Tesi di Perfezionamento in Matematica

Ordinary differential equations in Banach spaces and the spectral flow

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Introduction

Given a path $\{A(t) \mid t \in \mathbb{R}\}$, of linear operators on some Banach space E, we consider the differential operator

$$F_A u = \left(\frac{d}{dt} - A(t)\right) u$$

on suitable spaces of curves $u\colon \mathbb{R} \to E$. A classical question is whether the operator F_A is Fredholm and what is its index. If $A(t)$ is a path of unbounded operators the literature is rich. We recall the work of J. Robbin and D. Salamon, [RS95], where A is an asymptotically hyperbolic path of unbounded self-adjoint operators and defined on a common domain $W \subset H$ compactly included in a Hilbert space H. For such paths they prove that the differential operator

$$F_A\colon L^2(\mathbb{R}, W) \cap W^{1,2}(\mathbb{R}, H) \to L^2(\mathbb{R}, H), \quad u \mapsto u' - Au$$

is Fredholm. The index of F_A is minus the *spectral flow* of A, an integer which counts algebraically the eigenvalues of $A(t)$ crossing 0. The result applies to *Cauchy-Riemann* operators and it is widely used in Floer homology. This result has been generalized to Banach spaces with the *unconditional martingale difference* (UMD) property by P. Rabier in [Rab04]; the compact inclusion of the domain is still required. In this setting the spectral flow $\mathrm{sf}\,(A)$ is still well-defined and the identity

$$\mathrm{ind} F_A = -\,\mathrm{sf}\,(A). \tag{1}$$

still holds. Y. Latushkin and T. Tomilov in [LT05] proved the Fredholmness of the operator F_A for paths A with variable domain $D(A(t)) \subset E$ with E reflexive using *exponential dichotomies*. D. di Giorgio, A. Lunardi and R. Schnaubelt in [DGLS05] obtained the same results for *sectorial operators* in an arbitrary Banach space and give necessary and sufficient conditions on the *stable* and *unstable spaces* in order to have the Fredholmness of F_A.

For the bounded case the problem has been studied by A. Abbondandolo and P. Majer in [AM03b]. This setting is suggested by the Morse Theory on a Hilbert manifold M: given a vector field ξ on M and ϕ_t its flow, x and y hyperbolic zeroes of ξ the stable and unstable manifolds

$$W^s_\xi(x) = \left\{ p \in M \mid \lim_{t\to+\infty} \phi_t(p) = x \right\}$$

$$W^u_\xi(y) = \left\{ p \in M \mid \lim_{t\to-\infty} \phi_t(p) = y \right\}$$

are immersed submanifolds of M, in fact they are submanifolds if the vector field is the gradient of a Morse function on M. It is not hard to check that the intersection of the stable and unstable manifold of two different zeroes is a submanifold if, for every curve $u'(t) = \xi(u(t))$ such that $u(+\infty) = x$ and $u(-\infty) = y$, the differential operator

$$F_A(v) = v' - Av, \quad A(t) = D\xi(u(t))$$

is surjective and $\ker F_A$ splits. Since x and y are hyperbolic zeroes $A(+\infty)$ and $A(-\infty)$ are hyperbolic operators. In [AM03b] the study of the Fredholm index of such operator is carried out by considering the stable and unstable spaces

$$W_A^s = \left\{ x \in E \mid \lim_{t\to+\infty} X_A(t)x = 0 \right\}$$
$$W_A^u = \left\{ x \in E \mid \lim_{t\to-\infty} X_A(t)x = 0 \right\},$$

where X_A is the solution of the Cauchy problem $X' = AX$ with $X(0) = I$. If A is an *asymptotically hyperbolic* path on a Hilbert space the following facts hold:

Fact 1. The stable and unstable spaces W_A^s and W_A^u are closed in E and admit topological complements, PROPOSITION 1.2 of [AM03b].

Fact 2. The evolution of the stable space $X_A(t)W_A^s$ converges to the negative eigenspace of $A(+\infty)$, and any topological complement of W_A^s converges to the positive eigenspace of $A(+\infty)$, with a suitable topology on the set of closed linear subspaces of a Hilbert, see THEOREM 2.1 of [AM03b].

Fact 3. If two paths A and B have compact difference for every $t \in \mathbb{R}$ the stable space W_A^s is compact perturbation of W_B^s, THEOREM 3.6 of [AM03b].

Fact 4. The operator F_A is semi-Fredholm if and only if (W_A^s, W_A^u) is a semi-Fredholm pair; in this case $\operatorname{ind} F_A = \operatorname{ind}(W_A^s, W_A^u)$, THEOREM 5.1 of [AM03b]

In the bounded setting the spectral flow is defined in [Phi96] for paths in $\mathcal{F}^{sa}(E)$, the set of Fredholm and self-adjoint bounded operators. Unlike the unbounded case described in [RS95] and in [Rab04], given an asymptotically hyperbolic path in $\mathcal{F}^{sa}(E)$ the equality $\operatorname{ind} F_A = -\operatorname{sf}(A)$ does not hold in general. Examples are provided in §7 of [AM03b]. Our purpose is to generalize firstly these facts to an arbitrary Banach space E and, secondly, to define the spectral flow for suitable paths and prove that for a class of paths the relation (1) holds.

In the first chapter we define some metrics on the set of closed linear subspaces of E, the *Grassmannian* of E, denoted by $G(E)$, and the subset of closed and splitting subspaces, denoted by $G_s(E)$. This is done in order to have a definition of *convergence of subspaces*. Our main reference is the work of E. Berkson, [Ber63]. We also establish which pairs of closed subspaces (X, Y) are *compact perturbation one of each other* and the relative dimension for such pairs is defined. These definitions allow to state **Fact 1** and **Fact 3**.

In chapter 2 we recall some classical subspaces of a Banach algebra such as the *idempotent* elements and *square roots* of unit and the *Calkin algebra* $\mathcal{C}$, obtained as the quotient algebra of bounded operators by the compact ones. We call an operator A *essentially hyperbolic* if the spectrum of $[A] \in \mathcal{C}$ does not meet the imaginary axis. We denote by $e\mathcal{H}(E)$ the space of essentially hyperbolic operators. In Theorem 5.1.2 we prove that

$$e\mathcal{H}(E) = \{ A \in \mathcal{B}(E) \mid A = H + K,\ H \text{ hyperbolic } , K \text{ compact } \}.$$

In Theorem 2.1.4 we show that $e\mathcal{H}(E)$ has the homotopy type of the space of idempotent elements of the Calkin algebra. A homotopy equivalence is

$$\Psi \colon e\mathcal{H}(E) \to \mathcal{P}(\mathcal{C}), \quad A \mapsto P^+([A])$$

where P^+ denotes the eigenprojector relative to the positive spectrum. Our aim is to a define a group homomorphism on the fundamental group of $\mathcal{P}(\mathcal{C})$, the space of idempotent elements of $\mathcal{C}$, with values in $\mathbb{Z}$. In fact such homomorphism can be obtained as result of the *long exact sequence* of the fibre bundle

$$\mathcal{P}(E) \to \mathcal{P}(\mathcal{C}), \quad P \mapsto [P]$$

where $\mathcal{P}(E)$ is the space of projectors on E. If we say that two projectors are compact perturbation (one of each other) if their difference is compact, then the function that maps a projector to its range preserves the relation of compact perturbation of closed linear subspaces defined in chapter 1. Hence, given a projector P onto a closed subspace $X \subset E$, we can consider the equivalence class of P in the space of projectors, denoted by $\mathcal{P}_c(P;E)$, and the equivalence class of X in $G_s(E)$, denoted by $G_c(X;E)$. We denote by $\mathcal{P}_e(E)$ and $G_e(E)$ the quotient spaces respectively. The latter is called *essential Grassmannian*. The map $r(P) = P(E)$ induces the homotopy equivalences

$$\mathcal{P}_c(P;E) \to G_c(X;E),\ \mathcal{P}_e(E) \to G_e(E).$$

These equivalences are well known in Hilbert spaces ([AM03a] is our main reference). In order to extend these results to an arbitrary Banach space some techniques used by K. Gęba in [Gęb68] can be adapted. Using the *Leray-Schauder degree* we prove in Theorem 2.6.3 that the connected components of $\mathcal{P}_c(P;E)$ are in correspondence with $\mathbb{Z}$. Hence the homomorphism is defined as the composition

$$\pi_1(\mathcal{P}(\mathcal{C}),[P]) \xrightarrow{\partial} \pi_0(\mathcal{P}_c(P;E)) \longrightarrow \mathbb{Z}$$

where ∂ is the map induced by the long exact sequence of the fibre bundle $(\mathcal{P}(E), \mathcal{P}_c(P;E), \mathcal{P}(\mathcal{C}))$. This homomorphism will be denoted by φ or called sometimes *index* of the exact sequence. Given P in $\mathcal{P}(E)$ we give sufficient conditions to P is order to make an isomorphism of φ. Precisely these are

h1) P is connected to a projector Q such that $\dim(Q,P) = 1$

h2) the connected component of P in $\mathcal{P}(E)$ is simply-connected.

These properties are verified for an orthogonal projection in a Hilbert space with infinite-dimensional kernel and range. The most common Banach spaces such as $L^p(\Omega,\mu)$ and spaces of sequences l_p (see [Mit70, Sch98] for a richer list and references) fulfill these hypotheses. For an arbitrary Banach space none of them is true. We exhibit some example of space where φ is not surjective. This is the case of an infinite-dimensional undecomposable *undecomposable* space, where the only complemented subspaces have finite dimension or finite codimension, therefore $e\mathcal{H}(E)$ as is the union of two contractile components. In Proposition 2.8.2 we show a concrete example of projector such that the condition h2) holds. In section 2.9 we show an example of space where φ is not injective.

In chapter 3 and 4 we study the Cauchy problem for continuous paths of bounded operators on a Banach space E. Once a definition of a metric and compact perturbation are provided the proof of the four facts for a Banach space presents no difficulties because most of the ideas are the same as the Hilbert case.

In chapter 5 we define the spectral flow for paths in the space of essentially hyperbolic operators $e\mathcal{H}(E)$. The definition generalizes the one given for Fredholm and self-adjoint operators by J. Phillips in [Phi96]. By composition we define another homomorphism, namely $\mathrm{sf} \circ \Psi_*^{-1}$, on the fundamental group of $\mathcal{P}(\mathcal{C})$. In Theorem 5.3.1 we prove that the first differs from φ by a sign. Hence anything holds for the index φ is true for the spectral flow as well. Hence, given a projector P, when conditions h1) and h2) hold we have an isomorphism

$$\pi_1(e\mathcal{H}(E), 2P - I) \to \mathbb{Z};$$

when the hypotheses of Proposition 2.8.2 hold we have a surjective spectral flow and the examples of the chapter 2 show that in general the spectral flow is neither injective nor surjective.

In the last section we prove that for a suitable class of paths in $e\mathcal{H}(E)$, namely the *essentially splitting* and asymptotically hyperbolic ones, there holds

$$\operatorname{ind} F_A = -\,\mathrm{sf}\,(A).$$

We achieve this result in several steps: in Lemma 5.4.4 we prove that an asymptotically hyperbolic path, A, is essentially splitting if and only if the projectors of the set $\{\, P^+(A(t)) \mid t \in \mathbb{R} \,\}$ have pairwise compact difference. In Theorem 5.4.5 we compute the spectral flow for an essentially splitting path. Using the fact that the positive eigenprojectors are in the same equivalence class of compact perturbation we prove that

$$\mathrm{sf}\,(A) = -\dim(\operatorname{ran} P^-(A(+(\infty))), \operatorname{ran} P^-(A(-\infty))).$$

For such paths we compute the Fredholm index of F_A in Theorem 5.4.3 and the equality

$$\operatorname{ind} F_A = \dim(\operatorname{ran} P^-(A(+(\infty))), \operatorname{ran} P^-(A(-\infty)))$$

holds; thus for such paths we obtain the relation $\mathrm{sf}\,(A) = -\operatorname{ind} F_A$.

Chapter 1

Topology of the Grassmannian

Given a Banach space E we consider the set of the closed linear subspaces, called *Grassmannian* of E. In literature there are plenty of metrics that make the Grassmannian a complete metric space, see [Ber63] and [Ost94]; here we work with the *Hausdörff metric*. The subset of the linear subspaces that admit a topological complement it is also considered and endowed with the induced topology. We prove that natural applications, such as the one that associates an operator between two Banach spaces with its graph is continuous respect to this metric. The last two sections of the chapter deal with the definition of *relative dimension* of two closed linear subspaces. We recall briefly the definition of relative dimension for subspaces of a Hilbert space and generalize the concept to Banach spaces.

1.1 The Hausdörff metric

Let (X, d) be a metric space. Given two subsets of $A, B \subseteq X$ it is well defined the distance

$$\operatorname{dist}(a, B) = \inf_{b \in B} d(a, b).$$

We denote by $\mathscr{H}(X)$ the family of closed, nonempty and bounded subsets of X. It is possible to build a metric on $\mathscr{H}(X)$ as follows: let A, B be two closed and bounded subsets of X. Define

$$\rho_{\mathscr{H}}(A, B) = \sup_{a \in A} \operatorname{dist}(a, B), \;\; \delta_{\mathscr{H}}(A, B) = \max\{\rho_{\mathscr{H}}(A, B), \rho_{\mathscr{H}}(B, A)\};$$

the second is called *Hausdörff metric*. We show that it has all the properties of a metric. It is clearly symmetric; if $\rho_{\mathscr{H}}(A, B) = 0$ $A \subset B$ because B is closed. Thus $\delta_{\mathscr{H}}(A, B) = 0$ if and only if $A = B$. For the triangular inequality let $A, B, C \in \mathscr{H}(X)$ be closed and bounded subsets of X. Given $\varepsilon > 0$ there exists $a_1 \in A$ such that

$$\rho_{\mathscr{H}}(A, C) \leq \varepsilon + \operatorname{dist}(a_1, C) \leq \varepsilon + d(a_1, b) + d(b, c) \tag{1.1}$$

for any $(b,c) \in B \times C$. Taking $b_1 \in B$ such that $d(a_1,b_1) \le \varepsilon + \operatorname{dist}(a_1,B)$, (1.1) becomes

$$\rho_{\mathscr{H}}(A,C) \le 2\varepsilon + \operatorname{dist}(a_1,B) + d(b_1,c)$$

for any $c \in C$. Taking the infimum over C we find that $\rho_{\mathscr{H}}(A,C) \le \rho_{\mathscr{H}}(A,B) + \rho_{\mathscr{H}}(B,C)$. Finally, suppose that $\delta_{\mathscr{H}}(A,C) = \rho_{\mathscr{H}}(A,C)$. Therefore

$$\delta_{\mathscr{H}}(A,C) = \rho_{\mathscr{H}}(A,C) \le \rho_{\mathscr{H}}(A,B) + \rho_{\mathscr{H}}(B,C) \le \delta_{\mathscr{H}}(A,B) + \delta_{\mathscr{H}}(B,C).$$

The following proposition states a relation between the metric spaces (X,d) and $(\mathscr{H},\delta_{\mathscr{H}})$. The proof of this can also be found in [Kur92].

Proposition 1.1.1. *The application $\delta_{\mathscr{H}} \colon \mathscr{H} \times \mathscr{H} \to \mathbb{R}^+$ defines a complete metric in $\mathscr{H}(X)$ if and only if (X,d) is complete. Moreover if $\{A_n \mid n \in \mathbb{N}\}$ is a converging sequence its limit is the set of the limits of sequences $\{a_n\}$ such that $a_n \in A_n$.*

Proof. We have proved that $\delta_{\mathscr{H}}$ is a metric. Given $a,b \in X$ it follows from the definition that $\delta_{\mathscr{H}}(\{a\},\{b\}) = d(a,b)$; thus, for a Cauchy sequence $\{a_n\} \subset X$, the sequence $\{\{a_n\}\}$ converges to a closed and bounded subset of $S \subset X$. For every element $s \in S$ there holds

$$d(s,a_n) = \operatorname{dist}(s,\{a_n\}) \le \delta_{\mathscr{H}}(S,\{a_n\})$$

thus s is the limit of the sequence $\{a_n\}$. By uniqueness of the limit S consist of a single point, thus (X,d) is complete. To prove the converse let $\{A_n\}$ be a Cauchy sequence in $\mathscr{H}(X)$ and $\varepsilon > 0$; there exists $n(\varepsilon)$ such that for every $n \ge n(\varepsilon)$

$$\delta_{\mathscr{H}}(A_{n(\varepsilon)},A_n) < \varepsilon/2;$$

given $a \in A_{n(\varepsilon)}$ using induction we can build a sequence $\{a_k\}$ and $n_k \in \mathbb{N}$ such that

$$a_0 = a,\ a_k \in A_{n_k},\ n_0 = n(\varepsilon),\ n_{k+1} > n_k,\ d(a_{k+1},a_k) < 2^{-(k+2)}\varepsilon; \tag{1.2}$$

then $\{a_k\}$ is a Cauchy sequence in X and, since X is complete, converges to a limit, say x. Define L as the set of the elements that are limits of sequences $\{a_k\}$ such that $a_k \in A_{n_k}$. The construction above shows that L is nonempty. We prove now that A_n converges to L; first there exists $a_0 \in A_{n(\varepsilon)}$ such that

$$\rho_{\mathscr{H}}(A_{n(\varepsilon)},L) < \varepsilon/8 + \operatorname{dist}(a_0,L);$$

let $\{a_k\}$ be as in (1.2) and call x its limit. Let k be such that $d(a_k,a) < \varepsilon/8$. We have

$$\begin{aligned}\rho_{\mathscr{H}}(A_{n(\varepsilon)},L) &< \varepsilon/8 + d(a_0,a_k) + d(a_k,x) < \varepsilon/4 + \sum_{j=0}^{k-1} d(a_{j+1},a_j)\\ &< \varepsilon/4 + \varepsilon\sum_{j=2}^{\infty} 2^{-j} < \varepsilon/2;\end{aligned}$$

thus $\rho_{\mathscr{H}}(A_n, L) \leq \rho_{\mathscr{H}}(A_n, A_{n(\varepsilon)}) + \rho_{\mathscr{H}}(A_{n(\varepsilon)}, L) < \varepsilon$ for every $n \geq n(\varepsilon)$. Similarly there exists $x \in L$ such that

$$\rho_{\mathscr{H}}(L, A_{n(\varepsilon)}) < \varepsilon/8 + \operatorname{dist}(x, A_{n(\varepsilon)});$$

by definition of L there exists a sequence a_k converging to x such that $a_k \in A_{n_k}$. Choose $k(\varepsilon)$ such that, for every $k > k(\varepsilon)$, we have

$$d(x, a_k) < \varepsilon/4, \ n_k > n(\varepsilon);$$

by the triangular inequality, for every $n > n_{k(\varepsilon)}$, we have

$$\rho_{\mathscr{H}}(L, A_{n(\varepsilon)}) < \varepsilon/4 + \operatorname{dist}(x, A_n) + \rho_{\mathscr{H}}(A_n, A_{n(\varepsilon)}) < \varepsilon,$$

thus $\delta_{\mathscr{H}}(L, A_n) < \varepsilon$. This proves the completeness of $\mathscr{H}(X)$. To conclude the proof observe that, since $\rho_{\mathscr{H}}(L, A_n)$ is an infinitesimal sequence, given $x \in L$ there exists an infinitesimal sequence $\{\varepsilon_n\}$ and a_n such that

$$d(x, a_n) - \varepsilon_n < \operatorname{dist}(x, A_n) \leq \rho_{\mathscr{H}}(L, A_n);$$

taking the limit as $n \to \infty$ we prove that $\{a_n\}$ converges to x. □

1.2 Metrics on the Grassmannian

Let $(E, |\cdot|)$ be a Banach space, define $G(E)$ as the set of the closed linear subspaces of E, usually called *Grassmannian*. We want to build a complete metric on this set. For any subspace $Y \subset E$ we can always consider the following subsets

$$D(Y) = \{ y \in E \mid |y| \leq 1 \},$$
$$S(Y) = \{ y \in E \mid |y| = 1 \}, \ (Y \neq 0);$$

with the metric induced by the norm of E is a complete metric space, and there is a natural inclusion $i \colon G(E) \hookrightarrow \mathscr{H}(E), \ Y \mapsto D(Y)$. On $G(E)$ we consider the metric induced by the inclusion, that is

$$\rho(Y, Z) = \rho_{\mathscr{H}}(D(Y), D(Z)),$$
$$\delta(Y, Z) = \delta_{\mathscr{H}}(D(Y), D(Z)).$$

Proposition 1.2.1. *The subset $i(G(E))$ is closed in $\mathscr{H}(E)$, hence δ is complete.*

Proof. Let Y_n be a sequence in $G(E)$ such that $D_n = D(Y_n)$ converges to $D \subseteq E$, a nonempty, closed and bounded subset of E. Let Z be the linear vector subspace generated by D. First observe that D is the unit disc of the space Z. In fact we have the following properties:

p1) $0 \in D$;

p2) provided $Z \neq \{0\}$ we have $D \supset S(Z)$;

p3) D is *star-shaped* to 0, that is $tx \in D$ for every $t \in [0, 1]$ if $x \in D$.

All these properties are consequences of Proposition 1.1.1. For instance the first follows in that $0 \in D_k$ for every $k \in \mathbb{N}$. For the second let $z \in S(Z)$; since D generates Z there are constants t_i such that

$$z = t_1 y_1 + \cdots + t_n y_n, \quad y_i \in D.$$

Each of these elements are limits of a sequence $y_{i,k} \in D_k$, hence, for every $k \in \mathbb{N}$, we have

$$\begin{aligned} z_k &= t_1 y_{1,k} + \cdots + t_n y_{n,k} \in Y_k \\ z_k/|z_k| &= \hat{t}_1 y_{1,k} + \cdots + \hat{t}_n y_{n,k} \in D_k; \end{aligned}$$

applying the Proposition 1.1.1 to the second sequence we find $z \in D$. The proof of the third property is similar and we omit it. From p1)–p3) it follows easily that $D \supseteq D(Z)$: given $z \neq 0$ in $D(Z)$ the vector $\hat{z} = z/|z| \in D$ and, since D is star-shaped, $z \in D$. The inclusion $D \subseteq D(Z)$ it is just the definition of Z, hence $D(Z) = D$. To conclude the proof we show that Z is a closed subspace of E. Let $\{z_n\}$ be a sequence converging to $x \in E$; if $x = 0$ clearly $x \in Z$. If $x \neq 0$ for n large each term of the sequence is nonzero. We write

$$z_n = \hat{z}_n \cdot |z_n|, \;\; \hat{z}_n \in D;$$

since D is closed $\hat{x} \in D$. Thus $z = |x|\hat{x}$ belongs to vector space generated by D, hence $z \in Z$. We have proved that $D = i(Z)$. □

Similarly we can consider the inclusion of spheres given by $j\colon G(E) \setminus \{0\} \hookrightarrow \mathscr{H}(E),\ Y \mapsto S(Y)$ and define a metric on $G(E) \setminus \{0\}$ as follows

$$\begin{aligned} &\rho_S(Y,Z) = \rho_{\mathscr{H}}(S(Y), S(Z)), \\ &\delta_S(Y,Z) = \delta_{\mathscr{H}}(S(Y), S(Z)), \text{ if } Y, Z \neq 0; \end{aligned}$$

we extend it to a metric on $G(E)$ with $\rho_S(\{0\},\{0\}) = 0$ and $\rho_S(Y,\{0\}) = \rho_S(\{0\}, Z) = 1$. It is also called *opening metric* (see [Ber63], §2). As above we have the following

Proposition 1.2.2. *The subset $j(G(E) \setminus \{0\})$ is closed in $\mathscr{H}(E)$, hence δ_S is complete.*

The proof is similar to the previous one. It just takes to prove that limits of sequences of spheres is a sphere.

Proposition 1.2.3. *The metrics δ_S and $\delta_{\mathscr{H}}$ are equivalent. In particular the inequalities*

$$\begin{aligned} \rho_S(Y,Z) &\leq 2\rho(Y,Z) \\ \rho(Y,Z) &\leq \rho_S(Y,Z); \end{aligned}$$

hold.

Proof. To prove the first inequality we will use this fact: for any pair of vectors $x \in S(E)$ and $y \in E \setminus \{0\}$ we have $|x - \hat{y}| \leq 2|x - y|$ where $\hat{y} = y/|y|$. Let

$Y, Z \neq \{0\}$ and $\varepsilon > 0$. There exists $y \in S(Y)$ such that, for every $z \in S(Z)$ and $0 < r \leq 1$ there holds

$$\rho_{\mathscr{H}}(S(Y), S(Z)) \leq \varepsilon + |y - z| = \varepsilon + |y - \widehat{rz}| \leq \varepsilon + 2|y - rz|;$$

taking the infimum over $(0,1] \times S(Z)$ we find

$$\rho_{\mathscr{H}}(S(Y), S(Z)) \leq \varepsilon + 2\,\mathrm{dist}\,(y, D(Z) \setminus \{0\});$$

since $\rho_{\mathscr{H}}(S(Y), S(Z)) \leq 1 < 2$ we can write

$$\rho_{\mathscr{H}}(S(Y), S(Z)) \leq 2 \min\{1, \varepsilon/2 + \mathrm{dist}\,(y, D(Z) \setminus \{0\})\};$$

since $|y| = 1$ the second member of the inequality becomes

$$\begin{aligned} &2 \min\{1, \varepsilon/2 + \mathrm{dist}\,(y, D(Z) \setminus \{0\})\} \\ \leq &2 \min\{\varepsilon/2 + |y|, \varepsilon/2 + \mathrm{dist}\,(y, D(Z) \setminus \{0\})\}; \end{aligned}$$

the latter is equal to

$$2(\varepsilon/2 + \mathrm{dist}\,(y, D(Z)) \leq \varepsilon + 2\,\mathrm{dist}\,(y, D(Z)).$$

Taking the supremum over $S(Y)$ we obtain

$$\rho_{\mathscr{H}}(S(Y), S(Z)) \leq \varepsilon + 2\rho_{\mathscr{H}}(S(Y), D(Z)) \leq \varepsilon + 2\rho_{\mathscr{H}}(D(Y), D(Z)).$$

If $Y = \{0\}$ and $Z \neq 0$ we have $\rho_{\mathscr{H}}(\{0\}, S(Z)) = 1 = \delta_{\mathscr{H}}(D(Z), \{0\})$, thus we have proved that $\delta_S(Y, Z) \leq 2\delta(Y, Z)$.
We prove the second inequality in the case $Y, Z \neq \{0\}$ first. Suppose $\rho(Y, Z) \neq 0$ and pick $\varepsilon > 0$ such that $0 < 2\varepsilon < \rho(Y, Z)$. There exists $y \in D(Y)$ such that

$$\rho(Y, Z) < \varepsilon/2 + \mathrm{dist}\,(y, D(Z));$$

in fact this implies $y \neq 0$. Set $\hat{y} = y/|y|$; there exists $\nu \in S(Z)$ such that

$$d(\hat{y}, \nu) < \varepsilon/2 + \mathrm{dist}\,(\hat{y}, S(Z)).$$

Hence the second term of the first inequality is bounded by $d(y, |y|\nu)$ which is equal to $|y| d(\hat{y}, \nu)$, thus

$$\begin{aligned} \rho(Y, Z) &< \varepsilon/2 + |y| d(\hat{y}, \nu) \leq \varepsilon/2 + d(\hat{y}, \nu) \\ &< \varepsilon + \mathrm{dist}\,(\hat{y}, S(Z)) \leq \varepsilon + \rho_S(Y, Z). \end{aligned}$$

If one among Y and Z is $\{0\}$ we have $\rho(Y, \{0\}) = 1 = \rho_S(Y, \{0\})$. □

By technical reasons we also define, for two closed subspaces Y, Z

$$\rho_1(Y, Z) = \sup_{y \in D(Y)} \mathrm{dist}\,(y, Z), \;\; \delta_1(Y, Z) = \max\{\rho_1(Y, Z), \rho_1(Z, Y)\}.$$

The triangular inequality does not hold for ρ_1 (see [Ber63], §3 for a counterexample). However the *weakened triangular inequality* holds, that is

$$\rho_1(X, Z) \leq \rho_1(Y, Z)(1 + \rho_1(X, Y)) + \rho_1(X, Y)$$

for every X, Y, Z (see [Kat95], Ch. IV, Lemma 2.2) [1].

[1] The inequality allows to consider $d_1(X, Y) = \log(1 + \delta_1(X, Y))$ which is a metric and induces the same topology as the *neighbourhood topology* generated by δ_1.

Proposition 1.2.4. *The topology generated by the neighbourhoods*

$$\{U(Y,r) \mid Y \in G(E),\ r > 0\},\ U(Y,r) = \{Z \mid \rho_1(Y,Z) < r\}$$

is equivalent to the one induced by the Hausdörff metric of the discs. More precisely for every Y, Z

$$1/2 \cdot \delta(Y,Z) \leq \delta_1(Y,Z) \leq \delta(Y,Z).$$

Proof. Given $y \in D(Y)$, $\operatorname{dist}(y,Z) \leq \operatorname{dist}(y,D(Z))$, then $\delta_1(Y,Z) \leq \delta(Y,Z)$. In order to prove the lower estimate suppose both Y, Z are different from the null space. Let $y \in S(Y)$; for every $z \in S(Z)$ and $r > 0$ we have

$$\operatorname{dist}(y,S(Z)) \leq |y - z| = |y - \widehat{rz}| \leq 2|y - rz|;$$

taking the infimum over $\mathbb{R}^+ \times S(Z)$ we find $\operatorname{dist}(y,S(Z)) \leq 2\operatorname{dist}(y, Z \setminus \{0\})$. Since $\operatorname{dist}(y,S(Z)) \leq 2$ we can write

$$\begin{aligned}\operatorname{dist}(y,S(Z)) &\leq 2\min\{1, \operatorname{dist}(y, Z\setminus\{0\})\} = 2\min\{|y|, \operatorname{dist}(y, Z\setminus\{0\})\} \\ &= 2\operatorname{dist}(y,Z).\end{aligned}$$

Then $\delta_S(Y,Z) \leq 2\delta_1(Y,Z)$. Since $\delta(Y,Z) \leq \delta_S(Y,Z)$ the proof is complete. □

We remark that the quantities introduced in this section such as δ, δ_1 and δ_S induce the same topology on $G(E)$. However, in literature, there are noticeable metrics that induce different topologies on $G(E)$. Of high interest it is the so called *Schäffer metric* or *operator opening*. It is defined as follows

$$\begin{aligned} r_0(X,Y) &= \inf\{\|T - I\|; T \in GL(E), TX = Y\}, \\ r(X,Y) &= \max\{r_0(X,Y), r_0(Y,Z)\}. \end{aligned}$$

It induces the same topology of δ_S on $G_s(E)$, but these topologies are different in $G(E)$. It is not hard to prove that with the Schäffer metric the subset of splitting subspaces is closed in $G(E)$ (see Theorem 4.1 of [Ber63]). We report in the following example an argument of V. I. Gurarii and A. S. Markus of [GM65].

Example 1. Let E and F be two Banach spaces, $X \subset E$ a splitting closed subspace and $Y \subset F$ a closed non-splitting subspace isomorphic to X. In the Banach space $E \oplus F$ the subspace $\{0\} \oplus Y$ does not have a topological complement. Let T be an isomorphism of Y onto X. Consider the family of subspaces

$$Y(\lambda) = \{(\lambda Ty, y) \mid y \in Y\},\ \lambda \in \mathbb{R};$$

since T is bounded these are closed subspaces; in fact, given a projector P with range X, the linear operator $P(\lambda)(v,w) = (\lambda Pv, T^{-1}Pv)$ is a projector with range $Y(\lambda)$. However $Y(\lambda)$ converges to $Y(0)$ as $\lambda \to 0$ in the Hausdörff topology, in fact given $y \in D(Y)$ we have

$$\operatorname{dist}((0,y), Y(\lambda)) = |\lambda||Ty| \leq |\lambda|\|T\|$$

hence $\rho_1(Y(0), Y(\lambda)) \leq |\lambda|\|T\|$. Similarly it can be proved that $\rho_1(Y(\lambda), Y(0))$ converges to zero as $\lambda \to 0$. Hence, a sequence in $G_s(E \oplus F)$, namely $\{Y(\lambda)\}$, converges to an uncomplemented subspace of $E \oplus F$, therefore $G_s(E \oplus F)$ is not closed in $G(E \oplus F)$ in the topology induced by δ_1. In the next section we will prove that $G_s(E)$ is open in the Hausdörff topology.

Since δ, δ_S and δ_1 induce the same topology we will choose time after time the one that most fits our settings.

1.3 Properties of the Hausdörff topology

Given Banach spaces E and F we denote by $\mathcal{B}(E, F)$ the space of linear and bounded applications. We call *general linear group* the set of invertible bounded operators of E with itself endowed with the topology of the norm and denote it by $GL(E)$. In this section we show that the choice of the Hausdörff metric makes continuous some natural operations on $G(E)$, such as the multiplication by an invertible operator.

Proposition 1.3.1. *Consider the set $GL(E) \times G(E)$ with the topology induced by the product metric $\|\cdot\| \times \delta$. The action of $GL(E)$ on $G(E)$ given by*

$$GL(E) \times G(E) \longrightarrow G(E), \quad (T, Y) \longmapsto T \cdot Y$$

is continuous.

Proof. We will prove that this map is locally Lipschitz. Fix $T \in GL(E)$ and let Y, Z be two closed subspaces in $G(E)$. Set $Ty = y' \in D(TY)$ and $r = \|T^{-1}\|$. Hence $|y| \leq r$. Thus, by Proposition 1.2.4, we have

$$\begin{aligned} \operatorname{dist}(y', D(TZ)) &\leq 2 \operatorname{dist}(y', TZ) = 2r \operatorname{dist}(y'/r, TZ) \leq 2r\|T\| \operatorname{dist}(y/r, Z) \\ &\leq 2\|T^{-1}\|\|T\|\rho_1(Y, Z) \leq 2\|T^{-1}\|\|T\|\rho(Y, Z) \end{aligned}$$

hence

$$\rho(TY, TZ) \leq 2\|T^{-1}\|\|T\|\rho(Y, Z). \tag{1.3}$$

Now fix $Y \in G(E)$, T and S invertible operators and $y' \in D(TY)$. As above $|y| \leq r$ and we have

$$\operatorname{dist}(y', D(SY)) \leq 2 \operatorname{dist}(y', SY) \leq 2\|T - S\||y| \leq 2\|T - S\|\|T^{-1}\|;$$

taking the supremum over $D(TY)$ and switching T and S we find

$$\delta(TY, SY) \leq 2\|T - S\| \max\{\|T^{-1}\|, \|S^{-1}\|\}. \tag{1.4}$$

Now choose a point $(T_0, Y_0) \in GL(E) \times G(E)$ and set $r_0 = \|T_0^{-1}\|$; given $\alpha < 1$ we claim that in the neighbourhood

$$U = B(T_0, \alpha r_0^{-1}) \times G(E)$$

the map is Lipschitz. It is not hard to prove that for such radius the norm of the inverse of every operator is bounded by a constant that depends only on α

and r_0. More precisely, using Von Neumann series, it is simple to find $r_0/(1-\alpha)$ as bound. Let (T,Y) and (S,Z) be two points in U. Hence

$$\begin{aligned}\delta(TY,SZ) &\le \delta(TY,SY)+\delta(SY,SZ)\\ &\le 2\max\{\|T^{-1}\|,\|S^{-1}\|\}\|T-S\|+2\|S\|\|S\|^{-1}\delta(Y,Z)\\ &\le \frac{2r_0}{1-\alpha}\|T-S\|+2\alpha r_0^{-1}\cdot\frac{r_0}{1-\alpha}\delta(Y,Z)\\ &\le \frac{2\max\{\alpha,r_0\}}{1-\alpha}\Big(\|T-S\|+\delta(Y,Z)\Big).\end{aligned}$$

□

Proposition 1.3.2. *If $\rho_1(Y,Z)<1$ and $Z\subseteq Y$ then $Z=Y$.*

Proof. If Y is the null space the proof is trivial. Otherwise let $\rho(Y,Z)=1-\varepsilon_0$ and suppose $S(Y)\setminus Z$ is not empty and contains an element, say y. Let $z\in Z$ be such that

$$\operatorname{dist}(y,Z)\ge|y-z|-\varepsilon_0/2;$$

define $y_0=z-y$. Since $Z\subseteq Y$, $y_0\in Y$. Thus $\operatorname{dist}(\hat{y_0},Z)\ge 1-\varepsilon_0/2$, thus

$$1-\varepsilon_0=\rho(Y,Z)\ge\operatorname{dist}(\hat{y_0},Z)\ge 1-\varepsilon_0/2$$

which is impossible, then $Y\subset Z$ and $Y=Z$. □

We define E^* as the space of bounded maps defined on E with real values. It is called *topological dual* of E and its elements are called *functionals*. For any subset $S\subset E$ we denote by $S^\perp$ the *annihilator* of S, that set of functionals whose kernel contains S. The annihilator is a closed subspace of E^*. The annihilator has a good behaviour respect to the topology of $G(E)$ as we will see in the next Proposition.

