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Extension problems in complex and CR-geometry

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Ph.D. Thesis

Alberto Saracco

Advisor: Prof. Giuseppe Tomassini

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Chapter 1

Introduction

That of extending objects is a recurring problem in mathematics, and is present in many different fields. May it be extending functionals, surfaces, curves, or simply functions, extension problems have proved absolutely non-trivial and interesting and gave birth to some new branches of mathematics. All boundary-values problems can be considered as extension problems: indeed elements of one cathegory can be seen as traces of objects in another cathegory. In complex and CR-geometry, we are interested in extending holomorphic or meromorphic functions, and, in general, all analytic objects, i.e. those defined starting from holomorphic functions. The Hartogs' Theorem (holomorphic functions "fill compact holes" in $\mathbb{C}^n$, $n \geq 2$) is the prototype of all these theorems.

The aim of this thesis work is to present some of the many different extension problems that were considered in complex and CR-geometry, starting from well-known classical ones, up to the very recent still unpublished results and to the still open problems. This is going to be a (partial, since the subject is so broad) survey on the extension problems. There is neither presumption of completeness, nor a judgement on the importance of the arguments chosen, since the choice of the arguments treaten, a very personal one, simply arose from my interests of research and my mathematical experience of the last few years.

The thesis is divided in three parts. In the first one (Chapters 1–3), after recalling basic notions in complex and CR-geometry, we give a survey of the main results about extension problems starting from the pioneer ones proved by Hans Lewy [61, 62]. In the second and third part are included the original results of the thesis obtained in my recent joint works with Giuseppe Tomassini [82, 83] and Giuseppe Della Sala [15, 16]. This last two parts on the surface may appear as being on two different subjects (the cohomology of semi-coronae, Chapters 4–5, and the non-compact boundary problem, Chapters 7–8), are indeed tightly linked together. While having some superficial connections (as being valid starting from dimension 3 and sharing in dimension 2 the same counterexample, see Example 4.1), the most important link is their nature of extension results in a “non-compact setting”.

This is indeed the interpretation key we would like to give to these results.

Contents

Let us examine the matter of the thesis more in details.

Chapters 1 and **2** are devoted to the really basic definitions of the notions of complex and CR-geometry that will be needed in the sequel, and and to the presentation of the classical extension results in one and several complex variables.

In **Chapter 3 — Extension of CR-function up to a Levi-flat boundary and of holomorphic maps** — we collect several local and global extension results for CR-functions. In particular, we discuss in details some extension theorems, proved in the Eighties by Guido Lupacciolu and Giuseppe Tomassini [63, 65], for CR-functions defined on the boundary of a bounded domain in $\mathbb{C}^n$ away from a Levi-flat part (Theorems 3.3 and 3.5), as well as the generalization to Stein manifolds proved by Christine Laurant-Thiébaut (Theorem 3.12, [57]). Finally, we consider the result on the extension of CR-functions defined in unbounded hypersurfaces (Theorem 3.13, [64]), where the Lupacciolu’s ($\star$) condition —referred as condition (3.29) in this chapter— (this condition will show up again in the results of part three) was first introduced. The last section of the chapter is devoted to a survey on the results obtained by Klas Diederich and Sergey Pinchuk [20, 22, 23, 72, 73] in the last 30 years on generalizations of Schwartz reflection principle and the extension of holomorphic maps.

Chapter 4 — Cohomology vanishing and extension problems for semi q-coronae — is mainly based on the joint paper with Giuseppe Tomassini [82], where for the first time were introduced semi q-coronae. *Semi q-coronae* are domains whose boundary contains a Levi-flat part, a q-pseudoconvex part and a q-pseudoconcave part. In this chapter cohomology techniques, and classical results by Andreotti and Grauert [3] are used in order to prove extension results for holomorphic functions, meromorphic functions, divisors (i.e. analytic subsets of codimension one), sections of coherent sheaves.

Chapter 5 — Cohomology of semi 1-coronae and extension of analytic subsets — is mainly based on the joint paper [83]. It is well-known vanishing or more generally finiteness of cohomology of analytic sheaves plays an important role for extension of analytic objects. So our first task is to study cohomology of semi q-coronae. In order to apply Andreotti-Grauert's method (bump lemma and approximation) to semi q-coronae, we have to manage boundary points which are at the same time "pseudoconvex and pseudoconcave". The bump lemma with open bumps is not true in general and so the idea is to use a bump lemma with closed bumps, using in a crucial way a local vanishing theorem proved by Christine Laurent-Thiébaut and Jürgen Leiterer. This allows us to prove an isomorphism theorem for cohomology groups (Theorem 5.5) and consequently a finiteness theorem for closed semi q-coronae. This leads to a somehow weaker result than that obtained in the situation of a full corona (see Remark 5.2). Anyhow the result obtained is enough to prove an extension theorem for analytic subsets and sheaves of an open semi 1-corona, whose depth is at least three (Theorem 5.16).

We conclude the chapter outlining some possible generalizations of the above results, to semi q-coronae in Stein spaces.

The third part of the thesis, Chapters 6–8, as already said, is devoted to the boundary problem in complex analysis.

In **Chapter 6 — The boundary problem —** we present the second main problem. Basically, the boundary problem is the following: given a real odd-dimensional submanifold in a complex manifold, which conditions are necessary and sufficient for it being the boundary of a "complex variety"? The problem has been intensively studied by several people essentially when the boundary is compact. The maximal complexity (for manifolds of positive CR-dimension) or moments condition (for curves) have been pointed out as necessary and sufficient conditions. In this chapter we state the main results obtained in the last fifty years, starting from John Wermer's result on compact curves [101] which dates 1958. Some sketches of the proofs or ideas lying underneath are presented. We also state: Stolzenberg's result on union of curves in $\mathbb{C}^n$ [91], Harvey-Lawson's theorem on compact manifolds of any dimension in $\mathbb{C}^n$ [38]; moreover the results in q-concave sets and the recent approaches to the problem in $\mathbb{CP}^n$ are surveyed.

Chapter 7 — Non-compact boundaries of complex analytic varieties — is mainly based on the joint research with Giuseppe Della Sala [15]. We approach the non-compact version of the boundary problem. What we get is a non-compact version of Harvey-Lawson extension theorem in strongly convex and strictly pseudoconvex domains satisfying Lupacciolu's $(\star)$ condition. The method used for the proof is basically the following: we first extend locally the odd dimensional maximally complex manifold by a complex strip, then we slice it into compact manifolds, to apply Harvey-Lawson's result to each slice. In order to get the solution we need to show that the obtained varieties are organized as a complex variety. This is a crucial step of the proof and is based on an integral representation of the Harvey-Lawson's solution (see Lemma 7.7). It has to be pointed out that this result is not valid for curves. Chapter 7 concludes with a section with some considerations on Lupacciolu's ($\star$) condition.

Finally, the thesis is concluded by **Chapter 8 — Semi-local extension of maximally complex submanifolds** — based on [16]. Here we deal with the boundary problem in the following semi-local setting: given an open subset A of the boundary of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, which is a natural domain $D \subset \Omega$ where all real odd-dimensional maximally complex submanifolds M of A can be extended? Using the methods of the previous chapter we give a (partial) answer to the problem (see Theorems 8.1, 8.10), which relates the domain D in which the extension exists with the hull of bA. The problem whether the above D is the maximal with this property, is still unsolved. The methods employed in this situation are essentially the same as those of the previous chapter, but with some sligth improvement. In the particular case $A = b\Omega$ we improve the result of [15].

The thesis is also endowed with an ample **Bibliography** which gives some references for further reading on the broad subject of extension of analytic objects.

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1.1 Definitions

In this section we recall the elementary definitions and fix some notations.

1.1.1 Holomorphic functions and tangent spaces

We will consider $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$, $z_k = x_k + iy_k$, unless otherwise stated. Let $D \subset \mathbb{C}^n$ be a domain, i.e. an open connected set. For a differentiable function $f : D \to \mathbb{C}$ we define

$$\begin{aligned} \partial f &= \sum_1^n \frac{\partial f}{\partial z_k} dz_k \\ \overline{\partial} f &= \sum_1^n \frac{\partial f}{\partial \overline{z}_k} d\overline{z}_k \\ df &= \partial f + \overline{\partial} f. \end{aligned}$$

Holomorphic functions on D are the differentiable functions $f : D \to \mathbb{C}$ which satisfy $\overline{\partial} f = 0$. Note that if $n = 1$ this reduces to the classical Cauchy-Riemann condition. As in the one variable case, a holomorphic function is of class $\mathcal{C}^\infty$. A function f is holomorphic if and only if it is separately holomorphic in each variable (Riemann).

We will denote the algebra of holomorphic functions on $D \subset \mathbb{C}^n$ by $\mathcal{O}(D)$, and the algebra of complex functions which are holomorphic on an open neighborhood of D by $\mathcal{O}(\overline{D})$.

Let $N \subset \mathbb{C}^n$ be a smooth connected real submanifold, and let $p \in N$. We denote by $T_p(N)$ the real tangent space of N at the point p, and by

$$H_p(N) = T_p(N) \cap iT_p(N)$$

the holomorphic tangent space of N at the point p.

A k-dimensional real submanifold $N \subset \mathbb{C}^n$ is said to be *totally real* if $H_p(N) = \{0\}$ at every point $p \in N$. In particular $k \leq n$.

1.1.2 Analytic subsets

Let $D \subset \mathbb{C}^n$ be a domain and $A \subset D$ a closed subset locally defined as zero-set of holomorphic functions $A = \{f_1 = \cdots f_k = 0\}$, $f_1, \ldots, f_k \in \mathcal{O}(D)$. A is said to be an *analytic subset*. Assume now that the $\mathbb{C}$-linear forms $\partial f_1, \ldots, \partial f_k$, are linearly independent at every point $z \in A$. Then A is a *complex submanifold* of D of *complex dimension* $n - k$, i.e. for every $z \in A$ there exists a neighbourhood U of z and a biholomorphism $f : U \to f(U)$ such that $f(U \cap A)$ is an open sets of a $n - k$ coordinate space.

Remark 1.1 In general, an analytic subset cannot be globally given as the zero locus of holomorphic functions.

Let $m \in \mathbb{N}$. An *m-branched analytic covering* of an analytic space A is a complex space $\widetilde{A}$ with a surjective projection $\pi : \widetilde{A} \to A$ with fiber of cardinality m outside of a discrete subset $E \subset A$ and such that for each point $z \in A \setminus E$ there is a neighborhood $U \ni z$ such that $\pi^{-1}(U)$ is the disjoint union of m copies of U. $\pi^{-1}(E)$ is called the *branching locus* of $\widetilde{A}$.

1.1.3 Hulls of compact sets

Let $D \subset \mathbb{C}^n$ be a domain, $K \subset D$ a compact subset, and $\mathcal{A}$ an algebra of functions on D. We can define the $\mathcal{A}$-hull of K as

$$\widehat{K}_{\mathcal{A}} = \bigcap_{\varphi \in \mathcal{A}} \left\{ z \in D \ : \ |\varphi(z)| \le \max_K |\varphi| \right\}.$$

Obviously $K \subset \widehat{K}_{\mathcal{A}}$. The subset K is said to be $\mathcal{A}$-convex if K coincides with its $\mathcal{A}$-hull, $\widehat{K}_{\mathcal{A}}$.

Remark 1.2 If $f, g \in \mathcal{A}$ coincide on K, then they must also coincide on $\widehat{K}_{\mathcal{A}}$. Indeed consider $f - g \in \mathcal{A}$. $f - g \equiv 0$ on K, hence on $\widehat{K}_{\mathcal{A}}$.

These definitions have a strong relation with the maximum principle, hence they are interesting definitions when the algebra $\mathcal{A}$ satisfies it, like in the case of holomorphic functions.

If $\mathcal{A} = \mathcal{O}(D)$, we will denote by $\widehat{K}_D$ the $\mathcal{O}(D)$-hull of K, and if $\mathcal{A} = \mathcal{O}(\overline{D})$, by $\widehat{K}_{\overline{D}}$ the $\mathcal{O}(\overline{D})$-hull of K.

1.1.4 Harmonic, pluriharmonic and plurisubharmonic functions

A $\mathcal{C}^2$-smooth function $\varphi : \mathbb{C}^n \to \mathbb{R}$ is said to be *harmonic* in $U \subset \mathbb{C}^n$ if

$$\Delta\phi(z) = \sum_1^n \left(\frac{\partial^2 \phi}{\partial x_k^2}(z) + \frac{\partial^2 \phi}{\partial y_k^2}(z) \right) = 0,$$

for all $z \in U$.

A $\mathcal{C}^2$-smooth function $\varphi : \mathbb{C}^n \to \mathbb{R}$ is said to be *pluriharmonic* in $U \subset \mathbb{C}^n$ if its restriction to every complex line L of $\mathbb{C}^n$ is harmonic in $L \cap U$. A pluriharmonic function is in particular harmonic.

A upper-semicontinuous function $\varphi : \mathbb{C}^n \to \mathbb{R}$ is said to be *plurisubharmonic* (or *psh*) in $U \subset \mathbb{C}^n$ if, for every $V \Subset U$ with smooth boundary bV and for every function ψ pluriharmonic in a neighborhood of $\overline{V}$, $\psi \geq \varphi$ on bV implies $\psi \geq \varphi$ on V. If φ is $\mathcal{C}^2$-smooth this definition can be given considering the eigenvalues of the Levi-form, thus enabling also a notion of strict plurisubharmonicity. We will give these definitions in Subsection 1.1.6.

1.1.5 CR-geometry

A $(2k+1)$-dimensional real submanifold $N \subset \mathbb{C}^n$, $k \geq 1$, is said to be a *CR-submanifold* if $\dim_{\mathbb{C}} H_p(N)$ is constantIf N is a CR-manifold,

$$H(N) = \bigcup_{p \in N} H_p(N)$$

is a subbundle of the tangent bundle $T(N)$. A totally real manifold is a trivial example of a CR-manifold. On the opposite side there is the case when $\dim_{\mathbb{C}} H_p(N)$ is the greatest possible, i.e. $\dim_{\mathbb{C}} H_p(N) = k$ for every p; in this case N is said to be *maximally complex*. If $k = 0$ the definition is an empty condition: all one-dimensional real manifolds are totally real, hence CR.

A $\mathcal{C}^\infty$ function $f : N \to \mathbb{C}$ is said to be a *CR-function* if for a (and hence for any) $\mathcal{C}^\infty$ extension $\widetilde{f} : U \to \mathbb{C}$ (U being a neighborhood of N) we have

$$\left(\overline{\partial}\widetilde{f}\right)\Big|_{H(N)} = 0. \tag{1.1}$$

In particular the restriction of a holomorphic function to a CR-submanifold is a CR-function. It is immediately seen that f is CR if and only if

$$df \wedge (dz_{j_1} \wedge \ldots \wedge dz_{j_k})|_N = 0, \tag{1.2}$$

for any $(j_1, \ldots, j_k) \in \{1, \ldots, n\}^k$. Similarly N is maximally complex if and only if

$$(dz_{j_1} \wedge \ldots \wedge dz_{j_{k+1}})|_N = 0$$

for any $(j_1, \ldots, j_{k+1}) \in \{1, \ldots, n\}^{k+1}$.

We can also define a notion of *w*eak CR-function, for function which are only $\mathcal{C}^0$-smooth. Indeed, let N be a CR-manifold of CR-dimension l. A function $f \in \mathcal{C}^0(N)$ is said to be a weak CR-function if for all smooth $(l, l-1)$-forms φ with compact support, such that $\overline{\partial}\varphi = 0$,

$$\int_N f\varphi = 0.$$

Observe that the boundary M of a complex submanifold W of dimension at least 2 is maximally complex. Indeed, for any $p \in bW = M$, $T_p(bW)$ is a real hyperplane of $T_p(W) = H_p(W)$ and so is $iT_p(bW)$. Hence the complex tangent $H_p(bW) = T_p(bW) \cap iT_p(bW)$ is of real codimension 2 in $H_p(W)$.

By the same argument, a $(2n-1)$-real submanifold of $\mathbb{C}^n$ is maximally complex.

If $\dim_{\mathbb{C}} W = 1$ and bW is compact then for any holomorphic $(1,0)$-form ω we have, due to Stokes theorem (see Theorem 2.2)

$$\int_M \omega \;=\; \iint_W d\omega \;=\; \iint_W \partial\omega \;=\; 0$$

since $\partial\omega|_W \equiv 0$. This condition for M is called *moments condition* (cfr. [38]).

1.1.6 The Levi-form and Levi-convexity

For this section we change the notations of the real variables setting

$$z_1 = x_1 + ix_{n+1}, \ldots, z_n = x_n + ix_{2n}.$$

Let $D \in \mathbb{C}^n$ be a domain and $\phi \in \mathcal{C}^\infty(D)$. For every point $z^0 \in D$ we denote by $\mathcal{H}(\phi; z^0)$ the real Hessian form of ϕ at z^0:

$$\mathcal{H}(\phi; z^0)(\xi) = \sum_{j,k=1}^{2n} \frac{\partial^2 \phi}{\partial x_j \partial x_k}(z^0)\xi_j\xi_k, \;\; \forall \xi = (\xi_1, \ldots, \xi_{2n}) \in \mathbb{R}^{2n}. \tag{1.3}$$

Denote by $Q(\phi; z^0)$ the $\mathbb{C}$-bilinear form of ϕ at z^0

$$Q(\phi; z^0)(\zeta) \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \phi}{\partial z_\alpha \partial z_\beta}(z^0)\zeta_j\zeta_k, \;\; \forall \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n. \tag{1.4}$$

and by $L(\phi; z^0)$ the hermitian form of ϕ at z^0

$$L(\phi; z^0)(\zeta) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \phi}{\partial z_\alpha \partial \overline{z}_\beta}(z^0)\zeta_j\overline{\zeta}_k, \;\; \forall \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n, \tag{1.5}$$

where

$$\zeta_\alpha = \xi_\alpha + i\xi_{n+\alpha},\; 1 \le \alpha \le n.$$

$L(\phi; z^0)$ given by (1.5) is called the *complex Hessian* or the *Levi-form* of ϕ at z^0.

Remark 1.3 If $U \subset \mathbb{C}^n$ is an open set, $f : U \to \mathbb{C}^n$ a holomorphic map, $f = (f_1, \dots, f_n)$ and $\phi : \mathbb{C}^n \to \mathbb{R}$ a differentiable function. Then

$$L(\phi \circ f; z^0)(\zeta_1, \dots, \zeta_n) = \tag{1.6}$$

$$= L(\phi; f(z^0)) \left(\sum_{\alpha=1}^{n} \frac{\partial f_1}{\partial z_\alpha}(z^0)\zeta_\alpha, \dots, \sum_{\alpha=1}^{n} \frac{\partial f_n}{\partial z_\alpha}(z^0)\zeta_\alpha \right).$$

The following properties hold (for a proof see [17])

(i) $4L(\phi; z^0)(\zeta) = \mathcal{H}(\phi; z^0)(\mathsf{Re}\,\zeta, \mathsf{Im}\,\zeta) + \mathcal{H}(\phi; z^0)(\mathsf{Re}\,i\zeta, \mathsf{Im}\,i\zeta)$;

(ii) $\mathsf{Re}\,Q(\phi; z^0)(\zeta) = \mathcal{H}(\phi; z^0)(\mathsf{Re}\,\zeta, \mathsf{Im}\,\zeta) - \mathcal{H}(\phi; z^0)(\mathsf{Re}\,i\zeta, \mathsf{Im}\,i\zeta)$;

(iii) $\mathcal{H}(\phi; z^0)(\xi) = 2L(\phi; z^0)(\zeta) + 2\mathsf{Re}\,Q(\phi; z^0)(\zeta)$;

(iv) the real symmetric form $\mathsf{Re}\,Q(\phi; z^0)(\zeta)$ has the same number of positive and negative eigenvalues;

(v) if $p(L)$ is the number of positive eigenvalues of $L(\phi; z^0)$ and $p(\mathcal{H})$ is the number of positive eigenvalues of $\mathcal{H}(\phi; z^0)$, then $p(L) \le p(\mathcal{H})$;

(vi) if $n(\mathcal{H})$ is the number of negative eigenvalues of $\mathcal{H}(\phi; z^0)$, then $p(L) + n(\mathcal{H}) \le 2n$.

A domain D in $\mathbb{C}^n$ is said to have a *differentiable boundary* if for every $z^0 \in \mathrm{b}D$ there exist both a neighborhood U of z^0 and a smooth function $\rho \in \mathcal{C}^\infty(U)$ such that

(a) $U \cap D = \{z \in U : \ \rho(z) < 0\}$;

(b) $d\rho(z) \neq 0$, for every $z \in U \cap \mathrm{b}D$.

In this situation we say that ρ *defines* $\mathrm{b}D$ (locally) at z^0.

Remark 1.4 If ρ_1, ρ_2 define $\mathrm{b}D$ at z^0, then there exists a positive smooth function h in a neighborhood of z^0 such that $\rho_1 = h\rho_2$.

Suppose that ρ defines $\mathrm{b}D$ at z^0. Let $T_{z^0}(\mathrm{b}D)$ be the real hyperplane tangent to $\mathrm{b}D$ at z^0 and

$$\sum_{j=1}^{2n} \frac{\partial \rho}{\partial x_j}(z^0)(x_j - x_j^0) = 0. \tag{1.7}$$

the equation of $T_{z^0}(\mathrm{b}D)$. Since

$$\mathsf{Re} \sum_{\alpha=1}^{n} \frac{\partial \rho}{\partial z_\alpha}(z^0)(z_\alpha - z_\alpha^0) = \sum_{j=1}^{2n} \frac{\partial \rho}{\partial x_j}(z^0)(x_j - x_j^0),$$

$T_{z^0}(\mathrm{b}D)$ contains the complex hyperplane $H_{z^0}(\mathrm{b}D)$ through z^0 defined by

$$\sum_{\alpha=1}^{n} \frac{\partial \rho}{\partial z_\alpha}(z^0)(z_\alpha - z_\alpha^0) = 0. \tag{1.8}$$

$H_{z^0}(\mathrm{b}D)$ is indeed the previously defined holomorphic tangent space of $\mathrm{b}D$ at z^0 (see Subsection 1.1.1). Let us consider the Hessian form of ρ in z^0, $\mathcal{H}(\rho; z^0)$. If h is smooth, then

$$\mathcal{H}(h\rho; z^0)(\xi) = h(z^0)\mathcal{H}(\rho; z^0)(\xi) + \tag{1.9}$$
$$+2\left(\sum_{j=1}^{2n} \frac{\partial h}{\partial x_j}(z^0)\xi_j\right)\left(\sum_{j=1}^{2n} \frac{\partial \rho}{\partial x_j}(z^0)\xi_j\right).$$

Therefore, if $h > 0$ and $\xi \in T_{z^0}(\mathrm{b}D)$, the forms

$$\mathcal{H}(\rho; z^0)|_{T_{z^0}(\mathrm{b}D)},\ \mathcal{H}(h\rho; z^0)|_{T_{z^0}(\mathrm{b}D)}$$

have the same signature. In particular, the condition

$$\mathcal{H}(\rho; z^0)|_{T_{z^0}(\mathrm{b}D)} \geq 0 \tag{1.10}$$

does not depend on the function ρ which defines $\mathrm{b}D$ at z^0. As it is well known, the inequality (1.10) at every point $z^0 \in \mathrm{b}D$ characterizes the domains with differentiable boundary which are convex.

For the Levi-convexity, the form $L(\rho; z^0)$ plays the role of $\mathcal{H}(\rho; z^0)$. In this setting equation (1.9) becomes

$$L(h\rho; z^0)(\zeta) = h(z^0)L(\rho; z^0)(\zeta) + \tag{1.11}$$
$$+2Re\left(\sum_{\alpha=1}^{n} \frac{\partial h}{\partial z_\alpha}(z^0)\zeta_\alpha\right)\left(\sum_{\alpha=1}^{n} \frac{\partial \rho}{\partial \overline{z}_\alpha}(z^0)\overline{\zeta}_\alpha\right)$$

and, if $h > 0$ and $\zeta \in H_{z^0}(\mathrm{b}D)$, we have

$$L(h\rho; z^0)(\zeta) = h(z^0)L(\rho; z^0)(\zeta),$$

i.e.

$$L(h\rho; z^0)|_{H_{z^0}(\mathrm{b}D)}$$

and

$$L(\rho; z^0)|_{H_{z^0}(\mathrm{b}D)}$$

have the same signature.

Let $D \subset \mathbb{C}^n$ be a domain with differentiable boundary. bD is called *Levi-convex* (or *pseudoconvex*), respectively *strongly Levi-convex* (or *strongly pseudoconvex*), if

$$L(\rho; z^0)|_{H_{z^0}(\mathrm{b}D)} \geq 0,$$

respectively

$$L(\rho; z^0)|_{H_{z^0}(\mathrm{b}D)} > 0,$$

for all $z^0 \in \mathrm{b}D$, ρ being a defining function for bD at z^0.

A strongly pseudoconvex domain is locally biholomorphic to a strongly convex domain. Indeed,

Lemma 1.1 (Narasimhan) *Let $\Omega \subset \mathbb{C}^n$ be a domain with $\mathcal{C}^2$-smooth boundary, adn $p \in b\Omega$ a point of strong pseudoconvexity. Then there is a neighborhood $U \ni p$ and a biholomorphic map $\Phi : U \to U'$ such that $\Phi(U \cap \Omega)$ is strongly convex.*

Proof. Since Ω is strongly pseudoconvex at p, there exist a local defining function ρ for Ω at p and a positive constant $C > 0$ satisfying

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k \geq C|w|^2, \quad \forall w \in \mathbb{C}^n. \tag{1.12}$$

Up to a rotation and a translation of coordinates, we may suppose $p = 0$ and that $\nu = (1, 0, \ldots, 0)$ is the normal outward-pointing vector at 0. The second-order Taylor expansion of ρ near 0 is

$$\begin{aligned}
\rho(w) &= \rho(0) + \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(0) w_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(0) w_j w_k + \sum_{j=1}^{n} \frac{\partial \rho}{\partial \overline{z}_j} + \\
&\quad + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial \overline{z}_j \partial \overline{z}_k}(0) \overline{w}_j \overline{w}_k + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k + o(|w|^2) = \\
&= 2\mathsf{Re} \left\{ \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(0) w_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(0) w_j w_k \right\} + \\
&\quad + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k + o(|w|^2) = \\
&= 2\mathsf{Re} \left\{ w_1 + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(0) w_j w_k \right\} + \\
&\quad + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k + o(|w|^2).
\end{aligned} \tag{1.13}$$

Define the mapping $w \mapsto w'$ by

$$\begin{aligned} w_1' &= \Phi_1(w) &= w_1 + \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(0) w_j w_k \\ w_j' &= \Phi_j(w) &= w_j \quad \forall j > 1. \end{aligned}$$

By the implicit function theorem, if w is in a sufficiently small neighborhood of 0, this is a well defined holomorphic change of coordinates. Equation (1.13) tells us that, in the coordinates w', the defining function becomes

$$\rho(w') = 2\mathsf{Re}\, w_1' + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} w_j' \overline{w}_k' + o(|w'|^2).$$

Thus the Hessian at 0 is the Levi-form at 0 and so it is positive. By continuity it is positive in a small neighbourhood U. In there $U \cap b\Omega$ is strongly convex as claimed. $\square$

Remark 1.5 Observe that if $\Omega_1, \ldots, \Omega_l \Subset \mathbb{C}^n$ $(l \leq n)$ are l domains strongly pseudoconvex at the point p where they intersect $\mathbb{C}$-transversally (i.e. their normal outward pointing vectors at p generate an l-dimensional affine complex space), then the domains can be locally convexified simultaneously, by a slight modification of the proof above.

The definitions of Levi-convexity can be given for differentiable hypersurfaces in $\mathbb{C}^n$. Let $S \subset \mathbb{C}^n$ be a differentiable hypersurface. Then S is called *Levi-convex* (or *pseudoconvex*), respectively *strongly Levi-convex* (or *strongly pseudoconvex*), if for every $z^0 \in S$ the hermitian form

$$L(\rho; z^0)|_{H_{z^0}(\mathrm{b}D)}$$

is positive (negative) definite, respectively strictly positive (negative) definite, $\rho = 0$ being a local equation for S at Z^0.

We remark that the notion of Levi-convexity is invariant under biholomorphisms (see Remark 1.3).

The hypersurfaces $S \subset \mathbb{C}^n$ which satisfy

$$L(\rho; z^0)|_{H_{z^0}(\mathrm{b}D)} = 0,$$

for every $z^0 \in S$ are called *Levi-flat*.

Levi-flat hypersurfaces have an interesting geometry. We just point out the following:

Remark 1.6 The zero set of a pluriharmonic function is a real analytic Levi-flat hypersurface. The converse holds locally.

Let D be a domain in $\mathbb{C}^n$. A $\mathcal{C}^2$ function $\varphi : D \to \mathbb{R}$ is called *strongly q-plurisubharmonic*, if the Levi-form $L(\varphi; z)$ has at least $n - q + 1$ positive eigenvalues for every $z \in D$; strongly 1-plurisubharmonic functions are called *strongly plurisubharmonic*. If $L(\varphi; z) \geq 0$ for every $z \in D$, the function φ is said to be *plurisubharmonic*. These notions coincide with those defined before and can therefore be extended to upper semicontinuous functions.

1.2 Representation formulas

Many extension results can be obtained via representation (or integral) formulas, which can be used to find the values of a holomorphic function of which only boundary values are know. Moreover representation formulas can be seen as criterions for holomorphicity.

