup of G. Let $p\colon G \to G/G^{00}$ be the projection map. By [HPP–08a] we have $A \cap G^{00} = A^{00}$ (see [Ber–07] for the non-abelian case). Therefore

$$p(A) = AG^{00}/G^{00} \cong A/A^{00}$$

via the natural homomorphism sending $aA^{00} \in A/A^{00}$ to $aG^{00} \in AG^{00}/G^{00}$.

Lemma 4.8.6. *Let G be a definably compact definably connected group. Write $G = Z^0(G) \times_\Gamma [G,G]$. Let $p\colon G \to G/G^{00}$ be the projection map. Then*

$$\begin{aligned} p(Z^0(G)) &= Z^0(G/G^{00}) \\ p([G,G]) &= [G/G^{00}, G/G^{00}] \\ G/G^{00} &= p(Z^0(G)) \times_{p(\Gamma)} p([G,G]) \\ &= Z^0(G/G^{00}) \times_{p(\Gamma)} [G/G^{00}, G/G^{00}] \end{aligned}$$

Proof. By Fact 4.8.1 we have $\dim(G) = \dim(Z(G)) + \dim([G,G])$ and similarly for G/G^{00}. By [HPP–08a] the dimension of A as a definable group equals the dimension of A/A^{00} as a Lie group. So p preserves dimensions. The equality $p([G,G]) = [G/G^{00}, G/G^{00}]$ is clear. The inclusion $p(Z^0(G)) \subset Z^0(G/G^{00})$ is also clear (using the fact the image under p of a definably connected set is connected). The result follows by counting dimensions. □

Theorem 4.8.7. *Let G_1 and G_2 be definably compact definably connected groups. Suppose that there is a Lie isomorphism $\psi\colon G_1/G_1^{00} \to G_2/G_2^{00}$. Then:*

i. *There is a definable homotopy equivalence $f\colon G_1 \to G_2$.*
ii. *Given a finite central subgroup Γ of G_1 we can choose f to be Γ-equivariant, or even Γ'-equivariant where $\Gamma' = \Gamma[G_1, G_1]$ (so in particular $f\restriction_{[G_1,G_1]}$ is an isomorphism onto $[G_2, G_2]$).*
iii. *Moreover, assuming saturation, we can ensure that $p^{G_2} \circ f\restriction_{\Gamma'} = \psi \circ p^{G_1}\restriction_{\Gamma'}$ where $p_{G_1}\colon G_1 \to G_1/G_1^{00}$ and $p^{G_2}\colon G_2 \to G_2/G_2^{00}$ are the projections.*

Proof of Theorem 4.8.7. We can write $G_1 = Z^0(G_1) \times_{\Gamma_0} [G_1, G_1]$ and we can assume $\Gamma \supset \Gamma_0$. Let $\Gamma_1 = \Gamma[G_1,G_1] \cap Z^0(G_1)$ and note $\Gamma_1[G_1,G_1] = \Gamma[G_1,G_1]$. Moreover Γ_1 is finite: take any $c \in \Gamma$, fix an $x \in c[G_1,G_1] \cap Z^0(G_1)$, then for any $y \in c[G_1,G_1] \cap Z^0(G_1)$ we have $xy^{-1} \in [G_1,G_1] \cap Z^0(G_1)$, which is finite. By Theorem 4.7.4 (and Lemma 4.8.6) there is a definable Γ_1-equivariant homotopy equivalence $f^Z\colon Z^0(G_1) \to Z^0(G_2)$ such that $p \circ f^Z\restriction_\Gamma = \psi \circ p\restriction_\Gamma$. By Theorem 4.6.5 (and Lemma 4.8.6) there is a definable isomorphism

$$f^{[G_1,G_1]}\colon [G_1,G_1] \to [G_2,G_2]$$

In particular both f^Z and $f^{[G_1,G_1]}$ are Γ_0-equivariant definable homotopy equivalences. So by Lemma 4.8.2 there is a Γ_0-equivariant definable homotopy equivalence $f^{G_1}\colon G_1 \to G_2$ such that $f^{G_1}(ab) = f^Z(a) f^{[G_1,G_1]}(b)$ for all $a \in Z^0(G_1)$ and $b \in [G_1,G_1]$. By construction, using equation (4.1) in Lemma 4.8.2, we have that f^{G_1} is in fact a $\Gamma_1[G_1,G_1]$-equivariant definable homotopy equivalence. If the o-minimal structure is sufficiently saturated (and we choose $f^{[G_1,G_1]}$ as in Theorem 4.6.5) we obtain $p^{G_2} \circ f\restriction_{\Gamma'} = \psi \circ p^{G_1}\restriction_{\Gamma'}$. □