Proposition 1.3.3. *Given two closed subspaces Y, Z and $Y^\perp$, $Z^\perp$ its annihilators, we have $\rho_1(Y,Z)=\rho_1(Z^\perp,Y^\perp)$.*

Proof. We prove that, for any closed subspace Y, a functional $\xi\in E^*$ and $x\in E$, the equalities

$$\operatorname{dist}(\xi,Y^\perp)=\sup_{D(Y)}|\langle\xi,y\rangle|=|\xi_{|Y}|, \tag{1.5}$$

$$\operatorname{dist}(x,Y)=\sup_{D(Y^\perp)}|\langle\eta,x\rangle| \tag{1.6}$$

hold. The proof of both uses Hahn-Banach theorems of extension of functionals, see [Bre83] details. Given ε there exists $\overline{y}\in D(Y)$ such that, for every $\eta\in Y^\perp$, we can write

$$|\xi_{|Y}|<\varepsilon+\langle\xi,\overline{y}\rangle=\varepsilon+\langle\xi-\eta,\overline{y}\rangle\le\varepsilon+|\xi-\eta|;$$

taking the infimum over $D(Y^\perp)$ we get $|\xi_{|Y}|\le\operatorname{dist}(\xi,Y^\perp)$. Conversely, given a functional ξ, by Hahn-Banach, there exists an extension ξ_1 of $\xi_{|Y}$ such that $|\xi_1|=|\xi_{|Y}|$. Thus $\eta=\xi-\xi_1$ annihilates Y and we can write

$$\operatorname{dist}(\xi,Y^\perp)\le|\xi-\eta|=|\xi_1|=|\xi_{|Y}|.$$

We prove the second equality. Let $\varepsilon > 0$. There exists $\eta_1 \in D(Y^\perp)$ such that, for every $y \in Y$

$$\sup_{D(Y^\perp)} |\langle \eta, x\rangle| < \varepsilon + |\langle \eta_1, x\rangle| = \varepsilon + |\langle \eta_1, x-y\rangle| \le \varepsilon + |x-y|;$$

taking the infimum over Y we find

$$\sup_{D(Y^\perp)} |\langle \eta, x\rangle| \le \varepsilon + \operatorname{dist}(x, Y).$$

To prove the opposite inequality we distinguish two cases. If $x \in Y$ the proof is trivial, because both terms of (1.6) are zero. Suppose $x \notin Y$. Let $0 \le \alpha < 1$. There exists $y_\alpha \in Y$ such that

$$\alpha|x - y_\alpha| < \operatorname{dist}(x, Y) \le |x - y_\alpha|;$$

since $x - y_\alpha \notin Y$ we can define a functional η_α such that its restriction to Y is zero and $\langle \eta_\alpha, x - y_\alpha\rangle = \alpha|x - y_\alpha|$. By Hahn-Banach for every α there exists an extension $\tilde{\eta}_\alpha$ of η_α such that $|\tilde{\eta}_\alpha| = |\eta_\alpha|$. It is clear by its definition that $\tilde{\eta}_\alpha \in Y^\perp$. Consider $z = \lambda(x - y_\alpha) + y$. We have

$$\begin{aligned} |z| &= |\lambda| \left| x - y_\alpha + \frac{y}{\lambda} \right| \ge |\lambda| \operatorname{dist}(x, Y) \ge \alpha|\lambda||x - y_\alpha| \\ &\ge |\lambda||\langle \eta_\alpha, x - y_\alpha\rangle| = |\langle \eta_\alpha, z\rangle| \end{aligned}$$

then $|\eta_\alpha| \le 1$ and $\tilde{\eta}_\alpha \in D(Y^\perp)$. Therefore

$$\alpha \operatorname{dist}(x, Y) \le \alpha|x - y_\alpha| = |\langle \tilde{\eta}_\alpha, x - y_\alpha\rangle| = |\langle \tilde{\eta}_\alpha, x\rangle| \le \sup_{D(Y^\perp)} |\langle \tilde{\eta}, x\rangle|.$$

The equality is proved as $\alpha \to 1$. Now we can prove the equality claimed in the statement. We have

$$\rho_1(Y, Z) = \sup_{D(Y)} \operatorname{dist}(y, Z) = \sup_{D(Y)} \sup_{D(Z^\perp)} |\langle \xi, y\rangle|$$

by (1.6). Here we switch the order of the supremums. By (1.5) the last term of the equality is

$$\sup_{D(Z^\perp)} \sup_{D(Y)} |\langle \xi, y\rangle| = \sup_{D(Z^\perp)} \operatorname{dist}(\xi, Y^\perp) = \rho(Z^\perp, Y^\perp).$$

□

Corollary 1.3.4. *The map $G(E) \to G(E^*)$ that associates a subspace with its annihilator is continuous.*

1.4 The Grassmannian of splitting subspaces

A closed subspace $Y \in G(E)$ is said to *split* if there exists $Z \in G(E)$ such that $Y \oplus Z = E$. We call the set

$$G_s(E) = \{ Y \in G(E) \mid Y \text{ splits } \}$$

Grassmannian of splitting subspaces. In $G_s(E)$ we consider the subspace topology induced by $G(E)$. The subspace Z is also called *topological complement* of Y. By the open mapping theorem, for each topological complement, there exists an unique bounded operator P such that $P^2 = P$ and $\operatorname{ran} P = Y$, $\ker P = Z$. We call it *projector onto* Y *along* Z and denote it by $P(Y, Z)$. Unless E is an Hilbert space $G_s(E) \subsetneq G(E)$, see [Bre83]. Our aim is to prove that $G_s(E)$ is an open subset of $G(E)$. For this purpose we need to introduce the notion of *minimum gap* between closed spaces (see also [Kat95], Ch. IV, §4). We recall that, for any closed subspace $Y \in G(E)$, the quotient space E/Y is endowed with the norm $|x+Y| = \operatorname{dist}(x, Y)$ that makes it a Banach space called *quotient space*. Moreover the projection to the quotient is a bounded operator between two Banach spaces.

Definition 1.4.1 (The minimum gap)**.** *Let* Y *and* Z *be two closed subspaces. Set*

$$\gamma(Y, Z) = \inf_{Y\setminus Z} \frac{\operatorname{dist}(y, Z)}{\operatorname{dist}(y, Y \cap Z)}$$

if $Y \neq 0$*,* $\gamma(Y, Z) = 1$ *otherwise. We define the* gap *by*

$$\hat{\gamma}(Y, Z) = \min\{\gamma(Y, Z), \gamma(Z, Y)\}.$$

Lemma 1.4.2. (cf. [Kat95], Theorem 4.2, Ch. IV) Let Y and Z be closed subspaces of E. Then $Y + Z$ is closed in E if and only if $\gamma(Y, Z) > 0$.

Proof. Suppose both spaces are different from $\{0\}$. We prove the statement when $Y \cap Z = \{0\}$. Suppose $X = Y \oplus Z$ is closed and call P the projector onto Y along Z. Since $Y \neq \{0\}$ the projector is not zero. Let $x = y + z$. Then

$$\|P\| = \sup_{y+z\neq 0} \frac{|y|}{|y+z|} = \sup_{y\neq 0} \frac{|y|}{\operatorname{dist}(y, Z)}; \tag{1.7}$$

taking the inverses in the equation we find then $\|P\|^{-1} = \gamma(Y, Z)$. Suppose, conversely, that $\gamma(Y, Z) > 0$. Let $\{y_n\} \subset Y$ and $\{z_n\} \subset Z$ be sequences such that $x_n = y_n + z_n \to x \in E$. If the sequence $\{x_n\}$ has a constant subsequence, then $x \in Z$, since both $\{y_n\}$ and $\{z_n\}$ are constants. Otherwise, up to extracting a subsequence we can suppose that $x_n \neq x_m$ whenever $n \neq m$. Then we can write

$$\begin{aligned} |y_n - y_m| =& \frac{|y_n - y_m|}{|x_n - x_m|} \cdot |x_n - x_m| \le \frac{|y_n - y_m|}{\operatorname{dist}(y_n - y_m, Z)} \cdot |x_n - x_m| \\ \le& \frac{|x_n - x_m|}{\gamma(Y, Z)}; \end{aligned}$$

since the last term of the inequality is a Cauchy sequence, $\{y_n\}$ (and thus $\{z_n\}$) converges and $x = \lim y_n + \lim z_n \in X$. Since both Y and Z are closed $x \in Y + Z$. For the general case consider the quotient space $E/(Y \cap Z)$ and call π the projection onto the quotient. Let $\tilde{Y} = \pi(Y)$ and $\tilde{Z} = \pi(Z)$; these are closed subspaces of F because π maps closed subspaces of E containing $\ker \pi$ onto closed subspaces. Moreover $\gamma(\tilde{Y}, \tilde{Z}) = \gamma(Y, Z)$, in fact

$$\operatorname{dist}(\tilde{y}, \tilde{Z}) = \inf_{z\in Z} \operatorname{dist}(y - z, Y \cap Z) = \operatorname{dist}(y, Z).$$

The proof carries on as follows: suppose $Y + Z$ is a closed subspace. Then $\pi(Y + Z) = \tilde{Y} + \tilde{Z}$ is closed in the quotient space. The space $\pi(Y)$ and $\pi(Z)$ have null intersection thus, by the first part of the proof, $\gamma(\pi(Y), \pi(Z)) > 0$ hence $\gamma(Y, Z) > 0$. The converse is completely similar. □

In the next proposition we prove that $G_s(E)$ is an open subset of $G(E)$. A proof of this is due to E. Berkson, [Ber63] THEOREM 5.2, when $G(E)$ has the topology induced by the Schäffer metric. However the same proof works for the metric of geometric opening.

Proposition 1.4.3. *Let $X \in G_s(E)$ be a proper subspace of E. Let $Y \in G_s(E)$ be a topological complement of X. Denote by P the projector onto X along Y. If $Z \in G(E)$ and*

$$\rho_S(X, Z) < \gamma(X, Y), \tag{1.8}$$

$$\rho_S(Z, X) < \gamma(Y, X) \tag{1.9}$$

then $Z \oplus Y = E$. If Q is the projector onto Z along Y the operator $I + Q - P$ is invertible and maps X onto Z. Moreover

$$\|P - Q\| \leq \|I - P\| \frac{\|P\| \rho_S(X, Z)}{1 - \|P\| \rho_S(X, Z)} \tag{1.10}$$

Proof. First we prove that $Z \cap Y = \{0\}$ and $Z + Y$ is closed. In fact, given $y \in Z \cap Y$, $|y| = 1$, from (1.9) we can write

$$\operatorname{dist}(y, X) \leq \operatorname{dist}(y, S(X)) \leq \rho_S(Z, X) < \gamma(Y, X) \leq \operatorname{dist}(y, X)$$

which is absurd. To prove that $Y + Z$ is closed it will suffice to show that $\gamma(Z, Y) > 0$, by Proposition 1.4.2. Let $z \in S(Z)$ and $1 < \alpha$; there exists $x_\alpha \in S(X)$ such that

$$\alpha \operatorname{dist}(x_\alpha, Z) \geq |x_\alpha - z|;$$

for any $y \in Y$ we can write

$$\begin{aligned} |z - y| &\geq |x_\alpha - y| - |x_\alpha - z| \geq \operatorname{dist}(x_\alpha, Y) - \alpha \operatorname{dist}(x_\alpha, Z) \\ &\geq \gamma(X, Y) - \alpha \rho_S(X, Z); \end{aligned}$$

if $\alpha - 1$ is small the last term is positive. Taking the infimum over Y and $S(Z)$ we get $\gamma(Z, Y) > 0$, hence $Y + Z$ is closed. We prove now that $Z + Y = E$ by showing that $X \subseteq Z + Y$. Let $x \in X$ and $\lambda > 1$; by induction we can build two sequences $\{x_n\} \subset X$, $\{z_n\} \subset S(Z)$, such that

$$x_0 = x, \ |x_n - z_n| \leq \lambda \rho_S(X, Z) |x_n|, \ x_{n+1} = P(x_n - z_n) \tag{1.11}$$

$$x = \sum_{k=0}^{n} (z_k + y_{k+1}) + x_{n+1} \tag{1.12}$$

where $y_{k+1} = (I - P)(x_k - z_k)$. For every $k \in \mathbb{N}$ we also have, by induction

$$|x_k| \leq (\lambda \|P\| \rho_S(X, Z))^k |x_0|; \tag{1.13}$$

by (1.7) $\|P\| = \gamma(X,Y)^{-1}$ and (1.8) allows us to choose a positive λ such that $\lambda\rho_S(X,Z)\gamma(X,Y)^{-1} < 1$. Then $x_k \to 0$. Taking the limit in (1.12) we find $x \in \overline{Z+Y} = Z+Y$. The operator $I+Q-P$ maps X into Z and fixes Y. Since Q and P project along the same space a direct computation shows that its inverse is $I-Q+P$, thus $(I+Q-P)X = Z$. Choose $\lambda > 1$ and $v \in E$. We apply the construction made above to $x = Pv$. By (1.11) we have

$$\begin{aligned} |y_{k+1}| &\leq \|I-P\||x_k - y_k| \leq \lambda\|I-P\|\rho_S(X,Z)|x_k| \\ &\leq \frac{\|I-P\|}{\|P\|}(\lambda\|P\|\rho_S(X,Z))^{k+1}|x| \end{aligned}$$

by (1.12). If $\lambda\|P\|\rho_S(X,Z) < 1$ we have

$$|(P-Q)Pv| \leq \sum_{k=0}^{\infty}|y_{k+1}| \leq \|I-P\|\frac{\lambda\rho_S(X,Z)}{1-\lambda\|P\|\rho_S(X,Z)}|Pv|.$$

Letting $\lambda \to 1$, since $(P-Q)v = (P-Q)Pv$, we obtain (1.10). □

Corollary 1.4.4. *The subset $G_s(E)$ is open in $G(E)$ with the topology induced by the geometric opening.*

As we showed in the preceding example there are Banach spaces where the subset of splitting subspaces is not closed in the Grassmannian of closed subspaces. For sake of completeness we provide an example of Banach E (non-isomorphic to a Hilbert) where $G_s(E)$ is both open and closed. This is the case of $l^\infty(\mathbb{C})$. It is known that the closed and splitting subspaces of $l^\infty(\mathbb{C})$ are the non-separable ones. If a closed subspace X is limit of a sequence of closed and splitting subspaces in a ball centered in X of radius smaller than $1/2$ there are splitting subspaces. We use now a result of E. Berkson (Theorem 2.2 of [Ber63]): if $\delta_S(X,Y) < 1/2$ then the minimum cardinality of a dense subset of X is the same as that of Y. Thus, if Y splits it is not separable, hence X is not even separable, thus X splits.

Definition 1.4.5. *We define the space of projectors the closed subset*

$$\{P \in \mathcal{B}(E) \mid P^2 = P\}$$

endowed with metric of the operator norm and denote it by $\mathcal{P}(E)$.

Let X, Y be Banach spaces and $S \in \mathcal{B}(X,Y)$. We denote by $\operatorname{graph}(S)$ the graph of S, that is $\{(x, Sx) \mid x \in E\}$. Another consequence of Proposition 1.4.3 is the following

Proposition 1.4.6. *Let X and Y be Banach spaces. The map $\mathcal{B}(X,Y) \to G_s(X \times Y)$ that associates an operator with its graph is a homeomorphism with the open subset $\{Z \in GL_s(X\times Y) \mid Z \oplus \{0\} \times Y = X \times Y\}$.*

Proof. Since S is bounded $\operatorname{graph}(S)$ is closed and it is a topological complement of $\{0\}\times Y$, then it is an element of $G_s(X\times Y)$. Hence the map is well defined. For any $S \in \mathcal{B}(X,Y)$ define $\check{S}(x,y) = (x, y+Sx)$; it is an invertible operator. Since $\operatorname{graph}(S) = \check{S}(X \times \{0\})$, by Proposition 1.3.1 the map is continuous and injective. To prove that it is also open let $\operatorname{graph}(S)$ be a point in the image.

We show that there exists $r > 0$ such that $B(\mathrm{graph}(S), r) \subset \mathrm{Im}(\mathrm{graph})$, with the metric induced by δ_S. We choose

$$r < \hat{\gamma}(\mathrm{graph}(S), \{0\} \times Y);$$

given $Z \in B(\mathrm{graph}(S), r)$, by Proposition 1.4.3 Z is a topological complement of $\{0\} \times Y$. Thus, for every $x \in X \times \{0\}$ there exists a unique $z \in Z$ such that $Pz = x$. Then P maps isomorphically Z onto X and

$$\mathrm{graph}((I - P)P^{-1}_{|Z}) = Z$$

which concludes the proof. □

Given $X \in G_s(E)$ and Y such that $X \oplus Y$ we can identify X with $X \times \{0\}$, the graph of the null operator. The subset of topological complements of $X \times \{0\}$ is open and homeomorphic to the Banach space $\mathcal{B}(X, Y)$ by Proposition 1.4.6. Thus we have proved that

Corollary 1.4.7. *$G_s(E)$ is a topological Banach manifold.*

Definition 1.4.8. *Define the space of splits the subset*

$$\{(X, Y) \in G_s(E) \times G_s(E) \mid X \oplus Y = E\}$$

with the product metric $\delta_S \times \delta_S$ and denote it by $\mathrm{Splt}(E)$.

We can associate to a pair $(X, Y) \in \mathrm{Splt}(E)$ the projector $P(X, Y)$.

Proposition 1.4.9. *The map $P\colon \mathrm{Splt}(E) \to \mathcal{P}(E)$, $(X, Y) \mapsto P(X, Y)$ is a homeomorphism with its image.*

Proof. First observe that P is a bijection. Its inverse maps P to $(\mathrm{ran}\, P, \ker P)$. Suppose $(X_0, Y_0) = (\mathrm{ran}\, P_0, \ker P_0)$ and $\varepsilon > 0$. We prove that there exists $\delta > 0$ such that $P(B((X_0, Y_0), \delta)) \subseteq B(P_0, \varepsilon)$. More precisely, in a suitable neighbourhood of (X_0, Y_0), for every (X, Y) we can choose continuously an invertible operator U that maps X_0 and Y_0 onto X and Y respectively and

$$\|UP_0U^{-1} - P_0\| < \varepsilon. \tag{1.14}$$

This completes the proof because UP_0U^{-1} is a projector with range X and kernel Y. Thus UP_0U^{-1} is the projector onto X along Y. We construct U and δ as follows: as first step we choose $\delta_0 < \hat{\gamma}(X_0, Y_0)$. If $\delta_S(X_0, X) < \hat{\gamma}(X_0, Y_0)$ the Proposition 1.4.3 provides us with an operator $T = I + P(X, Y_0) - P_0$ and a positive constant c such that

$$TX_0 = X,\ TY_0 = Y_0,\ \|T - I\| < c\delta_S(X_0, X). \tag{1.15}$$

As second step we build another invertible operator S that maps Y_0 onto $T^{-1}Y$ and fixes X_0, applying the same Proposition. Hence $U = TS$ fits our request. This can be done if, for instance, $\delta_S(T^{-1}Y, Y_0) < \hat{\gamma}(X_0, Y_0)$. Using the estimate (1.3) we write

$$\delta_S(T^{-1}Y, Y_0) = \delta_S(T^{-1}Y, T^{-1}Y_0) \leq 2\|T\|\|T^{-1}\|\delta_S(Y_0, Y); \tag{1.16}$$

if $c\delta_S(X, X_0) < 1$, using Von Neumann series, we can estimate $\|T^{-1}\|$ with $1/(1 - \|I - T\|)$. Then the (1.16) becomes

$$\delta_S(T^{-1}Y, Y_0) \leq 2\frac{1 + c\hat{\gamma}(X_0, Y_0)}{1 - c\hat{\gamma}(X_0, Y_0)}\delta_S(Y_0, Y). \tag{1.17}$$

Then, if we choose

$$\delta_S(Y_0, Y) < \frac{1 - c\hat{\gamma}(X_0, Y_0)}{2(1 + c\hat{\gamma}(X_0, Y_0))} \tag{1.18}$$

we have $\delta_S(T^{-1}Y, Y_0) < \hat{\gamma}(X_0, Y_0)$ and it is possible to apply 1.4.3 and such operator S exists. By (1.10) and (1.17) we can write the (1.18) as

$$\|I - S\| < k\delta_S(Y_0, Y). \tag{1.19}$$

If we choose $\delta_1 = \min\{\delta_0, 1, 1/8k, 1/4c\}$, using (1.15) and (1.19) we can estimate the norm of the operator $U - I$ from above by

$$\begin{aligned} \|T(S - I) + T - I\| &\leq k(1 + c\delta_S(X_0, X))\delta_S(Y_0, Y) + c\delta_S(X_0, X) \\ &\leq 2k\delta_S(Y_0, Y) + c\delta_S(X_0, X) \leq 1/2. \end{aligned} \tag{1.20}$$

We can write $UP_0U^{-1} - P_0$ as $(U - I)P_0U^{-1} + P_0(U^{-1} - I)$. By (1.20) the norm of $I - U$ is strictly smaller than 1. Hence $\|U^{-1}\|$ can be estimated by $1/(1 - \|I - U\|)$ which is smaller than 2, still by (1.20). Then

$$\|UP_0U^{-1} - P_0\| \leq 4\|P_0\|\|I - U\| \leq 4\|P_0\|(2k\delta_S(Y_0, Y) + c\delta_S(X_0, X)).$$

Finally we set

$$\delta = \min\left\{\delta_1, \frac{\varepsilon}{4(2k + c)\|P_0\|}\right\}.$$

The continuity of the inverse follows at once: given $P, Q \in \mathcal{P}(E)$

$$\begin{aligned} \delta_S \times \delta_S((\operatorname{ran} Q, \ker Q), (\operatorname{ran} P, \ker P)) =& \delta_S(\operatorname{ran} Q, \operatorname{ran} P) + \delta_S(\ker Q, \ker P) \\ \leq& 4\|P - Q\|; \end{aligned}$$

in fact is Lipschitz. □

1.5 Compact perturbation of subspaces

The purpose of this section is to build suitable relations of *compact perturbation* for pairs in $G(E)$, where E is a Banach space, and define an integer for these pairs, called *relative dimension*. If such pairs lie in $G_s(E)$ this definition is meant to generalize the relative dimension known in Hilbert spaces.

First we need some preliminary concepts about Fredholm operators and compact operators. We recall some basic definitions and state some useful results about Fredholm operators and Fredholm pairs. For more details we refer to the Appendix B.

Given a linear operator $T : E \to F$ we can always consider the vector spaces $\ker T$ and $F/\operatorname{ran} T$. We denote the second by $\operatorname{coker} T$.

Definition 1.5.1. *A bounded operator $T \in \mathcal{B}(E,F)$ is called* semi-Fredholm *if and only if* $\operatorname{ran} T$ *is closed and either* $\ker T$ *or* $\operatorname{coker} T$ *has finite dimension. We define its* index *as*

$$\operatorname{ind}(T) = \dim \ker T - \dim \operatorname{coker} T.$$

Here ∞ and $-\infty$ are allowed. If both spaces have finite dimension we say that T is Fredholm *and the index is a integer.*

Definition 1.5.2. *A pair (X,Y) of closed and linear subspaces is said semi-Fredholm if and only if $X+Y$ is closed and either $X \cap Y$ or $E/(X+Y)$ has finite dimension. We define its* index *as*

$$\operatorname{ind}(X,Y) = \dim X \cap Y - \operatorname{codim} X + Y.$$

Of course the values ∞ and $-\infty$ are allowed. If both $X \cap Y$ and $X+Y$ have finite dimension the pair is said Fredholm.

There is a strict relation between (semi)Fredholm pairs and (semi)Fredholm operators. Precisely, given closed subspaces (X,Y) the operator

$$F_{X,Y} \colon X \times Y \to E, \quad (x,y) \mapsto x - y \tag{1.21}$$

is (semi)Fredholm if and only if (X,Y) is (semi)Fredholm and $\operatorname{ind}(X,Y) = \operatorname{ind}(F_{X,Y})$. Given Banach spaces E,F we denote by $\mathcal{B}_c(E,F)$ the set of compact operators.

Definition 1.5.3. *An operator $T \colon E \to F$ is said* essentially invertible *if and only if there exists $S \in \mathcal{B}(F,E)$ and compact operators $K \in \mathcal{B}_c(E)$, $H \in \mathcal{B}_c(F)$ such that*

$$\begin{aligned} S \circ T &= I_E + K \\ T \circ S &= I_F + H. \end{aligned}$$

It is not hard to prove that an operator is Fredholm if and only if is essentially invertible, see Proposition B.3. We end this section with a strong result of perturbation theory.

Theorem 1.5.4. (cf. [Kat95], Ch. IV, §5). Let (X,Y) be a semi-Fredholm pair. Then there exists $\delta > 0$ such that, $\delta_S(X',X) < \delta$, $\delta_S(Y',Y) < \delta$ implies that (X',Y') is semi-Fredholm and $\operatorname{ind}(X',Y') = \operatorname{ind}(X,Y)$.

Definition 1.5.5. (cf. Definition 1.1 of [AM01]). Two closed subspaces X and Y of a Hilbert spaces are compact perturbation one of each other if the orthogonal projections P_X and P_Y have compact difference. This implies that $X \cap Y^\perp$ and $X^\perp \cap Y$ are finite dimensional subspaces and the relative dimension is defined as

$$\dim(X,Y) = \dim(X \cap Y^\perp) - \dim(X^\perp \cap Y).$$

Our first aim is to define the relative dimension for pairs of closed subspaces that do not necessarily split.

Definition 1.5.6 (commensurability). *Let $X, Y \in G(E)$. The pair (X, Y) is said* commensurable *if there are $F, G \in \mathcal{B}(E)$ such that*

$$GX \subset Y,\ G_{|X} = (I + H)_{|X}, \tag{1.22}$$
$$FY \subset X,\ F_{|Y} = (I + K)_{|Y} \tag{1.23}$$

where H and K are compact operators.

Being commensurable is an equivalence relation. Symmetry and reflectivity are obvious. The proof of transitivity reduces to check that products of compact perturbations of the identity is a compact perturbation of the identity. From now on when X is commensurable to Y we will call the pair (X, Y) commensurable.

Proposition 1.5.7. *Let (X, Y) be a commensurable pair and (F, G) as above. The restrictions of F and G to Y and X, denoted by f and g respectively, are the essential inverse, one of each other, hence, by Proposition B.3, are Fredholm operators. Moreover, if (F', G') is another pair*

$$\operatorname{ind} f = \operatorname{ind} f',\ \operatorname{ind} g = \operatorname{ind} g'. \tag{1.24}$$

Proof. For every $t \in [0, 1]$ consider the convex combinations $F_t = (1-t)F + tF'$, $G_t = (1 - t)G + tG'$. It is easy to check that

$$f_t g_t = F_t G_{t|X} = I_X + k(t),$$
$$g_t f_t = G_t F_{t|Y} = I_Y + h(t)$$

where h and k are continuous paths of compact operators on Y and X respectively. Thus, for every t the operators f_t and g_t are the essential inverse one of each other. Taking $t = 0$, we obtain the first part of the statement. By ii) of Proposition B.5 continuous paths of Fredholm operators have constant index. Hence

$$\operatorname{ind} f = \operatorname{ind} f_0 = \operatorname{ind} f_1 = \operatorname{ind} f',$$
$$\operatorname{ind} g = \operatorname{ind} g_0 = \operatorname{ind} g_1 = \operatorname{ind} g'.$$

□

Definition 1.5.8 (relative dimension). *Let (X, Y) and (F, G) be as in the preceding definition. We define the* relative dimension *of the pair* $\operatorname{ind} g$ *and denote it by* $\operatorname{Dim}(X, Y)$.

The proposition proved above says that this definition does not depend on the choice of the pair of operators (F, G). Given X, Y, Z such that (X, Y) and (Y, Z) are commensurable the properties

$$\operatorname{Dim}(X, X) = 0,$$
$$\operatorname{Dim}(X, Y) = -\operatorname{Dim}(Y, X),$$
$$\operatorname{Dim}(X, Z) = \operatorname{Dim}(X, Y) + \operatorname{Dim}(Y, Z)$$

follow from the properties of composition of Fredholm operators stated in Proposition B.4. We give now a definition of compact perturbation for pair of splitting subspaces, useful for building examples.

Definition 1.5.9 (compact perturbation). *Let $X, Y \in G_s(E)$. We say that they are* compact perturbation *(one of the each other) if, given two projectors P and Q with ranges X and Y respectively, the operators*

$$(I - P)Q, \ (I - Q)P \tag{1.25}$$

are compact.

When (X, Y) is a pair of elements of the Grassmannian of splitting spaces commensurability and compact perturbation are equivalent.

Proposition 1.5.10. *Let X and Y closed and complemented subspaces of E. Then (X, Y) is a commensurable pair if and only if X is compact perturbation of Y.*

Proof. Suppose X is compact perturbation of Y and let P and Q be two projectors with ranges X and Y. Clearly $QX \subset Y$ and $PY \subset X$. Moreover,

$$Qx = Qx - x + x = -(I - Q)Px + x$$
$$Py = Py - y + y = -(I - P)Qy + y;$$

we obtain two restrictions of compact perturbation of the identity, as the definition of commensurability requires. Conversely let F and G be as in Definition 1.5.6 and (P, Q) a pair of projectors with ranges X and Y. We check, for instance, that $(I - P)Q$ is compact.

$$(I - P)Q = (I - P)(Q - FQ) + (I - P)FQ = (I - P)KQ + 0.$$

Similarly $(I - Q)P$ is compact. □

For sake of simplicity we will sometimes use the notation $\dim(P, Q)$ instead of $\dim(\operatorname{ran} P, \operatorname{ran} Q)$. Let H be a Hilbert space and (X, Y) a pair of two closed subspaces that are compact perturbation one of each other. Call P_X and P_Y the orthogonal projections. By (1.25) $P_{Y^\perp} P_X$ and $P_{X^\perp} P_Y$ are compact operators. Therefore

$$P_X - P_Y = (P_Y + P_{Y^\perp})P_X - P_Y(P_X + P_{X^\perp}) =$$
$$= P_{Y^\perp} P_X - P_Y P_{X^\perp} = P_{Y^\perp} P_X - (P_{X^\perp} P_Y)^* \in \mathcal{B}_c(E).$$

Hence P_X and P_Y have compact difference and the Definition 1.5.9 coincides with the one known for Hilbert spaces. The relative dimension can be computed as

$$\operatorname{Dim}(X, Y) = \dim \ker P_{Y|X} - \operatorname{coker} P_{Y|X} = \dim(X \cap Y^\perp) - \dim(X^\perp \cap Y)$$

which coincides with the definition of relative dimension in Hilbert spaces. Henceforth we will write $\dim(X, Y)$ instead of $\operatorname{Dim}(X, Y)$. In the following example we compute the relative dimension in some special case.

Example 2. Let V_0 and W_0 be finite dimensional subspaces and V_1 and W_1 topological complements of V_0 and W_0 respectively. We prove, using the result of Proposition 1.5.10, that (V_0, W_0) and (V_1, W_1) are commensurable pairs and compute their relative dimension. Let P and Q be two projectors onto V_0 and

W_0. Denote by q the restriction of Q to V_0. It is a linear map between finite dimensional subspaces, hence

$$\dim V_0 = \dim\ker q + \dim\operatorname{ran} q = \dim\ker q + \dim W_0 - \operatorname{coker} q$$

and the Fredholm index of q is the difference of the dimensions of V_0 and W_0. Now consider the pairs (V_1, E) and (E, W_1) and the pairs of projectors $(I-P, I)$, $(I, I-Q)$ Thus

$$\begin{aligned}\dim(V_1, E) &= \operatorname{ind} I_{|V_1} = -\operatorname{codim} V_1\\ \dim(E, W_1) &= \operatorname{ind} Q = \operatorname{codim} W_1\end{aligned}$$

hence $\dim(V_1, W_1) = \operatorname{codim} W_1 - \operatorname{codim} V_1$.

Example 3. In general it is not true that topological complements of two commensurable subspaces are commensurable. Given two splittings of the space

$$X \oplus X' = E = Y \oplus Y', \quad P = P(X, X'), \quad Q = P(Y, Y')$$

with X and X' compact perturbations of Y and Y' respectively, from the relations (1.5.9) it follows that

$$P - Q = (I - Q)P + P(I - Q)$$

is a compact operator. This is unlikely to happen even when X and Y are the same space. For instance let $X \subset E$ be a splitting subspace with a topological complement X' such that $\mathcal{B}_c(X', X) \subsetneq \mathcal{B}(X', X)$. For any $L \in \mathcal{B}(X', X) \setminus \mathcal{B}_c(X', X)$ define

$$P(L)(x, y) = (x + Ly, 0);$$

it is easy to check that $P(L)$ is a projector with range X and $P(L) - P$ is not compact. However for a given pair of two commensurable splitting subspaces a pair of projectors with compact difference always exists and we prove it in the next theorem. This is equivalent to find topological commensurable complements.

In the next proposition we describe the relation between the relative dimension and the Fredholm index of Fredholm pairs.

Proposition 1.5.11. *If X is compact perturbation of Y and (Y, Z) is a Fredholm pair, then (X, Z) is Fredholm and* $\operatorname{ind}(X, Z) = \dim(X, Y) + \operatorname{ind}(Y, Z)$.