There are a lot of different integral formulas, and it is a continuous struggle for finding better ones. In domains of general type no nice formulas are known, while on domains with a lot of simmetries the theory is pretty well developed. In here we are not interested in this theory, however. We only furnish some of the most useful or "representative" integral formulas, without proof. Detailed proofs can be found in [13, 17, 55], as well as in other basic text of one or several complex variables.

1.2.1 The Cauchy kernel

Let D be a domain in $\mathbb{C}$, and $f : D \to \mathbb{C}$ be a holomorphic function. Let $z \in D$ and γ be a Jordan curve in D containing z in its inside and homotope to a point in D. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \; = \; f(z). \tag{1.14}$$

The function

$$K_C(z, \zeta) \; = \; \frac{1}{2\pi i(\zeta - z)}$$

is called the *Cauchy kernel* relative to the point z.

The proof of the Cauchy formula (1.14) is a straightforward application of the classical Stokes' theorem (see Theorem 2.2).

This result clearly points out that in order to know a holomorphic function, the only knowledge of boundary values is sufficient, due to the maximum principle.

1.2.2 The Bergman kernel

Let $D \subset \mathbb{C}^n$ be a domain. We denote by $L^2(D)$ the Hilbert space of the square integrable (with respect to the Lebesgue measure $d\ell$) functions on D, and by $H^2(D)$ the subspace of those functions which are holomorphic. $H^2(D)$ is a closed subspace of $L^2(D)$, in particular a Hilbert space.

Given a point $z \in D$ consider on $H^2(D)$ the functional e_z, the *evaluation at* z: $e_z(f) = f(z)$. The functional e_z is continuous, hence it is a scalar product by the Riesz Theorem; i.e. there exists an element $K_D(z, \cdot) \in H^2(D)$ such that for every $f \in H^2(D)$

$$f(z) = e_z(f) = \int_D f\overline{K}_D(z,\cdot)d\ell \tag{1.15}$$

and

$$||K_D(z,\cdot)||_{L^2} = ||e_z|| \leq \frac{1}{(\sqrt{\pi})^n d(z)^n}, \tag{1.16}$$

where $d(z) = \inf_{\zeta \in \mathbb{C}^n \setminus D} \| z - \zeta \|$.

The function $K_D = K_D(z,\zeta)$ is called the *Bergman kernel* of D, and (1.15) is the *Bergman representation formula* of f.

We observe that in the unit ball $\mathbb{B}^n$, the Bergman kernel has the following explicit form (for more about the Bergman kernel in the ball, see [80]):

$$K_{\mathbb{B}^n}(z,w) = \frac{1}{\ell(\mathbb{B}^n)(1 - \langle z, w \rangle)^{n+1}}.$$

The Bergman kernel K_D has the following properties

(i) $K_D(z,\zeta) = \overline{K_D(\zeta,z)}$;

(ii) for every $z \in D$

$$\sqrt{K_D(z,z)} = \|K_D(z,\cdot)\|_{L^2} = \max_{f \in LH^2(D)} \frac{|f(z)|}{\|f\|_{L^2}};$$

(iii) K_D is uniquely determined by the function $K_D(z,z)$;

(iv) $|K_D(z,\zeta)| \leq \sqrt{K_D(z,z)}\sqrt{K_D(\zeta,\zeta)}$;

(v) $2|K_D(z,\zeta)| \leq K_D(z,z) + K_D(\zeta,\zeta)$;

(vi) If D, D' are domains in $\mathbb{C}^n$ and $\Phi : D \to D'$ is a biholomorphism, $\Phi = (\Phi_1, \ldots, \Phi_n)$, then

$$K_{D'}(\Phi(z), \Phi(\zeta))\, \overline{J_z(\Phi)} J_\zeta(\Phi) = K_D(z,\zeta),$$

where

$$J_w(\Phi) = \det \frac{\partial(\Phi_1, \ldots, \Phi_n)}{\partial(w_1, \ldots, w_n)}(w).$$

(vii) Let D be domain in $\mathbb{C}^n$ and D' a subdomain of D. Then for every $z \in D'$

$$K_D(z,z) \leq K_{D'}(z,z). \tag{1.17}$$

(viii) Let $D_1 \subset \mathbb{C}^{n_1}$, $D_2 \subset \mathbb{C}^{n_2}$ be domains. Then

$$K_{D_1 \times D_2}(z,w,\zeta,v) = K_{D_1}(z,\zeta) K_{D_2}(w,v) \tag{1.18}$$

1.2.3 The Bochner-Martinelli kernel

The following theorem is due to, indipendently, Martinelli [66], May [67] and Bochner [10].

Theorem 1.2 *Let D be a bounded domain in $\mathbb{C}^n$ with a connected boundary of class C^1, whose orientation is determined by inward pointing normal vectors. Let f be holomorphic in D and continuous on* bD*. Then, for every $z \in D$*

$$f(z) = \frac{(n-1)!}{(2\pi i)^n} \int_{\mathrm{b}D} f(\zeta) \omega(z,\zeta), \tag{1.19}$$

where

$$\omega(z,\zeta) = \sum_{\alpha=1}^{n} (-1)^\alpha \frac{\overline{\zeta}_\alpha - z_\alpha}{|z-\zeta|^n} \, d\zeta_1 \wedge \ldots \wedge \, d\zeta_n \wedge \, d\overline{\zeta}_1 \wedge \ldots \hat{\alpha} \ldots \wedge \, d\overline{\zeta}_n. \tag{1.20}$$

The $(n, n-1)$-form

$$\omega_{BM}(z,\zeta) = \frac{(n-1)!}{(2\pi i)^n} \, \omega(z,\zeta) \tag{1.21}$$

is called the *Bochner-Martinelli kernel* relative to the point z. Unlike the Cauchy kernel in one variable, the Bochner-Martinelli kernel is not holomorphic.

1.2.4 The Henkin kernel

On a strongly pseudoconvex domain $D \subset \mathbb{C}^n$ (with $\mathcal{C}^4$-smooth boundary bD), however, a holomorphic reproducing kernel for holomorphic function, known as the Henkin kernel, does exist. The down-side of the medal is that the Henkin kernel, unlike the Bochner-Martinelli one, is not explicit at all.

The proof of this representation formula is due —independently— to Henkin [50, 51] and Ramirez [74].

Let $\varepsilon > 0$ and denote by $D_\varepsilon \subset \mathbb{C}^n$ the set of point in D whose distance from the boundary bD is greater than ε. Solving a $\overline{\partial}$-problem in D Henkin found a $\mathcal{C}^1$ singular function $\Phi : D_\varepsilon \times bD \to \mathbb{C}$, called the *Henkin singular function*, which can be written as

$$\Phi(z, \zeta) = \sum_{j=1}^{n} (\zeta_j - z_j) P_j(z, \zeta),$$

$z \in D_\varepsilon$, $\zeta \in bD$, with P_j holomorphic in the variable $z \in D_\varepsilon$ and $\mathcal{C}^1$-smooth in the variable $\zeta \in bD$.

For each j, define

$$w_j(z, \zeta) = \frac{P_j(z, \zeta)}{\Phi(z, \zeta)},$$

$$\eta(z, \zeta) = \sum_{j=1}^{n} (-1)^{j+1} w_j \, dw_1 \wedge \ldots \wedge dw_{j-1} \wedge dw_{j+1} \wedge \ldots \wedge dw_n,$$

and

$$\omega(\zeta) = d\zeta_1 \wedge \ldots \wedge d\zeta_n.$$

Theorem 1.3 (Henkin [50]**)** *For any $f \in \mathcal{A}^1(D)$ and every $z \in D_\varepsilon$,*

$$f(z) = \int_{bD} f(\zeta) \eta(z, \zeta) \wedge \omega(\zeta).$$

For a proof refer to [50, 51, 55].

Chapter 2

Classical extension theorems in one and several complex variables

The great rigidity of holomorphic functions (compared to continuous or smooth functions, for whom a partition of unit exists) together with their richness (compared with the "few" polynomial functions) give a big interest to all extension results, which are incredibly wide-spread and of different flavors. Indeed if an extension of a holomorphic function exists on a bigger connected domain, it is unique thanks to analytic continuation.

In this chapter we first give some basic results of Complex Analysis, then explore some classical extension theorem in one and several variables.

2.1 Basic theorems

While this thesis is devoted to extension results, we feel some really basic theorems, which will be used in the following, cannot be left unstaded. We recall them here, referring the reader to [13, 18, 79] for proofs.

Theorem 2.1 (Morera) *Let $D \subset \mathbb{C}$ be a domain. A continuous function $f : D \to \mathbb{C}$ is holomorphic if and only if, for each piecewise $\mathcal{C}^1$-smooth compact closed curve $\gamma \subset D$, homotope to a point in D ,*

$$\int_\gamma f(z)\, dz \;=\; 0.$$

Theorem 2.2 (Stokes) *Let $D \subset \mathbb{R}^k$ be a domain with smooth boundary bD. Let ω be a smooth k-form in an open neighborhood of $\overline{D}$. Then*

$$\int_{bD} \omega = \iint_D d\omega.$$

Observe that if $D \subset \mathbb{C}^n$ and ω is a holomorphic form, then the previous formula becomes (since $d = \partial + \overline{\partial}$ and $\overline{\partial}\omega = 0$)

$$\int_{bD} \omega = \iint_D \partial\omega.$$

This in particular implies the following generalization to several complex variables of Morera's theorem (for a proof of this result in even greater generality, see [11]).

Theorem 2.3 *Let $D \subset \mathbb{C}^n$ be a domain. A continuous function $f : D \to \mathbb{C}$ is holomorphic if and only if, for each n-cycle Γ with smooth boundary,*

$$\int_\Gamma f(z_1, \ldots, z_n)\, dz_1 \wedge \ldots \wedge dz_n = 0.$$

2.2 Extension theorems in one complex variable

While the realm of extension theorems for holomorphic functions is in several complex variables, some nice, classical extension theorems hold also in the one variable case.

2.2.1 Removable singularities for bounded holomorphic functions

Theorem 2.4 (Riemann) *Let $D \subset \mathbb{C}$ be a domain, $a \in D$. If $f : D \smallsetminus \{a\} \to \mathbb{C}$ is a holomorphic function bounded near a, then a is a removable singularity for f, i.e. there exists a (unique) holomorphic function $h : D \to \mathbb{C}$ that coincides with f on $D \smallsetminus \{a\}$.*

Proof. By hypothesis, on a corona $C = \{0 < |z - a| < \delta\}$ $|f| \le M$, for some δ and M. Thus the function

$$F(z) = (z - a)^2 f(z)$$

is defined and holomorphic in C and $\lim_{z\to a} F(z) = 0$, hence $F \in \mathcal{C}^0(\Delta_{a,\delta})$, $\Delta_{a,\delta}$ being the disc of radius δ centered in a. Moreover, for $z \neq a$

$$F'(z) = 2(z-a)f(z) + (z-a)^2 f'(z).$$

Hence Cauchy-Riemann estimate on C

$$|f'(z)| \leq \frac{M}{|z-a|}$$

implies that $\lim_{z\to a} F'(z) = 0$, and so $F \in \mathcal{C}^1(\Delta_{a,\delta})$. It follows that F is holomorphic on $\Delta_{a,\delta}$ and vanishes in a. Hence it can be written as

$$F(z) = (z-a)^k g(z)$$

with $k \in \mathbb{Z}^+$, $g \in \mathcal{O}(\Delta_{a,\delta})$ and $g(a) \neq 0$. Thus in C

$$(z-a)^2 f(z) = (z-a)^k g(z),$$

with $k \geq 2$, since f is bounded in C. This proves that $h(z) = (z-a)^{k-2} g(z)$ is the required holomorphic extension of f. □

2.2.2 Reflection principle

Theorem 2.5 (Schwarz) *Let $D \subset \mathbb{C}$ be a domain symmetric with respect to the real axis and let $D^+ = D \cap \{\mathsf{Im}\, z > 0\}$, $D^- = D \cap \{\mathsf{Im}\, z < 0\}$ and $D^0 = D \cap \{\mathsf{Im}\, z = 0\}$. Let $f : D^+ \cup D^0 \to \mathbb{C}$ be a continuous function, holomorphic on D^+, and real valued on D^0.*

Then there exists a unique holomorphic function $g : D \to \mathbb{C}$ extending f, i.e. such that $g(z) = f(z)$ for all $z \in D^+ \cap D^0$.

Proof. The uniqueness is immediate from the identity principle.

For the existence, let us define

$$g(z) = \begin{cases} f(z) & \text{if } z \in D^+ \cup D^0 \\ \overline{f(\overline{z})} & \text{if } z \in D^- \cup D^0 \end{cases}$$

The function g is continuous and holomorphic outside the real axis, hence it is immediate to check, using Morera's theorem, that g is holomorphic everywhere in D. □

2.3 Extension theorems in several complex variable

The most important theorem of the theory of several complex variables is Hartogs' theorem, which basically state that holomorphic functions fill compact holes.

Theorem 2.6 (Hartogs) *Let $D \subset \mathbb{C}^n$, $n > 1$, be a domain, $K \subset D$ a compact subset such that $D \setminus K$ is connected. Then the restriction homomorphism*

$$\mathcal{O}(D) \xrightarrow{res} \mathcal{O}(D \setminus K)$$

is an isomorphism.

2.3.1 Extension near small-dimensional sets

An equivalent of Riemann extension theorem for bounded holomorphic functions in one variable exists in several complex variables.

Theorem 2.7 *Let $U \subset \mathbb{C}^n$ be an open neighborhood of the origin, L a k-dimensional linear subspace and f a holomorphic function on $U \setminus L$. If $|f|$ is bounded, then f extends holomorphically on U.*

Proof. We may assume that L is the hyperplane $\{z_n = 0\}$ and U is a ball. Let $z' = (z_1, \ldots, z_{n-1})$ and $\varepsilon > 0$ be such that for every fixed z' with $|z'| \leq \varepsilon$, the boundary of the 1-dimensonal disc

$$D_\varepsilon(z') = \{(z', z_n) \in \mathbb{C}^n : \ |z_n| < \varepsilon\}$$

is contained in $U \setminus L$. By the classical Riemann Theorem it follows that

$$f|_{D_\varepsilon(z') \setminus \{z'\}}$$

extends holomorphically on the disc. Consider the function

$$\widetilde{f} : \{z \in \mathbb{C}^n : \ |z'| \leq \varepsilon, \ |z_n| \leq \varepsilon\} \to \mathbb{C}$$

defined by:

$$\widetilde{f} = \frac{1}{2\pi i} \int_{\mathrm{b}D_\varepsilon(z')} \frac{f(z', \zeta)}{\zeta - z_n} d\zeta.$$

Since f is continuous on $\{z \in \mathbb{C}^n : \ |z'| \leq \varepsilon, \ |z_n| = \varepsilon\}$, $\widetilde{f}$ is continuous on $\overline{D}_\varepsilon(z')$ and, differentiating under the integral sign, it follows that $\widetilde{f}$ is holomorphic separately in each of the variable $z_1, \ldots, z_n$, thus holomorphic.

Since $\widetilde{f}$ coincides with f on $D_\varepsilon(z') \setminus \{z'\}$ for every z', $\widetilde{f}$ is the required extension of f. $\square$

In the previous theorem, the boundedness hypotesis can be dropped if $\dim_{\mathbb{C}} L \leq n-2$, as a corollary of Hartogs theorem (see Theorem 2.6).

Corollary 2.8 *Let $D \subset \mathbb{C}^n$, $n > 1$ be a domain and L an affine complex subspace with $\dim_{\mathbb{C}} L \leq n-2$. Then any holomorphic function on $D \setminus L$ extends holomorphically to D.*

Proof. We may assume that

$$L = \{z \in \mathbb{C}^n : \ z_k = \cdots = z_n = 0\},$$

with $k \leq n-1$. We denote $z' = (z_1, \ldots, z_{k-1})$. For every $(z', 0) \in L$, the affine subspace

$$L_{z'}^{\perp} = \{z \in \mathbb{C}^n : \ z = (z', z_k, \ldots, z_n)\}$$

has dimension ≥ 2 and $L_{z'}^{\perp} \cap L = \{(z', 0)\}$.

For a fixed $(a', 0) \in L$, there exist $\varepsilon = \varepsilon(a')$, and $r = r(a') > 0$ such that the $(n-k+1)$-dimensional polydisc

$$Q = Q(z'; \varepsilon) = \{(z', z_k, \ldots, z_n) \in \mathbb{C}^n : \ |z_j| < \varepsilon, \ k \leq j \leq n\}$$

is relatively compact in D for $\|z' - a'\| < r$ (see figure 2.1).

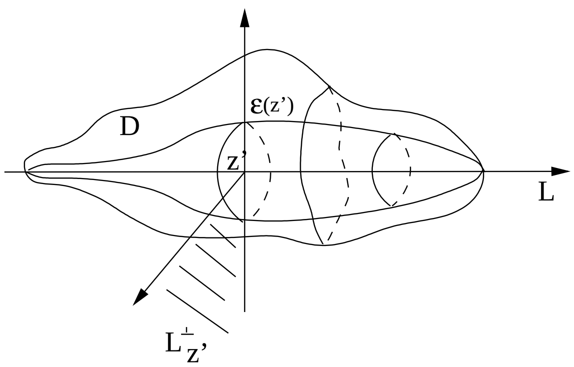


Figure 2.1:

By Theorem 2.6, $f|_{D \cap L_{z'}^{\perp} \setminus \{(z', 0)\}}$ extends holomorphically to $D \cap L_{z'}^{\perp}$. Let

$$U_L = \{z \in D : \ \|z' - a'\| < r, \ |z_j| < \varepsilon, \ k \leq j \leq n\}$$

and $\widetilde{f} : U_L \to \mathbb{C}$ be given by:

$$\widetilde{f}(z', z_k, \ldots, z_n) = \frac{1}{(2\pi i)^{n-k+1}} \int_{\check{S}(Q)} \frac{f(z', \zeta_k, \ldots, \zeta_n)}{(\zeta_k - z_k) \cdot \cdots \cdot (\zeta_n - z_n)} d\zeta_k \wedge \ldots \wedge d\zeta_n,$$

where $\check{S}(Q)$ is the Šilov boundary of the $(n-k+1)$-dimensional polydisc $Q = Q(z'; \varepsilon)$ (i.e. the product $\mathbb{S}^1_\varepsilon \times \cdots \times \mathbb{S}^1_\varepsilon$). Since f is continuous on bU_L, $\widetilde{f}$ is continuous on U_L and, differentiating under the integral sign, it follows that $\widetilde{f}$ is holomorphic separately in each of the variable $z_1, \ldots, z_n$, thus holomorphic. Since $\widetilde{f}$ coincides with f on $Q \setminus \{z'\}$ for every z', $\widetilde{f}$ is the required extension of f. $\square$

2.4 Edge of the wedge theorem

Let $0 \in U \subset \mathbb{C}^n$. Let $\rho_1(z), \ldots, \rho_n(z)$ be smooth real analytic functions, $\rho_j(0) = 0$ for all j, $M = \{z \in U \ : \ \rho_j(z) = 0, j = 1, \ldots, n\}$. Moreover suppose that $\overline{\partial}\rho_1 \wedge \cdots \wedge \overline{\partial}\rho_n \neq 0$, which in particular implies that the intersections of their zero-sets are $\mathbb{C}$-transversal:

$$(dz_1 \wedge \ldots \wedge dz_n)|_M \neq 0.$$

The set U is divided in 2^n parts (see Figure 2.2) by

$$\bigcup_{j=1}^{n} \{z \in U \ : \ \rho_j(z) = 0\}.$$

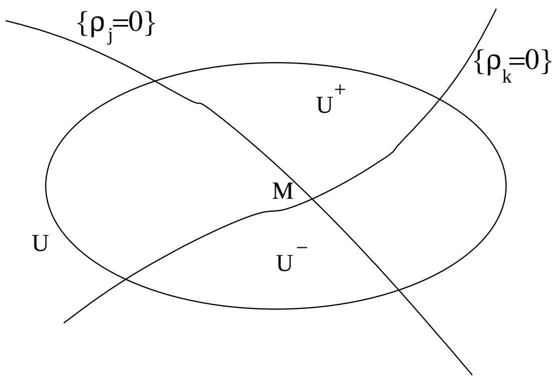


Figure 2.2: The open set U is divided in 2^n components

Let $M = \{z \in U : \rho_1(z) = \cdots = \rho_n(z) = 0\}$, and

$$U^{\pm} = \{z \in U \ : \ \pm\rho_j(z) > 0, \ j = 1, \ldots, n\}.$$

Theorem 2.9 (Edge of the Wedge theorem) *Let*

$$f^{\pm} \in \mathcal{O}\left(U^{\pm}\right) \cap \mathcal{C}^{0}\left(U^{\pm} \cup M\right)$$

be two holomorphic functions coinciding on M. Then there exist an open neighborhood W, $U \supset W \supset M$, and a holomorphic function $F \in \mathcal{O}(W)$ such that $F_{|_{U^{\pm}\cap W}} = f^{\pm}_{|_{U^{\pm}\cap W}}$. (see Figure 2.3)

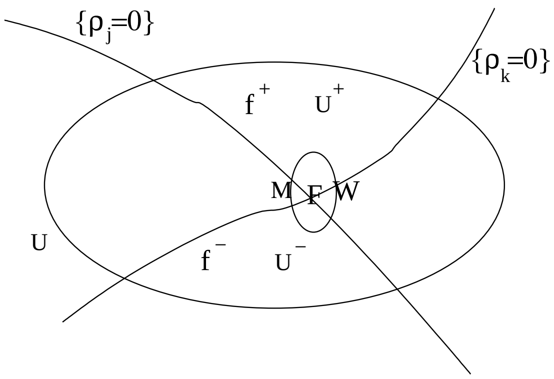


Figure 2.3: The functions $f^{\pm}$, defined on opposite wedges, and coinciding on the edge, can be holomorphically extended with a function F defined on an open neighborhood of the edge (Edge of the Wedge theorem)

For a proof of Edge of the Wedge theorem, refer to [55, 78, 90, 98, 105].

Example 2.1 The condition of $\mathbb{C}$-transversality of the smooth hypersurfaces $\{\rho_k(z)\}$ is necessary. $\mathbb{R}$-transversality is not enough. Indeed, let us consider $U = \mathbb{C}^2$ with coordinates $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, and $\rho_1(z) = x_1$, $\rho_2(z) = y_1$. Then

$$U^{+} = \{(z_1, z_2) \in \mathbb{C}^2 \mid x_1 > 0, y_1 > 0\}, \quad U^{-} = \{(z_1, z_2) \in \mathbb{C}^2 \mid x_1 < 0, y_1 < 0\},$$

their intersection M being $\{0\} \times \mathbb{C}_{z_2}$. The function

$$f(z) = \begin{cases} z_1 & \textit{if } z \in \overline{U}^{+}, \\ z_1^2 & \textit{if } z \in \overline{U}^{-} \end{cases}$$

is holomorphic in U^+ and U^- and continuous on the wedge M (and even on the boundaries of $U^{\pm}$, but of course there is no holomorphic function in a neighbourhood W of M such that $F|_{U^{\pm}\cap W} = f^{\pm}|_{U^{\pm}\cap W}$.

Example 2.2 In $U = \mathbb{C}^2$ as before, the functions $\rho_1(z) = y_1$, $\rho_2(z) = y_2$ satisfy the $\mathbb{C}$-transversality condition and the edge of the wedge theorem applies.

Chapter 3

Extension of CR-functions up to a Levi-flat boundary and of holomorphic maps

Hatogs' theorem (Theorem 2.6) enables to extend holomorphic functions through a compact hole. Using local extension results and representation formulas for holomorphic functions, it is possible also to fill some special kind of non-compact holes. A new kind of extension theorems (up to a Levi-flat boundary) was born in the middle 80's using this arguments (cfr. [63–65]).

In this chapter the local theorem and the non-compact extension results are stated and some sketches of the proofs are given.

The final section of the chapter is devoted to the extension of holomorphic maps across the boundary, providing some theorems similar to the classical Schwartz reflection principle in several complex variables.

3.1 Global extension for CR-functions

The following is the equivalent of Hartogs' theorem for CR-functions.

Theorem 3.1 *Let*

$$D = \{z \in \mathbb{C}^n \ : \ \rho(z) < 0\} \subset \mathbb{C}^n$$

be a domain, with defining function ρ of class $\mathcal{C}^2$. Every continuous weak CR-function in bD is the boundary value of a holomorphic function in D, continuous up to bD.

The idea of the proof is to consider the harmonic solution to the Dirichlet problem with boundary value. The weak CR-condition then ensures that the obtained solution is indeed holomorphic. For a proof, refer to [99].

3.2 Local extension for CR-functions

We just state, without proof, a very classical and useful local extension result for CR-functions.

Theorem 3.2 (Lewy [61]**, Kohn-Rossi** [53]**)** *Let*

$$D = \{z \in \mathbb{C}^n : \rho(z) < 0\} \subset \mathbb{C}^n$$

be a domain, with defining function ρ of class $\mathcal{C}^2$. If the Levi-form of ρ has at least one positive eingenvalue at a point $p \in bD$, then every continuous weak CR-function in a neighborhood $U \cap bD$ of p is the boundary value of a holomorphic function in $U' \cap D$ (U' smaller neighborhood of p), continuous up to $U' \cap bD$.

With this flavour there are other more precise results as those on monolateral and bilateral extension of CR-functions, see e.g. [75, 76].

3.3 Extension up to a Levi-flat boundary

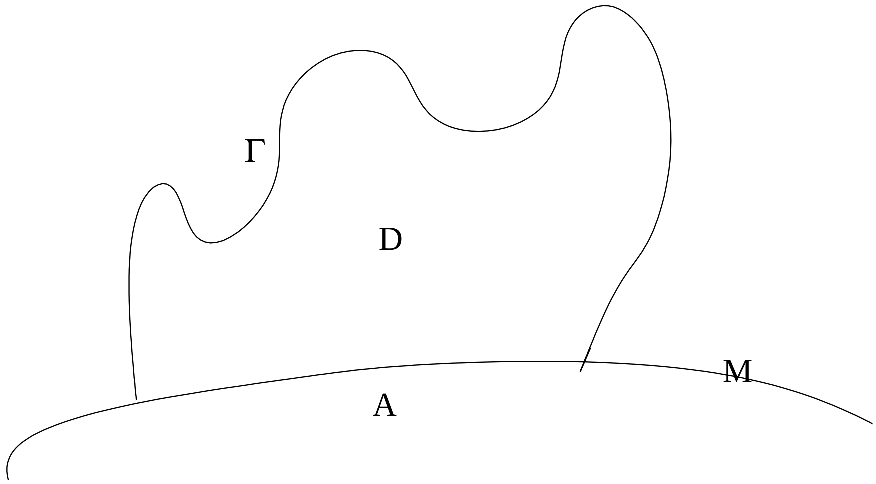
Let $\Gamma \subset \mathbb{C}^n$, $n > 2$, be a $\mathcal{C}^1$-smooth, compact, connected and orientable real hypersurface with boundary $b\Gamma$. Suppose Γ satisfies the following conditions

1. $b\Gamma \subset M = \{z \in \mathbb{C}^n : \rho(z) = 0\}$, where $\rho : \mathbb{C}^n \to \mathbb{R}$ is a pluriharmonic function;
2. $\Gamma \setminus b\Gamma \subset M^+ = \{z \in \mathbb{C}^n : \rho(z) > 0\}$;
3. $b\Gamma = bA$, the boundary of a bounded open set A of M.

Let $D \subset \mathbb{C}^n$ be the bounded domain with boundary $bD = \Gamma \cup A$ (see figure). In [65] the following theorem is proved.

Theorem 3.3 *Every locally Lipschitz CR-function f on $\Gamma \setminus b\Gamma$ extends uniquely to a function F holomorphic on D and continuous on $D \cup (\Gamma \setminus b\Gamma)$.*

Note that, when M is a hyperplane, this gives back Lewy's local extension theorem.



In the next Chapter we give a generalization of this theorem to Stein spaces (see Corollary 4.12).