Bibliography

[Ber–07] Alessandro Berarducci, *o-minimal spectra, infinitesimal subgroups and cohomology*, J. Symbolic Logic **72** (2007), no. 4, 1177–1193. MR MR2371198 (2008k:03075)

[Ber–08] ———, *Zero-groups and maximal tori*, Logic Colloquium 2004, Lect. Notes Log., vol. 29, Assoc. Symbol. Logic, Chicago, IL, 2008, pp. 33–46. MR MR2401858 (2009e:03069)

[Ber–09] ———, *Cohomology of groups in o-minimal structures: acyclicity of the infinitesimal subgroup*, J. Symbolic Logic **74** (2009), no. 3, 891–900. MR MR2548466

[BM–10] Alessandro Berarducci and Marcello Mamino, *On the homotopy type of definnable groups in an o-minimal structure*, 2010, `arXiv:0905.1069v2`.

[BMO–08] Alessandro Berarducci, Marcello Mamino, and Margarita Otero, *Higher homotopy of groups definable in o-minimal structures*, 2008, `arXiv:0809.4940v2`.

[BO–02] Alessandro Berarducci and Margarita Otero, *o-minimal fundamental group, homology and manifolds*, J. London Math. Soc. (2) **65** (2002), no. 2, 257–270. MR MR1883182 (2003b:03048)

[BO–09] Elias Baro and Margarita Otero, *On o-minimal homotopy groups*, 2009, `doi:10.1093/qmath/hap011`.

[BOPP–05] Alessandro Berarducci, Margarita Otero, Yaa'cov Peterzil, and Anand Pillay, *A descending chain condition for groups definable in o-minimal structures*, Ann. Pure Appl. Logic **134** (2005), no. 2-3, 303–313. MR MR2139910 (2006a:03052)

[Bor–61] Armand Borel, *Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes*, Tôhoku Math. J. (2) **13** (1961), 216–240. MR MR0147579 (26 #5094)

[Bro–68] Ronald Brown, *Elements of modern topology*, McGraw-Hill Book Co., New York, 1968. MR MR0227979 (37 #3563)

[Con–09] Annalisa Conversano, *On the connections between definable groups in o-minimal structures and real lie groups: the non-compact case*, Ph.D. thesis, Università di Siena, 2009.

[DvdD–88] J. Denef and L. van den Dries, p*-adic and real subanalytic sets*, Ann. of Math. (2) **128** (1988), no. 1, 79–138. MR MR951508 (89k:03034)

[Edm–06] Mário J. Edmundo, *Locally definable groups in o-minimal structures*, J. Algebra **301** (2006), no. 1, 194–223. MR MR2230327 (2008a:03061)

[EE–07] Mário J. Edmundo and Pantelis E. Eleftheriou, *The universal covering homomorphism in o-minimal expansions of groups*, MLQ Math. Log. Q. **53** (2007), no. 6, 571–582. MR MR2351581 (2009d:03085)

[EJP–06] Mário J. Edmundo, Gareth O. Jones, and Nicholas J. Peatfield, *Sheaf cohomology in o-minimal structures*, J. Math. Log. **6** (2006), no. 2, 163–179. MR MR2317425 (2008e:03063)

[EJP–07] ______, *Hurewicz theorem for definable groups, lie groups and their cohomologies*, 2007.

[EO–04] Mário J. Edmundo and Margarita Otero, *Definably compact abelian groups*, J. Math. Log. **4** (2004), no. 2, 163–180. MR MR2114966 (2005m:03073)

[Hat–02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR MR1867354 (2002k:55001)

[HM–98] Karl H. Hofmann and Sidney A. Morris, *The structure of compact groups*, de Gruyter Studies in Mathematics, vol. 25, Walter de Gruyter & Co., Berlin, 1998, A primer for the student—a handbook for the expert. MR MR1646190 (99k:22001)

[HP–09] Ehud Hrushovski and Anand Pillay, *On nip and invariant measures*, 2009, `arXiv:0710.2330v2`.

[HPP–08a] Ehud Hrushovski, Ya'acov Peterzil, and Anand Pillay, *Groups, measures, and the NIP*, J. Amer. Math. Soc. **21** (2008), no. 2, 563–596. MR MR2373360 (2008k:03078)

[HPP–08b] ______, *On central extensions and definably compact groups in o-minimal structures*, 2008, `arXiv:0811.0089v1`.