Proof. Let P and Q be projectors with ranges X and Y respectively. The restrictions p and q to Y and X are Fredholm operators; we have

$$\begin{aligned}F_{X,Z}(x, z) &= x - z = x - Qx + Qx - z\\ &= (I - Q)Px + Qx - z = (I - Q)Px + F_{Y,Z}(Qx, z)\\ &= ((I - Q)P, 0_Z) \cdot (x, z) + F_{Y,Z} \circ (q, I) \cdot (x, z).\end{aligned} \tag{1.26}$$

Since $F_{Y,Z}$ and (q, I) are Fredholm their composition is Fredholm; the first summand of the last equation is compact. Hence $F_{X,Z}$ is a compact perturbation of a Fredholm operator and therefore Fredholm by Proposition B.2 and

$$\operatorname{ind} F_{X,Z} = \operatorname{ind} F_{Y,Z} \circ (q, I) = \operatorname{ind} F_{Y,Z} + \operatorname{ind}(q, I) = \operatorname{ind}(Y, Z) + \dim(X, Y)$$

by Proposition B.4. □

Example 4. We use Proposition 1.5.11 with in example that shows that for commensurable pairs there is not a result like the Theorem 1.5.4, that is, they are not stable by small perturbation: consider a pair (X, Y) such that

i). X is isomorphic to Y,

ii). $X \oplus Y = E$ has infinite dimension;

let $f\colon Y \to X$ be an isomorphism and $\operatorname{graph}(f)$ its graph. For every integer n consider the sequence of subspaces

$$Y_n = \operatorname{graph}(nf);$$

since Y_n is graph of a bounded operator $X \oplus Y_n = E$. It is easy to check that Y_n converges to X. Thus there can be no open neighbourhood of X in $G_s(E)$ made of compact perturbations of X. In fact for n large Y_n would be contained in such neighbourhood and (X, Y_n) would be a commensurable pair; since (X, Y_n) is a Fredholm pair also, by Proposition 1.5.11 we would have proved that (X, X) is a Fredholm pair which happens only if $X \oplus Y$ has finite dimension, in contradiction with hypothesis ii).

The preceding Proposition suggests a definition of the relative dimension that involves the Fredholm index. Precisely, suppose X is compact perturbation of Y. Let Z be a topological complement of Y. Then (Y, Z) is a Fredholm pair. By Proposition 1.5.11 (X, Z) is a Fredholm pair and

$$\operatorname{ind}(X, Z) = \operatorname{ind}(Y, Z) + \dim(X, Y) = \dim(X, Y). \tag{1.27}$$

This definition, together with the Theorem 1.5.4 will allows us to state in the next chapter a stability result of the relative dimension for closed and splitting subspaces.

Theorem 1.5.12. *Let X be a splitting subspace, compact perturbation of Y. Then there are topological complements X' and Y' that are compact perturbation one of each other and*

$$\dim(X, Y) = -\dim(X', Y')$$

Proof. Let P and Q be projectors with ranges X and Y respectively. As consequence of the Proposition 1.5.11 the pair $(X, \ker Q)$ is a Fredholm. Let Z be a topological complement of $X \cap \ker Q$ in $\ker Q$ and $R \subset E$ a finite dimensional complement of $X + \ker Q$ in E. Then

$$X \oplus Z \oplus R = E, \quad P_X + P_Z + P_R = I_E;$$

we claim that P_X and Q have compact difference. We write

$$P_X - Q = (I - Q)P_X + (P_X - Q)P_Z + (P_X - Q)P_R;$$

the first term of the right member is compact by definition of compact perturbation, the second is 0, the third has finite rank. Hence

$$Q(I - P_X), \quad P_X(I - Q)$$

are compact operators. It is not hard to prove that for all the pairs of projectors (P', Q') onto X' and Y' respectively, compactness of (1.25) holds, thus $\ker P_X$ and $\ker Q$ are commensurable spaces. To compute the relative dimension we use restrictions of the operators Q and $I - Q$. We can write

$$\dim(X, Y) + \dim(X', Y') = \operatorname{ind} Q_{|X} + \operatorname{ind}(I - Q)_{|X'} = \operatorname{ind} I_E = 0.$$

□

Proposition 1.5.13. *Let $T, S \in \mathcal{B}(E, F)$ be operators with compact difference and closed images. If the kernels and the images split* [2] $\ker T$ *and* $\operatorname{ran} T$ *are compact perturbation of* $\ker S$ *and* $\operatorname{ran} T$ *respectively and the relation*

$$\dim(\ker T, \ker S) = -\dim(\operatorname{ran} T, \operatorname{ran} S).$$

holds.

Proof. Since kernels and images split we can write

$$\ker T \oplus Y(T) = E = \ker S \oplus Y(S)$$
$$Z(T) \oplus \operatorname{ran} T = F = Z(S) \oplus \operatorname{ran} S$$

Since T and S are isomorphism of $Y(T)$ with $\operatorname{ran} T$ and $Y(S)$ with $\operatorname{ran} S$ respectively, we can define operators T' and S' on F with values in E such that

$$T'T = P(Y(T), \ker T), \quad S'S = P(Y(S), \ker S)$$
$$TT' = P(\operatorname{ran} T, Z(T)), \quad SS' = P(\operatorname{ran} S, Z(S));$$

set $P(T) = P(\ker T, Y(T))$, $P(S) = P(\ker S, Y(S))$ and $K = T - S$. Then

$$(I - P(S))P(T) = S'SP(T) = S'(S - T)P(T) + S'TP(T) = S'KP(T) + 0$$

is a compact operator. Set $Q(T) = P(\operatorname{ran} T, Z(T))$, $Q(S) = P(\operatorname{ran} S, Z(S))$. Then

$$\begin{aligned}(I - Q(S))Q(T) &= (I - Q(S))TT' = (I - Q(S))(T - S)T' + (I - P(S))ST' \\ &= 0 + (I - P(S))KT'\end{aligned}$$

is compact. By Theorem 1.5.12, up to changing the topological complements of $\ker T$ and $\operatorname{ran} T$, we can suppose that our projectors have compact difference. Hence

$$\begin{aligned}\dim(\ker T, \ker S) &= -\dim(Y(S), Y(T)) \\ = -\operatorname{ind}(I - P(T))^{Y(T)}_{|Y(S)} &= -\operatorname{ind} T(I - P(T))^{\operatorname{ran} T}_{|Y(S)} \\ \dim(\operatorname{ran} T, \operatorname{ran} S) &= \operatorname{ind} Q(T)^{\operatorname{ran} T}_{|\operatorname{ran} S} = \operatorname{ind}(Q(T)S)^{\operatorname{ran} T}_{|Y(S)};\end{aligned}$$

observe that the operator

$$K_1 = T(I - P(T)) - Q(T)S = TT'T - TT'S = TT'(T - S)$$

[2] Although we cannot think to any good reason why this result shouldn't be true for operators with compact difference, no matter if the kernels and images split or not, we were not able to find a proof to achieve this improvement.

is compact. Therefore

$$\begin{aligned}\dim(\ker T, \ker S) &= -\operatorname{ind}(Q(T)S + K_1)^{\operatorname{ran} T}_{|Y(S)} \\ &= -\operatorname{ind}(Q(T)S)^{\operatorname{ran} T}_{|Y(S)} = -\dim(\operatorname{ran} T, \operatorname{ran} S).\end{aligned}$$

□

Chapter 2

Homotopy type of Grassmannians

The purpose of this chapter is to define the space of *essentially hyperbolic* operators on a Banach space E, that we will denote by $e\mathcal{H}(E)$, and prove the existence of a group homomorphism

$$\varphi\colon \pi_1(e\mathcal{H}(E), 2P - I) \to \mathbb{Z}$$

where P is a projector of E. The construction of such homomorphism is carried out as follows: as first step, in section §2.1, we define the *Calkin algebra*, $\mathcal{C}(E)$, as the quotient of the algebra of bounded operators $\mathcal{B}(E)$ with the closed ideal of compact operators $\mathcal{B}_c(E)$. Then we prove that $e\mathcal{H}(E)$ is homotopically equivalent to $\mathcal{P}(\mathcal{C}(E))$, the space of *idempotent* elements of the Calkin algebra. In section §2.4 we prove that, the map

$$\mathcal{P}(E) \to \mathcal{P}(\mathcal{C}), \quad P \mapsto \mathfrak{p}(P) = [P]$$

is surjective and induces a locally trivial fibre bundle. Using the *Leray-Schauder degree* we prove in section §2.6 that the typical fiber of such bundle has infinite numerable connected components. Hence the exact homotopy sequence induces maps

$$\pi_1(\mathcal{P}(E), P) \xrightarrow{\mathfrak{p}_*} \pi_1(\mathcal{P}(\mathcal{C}), \mathfrak{p}(P)) \xrightarrow{\partial} \pi_0(\mathfrak{p}^{-1}(\{[P]\}), P);$$

the last is not a group homomorphism, because we do not have group structure on π_0. However in section §2.8 we show that the composition of ∂ with a suitable bijection with $\mathbb{Z}$ gives a homomorphism that we denote by φ and called *index* of exact sequence or simply *index* when no ambiguity occurs. All these facts are proved without making assumptions on the Banach space E. Given a projector P we list two sufficient conditions

h1) P is connected to a projector Q such that $\dim(Q, P) = 1$

h2) the connected component of P in $\mathcal{P}(E)$ is simply-connected

in order to make the index an isomorphism. These hypotheses are verified from any projection of a Hilbert space with infinite dimensional range and kernel.

In the most common Banach spaces such as L^p spaces and spaces of sequences there are existence of such projectors in infinite dimensional Hilbert space and in the most common in Hilbert spaces and However In the last section we give some example where the homomorphism φ is an isomorphism. This happens, for instance, if E is an infinite dimensional Hilbert space.

2.1 The homotopy type of the space of projectors

Given a Banach algebra $\mathcal{B}$ with unit 1, we denote by $\mathcal{B}^*$ the set of invertible elements. If $x \in \mathcal{B}$ the *spectrum* of x is defined as the set $\{\, \lambda \in \mathbb{C} \mid x - \lambda \cdot 1 \notin \mathcal{B}^* \,\}$ and denoted it by $\sigma_{\mathcal{B}}(x)$ or simply $\sigma(x)$. Consider the following subsets endowed with the topology of the norm

$$\mathcal{P}(\mathcal{B}) = \left\{\, p \in \mathcal{B} \mid p^2 = p \,\right\}, \quad \mathcal{Q}(\mathcal{B}) = \left\{\, q \in \mathcal{B} \mid q^2 = 1 \,\right\},$$
$$\mathcal{H}(\mathcal{B}) = \{\, x \in \mathcal{B} \mid \sigma(x) \cap \mathrm{i}\mathbb{R} = \emptyset \,\};$$

We call the elements of these spaces *projectors* (or *idempotents*), *square roots of unity* and *hyperbolic* respectively. In literature hyperbolic operators are sometimes defined as those whose spectrum does not intersect the unit circle; in this case *infinitesimally hyperbolic* would be more appropriate. The spaces $\mathcal{P}(\mathcal{B})$ and $\mathcal{Q}(\mathcal{B})$ are analytic closed embedded submanifolds of $\mathcal{B}$, see [AM03a], Lemma 1.4 for a proof; $\mathcal{H}(\mathcal{B})$ is an open subset of $\mathcal{B}$. An analytical diffeomorphism between $\mathcal{P}(\mathcal{B})$ and $\mathcal{Q}(\mathcal{B})$ also exists, given by

$$\mathcal{P}(\mathcal{B}) \ni p \mapsto 2p - 1 \in \mathcal{Q}(\mathcal{B}).$$

We want to prove here that these three spaces have the same homotopy type. Since $\mathcal{P}$ and $\mathcal{Q}$ are diffeomorphic they have the same homotopy type; in the next proposition we build a homotopy equivalence between $\mathcal{Q}$ and $\mathcal{H}$. We need some preliminary notations and facts about elementary spectral theory for Banach algebras. Let x be an element of the algebra $\mathcal{B}$ and $\{A_i\}$ a finite open cover of the spectrum of pairwise disjoint sets. There are projectors p_i called *spectral projectors* such that

$$p_1 + \cdots + p_n = 1, \quad p_i p_j = \delta_{ij}, \quad \sigma_{\mathcal{B}_i}(p_i x p_i) = A_i$$

where $\mathcal{B}_i$ is the subalgebra of the elements $p_i x p_i$ with $x \in \mathcal{B}$. We denote p_i also by $p(x; A_i)$. These projectors can be obtained as integrals

$$p(x; A_i) = \frac{1}{2\pi i} \int_{\gamma_i} (\lambda - x)^{-1} d\lambda$$

where γ_i are closed paths such that in the *contour* $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ each γ_i *surrounds* $A_i \cap \sigma(x)$ in $\mathbb{C} \setminus \cup_{j \neq i} A_j$ in the sense of Definition C.1 of Appendix C. We will also denote $p(x; A_i)$ by $p_{\gamma_i}(x)$.

Proposition 2.1.1. *The space of idempotents is a deformation retract of the space of hyperbolic elements.*

Proof. If q is a square root of identity its spectrum is contained in $\{-1, +1\}$, hence q is hyperbolic. Call i the inclusion of the space of idempotents in the space of hyperbolic elements. We define a retraction map as follows: let x be a hyperbolic element of the algebra; then

$$\sigma(x) = (\sigma(x) \cap \{\operatorname{Re} z > 0\}) \cup (\sigma(x) \cap \{\operatorname{Re} z < 0\})$$

is a disconnection of the spectrum of x. Denote by $p^+(x)$ and $p^-(x)$ the spectral projectors $p(x; (\sigma(x) \cap \{\operatorname{Re} z > 0\}))$ and $p(x; (\sigma(x) \cap \{\operatorname{Re} z < 0\}))$ respectively. We define a retraction as

$$r(x) = p^+(x) - p^-(x);$$

r is continuous by Theorem C.3 and its values are square roots of identity. We prove that r is a left inverse of the inclusion i. To do this let q be a square root and $\zeta \in \mathbb{C} \setminus \sigma(q)$, then

$$(\zeta - q)^{-1} = \frac{\zeta}{\zeta^2 - 1} + \frac{q}{\zeta^2 - 1} = \frac{1}{2}\left(\frac{1}{\zeta + 1} + \frac{1}{\zeta - 1}\right) + \frac{1}{2}\left(\frac{1}{\zeta - 1} - \frac{1}{\zeta + 1}\right) q;$$

let γ_+ and γ_- be paths that surrounds 1 and -1 in $\mathbb{C} \setminus \{1\}$ and $\mathbb{C} \setminus \{-1\}$, respectively. If we integrate both sides around γ_+ and γ_- and multiply it by $(2\pi i)^{-1}$ we obtain

$$p^+(q) = (1 + q)/2, \quad p^-(q) = (1 - q)/2, \quad r(q) = p^+(q) - p^-(q) = q;$$

this proves that the square roots of identity are retraction of hyperbolic. Now consider the homotopy defined on $[0, 1] \times \mathcal{H}(\mathcal{B})$ as

$$F(t, x) \mapsto (1 - t)p^+xp^+ + tp^+ + (1 - t)p^-xp^- - tp^-; \tag{2.1}$$

clearly $F(t, x)$ is hyperbolic for every (t, x); we have $F(0, x) = x$, $F(x, 1) = i \circ r(x)$. Thus the map $i \circ r$ is homotopically equivalent to $id_{\mathcal{H}}$. ☐

Definition 2.1.2. *Given an operator $T \in \mathcal{B}(E)$ we call* essential spectrum*, and denote it by $\sigma_e(T)$, the set* $\{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not Fredholm}\}$.

Definition 2.1.3. *A bounded operator T is called* essentially hyperbolic *if and only if $\sigma_e(T) \cap i\mathbb{R} = \emptyset$. We denote by $e\mathcal{H}(E)$ the set of essentially hyperbolic operators endowed with the norm topology.*

The set of compact operators on a Banach space E is a closed ideal of the algebra of bounded operators. Thus the quotient has a structure of Banach algebra that makes the projection

$$p \colon \mathcal{B}(E) \to \mathcal{B}(E)/\mathcal{B}_c(E)$$

an algebra homomorphism. The quotient space is called *Calkin algebra* and we denote it by $\mathcal{C}(E)$ or, when no ambiguity occurs, simply $\mathcal{C}$. We characterize the essential spectrum in terms of the Calkin algebra: given $T \in \mathcal{B}(E)$ there holds

$$\sigma_e(T) = \sigma(T + \mathcal{B}_c(E)). \tag{2.2}$$

To prove the equality suppose $\lambda \notin \sigma_e(T)$, hence $T-\lambda$ is Fredholm. By Proposition B.3 there exists an essential inverse S such that

$$(T-\lambda)S - I \quad S(T-\lambda) - I$$

are compact operators. Hence $p(T-\lambda)$ is invertible in the Calkin algebra and $p(S)$ is its inverse. The converse is similar.

Theorem 2.1.4. *The space $e\mathcal{H}(E)$ has the homotopy type of $\mathcal{P}(\mathcal{C})$.*

Proof. We prove first that $e\mathcal{H}(E)$ is homotopically equivalent to $\mathcal{H}(C)$. By classical results of continuous selections there exists a continuous right inverse of p, call it s. It is a consequence of Theorem D.1 when the topological space T consists of a point. Using the characterization (2.2) it is easy to check that $e\mathcal{H}(E) = p^{-1}(\mathcal{H}(\mathcal{C}))$. We have an homeomorphism

$$\Psi\colon \mathcal{H}(\mathcal{C}) \times \ker p \to e\mathcal{H}(E),\ (x, K) \mapsto s(x) + K. \tag{2.3}$$

Since $\ker p = \mathcal{B}_c(E)$ is a vector subspace of $\mathcal{B}(E)$, thus contractible, using convex combinations it is easy build a homotopy between the homeomorphism s. Now, by Proposition 2.1.1, $\mathcal{H}(\mathcal{C})$ has the same homotopy type of $\mathcal{Q}(\mathcal{C})$ which is homeomorphic to $\mathcal{P}(\mathcal{C})$. An explicit homotopy equivalence between $e\mathcal{H}(E)$ and $\mathcal{P}(\mathcal{C})$ is given by

$$e\mathcal{H}(E) \to \mathcal{P}(\mathcal{C}), \quad A \mapsto P(A + \mathcal{B}_c; \{\operatorname{Re} z > 0\}).$$

□

2.2 The principal bundle over the space of idempotents

In this section we describe the principal bundle over the space of idempotents. The exact homotopy sequence of the bundle gives some relations between the homotopy type of $\mathcal{P}(\mathcal{B})$ and the one of $\mathcal{B}^*$. We denote by $\mathcal{B}^*_0$ the connected component of $\mathcal{B}^*$ which contains the identity. For a projector $p \in \mathcal{P}$ set $\overline{p} = 1-p$. Clearly $\overline{p}$ is idempotent.

Proposition 2.2.1. (cf. [PR87], PROPOSITION 4.2). If p is idempotent the ball of radius 1 in $\mathcal{P}(\mathcal{B})$ is arcwise connected. In particular $\mathcal{P}(\mathcal{B})$ is locally arcwise connected.

Proof. We prove that, given p, q two idempotents such that $\|p-q\| < 1$, there exists an invertible element g such that $gpg^{-1} = q$ and g lies in the connected component of $\mathcal{B}^*$ which contains the identity. A path is provided explicitly by the formula

$$g(t) = (1-t) + tL(p,q),\ t \in [0,1]$$

where $L(p,q) = p\cdot q + \overline{p}\cdot\overline{q}$. Observe that $L(p,q)$ and $L(q,p)$ commute. In fact $L(p,q)L(q,p) = 1-(p-q)^2$. It is easy to check that

$$(1-t+tL(p,q))(1-t+tL(q,p)) = 1 - t(2-t)(p-q)^2;$$

since $t(2-t) \leq 1$ and $\|p-q\| < 1$ the right term is invertible, thus $g(t)$ is invertible and $g(0) = 1$. Moreover

$$g(1)qg(1)^{-1} = L(p,q)qL(p,q)^{-1} = p, \; p(t) = g(t)qg(t)^{-1}, \tag{2.4}$$

then $B(p,1)$ is connected. □

The construction made in the preceding Proposition can be generalized as follows: given a continuous path γ in the space of idempotents we choose a partition of the unit interval $\{t_i; 0 \leq i \leq n\}$ such that $\|\gamma(t_i) - \gamma(t_{i+1})\| < 1$ for every i. Keeping the notation of [PR87] it is defined

$$\mathcal{B}_0^* \ni \Pi_\gamma(\{t_i\}) = \prod_{i=0}^{n-1} L(\gamma(t_{n-i}), \gamma(t_{n-i-1})).$$

Using induction and (2.4) it follows that $\Pi_\gamma \gamma(a) \Pi_\gamma^{-1} = \gamma(b)$. This proves a classical result:

Proposition 2.2.2. *Two idempotents p and q in the same connected component of $\mathcal{P}(\mathcal{B})$ are conjugated by an element of $\mathcal{B}_0^*$.*

Given a connected component C of the space of idempotents we denote by $\mathcal{B}_C^*$ the group $\{g \in \mathcal{B}^* \mid gCg^{-1} = C\}$. Fix $C \subset \mathcal{P}(\mathcal{B})$ and $p \in C$. Let $F_p = \{g \in \mathcal{B}^* \mid gp = pg\}$.

Theorem 2.2.3. (cf. [PR87], Section 7). We have a principal fibre bundle $(\mathcal{B}_C^*, f_p, C, F_p)$ defined as $f_p(g) = gpg^{-1}$ for every $g \in \mathcal{B}_C^*$. The group F_p acts on itself as the left multiplication.

Proof. It only takes to exhibit coordinate neighbourhoods. To do this local sections of f_p are build as follows. Fix $q \in C$ and let $g \in \mathcal{B}_0^*$ be as in (2.2). On the ball $B(q,1)$ we define a local section

$$s_q \colon B(q,1) \to \mathcal{B}_C^*, \; r \mapsto L(r,q)g. \tag{2.5}$$

Clearly $f_p(s_q(r)) = r$. Using local sections we can define coordinate neighbourhoods as $\phi(x,y) = s_q(x) \cdot y$. We have $f_p(\phi(x,y)) = f_p(s_q(x)) = x$. If two coordinate neighbourhoods, $B(q_1,1)$, $B(q_2,1)$, intersect we have sections s_i and a homeomorphism

$$\phi_{2,x}^{-1}\phi_{1,x} \colon F_p \to F_p, \; y \mapsto s_2(x)^{-1}s_1(x)y;$$

since $s_2(x)^{-1}s_1(x) \in F_p$ we have a *principal bundle* according to [Ste51], §8. □

For principal bundles we can write the *exact homotopy sequence*, see [Ste51], §17. The sequence

$$\pi_k(F_p,1) \xrightarrow{i_*} \pi_k(\mathcal{B}_C^*,1) \xrightarrow{f_{p,*}} \pi_k(C,p) \xrightarrow{\partial} \pi_{k-1}(F_p,1) \tag{2.6}$$

is exact for every $k \geq 1$.

2.3 The Grassmannian algebra

Given p, q idempotents of an algebra $\mathcal{B}$ we define the following equivalence relation

$$p \sim q \iff pq = q,\ qp = p. \tag{2.7}$$

Symmetry and reflectivity are obvious. If (p, q) and (q, r) are equivalent pairs then $pr = p(qr) = (pq)r = qr = r$, similarly $rp = p$.

Definition 2.3.1. *We denote by $Gr(\mathcal{B})$ the set of equivalence classes endowed with the quotient topology.*

H. Porta and L. Recht proved in [PR87] that the Grassmannian algebra is a metric space, the canonical projection $\pi(\mathcal{B})\colon \mathcal{P}(\mathcal{B}) \to Gr(\mathcal{B})$ is an open map and there exists a global continuous section of π on $Gr(\mathcal{B})$. In fact any global continuous section is a homotopy inverse of π (see [PR87], §3).

When $\mathcal{B}$ is the algebra of the bounded operators on a Banach space E two projectors are equivalent if and only if they have the same images. In fact the identity $PQ = Q$ exactly means that $\operatorname{ran} Q \subseteq \operatorname{ran} P$. Then we have a well defined bijection

$$Gr(\mathcal{B}(E)) \to G_s(E), \quad \pi(P) \mapsto \operatorname{ran} P.$$

In this section we prove that this map is a homeomorphism. To achieve this result we construct a continuous section on $G_s(E)$ of the map that associates a projector with its range.

Lemma 2.3.2. *There exists a continuous section of the map that associates a projector with its range. Every section is in fact a homotopy equivalence.*

Proof. Call r the map $\mathcal{P}(E) \ni P \mapsto \operatorname{ran} P$. This is a continuous map, with the opening metric. In fact, given $P, Q \in \mathcal{P}(E)$, it can be easily checked that

$$\delta_S(r(P), r(Q)) \leq 2\|P - Q\|.$$

We can build now a continuous section of r using the construction of [Gęb68] whose idea is the following: fix X a splitting subspace and choose Y a topological complement. By Proposition 1.4.3, for every $X' \in B(X, \hat{\gamma}(X, Y))$, we have $X \oplus Y = E$. We define

$$s\colon B(X, \hat{\gamma}(X, Y)) \to \mathcal{P}(E),\ X' \mapsto P(X', Y);$$

by Proposition 1.4.3 this is a continuous local section of the map r. Since $G_s(E)$ is metric, thus paracompact, we refine the open covering $\{B(X, \hat{\gamma}(X, Y))\}$ to a locally finite one, say $\mathcal{U} = \{U_i \mid i \in I\}$. Let $\{\varphi_i\}$ be a partition of unit subordinate to $\mathcal{U}$. Thus for every X in $G_s(E)$ define

$$s(X) = \sum_{i \in I} \varphi_i(X) s_i(X),\ s \in C(G_s(E), \mathcal{B}(E)).$$

To prove that $s(X)$ is a projector observe that if $X \in U_i \cap U_j$ $\operatorname{ran} P(X, Y_i) = \operatorname{ran} P(X, Y_j) = X$. This is equivalent to

$$s_i(X) s_j(X) = s_j(X),\ s_j(X) s_i(X) = s_i(X);$$

keeping in mind these relations it is easy to prove that $s(X)$ is a projector with range X. In fact

$$s(X)^2 = \sum_i \varphi_i s_i(X) \left(\sum_j \varphi_j s_j(X) \right) = \sum_i \varphi_i \left(\sum_j \varphi_j(X) s_i(X) s_j(X) \right)$$
$$= \sum_i \varphi_i s(X) = s(X).$$

This also proves that $r^{-1}(\{X\})$ is a convex, actually affine, subspace of $\mathcal{P}(E)$. By construction $r \circ s = id$. For every projector P we have

$$r(s \circ r(P)) = r(P), \quad tP + (1-t)s \circ r(P) \in \mathcal{P}(E)$$

for every $t \in [0,1]$. This defines a homotopy between $s \circ r$ and the identity map. □

As application of the preceding Lemma we state a result of stability of the relative dimension defined on Chapter I.

Theorem 2.3.3. *Let X and Y be continuous functions defined on a topological space M such that $X(t)$ and $Y(t)$ are closed and splitting subspaces and $X(t)$ is compact perturbation of $Y(t)$ for every t in M. Hence* $\dim(X(t), Y(t))$ *is locally constant.*

Proof. Let s be a continuous section on $G_s(E)$ of the map r defined in the Lemma 2.3.2. Then it is defined a continuous map

$$\nu \colon G_s(E) \to G_s(E), \quad X \mapsto \ker s(X).$$

By the identity (1.27) the relative dimension of the pair $(X(t), Y(t))$ is the Fredholm index of the pair $(X(t), \nu(Y(t)))$. Fix t_0 in M; by Theorem 1.5.4 there exists a open neighbourhood of t_0, say U, such that

$$\operatorname{ind}(X(t), \nu(Y(t))) = \operatorname{ind}(X(t_0), \nu(Y(t_0)))$$

for every $t \in U$. Therefore we conclude with (1.27). □

Theorem 2.3.4. *If $\mathcal{B}$ is the algebra of bounded operators on E then $Gr(\mathcal{B})$ with the quotient topology is homeomorphic to $G_s(E)$ with the topology induced by the metric δ_S.*

Proof. Let s and $s(\mathcal{B})$ be sections on $G_s(E)$ and $Gr(\mathcal{B})$ respectively. We prove that the maps $\pi \circ s$ and $r \circ s(\mathcal{B})$ are inverse one of each other. Let X be a closed splitting subspace. Then

$$s(\mathcal{B})((\pi \circ s)(X)) \sim s(X)$$

then $r(s(X)) = X$. Thus $(r \circ s(\mathcal{B})) \cdot (\pi \circ s) = id$. Similarly we have $(\pi \circ s) \cdot (r \circ s(\mathcal{B})) = id$. □

2.4 Fibrations of spaces of idempotents

Set $\mathcal{B} = \mathcal{B}(E)$; we recall that the Calkin algebra is defined as the quotient algebra $\mathcal{C} = \mathcal{B}(E)/\mathcal{B}_c(E)$ where $\mathcal{B}_c(E)$ is the ideal of compact operators on E. It is a Banach algebra with unit. The projection to the quotient $p\colon \mathcal{B} \to \mathcal{C}$ is a surjective homomorphism. Consider the restrictions

$$\mathfrak{p}\colon \mathcal{P}(E) \to \mathcal{P}(\mathcal{C})$$
$$\mathfrak{q}\colon \mathcal{Q}(E) \to \mathcal{Q}(\mathcal{C}).$$

The purpose of this section is to prove that these maps induce locally trivial bundle, with nonconstant fiber. First we need the following

Proposition 2.4.1. (cf. [AM03a], Proposition 6.1). The maps $\mathfrak{p}$ and $\mathfrak{q}$ are surjective and admit local sections.

Proof. It is enough to prove it for $\mathfrak{q}$, since $\mathcal{P}$ is homeomorphic to $\mathcal{Q}$. Since p is surjective, by Proposition D.3 there exists a right inverse of p on $\mathcal{C}$, call it s. Let q be a square root of identity in the Calkin algebra and set $Q = s(q)$. There exists a compact operator K such that $s(Q^2) = I + K$. The spectrum of $I + K$ is a countable subset of $\mathbb{C}$ with at most 1 as limit point. Since $\sigma(Q)^2 = \sigma(Q^2)$ the spectrum of Q is also countable with at most two limit points, -1 and $+1$. Let $0 < \delta < 1$ be such that the boundary of the open subset $U = \left\{ z \in \mathbb{C} \mid |1 - z^2| < \delta \right\}$ does not meet $\sigma(Q)$. Define

$$U^+ = U \cap \{\mathrm{Re}\, z > 0\},\ U^- = U \cap \{\mathrm{Re}\, z < 0\},\ V = \sigma(Q) \setminus U$$

and $P_0, P_\pm$ the spectral projectors. Since 0 is isolated in $\sigma(Q)$ the rank of P_0 is finite. There exists $\varepsilon > 0$ such that, for any $q' \in B(q, \varepsilon)$

$$\sigma(s(q')) \cap \partial U = \emptyset.$$

A local section is given by $\mathfrak{s}(q') = P_0(s(q')) + P^+(s(q')) - P^-(s(q'))$. □

Theorem 2.4.2. *The map $\mathfrak{p}\colon \mathcal{P}(E) \to \mathcal{P}(\mathcal{C})$ induce a locally trivial bundle.*

Proof. Given a connected component $D \subset \mathcal{P}(\mathcal{C})$ and x_0 in D we have a fibre bundle $(\mathcal{P}(E), \mathfrak{p}, \mathcal{P}(\mathcal{C}), \mathfrak{p}^{-1}(\{x_0\}))$. Given a point $x \in D$ we observe that if there exists $g \in \mathcal{C}^*$ and $G \in GL(E)$ such that

$$p(G) = g,\ gx_0g^{-1} = x$$

there is a homeomorphism between the fibers of p_0 and q that maps x in GxG^{-1}. In fact

$$p(GxG^{-1}) = p(G)p(x)p(G)^{-1} = gx_0g^{-1} = q.$$

This suggests how to construct coordinate neighbourhoods. We just have to choose such g and G continuously. As first step we define a trivialization map on a neighbourhood of x_0. By (2.5) on the ball centered in x_0 of radius 1 it is defined a continuous map

$$s\colon B(x_0, 1) \to \mathcal{C}_D^*,\quad s(x_0) = 1,\quad f_{x_0} \circ s = id.$$

By Proposition D.4 there exists an open neighbourhood of the unit of $\mathcal{C}_D^*$ and a local section

$$\Gamma\colon V \to GL(E),\ p\circ\Gamma = id,\ \Gamma(1) = 1.$$

As coordinate neighbourhood we choose $U_{x_0} = B(x_0,1)\cap s^{-1}(V)$. Setting $T = \Gamma\circ s$ a trivialization of U_{x_0} is given by

$$\phi\colon U_{x_0}\times p^{-1}(\{x_0\}) \to p^{-1}(U_{x_0}),\quad \phi(x,y) = T(x)yT(x)^{-1}.$$

To extend this construction to a neighbourhood of any point $z\in D$ we argue as follows. Let γ be a continuous path with endpoints x_0 and z and $\{t_i\}$ a partition of the unit interval such that

$$\|\gamma(t_i)-\gamma(t_{i+1})\| < 1,\quad 1\le i\le n-1;$$

this defines an invertible element Π_γ such that $\Pi_\gamma x_0\Pi_\gamma^{-1} = z$. If we refine the partition we can suppose that $L(\gamma(t_i),\gamma(t_{i+1}))$ belongs to V. Let

$$G = \prod_{i=1}^{n-1}\Gamma(L(\gamma(t_i),\gamma(t_{i+1})));$$

clearly $p(G) = \Pi_\gamma$. The subset $U_z = gU_{x_0}g^{-1}$ is a neighbourhood of z. We define a homeomorphism of the cartesian product $U_z\times p^{-1}(\{x_0\})$ with $p^{-1}(U_z)$ as

$$\phi(x,y) = GT(g^{-1}zg)yT(g^{-1}zg)^{-1}G^{-1}.$$

The left composition with p gives the projection on the first factor of the product, in fact

$$\begin{aligned}\mathfrak{p}\circ\phi(x,y) &= p(G)p(T(g^{-1}xg))yp(T(g^{-1}xg)^{-1})^{-1}p(G)^{-1}\\ &= gp[T(g^{-1}xg)yT(g^{-1}xg)^{-1}]g^{-1}\\ &= gg^{-1}x(gx_0g^{-1})xgg^{-1} = x^3 = x.\end{aligned}$$

□

2.5 The essential Grassmannian

In $\mathcal{P}(E)$ and $G_s(E)$ consider the relation of compact perturbation. We write $X\sim_c Y$ if and only if X is compact perturbation of Y in the sense of Definition 1.5.9 and $P\sim_c Q$ if and only if they have compact difference. Given $X\in G_s(E)$ and $P\in\mathcal{P}(E)$ we define

$$\begin{aligned}\mathcal{P}_c(P;E) &= \{Q\in\mathcal{P}(E)\mid P\sim_c Q\}\\ G_c(X;E) &= \{Y\in G_s(E)\mid X\sim_c Y\}\end{aligned}$$

endowed with the topology of subspace. We denote by $\mathcal{P}_e(E)$ and $G_e(E)$ the quotient spaces, endowed with the quotient topology. The latter is called *essential Grassmannian*, see [AM03a], §6. Let Π_e and π_e be the projections onto the quotient spaces. By Proposition 2.4.1 the map

$$\mathfrak{p}\colon\mathcal{P}(E)\to\mathcal{P}(\mathcal{C})$$

is open. Moreover two projector belong to the same class of compact perturbation if and only if their difference is compact, hence the map induced to the quotient

$$\mathfrak{p}_e\colon \mathcal{P}_e(E) \to \mathcal{P}(\mathcal{C}).$$

is a homeomorphism. As we will see in the next Proposition the same relation holds between the essential Grassmannian and the Grassmannian algebra of $\mathcal{C}(E)$. Define r_e the function on $\mathcal{P}_e(E)$ that maps a class of compact perturbation $[P]$ in the class $[r(P)]$.