Here we give the proof only with the stronger hypothesis for the function f of being $\mathcal{C}^1$-smooth (this simplifies the proof by removing technicalities, but does not change the ideas underlying in it):

Theorem 3.4 *Every $\mathcal{C}^1$-smooth CR-function f on $\Gamma \smallsetminus b\Gamma$ extends uniquely to a function F holomorphic on D and continuous on $D \cup (\Gamma \smallsetminus b\Gamma)$.*

Proof. *Uniqueness.* ρ is the real part of a uniquely determined (up to the addition of a constant) holomorphic function $\varphi : \mathbb{C}^n \to \mathbb{C}$. Suppose F_1 and F_2 coincide with f on $\Gamma \smallsetminus b\Gamma$. The maximum principle for $F_1 - F_2$ on the analytic submanifold $V_\zeta = \{z \in \mathbb{C}^n \ : \ \varphi(z) = \varphi(\zeta)\}$ implies

$$|F_1(\zeta) - F_2(\zeta)| \ \leq \ \max_{V_\zeta \cap \Gamma} |0| \ = \ 0,$$

for every $\zeta \in D$, i.e. $F_1 = F_2$. $\square$

Before going into the proof of the existence, we make some considerations. Let $\zeta \in \mathbb{C}^n$, and

$$M_\zeta = \{z \in \mathbb{C}^n \ : \ \rho(z) = \rho(\zeta)\}.$$

M_ζ is foliated in complex analytic hypersurfaces, the level sets of φ:

$$\{z \in \mathbb{C}^n \ : \ \varphi(z) = const\}.$$

$\mathbb{C}^n \smallsetminus V_\zeta$ is a domain of holomorphy, hence on $\mathbb{C}^n \smallsetminus V_\zeta$ there is a $(n, n-2)$-form $\Phi = \Phi(\cdot, z)$ such that $\overline{\partial}\Phi = \omega_{BM}$, ω_{BM} being the Bochner-Martinelli kernel in (1.21). To obtain the extension theorem, an explicit solution of such an equation, which depends analytically upon $\mathsf{Re}\,\zeta_i$, $\mathsf{Im}\,\zeta_i$, is needed. In [65] such a solution is proved to be

$$\Phi(z,\zeta) \;=\; \frac{1}{\varphi(z)-\varphi(\zeta)}\sum_{k=1}^{n} h_k(z,\zeta)(z_k-\zeta_k)\Omega_k(z,\zeta), \tag{3.1}$$

where $\Omega_k(z,\zeta)$ $(k=1,\ldots,n)$ are the following $(n,n-2)$-form

$$\Omega_k(z,\zeta) = C_k dz_1\wedge\ldots\wedge dz_n \wedge \left[\sum_{j\neq k}(-1)^{j-\max\left(0,\frac{j-k}{|j-k|}\right)}(\overline{z}_j-\overline{\zeta}_j)\bigwedge_{l\neq j,k} d\overline{z}_l\right],$$

$$C_k = \frac{(n-2)!}{(2\pi i)^n}\frac{(-1)^{\frac{n(n+1)}{2}+k}}{(z_k-\zeta_k)|z-\zeta|^{2n-2}}.$$

The forms Ω_k are defined on $\mathbb{C}^n\setminus\{\zeta\}$ and they satisfy

$$\frac{\partial\omega_{BM}}{\partial\overline{\zeta}_k} \;=\; \overline{\partial}\,\frac{\partial\Omega_k}{\partial\overline{\zeta}_k}, \tag{3.2}$$

for $k=1,\ldots,n$. We may now prove the existence.

Proof. *Existence.* If such a function F exists, we claim that

$$F(\zeta) \;=\; \int_\Gamma f\omega_{BM}(\cdot,\zeta) - \int_{b\Gamma} f\Phi(\cdot,\zeta), \tag{3.3}$$

for all $\zeta\in D$.

Indeed, consider $\Gamma_\varepsilon = \Gamma\cap\{z\in\mathbb{C}^n \;:\; \rho(z)\geq\varepsilon\}$, for $\varepsilon>0$ sufficiently small. For almost all ε, Γ_ε is a $\mathcal{C}^1$-smooth, compact, connected hypersurface with boundary $b\Gamma_\varepsilon = \Gamma\cap\{z\in\mathbb{C}^n \;:\; \rho(z)=\varepsilon\}$, which verifies conditions 1., 2. and 3. with respect to the Levi-flat (non-singular) hypersurface $M_\varepsilon = \{z\in\mathbb{C}^n \;:\; \rho(z)=\varepsilon\}$.

Again, let A_ε be the bounded open domain of M_ε with boundary $b\Gamma_\varepsilon$, and D_ε the bounded domain of $\mathbb{C}^n$ with boundary $\Gamma_\varepsilon\cup A_\varepsilon$. Applying to this domain Bochner-Martinelli's formula (1.19) we get

$$F(\zeta) \;=\; \int_{\Gamma_\varepsilon} f\omega_{BM}(\cdot,\zeta) + \int_{A_\varepsilon} F\omega(\cdot,\zeta) \tag{3.4}$$

for all $\zeta\in D_\varepsilon$.

Since, in a neighborhood U_ε of $\overline{A}_\varepsilon$, $V_\zeta\cap M_\varepsilon\cap U_\varepsilon=\emptyset$, then in D $F\omega_{BM}(\cdot,\zeta) = d(F\Phi(\cdot,\zeta))$. Green-Stokes' formula thus imply

$$F(\zeta) \;=\; \int_{\Gamma_\varepsilon} f\omega_{BM}(\cdot,\zeta) + \int_{b\Gamma_\varepsilon} f\Phi(\cdot,\zeta), \tag{3.5}$$

for all $\zeta \in D_\varepsilon$. From (3.5), when $\varepsilon \to 0$, we get (3.3), as claimed.

Hence to prove the existence, it sufficies to show that (3.3) actually is holomorphic and gives an extension of f. The functions

$$F_1(\zeta) \;=\; \int_\Gamma f\omega_{BM}(\cdot,\zeta) \tag{3.6}$$

$$F_2(\zeta) \;=\; \int_{\partial\Gamma} f\Phi(\cdot,\zeta) \tag{3.7}$$

are real analytic on $\mathbb{C}^n \smallsetminus \Gamma$ (respectively on $\mathbb{C}^n \smallsetminus M$). Let us denote by F_j^+ (respectively by F_j^-), the restriction on F_j to D (respectively to $\mathbb{C}^n \smallsetminus (\overline{D} \cup M)$), $j = 1, 2$. Then $F = F_1^+ - F_2^+$. $F_1^\pm$ both extend continuously to $\Gamma \smallsetminus b\Gamma$, and on $\Gamma \smallsetminus b\Gamma$ $F_1^+ - F_1^- = f$ (see [38, Appendix B]). Moreover, since F_2 is real analytic on $\mathbb{C}^n \smallsetminus M$, on $\Gamma \smallsetminus b\Gamma$ $F_2^+ - F_2^- = 0$.

Hence F extends continuously to $\Gamma \smallsetminus b\Gamma$, where

$$F \;=\; F_1^+ - F_2^+ \;=\; f + F_1^- - F_2^-. \tag{3.8}$$

Hence we must prove that $F_1^- - F_2^- = 0$ on $\Gamma \smallsetminus b\Gamma$.

Let

$$U = \left\{ \zeta \in \mathbb{C}^n \;:\; \rho(\zeta) \notin \left[0, \max_{\overline{D}} \rho\right] \right\},$$

and $\zeta \in U$. $\overline{D} \subset \mathbb{C}^n \smallsetminus M_\zeta$, thus $\omega(\cdot,\zeta)$ and $\Phi(\cdot,\zeta)$ are $\mathcal{C}^\infty$-smooth in a neighborhood of Γ. Thus, by Green-Stokes' formula

$$(F_1^- - F_2^-)(\zeta) \;=\; \int_\Gamma f\overline{\partial}\Phi(\cdot,\zeta) - \int_{b\Gamma} f\Phi(\cdot,\zeta) \;=\; 0, \tag{3.9}$$

for $\zeta \in U$. Since U intersects all connected components of $\mathbb{C}^n \smallsetminus (\overline{D} \cup M)$ and $F_1^- - F_2^-$ is real analytic on $\mathbb{C}^n \smallsetminus (\overline{D} \cup M)$ and continuous up to $\Gamma \smallsetminus b\Gamma$, (3.9) holds also for $\zeta \in \Gamma \smallsetminus b\Gamma$. So F is indeed an extension of f.

To show that F is holomorphic, consider its antiholomorphic derivatives $(k = 1, \ldots, n)$

$$\frac{\partial F}{\partial \overline{\zeta}_k}(\zeta) \;=\; \int_\Gamma f\overline{\partial}\,\frac{\partial \Omega_k}{\partial \overline{\zeta}_k}(\cdot,\zeta) - \int_{\partial\Gamma} f\,\frac{\partial \Phi}{\partial \overline{\zeta}_k}(\cdot,\zeta). \tag{3.10}$$

If $n = 2$, one easily verifies that

$$\frac{\partial \Omega_k}{\partial \overline{\zeta}_k} \;=\; \frac{\partial \Phi}{\partial \overline{\zeta}_k},$$

for $k = 1, 2$, hence Green-Stokes' formula implies

$$\frac{\partial F}{\partial \overline{\zeta}_k} \;=\; 0,$$

for $k = 1, 2$.

If $n > 2$ and $\zeta \in D$, fix $\varepsilon > 0$ s.t. $\zeta \notin V_\varepsilon = \{z \in \mathbb{C}^n \;:\; -\varepsilon < \rho(z) < \varepsilon\}$. The $(n, n-2)$-forms

$$\Psi_k(\cdot, \zeta) \;=\; \frac{\partial \Omega_k}{\partial \overline{\zeta}_k}(\cdot, \zeta) - \frac{\partial \Phi}{\partial \overline{\zeta}_k}(\cdot, \zeta), \tag{3.11}$$

for $k = 1, \ldots, n$, are $\mathcal{C}^\infty$-smooth and $\overline{\partial}$-closed on V_ε, which is an open set of holomorphy. Hence, there exist $\mathcal{C}^\infty$-smooth $(n, n-3)$-forms $\Theta_k(\cdot, \zeta)$ on U s.t.

$$\Psi_k(\cdot, \zeta) \;=\; \overline{\partial}\Theta_k(\cdot, \zeta), \tag{3.12}$$

for $k = 1, \ldots, n$. Let $\tilde{\Theta}_k(\cdot, \zeta)$ be $\mathcal{C}^\infty$-smooth $(n, n-3)$-form on $\mathbb{C}^n$, equal to $\Theta_k(\cdot, \zeta)$ in a neighborhood of $\partial\Gamma$, $k = 1, \ldots, n$. (3.10), (3.12) and Green-Stokes' formula thus imply

$$\frac{\partial F}{\partial \overline{\zeta}_k}(\zeta) = \int_\Gamma f \overline{\partial} \left[\frac{\partial \Omega_k}{\partial \overline{\zeta}_k}(\cdot, \zeta) - \overline{\partial}\tilde{\Theta}_k(\cdot, \zeta) \right] - \int_{\partial\Gamma} f \left[\frac{\partial \Omega_k}{\partial \overline{\zeta}_k}(\cdot, \zeta) - \overline{\partial}\tilde{\Theta}_k(\cdot, \zeta) \right] = 0,$$

$k = 1, \ldots, n$. So F is holomorphic. □

3.4 Extension out of holomorphic hulls

Theorem 3.3 was later generalized by Lupacciolu in [63].

Problem 3.1 *Consider a bounded domain $D \Subset \mathbb{C}^n$, $n \geq 2$, with boundary bD and a compact subset $K \subset bD$, such that $bD \setminus K$ is a $\mathcal{C}^1$-smooth real hypersurface in $\mathbb{C}^n \setminus K$. Given $f \in CR(bD \setminus K)$ does there exist a holomorphic extension $F \in \mathcal{O}(D \setminus K) \cap \mathcal{C}^0(\overline{D} \setminus K)$ such that*

$$F_{|_{bD \setminus K}} = f?$$

A positive answer was given in [63], if K is $\mathcal{O}(\overline{D})$-convex, and $bD \setminus K$ is connected:

Theorem 3.5 *Assume that K is $\mathcal{O}(\overline{D})$-convex, and $bD \setminus K$ is connected. Then every continuous CR-function f on $bD \setminus K$ has a unique extension*

$$F \in \mathcal{O}(D \setminus K) \cap \mathcal{C}^0(\overline{D} \setminus K).$$

Moreover

Theorem 3.6 *Assume that* $bD \smallsetminus \widehat{K}_{\overline{D}}$ *is a connected* C^1*-smooth real hypersurface in* $\mathbb{C}^n \smallsetminus \widehat{K}_{\overline{D}}$*. Then every continuous* CR*-function* f *on* $bD \smallsetminus \widehat{K}_{\overline{D}}$ *has a unique extension*

$$F \in \mathcal{O}(D \smallsetminus \widehat{K}_{\overline{D}}) \cap \mathcal{C}^0(\overline{D} \smallsetminus \widehat{K}_{\overline{D}}).$$

Setting $D' = D \smallsetminus \widehat{K}_{\overline{D}}$ and $K' = \overline{D}' \cap \widehat{K}_{\overline{D}}$, Theorem 3.5 with D' and K' implies Theorem 3.6. Hence the two statements are equivalent.

In the following chapter we will give a generalization of Theorem 3.5 to Stein spaces (see Corollary 4.13).

Before entering the details of the proof, we make some considerations. Let U be an open neighborhood of $\overline{D}$ and $\varphi \in \mathcal{O}(U)$. We denote by $L_\zeta(\varphi)$ the level set of φ through $\zeta \in U$, i.e.

$$L_\zeta(\varphi) \;=\; \{z \in U \;:\; \varphi(z) = \varphi(\zeta)\}\,.$$

It is known (see e.g. [37]) that there exist (possibly not unique) $h_1, \ldots, h_n \in \mathcal{O}(U \times U)$ such that

$$\varphi(z) - \varphi(\zeta) \;=\; \sum_{\alpha=1}^{n} h_\alpha(z,\zeta)(z_\alpha - \zeta_\alpha). \tag{3.13}$$

Using the n-uple $h_1, \ldots, h_n$ we can define a $\overline{\partial}$-primitive of the Bochner-Martinelli form ω_{BM} on $U \smallsetminus L_\zeta(\varphi)$. Indeed, setting

$$\Psi_\alpha^+ = \sum_{\beta=1}^{\alpha-1} (-1)^\beta (\overline{z}_\beta - \overline{\zeta}_\beta)\, d\overline{z}_1 \wedge \cdots \hat{\beta} \cdots \hat{\alpha} \cdots \wedge d\overline{z}_n,$$

$$\Psi_\alpha^- = \sum_{\beta=\alpha+1}^{n} (-1)^{\beta+1} (\overline{z}_\beta - \overline{\zeta}_\beta)\, d\overline{z}_1 \wedge \cdots \hat{\alpha} \cdots \hat{\beta} \cdots \wedge d\overline{z}_n,$$

$$\Omega_\alpha(\zeta) = \frac{(-1)^{n+\alpha}}{n-1} \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2\pi i)^n} \frac{dz_1 \wedge \cdots \wedge dz_n}{(z_\alpha - \zeta_\alpha)|z-\zeta|^{2n-2}} \wedge (\Psi_\alpha^+ + \Psi_\alpha^-),$$

$$\Phi_h(\zeta) = \frac{1}{\varphi(z) - \varphi(\zeta)} \sum_{\alpha=1}^{n} h_\alpha(z,\zeta)(z_\alpha - \zeta_\alpha)\Omega_\alpha(\zeta),$$

then $\Phi_h(\zeta)$ is a real analytic $\overline{\partial}$-primitive of the Bochner-Martinelli form $\omega_{BM}(\zeta)$ on $U \smallsetminus L_\zeta(\varphi)$.

If h and h' are two different n-uples of holomorphic functions satisfying (3.13), then the following properties hold (for a proof, refer to [63]):

1. if $n \geq 3$,

$$\Phi_h(\zeta) - \Phi_{h'}(\zeta) \;=\; \overline{\partial} X_{h,h'}(\zeta), \tag{3.14}$$

   where $X_{h,h'}(\zeta)$ is an $(n, n-3)$-form on $(U \setminus L_\zeta(\varphi)) \cap (U' \setminus L_\zeta(\varphi'))$;

2. if $n = 2$, on $(U \setminus L_\zeta(\varphi)) \cap (U' \setminus L_\zeta(\varphi'))$,

$$\Phi_h(\zeta) - \Phi_{h'}(\zeta) \;=\; -\frac{1}{(2\pi i)^2} \frac{(h_1 h_2' - h_2 h_1')\, dz_1 \wedge dz_2}{(\varphi(z) - \varphi(\zeta))(\varphi(z') - \varphi(\zeta'))}. \tag{3.15}$$

Before entering the proof of Theorem 3.5, we state for completeness the following result (which is a sharper version of Stokes' Theorem for CR-functions). We do not give a proof, as in here we are not much interested in a small-smoothness situation.

Theorem 3.7 *Let Σ be a $\mathcal{C}^1$-smooth real hypersurface in $\mathbb{C}^n$ without boundary. Let $f : \Sigma \to \mathbb{C}$ be a continuous function. Then $f \in CR(\Sigma)$ if and only if*

$$\int_{c_{n+q}} f \overline{\partial} \mu \;=\; \int_{bc_{n+q}} f \mu, \tag{3.16}$$

for every $1 \leq q \leq n-1$, and every singular $(n+q)$-chain c_{n+q} of Σ of class C^1 and every $(n, q-1)$-form μ of class C^1 on an open neighborhood of Σ.

We can now begin the actual construction of the function F. Let $V \supset K$ be an open neighborhood, and $\sigma : \mathbb{C}^n \to [0,1]$ a C^∞-smooth function such that

(a) $\operatorname{supp} \sigma \subset V$;

(b) $\sigma(K) = 1$;

(c) $\sigma(z) < 1$ if $z \notin K$.

Fix $\varepsilon > 0$ and set

$$D_\varepsilon \;=\; D \cap \{z \in \mathbb{C}^n \mid \sigma(z) < 1 - \varepsilon\};$$

$$\Gamma_\varepsilon \;=\; bD \cap bD_\varepsilon; \quad K_\varepsilon \;=\; \overline{D} \cap \{z \in \mathbb{C}^n \mid \sigma(z) = 1 - \varepsilon\}.$$

D can be exhausted by a numerable increasing sequence $D_s = D_{\varepsilon_s}$, $s \in \mathbb{N}$, of such domains, so that $D = \cup_s D_s$ and $bD \setminus K = \cup_s \Gamma_s$.

Fix a neighborhood U of $\overline{D}$ and $\varphi \in \mathcal{O}(U)$. For every $s \in \mathbb{N}$, we define the set

$$U_s(\varphi) \;=\; \left\{\zeta \in U \;\middle|\; |\varphi(\zeta)| > \max_{\overline{D \smallsetminus D_s}} |\varphi|\right\},$$

which is open in $U \smallsetminus \overline{D \smallsetminus D_s}$, and it is foliated by the level-sets $L_\zeta(\varphi)$. Since $\overline{D \smallsetminus D_s}$ is a decreasing sequence of compacts in $\overline{D}$ whose intersection is K, it follows that

$$\bigcup_{s=1}^{\infty} U_s(\varphi) \;=\; U(\varphi) \;=\; \left\{\zeta \in U \;\middle|\; |\varphi(\zeta)| > \max_K |\varphi|\right\}. \tag{3.17}$$

Since

$$\widehat{K}_{\overline{D}} \;=\; \bigcap_{U \supset \overline{D}} \widehat{K}_U,$$

the $\mathcal{O}(\overline{D})$-convexity of K implies that

$$\overline{D} \smallsetminus K \;=\; \left(\bigcup_{U \supset \overline{D}} \bigcup_{\varphi \in \mathcal{O}(U)} U(\varphi)\right) \cap \overline{D}. \tag{3.18}$$

Now, for every U, φ, s as above and $h_1, \ldots, h_n \in \mathcal{O}(U \times U)$ as in equation (3.13), consider the following function

$$F_h^s(\zeta) \;=\; \int_{\Gamma_s} f\omega(\zeta) - \int_{b\Gamma_s} f\Phi_h(\zeta), \tag{3.19}$$

(Γ_s being oriented as an open set in bD and $b\Gamma_s$ as the boundary of Γ_s), which is defined on $U_s(\varphi) \smallsetminus bD$.

Since $|\varphi_\zeta(\varphi)| > |\varphi(z)|$, for $\zeta \in U_s(\varphi)$ and $z \in b\Gamma_s$, the singular level-set $L_\zeta(\varphi)$ of $\Phi_h(\zeta)$ does not intersect $b\Gamma_s$. Hence F_h^s is indeed defined and real analytic on $U_s(\varphi) \smallsetminus \Gamma_s = U_s(\varphi) \smallsetminus bD$.

Proposition 3.8 *Suppose there is F satisfying the hypothesis of Theorem 3.5. Then for every U, φ, s, h, $F = F_h^s$ on $D \cap U_s(\varphi)$.*

Proof. Obviously $D \cap U_s(\varphi) \subset D_s$. By hypothesis $F \in \mathcal{C}^0(\overline{D}_s) \cap \mathcal{O}(D_s)$, and $F_{|\Gamma_s} = f_{|\Gamma_s}$. Therefore, Bochner-Martinelli formula implies

$$F_h^s(\zeta) \;=\; \int_{\Gamma_s} f\omega_{BM}(\zeta) + \int_{K_s} F\omega(\zeta).$$

Thus we have to prove that, for $\zeta \in D \cap U_s(\varphi)$,

$$\int_{K_s} F\omega_{BM}(\zeta) \;=\; -\int_{b\Gamma_s} f\Phi_h(\zeta). \tag{3.20}$$

Observe that the forms $F\omega(\zeta)$ and $f\Phi_h(\zeta)$ are analytic in $D \setminus L_\zeta(\varphi)$, continuous up to the boundary, and such that

$$F\omega_{BM}(\zeta) \;=\; d[f\Phi_h(\zeta)].$$

Stokes' Theorem (or its enhanced version Theorem 3.7) imply

$$\int_{K_s} F\omega_{BM}(\zeta) \;=\; \int_{bK_s} f\Phi_h(\zeta) \;=\; -\int_{b\Gamma_s} f\Phi_h(\zeta),$$

the last equality following from the fact that the boundaries of K_s and of Γ_s coincide with opposite orientations. □

Remark 3.1 Proposition 3.8 ensures the uniqueness of the function F and reduces the proof of the existence of F to show that the functions F_h^s "glue" together and the constructed F has f as boundary limit.

Proposition 3.9 *The functions F_h^s are each other coherent and holomorphic, i.e. they "glue" together to a holomorphic function.*

Proof. The coherence statement means that for all U, φ, h, s and U', φ', h', s'

$$F_h^s = F_{h'}^{s'}, \quad on\ (U_s(\varphi) \cap U'_{s'}(\varphi')) \setminus bD. \tag{3.21}$$

Without losing any generality, we may suppose that $s \geq s'$. Then (3.21) will be a consequence of the following

$$F_{h'}^s = F_{h'}^{s'}, \quad on\ \cap U'_{s'}(\varphi') \setminus bD, \tag{3.22}$$

$$F_h^s = F_{h'}^s, \quad on\ (U_s(\varphi) \cap U'_s(\varphi')) \setminus bD. \tag{3.23}$$

To prove (3.22) (in the case $s > s'$) consider the $(2n-1)$-chain of $bD \setminus K$, $c_{2n-1} = \Gamma_s - \Gamma_{s'}$. For $\zeta \in U'_{s'}(\varphi') \setminus bD$,

$$F_{h'}^s(\zeta) - F_{h'}^{s'}(\zeta) = \int_{c_{2n-1}} f\omega_{BM}(\zeta) - \int_{bc_{2n-1}} f\Phi_{h'}(\zeta). \tag{3.24}$$

Moreover, since $\mathsf{Supp}\, c_{2m-1} \subset \overline{D_s \setminus D_{s'}} \subset \overline{D \setminus D_{s'}}$ and $L_\zeta(\varphi') \subset U'_{s'}(\varphi') \subset U' \setminus \overline{D \setminus D_{s'}}$, it follows that

$$\mathsf{Supp}\, c_{2n-1} \subset U' \setminus L_\zeta(\varphi'),$$

where $\omega_{BM}(\zeta)$ and $\Phi_{h'}(\zeta)$ are both defined and satisfy $\omega_{BM}(\zeta) = \overline{\partial}\Phi_{h'}(\zeta)$. Hence by Stokes' Theorem the righthand side of (3.24) is zero. This proves (3.22).

To prove (3.23), note that, if $n \geq 3$, equation (3.14) implies that for $\zeta \in (U_s(\varphi) \cap U'_s(\varphi')) \setminus bD$

$$F^s_h(\zeta) - F^s_{h'}(\zeta) = -\int_{b\Gamma_s} f\overline{\partial}X_{h,h'}(\zeta). \tag{3.25}$$

Again Stokes' Theorem imply that the righthand side is zero, thus proving (3.23) when $n \geq 3$.

When $n = 2$, equation (3.15) is used to prove that $F^s_h = F^s_{h'}$.

So coherence is proved. It remains to prove the holomorphicity of F, i.e. that for each $\zeta \in U_s(\varphi) \setminus bD$, F^s_h is holomorphic:

$$\frac{\partial F^s_h}{\partial \overline{\zeta}_\alpha}(\zeta) \;=\; 0, \tag{3.26}$$

for $\alpha = 1, \ldots, n$. Clearly,

$$\frac{\partial F^s_h}{\partial \overline{\zeta}_\alpha}(\zeta) = \int_{\Gamma_s} f\frac{\partial \omega_{BM}}{\partial \overline{\zeta}_\alpha}(\zeta) - \int_{b\Gamma_s} f\frac{\partial \Phi_h}{\partial \overline{\zeta}_\alpha}(\zeta);$$

this can be rewritten as

$$\int_{\Gamma_s} f\overline{\partial}\left[\frac{\partial \Omega_\alpha}{\partial \overline{\zeta}_\alpha}(\zeta)\right] - \int_{b\Gamma_s} f\frac{\partial \Omega_\alpha}{\partial \overline{\zeta}_\alpha}(\zeta) + I_n, \tag{3.27}$$

where $I_2 = 0$ and, if $n \geq 3$

$$I_n = \int_{b\Gamma_s} f\overline{\partial}\Psi^\alpha_h(\zeta),$$

which is zero in view of Stokes' Theorem. By Stokes' Theorem also the difference in (3.27) is zero, hence F^s_h is holomorphic. □

Continuity at the boundary will end the proof. For this, we state without proof a useful lemma.

Lemma 3.10 *Let $V \supset bD \setminus K$ be an open neighborhood, $V \subset \cup U(\varphi)$ such that $V \setminus (bD \setminus K)$ has two connected components, V_+ in D and V_- in $\mathbb{C}^n \setminus \overline{D}$. Then $F = 0$ on V_-.*

Proposition 3.11 *For every point $z^0 \in bD \setminus K$,*

$$\lim_{D \ni \zeta \to z^0} F(\zeta) = f(z^0).$$

Proof. For every $z^0 \in bD \setminus K$, we first prove the existence of the perpendicular limit, i.e., denoting by $\nu(z^0)$ the inward pointing vector perpendicular to $T_{z^0}(bD)$, we prove that

$$\lim_{t \to 0^+} F(z^0 + t\nu(z^0)) = f(z^0), \tag{3.28}$$

and moreover that the limit is uniform on compact subsets of $bD \setminus K$. Fix $z^0 \in bD \setminus K$. We can find an open neighborhood $U \supset \overline{D}$, a function $\varphi \in \mathcal{O}(U)$ and $s \in \mathbb{N}$ such that $z^0 \in U_s(\varphi) \cap (\Gamma_s \setminus b\Gamma_s)$. Then, for $t > 0$ small enough,

$$z^0 + t\nu(z^0) \in U_s(\varphi) \cap D, \quad z^0 + t\nu(z^0) \in V_-,$$

where V_- is as in Lemma 3.10. So $F(z^0 - t\nu(z^0)) = 0$ by Lemma 3.10. Thus, for any $h \in \mathcal{O}^n_\varphi(U \times U)$,

$$F(z^0 + t\nu(z^0)) = F^s_h(z^0 + t\nu(z^0)) - F^s_h(z^0 - t\nu(z^0)) = I_1(z^0, t) - I_2(z_0, t),$$

where

$$I_1(z^0, t) = \int_{\Gamma_s} [\omega_{BM}(z^0 + t\nu(z^0)) - \omega_{BM}(z^0 - t\nu(z^0))] f,$$

$$I_2(z^0, t) = \int_{b\Gamma_s} [\Phi_h(z^0 + t\nu(z^0)) - \Phi_h(z^0 - t\nu(z^0))] f.$$

For every $f \in \mathcal{C}^0(\Gamma_s)$ Harvey and Lawson [38] proved that

$$\lim_{t \to 0^+} I_1(z^0, t) = f(z^0).$$

The function $f\Psi$ is defined and real analytic on all of $U_s(\varphi)$, hence for $z^0 \in U_s(\varphi) \cap (\Gamma_s \setminus b\Gamma_s)$,

$$\lim_{t \to 0^+} I_2(z^0, t) = f(z^0),$$

the limit being uniform on compact subsets of $U_s(\varphi) \cap (\Gamma_s \setminus b\Gamma_s)$. Thus (3.28) follows.

Continuity of F in D and triangular inequality suffice to conclude. □

3.5 Extension in Stein manifolds

For the sake of completeness, we cite here without proof an extension theorem for CR-functions in Stein manifold.

Theorem 3.12 (Laurent-Thiébaut [56]**)** *Let M be a Stein manifold of complex dimension at least 2, $D \Subset M$ a relatively compact open set and $K \subset bD$ a compact, such that:*

1. *there exists a holomorphic function* $\varphi : M \to \mathbb{C}$ *such that* $K \subset \{\mathsf{Re}\,\varphi > 0\}$;
2. $bD \setminus K$ *is* C^1*-smooth and connected.*

Then for each locally Lipschitz $f \in CR(bD \setminus K)$ *there is a (unique)* $F \in \mathcal{C}^0(\overline{D} \setminus K) \cup \mathcal{O}_M(D)$ *such that* $F_{|_{\overline{D} \setminus K}} = f$.

3.6 Extension on unbounded domains

The methods used in this chapter to prove the previous theorems (and the theorems themselves) lead to an extension result for CR-functions in unbounded domains (see [64]).

Theorem 3.13 *Let* $D \subset \mathbb{C}^n$, $n \geq 2$, *be an unbounded open domain with* $\mathcal{C}^1$*-smooth and connected boundary* bD. *If there exists an algebraic hypersurface* W *of* $\mathbb{C}^n$ *such that*

$$\overline{D}^{\infty} \cap \overline{W}^{\infty} \;=\; \emptyset, \tag{3.29}$$

where $\overline{}^{\infty}$ *denotes the closure in* $\mathbb{CP}^n$, *then for every* CR*-function* f *there exists a unique* $F \in \mathcal{O}(D) \cap \mathcal{C}^0(\overline{D})$

Condition (3.29) is equivalent to the existence of a polynomial $p : \mathbb{C}^n \to \mathbb{C}$ such that

$$D \subset \left\{ z \in \mathbb{C}^n \;:\; |p(z)|^2 > (1 + |z|^2)^{\deg p} \right\}.$$

Observe that if W is a non-algebraic holomorphic hypersurface in $\mathbb{C}^n$ then its closure in $\mathbb{CP}^n$ contains the hyperplane at infinity (see [85]), and thus intersects $\overline{D}^{\infty}$.