[Joh–01] Joseph Johns, *An open mapping theorem for o-minimal structures*, J. Symbolic Logic **66** (2001), no. 4, 1817–1820, `doi:10.2307/2694977`. MR MR1877024 (2003f:03051)

[Mam–10] Marcello Mamino, *Splitting definably compact groups in o-minimal structures*, 2010, `arXiv:1001.2229v1`.

[MS–92] James J. Madden and Charles M. Stanton, *One-dimensional Nash groups*, Pacific J. Math. **154** (1992), no. 2, 331–344. MR MR1159515 (93d:14087)

[NPR–91] A. Nesin, A. Pillay, and V. Razenj, *Groups of dimension two and three over* o*-minimal structures*, Ann. Pure Appl. Logic **53** (1991), no. 3, 279–296. MR MR1129781 (93b:03053b)

[OP–09] Margarita Otero and Ya'acov Peterzil, *G-linear sets and torsion points in definably compact groups*, Arch. Math. Logic **48** (2009), no. 5, 387–402. MR MR2505431

[OPP–96] Margarita Otero, Ya'acov Peterzil, and Anand Pillay, *On groups and rings definable in o-minimal expansions of real closed fields*, Bull. London Math. Soc. **28** (1996), no. 1, 7–14. MR MR1356820 (96i:12006)

[Ote–08] Margarita Otero, *A survey on groups definable in o-minimal structures*, Model theory with applications to algebra and analysis. Vol.2, London Math. Soc. Lecture Note Ser., vol. 350, Cambridge Univ. Press, Cambridge, 2008, pp. 177–206. MR MR2436142

[Pil–88] Anand Pillay, *On groups and fields definable in o-minimal structures*, J. Pure Appl. Algebra **53** (1988), no. 3, 239–255, doi:10.1016/0022-4049(88)90125-9. MR MR961362 (89i:03069)

[Pil–04] ———, *Type-definability, compact Lie groups, and o-minimality*, J. Math. Log. **4** (2004), no. 2, 147–162. MR MR2114965 (2006d:03064)

[PPS–00a] Y. Peterzil, A. Pillay, and S. Starchenko, *Definably simple groups in o-minimal structures*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4397–4419. MR MR1707202 (2001b:03036)

[PPS–00b] Ya'acov Peterzil, Anand Pillay, and Sergei Starchenko, *Simple algebraic and semialgebraic groups over real closed fields*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4421–4450 (electronic). MR MR1779482 (2001i:03080)

[PPS–02] Y. Peterzil, A. Pillay, and S. Starchenko, *Linear groups definable in o-minimal structures*, J. Algebra **247** (2002), no. 1, 1–23. MR MR1873380 (2002i:03043)

[PS–99] Ya'acov Peterzil and Charles Steinhorn, *Definable compactness and definable subgroups of o-minimal groups*, J. London Math. Soc. (2) **59** (1999), no. 3, 769–786, doi:10.1112/S0024610799007528. MR MR1709079 (2000i:03055)

[PS–00] Ya'acov Peterzil and Sergei Starchenko, *Definable homomorphisms of abelian groups in o-minimal structures*, Ann. Pure Appl. Logic **101** (2000), no. 1, 1–27. MR MR1729742 (2000m:03090)

[Raz–91] Vladimir Razenj, *One-dimensional groups over an* o*-minimal structure*, Ann. Pure Appl. Logic **53** (1991), no. 3, 269–277. MR MR1129780 (93b:03053a)

[Spa–66] Edwin H. Spanier, *Algebraic topology*, McGraw-Hill Book Co., New York, 1966. MR MR0210112 (35 #1007)

[Ste–51] Norman Steenrod, *The Topology of Fibre Bundles*, Princeton Mathematical Series, vol. 14, Princeton University Press, Princeton, N. J., 1951. MR MR0039258 (12,522b)

[Str–94a] Adam W. Strzebonski, *Euler characteristic in semialgebraic and other o-minimal groups*, J. Pure Appl. Algebra **96** (1994), no. 2, 173–201, doi:10.1016/0022-4049(94)90127-9. MR MR1303545 (95j:03067)

[Str–94b] ______, *One-dimensional groups definable in* o*-minimal structures*, J. Pure Appl. Algebra **96** (1994), no. 2, 203–214. MR MR1303546 (95j:03068)

[vdD–98] Lou van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998. MR MR1633348 (99j:03001)

[Whi–78] George W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York, 1978. MR MR516508 (80b:55001)

[Wil–96] A. J. Wilkie, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc. **9** (1996), no. 4, 1051–1094. MR MR1398816 (98j:03052)

[Woe–96] Arthur Woerheide, *O-minimal homology*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 1996.

Index of the definitions