Proposition 2.5.1. *There exists a homeomorphism g_e of the essential Grassmannian with the Grassmannian algebra of $\mathcal{C}(E)$ such that the diagram*

$$\begin{array}{ccc} \mathcal{P}_e(E) & \xrightarrow{\mathfrak{p}_e} & \mathcal{P}(\mathcal{C}) \\ \downarrow{\scriptstyle r_e} & & \downarrow{\scriptstyle \pi} \\ G_e(E) & \xrightarrow{g_e} & Gr(\mathcal{C}) \end{array}$$

commutes.

Proof. Let $[P]$ and $[Q]$ be points of $\mathcal{P}_e(E)$ such that $r_e([P]) = r_e([Q])$. By definition of r_e the subspace $r(P)$ is a compact perturbation of $r(Q)$. By Proposition 1.5.10 $(I-P)Q$ and $(I-Q)P$ are compact operators. Thus, by (2.7), $\pi(P+\mathcal{B}_c) = \pi(Q+\mathcal{B}_c)$. Then

$$\pi \circ \mathfrak{p}_e([P]) = \pi \circ \mathfrak{p}_e([Q]).$$

Given $[X] \in G_e(E)$ define $g_e(X)$ as $\pi(P+\mathcal{B}_c)$, where $r(P)$ is compact perturbation of X. Then g_e is well defined, injective and

$$g_e \circ r_e = \pi \circ \mathfrak{p}_e.$$

The continuity of g_e follows from the fact that $Gr(\mathcal{C})$ has the quotient topology. It is surjective because π and $\mathfrak{p}_e$ are surjective. We prove that g_e is an open map: let U be an open subset of the essential Grassmannian; since the Grassmannian algebra has the quotient topology and $\mathfrak{p}_e$ is a homeomorphism $g_e(U)$ is open if and only if

$$\mathfrak{p}_e^{-1}(\pi^{-1}(g_e(U)))$$

is open. This subset can be written as

$$\begin{aligned} \mathfrak{p}_e^{-1}(\pi^{-1}(g_e(U))) &= (\pi \circ \mathfrak{p}_e)^{-1}(g_e(U)) \\ =&(g_e \circ r_e)^{-1}(g_e(U)) = r_e^{-1}(g_e^{-1}(g_e(U))); \end{aligned}$$

since g_e is injective the last term is $r_e^{-1}(U)$ which is open by continuity of r_e. □

We recall that, by [PR87], §3, π is a homotopy equivalence. Hence as a corollary of the preceding result r_e is also a homotopy equivalence. A homotopy inverse of r_e is $\mathfrak{p}_e^{-1}s(\mathcal{C})g_e$ where $s(\mathcal{C})$ is a right inverse of π.
We conclude this section by showing that the spaces defined at the beginning of the section have the same homotopy type. This fact together with the previous proposition will allow us to switch safely from Grassmannians to spaces of idempotents without changing the homotopy type.

Proposition 2.5.2. *Let X be a splitting subspace of E and let P be a projector with range X. The restriction of r to $\mathcal{P}_c(P;E)$ takes values in $G_c(X;E)$ and is a homotopy equivalence.*

Proof. Let r_c be the restriction of r. To achieve this result we follow the same steps of Lemma 2.3.2. Fix X_0 compact perturbation of X. By Theorem 1.5.12 there exists a projector P_0 with range X_0 such that $P_0 - P$ is compact. Call Y_0 its kernel and define the local section

$$s_0 \colon B(X_0, \hat{\gamma}(X_0, Y_0)) \to P(E),\ X' \mapsto P(X', Y_0).$$

This is continuous by Proposition 1.4.3. Since $r(s_0(X')) = X'$, by Proposition 1.5.10 the operators $(I - s_0(X'))P_0$ and $(I - P_0)s_0(X')$ are compact. The relation $\ker s_0(X') = \ker P_0$ implies $s(X')(I - P_0) = 0$, therefore

$$P_0 - s(X') = (I - s(X'))P_0 + (P_0 - s(X'))(I - P_0) = (I - s(X'))P_0$$

which is compact. Then $s(X') - P$ is compact. Let $\mathcal{U} = \{U_i \mid i \in I\}$ be a locally finite refinement of $\{B(X_0, \hat{\gamma}(X_0, Y_0)) \mid X_0 \in G_c(X;E)\}$ and $\{\varphi_i \mid i \in I\}$ a partition of unit subordinate to $\mathcal{U}$. Then, for any $Y \in G_c(X;E)$

$$s(Y) - P = \sum_{i \in I} \varphi_i(Y)(s_i(Y) - P)$$

is a finite sum of compact operators. The convex combination of $s \circ r_c$ and id is a homotopy map. □

2.6 The Fredholm group

We call *Fredholm group* the set of invertible operator on a Banach space that can be written as the sum of the identity and a compact operator. It is a normal subgroup of $GL(E)$. The Fredholm group is endowed with the norm topology; we denote it by $GL_c(E)$.

Theorem 2.6.1. *If E is an infinite dimensional Banach space over $\mathbb{F}$, that is $\mathbb{R}$ or $\mathbb{C}$, the Fredholm group has the homotopy type of* $\mathrm{Lim}\, GL(n, \mathbb{F})$.

For the proof see, for instance, [Gęb68]. The homotopy groups of the Fredholm group are, in the real and complex case, respectively

$$\pi_i(GL(\infty, \mathbb{R})) \cong \begin{cases} \mathbb{Z}_2 & i \equiv 0, 1 \mod 8 \\ 0 & i \equiv 2, 4, 5, 6 \mod 8 \\ \mathbb{Z} & i \equiv 3, 7 \mod 8 \end{cases} \tag{2.8}$$

$$\pi_i(GL(\infty, \mathbb{C})) \cong \begin{cases} 0 & i \equiv 0 \mod 2 \\ \mathbb{Z} & i \equiv 1 \mod 2 \end{cases} \tag{2.9}$$

see Theorem II of [Bot59]. The spectrum of $T \in GL_c(E)$ is countable, and $\sigma(T) \setminus \{1\}$ is made of eigenvalues of finite multiplicity. When E is a real Banach space it is defined the *Leray-Schauder* degree as

$$\deg(T) = (-1)^{\beta(T)}$$

where $\beta(T)$ is the sum of the algebraic multiplicities of negative eigenvalues. It is well defined on the connected components of $GL_c(E)$ and defines a group isomorphism

$$\deg\colon \pi_0(GL_c(E)) \to \{-1,+1\} \cong \mathbb{Z}_2.$$

See [Llo78] for details. The L.S. degree will help us to determine the connected components of $G_c(X;E)$ when E is a real or complex Banach space. We will prove that $G_c(X;E)$ consists of infinitely numerable components; these are

$$G_k(X;E) = \{Y \in G_c(X;E) \mid \dim(X,Y) = k\}\,,\ k \in \mathbb{Z}. \tag{2.10}$$

Lemma 2.6.2. *The Fredholm group acts transitively on each $G_k(X;E)$ by the left multiplication. Moreover there are local sections of the action.*

The carrying out of the proof follows the same steps of the Hilbert case outlined in [AM03a], §5. A slight difficulty arises as we do not have a natural section as the orthogonal projection, but it can be overcome using any section that preserves the relation of compact perturbation of closed and complemented subspaces as the one found in Proposition 2.5.2.

Proof. Let $Y \in G_k$ and $T \in GL_c(E)$. Let t be the restriction of T to Y and $i\colon Y \hookrightarrow E$ the inclusion. Both $t, i \in \mathcal{B}(Y,E)$ are injective and $t-i$ is compact. Hence, by Proposition 1.5.13 $\operatorname{ran} t$ and $\operatorname{ran} i$ are compact perturbation one of each other and

$$\dim(Y,TY) = \dim(\operatorname{ran} i, \operatorname{ran} t) = \dim(\ker t, \ker i) = 0.$$

Hence $TY \in G_k$. Let $Y,Z \in G_k(X;E)$, hence $\dim(Y,Z) = 0$. Let s be a continuous right inverse of r_c as in Proposition 2.5.2. The operator $s(Z)-s(Y)$ is compact, call it K. Observe that the restriction of $s(Z)$ to Y, considered as operator with values in Z, is Fredholm. Similarly we can consider the restriction of $I-s(Z)$ to $Y' := \ker s(Y)$ with values in $\ker s(Z)$. For every y in Y and $y' \in Y'$ we can write

$$\begin{aligned} s(Z)y &= s(Y)y + Ky = (I+K)y \\ (I-s(Z))y' &= (I-s(Y))y' - Ky' = (I-K)y'. \end{aligned}$$

The Fredholm index of these operators is 0 by definition of relative dimension. Fredholm application of index 0 have a nice property: they are perturbation of an isomorphism by a finite rank operator. Then we can choose R_1 in $\mathcal{B}(Y,Z)$ and R_2 in $\mathcal{B}(Y', \ker s(Z))$ suitable finite rank operators. Call T the operator obtained as direct sum of the two isomorphisms $s(Z)_{|Y} + R_1$ and $(I-s(Z))_{|Y'} + R_2$. It is invertible, maps Y onto Z and can be written as

$$I + (K+R_1)s(Y) - (K-R_2)(I-s(Y))$$

hence belongs to the Fredholm group. This proves that the action is transitive. Given $Y \in G_k$ we build a local section around Y as follows: let s be a continuous section as in Proposition 2.5.2. There exists $\varepsilon > 0$ such that, for any $Z \in B(Y,\varepsilon)$ the operator $L(s(Z), s(Y))$ is invertible. Since

$$L(s(Z), s(Y)) = I + (2s(Z)-I)(s(Y)-s(Z)) \in I + \mathcal{B}_c(E)$$

$L(s(Z), s(Y)) \in GL_c(E)$ and $L(s(Z), s(Y))Y = Z$. Then a local section of the action is defined as

$$B(Y, \varepsilon) \to GL_c(E) \times G_k, \ Z \mapsto (L(s(Z), s(Y)), Y). \tag{2.11}$$

□

Theorem 2.6.3. *The connected components of $G_c(X; E)$ are $G_k(X; E)$ with k in $\mathbb{Z}$.*

Proof. Let $Y, Z \in G_c(X; E)$, connected by an arc, $k = \dim(X, Y)$. Given a map as in Proposition 2.5.2 there exists a path γ in $\mathcal{P}_c(P; E)$ that connects $s(Y)$ to $s(Z)$. Let $\{t_i\}$ be a partition of the unit interval such that $L(\gamma(t_{i+1}), \gamma(t_i))$ is invertible. Let Π_γ be the invertible element defined as (2.2)

$$\Pi_\gamma = \prod_{i=0}^{n-1} L(\gamma(t_{i+1}), \gamma(t_i)) \in GL_c(E), \ \Pi_\gamma Y = Z;$$

by the Lemma 2.6.2 the Fredholm group acts on G_k, hence Z also belongs to G_k. Conversely consider $Y, Z \in G_k$. Hence $\dim(Y, Z) = 0$ and, by Lemma 2.6.2, there exists $T \in GL_c(E)$ such that $TY = Z$. If E is a complex Banach space the Fredholm group is arcwise connected. Given a path α that connects I to T the path $\alpha(t)Y$ connects Y to Z. If E is a real Banach space we can choose $A \in GL_c(Y)$ such that $\deg(A) = \deg(T)$. Let $L = T(I_Y \oplus A)$. By elementary properties of the L.S. degree we have

$$LY = Z, \ L \ker s(Y) = \ker s(Z), \ \deg(L) = (\deg T)^2 = 1;$$

hence L is connected to I and we conclude as in the complex case. □

2.7 The Stiefel space

In this section we introduce the *Stiefel spaces* and compute its homotopy type in some case. Using the exact sequence of fibre bundle we determine the homotopy groups of $G_c(X; E)$ for some $X \in G_s(E)$.

Definition 2.7.1. *Let $X \in G_s(E)$. We define the* Stiefel space*, and denote it by $St(X; E)$, the set $\{ f \in \mathcal{B}_c(X, E) \mid f - i \text{ is compact}\}$, endowed topology of subspace, i is the inclusion of X into E.*

The Stiefel space is an analytical manifold because is an open subset of the affine space $I + \mathcal{B}_c(X, E)$. We recall some results on the homotopy type of $St(X; E)$.

Theorem 2.7.2. (cf. [DD63]). If X is a finite-dimensional subspace of E $St(X; E)$ is contractible.

Using the techniques of [Gęb68] it is possible to prove that when X has infinite dimension and infinite codimension $St(X; E)$ is contractible. Then, if X has infinite codimension $St(X; E)$ is always contractible. The following result is known for Hilbert space, see for example [AM03a] §5. The generalization to Banach spaces involves, as Lemma 2.6.2 does, the Proposition 2.5.2.

Theorem 2.7.3. *Let $r_{St}\colon St(X;E) \to G_0(X;E)$ be the continuous map defined as $r_{St} f = f(X)$. Then $(St(X;E), r_{St}, G_0(X;E), GL_c(X))$ is a principal fibre bundle. The action of $GL_c(X)$ onto itself is the left multiplication.*

Proof. As first step we build a local section around X. Consider a continuous map as in Proposition 2.5.2. Let U be an open neighbourhood of X where (2.11) is defined. Define

$$\gamma_0 \colon U \to St(X;E),\ Y \mapsto L(s(Y), s(X))_{|X}.$$

This suffices to build an open cover of coordinate neighbourhoods of G_0. Given $Y \in G_0$, by Lemma 2.6.2 there exists $T \in GL_c(E)$ such that $TX = Y$. Then a trivialization of $T(U)$ is given by

$$\phi \colon T(U) \times GL_c(X) \to r_{St}^{-1}(T(U)),\ (Y', g) \mapsto T\gamma_0(T^{-1}Y')g.$$

We have to check that whenever two coordinate neighbourhoods U_i, U_j intersect for every $X_0 \in U_i \cap U_j$ the transitions maps are left translations of $GL_c(X)$ onto itself. In fact, given T_i, T_j such that $T_i X = T_j X = X_0$ the transition map is

$$\phi_{j,X_0}^{-1} \phi_{i,X_0} g = \gamma_0(T_j^{-1} X_0)^{-1} T_j^{-1} T_i \gamma_0(T_i^{-1} X_0) g$$

is the left multiplication by an element of $GL_c(X)$. Then we have $GL_c(X)$ compatibility. ☐

When $X \subset E$ has infinite codimension and infinite dimension the exact sequence of the principal bundle $(St(X;E), r_{St}, G_0(X;E), GL_c(X))$ gives isomorphisms

$$\pi_i(G_0(X;E), X) \cong \pi_{i-1}(GL_c(X)) \cong \pi_{i-1}(GL(\mathbb{F}, \infty)),\ i \geq 1$$

where $\mathbb{F}$ is the real or complex field.

2.8 The index of the exact sequence

Using exact sequence of the fibre bundle $(\mathcal{P}(E), \mathcal{P}(\mathcal{C}), \mathfrak{p})$ we show how to associate an integer to a closed loop in the space of idempotents of $\mathcal{C}(E)$. In fact we define a group homomorphism on $\pi_1(\mathcal{P}(\mathcal{C}))$ denoted by φ. Since $\mathcal{P}(\mathcal{C})$ is homotopically equivalent to the space of essentially hyperbolic operators on E, we definitely have a group homomorphism on $\pi_1(e\mathcal{H}(E))$ obtained as the composition of φ with Ψ_*^{-1}.
Let P be any projector. By Theorem 2.4.2 the triple $(\mathcal{P}(E), \mathfrak{p}, \mathcal{P}(\mathcal{C}))$ is a locally trivial bundle. The typical fiber of $[P]$ is $\mathcal{P}_c(P;E)$. Then the exact homotopy sequence gives

$$\pi_1(\mathcal{P}(E), P) \xrightarrow{\mathfrak{p}_*} \pi_1(\mathcal{P}(\mathcal{C}), [P]) \xrightarrow{\partial} \pi_0(\mathcal{P}_c(P;E)) \xrightarrow{i_*} \pi_0(\mathcal{P}(E), P).$$

The map ∂ is not a group homomorphism because $\pi_0(\mathcal{P}_c(P;E))$ has no group structure. However, by Theorem 2.6.3 there is a bijection $\pi_0(\mathcal{P}_c(P;E)) \to \mathbb{Z}$ that maps the connected component of a projector Q in $\dim(Q, P)$.

Theorem 2.8.1. *There exists a group homomorphism $\varphi\colon \pi_1(\mathcal{P}(\mathcal{C}), [P]) \to \mathbb{Z}$ such that $\varphi(x) = \dim(\partial x, P)$.*

Proof. First observe that, given $\beta, \beta' \in C(I, \mathcal{P}(E))$ such that $\mathfrak{p} \circ \beta = \mathfrak{p} \circ \beta'$, $\dim(\beta(t), \beta'(t))$ is constant. This follows from the Theorem 2.3.3. Let a, b be two closed paths at the base point $[P]$. There are two lifting paths α, β such that

$$\alpha(0) = P, \ \mathfrak{p} \circ \alpha = a,$$
$$\beta(0) = P, \ \mathfrak{p} \circ \beta = b.$$

There also exists β' such that $\beta'(0) = \alpha(1)$ and $p \circ \beta' = b$. Define

$$\gamma = \alpha * \beta', \ \gamma(0) = \alpha(1)$$

which is a lifting path for $a * b$. Since β and β' are lifts of the same path b there exists $k \in \mathbb{Z}$ such that

$$\dim(\beta(t), \beta'(t)) = k, \quad \text{for every } t \in [0, 1].$$

We prove now that φ is a group homomorphism.

$$\begin{aligned}\varphi(a * b) &= \dim(\partial\gamma, P) = \dim(\beta'(1), \alpha(0)) = \dim(\beta'(1), \beta'(0)) + \dim(\alpha(1), \alpha(0)) \\ &= \dim(\beta'(1), \beta(1)) + \dim(\beta(1), \beta(0)) + \dim(\beta(0), \beta'(0)) + \varphi(a) \\ &= -k + \varphi(b) + k + \varphi(a) = \varphi(a) + \varphi(b).\end{aligned}$$

□

When P is a projector whose image has finite dimension or finite codimension its component in $\mathcal{P}(\mathcal{C})$ consists of a single point, hence φ is the null homomorphism. There are infinite-dimensional spaces, *undecomposable*, where the only complemented subspaces have finite dimension or finite-codimension; an example of such space was described by W. T. Gowers and B. Maurey in [GM93]. In that case $\mathcal{P}(\mathcal{C})$ consists of two points. In general, given a projector P, if the conditions

h1) P is connected to a projector Q such that $\dim(Q, P) = 1$,

h2) the connected component of P in $\mathcal{P}(E)$ is simply-connected

hold φ is an isomorphism. This follows straightforwardly from the definition: if β is a path with end-points P and Q the composition $a = \mathfrak{p} \circ \beta$ is a loop in $\mathcal{P}(\mathcal{C})$ and $\varphi(a) = 1$. The second condition applies to the exact sequence as $\ker \varphi = \operatorname{Im} \mathfrak{p}_* = \{0\}$, hence φ is also injective.
Here is a concrete example where the first conditions hold.

Proposition 2.8.2. *Let $E = X \oplus Y$ be a Banach space and X a closed complemented subspace isomorphic to its hyperplanes and to Y, complemented as well. Let P be the projector onto X along Y. Then P satisfies the condition* h1).

Proof. The proof relies on this fact: if two isomorphic subspaces have null intersection and their sum is complemented they are connected by a path of complemented subspaces. A proof of this can be found for instance in [PR87] also. Since X is isomorphic to its hyperplanes we can choose a subspace $R \subset X$ of dimension 1 such that $X \oplus R$ is isomorphic to X. Up to an isomorphism of E we can start from the decomposition $E = X \oplus R \oplus Y$. As base point of the loop we choose $P = P_X + P_R$. Since $X \oplus R$ is isomorphic to Y the projectors $P_X + P_R$ and P_Y are connected. Let σ be an isomorphism of Y with $X \oplus R$. We define the path

$$G_{\sigma,\theta}(x + y) = (\cos\theta x + \sin\theta\sigma y) + (-\sin\theta\sigma^{-1}x + \cos\theta y)$$

of invertible operators of E. Direct computations show that $G_\sigma(-\theta)$ is its inverse. Moreover $G_\sigma(0)$ is the identity and $G(\pi/2)$ conjugates the projector P to P_Y. Then the path

$$P_\theta = G_{\sigma,\theta} P G_{\sigma,-\theta}$$

connects P to P_Y. If we consider the subspace $X \oplus Y$ we can use the same argument in order to connect P_Y to P_X. It only takes to choose an isomorphism of X with Y and repeat the construction made above. Thus P satisfies the condition h1) because it is connected to P_X. □

The argument used to connect the two projectors P and P_X is a modification of the one used for Hilbert spaces by J. Phillips in Proposition 6 of [Phi96]: given the decomposition

$$E = X \oplus R \oplus Y$$

a shift operator s maps X and $R \oplus Y$ isomorphically onto $X \oplus R$ and Y respectively. Since the general linear group of a Hilbert space is contractible the projectors are connected. The isomorphism G_σ used in the proof are always connected to the identity regardless of whether $GL(E)$ is connected or not.

2.9 A space where φ is not injective

In this section we exhibit an example of Banach space E with a projector P of infinite dimensional range and kernel and a loop a in $\mathcal{P}(\mathcal{C})$ with base point $[P]$ such that $\varphi(a) = 0$ but not homotopically equivalent to the constant path.

Proposition 2.9.1. *Let $X \subset E$ be a closed complemented subspace isomorphic to its complement and P a projector such that $P(E) = X$. If $GL(X)$ is not connected the component of P in $\mathcal{P}(E)$ is not simply connected.*

Proof. Choose a topological complement Y and let $T \in GL(X)$ be such that there exists no path joining T to the identity. Let σ be an isomorphism of Y with X. Hence the invertible operator

$$T_1 = \begin{pmatrix} T & 0 \\ 0 & \sigma T^{-1}\sigma^{-1} \end{pmatrix}$$

lies in the connected component of $GL(E)$ of the identity. A path can be defined as $G_{\sigma,\theta} T_1 G_{\sigma,-\theta}$ where $G_{\sigma,\theta}$ is the operator defined in the preceding section. Call S such path and define $\alpha = SPS^{-1}$. Since T_1 commutes with P the path α is a loop with base point P. The group homomorphism

$$\Delta \colon \pi_1(\mathcal{P}(E), P) \to \pi_0(GL(X)) \times \pi_0(GL(Y))$$

induced by the fibre bundle $(GL(E), \phi_P, \mathcal{P}(E))$ maps α to T_1. Thus $\Delta\alpha \neq 0$, hence $\alpha \neq 0$. □

In order to find non-contractible loops with vanishing index we need some projector P such that the inclusion

$$j_*\colon \pi_1(\mathcal{P}_c(P;E)) \to \pi_1(\mathcal{P}(E),P)$$

is not surjective. We will prove that for some spaces the second group contains a subgroup isomorphic to $\mathbb{Z}$ while the first is trivial when E is a complex space and isomorphic to $\mathbb{Z}_2$ in the real case. Let F and G be such that

i). every bounded map $G \to F$ is compact,

ii). both F and G are isomorphic to their hyperplanes;

let $X = F \oplus G$. We have the following

Lemma 2.9.2. *There exists a continuous and surjective map $f\colon GL(X) \to \mathbb{Z}$. Thus $GL(X)$ is not connected.*

We sketch briefly the main idea of the construction of such map: an invertible operator T can be written block-wise using the projectors on F and G. Since every map from G to F is compact the diagonal blocks are Fredholm operators of F and G respectively. Define $f(T) = \operatorname{ind} T_{11}$. For a more detailed proof and references cf. [Mit70, Dou65]. Thus we have a surjective homomorphism obtained by composition

$$(f \times 0) \circ \Delta\colon \pi_1(\mathcal{P}(E),P) \to \mathbb{Z}.$$

Hence, given a loop $\alpha \notin j_*(\pi_1(\mathcal{P}_c(P;E)))$, we consider the element $a = \mathfrak{p}_*(\alpha)$. Since the sequence

$$\pi_1(\mathcal{P}(E),P) \longrightarrow \pi_1(\mathcal{P}(\mathcal{C}),[P]) \xrightarrow{\ \varphi\ } \mathbb{Z}$$

is exact $\varphi(a) = 0$ and $a \neq 0$. We conclude by showing that a pair of spaces with the properties i) and ii) exist. We have the following

Theorem 2.9.3. (cf. [Ban55]). *If $p_1 > p_2$ every bounded operator from l_{p_1} to l_{p_2} is compact.*

Thus a suitable Banach space is given by $(l_2 \oplus l_p) \oplus (l_2 \oplus l_p)$ with $p < 2$. Using Schauder bases isomorphisms with hyperplanes can be defined through *shift* operators.

Chapter 3

Linear equations in Banach spaces

We state and prove some general results about differential equations on a Banach algebra with unit, usually denoted by 1. We are mainly concerned of the Cauchy problem

$$u'(t) = A(t)u(t), \; u(0) = 1 \tag{3.1}$$

where A is a continuous path in a Banach algebra $\mathcal{B}$. Local existence and uniqueness hold. In fact these solutions admit a prolongation to the whole real line $\mathbb{R}$. Denote by X_A the solution of (3.1). Using local uniqueness we prove some properties of the solution X_A. When $\mathcal{B}$ is the algebra of bounded operators on a Banach space E two linear subspaces, the *stable* and *unstable space*, are defined

$$W_A^s = \left\{ x \in E \mid \lim_{t\to+\infty} X_A(t)x = 0 \right\}$$
$$W_A^u = \left\{ x \in E \mid \lim_{t\to-\infty} X_A(t)x = 0 \right\}.$$

If A is a bounded and asymptotically hyperbolic these are closed linear subspaces, admit a topological complement, and have the asymptotic behaviour

$$\lim_{t\to+\infty} X_A(t)W_A^s = E^-(A_0(+\infty)),$$
$$\lim_{t\to+\infty} X_A(t)Y = E^+(A_0(+\infty))$$

where $W_A^s \oplus Y = E$. The limits are taken in the topology of $G(E)$. In the last section we look at the effects of perturbation of an asymptotically hyperbolic path on its stable space. Precisely the stable space varies continuously in the topology of $G_s(E)$. If $A - B$ is a path of compact operators then W_A^s and W_B^s are compact perturbation one of each other.

3.1 The Cauchy problem

Let $\mathcal{B}$ be a Banach algebra and A a continuous path defined on the real line. Given $u, v \in \mathcal{B}$ we can always consider two *Cauchy problems*

$$X_{A,u}{}'(t) = A(t)X_{A,u}(t),\ X_{A,u}(0) = u \tag{3.2}$$

$$X^{A,v}{}'(t) = X^{A,v}(t)A(t),\ X^{A,v}(0) = v. \tag{3.3}$$

By Theorem A.1 unique local solutions always exist and the maximal solutions can extended, by Proposition A.6, to $\mathbb{R}$.

Proposition 3.1.1. *Let $u, v \in \mathcal{B}$. We have*

$$X^{-A,v}(t) \cdot X_{A,u}(t) = vu,$$
$$X_{A,u}(t) \cdot X^{-A,v}(t) = X_{A,1}(t) \cdot uv \cdot X^{-A,1}(t)$$

for every $t \in \mathbb{R}$. Moreover $X_{A,1}$ is invertible and its inverse is $X^{-A,1}$.

Proof. To prove the first equality consider the C^1 path $X^{-A,v} \cdot X_{A,u}$. By hypothesis the path is vu at $t = 0$ and its derivative is

$$X^{-A,v}{}' X_{A,u} + X^{-A,v} X'_{A,u} = -X^{-A,v} A X_{A,u} + X^{-A,v} A X_{A,u} = 0;$$

then $X^{-A,v} X_{A,u}(t) = vu$ for every t. To prove the second we argue similarly. The path $X_{A,v}(t) \cdot X^{-A,u}(t)$ has derivative

$$X_{A,v}{}' X^{-A,u} + X_{A,v} X^{-A,u}{}' = [A, X_{A,v} \cdot X^{-A,u}]$$

and is therefore solution of the Cauchy problem $X' = [A, X]$ with starting point at uv. By direct computation $X_{A,1} \cdot uv \cdot X^{-A,1}$ solves the same equation. By uniqueness the second equality holds. The first equality applied to $u = v = 1$ gives $X^{-A,1} \cdot X_{A,1} = 1$. Since $X_{A,1} \cdot X^{-A,1}$ and the constant path 1 solve the same equation, $X_{A,1}$ is invertible. □

Definition 3.1.2. *An element $u \in \mathcal{B}$ is a* left inverse *if there exists v, called* right inverse *for u, such that $uv = 1$. We denote the subsets of left and right inverses by $\mathcal{B}_l$ and $\mathcal{B}_r$ respectively.*

Proposition 3.1.3. *If $u \in \mathcal{B}_r$ (resp. $\mathcal{B}_l$) then $X_{A,u} \subset \mathcal{B}_r$ (resp. $\mathcal{B}_l$). If u is invertible then $X_{A,u}(t)^{-1} = X^{-A,u^{-1}}(t)$.*

Proof. Let $u \in \mathcal{B}_r$ and v be such that $vu = 1$. By the first equality of Proposition 3.1.1 $X_{A,u} \subset \mathcal{B}_r$. If $u \in \mathcal{B}^*$ let v be its inverse. The first and the second of 3.1.1 give $X^{-A,v} \cdot X_{A,u} = X_{A,u} \cdot X^{-A,v} = 1$. □

We will abbreviate the notation for the rest of this section: for curves with starting point 1 we write X_A instead of $X_{A,1}$.

Proposition 3.1.4. *$\mathcal{B}_r$ and $\mathcal{B}_l$ are open subsets of $\mathcal{B}$.*

Proof. We will prove that $\mathcal{B}_r$ is open. Let $u \in \mathcal{B}_r$ and v be such that $v \cdot u = 1$. Let $r_0 = 1/\|v\|$ and $h \in \mathcal{B}$. Then

$$v(u+h) = vu + vh = 1 + vh.$$

If $h \in B(u, r_0)$, by the Von Neumann series, $1+vh$ is invertible. Then $(1+vh)^{-1}v$ is a left inverse of $u+h$. Actually, in a neighbourhood of u, we have defined a smooth function

$$B(u, r_0) \to \mathcal{B}_l,\ u' \mapsto L_{u,r_0}(u') = [1 + v(u'-u)]^{-1}v \in \mathcal{B}_l. \tag{3.4}$$

such that $L_{u,r_0}(u') \cdot u = 1$. The same conclusions hold for $\mathcal{B}_l$. □

Proposition 3.1.5. *Let $X \in C^1(\mathbb{R}, \mathcal{B}_r)$. There exists $A \in C(\mathbb{R}, \mathcal{B})$ such that $X_{A,X(0)} = X$.*

Proof. As first step we prove that there exists a path Y with values in $\mathcal{B}_l$ such that $YX \equiv 1$. Let $t_0 \in \mathbb{R}$. Since $X(t_0) \in \mathcal{B}_r$ there exists $Y(t_0)$ such that $Y(t_0)X(t_0) = 1$ and the (3.4) provides us with a differentiable map defined in a neighbourhood $B(t_0, \varepsilon(t_0))$, namely $L_{X(t_0),\varepsilon(t_0)}$. By paracompactness of $\mathbb{R}$ we can extract a locally finite subcovering of $\{B(t,\varepsilon(t)) \mid t \in \mathbb{R}\}$, say $\mathcal{U} = \{U_i \mid i \in I\}$. Let $\sigma\colon I \to \mathbb{R}$ be a choice function and $\{\varphi_i \mid \operatorname{supp}\varphi_i \subseteq U_i\}$ a partition of unity subordinate to $\mathcal{U}$. Then set

$$Y = \sum_i \varphi_i Y_{\sigma(i)}.$$

Actually Y is infinitely differentiable. Its image lies in $\mathcal{B}_l$, in fact

$$Y(t)X(t) = \sum_i \varphi_i Y_{t_i}(t)X(t) = \sum_i \varphi_i(t)1 = 1.$$

Now, in the chain of equalities $X' = X' \cdot 1 = X' \cdot YX = (X'Y)X$ set $A = X'Y$ and obtain $X' = AX$. By uniqueness, $X = X_{A,X(0)}$ q.e.d. □

This proposition gives us a characterization of the solutions of $X' = AX$ when the starting point lies in $\mathcal{B}_r$ (resp. $\mathcal{B}_l$). They are just C^1 curves on $\mathcal{B}_r$ (resp. $\mathcal{B}_l$).

Proposition 3.1.6. *$\mathcal{B}^*$ is union of connected components of $\mathcal{B}_r$.*

Proof. Let $\mathcal{B}_r'$ be a connected component of $\mathcal{B}_r$ such that $\mathcal{B}^* \cap \mathcal{B}_r' \neq \emptyset$. Let $x \in \mathcal{B}_r'$. Since $\mathcal{B}_r'$ is an open set we may choose a path $\Gamma \in C^1([0,1], \mathcal{B}_r')$ such that $\Gamma(0) = g \in \mathcal{B}^*$ and $\Gamma(1) = x$. Then, by Proposition 3.1.5, $\Gamma = X_{A,\Gamma(0)}$ for some $A \in C([0,1], \mathcal{B})$. Since $\Gamma(0)$ is an invertible element of $\mathcal{B}$ Proposition 3.1.1 states that $\Gamma(t) \in \mathcal{B}^*$ for any $t \in [0,1]$. In particular $\Gamma(1) \in \mathcal{B}^*$ thus $\mathcal{B}_r' \subset \mathcal{B}^*$. □

The proofs of the following equalities are consequence of the uniqueness of the solutions of Cauchy problems. Given a path $A \in C(\mathbb{R}, \mathcal{B})$, $\tau \in \mathbb{R}$ we denote by A_τ the path $A(\cdot + \tau) = A(t+\tau)$.

Proposition 3.1.7. *Let A and B be two continuous paths. Then*

$$X_{A+B} = X_A \cdot X_{X_A{}^{-1}BX_A}$$
$$X_{A(\cdot+s)}(t)X_A(s) = X_A(t+s)$$

for any $t, s \in \mathbb{R}$.

Proof. Let $X = X_A X_{X_A{}^{-1}BX_A}$. Differentiating

$$\begin{aligned} X' &= X'_A \cdot X_{X_A{}^{-1}BX_A} + X_A \cdot X'_{X_A{}^{-1}BX_A} \\ &= (A+B)X_A \cdot X_{X_A{}^{-1}BX_A} = (A+B)X \end{aligned}$$

hence $X = X_{A+B}$. To prove the second equality let $Y = X_{A(\cdot+s)}(t)X_A(s)$. Differentiating we find that $Y'(t) = A(t+s)Y(t)$, $Y(0) = X_A(s)$. Since the same holds for $Z(t) = X_A(t+s)$ the second equality is proved. □

Proposition 3.1.8. *Let $A, B \in C(\mathbb{R}, \mathcal{B})$*

$$X_B(t) = X_A(t) + \int_0^t X_A(t)X_A(\tau)^{-1}(B-A)X_B(\tau)d\tau \tag{3.5}$$

Proof. Call X and Y respectively the left and right members of (3.5). We have $X(0) = Y(0) = 1$ at $t = 0$. We prove that both solve the Cauchy problem $u' = Au + (B-A)X_B$ with starting point 1. In fact

$$\begin{aligned} X' &= BX_B = AX_B + (B-A)X_B = AX + (B-A)X_B \\ Y' &= AX_A + A\int_0^t X_A(t)X_A(\tau)^{-1}(B-A)X_B(\tau) + (B-A)X_B \\ &= AY + (B-A)X_B. \end{aligned}$$

□

When $\mathcal{B}$ is the algebra of bounded operators on a Banach space E, given a path A in $\mathcal{B}(E)$ we can always consider the adjoint $A^* \in C(\mathbb{R}, \mathcal{B}(E^*))$. The relation

$$(X_A^{-1})^* = X_{-A^*} \tag{3.6}$$

holds. In fact the derivative of the left member is

$$(-(X_A)^{-1}AX_AX_A^{-1})^* = -A^*(X_A^{-1})^* = X'_{-A^*}.$$

3.2 Exponential estimate of X_A

In this section we denote by $C_b(\mathbb{R}, \mathcal{B})$ the space of bounded functions in $\mathcal{B}$. This space is endowed with the norm $\|A\|_\infty = \sup_{t\in\mathbb{R}} \|A(t)\|$ that makes it a Banach space.