Proof of Theorem 3.13. *Step 1.* W *is a hyperplane.* Consider a linear change of coordinates such that $\mathbb{CP}^{n-1}_{\infty} = \{\zeta_0 = 0\}$ and $\overline{W}^{\infty} = \{\zeta_n = 0\}$. Consider the linear map from $\mathbb{CP}^n$ to itself sending $\overline{W}^{\infty}$ to the hyperplane at infinity of $\mathbb{C}^n$. The image of D via this map is a domain satisfying the hypothesis of Theorem 3.5. So there is a unique F extending f.

Step 2. W *of order* ≥ 2. Let $W = \{P = 0\}$, with $\deg P = d > 1$,

$$P(z) = \sum_{|I|=d} a_I z^I$$

We use the Veronese map v to embed $\mathbb{CP}^n$ in a suitable $\mathbb{CP}^N$ in such a way that $v(\Sigma_0) = L_0 \cap v(\mathbb{CP}^n)$, where L_0 is a linear subspace. The Veronese map v is defined as follows: let d be the degree of P, and let

$$N \;=\; \binom{n+d}{d} - 1.$$

Then v is defined by

$$v(z) \;=\; v[z_0 : \ldots : z_n] \;=\; [\ldots : w_I : \ldots]_{|I|=d},$$

where $w_I = z^I$. We denote by H the hyperplane of equation

$$H \;=\; \left\{ \sum_{|I|=d} a_I w_I = 0 \right\}.$$

Hence $v(\overline{W}^{\infty}) \subset H$. We consider H as the new $\mathbb{CP}^{N-1}$ at infinity, i.e. consider the biholomorphism $\tau : \mathbb{CP}^N \smallsetminus H \to \mathbb{C}^N$. Define

$$M = \tau \circ v(\mathbb{CP}^n \smallsetminus \overline{W}^{\infty}), \quad D = \tau \circ v(\Omega), \quad K = \tau \circ v(\overline{\Omega}^{\infty} \cap \mathbb{CP}^{n-1}_{\infty}).$$

M is an n-dimensional algebraic submanifold of $\mathbb{C}^N$, hence a Stein manifold, $D \Subset M$ a relatively compact open set, $K \subset bD$, and $bD \smallsetminus K$ is a real hypersurface C^1-smooth and connected in $M \smallsetminus K$. In this situation the thesis is a direct consequence of a slight generalization of Theorem 3.12. □

3.7 Extension of holomorphic maps: the reflection principle in higher dimension

One of the possible extension results is that of showing boundary regularity of holomorphic maps under certain assumpion (i.e. they they are continuous, $\mathcal{C}^k$-smooth, or real analytic up to the boundary). In this direction, we are going to discuss here a generalization of the classical Schwarz reflection principle, both in one and in several complex variables. For a more detailed exposition see [20, 22, 23, 72, 73]. Namely, consider the following:

Problem 3.2 *Let $D, D' \Subset \mathbb{C}^n$ be relatively compact domains, with boundaries bD, bD' of class $\mathcal{C}^\omega$ (analytic), and let $f : D \to D'$ be a biholomorphic map. Does f extend holomorphically to a function $\tilde{f} \in \mathcal{O}(\overline{D})$?*

A positive answer to this question is known in the following cases:

1. D, D' are strictly pseudoconvex domains (Pinchuk [72], Lewy [60] '70s)

2. D, D' are (weakly) pseudoconvex domains (Bell & Catlin [9], Diederich & Fornæss [19], Baouendi & Rothschild [8] '80s)

3. $D, D' \subset \mathbb{C}^2$ (Diederich & Pinchuk [20] 1995)

4. $D, D' \subset \mathbb{C}^n$, $n \geq 2$, and f is continuous up to the boundary, i.e. $f \in A^0(D) = \mathcal{O}(D) \cap \mathcal{C}^0(\overline{D})$ (Diederich & Pinchuk [22, 23] 2003)

The previous results hold also when f is not a biholomorphism, but only a *proper* holomorphic map, i.e. if for any relatively compact set $K \Subset D'$, its pre-image $f^{-1}(K) \Subset D$ is relatively compact.

If $f = (f_1, \ldots, f_n)$, we denote its Jacobian determinant by

$$J_f(z) \ := \ \det \left(\frac{\partial f_\mu}{\partial z_\nu}(z) \right).$$

If f is a proper holomorphic map, both

$$E \ = \ \{z \in D \ : \ J_f(z) = 0\}$$

and $E' = f(E) \subset D'$ are analytic sets. Moreover, f is an m-branched analytic covering, and in particular for all $z' \in D' \setminus E'$, $\# f^{-1}(z') = m$.

To solve Problem 3.2, the most important tool is the reflection (or Schwarz) principle. First we state a generalization of the classical recflection principle in $\mathbb{C}$ and then we look at its generalizations in $\mathbb{C}^n$. Finally we state the main results of [22, 23] and outline the principal ideas underlying their proofs.

3.7.1 Reflection principle in $\mathbb{C}$

Let now $D, D' \subset \mathbb{C}$, and $f : D \to D'$ a biholomorphic (or proper holomorphic) function.

The extension problem being a local problem, we can assume the following situation. Let $\gamma \subset bD \subset \mathbb{C}$ be a real analytic curve in an open set U, $\gamma = \{z \in U \ : \ \rho(z) = 0\}$ (see Figure 3.1), where

$$\rho(z) \ = \ \rho(z, \overline{z}) \ = \ \sum_{k,l \geq 0} c_{kl} z^k \overline{z}^l,$$

with $c_{kl} = \overline{c_{lk}}$ and $d\rho \neq 0$ on γ. Similarly, we let ρ' be a local defining function for the domain D' on some open set $U' \subset \mathbb{C}$, so that $\gamma' = \{z \in U' : \rho'(z) = 0\}$ and, as sets, $f(\gamma) \subset \gamma'$. However, since f is biholomorphic (or proper), it follows that f extends continuously on $D \cup \gamma$.

To reflect the function f across γ, we can first complexify the equation of γ considering the equation:

$$\rho(z, \overline{w}) \ = \ \overline{\rho(w, \overline{z})} = 0.$$

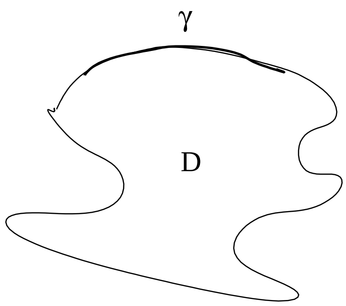


Figure 3.1: γ is a real analytic curve in the boundary of the domain D.

Then, from the implicit function theorem we get that if $\rho(z, \overline{w}) = 0$, then $z = \lambda(w)$ in a neighborhood U of γ, with λ being an antiholomorphic function of w. Thus w_0 is a point of the curve γ if and only if $\lambda(w_0) = w_0$. By definition λ switches $U \cap D$ and $U \setminus \overline{D}$. Observe also that $\lambda^2 = \mathrm{id}_U$, i.e. $\lambda^{-1} = \lambda$.

In a similar way we define the reflection λ' with respect to γ' and let $g = \lambda' \circ f \circ \lambda : U \setminus D \to U' \setminus D'$ (see Figure 3.2). It turns out that g is holomorphic and extends f through γ.

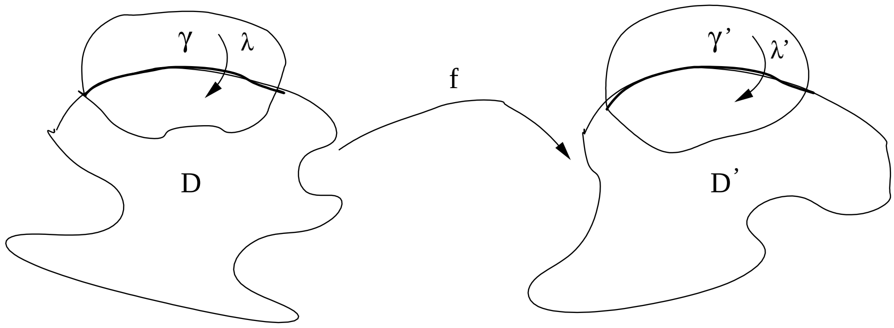


Figure 3.2: Reflection of the function f across γ

So the ingredients of the reflection principle in one variable are essentially

1. Reflection: antiholomorphic reflection with respect to a real analytic curve;

2 Continuation: if $D = D_+ \cup D_-$, $D_+ \cap D_- = \emptyset$, $\gamma = \overline{D}_+ \cap \overline{D}_-$, f_σ is holomorphic on D_σ, $\sigma = +, -$, and $f_+ = f_-$ on γ, then the function

$$F = \begin{cases} f_+ & on\ \overline{D}_+ \\ f_- & on\ \overline{D}_- \end{cases}$$

is holomorphic in D;

3 Initial boundary regularity of f.

We remark that the holomorphicity of the continuation (i.e. of the function F) can be proved, using continuity of f at the boundary by using Morera's theorem, as in the classical Schwarz theorem.

3.7.2 Reflection principle and extension theorems in $\mathbb{C}^n$, $n > 1$

Let $bD = \{\rho(z, \overline{z}) = 0\}$. We can consider again the complexification of ρ, $\rho(z, \overline{w}) = 0$. In several variables it is however no longer possible to apply the implicit function theorem to obtain z as a function of w (indeed, both $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ are vectors).

Before considering our case, let us consider the following situation. Let U be a domain in $\mathbb{C}^n$ containing the origin 0. Let $\rho_1(z, \overline{z}), \ldots, \rho_n(z, \overline{z})$ be real analytic function in U and let

$$M = \{z \in U \ : \ \rho_j(z, \overline{z}) = 0, \ j = 1, \ldots, n\}.$$

If $\overline{\partial}\rho_1 \wedge \cdots \wedge \overline{\partial}\rho_n \neq 0$ then M is a real analytic manifold of codimension n. Moreover M is a *totally real manifold*, i.e.

$$H_0(M) = T_0(M) \cap iT_0(M) = \emptyset.$$

From the equations

$$\begin{cases} \rho_1(z, \overline{w}) = 0 \\ \vdots \\ \rho_n(z, \overline{w}) = 0 \end{cases}$$

we may apply the implicit function theorem and get $z = \lambda(w)$ on M, with λ an antiholomorphic reflection.

Using the Edge of the Wedge theorem (see Theorem 2.9) we can prove the following reflection principle theorem.

Theorem 3.14 *Let $\rho_1, \ldots, \rho_n$ and $U^{\pm}$ be as before. Let $f : U^- \to \mathbb{C}^N$ ($N \geq n$) be a holomorphic map, and $M' \subset \mathbb{C}^N$ a totally real analytic manifold. Assume $f(M) \subset M'$ and $f \in \mathcal{O}(U^-) \cap \mathcal{C}^0(U^- \cup M)$. Then $f \in \mathcal{O}(U^- \cup M)$, i.e. f can be holomorphically extended across M.*

Proof. Reflect U^+ on U^- with antiholomorphic reflection function λ (and let λ' be the analogous antiholomorphic reflection function across M'), define $g = \lambda' \circ f \circ \lambda$, and apply the Edge of the Wedge theorem to f and g. $\square$

3.7.3 Extension in the strictly pseudoconvex case

Two preliminary results for holomorphic extension of functions in the pseudoconvex case are the following.

Theorem 3.15 (Fefferman 1974) *If D, D' are strictly pseudoconvex, bD, bD' are $\mathcal{C}^\infty$-smooth, and $f : D \to D'$ is a biholomorphism, then f extends to a diffeomorphism between $\overline{D}$ and $\overline{D}'$.*

Theorem 3.16 (Bell 1980) *If D, D' are weakly pseudoconvex, bD, bD' are $\mathcal{C}^\omega$-smooth, and $f : D \to D'$ is a biholomorphism, then $f \in \mathcal{C}^\omega(\overline{D})$.*

Assume that D and D' are strictly pseudoconvex domains in $\mathbb{C}^n$ with real analytic boundary. Let $a \in bD$. Assume $\frac{\partial\rho}{\partial z_n}(a) \neq 0$ (ρ being the defining function for bD) and define, for $j = 1, \ldots, n-1$,

$$T_j \;=\; \frac{\partial\rho}{\partial z_n}(z)\frac{\partial}{\partial z_j} \;-\; \frac{\partial\rho}{\partial z_j}(z)\frac{\partial}{\partial z_n}$$

The complex tangent space of bD at a is

$$H_a(\partial D) \;=\; \left\{ z \in \mathbb{C}^n \;:\; \sum_\nu \frac{\partial\rho}{\partial z_\nu} z_\nu = 0 \right\}.$$

Let $f : D \to D'$ be a biholomorphism. According to Fefferman's theorem f extends smoothly on bD. Since $\rho' \circ f(z) = 0$ if $z \in bD$, applying T_j ($j = 1, \ldots, n$) we get

$$\sum_{\nu=1}^{n} \frac{\partial\rho'}{\partial z'_\nu}(f(z))T_j f_\nu(z) \;=\; 0,$$

hence

$$\begin{cases} \rho'\left(f(z), \overline{f(z)}\right) = 0 & z \in \partial D \\ \sum\limits_{\nu=1}^{n} \frac{\partial\rho'}{\partial z'_\nu}\left(f(z), \overline{f(z)}\right) T_j f_\nu(z) = 0 & j = 1, \ldots, n-1 \end{cases} \tag{3.30}$$

Set $Tf = (T_1 f, \ldots, T_{n-1} f)$. Then choose a totally real manifold $M \subset bD$ ($a \in M$) and restrict f and Tf there. Since they are real analytic on M, we can reflect and use Edge of the Wedge theorem to prove that they extend holomorphically. Hence D, D' strongly pseudoconvex implies $f \in \mathcal{O}(\overline{D})$.

3.7.4 Holomorphic extension in dimension $n = 2$

Let us look more in details at what happens in dimension two. A strictly pseudoconvex domain D may be locally approximated, via a holomorphic change of coordinates, up to an $o(|z|^2)$ by the unit ball B^2 (which has the defining function $\rho(z) = |z_1|^2 + |z_2|^2 - 1$), for a proof refer to [68]. So, let us consider a holomorphic function $f : B^2 \to B^2$.

The system (3.30) becomes

$$\begin{cases} f_1(z)\overline{f_1(z)} + f_2(z)\overline{f_2(z)} = 1 & z \in \partial B^2 \\ T_1 f_1(z)\overline{f_1(z)} + T_1 f_2(z)\overline{f_2(z)} = 0 \end{cases} \tag{3.31}$$

The previous system can be considered as a linear system with respect to

$$\overline{f(z)} = \left(\overline{f_1(z)}, \overline{f_2(z)}\right)$$

Since its determinant

$$\Delta = \begin{vmatrix} f_1 & f_2 \\ T_1 f_1 & T_1 f_2 \end{vmatrix}$$

is non-zero (because f is a radial vector and $T_1 f$ is a tangent vector), then system (3.31) can be uniquely solved:

$$\overline{f(z)} = h\left(f(z), T_1 f(z)\right).$$

Let l_t be the complex line $\{z_2 = t\}$, where $t \in \mathbb{C}$ is near 1 and of modulus less than one (see Figure 3.3). Then

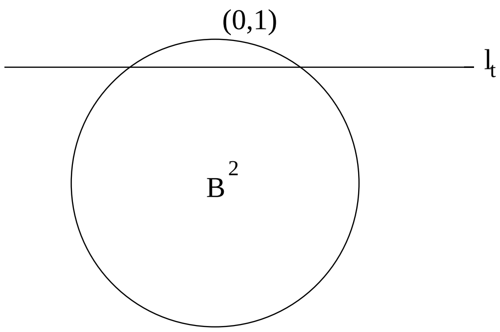


Figure 3.3: We restrict the function to the complex line $l_t = \{z_2 = t\}$.

$$l_t \cap bB^2 = \begin{cases} |z_1|^2 = 1 - |t|^2 \\ z_2 = t \end{cases}$$

Hence on $l_t \cap bB^2$

$$\overline{z}_1 = \frac{1-|t|^2}{z_1}$$

Using this last equality we obtain

$$T_1 f_j \;=\; \overline{t}\frac{\partial f_j}{\partial z_1} \;-\; \frac{1-|t|^2}{z_1}\frac{\partial f_j}{\partial z_2}$$

So $\overline{f} = h(f, T_1 f)$ becomes a rational function on $B^2 \cap l_t$, we can reflect and obtain a rational function on l_t. By changing l_t we obtain that f is rational. Since there are no poles in B^2 neither for f nor for f^{-1}, thus f is an automorphism of B^2.

3.7.5 Extension of proper holomorphic maps between strictly pseudoconvex $\mathcal{C}^\omega$-domains

To begin with, we want to avoid to rely on Fefferman's theorem to obtain regularity of the map up to the boundary. To this aim we have the following results:

Lemma 3.17 *Let $D, D' \subset \mathbb{C}^n$ be strictly pseudoconvex domains and let $f : D \to D'$ be a proper holomorphic map. Then f is $\frac{1}{2}$-Hölder ($f \in \mathcal{C}^{1/2}(\overline{D})$), i.e. there exists $c > 0$ such that for all $z', z'' \in D$ it follows*

$$|f(z') - f(z'')| \;\leq\; c|z' - z''|^{1/2}.$$

Proof. Step i) We remind that by Hardy-Littlewood lemma (for a proof see [54, 77]), for $0 < \alpha < 1$, if

$$\left|\frac{\partial f_\mu}{\partial z_\nu}(z)\right| \leq \frac{C}{(d(z, bD))^\alpha} \tag{3.32}$$

then $f \in \mathcal{C}^\alpha(\overline{D})$. Hence it suffices to prove the above inequality for f with $\alpha = 1/2$.

Step ii) We claim that $d(f(z), bD') \approx d(z, bD)$[1].

[1] by the notation $f \approx g$ we mean that there exist two positive constants $0 < c_1 < c_2$ such that

$$c_1 f \;\leq\; g \;\leq\; c_2 f$$

Indeed, if $\rho < 0$, ρ is plurisubharmonic in D, then (by Hopf lemma) $|\rho(z)| \geq Cd(z, bD)$. Moreover, since D' is strictly pseudoconvex, there exists a plurisubharmonic function ρ' on D' such that $|\rho'(z')| \approx d(z', bD')$. Then

$$d(f(z), bD') \ \approx \ |\rho'(f(z))| \ \geq \ Cd(z, bD).$$

If f is invertible we obtain the reverse inequality using f^{-1}. If f is proper, then it is a covering and the reverse inequality holds similarly.

Step iii) We remind the Schwarz lemma: if $\mathcal{U} \subset \mathbb{C}$ is the unit disc and $f : \mathcal{U} \to \mathcal{U}$ is holomorphic and $f(0) = 0$, then $|f'(0)| \leq 1$, $|f(z)| \leq |z|$ and for all $a \in \mathcal{U}$

$$|f'(a)| \ \leq \ \frac{1 - |f(a)|^2}{1 - |a|^2}.$$

It is easy to restate Schwarz lemma for other simply connected domains biholomorphic to the unit disc, e.g. let $\mathcal{U}_r \subset \mathbb{C}$ be the disc of radius r and V^- be the half plane of complex numbers with negative real part; the Schwarz lemma states that if $f : \mathcal{U} \to V^-$ is a holomorphic function with $f(0) = b$, $\mathsf{Re} b = -\delta$ then

$$|f'(0)| \ \leq \ \frac{2\delta}{r}.$$

In $\mathbb{C}^n$, with coordinates $z = ('z, z_n)$, $V_n^- = \mathbb{C}^{n-1} \times V^-$, consider a holomorphic function $f : \mathcal{U}_r \to V_n^-$ with

$$\mathsf{Re} f(0)_n = \mathsf{Re} b_n = -\delta, \quad f'(0) = v = ('v, v_n)$$

Then, by restricting f to the complex line

$$l_v \ = \ \{z = b + tv, \ v \in \mathbb{C}\}$$

and defining $\delta_V(b, v) = d(b, l_v \cap bV_n^-)$, then

$$\frac{|v|}{|v_n|} \ = \ \frac{\delta_V(b, v)}{\delta}, \quad |v| \ = \ \frac{|v_n|}{\delta}\delta_V(b, v)$$

and so, by Schwarz lemma

$$|v| \ = \ \frac{|v_n|}{\delta}\delta_V(b, v) \ \leq \ \frac{2\delta_V(b, v)}{r}$$

Step iv) We may now conclude the proof of the lemma by showing that inequality (3.32) holds. Let $a \in D$, $f(a) = b$, $|v| = 1$, $v \in T_a\mathbb{C}^n$, $v' = df_a(v)$ then

$$\left|\frac{\partial f}{\partial v}(a)\right| \ = \ |v'| \ \leq \ \frac{2\delta_{D'}(b, v')}{\delta_D(a, v)} \ \leq \ \frac{c\sqrt{d(b, bD')}}{d(a, bD)} \ \approx \ \frac{1}{(d(a, bD))^{1/2}}$$

the first inequality due to step iii), the second to the fact that we may assume D' to be convex (since it is strictly pseudoconvex and the statement is local), and the equivalence to step ii). □

3.7.6 Algebroid functions

Let as always $D, D' \subset \mathbb{C}^n$ be domains. Let $f : D \to D'$ be a proper holomorphic map, and $z' = f(z)$. Then the coordinates of z, $z_j = g_j(z')$ satisfy a polynomial equation

$$z_j^m + a_{j1}(z')z_j^{m-1} + \cdots + a_{jm}(z') \ = \ 0$$

with the coefficients $a_{ij} \in \mathcal{O}(D') \cap \mathcal{C}^0(\overline{D'})$. A function as $g_j(z')$, satisfying such a polynomial equation, is said an *algebroid function* of degree m.

It is a natural question whether algebroid functions satisfy some type of Schwarz lemma. The analogous for algebroid functions of the classical Schwarz lemma would say: *let $w = h(z)$ be an algebroid function of the unit disc $\mathcal{U} \subset \mathbb{C}$, with $h(0) = 0$, $|h(z)| < 1$. Then*

$$\Omega \Subset \mathcal{U} \quad \Rightarrow \quad \exists \alpha = \alpha(\Omega) \in (0, 1) \ |h(z)| < \alpha, \ \forall z \in \Omega?$$

This is trivially false, ad the function $w = \sqrt[m]{z}$ shows. We could allow the constant α to depend both on Ω and m, $\alpha = \alpha(\Omega, m)$, but again, the answer is no. Indeed, $(w - (1 - \varepsilon))(w - z) = 0$ is an algebroid function reaching all values up to module $1 - \varepsilon$. We could suppose that this is due to the fact that the equation satisfied by w is reducible, but also with a small modification to that algebroid function

$$(w - (1 - \varepsilon))(w - z) + \varepsilon^2 z^2 \ = \ 0$$

we can find irreducible algebroid functions whose module reaches values arbitrarily close to 1.

The main problem is that following analytic continuation around branching points outside Ω we follow different branches of the algebroid function which do not satisfy a Schwarz lemma (see Figure 3.4). Hence, to make a Schwarz lemma for algebroid functions hold true, we must consider only analytic continuation inside the relatively compact domain Ω, and not in $\mathcal{U}$, thus getting rid of branches that arise out of branching points near the boundary of the disc. We can now state the correct result, referring to [69] for more details.

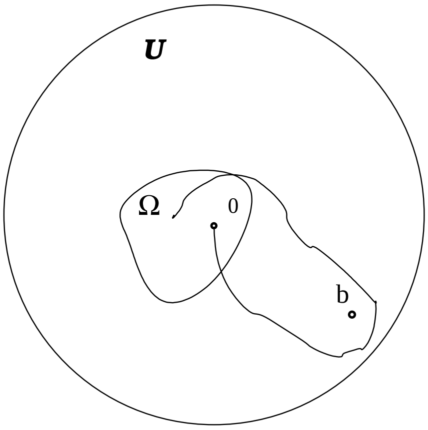


Figure 3.4: By analytic continuation around the branching point b outside of Ω, we get on branches of the algebroid function not satisfying a Schwarz lemma

Lemma 3.18 *Let $w = h(z)$ be an algebroid function of the unit disc $\mathcal{U} \subset \mathbb{C}$, of degree m, with $h(0) = 0$, $|h(z)| < 1$, and $\Omega \Subset \mathcal{U}$. Then, considering analytic continuation of h from 0 only inside Ω,*

$$\exists \alpha = \alpha(\Omega, m) \in (0,1) \ |h(z)| < \alpha, \ \forall z \in \Omega.$$

3.7.7 Edge of the Wedge theorem for the cotangent bundle

Let $D, D' \subset \mathbb{C}^n$ be domains with smooth boundaries, and $f : D \to D'$ be a biholomorphism. We can extend f to a holomorphic map $F : D \times \mathbb{CP}^{n-1} \to D' \times \mathbb{CP}^{n-1}$ defined by $F(z,p) = (f(z), df_z(p))$. We define

$$M \ = \ \{(z,p) \ : \ z \in bD, p \in H_z(bD)\} \ \subset \ b(D \times \mathbb{CP}^{n-1}).$$

Our aim will be to prove that M can be seen as the edge of a wedge, in order to apply the edge of the wedge theorem.
Claim. M is a totally real manifold of real dimension $2n-1$.
Proof. Up to a holomorphic change of coordinates, we may suppose that $0 \in bD$ and that the defining function of D is of the form $\rho(z) = 2x_n + |z|^2 + O(3)$. So $H_0(\partial D) = \{z_n = 0\}$.

$$H_a(\partial D) \ = \ \left\{ \sum_{\nu=1}^{n} \frac{\partial \rho}{\partial z_\nu}(a) z_\nu = 0 \right\}$$

hence in $H_a(\partial D)$

$$z_n = -\sum_{\nu=1}^{n} \frac{\frac{\partial\rho}{\partial z_\nu}(a)}{\frac{\partial\rho}{\partial z_n}(a)} z_\nu$$

Thus, using the equation of ρ, we get

$$p_\nu = -\frac{\frac{\partial\rho}{\partial z_\nu}(a)}{\frac{\partial\rho}{\partial z_n}(a)} = -\overline{z}_\nu + O(2)$$

Then

$$\begin{aligned} T_{(0,0)}M &= \{x_n = 0,\ p_\nu = -\overline{z}_\nu,\ \nu = 1, \ldots, n-1\} \\ iT_{(0,0)}M &= \{y_n = 0,\ p_\nu = \overline{z}_\nu,\ \nu = 1, \ldots, n-1\} \end{aligned}$$

So $T_{(0,0)}M \cap (iT_{(0,0)}M) = \{0\}$. $\square$

Thus we have $F : D \times \mathbb{CP}^{n-1} \to D' \times \mathbb{CP}^{n-1}$. We need to show that $F(M) \subset M'$, and that M is an edge of a wedge along which F is continuous at M.

By contradiction, suppose there exist a sequence of points $\{z_\nu\} \subset D$ and a sequence of complex hyperplanes $\{p_\nu\}$ such that $z_\nu \in p_\nu$ such that, when $\nu \to \infty$, $z_\nu \to 0 \in M$, $p_\nu \to H_0\partial D$, $f(z_\nu) \to f(0)$ (since we already proved that $f \in \mathcal{C}^{1/2}(\overline{D})$), but $df_{z_\nu}(p_\nu) \not\to H_{f(0)}\partial D'$. In order to continue the proof, we need to introduce the "scaling method".

3.7.8 Scaling method

Let us consider $\mathbb{C}^n$ with coordinates $z = ('z, z_n)$, and the biholomorphism

$$('z, z_n) \to \left(\frac{'z}{z_n+1}, \frac{z_n-1}{z_n+1}\right)$$

which sends $B^n = \{|z| < 1\}$ onto $\widetilde{B}^n = \{x_n + |'z|^2 < 0\}$, an unbounded representation of the ball.

Remark 3.2 $\widetilde{B}^n$ is invariant under rescaling $('z, z_n) \mapsto (\lambda' z, \lambda^2 z_n)$, $\lambda \in \mathbb{C}$.

Let $D \subset \mathbb{C}^n$ be of the type

$$D = \left\{z = ('z, z_n) \in \widetilde{B}^n \;\middle|\; \rho(z) = x_n + |z|^2 + O(3) < 0\right\}$$

and consider the sequence $Z^\nu = ('0, -\delta_n u)$, converging to zero normal to the tangent space of M in 0. For every $\nu \in \mathbb{N}$, consider the following change of coordinates

$$\begin{cases} 'z = \sqrt{\delta_\nu}\, '\tilde{z} \\ z_n = \delta_\nu\, \tilde{z} \end{cases}$$

which sends the point z_ν to the point $('0,-1)$ and the domain D to

$$D_\nu = \left\{ z = ('z, z_n) \in \widetilde{B}^n \;\middle|\; \frac{\rho(z)}{\delta_\nu} = \frac{\delta_\nu \tilde{x}_n + \delta_\nu |'\tilde{z}|^2 + \delta_\nu^2 |\tilde{z}_n|^2 + O(\delta_\nu^3, \tilde{z}^3)}{\delta_\nu} < 0 \right\}$$

Thus $D_\nu \to \widetilde{B}^n$ uniformely on compacts of $\mathbb{C}^n$ in the Hausdorff topology.

If the sequence $z^\nu \to 0$ not normal to the tangent space of M in 0, we first make a rotation to send the nearest point of the boundary γ^ν to 0, and then proceed as before.