Proposition 3.2.1. *If A is bounded $X_A(t)$ satisfies the exponential estimate*

$$\|X_A(t)X_A(s)^{-1}\| \leq ce^{\lambda|t-s|} \tag{3.7}$$

for some $c > 0$, $\lambda \in \mathbb{R}$ and any $t, s \in \mathbb{R}$.

Proof. Let $r = t - s$. By the Proposition 3.1.7 it is enough to prove that

$$\|X_{A(\cdot+s)}(r)\| \le ce^{\lambda|r|}$$

for every $r \in \mathbb{R}$. To achieve this inequality we apply the Gronwall's lemma to the function $\alpha(r) = \|X_{A(\cdot+s)}(r)\|$. In fact since

$$\alpha(r) \le 1 + \int_0^r \|A_{(\cdot+s)}(\tau)\|\alpha(\tau)d\tau$$

by the Gronwall's lemma (see Lemma A.5)

$$\alpha(r) \le 1 + \int_0^r e^{\|A\|_\infty(r-\tau)}d\tau.$$

Easy computations show that $c = 2\max\{1, 1 - 1/\|A\|_\infty\}$ and $\lambda = \|A\|_\infty$ fit our request. Repeating the same argument for $t < 0$ we complete the proof. □

Proposition 3.2.2. *Let $A, H \in C_b(\mathbb{R}, \mathcal{B})$. If $\|X_A(t)X_A(s)^{-1}\| \le ce^{\lambda(t-s)}$ for any $t \ge s \ge 0$ we have $\|X_{A+H}(t)X_{A+H}(s)^{-1}\| \le ce^{\mu(t-s)}$ where $\mu = \lambda + c\|H\|_\infty$.*

Proof. Applying the first equality of Proposition 3.1.7 to A and μ it easy to check that X_A satisfies the exponential estimate for any $t \ge s \ge 0$ with constants (c, λ) if and only if $A + \mu$ does the same with $(c, \lambda - \mu)$. In fact

$$X_{A+\mu I}(t) = X_A \cdot X_{X_A \cdot \mu I \cdot X_A^{-1}}(t) = X_A \cdot X_{\mu I}(t) = e^{\mu t}X_A(t).$$

Set $B = A + \mu I$. Hence we just have to prove that if $(c, \lambda - \mu)$ works with X_B then $(c, 0)$ works with X_{B+H}. Now fix $s \ge 0$. By the second of Proposition 3.1.7 $X_B(t)X_B(s)^{-1} = X_B(s + t - s)X_B(s)^{-1} = X_{B(\cdot+s)}(t - s)$ and the statement reduces to prove that

$$X_{B_s}(r) \le ce^{(\lambda-\mu)r} \Rightarrow X_{B_s+H_s}(r) \le c,\ r > 0, \tag{3.8}$$

where $B_s = B_{(\cdot+s)}$, $H = H_{(\cdot+s)}$. To prove (3.8) fix $t \in \mathbb{R}^+$ and consider the following map of $C_b([0, t], \mathcal{B})$ into itself

$$X \mapsto (fX)(r) = X_{B_s}(r)\left[1 + \int_0^r X_{B_s}(\tau)^{-1}H_s(\tau)Y(\tau)d\tau\right].$$

By (3.5) $X_{B_s+H_s}$ is a fixed point of f. We will prove that f is a contraction and that $\overline{B}(0, c)$ is invariant for f. Since every nonempty closed invariant subset for a contraction contains its fixed point this will conclude the proof. It is enough to prove that the linear application $L = f - X_{B_s}$ is bounded and $\|L\| < 1$. This will suffice to prove that L is a contraction, hence the affine map $L + X_B$ is also a contraction. Let X in $C([0, t], \mathcal{B})$

$$\|LX\|_\infty \le \frac{c\|H\|_\infty}{\mu - \lambda}\left(1 - e^{-(\mu-\lambda)t}\right)\|X\|_\infty,$$

hence f is a contraction. To prove that $\overline{B}(0,c)$ is invariant for f let $X \in \overline{B}(0,c)$ thus

$$\begin{aligned}\|(fX)(t)\| &= \left\| X_{B_s}(t)\left[1+\int_0^t X_{B_s}(\tau)^{-1}H_s(\tau)Y(\tau)d\tau\right]\right\| \\ &\leq ce^{(\lambda-\mu)t} + c^2\|H\|_\infty \int_0^t \|X_B(t)X_B(\tau)^{-1}\|d\tau \\ &\leq ce^{(\lambda-\mu)t}\left(1-\frac{c\|H\|_\infty}{\mu-\lambda}\right)+\frac{c^2\|H\|_\infty}{\mu-\lambda} = c.\end{aligned}$$

Then $\|fX\|_\infty \leq c$ and the proof is complete. □

3.3 Asymptotically hyperbolic paths

For the remainder of this chapter we restrict our attention to the algebra of bounded operators on a Banach space E. Given a continuous path A in the space of bounded operators, defined on $\mathbb{R}^+$ we define the *stable space* as

$$W_A^s = \left\{ x \in X \,\middle|\, \lim_{t\to+\infty} X_A(t)x = 0 \right\}.$$

Similarly, if A is a path defined on $\mathbb{R}^-$ we define the unstable space

$$W_A^u = \left\{ x \in X \,\middle|\, \lim_{t\to-\infty} X_A(t)x = 0 \right\}.$$

Using the equalities of Proposition 3.1.7, for every $t \geq 0$ and $s \leq 0$ we have

$$X_A(t)W_A^s = W_{A(\cdot+t)}^s, \;\; X_A(s)W_A^u = W_{A(\cdot+s)}^u. \tag{3.9}$$

We denote by $\mathbb{H}^+$ and $\mathbb{H}^-$ the semi-planes of $\mathbb{C}$ with positive and negative real part, respectively. Let A_0 be a hyperbolic operator, that is $\sigma(A_0) \cap \mathrm{i}\mathbb{R} = \emptyset$. Thus we have a decomposition of the spectrum

$$\sigma(A_0) = \sigma^+(A_0) \cup \sigma^-(A_0)$$

where $\sigma^\pm(A_0) = \sigma(A_0) \cap \mathbb{H}^\pm$. Let P^+, P^- be the spectral projectors of the decomposition, E^+ and E^- their range respectively. It is clear that the stable and unstable spaces of the constant path A_0 are E^- and E^+. In the following theorem we prove that if $A = A_0 + H$ is a small perturbation of A_0 the stable and unstable spaces of A are closed and admit a topological complement.

Proposition 3.3.1. (cf. [AM03b], Proposition 1.2). Let A_0 be a hyperbolic operator, with $\sigma^-(A_0)$ and $\sigma^+(A_0)$ nonempty, and a pair (c,λ), $\lambda > 0$ such that, for any $t \geq 0$

$$\|e^{tA_0}|_{E^-}\| \leq ce^{-\lambda t}, \;\; \|e^{-tA_0}|_{E^+}\| \leq ce^{-\lambda t}. \tag{3.10}$$

Let $M := \max\{\|P^+\|, \|P^-\|\}$. There are positive constants h, ν, b depending only on c and λ such that if

$$\|H\|_\infty \leq \frac{\lambda}{Mc(1+\sqrt{c})}$$

the following facts hold:

i). for every $t \geq 0$, $X_A(t)W_A^s$ is the graph of a bounded operator $S(t) \in \mathcal{B}(E^-, E^+)$,

ii). $\|S(t)\| \leq c^2 \int_t^\infty e^{-\nu(\tau-t)}\|H(\tau)\|d\tau$,

iii). the function S has much differentiability as X_A,

iv). for every $u_0 \in W_A^s$ and every $t \geq s \geq 0$ there holds

$$|X_A(t)u_0| \leq be^{-\nu(t-s)}|X_A(s)u_0|.$$

Proof. First we check what kind of differential equation satisfies $u = X_A \cdot u_0$, for any $u_0 \in E^- \oplus E^+$, in terms of the projectors $P^\pm$. Let $u = x+y$. Differentiating both sides we find that

$$\begin{cases} x' = A_- x + A_\mp y \\ y' = A_\pm x + A_+ y \end{cases} \tag{3.11}$$

where $A_\pm = P^+AP^-$, $A_- = P^-AP^-$ and so on. For every $r \geq t \geq s$ the system above can be rewritten as

$$\begin{aligned} x(t) &= X_{A_-}(t)X_{A_-}(s)^{-1}x(s) + \int_s^t X_{A_-}(t)X_{A_-}(\tau)^{-1}A_\mp(\tau)y(\tau)d\tau \\ y(t) &= X_{A_+}(t)X_{A_+}(r)^{-1}y(r) - \int_t^r X_{A_+}(t)X_{A_+}(\tau)^{-1}A_\pm(\tau)x(\tau)d\tau. \end{aligned} \tag{3.12}$$

By hypothesis $A_{0,-}$ fulfills the exponential estimate (3.7) with constants $(c, -\lambda)$. Thus A_-, by Proposition 3.2.2, also does it with constants c and $-\mu_- = -\lambda + c\|H_-\|$. Similarly, by (3.10) $-A_+^*$ fulfills the estimate (3.7) with constants c and $-\mu_+ = -\lambda + c\|H_+^*\| = -\lambda + c\|H_+\|$. By the equality (3.6) we have

$$\begin{aligned} &\|X_{A_+}(t)X_{A_+}(r)^{-1}\| = \|(X_{A_+}(t)X_{A_+}(r)^{-1})^*\| \\ =&\|X_{A_+}(r)^{-1^*}X_{A_+}(t)^*\| = \|X_{-A_+^*}(r)X_{-A_+^*}(t)^{-1}\| \end{aligned} \tag{3.13}$$

for $r \geq t \geq 0$. The first equation of (3.12) gives inequalities

$$\begin{aligned} \left|\int_s^t X_{A_-}(t)X_{A_-}(\tau)^{-1}H_\mp(\tau)y(\tau)d\tau\right| &\leq c\int_s^t e^{-\mu_-(t-s)}\|H_\mp(\tau)\||y(\tau)|d\tau \\ &\leq \frac{c\|H_\mp\|}{\mu_-}\left(1 - e^{-\mu_-(t-s)}\right)\|y\|_{\infty,[s,t]} \end{aligned} \tag{3.14}$$

and the second gives

$$\begin{aligned} \left|\int_t^r X_{A_+}(t)X_{A_+}(\tau)^{-1}H_\pm(\tau)x(\tau)d\tau\right| &\leq c\int_t^r e^{-\mu_+(\tau-t)}\|H_\pm(\tau)\|dt\,\|x\|_{\infty,[t,r]} \\ &\leq \frac{c\|H_\pm\|}{\mu_+}\left(1 - e^{-\mu_+(r-t)}\right)\|x\|_{\infty,[t,r]}. \end{aligned} \tag{3.15}$$

Since μ_+ and μ_- are positive, in the second of (3.12) we can take the limit as $r \to +\infty$. Set $s = 0$ in the first of (3.12). Therefore the equations (3.14) and (3.15) permit to define a continuous map on the Banach space $C_b(\mathbb{R}^+, E^- \oplus E^+)$

$$\varphi_{A,x_0} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = L_A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} X_{A_-}(\cdot)x_0 \\ 0 \end{pmatrix} \tag{3.16}$$

where

$$L_A \begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} \int_0^t X_{A_-}(t)X_{A_-}(\tau)^{-1}A_{\mp}(\tau)y(\tau)d\tau \\ -\int_t^\infty X_{A_+}(t)X_{A_+}(\tau)^{-1}A_{\pm}(\tau)x(\tau)d\tau \end{pmatrix} \tag{3.17}$$

By (3.14) and (3.15), the operator L_A is bounded. A bounded solution u of (3.11), with $P^- u(0) = x_0$ is a fixed point of φ_{A,x_0}. The estimate of $\|H\|_\infty$ in the hypothesis gives

$$(2c^3)^{1/2}\|H_{\mp}\| < \mu_-, \ (2c^3)^{1/2}\|H_{\pm}\| < \mu_+ \tag{3.18}$$

hence φ_{A,x_0} is a contraction. Clearly if $u_0 \in W_A^s$ the curve $X_A(t)u_0$ is a fixed point of φ_{A,x_0}. Using (3.14) and (3.15) we prove that if u is fixed point then $u(0) \in W_A^s$, hence u is not just bounded, but infinitesimal also. If $u(0) = 0$ it is clear. Suppose $u(0) \neq 0$. For any $t \geq s$

$$\begin{aligned} |x(t)| &\leq ce^{-\mu_-(t-s)}|x(s)| + \frac{c\|H_{\mp}\|}{\mu_-}\Big(1 - e^{-\mu_-(t-s)}\Big)\|y\|_{\infty,[s,t)} \leq \\ &\leq \max\{c|x(s)|, \frac{c\|H_{\mp}\|}{\mu_-}\|y\|_{\infty,[s,\infty)}\}, \end{aligned} \tag{3.19}$$

the supremum on the real axis is allowed since we know that both x and y are bounded. From (3.15)

$$|y(s)| \leq \frac{c\|H_{\pm}\|}{\mu_+}\|x\|_{\infty,[s,\infty)} \tag{3.20}$$

and, taking the sup on $[s, \infty)$

$$\|y\|_{\infty,[s,\infty)} \leq \frac{c\|H_{\pm}\|}{\mu_+}\|x\|_{\infty,[s,\infty)} \tag{3.21}$$

and we get

$$\|x\|_{\infty,[s,\infty)} \leq \max\{c|x(s)|, \frac{c^2\|H_{\pm}\|\|H_{\mp}\|}{\mu_-\mu_+}\|x\|_{\infty,[s,\infty)}\}; \tag{3.22}$$

the estimate of $\|H\|$ also implies that $c^2\|H_{\pm}\|\|H_{\mp}\| < \mu_-\mu_+$, therefore (3.22) allows to write

$$\|x\|_{\infty,[s,\infty)} \leq c|x(s)|, \tag{3.23}$$

and, by (3.20) we get the final estimate

$$|y(s)| \leq \frac{c^2\|H_{\pm}\|}{\mu_+}|x(s)|. \tag{3.24}$$

It is easy to check that x does not vanish at any point of $\mathbb{R}^+$ for, if such $t \in \mathbb{R}^+$ exists (3.24) implies $y(t) = 0$, thus $0 = x(t) + y(t) = u(t) = X_A(t)u_0$. Since $X_A(t)$ is invertible we had $u_0 = 0$ in contradiction with the hypothesis. If E is a Hilbert space it is easy to build a continuous path $U(t)$ of operators in $\mathcal{B}(E^-, E^+)$ that maps $x(t)$ to $y(t)$ and $\|U(t)\| = |y(t)|/|x(t)|$. Just define

$$U(t)z = \frac{(x(t), z)}{|x(t)|^2} y(t)$$

where $(\cdot, \cdot)$ denotes the scalar product of the Hilbert space. For Banach spaces we need some results of continuous selection such as THEOREM 4 of [BK73]. By Corollary D.2 of Appendix D there exists a path U_ε continuous and bounded in $\mathcal{B}(E^-, E^+))$ such that

$$U_\varepsilon(t)x(t) = y(t), \ \|U_\varepsilon(t)\| \leq (1+\varepsilon)\frac{c^2\|H_\pm\|}{\mu_+} + \varepsilon$$

for every $\varepsilon > 0$. Then we can write the first of (3.11) as

$$x' = [A_-(t) + A_\mp(t)U_\varepsilon(t)]x$$

Since $A_\mp(t)U_\varepsilon(t)$ is a bounded operator in $\mathcal{B}(E^-)$ we can apply the Proposition 3.2.2: in fact A_- satisfies an exponential estimate with constants $(c, -\mu_-)$, then the path $A_-(t) + A_\mp(t)U_\varepsilon(t)$ does it with constants $(c, -\nu_\varepsilon)$ where

$$-\nu_\varepsilon = -\mu_- + c\|H_\mp U_\varepsilon\| \leq -\mu_- + c\|H_\mp\| \cdot (1+\varepsilon)\frac{c^2\|H_\pm\|}{\mu_+} + c\varepsilon\|H_\mp\|.$$

Let $\nu = \nu_0$. We have $-\mu_+\nu = -\mu_-\mu_+ + c^3\|H_\pm\|\|H_\mp\|$. By (3.18) $-\mu_+\nu < 0$, hence $-\nu < 0$. Then, if we choose ε small enough $-\nu_\varepsilon < 0$ and

$$|x(t)| \leq e^{-\nu_\varepsilon(t-s)}|x(s)|, \quad t \geq s \geq 0.$$

Taking the limit as $\varepsilon \to 0$ we obtain

$$\begin{aligned} |x(t)| &\leq ce^{-\nu(t-s)}|x(s)| \\ |y(t)| &\leq \frac{c^3\|H_\pm\|}{\mu_+} e^{-\nu(t-s)}|x(s)| \end{aligned} \tag{3.25}$$

and x and y vanish at infinity. Thus the fixed point u of φ_{A,x_0} can be characterized as a curve that solves (3.11) such that

$$u(+\infty) = 0, \quad P^-u(0) = x_0. \tag{3.26}$$

An application S in $\mathcal{B}(E^-, E^+)$ whose graph is W^s_A is defined as follows: given x_0 in E^- there exists a unique fixed point of φ_{A,x_0}, call it u. Thus $u(0) \in W^s_A$. We define $Sx_0 = P^+u(0)$ and we have

$$u(0) = P^-u(0) + P^+u(0) = x_0 + Sx_0$$

hence graph$(S) \subseteq W^s_A$. Conversely, given $u_0 \in W^s_A$ the curve $v(t) = X_A(t) \cdot u_0$, by the characterization in (3.26), is the fixed point of φ_{A,P^-u_0}, hence $P^+u_0 = SP^-u_0$. Then graph$(S) = W^s_A$. We can write explicitly S

$$S = P^+ \circ ev_0 \circ (I - L_A)^{-1} \cdot \begin{pmatrix} X_{A_-}(\cdot)x_0 \\ 0 \end{pmatrix}, \tag{3.27}$$

where ev_0 is defined on C_b as the evaluation at $t = 0$. Then S is bounded and W^s_A is closed and $E = E^+ \oplus W^s_A$. Since $A_t = A_0 + H(\cdot + t)$ the same constants work to show that $W^s_{A_t} = X_A(t) W^s_A$ is graph of an unique bounded operator, say $S(t)$ and i) is proved. By direct computation

$$S(t) = P^+ X_A(t)(I_{E_-} + S) \cdot [P^- X_A(t)(I_{E_-} + S)]^{-1}. \tag{3.28}$$

Hence $S \in C(\mathbb{R}, \mathcal{B}(E^-, E^+))$ inherits the regularity of X_A and ii) follows. Taking the limit as $r \to +\infty$ and $t = 0$ in (3.15)

$$\begin{aligned} |Sx_0| = |y_0| &\le c \left(\int_0^\infty e^{-\mu_+ \tau} \|H_\pm(\tau)\| d\tau \right) \|x\|_\infty \\ &\le c^2 \left(\int_0^\infty e^{-\nu\tau} \|H(\tau)\| d\tau \right) |x_0| \end{aligned}$$

since $\nu < \mu_+$. For the general case consider the shifted path $A(\cdot + t)$. Then

$$\begin{aligned} |S(t)x_0| &\le c^2 \left(\int_0^\infty e^{-\mu_+ \tau'} \|H_{(\cdot+t)\pm}(\tau')\| d\tau' \right) |x_0| \\ &= c^2 \left(\int_t^\infty e^{-\mu(\tau - t)} \|H_\pm(\tau)\| d\tau \right) |x_0| \end{aligned}$$

where $\tau = t + \tau'$. This proves iii). Finally let $u_0 \in W^s_A$. By (3.25) and (3.18) we can write

$$\begin{aligned} |X_A(t)u_0| = |x(t) + y(t)| \le |x(t)| + |y(t)| &\le ce^{-\nu(t-s)} \left(1 + \frac{c^2 \|H_\pm\|}{\mu_+} \right) |x(s)| \\ &\le (c + c^2) \|P^-\| e^{-\nu(t-s)} |u(s)| \le be^{-\nu(t-s)} |X_A(s)u_0|. \end{aligned}$$

where $b = c(1 + c)\|P^-\|\|P^+\|$. The proof is complete. □

Proposition 3.3.2. (cf. [AM03b], Proposition 1.2). With the same hypotheses of the preceding statement we have

i). for every $t \ge 0$, $X_A(t)E^+$ is the graph of an operator $T(t) \in \mathcal{B}(E^+, E^-)$,

ii). $\|T(t)\| \le c^2 \int_0^t e^{-\nu(t-\tau)} \|H(\tau)\| \, d\tau$,

iii). T is as much differentiable as X_A,

iv). for every $y_0 \in E^+$, $t \ge s \ge 0$ the inequality

$$|X_A(t)y_0| \ge b^{-1} e^{\nu(t-s)} |X_A(s)y_0|$$

holds.

Proof. Let $\bar{t} \in \mathbb{R}^+$ and $\bar{y} \in E^+$. In (3.12) let $r = \bar{t}$ and $s = 0$. Then we have a continuous map, on $C([0, \bar{t}], E^- \oplus E^+)$ into itself

$$\psi_{A,\bar{y}} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = R_A \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ X_{A^+}(\cdot) X_{A^+}(\bar{t})^{-1} \bar{y}. \end{pmatrix} \tag{3.29}$$

where R_A is a bounded operator defined as

$$R_A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \int_0^t X_{A_-}(t)X_{A_-}(\tau)^{-1}A_{\mp}(\tau)y(\tau)d\tau \\ -\int_t^{\bar{t}} X_{A_+}(t)X_{A_+}(\tau)^{-1}A_{\pm}(\tau)x(\tau)d\tau \end{pmatrix}$$

The map is continuous because R_A is bounded. If $\|H\|_\infty$ is estimated by the same constant of the preceding proposition $\|R_A\| < 1$, hence $\psi_{A,\overline{y}}$ is a contraction. The fixed point v is a solution of (3.11) characterized by the property

$$P^- v(0) = 0, \quad P^+ v(\bar{t}) = \overline{y} \tag{3.30}$$

Let $\overline{y} \in E^+$ and let u be the fixed point of (3.29). We define $T(\bar{t}) \cdot \overline{y} = P^- u(\bar{t})$. By (3.29) $u(0) \in E^+$ and $P^+ u(\bar{t}) = \overline{y}$, hence

$$\overline{y} + T(\bar{t})\overline{y} = P^+ u(\bar{t}) + P^- u(\bar{t}) = u(\bar{t}) = X_A(\bar{t})u(0)$$

thus $\operatorname{graph}(T(\bar{t})) \subset X_A(\bar{t})E^+$. Conversely, let $z \in X_A(\bar{t})E^+$ and $y \in E^+$ be such that $z = X_A(\bar{t})y$. The curve $u = X_A(\cdot)y$ has the property (3.30), thus coincides with the fixed point of ψ_{A,P^+z}. Hence

$$z = P^+z + P^-z = P^+z + P^-u(\bar{t}) = P^+z + P^-v(\bar{t}) = P^+z + T(\bar{t})P^+z.$$

Hence $X_A(\bar{t})E^+ = \operatorname{graph}(T(\bar{t}))$. The map can also be written as

$$T(\bar{t})y = P^- \circ ev_0(I - R_A)^{-1} \cdot \begin{pmatrix} 0 \\ X_{A^+}(\cdot)X_{A^+}(\bar{t})^{-1}\overline{y}. \end{pmatrix}$$

and i) is proved. For every $t \geq 0$

$$T(t) = P_-(P_+|_{X_A(t)E^+})^{-1} = P_- X_A(t)[P_+ X_A(t)]^{-1} \tag{3.31}$$

and iii) follows. Now let $0 \leq t \leq r \leq \bar{t}$. If (x, y) is the fixed point of ψ_A we find

$$|x(t)| \leq \frac{c\|H_\mp\|}{\mu_-}\|y\|_{\infty,[0,t]} \tag{3.32}$$

still from (3.14) and (3.15) we can write

$$\begin{aligned} |y(t)| &\leq ce^{-\mu_+(r-t)}|y(r)| + \frac{c\|H_\pm\|}{\mu_+}\Big(1 - e^{-\mu_+(r-t)}\Big)\|x\|_{\infty,[0,r)} \leq \\ &\leq \max\{c|y(r)|, \frac{c\|H_\pm\|}{\mu_+}\|x\|_{\infty,[0,r]}\}; \end{aligned} \tag{3.33}$$

from (3.32) we write

$$\|x\|_{\infty,[0,t]} \leq \frac{c\|H_\mp\|}{\mu_-}\|y\|_{\infty,[0,t]} \tag{3.34}$$

for any $0 \leq t \leq r$. By (3.33) and (3.34)

$$\|y\|_{\infty,[0,r]} \leq \max\{c|y(r)|, \frac{c^2\|H_\pm\|\|H_\mp\|}{\mu_-\mu_+}\|y\|_{\infty,[0,r]}\}. \tag{3.35}$$

Since $c^2\|H_\pm\|\|H_\mp\| < \mu_-\mu_+$ we have

$$\|y\|_{\infty,[0,r]} \leq c|y(r)|. \tag{3.36}$$

Setting $t = r$ in (3.32) from (3.36) it follows that

$$|x(r)| \leq \frac{c^2\|H_\mp\|}{\mu_-}|y(r)| \tag{3.37}$$

As we have done for the preceding Proposition for every $\varepsilon > 0$ the Corollary D.2 provides us with $V_\varepsilon \in C([0,\bar{t}], \mathcal{B}(E^+, E^-))$ such that

$$V_\varepsilon(r)y(r) = x(r), \ \|V_\varepsilon\| \leq (1+\varepsilon)\frac{c^2\|H_\mp\|}{\mu_-} + \varepsilon$$

hence $y' = (A_+ + H_\pm V_\varepsilon)y$. Applying the Proposition 3.2.2 to $-A_+^*$ for every $\varepsilon > 0$ and $r \geq t \geq 0$ there holds $|y(r)| \geq c^{-1}e^{\nu_\varepsilon(r-t)}|y(t)|$. Taking the limit as $\varepsilon \to 0$

$$|y(r)| \geq c^{-1}e^{\nu(r-t)}|y(t)|. \tag{3.38}$$

for every $r \geq t \geq 0$. By (3.37)

$$|y(s)| = \frac{1}{1+c}\Big(c|y(s)| + |y(s)|\Big) \geq \frac{1}{1+c}\Big(|x(s)| + |y(s)|\Big) \geq \frac{1}{1+c}|u(s)|. \tag{3.39}$$

Given $u_0 \in E^+$, using (3.39) and the fact that the norm of a projector is at least 1 we can write

$$\begin{aligned} |X_A(r)u_0| &\geq |y(r)|\|P^+\|^{-1} \geq (c\|P^+\|)^{-1}e^{\nu(r-t)}|y(s)| \\ &\geq \frac{e^{\nu(r-t)}|u(t)|}{c(1+c)\|P^+\|} \geq b^{-1}e^{\nu(r-t)}|X_A(t)u_0|. \end{aligned} \tag{3.40}$$

and (iv) follows. Finally

$$\begin{aligned} |T(\bar{t})\bar{y}| = |x(\bar{t})| &\leq c\left(\int_0^{\bar{t}} e^{-\mu_+(\bar{t}-\tau)}\|H_\mp\|\right)\|y\|_{\infty,[0,\bar{t}]} \\ &\leq c^2\left(\int_0^{\bar{t}} e^{-\nu(\bar{t}-\tau)}\|H_\mp\|\right)|\bar{y}|; \end{aligned}$$

the last estimate follows from (3.36) with $r = \bar{t}$ and (ii) is proved. □

Remark 3.3.3. The statements and the proofs of the two theorems regard only the stable space. To obtain the same conclusions for the unstable space defined on the negative real line just set $\check{A}(t) = A(-t)$. Using argument of uniqueness of Cauchy problems we obtain

$$X_A(-t) = X_{-\check{A}}(t), \ W_A^u = W_{-\check{A}}^s, \ -\check{A}(+\infty) = -A(-\infty).$$

Thus we can apply Propositions 3.3.1 and 3.3.2 to $-\check{A}$ on the positive real line.

3.4 Properties of W_A^s and W_A^u

In the preceding section it has been proved that W_A^s (as W_A^u) is a splitting space if A is close, in the uniform topology, to a constant hyperbolic path A_0. We prove that it is true for any asymptotically hyperbolic path. Conversely we provide, for any pair (X, Y) in $G_s(E)$, a path A such that $(W_A^s, W_A^u) = (X, Y)$.

Theorem 3.4.1. (cf. [AM03b], Theorem 2.1). Let A be an asymptotically hyperbolic path of operators defined on $\mathbb{R}^+$. Let $A_0 = A(+\infty)$, $E^+ \oplus E^-$ the spectral decomposition. Then W_A^s is a splits

i). W_A^s is the only closed subspace W such that $X_A(t)W \to E^-$,

ii). $\|X_A(t)_{|W_A^s}\| \leq ce^{-\lambda(t-s)}\|X_A(s)_{|W_A^s}\|$ for suitable $c, \lambda > 0$ and every $t \geq s \geq 0$,

iii). for every $V \in G_s(E)$ such that $V \oplus W_A^s = E$ $\rho(X_A(t)V, E^+) \to 0$,

iv). $\inf\limits_{\substack{v\in V\\|v|=1}} |X_A(t)v|$ grows at exponential rate,

v). $W_{-A^*}^s = (W_A^s)^\perp$.

Proof. Let $A(+\infty) = A_0$. Since A_0 is a hyperbolic operator there exist c and λ such that the condition (3.10) holds. Let $H = A_0 - A$. If τ is large enough $\|H_{(\cdot+\tau)}\|$ is smaller than the constant of Proposition 3.3.1 then

$$W_{A(\cdot+\tau)}^s = X_A(\tau)W_A^s$$

is a topological complement of E^+ and, since $X_A(\tau)$ is invertible, W_A^s is closed too and

$$X_A(\tau)W_A^s \oplus E^+ = E = W_A^s \oplus X_A(\tau)^{-1}E^+.$$

Now for $t \geq \tau$ the Proposition 3.3.1 says that $X_{A(\cdot+\tau)}(t)W_{A(\cdot+\tau)}^s$ is the graph of a bounded linear map $S(t)\colon E^- \to E^+$ and

$$\|S(t)\| \leq c^2 \int_t^\infty e^{-\nu(t-\tau')}\|H(\tau+\tau')\|d\tau'.$$

This implies that $S(t)$ converges to the null operator as $t \to +\infty$. By Proposition 1.3.1, graph$\,(S(t))$ converges to graph$\,(0) = E^-$, hence $X_A(t+\tau)W_A^s = X_{A(\cdot+\tau)}(t)W_{A(\cdot+\tau)}^s \to E^-$.

The ii) follows from iv) of Proposition 3.3.1 taking the supremum over the unit sphere of W_A^s on both sides of the inequality.

Let V be a closed subspace of E. Up to a time shift we can suppose that V is graph of a bounded operator $L \in \mathcal{B}(E^+, W_A^s)$. First we prove that $\rho(X_A(t)E^+, X_A(t)V)$ converges to 0. Let $v \in X_A(t)V$ and $y \in E^+$ be such that $v = X_A(t)\cdot(y + Ly)$. Set $u = X_A(t)y$. Then

$$\begin{aligned}|v-u| = |X_A(t)Ly| &\leq be^{-\nu t}\|L\||y| \leq b^2e^{-2\nu t}\|L\||X_A(t)y| \\ &= b^2e^{-2\nu t}\|L\||u| \leq b^2e^{-2\nu t}\|L\|(|v| + |v-u|)\end{aligned}$$

since $\alpha(t) := b^2 e^{-2\nu t}\|L\|$ is an infinitesimal sequence, for $t \geq \bar{t}$ we have $\alpha(t) < 1$ and the above inequality becomes

$$|v-u| \leq \alpha(t)(|v| + |v-u|) \Rightarrow |v-u| \leq \frac{\alpha(t)}{1-\alpha(t)}|v|$$

and we conclude that $\rho(X_A(t)Y, X_A(t)E^+) \to 0$ as $t \to +\infty$. On other hand

$$\begin{aligned}|u-v| &= |X_A(t)Ly| \leq be^{-\nu t}\|L\||y| \leq b^2 e^{-2\nu t}\|L\||X_A(t)y| \\ &= b^2 e^{-2\nu t}\|L\||u| = \alpha(t)|u|\end{aligned}$$

and $\rho(X_A(t)E^+, X_A(t)V) \leq \alpha(t)$. The proof is complete using the fact that $\rho(X_A(t)E^+, E^+) \to 0$ which follows from i) and ii) of Theorem 3.3.2.

To prove the converse of i) let $W \subseteq E$ be a closed subspace such that $X_A(t)W \to E^-$. By iii) for every topological complement of W^s_A, say V, we have $V \cap W = \{0\}$, hence $W \subset W^s_A$. There exists $t_0 > 0$ such that, $\rho(X_A(t_0)W, X_A(t_0)W^s_A) < 1$ and, by Proposition 1.3.2, $X_A(t_0)W = X_A(t_0)W^s_A$ hence $W = W^s_A$ and i) is proved.

In order to prove the iv) we can suppose, up to a time shift, that $V \oplus W^s_A = E = W^s_A \oplus E^+$. Again $V = \operatorname{graph}(L)$, $L \in \mathcal{B}(E^+, W^s_A)$. Then

$$\begin{aligned}|X_A(t)v| &= |X_A(t)y + X_A(t)Ly| \geq |X_A(t)y| - |X_A(t)Ly| \\ &\geq b^{-1}e^{\nu t}|y| - be^{-\nu t}\|L\||y| = (b^{-1}e^{\nu t} - be^{-\nu t}\|L\|)|y| \\ &\geq 1/(1+\|L\|)\,(b^{-1}e^{\nu t} - be^{-\nu t}\|L\|)|v|\end{aligned}$$

and iv) follows by taking the infimum over $S(V)$. By (3.6) we have the chain of equalities

$$X_{-A^*}(t)(W^s_A)^\perp = (X_A(t)^{-1})^*(W^s_A)^\perp = (X_A(t)W^s_A)^\perp. \tag{3.41}$$

Since $X_A(t)W^s_A$ converges to E^- and E^- splits the Proposition 1.3.3 allows us to take the limit in (3.41) which is $(E^-)^\perp$. Since $(E^-)^\perp = E^-(-A^*)$, by i)

$$X_{-A^*}(t)(W^s_A)^\perp \to E^-(-A^*)$$

implies $(W^s_A)^\perp = W^s_{-A^*}$. □

Analogous statements hold for the unstable space W^u_A by considering the path $-\check{A}$.