We apply the scaling method to both D and D'. Now we can choose a wedge in $D \times \mathbb{CP}^{n-1}$ so that, after the change of coordinates

$$z_n = \sum_{j=1}^{n-1} p_j^\nu z_j \;\longrightarrow\; \tilde{z}_n = \sum_{j=1}^{n-1} \frac{p_j^\nu}{\sqrt{\delta_\nu}} \tilde{z}_j$$

we can find a wedge (to do this $\frac{1}{2}$-Hölder continuity is needed) where all $\frac{p_j^\nu}{\sqrt{\delta_\nu}}$ remain bounded, so that the planes converge to

$$\sum_{j=1}^{n-1} p_j' \tilde{z}_j' \;=\; 0,$$

i.e. the planes in D' become vertical (since scaling is bigger in one direction rather than in the other, see figure 3.5). Hence some boundary point (not at infinity) a of bD is sent to a point at infinity by any limit map of the sequence given by f composed with the scalings. However, such a limit map is an automorphism of the (unbounded) ball which fixes infinity and thus we reach a contradiction. So we proved that F is continuous along a wedge at the edge M.

The scaling method may also be used to prove some other useful results such as

Theorem 3.19 (Wong-Rosay) *Let $D \Subset \mathbb{C}^n$, $f^\nu \in \mathrm{Aut}(D)$, $\nu \in \mathbb{N}$, $a \in D$, $a^\nu = f^\nu(a) \to a^0 \in bD$. If a^0 is a strictly pseudoconvex point of bD, then $D \approx \mathbb{B}^n$.*

Theorem 3.20 (Poincaré-Alexander) *All proper holomorphic maps $\mathbb{B}^n \mapsto \mathbb{B}^n$ are biholomorphisms (i.e. holomorphic automorphisms).*

The statement of the previous theorem is not true even in the simple case when the unit ball is replaced by the unit polidisc Δ^n: e.g. the map defined by $(z_1, \ldots, z_n) \mapsto (z_1^{m_1}, \ldots, z_n^{m_n})$ is a proper holomorphic map, but not a biholomorphism.

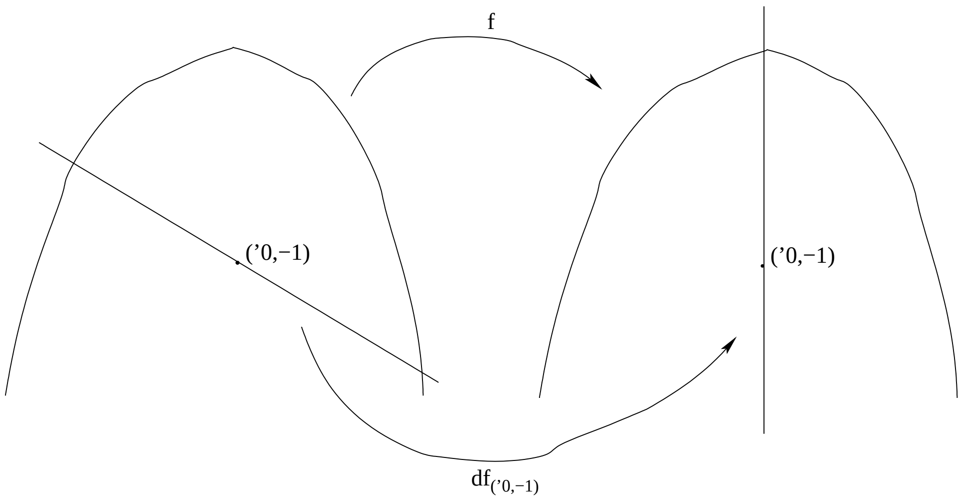


Figure 3.5: The limit plane through $('0,-1)$ is sent by the limit map $df_{('0,-1)}$ to a vertical plane through $('0,-1)$

3.7.9 Non-pseudoconvex case

We are now interested in the non-pseudoconvex case. In this setting, a very important notion is that of Segre variety.

Definition: Let $D \subset \mathbb{C}^n$ be a domain with smooth boundary, defined by $D = \{\rho(z,\overline{z}) < 0\}$. For $w \in \mathbb{C}^n$, the set

$$Q_w \;=\; \{z \in \mathbb{C}^n \;:\; \rho(z,\overline{w}) = 0\}$$

is called the *Segre variety* with respect to w.

3.7.10 Main Theorems

Definition: Let M be as before and let $p \in M$. The manifold M is said to be of *D'Angelo finite type* at p if there exists a constant $C > 0$ such that for all non-constant one-dimensional holomorphic curves

$$\varphi : \mathcal{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\} \to \mathbb{C}^n$$

such that $\varphi(0) = p$ it follows that $v(\rho \circ \varphi)/v(\varphi) \leq C$, where, for a function h defined near $0 \in \mathcal{U}$, the symbol $v(h)$ denotes the multiplicity of h at 0. The manifold M is said to be of *D'Angelo finite type* if it is of D'Angelo finite type at each point $p \in M$.

Definition: Let $M \subset \mathbb{C}^n$ be a real hypersurface and $p \in M$. M is said to be *minimal at p in the sense of Trépreau* if there does not exist a neighborhood

$V \ni p$ and a complex hypersurface $W \subset V$ passing throught p such that $W \subset M$. M is said to be *minimal in the sense of Trépreau* if it is minimal at each $p \in M$.

Remark 3.3 If M is of D'Angelo finite type, it is minimal in the sense of Trépreau. Hence every point $z_0 \in M$ has a neighborhood V s.t. every CR-function on M extends holomorphically either on V^+ or V^-, where

$$V^{\pm} = \{z \in V \mid \pm\rho(z, \overline{z}) > 0\},$$

and whether it extends on V^+ or on V^- does not depend upon the function to be extended.

Theorem 3.21 (Diederich-Pinchuk [22, 23]**)** *Let $D, D' \Subset \mathbb{C}^n$ be domains with $\mathcal{C}^\omega$-boundaries. Let $f : D \to D'$ be a proper holomorphic map continuous up to the boundary. Then $f \in \mathcal{O}(\overline{D})$.*

Remark 3.4 If bD is $\mathcal{C}^\omega$ then it is of D'Angelo finite type.

Theorem 3.22 (Diederich-Pinchuk [22, 23]**)** *Let M, M' be real analytic hypersurfaces in $\mathbb{C}^n$. Let $f : M \to M'$, $f \in \mathcal{C}^0(M) \cap CR(M)$. Then $f \in \mathcal{O}(M)$.*

Observe that Theorem 3.22 implies Theorem 3.21 (it suffices to consider $f_{|_{bD}}$). We shall give only the main ideas of the proof of Theorem 3.21 and Theorem 3.22.

Let $M = \partial D$, $M' = \partial D'$.

Claim. Let

$$\Sigma = \{z \in M \mid f \in \mathcal{O}(\{z\})\} \subset M$$

be the maximal set at which points f extends holomorphically. Then Σ is dense[2] in M.

Proof. Obviously if M is strictly pseudoconvex, then $\Sigma = M$ and there is nothing to prove. So let us decompose the boundaries

$$M = M^+ \cup M^- \cup M^0; \quad M' = M'^+ \cup M'^- \cup M'^0,$$

where M^+ are the strictly pseudoconvex points of M, M^- are the points of M with eigenvalues of different signature in the Levi-form, and M^0 are the points of M with degenerate Levi-form (same definitions for M'^+, M'^-, M'^0).

Points in M^- are in the holomorphic convex hull of D, hence all holomorphic functions extend beyond them: $M^- \subset \Sigma$.

[2]The final aim will be to prove that $\Sigma = M$.

Points in $M^+ \cap f^{-1}(M'^+)$ are strictly pseudoconvex points mapped to strictly pseudoconvex points, hence the (local) theorems of the previous section imply extension holds and $M^+ \cap f^{-1}(M'^+) \subset \Sigma$. There are no points in $M^+ \cap f^{-1}(M'^-)$ since strictly pseudoconvex points may not be mapped to points where the Levi-form has negative eigenvalues. We need to show what happens at points in $M^+ \cap f^{-1}(M'^0)$. If this set has non-empty interior, then it follows that f is constant (hence not proper, contradiction). Thus Σ is dense in M. $\square$

Let us consider the local setting: $0 \in M$, $0' \in M'$, $f(0) = 0'$, $\mathcal{U} \ni 0$, $\mathcal{U}' \ni 0'$ open neighborhoods. We define

$$\mathcal{U}^{\pm} \;=\; \{z \in \mathcal{U} \;:\; \pm\rho(z) > 0\}$$

$f \in \mathcal{C}^{\omega}(\mathcal{U}) \cap \mathcal{O}(\mathcal{U}^-)$. Fixed $w \in \mathcal{U}$, let

$$Q_w \;=\; \{z \in \mathcal{U} \mid \rho(z, \overline{w}) = 0\}$$

be the (local) Segre variety for w. Up to a change of coordinates, we may assume $\frac{\partial \rho}{\partial z_n}(0) \neq 0$ and hence, in view of the implicit function theorem, up to shrinking the neighborhood $\mathcal{U}$,

$$Q_w \;=\; \{('z, z_n) \in \mathbb{C}^n \mid z_n = h('z, \overline{w})\}\,.$$

Let $g : \mathcal{U} \to \mathcal{U}'$ be a biholomorphism. Then

$$\rho'\left(g(z), \overline{g(z)}\right) \;=\; \alpha\,(z, \overline{z})\,\rho(z, \overline{z}),$$

where $\alpha\,(z, \overline{z})$ is a real analytic never-vanishing function on $\mathcal{U}$. Hence

$$\rho\left(g(z), \overline{g(w)}\right) \;=\; \alpha\,(z, \overline{w})\,\rho\,(z, \overline{w})$$

and so $z \in Q_w \;\Rightarrow\; g(z) \in Q_{g(w)}$, i.e. $g(Q_w) \subset Q_{g(w)}$. Since g is a biholomorphism, the opposite inclusion holds true, and we have shown that Segre varieties are invariant under biholomorphisms.

We denote by Q^c_w the canonical component of $Q_w \cap \mathcal{U}^-$ (i.e. the one containing the reflection w^s of w across M, see Figure 3.6). We want to extend the graph of f. Define

$$F \;=\; \left\{(w, w') \in \mathcal{U}^+ \times \mathcal{U}' \;\mid\; f(Q^c_w) \subset Q_{w'}\right\}$$

Proposition 3.23 *F is an analytic set in $\mathcal{U}^+ \cap \mathcal{U}'$.*

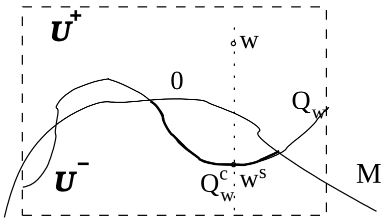


Figure 3.6: The simmetric point w^s of w is the intesection of the vertical line by w and the Segre variety Q_w

Proof. Q_w is defined by $z_n = h('z, \overline{w})$.

$$f(z) \in Q_{w'} \quad \Leftrightarrow \quad \rho'(f('z, h('z, \overline{w})), \overline{w}') \;=\; 0$$

thus F is defined by infinitely many analytic equations and hence only a finite number of equations are needed in order to define the set F which, in particular, is analytic. □

3.7.11 Properties of Segre varieties

It is worthwhile observing the following properties of Segre varieties:

1. Since $\rho(z, \overline{w}) = \overline{\rho(w, \overline{z})}$,

$$z \in Q_w \quad \Longleftrightarrow \quad w \in Q_z$$

2. Since $\rho(z, \overline{z}) = 0$ defines M,

$$z \in Q_z \quad \Longleftrightarrow \quad z \in M$$

Example 3.1 Let us consider, in $\mathbb{C}^2$, the function $\rho(z) = 2x_2 + |z_1|^2$ (defining function for a 3-dimensional sphere). Then

$$Q_w \;=\; \left\{ z = (z_1, z_2) \in \mathbb{C}^2 \;:\; \rho(z, \overline{w}) = z_2 + \overline{w}_2 + z_1 \overline{w}_1 = 0 \right\}.$$

Different w's give different Segre varieties.

Example 3.2 The previous is not always the case. Indeed, consider —again in $\mathbb{C}^2$— the function $\rho(z) = 2x_2 + |z_1|^4$. Then

$$Q_w \;=\; \left\{ z = (z_1, z_2) \in \mathbb{C}^2 \;:\; \rho(z, \overline{w}) = z_2 + \overline{w}_2 + z_1^2 \overline{w}_1^2 = 0 \right\}.$$

The points (w_1, w_2) and $(-w_1, w_2)$ correspond to the same Segre variety.

Keeping in mind the last example, for every point $w \in \mathcal{U}$ (or $w \in \mathbb{C}^n$, if our interest is global), we define

$$I_w = \{z \in \mathcal{U} \mid Q_z = Q_w\}.$$

By definition and by property 1. above, it follows

$$I_w = \bigcap_{\xi \in Q_w} Q_\xi.$$

Indeed, if $z \in I_w$, $Q_z = Q_w$, then for all $\xi \in Q_w = Q_z$, $z \in Q_\xi$.

Moreover, by definition and property 2. above

$$w \in M \implies I_w \subset M.$$

Indeed if $z \in I_w$, $Q_z = Q_w \ni w$, $z \in Q_w = Q_z$, and so $z \in M$.

In the hypothesis of Theorem 3.21 and Theorem 3.22 M is of finite type, so there are no analytic varieties in M. Hence I_w is discrete, for all $w \in M$.

3.7.12 Complex structure of the set of Segre varieties

Let $\lambda : w \to Q_w$ be the *Segre map*. We can always choose coordinates, called *normal coordinates* such that

$$\rho(z, \overline{z}) = 2x_n + \sum_{|k|,|l| \geq 1} b_{kl}(y_n)\, {}'z^k\, {}'\overline{z}^l$$

Hence Q_w is defined by

$$\rho(z, \overline{w}) = z_n + \sum_{|k|,|l| \geq 1} b_{kl}\left(\frac{z_n - \overline{w}_n}{2i}\right) {}'z^k\, {}'\overline{w}^l = 0.$$

By the implicit function theorem Q_w is locally defined by

$$z_n = \sum_{k \in \mathbb{N}} \overline{\lambda(w)}\, {}'z^k$$

(of course $\lambda_0(w) = -\overline{w}_n$) Thus $Q_w \cong \{\lambda_k(w)\}_{k \in \mathbb{N}^{n-1}}$. Hence

$$Q_w = Q_z \iff \lambda_k(z) = \lambda_k(w)\ \forall k \in \mathbb{N}^{n-1}$$

So, let us consider in $\mathcal{U} \times \mathcal{U}$ the equation

$$\lambda_k(z) - \lambda_k(w) = 0.$$

There is an $l_0 > 0$ (indipendent of z and w) such that if $\lambda_k(z) = \lambda_k(w)$ for all $|k| \leq l_0$, then $\lambda_k(z) = \lambda_k(w)$ for all $k \in \mathbb{N}^{n-1}$.

So, if we define

$$N \;=\; \#\left\{k \in \mathbb{N}^{n-1} \;:\; |k| \leq l_0\right\},$$

we can consider a map $\lambda : \mathcal{U} \to \mathbb{C}^N$

$$\lambda(w) \;=\; (\lambda_k(w))_{|k|\leq l_0}\,.$$

The map λ is a local proper holomorphic map with image in a relatively compact open set V of $\mathbb{C}^N$, and so induces a complex structure on the set of Segre varieties.

3.7.13 Extending the graph of f

We get back to our problem. The function $f : M \to M'$ (see Figure 3.7) is

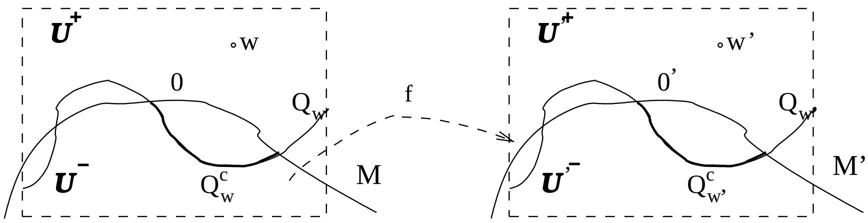


Figure 3.7: We are willing to extend holomorphically the function f to a function defined on the neighborhood $\mathcal{U}$ with the aid of Segre varieties

continuous and CR, $f(0) = 0'$, $\mathcal{U} \ni 0$, $\mathcal{U}' \ni 0'$, $f(\mathcal{U}^-) \subset \mathcal{U}'$. Our aim is to extend the graph of f, Γ_f, in $\mathcal{U} \times \mathcal{U}'$. The set

$$F \;=\; \left\{(w, w') \in \mathcal{U}^+ \times \mathcal{U}' \;:\; f(Q^c_w) \subset Q_{w'}\right\}$$

is such that $F \supset \Gamma_f$ near points of extendability.

Indeed, the set $\Sigma = \{z \in M \;:\; f \text{ is holomorphic at } z\}$ is dense in M, thus f extends holomorphically on $\mathcal{U}$ and $\Gamma_f \subset F$. Moreover this implies $\dim_{\mathbb{C}} F \geq n$.

Since M' is of finite type, $\lambda' : w' \mapsto Q_{w'}$ is a finite map,

$$\alpha \;=\; \left\{(w, w') \in \mathcal{U}^+ \times \mathcal{U}' \;:\; J_{f|_{Q^c_w}} \equiv 0\right\}$$

is a discrete set and $f(Q^c_w) \subset Q'_{w'}$ completely determines the Segre set $Q'_{w'}$. Hence $\dim_{\mathbb{C}} F \leq n$. Thus $\dim_{\mathbb{C}} F = n$.

So the set F is a good candidate for being the extension of the graph of f.

Definition: Let $V \subset \mathbb{C}^n$, $V' \subset \mathbb{C}^n$ be domains, and $A \subset V \times V'$ a relation such that $\dim_{\mathbb{C}} A = n$ and $\pi_1 : A \to V$ is proper. Such an A is called a *holomorphic correspondence*. The natural map $\widehat{A} = \pi_2 \circ \pi_1^{-1} : V \to V'$ is multivalued.

If we show that $\pi_1 : F \to \mathcal{U}^+$ is proper, then F is a holomorphic correspondence and from this it follows that F extends to a correspondence $\widehat{F}$ in $\mathcal{U} \times \mathcal{U}'$. $\widehat{F}$ satisfies the following *invariance property*: $\widehat{F}(Q_w \cap \mathcal{U}) \subset Q'_{w'}$ for all $(w, w') \in F$. The invariance property implies the existence of a map $\Phi : \mathcal{S} \to \mathcal{S}'$ between the spaces of Segre varieties in $\mathcal{U}$ and in $\mathcal{U}'$ which makes the diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\Phi} & \mathcal{S}' \\ {\scriptstyle\lambda}\uparrow & & \uparrow{\scriptstyle\lambda'} \\ \mathcal{U} & \xrightarrow[\widehat{F}]{} & \mathcal{U}' \end{array}$$

commutative. One can prove that $\widehat{F}$ is holomorphic near 0, i.e. that it is the seeked extension of f, but the proof is quite technical (see [22, 23] for details).

3.7.14 Conclusion of proof

We have already seen that $M = \Sigma \cup E$, where Σ is a dense set in which there is extension. We need to prove that the bad set E is empty.

Fix a point $0 \in E$. Then there is a neighborhood $\mathcal{U} \ni 0$ and a sequence of analytic sets $\sigma_\nu \subset \mathcal{U}$ ($\dim_{\mathbb{C}} \sigma_\nu \geq 1$) such that their cluster sets are in E: $\mathrm{cl}(\sigma_\nu) \subset E \subset M$. Searching for a contradiction, the following problem naturally arises:

Problem 3.3 *Let $N \subset \mathcal{U} \subset \mathbb{C}^n$ be a real analytic surface of finite type, and let $A_\nu \subset \mathcal{U}$ be analytic sets such that* $\mathrm{cl}(A_\nu) \neq \emptyset$. *Does this imply that* $\mathrm{cl}(A_\nu) \not\subset N$*?*

The most general such result known is the following:

Theorem 3.24 *Let $N \subset \mathcal{U} \subset \mathbb{C}^n$ be a closed set, $a \in N$. Suppose there is a plurisubharmonic peak function φ for a in N (i.e. $\varphi(a) = 0$, $\varphi < 0$ on $N \setminus \{a\}$). Let $A_\nu \subset \mathcal{U}$ ($a \in A_\nu$) be analytic sets,* $\dim_{\mathbb{C}} A_\nu \geq 1$. *Then if* $\mathrm{cl}(A_\nu) \neq \emptyset$, $\mathrm{cl}(A_\nu) \not\subset N$.

In the following cases N verifies the condition of Theorem 3.24:

1. N is a totally real manifold;
2. N is a strictly pseudoconvex hypersurface;
3. N is a pseudoconvex hypersurface (see [19]);
4. $N \subset \mathbb{C}^2$ is real analytic of finite type.

Thanks to Theorem 3.24 we can conclude our proof. Indeed, if $E \neq \emptyset$, then $\max_{z\in E}|z| = |a|$, $a \in E$ (E is closed). Then $\varphi(z) = \mathsf{Re} < z, a > -|a|^2$ is a peak function for a on E, and Theorem 3.24 gives a contradiction. $\square$

3.7.15 Final considerations

Let the sets A_ν (of codimension 1) be defined in $\mathcal{U}$ by

$$A_\nu \;=\; \{z \in \mathcal{U} \;:\; g_\nu(z) = 0\}$$

where $g_\nu \in \mathcal{O}(\mathcal{U})$. Up to shrinking $\mathcal{U}$, we may suppose all g_ν to be bounded. Multiplying them by small constants, we may suppose $|g_\nu| \leq 1$. Up to choosing a subsequence, the sequence converges: $g_\nu \to g_0 \in \mathcal{O}(\mathcal{U})$. Two cases arise:

1. $g_0 \not\equiv 0$. Then $A_\nu \to A_0 = \{z \in \mathcal{U} \;:\; g_0 = 0\}$.
2. $g_0 \equiv 0$. Bad case.

More generally, if the sets A_ν are of codimension r:

$$A_\nu \;=\; \{z \in \mathcal{U} \;:\; g_{\nu,j}(z) = 0,\; j = 1, \ldots, r\}$$

again we may suppose the functions $g_{\nu,j}$ equibounded ($|g_{\nu,j}| \leq 1$) and converging ($g_{\nu,j} \to g_j$). Define

$$A_0 \;=\; \{z \in \mathcal{U} \;:\; g_j(z) = 0,\; j = 1, \ldots, r\}$$

If $\dim_{\mathbb{C}} A_0 = n - r$ (good case) then $A_\nu \to A_0$. What happens if the codimension is smaller (i.e. some of the g_j's are proportional or are identically zero)?

Example 3.3 In $\mathbb{C}^3$ consider two families

$$\pi_{\nu,1} \;=\; \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \;\middle|\; \frac{1}{\nu} z_1 + z_3 = 0\right\}$$

and

$$\pi_{\nu,2} \;=\; \left\{(z_1,z_2,z_3)\in\mathbb{C}^3 \;\middle|\; \frac{1}{\nu}z_2+z_3=0\right\}$$

of complex planes which intersects in the complex lines

$$l_\nu \;=\; \pi_{\nu,1}\cap\pi_{\nu,2} \;=\left\{(t,t,z_3)\in\mathbb{C}^3 \;\middle|\; \frac{1}{\nu}t+z_3=0\right\}.$$

Then the lines l_ν converge to the complex line

$$l_\infty \;=\; \left\{(t,t,0)\in\mathbb{C}^3\right\}$$

while the set defined by the limit function is the complex plane

$$l_0 \;=\; \left\{(z_1,z_2,0)\in\mathbb{C}^3\right\}$$

Theorem 3.25 *Let $\mathcal{U}\Subset\mathbb{C}^n$, $\mathcal{V}\Subset\mathbb{C}^m$ be open sets. Let $g_j(z,w)\in\mathcal{O}(\mathcal{U}\times\mathcal{V})$, $j=1,\dots,r$, and*

$$A_w \;=\; \{w\in\mathcal{U} \;:\; g_j(z,w)=0,\; j=1,\dots,r\}\,,$$

$$\mathcal{S} \;=\; \{w\in\mathcal{V} \;:\; \dim_{\mathbb{C}} A_w > n-r\}\,.$$

Then for every $\mathcal{U}_1\Subset\mathcal{U}$ and $\mathcal{V}_1\Subset\mathcal{V}$, there exists a constant $c=c(\mathcal{U}_1,\mathcal{V}_1)$ such that

$$\mathrm{vol}_{2(n-r)}\left(A_w\cap\mathcal{U}_1\right) \;<\; c, \qquad \forall w\in\mathcal{V}_1\setminus\mathcal{S}.$$

Proof. The proof is by induction on r.

Base step ($r=1$). Let $g\in\mathcal{O}(\mathcal{U}\times\mathcal{V})$ and

$$A_w \;=\; \{z\in\mathcal{U} \;:\; g(z,w)=0\}$$

If $0\in\mathcal{S}$, $w^\nu\notin\mathcal{S}$, $w^\nu\to 0$, then $g(z,w)=\sum_k \alpha_k(w)z^k$, with $\alpha_k(0)=0$ for all k (since $g(z,0)\equiv 0$) and $g(z,w^\nu)\not\equiv 0$ for all ν. By shrinking the neighborhoods, we may suppose $g\in\mathcal{O}\left(\overline{\mathcal{U}_1\times\mathcal{V}_1}\right)$. Since this is a Nötherian ring, there is an $N\in\mathbb{N}$ such that

$$\alpha_k(w) \;=\; \sum_{j\le N} h_{jk}(w)\alpha_j(w) \qquad \forall k\in\mathbb{N}. \tag{3.33}$$

Moreover, up to extracting a subsequence, there is $l\in\mathbb{N}$ such that

$$|\alpha_j(w^\nu)| \;\le\; |\alpha_l(w^\nu)| \qquad \forall j\le N. \tag{3.34}$$

From (3.33) and (3.34) it follows that for each $k \in \mathbb{N}$ there exists a constant c_k such that

$$\frac{|\alpha_k(w^\nu)|}{|\alpha_l(w^\nu)|} \leq c_k \quad \forall k, \nu.$$

Hence

$$\widetilde{g}(z,w) = \frac{g(z,w)}{\alpha_l(w)} = \sum_{k=0}^{\infty} \frac{\alpha_k(w)}{\alpha_l(w)} z^k = \sum_{k=0}^{\infty} \widetilde{\alpha}_k(w) z^k$$

Since $\widetilde{\alpha}_l \equiv 1$, $\widetilde{g} \not\equiv 0$, and the sets converge to a set of the right dimension.

Inductive step. All is needed in the inductive step is to resolve the singularity in zero. □

The answer to Problem 3.3 is positive if the dimension of A_ν is greater than or equal to something. How big is needed? The question has positive answer if

$$\dim_{\mathbb{C}} A_\nu \geq \frac{n}{2}$$

But what happens if the dimension of A_ν is less than that? An important step toward answering this question is the following.

Theorem 3.26 (Tumanov) *Let $N \subset \mathbb{C}^n$ be a real analytic manifold of finite type. Then N is stratified: $N = \cup S_j$, with $\dim_{\mathbb{C}} S_j = j$ such that for every j there exists a real analytic hypersurface $N_j \supset S_j$ with non-degenerate Levi-form.*

Chapter 4

Cohomology vanishing and extension problems for semi q-coronae

This chapter is based on [82].

4.1 Introduction

Let X be a (connected and reduced) complex space. We recall that X is said to be *strongly q-pseudoconvex* in the sense of Andreotti-Grauert [3] if there exists a compact subset K and a smooth function $\varphi : X \to \mathbb{R}$, $\varphi \geq 0$, which is strongly q-plurisubharmonic on $X \setminus K$ and such that:

a) $0 = \min_X \varphi < \min_K \varphi$;

b) for every $c > \max_K \varphi$ the subset

$$B_c = \{x \in X : \varphi(x) < c\}$$

is relatively compact in X.

If $K = \emptyset$, X is said to be *q-complete*. We remark that, for a space, being 1-complete is equivalent to being Stein.

Replacing the condition b) by

b') for every $0 < \varepsilon < \min_K \varphi$ and $c > \max_K \varphi$ the subset

$$B_{\varepsilon,c} = \{x \in X : \varepsilon < \varphi(x) < c\}$$

is relatively compact in X,

we obtain the notion of *q-corona* (see [3, 5]).

A q-corona is said to be *complete* whenever $K = \emptyset$.

The extension problem for analytic objects defined on q-coronae was studied by many authors (see e.g. [33, 84, 88, 89, 94]). In this chapter (and the following one) we deal with the larger class of the semi q-coronae which are defined as follows. Consider a strongly q-pseudoconvex space (or, more generally, a q-corona) X, and a smooth function $\varphi : X \to \mathbb{R}$ displaying the q-pseudoconvexity of X. Let $B_{\varepsilon,c} \subset X$ and let $h : X \to \mathbb{R}$ be a pluriharmonic function (i.e. locally the real part of a holomorphic function) such that $K \cap \{h = 0\} = \emptyset$. A connected component of $B_{\varepsilon,c} \smallsetminus \{h = 0\}$ is, by definition, a *semi q-corona*.

Another type of semi q-corona is obtained by replacing the zero set of h with the intersection of X with a Levi-flat hypersurface. More precisely, consider a closed strongly q-pseudoconvex subspace X of an open subset of $\mathbb{C}^n$ and the q-corona $C = B_{\varepsilon,c} = B_c \smallsetminus \overline{B}_\varepsilon$. Let H be a Levi-flat hypersurface of a neighborhood U of $\overline{B}_c$ such that $H \cap K = \emptyset$. The connected components C_m of $C \smallsetminus H$ are called semi q-coronae.