Lemma 3.4.2. *Let A be a asymptotically hyperbolic path of operators on $\mathbb{R}^+$. Then $X_A(t)W^s_A = E^-$ for every $t \geq 0$ if and only if $A(t)E^- \subseteq E^-$*

Proof. For any $W \subseteq E$ such that $X_A(t)W = E^-(A(+\infty))$ we can set $t = 0$ to get $W = E^-(A(+\infty))$, hence

$$X_A(t)E^- = E^- \tag{3.42}$$

for any $t \geq 0$. Now, fix $\bar{t} \in \mathbb{R}^+$ and let $x \in E^-$, $\bar{x} = X_A(\bar{t})^{-1}x$. By the (3.42) the curve $u(t) = X_A(t)\bar{x}$ is C^1 and takes values in E^-, therefore $u'(t) \in E^-$ for any $t \in \mathbb{R}^+$. Hence

$$E^- \ni u'(\bar{t}) = A(\bar{t})X_A(\bar{t})\bar{x} = A(\bar{t})X_A(\bar{t})X_A(\bar{t})^{-1}x = A(\bar{t})x.$$

Conversely, assume that the second condition is true for any $t \in \mathbb{R}^+$. First we prove that $X_A(t)E^- \subseteq E^-$. Let $x \in E^-$ and let $u(t) = X_A(t)x$. In the second of (3.11) we have $A_\pm x = 0$ by hypothesis, thus $y' = A_+ y$. Hence

$$P^+u(t) = X_{A_+}(t)P^+u(0);$$

since $P^+u(0) = 0$ we have $P^+u = 0$ and from the first of (3.11) we obtain $u(t) \in E^-$. Now, X_A sets a continuous path of semi-Fredholm operators on E^-. By Proposition B.5 these operators have the same index for any $t \in \mathbb{R}^+$. Since $X_A(0) = Id$ the index of these operators is zero. Since every $X_A(t)$ is restriction of an invertible operator they are injective, thus surjective, that is $X_A(t)E^- = E^-$. In particular $X_A(t)E^-$ converges to E^-. By i) of Theorem 3.4.1 $E^- = W^s_A$. □

Proposition 3.4.3. *Given a pair of splitting subspaces (X, Y) in E there exists a path A, continuous and asymptotically hyperbolic on $\mathbb{R}$, such that $W^s_A = X$, $W^u_A = Y$.*

Proof. Let P, Q be two projectors on X and Y respectively. We build first a path A^s on $\mathbb{R}^+$ such that $W^s_{A^s} = X$. Let A^s be the constant path $I - 2P$ which is hyperbolic because $(I - 2P)^2 = I$. The spectral projector on the negative and positive eigenprojectors are, respectively, P and $I - P$. A solution $x + y$ of (3.11) satisfies

$$\begin{aligned} x' &= A^s_- x + A^s_\mp y = -x \\ y' &= A^s_+ y + A^s_\pm x = y. \end{aligned}$$

Thus $X_{A^s}(t) = e^{-t}P + e^t(I - P)$ and the stable space is X. Similarly we can define $A^u(t) = 2Q - I$ for $t < 0$. The joint path $A^u \# A^s$ is piecewise continuous. In order to find a smooth path consider a smooth function φ such that $\varphi([-1/2, 1/2]) = 1$ and $\varphi({}^c(-1,1)) = -1$. Thus the path

$$A = \begin{cases} \varphi(t)P + (I - P) & t \geq 0 \\ \varphi(t)(I - Q) + Q & t < 0 \end{cases}$$

is smooth. The solution of (3.11) with starting point $x(0) + y(0)$ is

$$\begin{cases} x(t) &= e^{\Phi(t)}x(0) \\ y(t) &= e^t y(0) \end{cases}$$

where Φ is the smooth function such that $\Phi(0) = 0$ and $\Phi'(t) = \varphi(t)$. Since Φ diverges to $-\infty$ as $t \to +\infty$ the stable space is X. Since Φ diverges to $+\infty$ as $t \to -\infty$, hence the unstable space is Y. □

3.5 Perturbation of the stable space

In the previous sections we have defined the stable (and unstable) space and proved that is an element of $G_s(E)$, the Grassmannian of splitting subspaces. Thus, in the set

$$C_h(\mathbb{R}^+, \mathcal{B}(E)) = \left\{ A \in C(\overline{\mathbb{R}}^+, \mathcal{B}(E)) \mid \sigma(A(+\infty)) \cap \mathrm{i}\mathbb{R} = \emptyset \right\}$$

endowed with the uniform topology it is defined an application that maps A to W^s_A. In the next two theorems we prove that it is continuous and that if two paths differ by a path of compact operators then the stable spaces are compact perturbation one of each other.

Theorem 3.5.1. (cf. [AM03b], THEOREM 3.1). The map $A \mapsto W^s_A$ is continuous.

Proof. Since $C_h(\mathbb{R}^+, \mathcal{B}(E))$ is a metric space it is enough to prove that the map is sequentially continuous. Let $\{A_n \,|\, n \in \mathbb{N}\}$ be a sequence in $C_h(\mathbb{R}^+, \mathcal{B}(E))$ converging to an asymptotically hyperbolic path A. Let $A(+\infty) = A_0$. Call $P^\pm$ the spectral projectors on $E^-(A_0)$ and $E^+(A_0)$ respectively. Since A_0 is hyperbolic, there exist a pair (c, λ) such that

$$\|e^{tA_0}P^-\| \leq ce^{-\lambda t}, \quad \|e^{-tA_0}P^+\| \leq ce^{-\lambda t}, \quad t \geq 0.$$

The sequence $\{A_n\}$ converges to A_0 uniformly as $n \to \infty$. Moreover $A(t)$ converges to A_0, as $t \to +\infty$. Using triangular inequalities we can find $\tau \in \mathbb{R}^+$ and $N \in \mathbb{N}$ such that, for every $t \geq \tau$ and $n \geq N$

$$\|A_n(t) - A_0\| \leq \frac{\lambda}{Mc(1+\sqrt{c})}. \tag{3.43}$$

where $M = \max\{\|P^+\|, \|P^-\|\}$. Therefore for every $n \geq N$ the paths $A_{n,\tau}$, together with A_τ, fulfill the conditions of Proposition 3.3.1. In particular there are $S_n, S \in \mathcal{B}(E^-, E^+)$ such that

$$X_{A_n}(\tau)W^s_{A_n} = W^s_{A_{n,\tau}} = \mathrm{graph}\,(S_n), \quad X_A(\tau)W^s_A = \mathrm{graph}\,(S).$$

It is enough to prove that S_n converges to S. In fact, by Proposition 1.4.6, this implies that $X_{A_n}(\tau)W^s_{A_n}$ converges to $X_A(\tau)W^s_A$ and the conclusion follows because X_{A_n} converges to X_A pointwise. For the remainder of the proof we omit the subscript τ from the paths. We recall that, by (3.27), given $x \in E^-$

$$S_n x = P^+ ev_0(I - L_{A_n})^{-1}(X_{A_{n-}}(\cdot)x) = P^+ \sum_{k=0}^{\infty} ev_0[L^k_{A_n}(X_{A_{n-}}(\cdot)x)]. \tag{3.44}$$

Since the estimate (3.43) holds for every $n \geq N$ we can apply the Proposition 3.2.2 to A_{n-} and A_{n+} in order to obtain uniform exponential estimates

$$\|X_{A_{n-}}(t)X_{A_{n-}}(s)^{-1}x\| \leq ce^{-\mu_-(t-s)}|x|$$
$$\|X_{A_{n+}}(t)X_{A_{n+}}(r)^{-1}x\| \leq ce^{\mu_+(t-r)}|x|$$

where μ_- and μ_+ are the same constants defined in Proposition 3.3.1. By (3.14) and (3.15) there exists $0 < \alpha < 1$ such that $\|L_{A_n}\| \leq \alpha$ for every $n \geq N$. Then

$$|[L^k_{A_n}X_{A_{n-}}(\cdot)x]| \leq c\alpha^k|x|. \tag{3.45}$$

In order to prove that S_n converges to S we show, by induction on $k \in \mathbb{N}$, that $L^k_{A_n}X_{A_{n-}}(\cdot)x$ converges to $L^k_A X_{A_-}(\cdot)x$ pointwise. Therefore the series

$$\sum_{k=0}^{\infty} ev_0[L^k_{A_n}(X_{A_{n-}}(\cdot)x)]$$

converges pointwise and, by (3.45), is dominated uniformly on $\mathbb{N}$ by the series of the sequence $\{\alpha^k\}$. This is enough to obtain the convergence of series to the pointwise limit. We claim that for every $t \geq 0$

$$\lim_{n\to\infty} L^k_{A_n} X_{A_{n-}}(t)x = L^k_{A_0} X_{A_{0-}}(t)x,$$
$$L^k_{A_n} X_{A_{n-}}(t)x \in E^-, \text{ if } k \text{ is even},$$
$$L^k_{A_n} X_{A_{n-}}(t)x \in E^+, \text{ if } k \text{ is odd}.$$

If $k = 0$ the thesis follows since $x \in E^-$ by hypothesis. Suppose it is true for $k \in \mathbb{N}$. If k is odd, by (3.17)

$$L^{k+1}_{A_n} X_{A_{n-}}(t)x = \int_0^t X_{A_{n-}}(t)X_{A_{n-}}(\tau)^{-1}A_{n\mp}(\tau)L^k_{A_n}X_{A_{n-}}(\tau)x d\tau \qquad (3.46)$$

which belongs to E^-. The last term converges to $L^k_{A_n} X_{A_{n-}}(t)x$ by inductive hypothesis. The other converges by Proposition 3.1.8 and the fact that A_n converges to A. The integrand of (3.46) is bounded in $[0,t]$ by

$$c^2 e^{-\mu_-(t-\tau)} \sup_n \|A_n\|_\infty \alpha^k |x|.$$

Then, by the dominate convergence theorem, the left member of (3.46) converges pointwise. If k is even, by (3.17)

$$L^{k+1}_{A_n} X_{A_{n-}}(t)x = -\int_t^\infty X_{A_{n+}}(t)X_{A_{n+}}(\tau)^{-1}A_{n\pm}(\tau)L^k_{A_n}X_{A_{n-}}(\tau)x d\tau. \quad (3.47)$$

Similarly the integrand converges pointwise and is dominated by

$$c^2\alpha^k e^{\mu_+(t-\tau)}|x| \sup_n \|A_n\|_\infty \in L^1(\mathbb{R}^+).$$

Again, by the dominate convergence theorem, we clinch the pointwise convergence of (3.47) and the inductive step is concluded. Thus

$$\lim_{n\to\infty} ev_0[L^k_{A_n} X_{A_{n-}}(\cdot)x] = ev_0[L^k_{A_0} X_{A_{0-}}(\cdot)x],$$
$$|ev_0[L^k_{A_n} X_{A_{n-}}(\cdot)x]| \leq c\alpha^k|x|$$

for every $k \in \mathbb{N}$ we have convergence of the series. □

We state without proof a couple of facts on compactness useful for the next theorem.

Lemma 3.5.2. *Let J be an interval of the real line, $K \in L^1(J, \mathcal{B}(E))$ such that $K(t) \in \mathcal{B}_c(E)$ almost everywhere. Then the map*

$$C_b(J,E) \ni u \longmapsto \int_J K(\tau)u(\tau)d\tau \in E$$

is a compact operator.

Proof. When K is constant the map is obtained by composition on the left with a compact operator. If K is a characteristic function on J it is sum of compact operators. We conclude with the density of characteristic functions in $L^1(J, \mathcal{B}(E))$ and closeness of compact operators. □

Theorem 3.5.3 (Ascoli-Arzelà). *Let X be a compact metric space, E a Banach space. A bounded subset $\mathcal{W} \subset C(X,E)$ is relatively compact if and only is equicontinuous and, for every $x \in X$, the set $\mathcal{W}(x) = \{ f(x) \mid f \in \mathcal{W}\}$ is relatively compact in E.*

For a proof see [Die87], pp. 142–143.

Theorem 3.5.4. (cf. [AM03b], Theorem 3.6). Let $A, B \in C_h(\mathbb{R}^+, \mathcal{B}(E))$ be such that $K = B - A$ is a compact operator for every t. Then W_A^s is a compact perturbation of W_B^s and

$$\dim(W_A^s, W_B^s) = \dim(E^-(A(+\infty)), E^-(B(+\infty))).$$

Proof. Up to a time shift we can assume that A and B satisfy the conditions of the Proposition 3.3.1. Then W_A^s and W_B^s are graph of operators

$$S_A \in \mathcal{B}(E^-(A(+\infty)), E^+(A(+\infty))),$$
$$S_B \in \mathcal{B}(E^-(B(+\infty)), E^+(B(+\infty))).$$

Let $P^-(A)$ and $P^-(B)$ be the spectral projectors of the negative eigenspaces. Observe that

$$W_A^s = \ker(P^+(A) - S_A P^-(A)),\ W_B^s = \ker(P^+(B) - S_B P^-(B)).$$

The differences $P^\pm(A) - P^\pm(B)$ are compact operators; we wish to prove that $S_A P^-(A) - S_B P^-(B)$ is also compact. Therefore W_A^s is a compact perturbation of W_B^s and, by Proposition 1.5.13,

$$\begin{aligned}\dim(W_A^s, W_B^s) &= \dim(\ker(P^+(A) - S_A P^-(A)), \ker(P^+(B) - S_B P^-(B)))\\ &= \dim(\operatorname{ran}(P^+(B) - S_B P^-(B)), \operatorname{ran}((P^+(A) - S_A P^-(A)))\\ &= \dim(E^+(B(+\infty)), E^+(A(+\infty)))\\ &= \dim(E^-(A(+\infty)), E^-(B(+\infty))),\end{aligned}$$

which is the thesis when W_A^s and W_B^s are graphs. In the general case there exists a real τ such that $A(\cdot+\tau)$ and $B(\cdot+\tau)$ satisfy the conditions of Proposition 3.3.1. Then

$$\begin{aligned}\dim(W^s_{A(\cdot+\tau)}, W^s_{B(\cdot+\tau)}) &= \dim(X_A(\tau)W_A^s, X_B(\tau)W_B^s)\\ &= \dim(W_A^s, X_A(\tau)^{-1}X_B(\tau)W_B^s)\\ &= \dim(W_A^s, W_B^s) + \dim(W_B^s, X_A(\tau)^{-1}X_B(\tau)W_B^s)\end{aligned}$$

The last term of the equality is 0 because $X_A(\tau)^{-1}X_B(\tau)$ can be written as $I + (X_A(\tau)^{-1} - X_B(\tau)^{-1})X_B(\tau)$ which is an invertible operator of the Fredholm group. Then the conclusion follows from Proposition 1.5.13. Now we write, by (3.27)

$$S_A P^-(A) = P^+(A) ev_0[(I - L_A)^{-1} X_{A_-}(\cdot) P^-(A)],$$
$$S_B P^-(B) = P^+(B) ev_0[(I - L_B)^{-1} X_{B_-}(\cdot) P^-(B)].$$

Using the Theorem of Ascoli–Arzelà we prove first that $L_A - L_B$ is a compact operator on $C_b(\mathbb{R}^+, E)$. In fact let $\mathcal{W}$ be a bounded subset of $C_b(\mathbb{R}^+, E)$. Given $u \in \mathcal{W}$ for every $t \in \mathbb{R}^+$ we have

$$(L_A u)'(t) = [P^+(A)A(t)P^+(A) + P^-(A)A(t)P^-(A)](L_A - I)u(t) + A(t)u(t)$$
$$(L_B u)'(t) = [P^+(B)B(t)P^+(B) + P^-(B)B(t)P^-(B)](L_B - I)u(t) + B(t)u(t).$$

Since A and B are bounded the set $\{(L_A - L_B)u(t) \mid u \in \mathcal{W}\}$ is bounded by a constant that depends on t at most. Then $(L_A - L_B)\mathcal{W}$ is equicontinuous. Now we prove that the set

$$\{(L_A - L_B)u(t) \mid u \in \mathcal{W}\}$$

is relatively compact. The prove is carried on interpolating L_A and L_B and applying Lemma 3.5.2 to the differences as follows

$$\begin{aligned}(L_A - L_B)u(t) = {} & P^-(A)\int_0^t X_{A_-}(t)X_{A_-}(\tau)^{-1}P^-(A)A(\tau)P^+(A)u(\tau)d\tau \\ & - P^-(B)\int_0^t X_{B_-}(t)X_{B_-}(\tau)^{-1}P^-(B)B(\tau)P^+(B)u(\tau)d\tau \\ & - P^+(A)\int_t^\infty X_{A_+}(t)X_{A_+}(\tau)^{-1}P^+(A)A(\tau)P^-(A)u(\tau)d\tau \\ & + P^+(B)\int_t^\infty X_{B_+}(t)X_{B_+}(\tau)^{-1}P^+(B)B(\tau)P^-(B)u(\tau)d\tau.\end{aligned}$$

Since $X_A(t) - X_B(t)$ and $A(t) - B(t)$ are compact by interpolation we obtain the sum of two integrals on $[0, t]$ and $[t, +\infty)$ with compact integrands. We conclude by applying Lemma 3.5.2 to the two integrands. By composition $S_A P^-(A) - S_B P^-(B)$ is compact. $\square$

Chapter 4

Ordinary differential operators on Banach spaces

Given a path $A \in C(\mathbb{R}, \mathcal{B}(E))$ we study the properties of the differential operator $F_A u = u' - Au$. When E is a Hilbert space the operator can be defined in $H^1(\mathbb{R}, E)$ with values in $L^2(\mathbb{R}, E)$. By THEOREM 5.1 of [AM03b] the operator F_A is Fredholm if and only if the pair (W_A^s, W_A^u) is a Fredholm pair and

$$\operatorname{ind} F_A = \operatorname{ind}(W_A^s, W_A^u).$$

In this chapter we prove the same result when E is a Banach space and the operator F_A is defined on $C_0^1(\mathbb{R}, E)$ and takes values in $C_0(\mathbb{R}, E)$, where

$$C_0(\mathbb{R}, E) = \left\{ u \in C(\mathbb{R}, E) \mid \lim_{t \to \pm\infty} u(t) = 0 \right\}$$
$$C_0^1(\mathbb{R}, E) = \left\{ u \in C^1(\mathbb{R}, E) \mid \lim_{t \to \pm\infty} u(t) = 0, \ \lim_{t \to \pm\infty} u'(t) = 0 \right\}.$$

We remark that the result also holds when F_A is defined on the Sobolev space $W^{1,p}(\mathbb{R}, E)$ with values in $L^p(\mathbb{R}, E)$ with $p \geq 1$.

4.1 The operators F_A^+ and F_A^-

Consider the spaces

$$C_0(\mathbb{R}^+, E) = \left\{ u \in C^1(\mathbb{R}^+, E) \mid \lim_{t \to +\infty} u(t) = 0 \right\}$$
$$C_0^1(\mathbb{R}^+, E) = \left\{ u \in C^1(\mathbb{R}^+, E) \mid \lim_{t \to +\infty} u(t) = 0, \ \lim_{t \to +\infty} u'(t) = 0 \right\};$$

we define the operator

$$F_A^+ \colon C_0^1(\mathbb{R}^+, E) \to C_0(\mathbb{R}^+, E), \ u \mapsto u' - Au$$

and similarly F_A^- on $C_0^1(\mathbb{R}^-, E)$. We wish to prove that when A is asymptotically hyperbolic F_A^+ has a right inverse. First observe that in special case $A \equiv A_0$

the operator F_{A_0} is invertible and its inverse is given by

$$R_{A_0}h = G_{A_0} * h \tag{4.1}$$

for any $h \in C_0^1(\mathbb{R}, E)$, where

$$G_{A_0}(t) = e^{tA_0}\left[P^-(A_0)1_{\mathbb{R}^+} - P^+(A_0)1_{\mathbb{R}^-}\right] \tag{4.2}$$

where $P^-(A_0)$ and $P^+(A_0)$ are the spectral projectors of A_0 relative to decomposition $\sigma(A_0) = \sigma^+ \cup \sigma^-$ and $1_{\mathbb{R}^+}$ and $1_{\mathbb{R}^-}$ are the characteristic functions of the subsets $\mathbb{R}^+$ and $\mathbb{R}^-$. Exponential estimates of G_{A_0} makes $G_{A_0} * h$ a continuously differentiable function in $C_0^1(\mathbb{R}, E)$. Moreover

$$\begin{aligned} F_{A_0}(G_{A_0} * h)(t) &= (G_{A_0} * h)' - A_0(G_{A_0} * h) \\ &= A_0(G_{A_0} * h) + P^-h(t) + P^+h(t) - A_0(G_{A_0} * h) = h; \end{aligned}$$

hence R_{A_0} is a right inverse of F_{A_0}. Otherwise

$$G_{A_0} * F_{A_0}u = \int_{-\infty}^t e^{(t-\tau)A_0}P^-(u' - A_0u)d\tau - \int_t^{+\infty} e^{(t-\tau)A_0}P^+(u' - A_0u)d\tau$$

integration by parts lead to

$$\begin{aligned} \int_{-\infty}^t e^{(t-\tau)A_0}P^-(u' - A_0u)d\tau &= P^-u(t) \\ -\int_t^{+\infty} e^{(t-\tau)A_0}P^+(u' - A_0u)d\tau &= P^+u(t) \end{aligned}$$

taking the sum we conclude. If A is a asymptotically hyperbolic path we know that W_A^s and W_A^u are closed and have topological complements. Choose X_s and X_u such that $X_s \oplus W_A^s = E = X_u \oplus W_A^u$ and let $P_s = P(W_A^s, X_s)$, $P_u = P(W_A^u, X_u)$. Define

$$G^+_{A,P_s}(t,\tau) = X_A(t)\left[P_s 1_{\mathbb{R}^+} - (I - P_s)1_{\mathbb{R}^-}\right] X_A(\tau)^{-1} \tag{4.3}$$

$$G^-_{A,P_u}(t,\tau) = X_A(t)\left[(I - P_u)1_{\mathbb{R}^+} - P_u 1_{\mathbb{R}^-}\right] X_A(\tau)^{-1} \tag{4.4}$$

Proposition 4.1.1. *If A is an asymptotically hyperbolic path there are positive constants (c, λ) such that*

$$\|G^+_{A,P_s}(t,\tau)\| \le ce^{-\lambda|t-\tau|} \tag{4.5}$$

for every $(t,\tau) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Proof. By the Theorem 3.4.1, if P_s is a projector on W_A^s, $I - P_s^*$ is a projector on $(W_A^s)^\perp = W^s_{-A^*}$. Hence $(G^+_{A,P_s}(t,\tau))^* = G_{-A^*, I-P_s^*}(\tau, t)$ and it's enough to prove the statement for $t \ge \tau \ge 0$. We have

$$\begin{aligned} \|G^+_{A,P_s}(t,\tau)\| &\le \|X_A(t)P_sX_A(t)^{-1}\| \cdot \|X_A(t)X_A(\tau)^{-1}\| \\ &\le c'e^{-\lambda(t-\tau)}\|X_A(t)P_sX_A(t)^{-1}\|. \end{aligned} \tag{4.6}$$

For every $t \in \mathbb{R}^+$ $P(t) = X_A(t)P_sX_A(t)^{-1}$ is a projector onto $X_s(t) = X_A(t)W_A^s$ and $I - P(t)$ onto $X_u(t) = X_A(t)X_u$. By Theorem 3.4.1, i) and iii), $X_s(t)$ converges to $E^-(A(+\infty))$, and $X_u(t)$ to $E^+(A(+\infty))$. Then by Proposition 1.4.9 the $P(t)$ is bounded (in fact converges to a projector). Then the last term of (4.6) is estimated by $Mc'e^{-\lambda(t-\tau)}$. □

This allows us to prove the following

Proposition 4.1.2. *Let A be a bounded continuous path on $\mathbb{R}^+$. Then $F_{A,+}$ is a bounded operator. Moreover if A is asymptotically hyperbolic F_A^+ has right inverse also and one is given by*

$$R^+_{A,P_s}h(t) = \int_{\mathbb{R}} G^+_{A,P_s}(t,\tau)h(\tau)1_{\mathbb{R}^+}(\tau)d\tau. \tag{4.7}$$

where P_s is a projector onto the stable space.

Proof. That F_A^+ is bounded it's clear from the definition. Let's prove that R^+_{A,P_s} maps $C_0(\mathbb{R}^+, E)$ in $C^1_0(\mathbb{R}^+, E)$. In fact if $h \in C_0$ then $R^+_{A,P_s}h(t)$ is

$$\int_0^t X_A(t)P_sX_A(\tau)^{-1}h(\tau)d\tau - \int_t^{+\infty} X_A(t)(I-P_s)X_A(\tau)^{-1}h(\tau)d\tau$$

hence is continuous and continuously differentiable. By the (4.5) we have

$$\|R^+_{A,P_s}h(t)\| \le \int_{\mathbb{R}^+} ce^{-\lambda|t-\tau|}|h(\tau)|d\tau \le \|h\|_\infty \int_{\mathbb{R}^+} e^{-\lambda|t-\tau|}d\tau \le \frac{\|h\|_\infty}{\lambda}e^{-\lambda t} \tag{4.8}$$

hence $R^+_{A,P_s}h \in C_0(\mathbb{R}^+, E)$. Since its derivative is

$$(R^+_{A,P_s}h)' = AR^+_{A,P_s}h + h \tag{4.9}$$

and A is bounded, we have $R^+_{A,P_s}h \in C^1_0(\mathbb{R}^+, E)$. Actually (4.8) and (4.9) say that R_{A,P_s} is a bounded operator. Still from (4.9)

$$F^+_A R^+_{A,P_s}h = (R^+_{A,P_s}h)' - AR^+_{A,P_s}h = AR^+_{A,P_s}h + h - AR^+_{A,P_s}h = h.$$

Then R^+_{A,P_s} is a right inverse of F^+_A. □

Similarly we have

Proposition 4.1.3. *If A is a bounded continuous path on $\mathbb{R}^-$ the operator $F_{A,-}$ is bounded and admits a right inverse if A is asymptotically hyperbolic. One is given by*

$$R^-_{A,P_u}h(t) = \int_{\mathbb{R}} G^-_{A,P_u}(t,\tau)h(\tau)1_{\mathbb{R}^-}(\tau)d\tau.$$

where P_u is a projector onto the unstable space.

The proof is completely similar and we omit it.

Example 5. Notice that if A_0 is invertible but not hyperbolic these operators can be non surjective. For example let E be the Euclidean space $\mathbb{R}^2$ and define

$$A_0 = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix},\ e^{A_0} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R_\theta.$$

First observe that $F^+_{A_0}$ is injective: given u in $C^1_0(\mathbb{R}^+, E)$ such that $F^+_{A_0}u = 0$. We have $u(t) = R_{t\theta}u(0)$ by uniqueness of the solutions of (3.2). Since R_θ is an isometry $|u(t)| = |u(0)|$ for every $t \geq 0$. Taking the limit as $t \to +\infty$ we obtain $u(0) = 0$, hence u is zero. Now let h be a continuous function on $\mathbb{R}^+$ that vanishes at $+\infty$ and u in $C_0^1(\mathbb{R}^+, E)$ such that $F_{A_0}^+ u = h$. Since $F_{A_0}^+$ is injective

$$u(t) = e^{tA_0}\left(\int_0^t e^{-sA_0}h(s)ds + u(0)\right) \tag{4.10}$$

is the only solution of the problem. Fix v_0 in $E \setminus \{0\}$ and α in $C_0(\mathbb{R}^+, \mathbb{R}^+)$ not integrable. Let $h(s) = \alpha(s)R_{s\theta}v_0$. Since R_θ is an isometry, the norm of $u(t)$ is equal to the one of

$$\int_0^t R_{-s\theta}h(s)ds + u(0) = \int_0^t \alpha(s)R_{-s\theta}(R_{s\theta})v_0 ds + u(0) = \int_0^t \alpha(s)ds\, v_0 + u(0). \tag{4.11}$$

Since the last term of (4.11) does not converge to 0 as $t \to +\infty$ the function h is not in the image of $F_{A_0}^+$.

Given a continuous function h in $C_0(\mathbb{R}^+, E)$ evaluating $R_{A,P_s}^+ h$ at $t = 0$ we obtain a vector of $\ker P_s$. Similarly we can evaluate $R_{A,P_u}^- h$ and we have a continuous functions

$$r_{A,P_s}^+ : C_0(\mathbb{R}^+, E) \to X_s,\ h \mapsto ev_0 R_{A,P^s}^+ h$$
$$r_{A,P_u}^- : C_0(\mathbb{R}^-, E) \to X_u,\ h \mapsto ev_0 R_{A,P^u}^- h.$$

When no ambiguity occurs on the choice of the path A and the projectors we simply denote them by r^+ and r^- respectively. We have the following

Proposition 4.1.4. (cf. [AM03b], Lemma 4.2). The functions r^+ and r^- are linear and continuous applications and map $C_c^\infty((0, +\infty), E)$ onto X_s and $C_c^\infty((-\infty, 0), E)$ onto X_u.

Proof. We prove the assertion for r^+. Since R_{A,P_s}^+ is bounded, r^+ is bounded. Let v be a vector of E and $\varphi \in C_c^\infty((0, +\infty), \mathbb{R})$ a smooth function such that

$$U = -\int_{\mathbb{R}} \varphi(\tau)X_A(\tau)^{-1}d\tau$$

is an invertible operator on E. We choose $h = \varphi \cdot U^{-1}v$

$$\begin{aligned} r^+h &= -(I - P_s)\int_0^{+\infty} X_A(\tau)^{-1}\varphi(\tau)U^{-1}v d\tau \\ &= -(I - P_s)\int_0^{+\infty} X_A(\tau)^{-1}\varphi(\tau)d\tau U^{-1}v = (I - P_s)v. \end{aligned}$$

□

In the above proof one could remark that choosing a smooth compact supported function ψ on $\mathbb{R}^+$ such that $\int \psi = 1$, for every $v \in X_s$ the function $h(t) = -\psi(t)X_A(t) \cdot v$ still works. However h is at most as regular as X_A.

4.2 Fredholm properties of F_A

We consider now the operator $F_A : C^1_0(\mathbb{R}, E) \to C_0(\mathbb{R}, E)$ and investigate whenever it is invertible or not and, more generally, if it is Fredholm. As in the Hilbert setting we find that F_A is Fredholm if and only if (W^s_A, W^u_A) is a Fredholm pair.

Lemma 4.2.1. (cf. [AM03b], Proposition 5.2). We have the following characterizations of $\ker F_A$ and $\operatorname{ran} F_A$:

$$\ker F_A = \left\{ u \in C^1_0 \mid u(0) \in W^s_A \cap W^u_A \right\} \tag{4.12}$$

$$\operatorname{ran} F_A = \left\{ h \in C_0 \mid r^+_{A,P_s} h - r^-_{A,P_u} h \in W^s_A + W^u_A \right\} \tag{4.13}$$

$$\overline{\operatorname{ran} F_A} = \left\{ h \in C_0 \mid r^+_{A,P_s} h - r^-_{A,P_u} h \in \overline{W^s_A + W^u_A} \right\} \tag{4.14}$$

Proof. We omit the proof of (4.12) that comes straightforwardly from the definition of stable and unstable subspaces. Let $h \in \operatorname{ran} F_A$ and $u \in C^1_0$ such that $F_A u = h$. By Proposition 4.1.2 we have a decomposition $C^1_0(\mathbb{R}^+, E) = \ker F^+_A \oplus \operatorname{ran} R^+_{A,P_s}$. Thus

$$\begin{aligned} u^+ &= X_A(t) u_0 + R^+_{A,P_s} h^+ \\ u^- &= X_A(t) v_0 + R^-_{A,P_u} h^- \end{aligned} \tag{4.15}$$

where u^+ and u^- are the restrictions of u to the positive (respectively negative) real line. Evaluating in 0 and taking the difference of the two equations we obtain

$$W^s_A + W^u_A \ni u_0 - v_0 = r^-_{A,P_u} h - r^+_{A,P_s} h.$$

To prove the converse let $h \in C_0$ such that $r^+ h - r^- h \in W^s_A + W^u_A$. By Propositions 4.1.2 and 4.1.3 we have u^+ and u^- such that

$$F^+_A u^+ = h^+, \quad F^-_A u^- = h^-. \tag{4.16}$$

In order to exhibit an element of C^1_0 such that $F_A u = h$ we want to find suitable u^+ and u^- such that $u^- \# u^+$ is a continuous function and continuously differentiable. Hence it's enough to choose u_0 and v_0 in (4.15) such that

$$u^+(0) = u^-(0) \tag{4.17}$$

$$u^{+\prime}(0) = u^{-\prime}(0), \tag{4.18}$$

as before evaluate (4.15) in 0 and set (4.17) in the left sides. If we choose u_0 and v_0 such that $u_0 - v_0 = r^+ h - r^- h = w$ the joint function $u_- \# u_+$ is continuous. Differentiating the (4.15)

$$\begin{aligned} u^{+\prime}(t) &= A(t) X_A(t) u_0 + A(t) R^+_{A,P_s} h^+(t) + h^+(t) \\ u^{-\prime}(t) &= A(t) X_A(t) v_0 + A(t) R^-_{A,P_u} h^-(t) + h^-(t) \end{aligned}$$

we get $A(0)(u_0 - v_0 - w) = 0$, hence any choice in $W^s_A \times W^u_A$ that makes $u^- \# u^+$ continuous it also makes it C^1.

The proof of the left inclusion of (4.14) is completely similar to the above step. Conversely suppose that h belongs to the right set of the (4.14). Let $\varepsilon > 0$

and $\delta = 1/(\|I - P_s\| \cdot \|U^{-1}\|)$ where U is the operator defined in (4.1). Set $w = r^+_{A,P_s} h - r^-_{A,P_u} h$. There exists $x \in W^s_A + W^u_A$ such that $|w - x| < \delta$. By Proposition 4.1.4

$$r^+ h_\delta = (I - P_s)(w - x), \quad h_\delta = -\varphi U^{-1}(w - x)$$

and $\|h_\delta\| < \varepsilon$. Since h_δ has compact support in $(0, +\infty)$ it can be extended on $\mathbb{R}^-$ with the constant value 0. Thus

$$r^+(h - h_\delta) - r^-(h - h_\delta) = w - r^+ h_\delta = x + P_s(w - x)$$

is an element of $W^s_A + W^u_A$ hence, by (4.13), $h - h_\delta$ is in the image of F_A. □

We conclude the chapter with the relationship between the Fredholm properties of F_A and the Fredholm properties of the pair (W^s_A, W^u_A).

Theorem 4.2.2. (cf. [AM03b], Theorem 5.1). If A is an asymptotically hyperbolic path the following facts hold:

i). F_A has closed range if and only if $W^s_A + W^u_A$ is closed,

ii). F_A is onto if and only if $W^s_A + W^u_A = E$,

iii). F_A is semi-Fredholm if and only (W^s_A, W^u_A) is a semi-Fredholm pair; in this case we also have $\operatorname{ind} F_A = \operatorname{ind}(W^s_A, W^u_A)$.

Proof. If $W^s_A + W^u_A$ is closed the two sets on the right of (4.12) and (4.13) are equal, hence $\operatorname{ran} F_A$ coincides with its closure. Conversely, suppose $\operatorname{ran} F_A$ is closed and let w be an element of $\overline{W^s_A + W^u_A}$. By Proposition 4.1.4, there exists h smooth with compact support such that

$$w = P_s w + (I - P_s)w = P_s w + r^+ h - r^- h$$

hence $r^+h - r^-h$ is in the closure of $W^s_A + W^u_A$. Then, by hypothesis $r^+h - r^-h \in W^s_A + W^u_A$, hence $w \in W^s_A + W^u_A$ and i) is proved. Suppose F_A is onto, that is the range of F_A is closed. By i) $W^s_A + W^u_A$ is also closed and there is an isomorphism of Banach spaces

$$C_0/\operatorname{ran} F_A \to E/W^s_A + W^u_A, \quad h + \operatorname{ran} F_A \mapsto r^+h - r^-h. \tag{4.19}$$

It is injective by (4.13). Given $x \in E$ the element $h + \operatorname{ran} F_A$ such that $r^+h - r^-h = (I - P^s)x$ is in the counter-image of $x + W^s_A + W^u_A$, therefore is surjective. The continuity follows straightforwardly from the definition of the norm for a quotient space. In fact, for every $u \in C^1_0$, we have

$$\begin{aligned}\operatorname{dist}(r^+h - r^-h, W^s_A + W^u_A) &\le \operatorname{dist}(r^+h - r^-h, r^+F_A u - r^-F_A u)\\ &\le (\|r^+\| + \|r^-\|)|h - F_A u|.\end{aligned}$$

Taking the infimum over C^1_0 we prove that the application is bounded. We conclude with the open mapping theorem. If F_A is onto the quotient spaces $C_0/\operatorname{ran} F_A$ is the null space, then, by (4.19) $W^s_A + W^u_A = E$ and the converse is similar, hence ii) is proved. If F_A is semi-Fredholm $\operatorname{ran} F_A$ is closed, hence $W^s_A + W^u_A$ is also closed. By (4.12) and (4.19) the index of F_A and the one of the pair (W^s_A, W^u_A) coincide, this proves iii). □

Chapter 5

Spectral flow

Given a continuous path of essentially hyperbolic operators we can define an integer called *spectral flow*. The definition we provide in this chapter generalizes the one given by J.Phillips for paths of Fredholm and self-adjoint operators.
We show that the definition depends only on the class of fixed-endpoints homotopy of a path. Moreover the spectral flow of the concatenation of two paths is the sum of the spectral flows of the paths, hence we have a well defined group homomorphism

$$\mathrm{sf} : \pi_1(e\mathcal{H}(E), A_0) \to \mathbb{Z}.$$

In chapter 2 we established a homotopy equivalence between the space of essentially hyperbolic operators $e\mathcal{H}(E)$ and the space of idempotents $\mathcal{P}(\mathcal{C})$ of the Calkin algebra, we denoted it by Ψ and defined it as

$$\Psi(A) = P^+([A])$$

where $P^+([A])$ is the eigenprojector relative to the positive complex half-plane.
In Theorem 5.3.1 we prove that there is a strict relation between the spectral flow and the homomorphism φ defined through the exact sequence of the bundle $(\mathcal{P}(E), \mathcal{P}(\mathcal{C}), \mathfrak{p})$. Precisely

$$\mathrm{sf} \circ \Psi_*^{-1} = -\varphi.$$

Thus the spectral flow inherits all the properties of the index φ. The equality holds for every Banach space and gives a characterization of the paths whose spectral flow is zero and necessary and sufficient conditions in order to have nontrivial spectral flow.
In the last section we extend the definition of spectral flow to asymptotically hyperbolic and essentially hyperbolic paths. We prove that if A is also an *essentially splitting path* the differential operator F_A is Fredholm and

$$\mathrm{ind}\, F_A = -\,\mathrm{sf}\,(A) = \dim(E^-(A(+\infty)), E^-(A(-\infty))).$$

In general none of the these equalities holds. Counterexamples are known even in the Hilbert spaces.

5.1 Essentially hyperbolic operators

We recall that an operator A is said *essentially hyperbolic* if $A+\mathcal{B}_c$ is a hyperbolic element of the Calkin algebra $\mathcal{C}$. We denote by $e\mathcal{H}(E)$ the set of the essentially hyperbolic operators.

Lemma 5.1.1 (Structure of the spectrum)**.** *Let A be a bounded operator, D the set of isolated points of $\sigma(A)$. Then $\partial\sigma(A) \setminus D \subset \sigma_e(A)$.*

Proof. We argue by contradiction: let $\lambda_0 \in \sigma(A) \setminus D$. If $\lambda_0 \notin \sigma_e(A)$ $A - \lambda_0$ is Fredholm of index k. There exists $r > 0$ such that for every $\lambda \in B(\lambda_0, r) \setminus \{\lambda_0\}$ the operator $A - \lambda$ is Fredholm of the same index and $\dim\ker(A - \lambda)$ and $\dim\operatorname{coker}(A - \lambda)$ have constant dimension, by Theorem B.6. Since λ_0 is a boundary point there are $z, w \in B(\lambda_0, r) \setminus \{\lambda_0\}$ such that $z \in \sigma(A)$ and $w \in \rho(A)$. But $A - w \in GL(E)$ implies that $B(\lambda_0, r) \setminus \{\lambda_0\} \subset \rho(A)$, hence $z \in \rho(A)$ and we get a contradiction. □

Theorem 5.1.2. *An operator B is essentially hyperbolic if and only if $B = A + K$, $K \in \mathcal{B}_c(E)$, A hyperbolic.*

Proof. Let A be a hyperbolic operator. We want to prove that $A + K$ is essentially hyperbolic, in fact, by Proposition B.2 we have $\sigma_e(A+K) = \sigma_e(A)$. Since A is hyperbolic its spectrum does not meet the imaginary axis. Suppose B is essentially hyperbolic. We show that $F = \sigma(B) \cap \mathrm{i}\mathbb{R}$ is an isolated set in $\sigma(B)$ and therefore finite (since is compact). We argue by contradiction. Suppose λ is not isolated. By hypothesis $B - \lambda$ is Fredholm. Let C be the connected component of λ in $\sigma(B) \cap i\mathbb{R}$. It is a closed interval of the imaginary axis. Let

$$J = -i(C \cap \mathrm{i}\mathbb{R}), \; a = \max J.$$

By Proposition B.5 $B-a$ is Fredholm with the same index as $B-\lambda$. By Theorem B.6 there exists $r > 0$ such that, for every $w \in B(ia, r)$ the operator $B - w$ is Fredholm and

$$\dim\ker(B - w), \quad \dim\operatorname{coker}(B - w)$$

are constants, for every $w \in B(ia, r) \setminus \{ia\}$. Since a connected component is maximal respect to the inclusion ia is not an internal point of $\sigma(B) \cap i\mathbb{R}$, hence there exists $0 < t < r$ such that $i(a + t)$ is not in the spectrum of B, hence $B - i(a + t)$ is invertible and its kernel and cokernel are the null space, hence $B - i(a - t)$ is also invertible, thus the connected component of λ consists of $\{\lambda\}$. This proves that λ is not an internal point of $\sigma(B)$; it is not isolated neither, by hypothesis. Therefore Lemma 5.1.1 allows us to conclude that $\lambda \in \sigma_e(B)$ which contradicts the hypothesis.
Now we can write the spectrum as $\sigma(B) = \sigma^+ \cup \sigma^- \cup \{\lambda_1, \dots, \lambda_n\}$ and choose a family of paths that surrounds $\sigma(B)$ in $\mathbb{C}$, say $\Gamma = \{\gamma^+, \gamma^-, \gamma_1, \dots, \gamma_n\}$. We have projectors $\{P^+, P^-, P_i\}$. Since all the points of $\sigma(B) \cap \mathrm{i}\mathbb{R}$ are isolated *eigenvalues* of B, each $B - \lambda_i$ is a Fredholm operator of index 0. By Theorem 5.28, Ch. IV, §5.4 of [Kat95], each eigenprojector P_i has finite rank. Thus

$$B = \left[B(P^+ + P^-) + \sum_{i=1}^{n} P_i \right] - (I - B) \sum_{i=1}^{n} P_i. \tag{5.1}$$

□

The space $e\mathcal{H}(E)$ is an open subset of $\mathcal{B}(E)$ hence is locally arcwise connected. Theorem 5.1.2 and Proposition 2.1.1 allow us to connect the operator B to the square root of unit

$$P^+(B) - P^-(B) + \sum_{i=1}^{n} P_i.$$

Moreover, if there exists a path that connects $2P - I$ and $2Q - I$ in $e\mathcal{H}(E)$, by Theorem 2.4.2, there exists T invertible such that $TPT^{-1} - Q$ is a compact operator. For instance, if P is a finite rank projector and E is an infinite dimensional space we always have at least the components: the one that contains $2P - I$ and the one of $2(I - P) - I$. We denote them by $e\mathcal{H}_+(E)$ and $e\mathcal{H}_-(E)$ respectively. By Theorem 5.1.2 we have

$$\begin{aligned} e\mathcal{H}_+(E) &= \{ A \in \mathcal{H}(E) \mid \operatorname{Re} z > 0 \ \forall z \in \sigma_e(A) \} \\ e\mathcal{H}_-(E) &= \{ A \in \mathcal{H}(E) \mid \operatorname{Re} z < 0 \ \forall z \in \sigma_e(A) \} . \end{aligned}$$

These are star-shaped to I and $-I$ respectively, hence contractible. There are infinite dimensional Banach spaces (see COROLLARY 19 of [GM93]) where the only complemented subspaces are the finite dimensional and the closed infinite dimensional. For such spaces $e\mathcal{H}_+(E)$ and $e\mathcal{H}_-(E)$ are the only connected components of $e\mathcal{H}(E)$.

5.2 The spectral flow in Banach spaces

We state a fact that we have used more than once in the previous sections and frequently later on.

Proposition 5.2.1. *Let $\mathcal{B}$ be a Banach algebra, $x \in \mathcal{B}$, $\sigma(x) \subset \Omega$ an open subset on $\mathbb{C}$. Then there exists $\varepsilon = \varepsilon(x) > 0$ such that $\|x' - x\| < \varepsilon \Rightarrow \sigma(x') \subset \Omega$.*

It is also known as *lower semi-continuity* of the spectrum. Before stating the next Theorem we remark that given a bounded operator A and two open disjoint subsets of the complex plane U and V such that $U \cup V \supset \sigma(A)$ we denote by $P(A;U)$ and $P(A;V)$ the spectral projectors. If γ is a closed path that surrounds $U \cap \sigma(x)$ in $\mathbb{C} \setminus U$ we also denote $P(A;U)$ by $P_\gamma(A)$.

Theorem 5.2.2. *Let $A \in e\mathcal{H}(E)$. There exists a neighbourhood N of A in $\mathcal{B}(E)$, a closed square $Q(N) = [-a, a] \times [-b, b]$ and $\delta > 0$ such that, for any S in N, the following conditions hold:*

i). $\sigma_e(S) \cap J_a \times \mathbb{R} = \emptyset$,

ii). $\sigma(S) \cap \{|\operatorname{Im} z| \geq b\} = \emptyset$,

iii). $\operatorname{dist}(\sigma(S), \partial Q) \geq \delta$,

where J_a is the closed interval $[-a, a]$. If γ is a simple closed curve which does not intersect ∂Q the second condition allows us to apply the spectral decomposition theorem to define a continuous map

$$N \ni S \longmapsto P_\gamma(S) = \frac{1}{2\pi i} \int_\gamma (\lambda - S)^{-1} d\lambda$$

such that $P_\gamma(S)$ has constant finite rank as S varies in N.

Proof. There exists $a_1 > 0$ such that $\sigma(A) \cap J_{a_1} \times \mathbb{R}$ is finite. To see this choose a_1 such that $\sigma_e(A) \cap J_{a_1} \times \mathbb{R} = \emptyset$, which is possible from Proposition 5.2.1 applied to $A + \mathcal{B}_c$ in the Calkin algebra. A complex $\lambda \in \sigma(A) \cap J_{a_1} \times \mathbb{R}$ is isolated for, if it wasn't $i\,\mathrm{Im}(\lambda)$ would not be isolated in $\sigma(A - \mathrm{Re}\lambda)$ and this, by Theorem 5.1.2, contradicts the fact that $A - \mathrm{Re}\lambda$ is essentially hyperbolic. For a small perturbation of a_1, say $a < a_1$, we also have $\sigma(A) \cap \partial J_a \times \mathbb{R} = \emptyset$. Finally, since $\sigma(A)$ is compact, there exists $b' > 0$ such that $\sigma(A) \subset \mathbb{R} \times J_{b'}$. Choose any $b > b'$ and set $Q = J_a \times J_b$. Since $\sigma(A) \cap \partial Q = \emptyset$ there exists $\delta > 0$ such that $\mathrm{dist}\,(\sigma(A), \partial Q) \geq \delta$. Then we have proved that the three conditions hold for A. In order to extend these properties in a neighbourhood of A we use Proposition 5.2.1 applied with the

$$\Omega_1 = (J_a \times \mathbb{R})^c, \ \Omega_2 = \{|\mathrm{Im}\, z| < b\}, \ \Omega_3 = \overline{B(\partial Q, \delta)}^c,$$

as open subsets and the spectra $\sigma(p(S))$ and $\sigma(S)$ in the algebras $\mathcal{C}(E)$ and $\mathcal{B}(E)$. Hence there exists $r > 0$ such that $\|S - A\| < r$ implies

$$\sigma_e(S) \subset \Omega_1, \ \sigma(S) \subset \Omega_2 \cap \Omega_3;$$

as we have seen the first condition says that $\sigma(S) \cap J_a \times \mathbb{R}$ is finite, the second tells us that none of these spectra hits the border of the square $J_a \times J_b$ as S varies in $N = B(A, r)$. Let γ be a closed path with support equal to ∂Q and $\mathrm{ind}\,(\gamma, 0) = 1$; we have a continuous map of projectors of finite dimensional range

$$N \ni S \longmapsto P_\gamma(S);$$

arguing by compactness and using the fact that two subspaces are isomorphic as soon as two projectors have distance smaller than 1 (see Proposition 2.2.1) we can conclude that $P_\gamma(S)$ has the same rank for any $S \in N$. ☐

Definition 5.2.3. *A neighbourhood $N = N(a, b) \subseteq e\mathcal{H}$ such that the three conditions hold is called* fundamental.

If A is a continuous path on the closed interval $[x_0, x_1]$ such that $A(t)$ belongs to a fixed fundamental neighbourhood N we define the spectral flow

$$\mathrm{sf}\,(A, N, [x_0, x_1]) = \dim(P(A(t_1); \mathbb{H}^+ \cap Q)) - \dim(P(A(t_0); \mathbb{H}^+ \cap Q)).$$

This definition extends to an arbitrary path $A \in C([0,1], e\mathcal{H})$ as follows: choose a finite partition of $[0,1]$, $\mathscr{P} = \{t_i \mid 0 \leq i \leq n\}$ with $t_0 = 0$ and $t_1 = 1$, such that for any $s \in [t_i, t_{i+1}]$ $A(t)$ belongs to a fundamental neighbourhood $N(a_i, b_i)$. We call $\mathscr{N} = \{N(a_i, b_i)\}$ a *family of neighbourhoods ordered with* $\mathscr{P}$. Define

$$\mathrm{sf}\,(A, \mathscr{N}, \mathscr{P}) = \sum_{i=0}^{n-1} \mathrm{sf}\,(A, N_i, [t_i, t_{i+1}]).$$

In the next Proposition we prove that the construction does not depend on the choice of the partition $\mathscr{P}$ and the family of neighbourhoods $\mathscr{N}$.

Proposition 5.2.4. *Given a path A of essentially hyperbolic operators defined on a closed interval $[x_0, x_1]$, the spectral flow does not depend on the choice of neighbourhoods and projectors.*

Proof. We exploit the proof in two steps; first we fix a partition and change neighbourhoods, then we fix a family of neighbourhoods and change the partition. Define $\mathscr{P} = \{t_i \mid 0 \le i \le n\}$, $J_i = [t_i, t_{i+1}]$. Let $\mathscr{N}$ and $\mathscr{M}$ be two families of neighbourhoods ordered with the given partition. Then $A(J_i) \subseteq N_i \cap M_i$. If $Q(N_i) = Q(M_i)$ the proof is simple because we have the same projectors. Without loss of generality we can suppose that $a(N_i) > a(M_i)$; by iii) and ii) of Proposition 5.2.2 we have

$$\sigma(A(s)) \cap (\partial Q(N_i) \cup \partial Q(M_i)) = \emptyset,$$
$$\sigma(A(s)) \cap \overline{Q(M_i) \setminus Q(N_i)} = \emptyset$$

for $s \in J_i$. Thus $\sigma(A(s))$ does not hit the boundary of the set

$$C^+ = \overline{Q(N_i) \setminus Q(M_i)} \cap \mathbb{H}^+;$$

then, by Lemma C.2, the map $s \mapsto P(A(s); C^+)$ is continuous in J_i. Using Proposition 2.2.1 for every s we have $\dim P(A(s); C^+) = m$. Then

$$\begin{aligned}\mathrm{sf}\,(A, N_i, J_i) &= \dim P(A(t_{i+1}); Q^+(N_i)) - \dim P(A(t_i); Q^+(N_i)) \\ &= \dim P(A(t_{i+1}); Q^+(M_i)) + m - \dim P(A(t_i); Q^+(M_i)) - m \\ &= \mathrm{sf}\,(A, M_i, J_i).\end{aligned}$$

Taking the sum over i we find $\mathrm{sf}\,(A, \mathscr{N}, \mathscr{P}) = \mathrm{sf}\,(A, \mathscr{M}, \mathscr{P})$. Now let $(\mathscr{N}, \mathscr{P})$ and $(\mathscr{M}, \mathscr{Q})$ be two pairs where $\mathscr{P} = \{t_i; 0 \le i \le n\}$, $\mathscr{Q} = \{s_j; 0 \le j \le m\}$. Let $\mathscr{R} = \mathscr{P} \cup \mathscr{Q}$ be a refinement of both; if $S_i = \{j \mid t_i \le s_j \le t_{i+1}\}$ we have

$$\begin{aligned}\mathrm{sf}\,(A, \mathscr{N}, \mathscr{P}) &= \sum_{i=0}^{n} \mathrm{sf}\,(A, N_i, [t_i, t_{i+1}]) = \sum_{i=0}^{n} \sum_{j \in S_i} \mathrm{sf}\,(A, N_i, [s_j, s_{j+1}]) \\ &= \sum_{i=0}^{n} \sum_{j \in S_i} \mathrm{sf}\,(A, M_j, [s_j, s_{j+1}]) = \sum_{0 \le k \le m} \mathrm{sf}\,(A, M_k, [s_k, s_{k+1}]) \\ &= \mathrm{sf}\,(A, \mathscr{M}, \mathscr{Q}).\end{aligned}$$

□

Corollary 5.2.5. *Two paths with the same endpoints contained in a fundamental neighbourhood have the same spectral flow.*

Proposition 5.2.6. *If $f : [x_0, x_1] \to [y_0, y_1]$ is a homeomorphism such that $f(x_0) = y_0$, A a path on $[y_0, y_1]$ we have* $\mathrm{sf}\,(A \circ f) = \mathrm{sf}\,(A)$.

Proof. In fact let $\mathscr{Q} = \{s_i \mid A([s_i, s_{i+1}]) \subseteq M_i\}$ be a partition of $[y_0, y_1]$. Then $\mathscr{P} = f^{-1}(\mathscr{Q}) = \{t_i = f^{-1}(s_i)\}$ is a partition of $[x_0, x_1]$ and $A \circ f([t_i, t_{i+1}]) \subseteq M_i$. To conclude we notice that $P(A(s_i); \mathbb{H}^+) = P(A \circ f(t_i); \mathbb{H}^+)$. □

Lemma 5.2.7. *Let $A \in C([x_0, x_1], e\mathcal{H}(E))$, $B \in C([x_1, x_2], e\mathcal{H}(E))$ be two paths such that $A(x_1) = B(x_1)$. The continuous path on $[x_0, x_2]$ such that*

$$A\#B(t) = \begin{cases} A(t) & \text{if } t \in [x_0, x_1] \\ B(t) & \text{if } t \in [x_1, x_2] \end{cases}$$

and $\mathrm{sf}\,(A\#B, [x_0, x_2]) = \mathrm{sf}\,(A, [x_0, x_1]) + \mathrm{sf}\,(B, [x_1, x_2])$.

Proof. Choose two partitions $(\mathscr{P}(A), \mathscr{N}(A))$ for the path A and $(\mathscr{P}(B), \mathscr{N}(B))$ for B. We wish to build a partition and a family of neighbourhoods for $A\#B$. The only difficulty of the proof arises in the choice of the partition in x_1. Since $A(x_1) \in M_0$ and $B(x_1) \in N_{n-1}$ there exists $\varepsilon > 0$ such that $t_{n-1} < x_1 - \varepsilon < x_1$ and $A([x_1 - \varepsilon, x_1]) \subseteq M_0$, therefore $A\#B([x_1 - \varepsilon, x_1]) \subseteq M_0 \cap N_{n-1}$. Choose

$$\mathscr{P}(A\#B) = \{\, t_i, x_1 - \varepsilon, x_1, s_j \mid 0 \le i \le n-2,\ 0 \le j \le m-1 \,\}$$
$$\mathscr{N}(A\#B) = \{\, N_i, N_{n-1} \cap M_0, M_j \mid 0 \le i \le n-2,\ 0 \le j \le m-1 \,\};$$

we are now able to compute the spectral flow;

$$\begin{aligned}\mathrm{sf}\,(A\#B) = &\sum_{i=0}^{n-2} \mathrm{sf}\,(A, N_i, [t_i, t_{i+1}]) + \mathrm{sf}\,(A, N_{n-1}, [t_{n-1}, x_1 - \varepsilon]) \\ &+ \mathrm{sf}\,(A, N_{n-1} \cap M_0, [x_1 - \varepsilon, x_1]) + \sum_{j=0}^{m-1} \mathrm{sf}\,(B, M_j, [s_j, s_{j+1}])\end{aligned}$$

by Proposition 5.2.4 it is equal to

$$\begin{aligned}&\sum_{i=0}^{n-2} \mathrm{sf}\,(A, N_i, [t_i, t_{i+1}]) + \mathrm{sf}\,(A, N_{n-1}, [t_{n-1}, x_1 - \varepsilon]) \\ &\quad + \mathrm{sf}\,(A, N_{n-1}, [x_1 - \varepsilon, x_1]) + \mathrm{sf}\,(B)\end{aligned}$$

the statement is trivially true for concatenation of path that lie in the same fundamental neighbourhood, hence the last expression is

$$\begin{aligned}&\sum_{i=0}^{n-2} \mathrm{sf}\,(A, N_i, [t_i, t_{i+1}]) + \mathrm{sf}\,(A, N_{n-1}, [t_{n-1}, x_1]) + \mathrm{sf}\,(B) \\ &= \mathrm{sf}\,(A) + \mathrm{sf}\,(B).\end{aligned}$$

□

Lemma 5.2.8 (Homotopy equivalence)**.** (cf. [Phi96]). *The spectral flow is invariant for fixed endpoints homotopy in $e\mathcal{H}(E)$.*

Proof. Let $A, B \in C([x_0, x_1], e\mathcal{H}(E))$ be two homotopically equivalent paths and $H : [0,1] \times [x_0, x_1] \to e\mathcal{H}$ an homotopy such that $H(t, 0) = A(t)$, $H(t, 1) = B(t)$, $H(x_0, s) \equiv A(0)$, $H(x_1, s) \equiv A(1)$. Thus, if $A_s = H(\cdot, s)$ we have $A_0 = A$ and $A_1 = B$. Let $K = H(I \times I)$ and a finite cover $\{\, N_h \mid 0 \le h \le k \,\}$ of neighbourhoods, hence $\mathcal{U} = \left\{\, H^{-1}(N_h) \mid 0 \le h \le k \,\right\}$ is a finite cover of $[0,1] \times [x_0, x_1]$. By paracompactness there exists a Lebesgue number of the cover $\mathcal{U}$, say ε; then every subset of $[0,1] \times [x_0, x_1]$ with diameter smaller than ε is contained in $H^{-1}(N_h)$ for some h. Choose partitions of the unit interval and $[x_0, x_1]$

$$\mathscr{P} = \{0, s_1, \dots, s_{m-1}, 1\}$$
$$\mathscr{Q} = \{x_0, t_1, \dots, t_{n-1}, x_1\}$$

such that $s_{j+1} - s_j = a = t_{i+1} - t_i$ for every i, j and $a < \varepsilon/\sqrt{2}$. Set $I_i = [t_i, t_{i+1}]$ and $J_j = [s_j, s_{j+1}]$ and let h be a choice function such that $H(I_i \times J_j) \subseteq N_{h(i,j)}$. For every $1 \le j \le m$ we want to prove that $\mathrm{sf}\,(A_{s_j}) = \mathrm{sf}\,(A_{s_{j-1}})$. First define the paths $\Gamma_{ij} = (H(\cdot, s_j), [t_i, t_{i+1}])$, $B_{ij} = (H(t_i, \cdot), [s_j, s_{j-1}])$ for every $1 \le j \le m$, $0 \le i \le n$. Notice that

$$\mathrm{sf}\,(B_{ij} \# \Gamma_{i,j-1}) = \mathrm{sf}\,(\Gamma_{ij} \# B_{i+1,j}), \tag{5.2}$$

$$\mathrm{sf}\,(B_{nj}) = \mathrm{sf}\,(B_{0j}) = 0; \tag{5.3}$$

the first follows from Corollary 5.2.5. In fact they have the same endpoints and are contained in the same fundamental neighbourhood $N_{h(i,j-1)}$. The second is even simpler: the spectral flow of a constant path is 0. We will prove that for every $0 \le i \le n$ the paths

$$P_{ij} = (A_{s_j}, [x_0, t_i]) \# B_{ij} \# (H(\cdot, s_{j-1}), [t_i, x_1])$$

have the same spectral flow. In fact, using Lemma 5.2.7 and (5.2) we have

$$\begin{aligned}
\mathrm{sf}\,(P_{ij}) =& \mathrm{sf}\,(A_{s_j}, [x_0, t_{i-1}]) + \mathrm{sf}\,(\Gamma_{i-1,j} \# B_{ij}) + \mathrm{sf}\,(H(\cdot, s_{j-1}), [t_i, x_1]) \\
=& \mathrm{sf}\,(A_{s_j}, [x_0, t_{i-1}]) + \mathrm{sf}\,(B_{i-1,j} \# \Gamma_{i-1,j-1}) + \mathrm{sf}\,(H(\cdot, s_{j-1}), [t_i, x_1]) \\
=& \mathrm{sf}\,(A_{s_j}, [x_0, t_{i-1}]) + \mathrm{sf}\,(B_{i-1,j}) + \mathrm{sf}\,(H(\cdot, s_{j-1}), [t_{i-1}, x_1]) \\
=& \mathrm{sf}\,(P_{i-1,j}).
\end{aligned}$$

Using induction on i and (5.3) we can write

$$\mathrm{sf}\,(A_{s_j}) = \mathrm{sf}\,(A_{s_j} \# B_{nj}) = \mathrm{sf}\,(P_{nj}) = \mathrm{sf}\,(P_{0j}) = \mathrm{sf}\,(A_{s_{j-1}})$$

and apply induction on j in order to get $\mathrm{sf}\,(A_{s_m}) = \mathrm{sf}\,(A_{s_0})$. □

Lemma 5.2.8 sets a well defined application $\pi_1(e\mathcal{H}(E), A_0) \to \mathbb{Z}$ for every $A_0 \in e\mathcal{H}$. More precisely

Lemma 5.2.9. *The map* $\mathrm{sf} : \pi_1(e\mathcal{H}(E), A_0) \to \mathbb{Z}$ *is a group homomorphism.*

Proof. Let A, B two continuous loops on the base point A_0. Their multiplication is defined as

$$A * B(t) = \begin{cases} A(2t) & \text{if } 0 \le t \le 1/2 \\ B(2t-1) & \text{if } 1/2t \le 1 \end{cases};$$

call r_a, r_b the parametrization of $[0,1]$ to $[0,1/2]$ and $[1/2,1]$ respectively. Then, by Lemma 5.2.7 and Proposition 5.2.6

$$\mathrm{sf}\,(A * B) = \mathrm{sf}\,(A \circ r_a) + \mathrm{sf}\,(B \circ r_b) = \mathrm{sf}\,(A) + \mathrm{sf}\,(B).$$

□

We end this section with a simple case in which is very easy to compute the spectral flow. We denote by $\mathcal{H}(E)$ the set of hyperbolic operators.

Proposition 5.2.10. *Let* $A \in C([0,1], \mathcal{H}(E))$*; then* $\mathrm{sf}\,(A) = 0$.

Proof. As usual let $(\mathscr{P}, \mathscr{N})$ be a partition of $[0,1]$, and a family of neighbourhoods. Since $\sigma(A(t)) \cap \mathrm{i}\mathbb{R} = \emptyset$ we have for every $0 \le i \le n-1$

$$P(A(s); Q(N_i)) = P(A(s); Q(N_i) \cap \mathbb{H}^+) + P(A(s); Q(N_i) \cap \mathbb{H}^-)$$

for $s \in [t_i, t_{i+1}]$. Since $\sigma(A(s))$ does not meet the imaginary axis the two terms on the right member are continuous, then the dimension of their images is constant. Thus $\mathrm{sf}\,(A, [t_i, t_{i+1}]) = 0$. Taking the sum over $0 \le i \le n-1$ we conclude. □

5.3 Spectral flow and index of exact sequence

Given a projector P of E we consider the connected component of $e\mathcal{H}(E)$ of the hyperbolic element $2P-I$. We have defined the spectral flow on $\pi_1(e\mathcal{H}(E),P)$. Using the homotopy equivalence of Theorem 2.1.4

$$\Psi\colon e\mathcal{H}(E)\to\mathcal{P}(\mathcal{C}),\quad A\mapsto P([A];\{\operatorname{Re}z>0\})$$

we can define a group homomorphism on $\pi_1(\mathcal{P}(\mathcal{C}),[P])$ as the composition $\mathrm{sf}\circ\Psi_*^{-1}$. We recall that another homomorphism is defined in Theorem 2.8.1 through the exact homotopy sequence of the fibre bundle $(\mathcal{P}(E),\mathcal{P}(\mathcal{C}),\mathfrak{p})$. We wish to prove that these homomorphisms differ by a sign.

Theorem 5.3.1. *For every x in $\pi_1(\mathcal{P}(\mathcal{C}))$ we have $\varphi(x)=-\mathrm{sf}\circ\Psi_*^{-1}(x)$.*

Proof. In order to abbreviate the notations we set $\psi=\mathrm{sf}\circ\Psi_*^{-1}$. If $\varphi(x)=0$ there exists a loop γ in $\mathcal{P}(E)$ such that $[\mathfrak{p}\circ\gamma]=x$. Following the definition of Ψ it is easy to check that

$$[\Psi(2\gamma-I)]=x.$$

By Proposition 5.2.10 $\mathrm{sf}\,(2\gamma-I)=0$, hence $\psi(x)=0$. Suppose $k=\varphi(x)$ with $k\neq 0$. By definition of φ there exists a continuous path β in $\mathcal{P}(E)$ such that $\beta(0)=P$ and $\dim(\beta(1),P)=k$. If $k>0$ let R be a projector such that

$$\dim R=k,\quad RP=PR=0$$

hence $Q=P+R$ is a projector also; it is a compact perturbation of $\beta(1)$ and the relative dimension is zero. Thus, by Theorem 2.6.3 and Proposition 2.5.2 there exists a continuous path α in $\mathcal{P}(E)$ with endpoints Q and $\beta(1)$. In order to build a closed loop of the required spectral flow we choose as base point $P-R-(I-Q)$. First consider the path

$$A_0(t)=P-(1-2t)R-(I-Q),\quad A_0(1)=Q-(I-Q),\quad \mathrm{sf}\,(A_0)=k;$$

this is clearly an essentially hyperbolic path, in fact it can be written as $P-(I-P)+2tR$, that is a perturbation of a hyperbolic operator by a finite rank operator. It has spectral flow k straightforwardly from the definition. Since Q is connected to P through the path $\overline{\beta}*\alpha$ we define a path A_1 as

$$A_1(t)=\overline{\beta}*\alpha(t)+(I-\overline{\beta}*\alpha(t)),\quad A_1(1)=P+(I-P)=A_0(0);$$

since A_1 is a path of hyperbolic operators it has spectral flow equal to zero by Proposition 5.2.10. Then the loop $A=A_0*A_1$ has spectral flow k. It is not hard to prove that

$$\mathfrak{p}_*(\overline{\beta})=\Psi_*(A)$$

hence $\psi(\mathfrak{p}_*(\overline{\beta}))=\varphi(x)$. If $k<0$ the same steps can be repeated with x^{-1}. □

The theorem says, in particular, that the homomorphisms have the same kernel. Hence we have a characterization of the kernel of the spectral flow.

Proposition 5.3.2. *A path A has spectral flow equal to zero if and only if there exists a continuous loop β in $\mathcal{P}(E)$ such that*

$$\beta(t) - P(A(t); \mathbb{H}^+)$$

is compact for every $t \in [0,1]$.

The theorem states also that they have the same images. Thus we have a characterization of the image of the spectral flow also.

Proposition 5.3.3. *Given a Banach space E and a projector P there exists a loop of essentially hyperbolic operators based on $2P - I$ with spectral flow k if and only if the projector P is connected to a projector Q such that $P - Q$ is compact,* $\dim(Q,P) = k$ *and*

$$\dim(Q,P) = k.$$

In general all the facts proved for the index φ are true for the spectral flow: if $P \in \mathcal{P}(E)$ and the hypotheses h1) and h2) hold the spectral flow is an isomorphism on $\pi_1(e\mathcal{H}(E), 2P - I)$ with $\mathbb{Z}$. If E satisfies the hypotheses of Proposition 2.8.2 it is surjective.

5.4 The Fredholm index and the spectral flow

Given an asymptotically hyperbolic path A in $e\mathcal{H}(E)$ the spectral flow can defined as follows: since $\mathcal{H}(E)$ is an open subset of $\mathcal{B}(E)$ there exists $\delta > 0$ such that $A((-\infty, -\delta] \cup [\delta, +\infty)) \subset \mathcal{H}(E)$. Then define

$$\mathrm{sf}\,(A) = \mathrm{sf}\,(A, [-\delta, \delta]). \tag{5.4}$$

That the definition does not depend on the choice of δ follows from Proposition 5.2.10.

Definition 5.4.1. *A splitting $E = E_1 \oplus E_2$ is called* essential *for an operator T if there exists a compact perturbation T_0 of T such that $T_0(E_i) \subset E_i$.*

In fact it is easy to check that the above splitting is essential for an operator T if and only if $[T, P(E_1, E_2)]$ is a compact operator. Given an asymptotically hyperbolic path A we denote by $E^+(+\infty)$ and $E^-(+\infty)$ the images of the spectral projectors of $A(+\infty)$. Similarly we define $E^+(-\infty)$ and $E^-(-\infty)$.

Definition 5.4.2. *An a.h. path is called* essentially splitting *if and only if the following conditions hold:*

i). the splittings $E = E^+(+\infty) \oplus E^-(+\infty)$ and $E = E^+(-\infty) \oplus E^-(-\infty)$ are essential for $A(t)$, $t > 0$ and $t \leq 0$ respectively;

ii). $E^-(-\infty)$ is compact perturbation of $E^-(+\infty)$.