In both cases the semi q-coronae are differences $A_c \smallsetminus \overline{A}_\varepsilon$ where A_c, A_ε are strongly q-pseudoconvex spaces. Indeed, the function $\psi = -\log h^2$ (respectively $\psi = -\log \delta_H(z)$, where $\delta_H(z)$ is the distance of z from H) is plurisubharmonic in $W \smallsetminus \{h = 0\}$ (respectively $W \smallsetminus H$) where W is a neighborhood of $B_c \cap \{h = 0\}$ (respectively $B_c \cap H$). Let $\chi : \mathbb{R} \to \mathbb{R}$ be an increasing convex function such that $\chi \circ \varphi > \psi$ on a neighborhood of $B_c \smallsetminus W$. The function $\Phi = \sup(\chi \circ \varphi, \psi) + \varphi$ is an exhaustion function for $B_c \smallsetminus \{h = 0\}$ (for $B_c \smallsetminus H$) and it is strongly q-plurisubharmonic in $B_c \smallsetminus (\{h = 0\} \cup K)$ (in $B_c \smallsetminus H \cup K$).

The interest for domains whose boundary contains a “Levi-flat part” originated from Theorem 3.3 on the extension for CR-functions proved in [65] (see also [57, 59, 93]).

Using cohomological techniques developped in [3, 5–7, 12] we prove that, under appropriate regularity conditions, holomorphic functions defined on a complete semi 1-corona “fill in the hole” (Corollaries 4.4 and 4.6). Meanwhile we also obtain more general extension theorems for sections of coherent sheaves (Theorems 4.3 and 4.5). As an application, we finally obtain an extension theorem for divisors (Theorems 4.16 and 4.20) and for analytic sets of codimension one (Theorem 4.18).

We remark that this approach fails in the case when the objects to be extended are not sections of a sheaf defined on the whole B_c; in particular, this applies for analytic sets of higher codimension. This is intimately related with the general, definitely more difficult, problem of extending analytic objects assigned on some $B_{\varepsilon,c}$ when the subsets B_c are not relatively compact in X i.e. when X is a genuine q-corona.

4.2 Cohomology and extension of sections

4.2.1 Closed q-coronae

Let X be a strictly q-pseudoconvex space (respectively $X \subset \mathbb{C}^n$ be a strictly q-pseudoconvex open set) and $H = \{h = 0\}$ (respectively H Levi-flat), and $C = B_{\varepsilon,c} = B_c \smallsetminus \overline{B}_\varepsilon$ a q-corona.

We can suppose that $B_c \smallsetminus H$ has two connected components, B_+ and B_-, and define $C_+ = B_+ \cap C$, $C_- = B_- \cap C$.

We recall that if $\mathcal{F}$ is a coherent sheaf on a domain $U \subset \mathbb{C}^n$, $z \in U$ and

$$0 \to \mathcal{O}_z^{m_k} \to \cdots \to \mathcal{O}_z^{m_0} \to \mathcal{F}_z \to 0$$

is a minimal resolution of $\mathcal{F}_z$, then the *depth* of $\mathcal{F}$ at the point z is the integer $p(\mathcal{F}_z) = n - k$. The *depth* of $\mathcal{F}$ in $K \subset U$ is

$$p_K(\mathcal{F}) = \inf_{z \in K} p(\mathcal{F}_z).$$

If $\mathcal{F} \in \mathsf{Coh} B_c$, we denote $p(\mathcal{F}) = p_{B_c}(\mathcal{F}$, and if $\mathcal{F} = \mathcal{O}$, the structure sheaf of X, we denote $p(B_c) = p(\mathcal{O})$.

Theorem 4.1 *Let $\mathcal{F} \in \mathsf{Coh} B_c$. Then the image of the homomorphism*

$$H^r(\overline{B}_+, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F}) \longrightarrow H^r(\overline{C}_+, \mathcal{F})$$

(all closures are taken in B_c), defined by $(\xi \oplus \eta) \mapsto \xi_{|\overline{C}_+} - \eta_{|\overline{C}_+}$ has finite codimension provided that $q - 1 \le r \le p(\mathcal{F}) - q - 2$.

Proof. Consider the Mayer-Vietoris sequence applied to the closed sets $\overline{B}_+$ and $\overline{C}$

$$\begin{aligned} \cdots \ \to \ & H^r(\overline{B}_+ \cup \overline{C}, \mathcal{F}) \to H^r(\overline{B}_+, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F}) \xrightarrow{\delta} \\ \xrightarrow{\delta} \ & H^r(\overline{C}_+, \mathcal{F}) \to H^{r+1}(\overline{B}_+ \cup \overline{C}, \mathcal{F}) \to \cdots \end{aligned} \tag{4.1}$$

$\delta(a \oplus b) = a_{|\overline{C}_+} - b_{|\overline{C}_+}$. $\overline{B}_+ \cup \overline{C} = B_c \smallsetminus U$ where $U = B_- \cap B_\varepsilon$. U is q-complete, so the groups of compact support cohomology $H^r_c(U, \mathcal{F})$ are zero for $q \le r \le p(\mathcal{F}) - q$.

From the exact sequence of compact support cohomology

$$\begin{aligned} \cdots \ \to \ & H^r_c(U, \mathcal{F}) \to H^r(B_c, \mathcal{F}) \to \\ \to \ & H^r(B_c \smallsetminus U, \mathcal{F}) \to H^{r+1}_c(U, \mathcal{F}) \to \cdots \end{aligned} \tag{4.2}$$

it follows that

$$H^r(B_c, \mathcal{F}) \xrightarrow{\sim} H^r(B_c \smallsetminus U, \mathcal{F}), \tag{4.3}$$

for $q \leq r \leq p(\mathcal{F}) - q - 1$.

Since B_c is q-pseudoconvex,

$$\dim_{\mathbb{C}} H^r(B_c, \mathcal{F}) < \infty$$

for $q \leq r$ [3, Théorème 11], and so

$$\dim_{\mathbb{C}} H^r(B_c \smallsetminus U, \mathcal{F}) < \infty$$

for $q \leq r \leq p(\mathcal{F}) - q - 1$.

From (4.1) we see that $\dim_{\mathbb{C}} H^r(B_c \smallsetminus U, \mathcal{F}) = \dim_{\mathbb{C}} H^r(\overline{B}_+ \cup \overline{C}, \mathcal{F})$ is greater than or equal to the codimension of the homomorphism δ. □

Corollary 4.2 *Under the same assumption of Theorem 4.1, if $K \cap H = \emptyset$,*

$$\dim_{\mathbb{C}} H^r(\overline{C}_+, \mathcal{F}) < \infty$$

for $q \leq r \leq p(\mathcal{F}) - q - 2$.

Proof. Since $K \cap H = \emptyset$, $\overline{B}_+$ is a q-pseudoconvex space, and by virtue of [3, Théorème 11] we have

$$\dim_{\mathbb{C}} H^r(\overline{B}_+, \mathcal{F}) < \infty$$

for $r \geq q$. On the other hand, $\overline{C}$ is a q-corona, thus we obtain

$$\dim_{\mathbb{C}} H^r(\overline{C}, \mathcal{F}) < \infty$$

for $q \leq r \leq p(\mathcal{F}) - q - 1$ in view of [5, Theorem 3]. By Theorem 4.1 we then get that for $q \leq r \leq p(\mathcal{F}) - q - 1$ the vector space $H^r(\overline{B}_+, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F})$ has finite dimension and for $q - 1 \leq r \leq p(\mathcal{F}) - q - 2$ its image in $H^r(\overline{C}_+, \mathcal{F})$ has finite codimension. Thus $H^r(\overline{C}_+, \mathcal{F})$ has finite dimension for $q \leq r \leq p(\mathcal{F}) - q - 2$. □

Theorem 4.3 *If $\overline{B}_+$ is a q-complete space, then*

$$H^r(\overline{C}, \mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}_+, \mathcal{F})$$

for $q \leq r \leq p(\mathcal{F}) - q - 2$ and the homomorphism

$$H^{q-1}(\overline{B}_+, \mathcal{F}) \oplus H^{q-1}(\overline{C}, \mathcal{F}) \longrightarrow H^{q-1}(\overline{C}_+, \mathcal{F}) \tag{4.4}$$

is surjective for $p(\mathcal{F}) \geq 2q + 1$.

If $\overline{B}_+$ *is a* 1*-complete space and* $p(\mathcal{F}) \geq 3$*, the homomorphism*

$$H^0(\overline{B}_+, \mathcal{F}) \longrightarrow H^0(\overline{C}_+, \mathcal{F})$$

is surjective.

Proof. Since by hypothesis $\overline{B}_+$ is a q-complete space, $H^r(\overline{B}_+, \mathcal{F}) = \{0\}$ for $q \leq r$ [3, Théorème 5]. From (4.3) it follows that $H^r(\overline{B}_+ \cup \overline{C}_+, \mathcal{F}) = \{0\}$ for $q \leq r \leq p(\mathcal{F}) - q - 1$. Thus, the Mayer-Vietoris sequence (4.1) implies that $H^r(\overline{C}, \mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}_+, \mathcal{F})$ for $q \leq r \leq p(\mathcal{F}) - q - 2$ and that the homomorphism (4.4) is surjective if $p(\mathcal{F}) \geq 2q + 1$.

In particular, if $q = 1$ and $p(\mathcal{F}) \geq 3$ the homomorphism

$$H^0(\overline{B}_+, \mathcal{F}) \oplus H^0(\overline{C}, \mathcal{F}) \longrightarrow H^0(\overline{C}_+, \mathcal{F})$$

is surjective, i.e. every section $\sigma \in H^0(\overline{C}_+, \mathcal{F})$ is a difference $\sigma_1 - \sigma_2$ of two sections $\sigma_1 \in H^0(\overline{B}_+, \mathcal{F})$, $\sigma_2 \in H^0(\overline{C}, \mathcal{F})$. Since B_ε is Stein, the cohomology group with compact supports $H^1_k(B_\varepsilon, \mathcal{F})$ is zero, and so the Mayer-Vietoris compact support cohomology sequence implies that the restriction homomorphism

$$H^0(\overline{B}_c, \mathcal{F}) \longrightarrow H^0(\overline{B}_c \setminus B_\varepsilon, \mathcal{F}) = H^0(\overline{C}, \mathcal{F})$$

is surjective, hence $\sigma_2 \in H^0(\overline{C}, \mathcal{F})$ is restriction of $\widetilde{\sigma}_2 \in H^0(B_c, \mathcal{F})$. So σ is restriction to $\overline{C}_+$ of $(\sigma_1 - \widetilde{\sigma}_{2|\overline{B}_+}) \in H^0(\overline{B}_+, \mathcal{F})$, and the restriction homomorphism is surjective. □

Corollary 4.4 *Let* $\overline{B}_+$ *be a* 1*-complete space and* $p(B_c) \geq 3$*. Then every holomorphic function on* $\overline{C}_+$ *extends holomorphically on* $\overline{B}_+$*.*

4.2.2 Open q-coronae

Most of the Theorems and Corollaries of the previous section still hold in the open case and their proofs are very similar. First we give the proof of the extension results using directly Theorem 4.3. We have to assume that H is the zero set of a pluriharmonic function h and we define B_c, C, B_+, B_-, C_+ and C_- as we did before.

Let us suppose B_+ is 1-complete and $p(\mathcal{F}) \geq 3$. Let $s \in H^0(C_+, \mathcal{F})$. For all $\epsilon > 0$, we consider the closed semi 1-corona

$$\overline{C}_\epsilon = \overline{B_{\varepsilon+\epsilon,c} \cap \{h > \epsilon\}} \subset C_+$$

Let $\sigma_\epsilon = s_{|\overline{C}_\epsilon}$. By Theorem 4.3 (applied to $\overline{C}_\epsilon$, $H_\epsilon = \{h = \epsilon\}$), we obtain that σ_ϵ extends to a section $\widetilde{\sigma}_\epsilon \in H^0(\overline{B}_\epsilon, \mathcal{F})$, where $\overline{B}_\epsilon = \overline{B_+ \cap \{h > \epsilon\}}$. Since $B_+ = \cup_\epsilon \overline{B}_\epsilon$, if for all $\epsilon_2 > \epsilon_1 > 0$,

$$\widetilde{\sigma}_{\epsilon_1|_{\overline{B}_{\epsilon_2}}} = \widetilde{\sigma}_{\epsilon_2} \tag{4.5}$$

the sections $\widetilde{\sigma}_\epsilon$ can be glued toghether to a section $\sigma \in H^0(B_+, \mathcal{F})$ extending s.

Let ϵ_1, ϵ_2, $\epsilon_2 > \epsilon_1 > 0$, be fixed. We have to show that (4.5) holds. By definition,

$$\left(\widetilde{\sigma}_{\epsilon_1|_{\overline{B}_{\epsilon_2}}} - \widetilde{\sigma}_{\epsilon_2}\right)_{|\overline{C}_{\epsilon_2}} = s - s = 0.$$

Thus, the support of $\widetilde{\sigma}_{\epsilon_1|_{\overline{B}_{\epsilon_2}}} - \widetilde{\sigma}_{\epsilon_2}$, S, is an analytic set contained in $\overline{B}_{\epsilon_2} \smallsetminus C_{\epsilon_2}$. Let us consider the family

$$(\phi_\lambda = \lambda(\varphi - \epsilon_2) + (1 - \lambda)(h - \epsilon_2))_{\lambda \in [0,1]}$$

of strictly plurisubharmonic functions. Let $\overline{\lambda}$ be the smallest value of λ for which $\{\phi_\lambda = 0\} \cap S \neq \emptyset$. Then $\{\phi_{\overline{\lambda}} < 0\} \cap B_+ \subset B_+$ is a Stein domain in which the analytic set S intersects the boundary; so the maximum principle for plurisubharmonic functions and the strict plurisubharmonicity of $\phi_{\overline{\lambda}}$ toghether imply that $\{\phi_{\overline{\lambda}} = 0\} \cap S$ is a set of isolated points in S. By repeating the argument, we show that S has no components of positive dimension. Hence $\widetilde{\sigma}_{\epsilon_1|_{\overline{B}_{\epsilon_2}}} - \widetilde{\sigma}_{\epsilon_2}$ is zero outside a set of isolated points. Since $p(\mathcal{F}) \geq 3$, the only section of $\mathcal{F}$ with compact support is the zero-section [6, Théorème 3.6 (a), p. 46], and so $\widetilde{\sigma}_{\epsilon_1|_{\overline{B}_{\epsilon_2}}} - \widetilde{\sigma}_{\epsilon_2}$ is zero.

Hence, there exists a section $\sigma \in H^0(B_+, \mathcal{F})$ such that $\sigma_{|C_+} = s$. Thus we have proved the following

Theorem 4.5 *If a B_+ is 1-complete space, $\mathcal{F}$ a coherent sheaf on B_+ with $p(\mathcal{F}) \geq 3$, the homomorphism*

$$H^0(B_+, \mathcal{F}) \longrightarrow H^0(C_+, \mathcal{F})$$

is surjective.

In particular,

Corollary 4.6 *If B_+ is a 1-complete space and $p(B_c) \geq 3$, every holomorphic function on C_+ can be holomorphically extended on B_+.*

Theorem 4.7 *Let* $\mathsf{Sing}B_c = \emptyset$. *Let* $\mathcal{F} \in \mathsf{Coh}B_c$. *Then the image of the homomorphism*

$$H^r(B_+, \mathcal{F}) \oplus H^r(C, \mathcal{F}) \xrightarrow{\delta} H^r(C_+, \mathcal{F})$$

defined by $(\xi, \eta) \mapsto \xi_{|C_+} - \eta_{|C_+}$ *has finite codimension for* $q - 1 \leq r \leq p(\mathcal{F}) - q - 2$. *For* $q = 1$ *the thesis holds true also dropping the assumption* $\mathsf{Sing}B_c = \emptyset$.

Proof. Consider the Mayer-Vietoris sequence applied to the open sets B_+ and C

$$\begin{aligned} \cdots \ \to \ & H^r(B_+ \cup C, \mathcal{F}) \to H^r(B_+, \mathcal{F}) \oplus H^r(C, \mathcal{F}) \xrightarrow{\delta} \\ \xrightarrow{\delta} \ & H^r(C_+, \mathcal{F}) \to H^{r+1}(B_+ \cup C, \mathcal{F}) \to \cdots, \end{aligned} \tag{4.6}$$

$\delta(a \oplus b) = a_{|C_+} - b_{|C_+}$. $B_+ \cup C = B_c \setminus K_0$ where $K_0 = \overline{B}_- \cap \overline{B}_\varepsilon$. K_0 has a q-complete neighborhoods system and so the local cohomology groups $H^r_{K_0}(B_c, \mathcal{F})$ are zero for $q \leq r \leq p(\mathcal{F}) - q$ [12] (in the general case for $q = 1$, see [6, Lemme 2.3, p. 29]).

Then, from the local cohomology exact sequence

$$\begin{aligned} \cdots \ \to \ & H^r_{K_0}(B_c, \mathcal{F}) \to H^r(B_c, \mathcal{F}) \to \\ \to \ & H^r(B_c \setminus K_0, \mathcal{F}) \to H^{r+1}_{K_0}(B_c, \mathcal{F}) \to \cdots \end{aligned} \tag{4.7}$$

follows that

$$H^r(B_c, \mathcal{F}) \xrightarrow{\sim} H^r(B_c \setminus K_0, \mathcal{F}), \tag{4.8}$$

for $q \leq r \leq p(\mathcal{F}) - q - 1$.

Since B_c is q-pseudoconvex,

$$\dim_{\mathbb{C}} H^r(C, \mathcal{F}) < \infty$$

for $q \leq r$ [3, Théorème 11], and so

$$\dim_{\mathbb{C}} H^r(B_c \setminus K_0, \mathcal{F}) < \infty$$

for $q \leq r \leq p(\mathcal{F}) - q - 1$.

From (4.6) we see that $\dim_{\mathbb{C}} H^r(B_c \setminus K_0, \mathcal{F}) = \dim_{\mathbb{C}} H^r(B_+ \cup C, \mathcal{F})$ is greater than or equal to the codimension of the homomorphism δ. $\square$

Corollary 4.8 *Under the same assumption of Theorem 4.7, if* $K \cap H = \emptyset$,

$$\dim_{\mathbb{C}} H^r(C_+, \mathcal{F}) < \infty$$

for $q \leq r \leq p(\mathcal{F}) - q - 2$.

Proof. The proof is similar to that of Corollary 4.2. □

Theorem 4.9 *Suppose that* $\mathsf{Sing} B_c = \emptyset$ *and* B_+ *is a* q*-complete space, then*

$$H^r(C, \mathcal{F}) \xrightarrow{\sim} H^r(C_+, \mathcal{F})$$

for $q \leq r \leq p(\mathcal{F}) - q - 2$ *and the homomorphism*

$$H^{q-1}(B_+, \mathcal{F}) \oplus H^{q-1}(C, \mathcal{F}) \longrightarrow H^{q-1}(C_+, \mathcal{F}) \tag{4.9}$$

is surjective if $p(\mathcal{F}) \geq 2q+1$*. If* $q = 1$*, both results hold true for an arbitrary complex space* B_c*.*

Proof. The proof is similar to that of Theorem 4.3. □

4.2.3 Corollaries of the extension theorems.

From now on, unless otherwise stated, by B, B_+, B_ε, C and C_+ we denote both the open sets and their closures, and we suppose that $H = \{h = 0\}$, h pluriharmonic.

Extension of non-vanishing holomorphic functions

Let $f \in H^0(C_+, \mathcal{O}^*)$. In the hypothesis of Corollaries 4.4 and 4.6, both f and $1/f$ extend holomorphically on B_+ Hence:

Corollary 4.10 *If* B_+ *is a* 1*-complete space and* $p(B_c) \geq 3$*, the restriction homomorphism*

$$H^0(B_+, \mathcal{O}^*) \longrightarrow H^0(C_+, \mathcal{O}^*)$$

is surjective.

Extension of meromorphic functions

Corollary 4.11 *If* B_+ *is a* 1*-complete space,* $p(B_c) \geq 3$*, and* X *is locally factorial, then every meromorphic function on* C_+ *is a quotient of holomorphic functions and thus extends on* B_+*.*

Proof. Let $f \in \mathcal{M}(C_+)$. Since X is locally factorial, locally (in an open set U) $f = p/q$, where $p, q \in \mathcal{O}(C_+)$ are coprime in U. Define

$$I_f = \{\lambda \in \mathcal{O} \ : \ \lambda f \in \mathcal{O}\}\,.$$

I_f is a locally free sheaf of rank one. Since p and q are coprime, $\lambda p/q \in \mathcal{O}$ implies that $\lambda = \mu q$, for some $\mu \in \mathcal{O}$. Hence $I_f \approx \mathcal{O}$, and Cartan's theorem A holds for I_f, i.e. there is a global section of I_f and $f = \alpha/\beta$, globally. □

Finiteness and vanishing of cohomology

In Theorems 4.3 and 4.5 we have estabilished the isomorphism

$$H^r(C, \mathcal{F}) \xrightarrow{\sim} H^r(C_+, \mathcal{F}).$$

In some special cases this leads to finiteness or vanishing-cohomology theorems for C_+.

An example is given by the theorems by Andreotti and Tomassini in [5]. Theorem 3 in there implies that if C is a q-corona and $\mathcal{F}$ is a coherent sheaf on C, then

$$\dim H^r(C_+, \mathcal{F}) = \dim H^r(C, \mathcal{F}) < +\infty,$$

for $q < r < p(\mathcal{F}) - q - 1$.

We briefly recall that (see [5] for details) a bundle F on a complex space X is called *metrically pseudoconvex* if there is a hermitian metric on the fibres of F can be chosen which is strongly pseudoconvex outside the zero-section of F.

Theorem 5 in [5] implies that if F is a metrically pseudoconvex line bundle over the q-corona C, and $\mathcal{F}$ is a coherent sheaf on C, then there exists $k_0 = k_0(\mathcal{F}, F) \in \mathbb{Z}$ such that

$$H^r(C_+, \mathcal{F} \otimes \Omega(F^k)) \ = \ 0,$$

for every $k \geq k_0$ and $q < r < p(\mathcal{F}) - q - 1$.

Extension of CR-functions

Let X be a Stein space. Let $H = \{h = 0\} \subset X$ be the zero set of a pluriharmonic function, and let M be a real hypersurface of X with boundary, such that $M \cap H = bM = bA$, where A is an open set in H. Let $D \subset X$ be the relatively compact domain bounded by $M \cup A$. We recall that, by Theorem 3.3 [65], for $X = \mathbb{C}^n$, Lipshitz CR-functions on M extend holomorphically to D. As a corollary of the previous theorems, we can obtain a similar result for section of a coherent sheaf on an arbitrary Stein space X.

Let us consider the connected component Y of $X \setminus H$ containing D, the closure $\overline{D}$ of D in Y, and let be $F = Y \setminus D$ and $M_Y = M \cap Y$. For every coherent sheaf $\mathcal{F}$ on X, with $p(\mathcal{F}) \geq 3$ we have the Mayer-Vietoris exact sequence

$$\cdots \ \rightarrow \ H^0(\overline{D}, \mathcal{F}) \oplus H^0(F, \mathcal{F}) \ \rightarrow \ H^0(M_Y, \mathcal{F}) \ \rightarrow \ H^1(Y, \mathcal{F}) \ \rightarrow \ \cdots$$

Since Y is Stein, $H^1(Y, \mathcal{F})$ is zero, and every section σ on M_Y is a difference $s_1 - s_2$, where $s_1 \in H^0(\overline{D}, \mathcal{F})$ and $s_2 \in H^0(F, \mathcal{F})$. By choosing an ε big enough so that M is contained in the ball $B_\varepsilon(x_0)$ of radius ε of X centered in x_0, we can apply Theorem 4.5 to the semi 1-corona $C_+ = Y \smallsetminus (B_\varepsilon \cap Y)$, to extend $s_{2|_{C_+}}$ to a section $\tilde{s}_2$ defined on Y. In order to conclude that $s_1 - \tilde{s}_{2|_{\overline{D}}}$ extends the section σ, we have to prove that $s_{2|_F} - \tilde{s}_{2|_F} = 0$.

As before, we consider the set $\Sigma = \left\{ s_{2|_F} - \tilde{s}_{2|_F} \neq 0 \right\} \subset B_\varepsilon \cap Y$ and conclude that Σ is a set of isolated points. Since $p(\mathcal{F}) \geq 3$, $\mathcal{F}$ has no non-zero section with compact support [6, Théorème 3.6 (a), p. 46]. Thus $\Sigma = \emptyset$ and we have obtained the following:

Corollary 4.12 *Let X be a Stein space. Let $H = \{h = 0\} \subset X$ be the zero set of a pluriharmonic function, and M be a real hypersurface of X with boundary, such that $M \cap H = bM = bA$, where A is an open set in H. Let $D \subset X$ be the relatively compact domain bounded by $M \cup A$ and $\mathcal{F}$ be a coherent sheaf with $p(\mathcal{F}) \geq 3$. All sections of $\mathcal{F}$ on M extend (uniquely) to D.*

We can go further:

Corollary 4.13 *Let X be a Stein manifold, $\mathcal{F}$ a coherent sheaf on X such that $p(\mathcal{F}) \geq 3$ and D be a bounded domain and K a compact subset of bD such that $bD \smallsetminus K$ is smooth. Assume that K is $\mathcal{O}(\overline{D})$-convex, i.e.*

$$K = \left\{ z \in \overline{D} \; : \; |f(z)| \leq \max_K |f| \right\}.$$

Then every section of $\mathcal{F}$ on $bD \smallsetminus K$ extends to D.

Proof. We recall that since U is an open subset of a Stein manifold there exists an envelope of holomorphy $\widetilde{U}$ of U (cfr. [26]): $\widetilde{U}$ is a Stein domain $\pi_U : \widetilde{U} \to X$ over X and there exists an open embedding $j : U \to \widetilde{U}$ such that $\pi_U \circ j = \mathsf{id}_U$ and $j^* : \mathcal{O}(\widetilde{U}) \to \mathcal{O}(U)$ is an isomorphism. In particular $\pi_U^* \mathcal{F}$ is a coherent sheaf with the same depth as $\mathcal{F}$, which extends $\mathcal{F}_{|U}$.

Let us fix an arbitrary point $x \in D$. We need to show that any given section $\sigma \in H^0(bD \smallsetminus K, \mathcal{F})$ extends to a neighborhood of x. Since $x \notin K = \widehat{K}$, there exists an holomorphic function f, defined on a neighborhood U of $\overline{D}$, such that $|f(x)| > \max_K |f(z)|$.

Then σ extends to a section $\widetilde{\sigma} \in H^0(\pi^{-1}(D \smallsetminus K), \mathcal{F})$. Let $\widetilde{f}$ be the holomorphic extension of f to $\widetilde{U}$. The hypersurface

$$H = \left\{ z \in \widetilde{U} : |\widetilde{f}(z)| = \max_K |\widetilde{f}| \right\}$$

is the zero-set of a pluriharmonic function and, by construction,

$$x \in \widetilde{D}_+ = \left\{ z \in \widetilde{U} : |\widetilde{f}(z)| > \max_K |\widetilde{f}| \right\}.$$

Now we are in the situation of Corollary 4.12 so $\widetilde{\sigma}$ extends to a section on $\widetilde{D}_+$. Since $x \in \widetilde{D}_+$, this ends the proof. □

4.3 Extension of divisors and analytic sets of codimension one

First of all, we give an example in dimension $n = 2$ of a regular complex curve of C_+ which does not extend on B_+. Hence, not every divisor on C_+ extends to a divisor on B_+.

Example 4.1 Using the same notation as before, in $\mathbb{C}^2$ let B_c and B_ε be the balls

$$B_c = \left\{|z_1|^2 + |z_2|^2 < c\right\}, \quad B_\varepsilon = \left\{|z_1|^2 + |z_2|^2 < \varepsilon\right\}, \quad c > \varepsilon > 2.$$

Consider the connected irreducible analytic set of codimension one

$$A = \{(z_1, z_2) \in B_+ \ : \ z_1 z_2 = 1\}$$

and its restriction A_C to C_+. If A_C has two connected components, A_1 and A_2, if we try to extend A_1 (analytic set of codimension one on C_+) to B_+, its restriction to C_+ will contain also A_2. So A_1 is an analytic set of codimension one on C_+ that does not extend on B_+.

So, let us prove that A_C has indeed two connected components. A point of A (of A_C) can be written as $z_1 = \rho e^{i\theta}$, $z_2 = \frac{1}{\rho} e^{-i\theta}$, with $\rho \in \mathbb{R}^+$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Hence, points in A_C satisfy

$$2 < \varepsilon < \rho^2 + \frac{1}{\rho^2} < c \ \Rightarrow \ 2 < \sqrt{\varepsilon + 2} < \rho + \frac{1}{\rho} < \sqrt{c + 2}.$$

Since $f(\rho) = \rho + 1/\rho$ is monotone decreasing up to $\rho = 1$ (where $f(1) = 2$), and then monotone increasing, there exist a and b such that the inequalities are satisfied when $a < \rho < b < 1$, or when $1 < 1/b < \rho < 1/a$. A_C is thus the union of the two disjoint open sets

$$A_1 = \left\{ \left(\rho e^{i\theta}, \tfrac{1}{\rho} e^{-i\theta}\right) \in \mathbb{C}^2 \ \middle| \ a < \rho < b, \ -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \right\};$$

$$A_2 = \left\{ \left(\rho e^{i\theta}, \tfrac{1}{\rho} e^{-i\theta}\right) \in \mathbb{C}^2 \ \middle| \ a < \tfrac{1}{\rho} < b, \ -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \right\}.$$

The aim of this section is to prove an extension theorem for divisors, i.e. to prove that, under certain hypothesis, the homomorphism

$$H^0(B_+, \mathcal{D}) \to H^0(C_+, \mathcal{D}) \tag{4.10}$$

is surjective.