We can prove the following

Theorem 5.4.3. (cf. Theorem 6.3, [AM03b]). If A is asymptotically hyperbolic and essentially splitting the operator F_A is Fredholm and $\mathrm{ind}\, F_A = \dim(E^-(A(+\infty)), E^-(A(-\infty)))$.

Proof. Denote by $P^{\pm}(+\infty)$ and $P^{\pm}(-\infty)$ the spectral projectors of $A(\pm\infty)$. The following paths

$$A_+(t) = A(t) - [A(t), P^-(+\infty)] \quad \text{if } t > 0$$
$$A_-(t) = A(t) - [A(t), P^+(-\infty)] \quad \text{if } t \le 0$$

are compact perturbations of A and leave respectively $E^{\pm}(+\infty)$ and $E^{\pm}(-\infty)$ invariant. Since $A_+(+\infty) = A(+\infty)$ by Lemma 3.4.2 we have

$$W^s_{A_+} = E^-(+\infty),\ W^u_{A_-} = E^+(-\infty).$$

By Theorem 3.5.4 W^s_A and W^u_A are compact perturbation of $E^-(+\infty)$ and $E^+(-\infty)$. respectively. By hypothesis $(E^-(+\infty), E^+(-\infty))$ is a Fredholm pair. BY Proposition 1.5.11 the pair (W^s_A, W^u_A) is Fredholm, hence, by Theorem 4.2.2, F_A is Fredholm and

$$\begin{aligned}\operatorname{ind} F_A &= \dim(W^s_A, W^u_A) = \dim(W^s_A, E^-(+\infty)) + \operatorname{ind}(E^-(+\infty), E^+(-\infty)) \\ &\quad + \dim(E^+(-\infty), W^u_A) = \dim(E^-(+\infty), E^-(-\infty)).\end{aligned}$$

□

For essential splitting path we are able to compute the spectral flow. First we need the following

Lemma 5.4.4. *Let A be an asymptotically hyperbolic and essentially hyperbolic path. It is essentially splitting also if and only if the set $\{P^+(A(t)) \mid t \in \mathbb{R}\}$ is contained in the same class of compact perturbation.*

Proof. Suppose A is essentially splitting and consider the restriction on half line $\mathbb{R}^+$; hence, using the decomposition $E = E^+ \oplus E^-$, we can write

$$A(t) = \begin{pmatrix} A_+ & K_\pm \\ K_\mp & A_- \end{pmatrix}$$

where $K_\pm$ and $K_\mp$ are compact operators because A is essentially splitting. Since $A_+(+\infty)$ is hyperbolic there exists $t_+ > 0$ such that $A_+([t_+, +\infty)) \subset \mathcal{H}(E^+)$ and

$$\|P^+(A_+(t)) - P^+(A_+(+\infty))\| < 1.$$

But $A_+(+\infty)$ has positive spectrum, hence $P^+(A_+(+\infty)) = I$. It follows from Proposition 2.2.1 that $P^+(A_+(t))$ are the identity on E^+ if $t \in [t_+, +\infty)$. Since A is essentially hyperbolic on E A_+ is also essentially hyperbolic on E^+ and we have a path in $[0, t^+]$

$$A_+ \colon [0, t_+] \to e\mathcal{H}(E^+),\ A_+(t_+) \in e\mathcal{H}_+(E^+);$$

since $e\mathcal{H}_+(E^+)$ is a connected component $A_+([0, t^+])$ is contained in $e\mathcal{H}_+(E^+)$. Thus the positive eigenspaces have finite codimension for every $t > 0$. It is easy to check that two projectors $P^+(A_+(s))$ and $P^+(A_+(s'))$ with ranges of finite codimension have compact difference: the operator

$$P^+(A_+(s)) - P^+(A(s')) = (P^+(A_+(s)) - I) + I - P^+(A(s'))$$

is sum of finite rank operators. Similarly $P^+(A_-(+\infty)) = 0$ and there exists $t_- > 0$ such that the positive projector of $A_-(t)$ is zero for $t \geq t_-$. Thus $A_-(t)$ for $t \leq t_-$ is a path of continuous essentially hyperbolic operators that intersect a connected component, that is $e\mathcal{H}_-(E^-)$; by continuity of A the whole path lies $e\mathcal{H}_-(E^-)$. If $t_0 \geq \max\{t_+, t_-\}$ we can write for every $t \geq 0$

$$\begin{aligned} P^+(A(t)) &\sim_c P^+(A_+(t)) + P^+(A_-(t)) \sim_c P^+(A_+(t_0)) + P^+(A_-(t_0)) \\ &= I_{E^+} \oplus 0_{E^-} = P^+(+\infty) \end{aligned}$$

where $\sim_c$ denotes the relation of compact perturbation. Similarly we can prove that $P^+(A(t))$ is compact perturbation of $P^+(-\infty)$ for every $t \leq 0$. By hypothesis $P^+(+\infty) - P^+(-\infty)$ is compact, hence all the positive projectors (and thus the negative) are compact perturbation one of each other. Conversely if $\{P^+(A(t) \mid t \in \mathbb{R}\}$ is in the same class of compact perturbation we have

$$[A(t), P^-(+\infty)] = [A(t), P^-(A(t))] - [A(t), P^-(A(t)) - P^-(+\infty)]$$

for $t > 0$. The first term of the second member is 0, the last is compact by hypothesis. The proof for $t \leq 0$ is similar. □

We conclude the chapter with the proof that for an a.h. path which is essentially splitting and essentially hyperbolic there holds $\mathrm{sf}(A) = -\operatorname{ind} F_A$.

Theorem 5.4.5. *Let A be an asymptotically hyperbolic path and essentially hyperbolic such that $\{P^+(A(t)) \mid t \in \mathbb{R}\}$ are compact perturbation one of each other. Then*

$$\mathrm{sf}(A) = -\dim(E^-(A(+(\infty))), E^-(A(-\infty)))$$

Proof. Let $\delta > 0$ such that $A((-\infty, -\delta] \cup [\delta, +\infty)) \subset \mathcal{H}(E)$ and $\{t_i \mid 0 \leq i \leq n\}$ a partition of $[-\delta, \delta]$ such that all the conditions of Theorem 5.2.2 are fulfilled. Let $J_i = [t_i, t_{i+1}]$. By basic properties of relative dimension we can write

$$\begin{aligned} &\dim(P^+(A(t_{i+1})), P^+(A(t_i))) \\ &= \dim(P(A(t_{i+1}); Q_i), P(A(t_i); Q_i)) \\ &+ \dim(P(A(t_{i+1}); \mathbb{H}^+ \setminus Q_i), P(A(t_i); \mathbb{H}^+ \setminus Q_i)); \end{aligned}$$

the map $J_i \ni s \mapsto \operatorname{ran} P(A(s); \mathbb{H}^+ \setminus Q_i)$ is continuous because $\sigma(A(s))$ does not meet the boundary of $\mathbb{H}^+ \setminus Q_i$ and takes values in $\mathcal{P}(E^+(A(t_i)); E)$. By 2.6.3 the path lies in $\mathcal{P}_0(E^+(A(t_i)); E)$, thus the second summand of the equation is 0. Taking the sum we can write

$$\sum_{i=0}^{n-1} \dim(P^+(A(t_{i+1})), P^+(A(t_i))) = \sum_{i=0}^{n-1} \dim(P(A(t_{i+1}); Q_i), P(A(t_i); Q_i));$$

the terms on the left member cancel in pairs and $\dim(P^+(A(\delta)), P^+A(-\delta))$ is their sum. On the right member we have $\mathrm{sf}(A)$. Since A is hyperbolic in $(-\infty - \delta] \cup [\delta, +\infty)$ the path $P^+(A(t))$ is continuous on this subset. By Theorem 2.3.3

$$\dim(E^-(A(+\infty)), E^-(A(-\infty))) = -\dim(P^+(A(\delta)), P^+(A(-\delta))) = -\mathrm{sf}(A).$$

□

Thus Theorems 5.4.5 and 5.4.4 give for essentially splitting paths in $e\mathcal{H}(E)$ the equality $\operatorname{ind} F_A = -\operatorname{sf}(A)$. If A is not essentially splitting counterexamples are known even in a Hilbert space; here we describe the EXAMPLE 7 of [AM03b], Ch. 7.

Example 6. In Proposition 3.4.3 we showed how to patch a discontinuity of a path A without changing the stable space of A_+ and the unstable space of A_-. Here we describe another method; let X and Y be closed isomorphic subspaces that admit isomorphic topological complement X' and Y'. Define $P = P(X, X')$ and $Q = P(Y, Y')$. We have a piecewise continuous path

$$A(t) = \begin{cases} 2P - I & t \geq 1 \\ 2Q - I & t \leq -1 \end{cases} ;$$

call A^s and A^u the restrictions of A to the positive and negative half-line; by Proposition 3.4.3 we know that $W^s_{A^s} = X$, $W^u_{A^u} = Y$. There exists an invertible operator T such that $TQT^{-1} = P$ which means, in particular, that $TY = X$. If $GL(E)$ is connected there also exists a path U that $U(-1) = I$ and $U(1) = T$. Define

$$A_U(t) = \begin{cases} 2P - I & t \geq 1 \\ U(t)(2P - I)U(t)^{-1} & -1 \leq t \leq 1 \\ 2Q - I & t \leq -1 \end{cases} ;$$

the path A_U is continuous and hyperbolic, hence, by Proposition 5.2.10 $\operatorname{sf}(A_U)$ is zero. By iii) of Theorem 4.2.2 the operator F_A is Fredholm if and only if the pair (X, Y) is Fredholm. Thus

$$\operatorname{sf}(A_U) \neq -\operatorname{ind} F_{A_U}$$

if (X, Y) is a Fredholm pair of index $k \neq 0$.

Appendix A

The Cauchy problem

Let E be a Banach space and let f be a function defined on a open subset $\Omega \subseteq \mathbb{R} \times E$ with values in E. We denote by $\Omega_t = \{u \in E \mid (t, u) \in \Omega\}$. We require f to have these properties:

i). f is continuous

ii). for any $t \in \mathbb{R}$ such that $\Omega_t \neq \emptyset$ there exists an open subset $\mathbb{R} \supseteq U_t \ni t$ and a constant M such that $f(t', \cdot)$ is a Lipschitz function with constant M for every t' in U_t.

Theorem A.1 (Cauchy)**.** *Let f and Ω be as above. Then for every $(t_0, u_0) \in \Omega$ there exists an open ball $B(t_0, r)$ and $u \in C^1(B(t_0, r), E)$ such that $(t, u(t)) \in \Omega$ for every $t \in B(t_0, r)$ and*

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(t_0) = u_0 \end{cases} ;$$

moreover, if there exists an open interval $J \ni t_0$ and $v \in C^1(J, E)$ satisfying the same conditions as $(u, B(t_0, r))$ u and v coincide in the intersection $B(t_0, r) \cap J$.

Proof. Set $z_0 = (t_0, u_0)$. There exists an open neighbourhood of z_0, $D(t_0, a) \times B(u_0, b') \subseteq \Omega$. By compactness of $D(t_0, a)$ we can find a open ball $B(u_0, b)$ such that $f(D(t_0, a) \times B(u_0, b))$ is bounded, call m its bound. For any $r \leq a$ let E_r be the space $C(J_r, B(u_0, b))$ endowed with the supremum topology. If $v \in E_r$ $(t, v(t)) \in J_r \times B(u_0, b) \subseteq \Omega$, thus we can define

$$\Phi_f(v) = u_0 + \int_{t_0}^{t} f(s, v(s))ds.$$

Since $\left|\int_{t_0}^{t} f(s, v(s))ds\right| \leq rM$ for every $t \in J_r$ we have

$$\Phi_f(v)(t) \in B(u_0, mr).$$

Still by compactness of $D(t_0, a)$, by property iii), there exists $k \in \mathbb{R}^+$ such that for every $t \in D(t_0, a)$ the function $f(t, \cdot)$ is Lipschitz with constant k in Ω_t. Let $v, w \in E_r$. Hence

$$\|\Phi_f(v) - \Phi_f(w)\| \leq kr\|v - w\|.$$

If we choose $rm < b$ and $kr < 1$ we make Φ_f a contraction of E_r into itself. Hence Φ_f has a unique fixed point u. Then $(u, B(t_0, r))$ fulfills the requirements. □

Proposition A.2. *Suppose f and Ω as in the theorem. If u and v are two solutions defined on a connected open interval J and coincide in $t_0 \in J$ then u and v coincide in J.*

Proof. Let $A = \{t \in J \mid u(t) = v(t)\}$. Since u and v are continuous A is a closed subset of J. By hypothesis we know that is nonempty. We prove that A is also open (hence $A = J$). Let $t' \in A$, $u_0 = u(t') = v(t')$. By Theorem A.1 there exists a solution $w \in C^1(B(t', r_0), E)$ such that $w(t') = u_0$. By uniqueness of local solutions $B(t', r_0) \subseteq A$. □

Definition A.3. *Let (u, J) be a solution. Then (v, J') is a* prolongation *of (u, J) if $J \supseteq I$ and $v(t) = u(t)$ for every $t \in J$.*

Using Zorn's Lemma it is easy to prove that for a solution (u, J) there exists a unique maximal prolongation (v, J'). There many criterions to establish when a solution (u, J) can be extended to a bigger interval J'. Here's an example:

Lemma A.4. *Let $(u, B(t_0, r))$ be a solution of (f, Ω) and suppose that the set $\{f(t, v(t)) \mid t \in B(t_0, r)\}$ is bounded in E and I_{t_0+r} and I_{t_0-r} are nonempty. Then there is a prolongation $(w, B(t_0, r'))$, $r' > r$.*

The Lemma can be used to prove the existence of global maximal solution in some particular case. First we need the

Lemma A.5 (Gronwall)**.** *Let w, ϕ, ψ be continuous real valued functions on the compact interval $[a, b]$ such that the estimate*

$$w(t) \leq \phi(t) + \int_a^t w(s)\psi(s)ds;$$

for every $a \leq t \leq b$. Then for every t in the interval the estimate

$$w(t) \leq \phi(t) + \int_a^t \phi(s) \left(\exp \int_s^t \psi(\xi) d\xi \right) ds$$

also holds.

Using Gronwall's Lemma we can prove the following statement.

Proposition A.6. *Suppose Ω is the product $J \times E$ where J an open connected interval of $\mathbb{R}$. If for every $t_0 \in J$ there exists a function $k \in C(J, E)$ such that*

$$|f(t, u) - f(t, v)| \leq k(|t - t_0|)|u - v|, \quad t_0 \in J;$$

then every solution admits a prolongation to the whole interval J.

It is easy to check that the pair (f, Ω) satisfies the three conditions of the Theorem A.1. Thus, given (t_0, u_0), there exists a maximal solution $(u, B(t_0, r))$. Since the domain Ω is a product the sets I_{t_0+r} and I_{t_0-r} are nonempty. Moreover, for every $t \in B(t_0, r)$ we have the estimate

$$|u(t)| \leq |u_0| + \int_{t_0}^t k(|s - t_0|)|u(s) - u_0| ds;$$

applying the Gronwall's Lemma we can conclude that u is bounded, hence admits a prolongation by Lemma A.4.

The Proposition A.6 applies to the particular case: let $\Omega = J \times E$ be the domain of f and $A \in C(J, \mathcal{B}(E))$, $b \in C(J, E)$ be two continuous functions. The Cauchy problem

$$f(t, u) = A(t)u + b(t), \quad \Omega = J \times E$$

admits unique global solutions defined on J. We conclude by remarking that the theorems of existence, prolongation and the related results can be restated in a more general setting: by *step function* we mean a finite sum of characteristic functions. Let $\mathscr{C}(J, E)$ be the vector space of step function. As a subset of $L^\infty(J, E)$ we can consider the closure $\overline{\mathscr{C}}$.

Definition A.7. *An element of* $\overline{\mathscr{C}}$ *is called* regulated function.

Here are the hypotheses of the Theorem A.1 for regulated functions: we f and Ω to solve the conditions

i). for every $w \in C(J, E)$ such that $\{(t, w(t))\} \subseteq \Omega$ $f(t, w(t))$ is regulated,

ii). for any point $(t, u) \in \Omega$ there are an open neighbourhood $B(t, r) \times B(u, b)$ and $M \in \mathbb{R}^+$ such that f is bounded $B(t, r) \times B(u, b)$, and $f(s, \cdot)$ is Lipschitz with constant M.

For the proofs and more details see [Die87].

Appendix B

Fredholm operators

Given an operator $T\colon E \to F$ we can consider the spaces $\ker T$ and $E/\mathrm{ran}T$. The latter is called cokernel and is denoted by $\mathrm{coker}T$.

Definition B.1. *An operator $T \in \mathcal{B}(E,F)$ is called semi-Fredholm if $\ker T$ and $\mathrm{ran}T$ are closed and at least one of $\ker T$ and $\mathrm{coker}T$ has finite dimension. It is said Fredholm if both have finite dimension.*

The *Fredholm index* of a (semi)Fredholm operator is $\mathrm{ind}T = \dim\ker T - \dim \mathrm{coker}T$. We denote by $\mathcal{F}(E,F)$ the set of Fredholm operators.

Proposition B.2. *If $T\colon E \to F$ is a Fredholm operator and K a compact operator then $T+K$ is Fredholm operator and $\mathrm{ind}(T+K) = \mathrm{ind}T$.*

Proposition B.3. *An operator $T \in \mathcal{B}(E,F)$ is Fredholm if and only if is essentially invertible, that is, there exists $S \in \mathcal{B}(F,E)$ such that*

$$ST = I + K$$
$$TS = I + H$$

where K and H are compact operators on E and F respectively.

Proof. Since $\ker T$ and $\mathrm{ran}T$ are complemented subspaces of E and F respectively there are $X \subset E$ and $Y \subset F$ such that $E = \ker T \oplus X$ and $F = Y \oplus \mathrm{ran}T$. The restriction of T to X maps isomorphically X onto $\mathrm{ran}T$, let σ be its inverse. Hence, given a pair (y,r) in F we have

$$T \circ (0 \oplus \sigma)(y,r) = r;$$

hence

$$T \circ (0 \oplus \sigma) = P(\mathrm{ran}T, Y) = I - P(Y, \mathrm{ran}T)$$

where the last term denotes the projector onto Y along $\mathrm{ran}T$. Since Y has finite dimension it is a perturbation of the identity by a finite-rank operator, hence compact. Similarly

$$(0 \oplus \sigma) \circ T = P(X, \ker T) = I - P(\ker T, X)$$

is a compact perturbation of the identity. Hence we can choose $S = 0 \oplus \sigma$. In order to prove the converse observe that if S is an essential inverse of T we have the inclusions

$$\begin{aligned} \ker T &\subset \ker S \circ T = \ker(I + K), \\ \operatorname{ran} T &\supset \operatorname{ran} T \circ S = \operatorname{ran}(I + H) \end{aligned}$$

where the right members have finite dimension and finite codimension because by Proposition B.2 a compact perturbation of the identity is Fredholm. $\square$

Proposition B.4. *Let $A \in \mathcal{B}(E, F)$ and $B \in \mathcal{B}(F, G)$ be two Fredholm operators. Then BA is Fredholm and its index is* $\operatorname{ind} B + \operatorname{ind} A$.

Proof. For the sake of simplicity we denote by k and c the dimension of the kernel and the cokernel respectively. Set $T = BA$. Since A is Fredholm there exists a finite-dimensional subspace $X \subset E$ such that

$$\ker T = \ker A \oplus X;$$

the restriction of A to X is an isomorphism with $\ker B \cap \operatorname{ran} A$. Thus

$$k(T) = k(A) + \dim \ker B \cap \operatorname{ran} A. \tag{B.1}$$

The image of T is $B(\operatorname{ran} A)$. Consider the inclusion of subspaces

$$B(\operatorname{ran} A) \subset \operatorname{ran} B \subset G;$$

the codimension of $B(\operatorname{ran} A)$ in $\operatorname{ran} B$ can be computed as the codimension of $\operatorname{ran} A + \ker B$ in F, hence

$$\begin{aligned} c(T) &= c(B) + \operatorname{codim}(\operatorname{ran} A + \ker B) \\ &= c(B) + \operatorname{codim} \operatorname{ran} A - (k(B) + \dim \operatorname{ran} A \cap \ker B). \end{aligned} \tag{B.2}$$

Thus adding the results of (B.1) and (B.2) we obtain

$$\begin{aligned} \operatorname{ind} T = k(T) - c(T) &= k(A) + \dim \ker B \cap \operatorname{ran} A - c(B) - c(A) \\ &- k(B) - \dim \operatorname{ran} A \cap \ker B = \operatorname{ind} A + \operatorname{ind} B. \end{aligned}$$

$\square$

Proposition B.5. *The subset $\mathcal{F}(E, F) \subset \mathcal{B}(E, F)$ is open and the Fredholm index is a locally constant function with values in $\mathbb{Z}$.*

Proof. We use the Proposition B.3. Let T be a Fredholm operator and S be an essential inverse, that is $TS - I$ is a compact operator. For every operator H such that $\|H\| < \|S\|^{-1}$ we have

$$(T + H)S = TS + HS = I + K + HS = (I + HS) + K$$

where K is a compact operator; since $I + HS$ is invertible we can multiply both terms by its inverse and obtain

$$(T + H)S(I + HS)^{-1} = I + K(I + HS)^{-1}$$

hence $S(I_F + HS)^{-1}$ is an essential right inverse for $T + H$. Similarly we can write $S(T+H) = I + SH + K'$ where K' is compact. Since $I + SH$ is invertible we obtain

$$(I + SH)^{-1}S(T + H) = I + (I + SH)^{-1}K'$$

and prove that $T + H$ has an essential left inverse also. Hence $B(T, \|S\|^{-1}) \subset \mathcal{F}(E, F)$. We compute the index of $T + H$ using the Propositions B.4 and B.2

$$\begin{aligned}\operatorname{ind}(T + H) &= -\operatorname{ind} S(I + HS)^{-1} = -\operatorname{ind} S - \operatorname{ind}(I + HS)^{-1}\\ &= -\operatorname{ind} S = \operatorname{ind} T.\end{aligned}$$

□

The preceding statement and the Proposition B.2 say that the index of a Fredholm operator is stable under small or compact perturbations. Here we state a more specific result regarding the dimension of the kernel and the cokernel

Theorem B.6. (cf. THEOREM 5.31, ch. IV §5.5 of [Kat95].) *Let T be a semi-Fredholm operator from E to F and A bounded. There exists $\delta > 0$ such that, for every $0 < |\lambda| < \delta$ the quantities*

$$\dim\ker(T + \lambda A), \quad \dim\operatorname{coker}(T + \lambda A)$$

are constants.

In order to prove the theorem we need the following lemma.

Lemma B.7. *Let T be an operator with finite-dimensional kernel from E to F and $X \subset E$ a closed subspace. Then $T(X) \subset F$ is closed.*

Proof. We use the fact that an open linear operator maps closed subspaces containing the kernel in closed subspaces. The purpose is to show that there exists $Y \subset E$ closed such that $T(Y) = T(X)$ and $Y \supset \ker T$. Such space can be taken as $Y = \ker T + X$ which is closed because the kernel has finite dimension. □

We are now able to prove the theorem. First we show that the theorem cannot be extended to a neighbourhood of zero. Let P be a projector of finite codimension non surjective, hence it is a Fredholm operator and let $A = I - P$. Let $x \in \ker(P + \lambda A)$ with $\lambda \neq 0$. We can write

$$Px = -\lambda(I - P)x$$

hence both $-\lambda(I - P)x$ and Px are zero. Since $\lambda \neq 0$ we also have $(I - P)x = 0$ thus $x = Px + (I - P)x = 0$. We have proved that $P + \lambda A$ is injective, but P is not injective.
Suppose first that $\ker T$ has finite dimension. Using induction we can build two decreasing sequences of closed subspaces $\{E_n\}$, $\{F_n\}$ of E and F respectively as follows

$$\begin{cases} E_0 = E \\ E_{n+1} = A^{-1}(TE_n) \end{cases} \qquad \begin{cases} F_0 = F \\ F_{n+1} = TE_n \end{cases}$$

these are all closed spaces by the previous lemma. We have $AE_n \subset F_n$ and $TE_n = F_{n+1}$ for any $n \in \mathbb{N}$. Let

$$E_\omega = \bigcap_{n\geq 0} E_n$$
$$F_\omega = \bigcap_{n\geq 0} F_n;$$

If $x \in \ker(T + \lambda I)$ and $\lambda \neq 0$ using induction on the equality $\lambda^{-1}Tx = -Ax$ it is easy to check that $x \in E_\omega$. It is clear that $T(E_\omega) \subset F_\omega$; we prove now that $T(E_\omega) = F_\omega$. Given $y \in E_\omega$

$$T^{-1}(\{y\}) \cap E_\omega = T^{-1}(\{y\}) \cap \big(\bigcap_{n\geq 1} E_n\big) = \bigcap_{n\geq 1} \big(T^{-1}(\{y\}) \cap E_n\big);$$

since $F_{n+1} = T(E_n)$ for $n \geq 1$ the last member is an decreasing intersection of finite-dimensional, since $\ker T$ has finite dimension, of affine subspaces. Hence the intersection is nonempty. Call T_ω the restriction of T to E_ω. We have proved that T_ω is surjective and, of course, is Fredholm. By Proposition B.5 there exists $\delta > 0$ such that the operator $T_\omega + \lambda A_\omega$ is Fredholm, of constant index, and surjective. If $|\lambda| < \delta$ and $\lambda \neq 0$

$$\operatorname{ind}(T_\omega + \lambda A_\omega) = \dim\ker(T + \lambda A).$$

and is still constant as long as $\lambda \neq 0$. If $\operatorname{coker} T$ has finite dimension the same steps can be repeated for T^*.

Appendix C

Spectral decomposition

We recall some basic results on spectral theory and decomposition of the of the spectrum. Let K is a compact subset of an open set $\Omega \subset \mathbb{C}$, Γ a collection of oriented curves $\gamma_1, \ldots, \gamma_n$ in Ω such that $\gamma_i \cap K = \emptyset$.

Definition C.1. *We say that* Γ surrounds K *in* Ω *if*

$$\mathrm{Ind}_\Gamma(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{d\lambda}{\lambda - \zeta} = \begin{cases} 1 & \text{if } \zeta \in K \\ 0 & \text{if } \zeta \notin \Omega \end{cases}$$

Lemma C.2. *Suppose* $\mathcal{A}$ *is a Banach algebra,* $x \in \mathcal{A}$, $\alpha \in \mathbb{C}$, $\alpha \notin \sigma(x)$ *and* Γ *surrounds* $\sigma(x)$ *in* Ω. *Then*

$$\frac{1}{2\pi i} \int_\Gamma (\alpha - \lambda)^n (\lambda - x)^{-1} d\lambda = (\alpha - x)^n.$$

for every $n \in \mathbb{Z}$.

The proof is made by induction on n. The case $n = 0$ is provided by Von Neumann series (see [Rud91], Lemma 10.24).

Theorem C.3. *Let* C^+ *and* C^- *closed subsets of* $\sigma(x)$ *such that* $\sigma(x) = C_- \cup C_+$. *Then we can always find two curves* $\Gamma = \{\gamma^-, \gamma^+\}$ *such that* Γ *surrounds* $\sigma(x)$ *in* $\mathbb{C}$. *Then the integrations*

$$p^+(x) = \frac{1}{2\pi\, i} \int_{\gamma^+} (\lambda - x)^{-1} d\lambda$$
$$p^-(x) = \frac{1}{2\pi\, i} \int_{\gamma^-} (\mu - x)^{-1} d\mu.$$

are projectors of $\mathcal{A}$ *and are called* spectral projectors. *In the Banach algebras* $p^+\mathcal{A}p^+$ *and* $p^-\mathcal{A}p^-$ *the elements* xp^+ *and* xp^- *have spectrum* C^+ *and* C^- *respectively.*

Using the *Fubini-Tonelli* theorem it can be checked that $p^+p^- = p^-p^+ = 0$. Applying the previous lemma with $n = 0$ we also have $p^+(x) + p^-(x) = 1$. Hence ${p^+}^2 = p^+$ and ${p^-}^2 = p^-$.

Appendix D

Continuous sections of linear maps

We recall some classical theorem that regards continuous selection We begin with the result of Bartle and Graves. Let X and Y be Banach spaces and let $L\colon E \to F$ be a linear surjective application. We do not require L to be bounded. Define

$$I(L) = \sup_{|y|=1} \inf_{Lx=y} |x|.$$

It is easy to check that if L is injective also and L^{-1} is bounded $I(L) = \|L^{-1}\|$. Let T be a paracompact Hausdörff space. The conditions of the theorem are the following: for every $t \in T$ we are given a bounded surjective operator $S(t) \in \mathcal{B}(X, Y)$ which is *strongly continuous*. Define

$$M_0(S) = \sup_{t\in T} \|S(t)\|,\ N_0(S) = \sup_{t\in T} I(S(t))$$

the map $s\colon C(T, X) \to C(T, Y),\ x \mapsto sx(t) = S(t)x(t)$ is well defined. Structures of Banach space on $C(T, X)$ and $C(T, Y)$ are not required.

Theorem D.1. *Suppose both M_0 and N_0 are finite. Fix $N > N_0$ and $\varepsilon > 0$. For every $y \in C(T, Y)$ there exists $x \in C(T, X)$ such that $sx = y$ and*

$$|x(t)| \le N|y(t)| + \varepsilon. \tag{D.1}$$

for every $t \in T$.

For the proof see [BK73], Theorem 4. As application of this results consider the situation of two Banach spaces E, F. Let T be a topological space and $y \in C(T, F)$ and $x \in C(T, E)$ such that $x(t) \neq 0$ for every $t \in T$. Let $\hat{x} = x/|x|$

Corollary D.2. *For every $\delta, \varepsilon > 0$ there exists $U^\varepsilon_\delta \in C(T, \mathcal{B}(E, F))$ such that $U^\varepsilon_\delta(t)x(t) = y(t)$ and*

$$\|U^\varepsilon_\delta(t)\| \le (1 + \delta)\frac{y(t)}{|x(t)|} + \varepsilon$$

Proof. We briefly check that the conditions of the theorem are fulfilled. As Banach spaces we choose $X = \mathcal{B}(E, F)$ and $Y = F$. Since $x(t) \neq 0$ for every $t \in T$ we have a map

$$S\colon C(T, \mathcal{B}(E, F)) \to C(T, F),\ U \mapsto U \cdot (x/|x|).$$

Strong continuity is trivial. Let $t \in T$ and $y \in F$. By Hahn-Banach there exists $\xi \in E^*$ such that $\langle \xi, \hat{x}(t) \rangle = 1$, $|\xi| = 1$. Then the operator

$$U \cdot z = \langle \xi, z \rangle y$$

maps $\hat{x}(t)$ in y and $\|U\| = |y|$. On the other side there can be no operator U such that $U\hat{x}(t) = y$ and $\|U\| < |y|$. This proves that $s(t)$ is surjective and $I(s(t)) = 1$. Thus $N_0(S) = 1$ and clearly $M_0(S) = 1$. Fix $\delta, \varepsilon > 0$. Let $y \in C(T, F)$ be a continuous function. Since $1 + \delta > N_0$ there exists $U \in C(T, \mathcal{B}(E, F))$ such that

$$U(t)\hat{x}(t) = y(t)/|x(t)|,\ \|U(t)\| \leq (1 + \delta)\frac{|y(t)|}{|x(t)|} + \varepsilon.$$

Thus $U(t)x(t) = y(t)$ for every $t \in T$. □

Proposition D.3. *Let E, F Banach spaces and $f \in \mathcal{B}(E, F)$ a bounded surjective operator. There exists a continuous map $s \in C(F, E)$ such that $f \circ s = id$.*

Proof. The Theorem D.1 can be applied as follows: since F is metric is a paracompact space. For every $x \in F$ we define

$$L(x)\colon C(F, E) \to C(F, F),\ s \mapsto f \circ s.$$

Since L is constant on F is clearly strongly continuous, in fact is bounded. Then there exists $s \in C(F, E)$ such that $Ls = id$, thus $f \circ s = id$. □

Proposition D.4. *Let $\mathcal{A}$ and $\mathcal{B}$ Banach algebras, $\varphi\colon \mathcal{A} \to \mathcal{B}$ a surjective homomorphism. There are local section of $\varphi\colon \mathcal{A}^* \to \varphi(\mathcal{A}^*)$.*

Proof. First let s be a continuous right inverse of $\varphi\colon \mathcal{A} \to \mathcal{B}$. Such a section exists by Proposition D.3. Let y_0 in $\varphi(A^*)$ and $x_0 \in A^*$ such that $\varphi(x_0) = y_0$. We can define another right inverse of φ such that

$$S(y) = s(y) + x_0 - s(y_0),\ S(y_0) = x_0.$$

Since $\mathcal{A}^*$ is an open subset of $\mathcal{A}$ there exists $\delta > 0$ such that $B(x_0, \delta) \subset \mathcal{A}^*$. Thus $S^{-1}(B(x_0, \delta)) \subset \varphi(A^*)$ and the restriction of S to $S^{-1}(B(x_0, \delta))$ is a local section on a neighbourhood of y_0. □

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Glossary

$\check{A}$	the path defined as $-A(-t)$	**52**
A_τ	the translation path defined as of $A(t+\tau)$	**43**
$\mathcal{B}(E,F)$	space of bounded linear maps from E to F	**7**
$\mathcal{B}_0^*$	connected component of the unit	**26**
$\mathcal{B}_c(E,F)$	compact operators from E to F	**15**
$\mathcal{B}_C^*$		**27**, **30**
$\mathcal{B}_l$	left inverses of $\mathcal{B}$	**42**
$\mathcal{B}_r$	right inverses of $\mathcal{B}$	**42**
$\mathcal{C}(E)$	Calkin algebra	**25**
$1_{\mathbb{R}+}$	characteristic function of $\mathbb{R}^+$	**62**
$1_{\mathbb{R}-}$	characteristic function of $\mathbb{R}^-$	**62**
$C_0(\mathbb{R},E)$	continuous function vanishing at infinity	**61**
$C_0(\mathbb{R}^+,E)$	functions of C_0 defined on the half-line	**61**
$C_0^1(\mathbb{R},E)$	smooth function vanishing at infinity with their derivative	**61**
$C_0^1(\mathbb{R}^+,E)$	functions of C_0^1 defined on the half-line	**61**
$C_h(\mathbb{R}^+,\mathcal{B}(E))$	space of continuous and asymptotically hyperbolic paths	**55**
$D(Y)$	unit disc	**3**
$\mathrm{Dim}(X,Y)$	relative dimension	**16**
$\deg T$	Leray-Schauder degree	**33**
$\delta(Y,Z)$	Grassmannian metric	**3**
$\delta_1(Y,Z)$	geometric opening metric	**5**
$\delta_S(Y,Z)$	sphere opening metric	**4**
$\delta_{\mathscr{H}}(A,B)$	Hausdörff metric	**1**
$\dim(P,Q)$	relative dimension of the images of P and Q	**17**, **36**, **74**
$\dim(X,Y)$	relative dimension in Hilbert spaces	**15**, **34**
$\mathrm{dist}\,(a,B)$	distance point from set	**1**
E/Y	quotient space	**9**
E^*	topological dual space	**8**
$e\mathcal{H}(E)$	space of essentially hyperbolic operators	**25**, **67**
$e\mathcal{H}_-(E)$	essentially hyperbolic with negative essential spectrum	**69**
$e\mathcal{H}_+(E)$	essentially hyperbolic with positive essential spectrum	**69**
ev_0	evaluation at the point $t=0$	**49**