In order to get this result, we observe that from the exact sequence

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0 \tag{4.11}$$

we get the commutative diagram (horizontal lines are exact)

$$\begin{array}{ccccccc}
H^0(B_+, \mathcal{M}^*) & \longrightarrow & H^0(B_+, \mathcal{D}) & \longrightarrow & H^1(B_+, \mathcal{O}^*) & \longrightarrow & H^1(B_+, \mathcal{M}^*) \\
\alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow \\
H^0(C_+, \mathcal{M}^*) & \longrightarrow & H^0(C_+, \mathcal{D}) & \longrightarrow & H^1(C_+, \mathcal{O}^*) & \longrightarrow & H^1(C_+, \mathcal{M}^*)
\end{array}$$

Thus, in view of the "five lemma", in order to conclude that β is surjective it is sufficient to show that α and γ are surjective, and δ is injective.

Lemma 4.14 *If* $\mathsf{Sing} B_+ = \emptyset$*,* B_c *is* 1*-complete and* $p(B_c) \geq 3$*, then* α *is surjective.*

Proof. Let f be a meromorphic invertible function on C_+. Since C_+ is an open set of the Stein manifold B_+, $f = f_1 f_2^{-1}$, $f_1, f_2 \in H^0(C_+, \mathcal{O})$. By Corollary 4.4 (respectively, Corollary 4.6), f_1 and f_2 extend to holomorphic functions on B_+ and consequently f extends on B_+ as well. $\square$

Lemma 4.15 *Assume that the restriction* $H^2(B_+, \mathbb{Z}) \to H^2(C_+, \mathbb{Z})$ *is surjective. If* B_c *is* 1*-complete and* $p(B_c) \geq 4$*, then* γ *is surjective.*

We remark that if $H^2(C_+, \mathbb{Z}) = \{0\}$ the first condition is satisfied.

Proof. From the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0 \tag{4.12}$$

we get the commutative diagram (horizontal lines are exact)

$$\begin{array}{ccccccc}
H^1(B_+, \mathcal{O}) & \longrightarrow & H^1(B_+, \mathcal{O}^*) & \longrightarrow & H^2(B_+, \mathbb{Z}) & \longrightarrow & H^2(B_+, \mathcal{O}) \\
f_2 \downarrow & & \gamma \downarrow & & f_4 \downarrow & & f_5 \downarrow \\
H^1(C_+, \mathcal{O}) & \longrightarrow & H^1(C_+, \mathcal{O}^*) & \longrightarrow & H^2(C_+, \mathbb{Z}) & \longrightarrow & H^2(C_+, \mathcal{O})
\end{array}$$

where $H^1(B_+, \mathcal{O}) = H^2(B_+, \mathcal{O}) = \{0\}$ because B_+ is Stein, and f_4 is surjective by hypothesis. Thus in order to prove that γ is surjective by the "five lemma" it is sufficient to show that f_2 is surjective, i.e. that $H^1(C_+, \mathcal{O}) = \{0\}$.

Since $p(B_c) \geq 4$, by Theorem 4.3 (respectively, by Theorem 4.9) it follows that

$$H^1(C, \mathcal{O}) \xrightarrow{\sim} H^1(C_+, \mathcal{O}). \tag{4.13}$$

Consider the local, respectively compact support, cohomology exact sequence

$$H^1_{\overline{B}_\varepsilon}(B_c, \mathcal{O}) \longrightarrow H^1(B_c, \mathcal{O}) \longrightarrow H^1(C, \mathcal{O}) \longrightarrow H^2_{\overline{B}_\varepsilon}(B_c, \mathcal{O})$$

$$H^1_k(B_\varepsilon, \mathcal{O}) \longrightarrow H^1(B_c, \mathcal{O}) \longrightarrow H^1(C, \mathcal{O}) \longrightarrow H^2_k(B_\varepsilon, \mathcal{O})$$

Since B_c is Stein, $H^1(B_c, \mathcal{O}) = \{0\}$ and $H^r_k(B_\varepsilon, \mathcal{O}) = H^r_{\overline{B}_\varepsilon}(B_c, \mathcal{O}) = \{0\}$ for $1 \leq r \leq p(B_\varepsilon) - 1$ [12]. In particular, since $p(B_\varepsilon) \geq p(B_c) \geq 4$, it follows that

$$\{0\} = H^1(B_c, \mathcal{O}) \xrightarrow{\sim} H^1(C, \mathcal{O}). \tag{4.14}$$

(4.13) and (4.14) give

$$\{0\} = H^1(B_c, \mathcal{O}) \xrightarrow{\sim} H^1(C, \mathcal{O}) \xrightarrow{\sim} H^1(C_+, \mathcal{O}).$$

and this proves the lemma. □

In the case $H^2(C_+, \mathbb{Z}) = \{0\}$ we remark that from the proof of Lemma 4.15 it follows that the sequence

$$\{0\} \longrightarrow H^1(C_+, \mathcal{O}^*) \longrightarrow \{0\}$$

is exact, that is $H^1(C_+, \mathcal{O}^*) = \{0\}$. Hence, the commutative diagram relative to (4.11) becomes (horizontal lines are exact)

$$\begin{array}{ccccc} H^0(B_+, \mathcal{M}^*) & \longrightarrow & H^0(B_+, \mathcal{D}) & \longrightarrow & H^1(B_+, \mathcal{O}^*) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ H^0(C_+, \mathcal{M}^*) & \longrightarrow & H^0(C_+, \mathcal{D}) & \longrightarrow & \{0\} \end{array} \tag{4.15}$$

and it is then easy to see that a divisor on C_+ can be extended to a divisor on B_+.

Thus we have proved the following:

Theorem 4.16 *Let B_c be 1-complete, $p(B_c) \geq 4$, and C_+ satisfy the topological condition $H^2(C_+, \mathbb{Z}) = \{0\}$. Then, if $\mathrm{Sing}(B_+) = \emptyset$, all divisors on C_+ extend (uniquely) to divisors on B_+.*

Corollary 4.17 *Let B_c be 1-complete, $p(B_c) \geq 4$, $\mathrm{Sing}(B_+) = \emptyset$, and ξ be a divisor on C_+ with zero Chern class in $H^2(C_+, \mathbb{Z})$. Then ξ extends (uniquely) to a divisor on B_+.*

Proof. Use diagram (4.15). $\square$

Theorem 4.18 *Assume that $H^2(C_+, \mathbb{Q}) = \{0\}$. If $\mathrm{Sing}(B_+) = \emptyset$, B_c is 1-complete and $p(B_c) \geq 4$, then all analytic sets of codimension 1 on C_+ extend to analytic sets on B_+.*

Proof. Let A be an analytic set of codimension 1 on C_+. Since B_+ is a Stein manifold, C_+ is locally factorial, and so there exists a divisor ξ on C_+ with support A. Since $H^2(C_+, \mathbb{Q}) = \{0\}$, there exists $n \in \mathbb{N}$ such that $nc_2(\xi) = 0 \in H^2(C_+, \mathbb{Z})$. Hence $n\xi$ has zero Chern class in $H^2(C_+, \mathbb{Z})$, and so, by Corollary 4.17 $n\xi$ can be extended to a divisor $\widetilde{n\xi}$ on B_+. The support of $\widetilde{n\xi}$ is an analytic set $\widetilde{A}$ which extends to B_+ the support A of $n\xi$. $\square$

In Theorem 4.16 the condition $H^2(C_+, \mathbb{Z}) = \{0\}$ can be relaxed and replaced by the weaker one: the restriction map $H^2(B_+, \mathbb{Z}) \to H^2(C_+, \mathbb{Z})$ is surjective. We need the following

Lemma 4.19 *δ is injective.*

Proof. First we prove lemma for C_+ closed. Let $\xi \in H^1(\overline{B}_+, \mathcal{M}^*)$ be such that $\xi_{|\overline{C}_+} = 0$. Consider the set

$$A = \{\eta \in [0, \varepsilon] \ : \ \xi_{|\overline{B}_+ \smallsetminus \overline{B}_\eta} = 0\}.$$

If we prove that $0 \in A$, we are done, because $0 = \xi_{|\overline{B}_+ \smallsetminus \overline{B}_0} = \xi_{|\overline{B}_+} = \xi$. Obviously $\eta_0 \in A$ implies $\forall \eta > \eta_0$, $\eta \in A$.

$A \neq \emptyset$. Since $C_+ = B_+ \smallsetminus \overline{B}_\varepsilon$ and $\xi_{|\overline{C}_+} = 0$, $\varepsilon \in A$.

A is closed. If $\eta_n \in A$, for all n, and $\eta_n \searrow \eta_\infty$, $\overline{B}_+ \smallsetminus \overline{B}_{\eta_\infty} = \cup_n (\overline{B}_+ \smallsetminus \overline{B}_{\eta_n})$, hence $\xi_{|\overline{B}_+ \smallsetminus \overline{B}_{\eta_n}} = 0$ for all n implies $\xi_{|\overline{B}_+ \smallsetminus \overline{B}_{\eta_\infty}} = 0$, i.e. $\eta_\infty \in A$.

A is open. Suppose $0 < \eta_0 \in A$. We denote $C_{\eta_0} = \overline{B}_+ \smallsetminus \overline{B}_{\eta_0}$. Let $\mathcal{A}$ be the family of open covering $\{U_i\}_{i \in I}$ of $\overline{B}_+$ such that:

$\alpha)$ U_i is isomorphically equivalent to an holomorphy domain in $\mathbb{C}^n$;

β) If $U_i \cap bB_{\eta_0} \neq \emptyset$, the restriction homomorphism

$$H^0(U_i, \mathcal{O}) \to H^0(U_i \cap C_{\eta_0}, \mathcal{O})$$

is bijective;

γ) $U_i \cap U_j$ is simply connected.

$\mathcal{A}$ is not empty and it is cofinal in the set of open coverings of $\overline{B}_+$ [3, Lemma 2, p. 222]. Let $\mathcal{U} = \{U_i\}_{i \in I} \in \mathcal{A}$, and $\{f_{ij}\} \in Z^1(\mathcal{U}, \mathcal{M}^*)$ be a representative of ξ. Let $W_i = U_i \cap C_{\eta_0}$. Since $\eta_0 \in A$, if $W_i \cap W_j \neq \emptyset$, $f_{ij|W_i \cap W_j} = f_i f_j^{-1}$ ($f_\nu \in H^0(W_\nu, \mathcal{M}^*)$). By α), $f_\nu = p_\nu q_\nu^{-1}$, $p_\nu, q_\nu \in H^0(W_\nu, \mathcal{O})$. By β), both p_ν and q_ν can be holomorphically extended on U_ν, with $\widetilde{p}_\nu$ and $\widetilde{q}_\nu$. Hence we have $f_{ij} = \widetilde{p}_i \widetilde{q}_i^{-1} (\widetilde{p}_j \widetilde{q}_j^{-1})^{-1}$ on $U_i \cap U_j$ (which is simply connected, so that there is no polidromy). So $\xi = 0$ in an open neighborhood U of C_{η_0} and, by compactness, there exists $\epsilon' > 0$ such that $C_{\eta_0 - \epsilon'} \subset U$. So $\eta_0 - \epsilon' \in A$ and consequently A is open.

Thus $A = [0, \varepsilon]$, and the lemma is proved if C_+ is closed.

If C_+ is open, we consider C_+ as a union of the closed semi 1-coronae

$$\overline{C}_\epsilon = \overline{B_{\varepsilon+\epsilon',c} \cap \{h > \epsilon'\}} \subset C_+.$$

Let $\xi \in H^1(B_+, \mathcal{M}^*)$ be such that $\xi_{|C_+} = 0$. Then $\xi_{|\overline{C}'_\epsilon} = 0$, for all $\epsilon' > 0$. Consequently from what we have already proved $\xi_{|\overline{B}'_\epsilon} = 0$, where $\overline{B}_\epsilon = \overline{B_+ \cap \{h > \epsilon'\}}$. Since $\cup'_\epsilon \overline{B}'_\epsilon = B_+$, $\xi = 0$ and the lemma is proved. □

Lemma 4.14, Lemma 4.15 and Lemma 4.19 lead to the following generalization of Theorem 4.16:

Theorem 4.20 *Assume that the restriction $H^2(B_+, \mathbb{Z}) \to H^2(C_+, \mathbb{Z})$ is surjective. If* $\mathrm{Sing}(B_+) = \emptyset$, *$B_c$ is 1-complete and $p(B_c) \geq 4$, then all divisors on C_+ extend to divisors on B_+.*

Chapter 5

Cohomology of semi 1-coronae and extension of analytic subsets

This chapter is based on [83].

5.1 Introduction

The results on the cohomology of coherent sheaves on semi q-coronae obtained in the previous chapter (see also [82]) were under the hypothesis that the sheaves are defined on the larger set B_b.

The aim of this chapter is to give a generalization for coherent sheaves $\mathcal{F}$ defined only on the semi q-corona. For the sake of simplicity we restrict ourselves to the case of smooth semi 1-coronae.

Following Andreotti-Grauert (see [3]), given a semi 1-corona

$$C_{a,b}^{+} = C_{a,b} \cap \{h > 0\},$$

where h is pluriharmonic, and a coherent sheaf $\mathcal{F}$ on $C_{a,b}^{+}$ we consider the strongly plurisubharmonic functions $P_\varepsilon(z) = \varepsilon|z|^2 - h$, $\varepsilon > 0$, and an exhaustion of $C_{a,b}^{+}$ by the following relatively compact domains

$$C_{\varepsilon}^{+} = \{z \in \mathbb{C}^n : P_\varepsilon(z) < 0\} \bigcap \overline{C}_{a+\varepsilon,b-\varepsilon}.$$

The idea is to prove for the domains C_{ε}^{+} a bump lemma and an approximation theorem as in the classical case of coronae. Here the situation is more complicated because of the presence of a non-empty pseudoconvex-pseudoconcave part in the boundary of each C_{ε}^{+}. In order to circumvent this difficulty, we work with the closed sets $\overline{C}_\varepsilon^+$ using in a crucial way a regularity result on the $\overline{\partial}$-equation due to Laurent-Thièbaut and Leiterer (see Section 5.3). This enables us to prove the following results: assume that depth $\mathcal{F}_z \geq 3$ for z near to the pseudoconcave part of the boundary of $C_{a,b}^+$; then

1) if ε is sufficiently small and $\varepsilon' < \varepsilon$ is near ε

$$H^1(\overline{C}_{\varepsilon'}^+, \mathcal{F}) \simeq H^1(\overline{C}_\varepsilon^+, \mathcal{F});$$

2) the cohomology spaces $H^1(\overline{C}_\varepsilon^+, \mathcal{F})$ are finite dimensional.

(see Lemma 5.6, Lemma 5.12 and Proposition 5.13).

Thus the function

$$d(\varepsilon) = \dim_{\mathbb{C}} H^1(\overline{C}_\varepsilon^+, \mathcal{F})$$

is piecewise constant, but, in general, it could have frequently a "jump-discontinuity" and it could happen that $d(\varepsilon) \to +\infty$ (see Remark 5.2). Nevertheless, the isomorphism 1) allows us to prove in the last section:

1. the fact that Oka-Cartan-Serre Theorem A holds in semi 1-coronae for sheaves which satisfy the condition of Theorem 5.4 (see Theorem 5.16);
2. an extension theorem for analytic subsets (see Corollary 5.17).

Remind that an extension theorem for codimension one analytic subsets of a non-singular semi 1-corona was proved in the previous chapter (Theorem 4.18, see also [82]) and for higher codimensions, using different methods based on Harvey-Lawson's theorem [38], will be analized in Chapter 8 (see [16]).

5.2 Remarks on the proofs of theorems in Chapter 4

Let X be a complex space. For every coherent sheaf $\mathcal{F}$ on X and every subset A of X we set

$$p(A; \mathcal{F}) = \inf_{x \in A} \operatorname{depth}(\mathcal{F}_x)$$

$$p(A) = p(A; \mathcal{O}).$$

Let $C = C_{a,b}$ be a q-corona of X. All the results in Chapter 4 (see [82]) on finite and/or vanishing cohomology results for q-coronae and semi q-coronae are obtained using Andreotti-Grauert methods. They consist of two main points

i) the bump lemma;

ii) for every corona $C_{a',b'} \Subset C$ there exists a corona $C_{a'+\varepsilon,b'+\varepsilon} \Subset C$, $\varepsilon > 0$ such that the homomorphism

$$H^r(C_{a'-\varepsilon,b'+\varepsilon}, \mathcal{F}) \longrightarrow H^r(C_{a',b'}, \mathcal{F})$$

is bijective for $q \le r \le p(C; \mathcal{F})$.

As a matter of fact the method of proof shows that the condition on the depth is needed only in $C_{a,a'}$ i.e. the homomorphism

$$H^r(C_{a'-\varepsilon,b'+\varepsilon}, \mathcal{F}) \longrightarrow H^r(C', \mathcal{F})$$

is bijective for $q \le r \le p(C_{a,a'}; \mathcal{F})$.

Let X be a strongly q-pseudoconvex space (respectively $X \subset \mathbb{C}^n$ be a strongly q-pseudoconvex open set) and $H = \{h = 0\}$ where h is pluriharmonic in X (respectively H Levi-flat), and $C = C_{a,b} = B_b \setminus \overline{B}_a$ a q-corona. We can suppose that $B_b \setminus H$ has two connected components, B^+ and B^-, and we define $C^+ = B^+ \cap C$, $C^- = B^- \cap C$.

From the above remark we derive the following improvements of Theorem 4.1, Corollary 4.2 and Theorem 4.3.

Theorem 5.1 *Let $\mathcal{F} \in \mathsf{Coh}(B_b)$. Then the image of the homomorphism*

$$H^r(\overline{B}^+, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F}) \longrightarrow H^r(\overline{C}^+, \mathcal{F})$$

(all closures are taken in B_b), defined by $(\xi \oplus \eta) \mapsto \xi_{|\overline{C}^+} - \eta_{|\overline{C}^+}$ has finite codimension provided that $q - 1 \le r \le p(\overline{B}_a; \mathcal{F}) - q - 2$.

Corollary 5.2 *If $K \cap H = \emptyset$, under the same assumption of Theorem 5.1*

$$\dim_{\mathbb{C}} H^r(\overline{C}^+, \mathcal{F}) < \infty$$

for $q \le r \le p(\overline{B}_a; \mathcal{F}) - q - 2$.

Theorem 5.3 *If $\overline{B}_+$ is a q-complete space, then*

$$H^r(\overline{C}, \mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}^+, \mathcal{F})$$

for $q \le r \le p(\overline{B}_a; \mathcal{F}) - q - 2$, and the homomorphism

$$H^{q-1}(\overline{B}^+, \mathcal{F}) \oplus H^{q-1}(\overline{C}, \mathcal{F}) \longrightarrow H^{q-1}(\overline{C}^+, \mathcal{F}) \tag{5.1}$$

is surjective for $p(\overline{B}_a; \mathcal{F}) \ge 2q + 1$.

If $\overline{B}^+$ is a 1-complete space and $p(\overline{B}_a; \mathcal{F}) \ge 3$, then

$$H^0(\overline{B}^+, \mathcal{F}) \xrightarrow{\sim} H^0(\overline{C}^+, \mathcal{F}).$$

This implies the following. Let $C_1 = B_{b_1} \smallsetminus \overline{B}_{a_1} \Subset C_2 = B_{c_2} \smallsetminus \overline{B}_{a_2}$. Then

$$H^r(\overline{C}_1^+, \mathcal{M}_{\{x\}}\mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}_2^+, \mathcal{F}),$$

for $q \le r \le p(\overline{B}_{a_1}; \mathcal{F})$.

In particular, if $x \in C_2 \smallsetminus \overline{B}_{a_1}$ and $\mathcal{M}_{\{x\}}$ denotes the sheaf of ideals of $\{x\}$, then

$$H^r(\overline{C}_2^+, \mathcal{M}_{\{x\}}\mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}_1^+, \mathcal{F}),$$

for $q \le r \le p(\overline{B}_{a_1}; \mathcal{F})$.

5.3 An isomorphism theorem for semi 1-coronae

Our aim is to give a generalization of the above results for sheaves defined only on the semi q-coronae, i.e. for the case when the "hole" is real. For the sake of simplicity we will consider only complete 1-coronae in $\mathbb{C}^n$ with $n \ge 3$. So we consider connected 1-coronae of the form

$$C = \{z \in \mathbb{C}^n \ : \ 0 < \varphi(z) < 1\} \Subset \mathbb{C}^n,$$

where $\varphi : \mathbb{C}^n \to \mathbb{R}$ is a smooth strongly plurisubharmonic function in a Stein neighborhood U of $\{0 \le \varphi \le 1\}$, $d\varphi \ne 0$ on $\varphi = 0, 1$. Let h be a pluriharmonic function on U and H the zero set of h. We assume that H is smooth and transversal to the hypersurfaces $\{\varphi = 0\}$, $\{\varphi = 1\}$, that $U \smallsetminus H$ has two connected components $U^\pm$ and $h > 0$ on U^+. For $0 < a < b < 1$ we set

$$B_b = \{z \in U : \varphi < b\},\ B_b^+ = B_b \cap U^+,$$

$$C_{a,b} = \left(B_b \smallsetminus \overline{B}_a\right),\ C_{a,b}^+ = C_{a,b} \cap U^+.$$

Let $P_\varepsilon(z) = \varepsilon|z|^2 - h$; then there is ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$ the hypersurfaces $\{\varphi = \varepsilon\}$, $\{\varphi = 1-\varepsilon\}$ meet $\{P_\varepsilon = 0\}$ transversally. Finally we define the following subsets (which are locally 1-*convex*, 1-*concave*, see [58] and Remark 5.1 below)

$$\overline{C}_\varepsilon^+ = \{z \in \mathbb{C}^n : P_\varepsilon(z) \le 0\} \bigcap \overline{C}_{\varepsilon, 1-\varepsilon}.$$

We want to prove the following

Theorem 5.4 *Let C^+ be a semi 1-corona in $\mathbb{C}^n$. Then for every $\varepsilon \in (0, \varepsilon_0)$ there exists $\overline{\varepsilon} \in [0, \varepsilon)$ such that for every $\mathcal{F} \in \mathsf{Coh}(C^+)$ satisfying*

$$\overline{\{z \in C^+ : \mathrm{depth}(\mathcal{F}_z) < 3\}} \bigcap B_{\varepsilon_0} = \emptyset.$$

and every $\varepsilon' \in (\bar{\varepsilon}, \varepsilon)$ *the homomorphism*

$$H^1(C^+_{\varepsilon'}, \mathcal{F}) \longrightarrow H^1(\overline{C}^+_{\varepsilon}, \mathcal{F})$$

is an isomorphism.

The main ingredients for the proof are the bump lemma and a density theorem as in Andreotti-Grauert [3]. Due to the presence of points in the pseudoconvex-pseudoconcave part of the boundary we cannot work with open bumps as in the Andreotti-Grauert's paper. Instead, we work with closed bumps using the following result due to Laurent-Thièbaut and Leiterer (see [58, Proposition 7.5]):

Proposition 5.5 *Let* $D \Subset \mathbb{C}^n$ *be a* 1*-concave,* 1*-convex domain of order* 1 *of special type, and suppose that* $n \geq 3$*. If* f *is a continuous* (n, r)*-form in some neighborhood* $U_{\overline{D}}$ *of* $\overline{D}$*,* $1 \leq r \leq n-2$*, such that* $\overline{\partial} f = 0$ *in* $U_{\overline{D}}$*, then there exists a form* $u \in \bigcap_{\varepsilon>0} C^{1/2-\varepsilon}_{n,r-1}(\overline{D})$ *such that* $\overline{\partial} u = f$ *in* D*.*

Remark 5.1 Proposition 7.5 in [58] is much more general, but we state it this way, since the semi 1-coronae we consider are locally 1-concave, 1-convex domain of order 1 of special type, i.e. they are locally biholomorpic to the set-difference of two convex domains.

The proof of Theorem 5.4 is a consequence of several intermediate results.

5.3.1 Bump lemma: surjectivity of cohomology

With the same notations as above let $\overline{D} = \overline{C}^+_{\varepsilon}$, $0 < \varepsilon < \varepsilon_0$ where $\varepsilon_0 < b$ is so chosen that for all $\varepsilon \in (0, \varepsilon_0)$ the hypersurfaces $\{\varphi = \varepsilon\}$, $\{\varphi = 1 - \varepsilon\}$ are $\mathbb{C}$-transversal to $\{P_\varepsilon = 0\}$. Let Γ_1, Γ_2 be respectively the pseudoconvex and the pseudoconcave part of the boundary bD of $\overline{D}$. Thus b$D = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ and $\overline{\Gamma}_2$ is contained in the smooth hypersurface $\{\varphi = \varepsilon\}$.

Lemma 5.6 (bump lemma) *There exists a finite open covering* $\mathcal{U}$ *of* bD*,* $\mathcal{U} = \{\overline{U}_j\}_{1 \leq j \leq m}$*, and relatively compact open subsets* $D_1, \ldots, D_m \Subset C^+$ *such that*

(i) $\overline{D} = \overline{D}_0 \subset \overline{D}_1 \subset \cdots \subset \overline{D}_m$*;*

(ii) $\overline{D} \subset D_m$*;*

(iii) $\overline{D}_j \setminus \overline{D}_{j-1} \subset \overline{U}_j$ *for* $1 \leq j \leq m$*;*

(iv) if $\mathcal{F} \in \mathsf{Coh}(C^+)$ then

$$H^r(\overline{U}_j \cap \overline{D}_k, \mathcal{F}) = 0$$

for every j, k and $1 \leq r \leq p(\overline{D}; \mathcal{F}) - 2$.

Moreover, the family of the coverings $\mathcal{U}$ as above is cofinal in the family of all finite coverings of bD.

Proof. If $z^0 \in \Gamma_1 \cup \Gamma_2$ i.e. z^0 is a point of pseudoconvexity or pseudoconcavity we argue as in the proof of the classical Andreotti-Grauert bump lemma.

Assume that $z^0 \in \overline{\Gamma}_1 \cap \overline{\Gamma}_2$. There exists a a sufficiently small closed ball $\overline{B}$ of positive radius, centered at z^0 and a biholomorphism on $\Phi : \overline{B} \to \Phi(\overline{B})$ which trasnsform $\overline{B} \cap \{\varphi \geq \varepsilon\}$ and $\overline{B} \cap \{P_\varepsilon \geq 0\}$ respectively in a strictly concave and strictly convex set (see Remark 1.5). We may also assume that $\mathcal{F}_{|\overline{B}}$ has a homological resolution

$$0 \to \mathcal{O}^{p_k} \to \cdots \to \mathcal{O}^{p_0} \to \mathcal{F} \to 0 \tag{5.2}$$

with $n - k \geq 3$. Choose a smooth function $\varrho \in C_0^\infty(B)$ such that $\varrho \geq 0$ and $\varrho(z^0) \neq 0$ and a positive number λ such that the closed domains

$$\overline{B}_1 = \{\varphi - \varepsilon - \lambda\rho \leq 0\} \cap \overline{B},\ \overline{B}_2 = \{P_\varepsilon + \lambda\rho \leq 0\} \cap \overline{B}$$

are respectively strictly concave and strictly convex and contain z^0 as an interior point. Set $\overline{B}_3 = \overline{B}_1 \cap \overline{B}_2$ and $\overline{D}_1 = \overline{C}^+_\lambda \cup \overline{B}_3$; z^0 is an interior point of $\overline{D}_1$ and b$\overline{B}_1 \smallsetminus$ b$\overline{B}_2 \Subset B$. By construction $\overline{D} \cap \overline{D}_1 \cap \overline{B} = \overline{D} \cap \overline{B}$ and $D \cap B$ is an intersection of two strictly convex domains with smooth boundaries thus applying Proposition 5.5 we obtain

$$H^r(\overline{D} \cap \overline{B}, \mathcal{O}) = \{0\}$$

for $1 \leq r \leq n - 2$ and consequently, in view of (5.2), the vanishing

$$H^r(\overline{D} \cap \overline{B}, \mathcal{F}) = \{0\}. \tag{5.3}$$

Iterating this procedure we get the conclusion. □

Proposition 5.7 *For every $\varepsilon \in (0, \varepsilon_0)$ there exists $\varepsilon' < \varepsilon$ such that the homomorphism*

$$H^r(\overline{C}^+_{\varepsilon'}, \mathcal{F}) \longrightarrow H^r(\overline{C}^+_\varepsilon, \mathcal{F})$$

is onto for $1 \leq r \leq p(\overline{C}^+_\varepsilon; \mathcal{F}) - 2$.

Proof. Keeping the notations of Lemma 5.6 we apply the Mayer-Vietoris exact sequence for closed sets to $\overline{D}_1 = \overline{D} \cup (\overline{D}_1 \cap \overline{B})$. We get

$$\cdots \to H^r(\overline{D}_1, \mathcal{F}) \to H^r(\overline{D}, \mathcal{F}) \oplus H^r(\overline{D}_1 \cap \overline{B}, \mathcal{F}) \to H^r(\overline{D} \cap \overline{D}_1 \cap \overline{B}, \mathcal{F}) \to \cdots$$

thus in view of (5.3) the homomorphism

$$H^r(\overline{D}_1, \mathcal{F}) \to H^r(\overline{D}, \mathcal{F})$$

is onto for $1 \le r \le n-2$. By induction, we obtain that the homomorphism

$$H^r(\overline{D}_m, \mathcal{F}) \to H^r(\overline{D}, \mathcal{F})$$

is onto for $1 \le r \le p(\overline{C}_\varepsilon^+; \mathcal{F}) - 2$. Since $\overline{C}_\varepsilon^+ \subset D_m$ if $\varepsilon' < \varepsilon$ is near ε one has $\overline{C}_\varepsilon^+ \subset C_{\varepsilon'}^+ \Subset D_m$, whence the homomorphism

$$H^r(\overline{C}_{\varepsilon'}^+, \mathcal{F}) \to H^r(\overline{C}_\varepsilon^+, \mathcal{F})$$

is onto for $1 \le r \le p(\overline{C}_\varepsilon^+; \mathcal{F}) - 2$. In particular, the canonical homomorphism

$$H^r(C_{\varepsilon'}^+, \mathcal{F}) \xrightarrow{\delta} H^r(\overline{C}_\varepsilon^+, \mathcal{F}) \tag{5.4}$$

is onto for $1 \le r \le p(\overline{C}_\varepsilon^+; \mathcal{F}) - 2$. □

From Proposition 5.7 we derive

Proposition 5.8 *For every $\varepsilon \in (0, \varepsilon_0)$ there exists an $\overline{\varepsilon} < \varepsilon$ such that for every $\varepsilon' \in [\overline{\varepsilon}, \varepsilon)$ the homomorphism*

$$H^r(C_{\varepsilon'}^+, \mathcal{F}) \xrightarrow{\delta} H^r(\overline{C}_\varepsilon^+, \mathcal{F}) \tag{5.5}$$

is onto for $1 \le r \le p(\overline{C}_\varepsilon^+; \mathcal{F}) - 2$.

Proof. We fix ε_0 as in Lemma 5.6. Let Λ be the (non-empty) set of the positive numbers $\varepsilon' < \varepsilon$ such that the homomorphism (5.4) is onto and $\overline{\varepsilon} = \inf \Lambda$. It follows (cfr. [3, Lemma pag. 241] for closed subsets) that the homomorphism (5.5) is onto. □

A second consequence of Proposition 5.7 is the following finiteness theorem

Theorem 5.9 *Under the conditions of Theorem 5.4, there exists $\varepsilon_1 \le \varepsilon_0$ such that*

$$\dim_{\mathbb{C}} H^1(\overline{C}_\varepsilon^+, \mathcal{F}) < +\infty.$$

for every $\varepsilon \in (0, \varepsilon_1)$.

Proof. We first observe the following. Let $\Omega \subset \mathbb{C}^n$ be a domain, $K \subset \Omega$ a compact subset. It is known that $\mathcal{F}(\Omega)$ is a Fréchet space. The space $\mathcal{F}(K)$ is an $\mathcal{LF}$-space i.e. a direct limit of Fréchet spaces and its topology is complete (cfr. [36, pag. 315]). Moreover, the restriction

$$\mathcal{F}(\Omega) \xrightarrow{\delta} \mathcal{F}(K)$$

is a compact map i.e. there exists a neighborhood U of the origin in $\mathcal{F}(\Omega)$ such that $\overline{\delta(U)}$ is a compact subset of $\mathcal{F}(K)$. This is a consequence of the following well known fact: if Ω' is a relatively compact subdomain of Ω then the restriction $\mathcal{F}(\Omega) \to \mathcal{F}(\Omega')$ is a compact map. Take ε_0 as in Lemma 5.6. The proof is similar to that of Théorème 11 in [3] taking into account the following

1) Leray theorem for acyclic closed coverings (see Théorème 5.2.4 and Corollaire in [35]);

2) the theorem of L. Schwartz on compact perturbations $u+v$ of a surjective linear operator $u : E \to F$ where E is a Fréchet (see [36, Corollaire 1]).

□

We remark that, up to some modifications in the technical details of the proof, the finiteness result holds for all cohomology groups:

Theorem 5.10 *Under the conditions of Theorem 5.4 there exists $\varepsilon_1 \le \varepsilon_0$ such that*

$$\dim_{\mathbb{C}} H^r(\overline{C}^+_\varepsilon, \mathcal{F}) < +\infty,$$

for every $\varepsilon \in (0, \varepsilon_1)$ and $1 \le r \le p(\overline{C}^+_\varepsilon; \mathcal{F}) - 2$.

5.3.2 Approximation

This subsection is devoted to approximation by global sections.

Lemma 5.11 *Let $\operatorname{depth}(\mathcal{F}_z) \ge 4$ for every $z \in \{\varphi = \varepsilon\}$, $\varepsilon \in (0, \varepsilon_0)$. Then, for every $z^0 \in \mathrm{b}C^+_\varepsilon$ there exists a closed neighbourood $\overline{U}$ of z^0 such that the homomorphism*

$$H^0(\overline{U}, \mathcal{F}) \longrightarrow H^0(\overline{U} \cap \overline{C}^+_\varepsilon, \mathcal{F})$$

is dense image.

Proof. This is known if $z^0 \in \Gamma_1 \cup \Gamma_2$ i.e. when z^0 is a point of pseudoconvexity or pseudoconcavity (see [3]), thus we may assume that $z^0 \in \overline{\Gamma}_1 \cap \overline{\Gamma}_2$. First we consider the case $\mathcal{F} = \mathcal{O}$. We may suppose that there exists a sufficiently small closed ball $\overline{B}$ of positive radius, centered at z^0 such that $\overline{B} \cap \{\varphi \geq \varepsilon\}$ and $\overline{B} \cap \{P_\varepsilon \leq 0\}$ respectively are strictly concave and strictly convex (again, locally, up to a biholomorphism). Take a real hyperplane with equation $l = 0$ such that $z^0 \in \{l > 0\}$ and $\{l = 0\} \cap \{\varphi \leq \varepsilon\} \Subset B$. Let $\psi = \alpha\varphi - \varepsilon + \beta l$, α, β positive real numbers; ψ is strongly plurisubharmonic. For α, β sufficiently small the hypersurface $\{\psi = 0\} \cap \{l < 0\}$ is a portion of a compact smooth hypersupersurface which bounds a domain $D \Subset B$. Set

$$\overline{V} = \{P_\varepsilon \leq 0\} \cap \overline{D},\ \overline{W} = \overline{D} \setminus \{\varphi < \varepsilon\}$$

and $\overline{U}' = \overline{V} \cap \overline{W}$. We are going to prove that $H^1(\overline{V} \cup \overline{W}, \mathcal{O}) = 0$. Let $R = \overline{D} \setminus \overline{V} \cup \overline{W}$. Since $\overline{D}$ is a Stein compact, from the exact sequence of cohomology relative to the closed subspace $\overline{V} \cup \overline{W}$ we get the isomorphism

$$H^r(\overline{V} \cup \overline{W}, \mathcal{O}) \simeq H_c^{r+1}(R, \mathcal{O}). \tag{5.6}$$

for $r \leq n-2$. R is an open subset of $S = \overline{D} \cap \{\varphi < \varepsilon\}$. Set $R' = S \setminus R$. Again, by the exact sequence of cohomology with compact supports relative to the closed subspace $R' = S \setminus R$ we get the exact sequence of groups

$$\cdots \longrightarrow H_c^r(S, \mathcal{O}) \longrightarrow H_c^r(R', \mathcal{O}) \longrightarrow$$

$$\longrightarrow H_c^{r+1}(R, \mathcal{O}) \longrightarrow H_c^{r+1}(S, \mathcal{O}) \longrightarrow \cdots .$$

Since S and R' have a fundamental system of Stein neighborhoods (see [95]) and $n \geq 3$, we have

$$H_c^r(S, \mathcal{O}) = H_c^r(R', \mathcal{O}) = 0$$

for $1 \leq r \leq n-2$ and consequently $H_c^r(R, \mathcal{O}) = 0$ for $1 \leq r \leq n-2$. In view of the isomorphism (5.6) we obtain

$$H^r(\overline{V} \cup \overline{W}, \mathcal{O}) = 0$$

for $1 \leq r \leq n-2$. In particular, since $n \geq 3$, (5.6) implies that

$$H^1(\overline{V} \cup \overline{W}, \mathcal{O}) = 0,$$

thus that every function $f \in \mathcal{O}(\overline{U}')$ is a difference of two functions $f_1 - f_2$ where $f_1 \in \mathcal{O}(\overline{V})$, $f_2 \in \mathcal{O}(\overline{W})$. Since $\overline{V}$ is Runge in $\overline{D}$ there exists a sequence of holomorphic functions $f_\nu \in \mathcal{O}(\overline{D})$ such that $f_\nu \to f_1$ in $\mathcal{O}(\overline{V})$. Moreover, by Theorem 3.3 (see [65]) the function f_2 extends holomorphically to $W \cap \{l \leq$

$0\}$. Choose a smooth function $\varrho \in \mathcal{C}_0^\infty(D)$ such that $\varrho \geq 0$ and $\varrho(z^0) \neq 0$ and a positive number λ such that the closed domains

$$\overline{D}_1 = \{\varphi - \varepsilon + \lambda\varrho \leq 0\} \cap \overline{D},\ \overline{D}_2 = \{P_\varepsilon - \lambda\rho \leq 0\} \cap \overline{D}$$

are respectively strongly pseudoconcave and strongly pseudoconvex, both contain z^0 as an interior point, $\mathrm{b}D_1 \smallsetminus \{\varphi = \varepsilon\} \cap D$ is relatively compact in $D \cap \{l > 0\}$ and $\mathrm{b}D_2 \smallsetminus \{P_\varepsilon = 0\}$ is relatively compact in D. Then we define $\overline{U} = \overline{D}_1 \cap \overline{D}_2$.

Observe that, by construction, Proposition 5.5 applies, thus $H^r(\overline{U} \cap C_\varepsilon^+, \mathcal{O}) = 0$ for $1 \leq r \leq n-2$.

In the general case, since $\overline{D}$ is Stein, we have on $\overline{D}$ an exact sequence

$$0 \longrightarrow \mathcal{H} \xrightarrow{\ \alpha\ } \mathcal{O}^q \xrightarrow{\ \beta\ } \mathcal{F} \longrightarrow 0\,.$$

Consider the following commutative diagram of continuous maps

$$\begin{array}{ccc}
H^0(\overline{U}, \mathcal{O}^q) & \xrightarrow{\quad\alpha\quad} & H^0(\overline{U}, \mathcal{F}) \\
\Big\downarrow{\scriptstyle \mathsf{r}} & & \Big\downarrow{\scriptstyle \mathsf{r}} \\
H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{O}^q) & \xrightarrow{\ \beta\ } & H^1(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{F})
\end{array}$$

where r denotes the natural restriction. Then, since $\mathrm{depth}(\mathcal{F}_z) \geq 4$ for every $z \in D$, we have $\mathrm{depth}(\mathcal{H}_z) \geq 5$ for every $z \in D$. Again by Proposition 5.5 we have $H^1(\overline{U} \cap C_\varepsilon^+, \mathcal{F}) = 0$, whence the homomorphism

$$H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{O}^q) \longrightarrow H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{F})$$

is onto. Let $\sigma \in H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{F})$ and N a neighborhood of σ. Let $g \in H^0(\overline{U}, \mathcal{O}^q)$ be such that $\beta(g) = \sigma$. Since the homorphism

$$H^0(\overline{U}, \mathcal{O}^q) \longrightarrow H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{O}^q)$$

is dense image there exists $h \in H^0(\overline{U}, \mathcal{O}^q)$ such that $\mathsf{r}(h) \in \beta^{-1}(N)$. Then $\mathsf{r}(\alpha(h)) \in N$ with $\alpha(h) \in H^0(\overline{U}, \mathcal{F})$. This shows that the homomorphism

$$H^0(\overline{U}, \mathcal{O}^q) \longrightarrow H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{O}^q)$$

is dense image. $\square$

Lemma 5.12 *Let $\mathcal{F}$ and ε_0 be as in Lemma 5.11. Then for every $\varepsilon \in (0, \varepsilon_0)$ there exists $\varepsilon_2 \in (0, \varepsilon)$ such that for every $\varepsilon' \in (\varepsilon_2, \varepsilon)$ the homomorphism*

$$H^0(\overline{C}_{\varepsilon'}^+, \mathcal{F}) \longrightarrow H^0(\overline{C}_\varepsilon^+, \mathcal{F})$$

is dense image.

Proof. With the notations of Lemma 5.6 we have

$$\overline{D} = \overline{C}_{\varepsilon}^{+},\ \overline{D}_1 = \overline{D} \cup \overline{B},\ \overline{D}_1 = \overline{D} \cup (\overline{D}_1 \cap \overline{B})$$

and we set $\overline{V} = \overline{D}_1 \cap \overline{B}$. In view of Lemma 5.11 we may assume that the homomorphism

$$H^0(\overline{V}, \mathcal{F}) \longrightarrow H^0(\overline{V} \cap \overline{D}, \mathcal{F})$$

is dense image. Moreover, $H^1(\overline{V}, \mathcal{F}) = 0$. Let $\overline{\mathcal{U}}$ be the closed covering $\{\overline{D}, \overline{V}\}$ of $\overline{D}_1$, $\mathsf{Z}^1(\overline{\mathcal{U}}, \mathcal{F})$ and $\mathsf{B}^1(\overline{\mathcal{U}}, \mathcal{F})$ respectively the space of cocycles and coboundaries of $\overline{\mathcal{U}}$ with values in $\mathcal{F}$. Since $H^1(\overline{\mathcal{U}}, \mathcal{F})$ is a subgroup of $H^1(\overline{D}_1, \mathcal{F})$ which is of finite dimension (cfr. Theorem 5.9) we have

$$\dim_{\mathbb{C}} H^1(\overline{\mathcal{U}}, \mathcal{F}) < +\infty.$$

It follows that $H^1(\overline{\mathcal{U}}, \mathcal{F})$ is of finite dimension in the $\mathcal{LF}$-space $\mathsf{Z}^1(\overline{\mathcal{U}}, \mathcal{F})$, thus an $\mathcal{LF}$-space for the induced topology. Moreover, in view of the Banach open mapping theorem the suriective map

$$H^0(\overline{D}, \mathcal{F}) \oplus H^0(\overline{V}, \mathcal{F}) \longrightarrow \mathsf{B}^1(\overline{\mathcal{U}}, \mathcal{F})$$

given by $s \oplus \sigma \mapsto s_{|\overline{D} \cap \overline{V}} - \sigma_{|\overline{D} \cap \overline{V}}$ is a topological homomorphism.

Let $s \in H^0(\overline{D}, \mathcal{F})$; $s_{|\overline{V} \cap \overline{D}} \in \mathsf{B}^1(\overline{\mathcal{U}}, \mathcal{F})$. By Lemma 5.11, there exists a sequence $\{s_\nu\} \subset H^0(\overline{V}, \mathcal{F})$ such that

$$s_{\nu|\overline{V} \cap \overline{D}} - s_{|\overline{V} \cap \overline{D}} \longrightarrow 0.$$

In view of Banach theorem there exist two sequences $\sigma_\nu^1 \in H^0(\overline{D}, \mathcal{F})$, $\sigma_\nu^2 \in H^0(\overline{V}, \mathcal{F})$ such that

$$\sigma^1_{\nu|\overline{D} \cap \overline{V}} - \sigma^2_{\nu|\overline{D} \cap \overline{V}} = s_{\nu|\overline{D} \cap \overline{V}} - s_{|\overline{D} \cap \overline{V}},$$

$$\sigma_\nu^1 \to 0,\ \ \sigma_\nu^2 \to 0.$$

It follows that for every ν

$$\tilde{s}_\nu = \begin{cases} s - \sigma_\nu^1 & \text{on } \overline{D} \\ s_\nu - \sigma_\nu^2 & \text{on } \overline{V} \end{cases}$$

is a section of $\mathcal{F}$ on $\overline{D}_1$ and that $\tilde{s}_\nu \to s$. In order to end the proof we apply this procedure a finite numbers of times. $\square$

As a corollary we get the following

Proposition 5.13 *Let $\mathcal{F}$ and ε_0 be as in Theorem 5.4. Then for every $\varepsilon \in (0, \varepsilon_0)$ there exists $\overline{\varepsilon}_0 \in [0, \varepsilon)$ such that for every $\varepsilon' \in (\overline{\varepsilon}_0, \varepsilon]$ the homomorphism*

$$H^0(C^+_{\varepsilon'}, \mathcal{F}) \longrightarrow H^0(\overline{C}^+_{\varepsilon}, \mathcal{F})$$

is dense image.

Proof. Let $I \subset (0, \varepsilon_0)$ be the (non-empty) set of $\varepsilon' < \varepsilon$ such that the homomorphism

$$H^0(\overline{C}^+_{\varepsilon'}, \mathcal{F}) \longrightarrow H^0(\overline{C}^+_{\varepsilon}, \mathcal{F})$$

is dense image. Let $\overline{\varepsilon} = \inf I$ and $\{\varepsilon_\nu\}$ be a decreasing sequence with $\varepsilon_0 = \varepsilon$, $\varepsilon_\nu \to \overline{\varepsilon}$ and set $F_\nu = H^0(\overline{C}^+_{\varepsilon_\nu}, \mathcal{F})$. The topology of F_ν can be defined by an increasing sequence $\{p^{(\nu)}_j\}_{j\in\mathbb{N}}$ of translation invariant seminorms. Let for $\nu \geq 1$

$$\mathsf{r}_\nu : F_\nu \longrightarrow F_{\nu-1}$$

be the restriction map; then

$$H^0(\overline{C}^+_{\overline{\varepsilon}}, \mathcal{F}) = \varprojlim_{\{\mathsf{r}_\nu\}} F_\nu$$

and denote by $\pi_\nu : H^0(\overline{C}^+_{\varepsilon}, \mathcal{F}) \to F_\nu$ the natural map. We have to show that π_0 is dense image.

Let $s \in F_0 = H^0(\overline{C}^+_{\varepsilon}, \mathcal{F})$ and N a neighborhood of s_0. We may assume that

$$N = \left\{ s \in F_0 : p^{(0)}_0(s - s_0) < \varepsilon \right\}.$$

Since the maps r_ν are continuous and dense image we can choose elements $s_\nu \in F_\nu$, for $\nu \geq 0$, satisfying the following conditions:

$$s_1 \in F_1 \quad p^{(0)}_0\left(\mathsf{r}_1(s_1) - s_0\right) < \varepsilon/2$$

$$s_2 \in F_2 \quad p^{(1)}_0\left(\mathsf{r}_2(s_2) - s_1\right) < \varepsilon/2$$

$$p^{(0)}_1\left(\mathsf{r}_1\mathsf{r}_2(s_2) - \mathsf{r}_1(s_1)\right) < \varepsilon/2^2$$

$$s_3 \in F_3 \quad p^{(2)}_0\left(\mathsf{r}_3(s_3) - s_2\right) < \varepsilon/2$$

$$p^{(1)}_1\left(\mathsf{r}_2\mathsf{r}_3(s_3) - \mathsf{r}_2(s_2)\right) < \varepsilon/2^2$$

$$p^{(0)}_2\left(\mathsf{r}_1\mathsf{r}_2\mathsf{r}_3(s_3) - \mathsf{r}_1\mathsf{r}_2(s_2)\right) < \varepsilon/2^3$$

and so on. Then, for every $\nu \in \mathbb{N}$, the series

$$s_\nu + (\mathsf{r}_{\nu+1}(s_{\nu+1}) - s_\nu) + (\mathsf{r}_{\nu+1}\mathsf{r}_{\nu+2}(s_{\nu+2}) - \mathsf{r}_{\nu+1}(s_{\nu+1})) + \dots$$

is convergent in F_ν and $\mathsf{r}_\nu(\sigma_\nu) = \sigma_{\nu-1}$. Hence $\sigma = \{\sigma_\nu\}_{\nu\in\mathbb{N}}$ belongs to $H^0(C^+_{\overline{\varepsilon}}, \mathcal{F})$ and, by definition $p_0^{(0)}(\sigma_0 - s_0) < \varepsilon$, i.e. $\pi_0(\sigma_0) \in N$. $\square$

Proof of Theorem 5.4. The proof uses Corollary 5.7 and Lemma 5.11. With the notations of Lemma 5.6 we have

$$\overline{D} = \overline{C}^+_\varepsilon,\ \overline{D}_1 = \overline{D} \cup \overline{B},\ \overline{D}_1 = \overline{D} \cup (\overline{D}_1 \cap \overline{B}),\ \overline{D} \cap (\overline{D}_1 \cap \overline{B}) = \overline{D} \cap \overline{B}.$$

We may assume that the homomorphism

$$H^1(\overline{D}_1, \mathcal{F}) \longrightarrow H^1(\overline{D}, \mathcal{F})$$

is onto and

$$H^0(\overline{D}_1, \mathcal{F}) \longrightarrow H^0(\overline{D}, \mathcal{F})$$

is dense image. Moreover, $H^1(\overline{B} \cap \overline{D}_1, \mathcal{F}) = 0$. Thus it is sufficient to show that the homomorphism

$$H^1(\overline{D}_1, \mathcal{F}) \longrightarrow H^1(\overline{D}, \mathcal{F})$$

is injective.

Since $H^1(\overline{D}_1 \cap \overline{B}, \mathcal{F}) = 0$ the Mayer-Vietoris exact sequence applied to $\overline{D}_1 = \overline{D} \cup (\overline{D}_1 \cap \overline{B})$ gives the exact sequence

$$H^0(\overline{D} \cap \overline{B}, \mathcal{F}) \xrightarrow{\mathsf{a}} H^1(\overline{D}_1, \mathcal{F}) \xrightarrow{\mathsf{b}} H^1(\overline{D}, \mathcal{F}) .$$

Let $\xi \in \mathsf{Ker}\,\mathsf{b} = \mathsf{Im}\,\mathsf{a}$, $\xi = \mathsf{a}(\eta)$ with $\eta \in H^0(\overline{D} \cap \overline{B}, \mathcal{F})$. By Lemma 5.11 η is approximated by a sequence $\{\eta_\nu\} \subset H^0(\overline{D}_1 \cap \overline{B}, \mathcal{F})$. Each η_ν is a 1-coboundary of the closed covering $\mathcal{U} = \left\{\overline{D}, \overline{D}_1 \cap \overline{B}\right\}$ with values in $\mathcal{F}$ and such a space is closed in the space $\mathsf{Z}^1(\mathcal{U}, \mathcal{F})$ of the 1-cocycles. This proves that η is a 1-coboundary of $\{\mathcal{U}, \mathcal{F}\}$, whence $\xi = \mathsf{a}(\eta) = 0$. $\square$

Remark 5.2 In the full q-corona the cohomology of all coronae are isomorphic (see [3]). Differently, in the semi 1-corona case the cohomology groups are isomorphic up to a critical $\overline{\varepsilon}$, where the dimension of the cohomology spaces jumps, then they are again all isomorphic up to a second critical value, and so on. They must not be all isomorphic, nor they dimensions must be bounded.

5.4 Extension of coherent sheaves and analytic subsets

An interesting consequence is that on a semi 1-corona $C^+ = C^+_{0,1}$ Theorem A of Oka-Cartan-Serre holds for a coherent sheaf $\mathcal{F}$ satisfying the conditions of Theorem 5.4. We first prove the following

Lemma 5.14 *Let X be a complex space, $\mathcal{F} \in \mathsf{Coh}(X)$ satisfying the following property: for every $x \in X$ there exists a subset $Y \not\ni x$ of X such that:*

i) $H^1(X, \mathcal{F}) \simeq H^1(Y, \mathcal{F})$

ii) if $\mathcal{M}_{[x]}$ denotes the ideal of $\{x\}$ the homomorphism

$$H^1(X, \mathcal{M}_{[x]}\mathcal{F}) \longrightarrow H^1(Y, \mathcal{M}_{[x]}, \mathcal{F})$$

is injective.

Then, for every $x \in X$ the space $H^0(X, \mathcal{F})$ of the global sections of $\mathcal{F}$ generates $\mathcal{F}_x$ over $\mathcal{O}_{X,x}$.

Proof. Let $x \in X$ and Y satisfying the conditions of the lemma. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{M}_{[x]}\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{M}_{[x]}\mathcal{F} \longrightarrow 0$$

and the associated diagram

$$\begin{array}{ccccccc} H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}/\mathcal{M}_{[x]}\mathcal{F}) \to & H^1(X, \mathcal{M}_{[x]}\mathcal{F}) & \xrightarrow{\delta} & H^1(X, \mathcal{F}) \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ & H^1(Y, \mathcal{M}_{[x]}\mathcal{F}) & \xrightarrow{\gamma} & H^1(Y, \mathcal{F}). \end{array}$$

The homomorphism is injective by hypothesis and β is an isomorphism since $\mathcal{M}_{[x]|Y} \simeq \mathcal{F}_{|Y}$, thus γ is an isomorphism. It follows that δ is injective and consequently that the homomorphism

$$H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}/\mathcal{M}_{[x]}\mathcal{F}) \simeq \mathcal{F}_x/\mathcal{M}_{[x],x}\mathcal{F}_x$$

is onto. Then the Lemma of Nakayama implies that

$$H^0(X, \mathcal{F}) \longrightarrow \mathcal{F}_x$$

is onto and this proves the lemma. $\square$

Keeping the notations of the proof of Theorem 5.4, we deduce the following

Corollary 5.15 *Under the conditions of Theorem 5.4 for every compact subset*

$$K \subset C^+_{\varepsilon'} \cap \{\varphi > \varepsilon'\} \cap \{P_{\varepsilon'} < 0\}$$

there exist sections $s_1, \ldots, s_k \in H^0(C^+_{\varepsilon'}, \mathcal{F})$ *which generate* $\mathcal{F}_z$ *for every* $z \in K$.

Theorem 5.16 *Let* $C^+ = (B_1 \smallsetminus B_0) \cap \{h \geq 0\}$ *and* $\mathcal{F} \in \mathsf{Coh}(C^+)$. *If* $\operatorname{depth}(\mathcal{F}_z) \geq 3$ *on* $\{\varphi = 0\}$ *then for every* $a > 0$ *near* 0 $\mathcal{F}_{|B_1 \smallsetminus \overline{B}_a}$ *extends on* $B_1 \cap \{h \geq 0\}$ *by a coherent sheaf* $\widetilde{\mathcal{F}}_a$.

Proof. With the usual notations choose $\varepsilon_0 \in (0, a)$, and $c_0 > 0$ such that

i) $\mathcal{F}$ is defined on the semi 1-corona $(B_1 \smallsetminus B_{-\varepsilon}) \cap \{h > -c\}$

ii) $\{z \in B_1 : h(z) \geq c\} \Subset \{z \in B_1 : P_\varepsilon(z) < 0\}$

iii) for every $\varepsilon \in (0, \varepsilon_0)$, $c \in (0, c_0)$ the hypersurfaces $\{P_\varepsilon = -c\}$, $P_\varepsilon(z) = \varepsilon|z|^2 - h$, meet the hypersurfaces $\{\varphi = \varepsilon\}$, $\{\varphi = -\varepsilon\}$ transversally.

Let $Y^+_{\alpha,\beta}$ denote the semi 1-corona $\{\alpha < \varphi < \beta\} \cap \{h > c\}$, with $\alpha < \beta < \varepsilon$. In view of Corollary 5.15 applied to the semi 1-corona $Y^+_{\varepsilon,a}$ there exist $\alpha, \beta, \gamma \in (0, a)$ with $\alpha < \beta < \gamma$ such that $H^0(Y^+_{\alpha,\gamma}, \mathcal{F})$ generates $\mathcal{F}$ on $K_{\beta,\gamma} = \overline{Y}^+_{\beta,\gamma} \cap \{h \geq 0\}$. Thus on $K_{\beta,\gamma}$ there exists an exact sequence

$$\mathcal{O}^p \xrightarrow{\ \beta\ } \mathcal{F} \longrightarrow 0\,.$$

Since, by hypothesis, $\operatorname{depth}(\mathcal{F}_z) \geq 3$ for every $z \in K_{\beta,\gamma}$ we have

$$\operatorname{depth}(\mathsf{Ker}\,\alpha) \geq 4$$

on $K_{\beta,\gamma}$ (cfr. [12]). Again by Corollary 5.15 there exist $\beta_1, \gamma_1 \in (\beta, \gamma)$, $\beta_1 < \gamma_1$ and sections $\sigma_1, \ldots, \sigma_l$ on $K_{\beta_1,\gamma_1} = \overline{Y}^+_{\beta_1,\gamma_1} \cap \{h \geq 0\}$ which generate $(\mathsf{Ker}\,\alpha)_z$ for every $z \in V$. Since $\mathsf{Ker}\,\alpha$ is a subsheaf of $\mathcal{O}^p$, by the theorem in [65] the sections $\sigma_1, \ldots, \sigma_l$ extend holomorphically on

$$\{\varphi \leq \gamma_1\} \cap \{h \geq 0\}$$

and their extensions $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_l$ generate a coherent sheaf $\mathcal{H}$ on

$$\{\varphi \leq \gamma_1\} \cap \{h \geq 0\}\,.$$

Let $\widetilde{\mathcal{F}'}_a$ be the sheaf defined by

$$\widetilde{\mathcal{F}'}_{a,z} = \begin{cases} \mathcal{F}_z & \text{for } z \in \{\varphi > \gamma_1\} \cap \{h \geq 0\} \\ \mathcal{O}_z/\mathcal{H}_z & \text{for } z \in \{\varphi \leq \gamma_1\} \cap \{h \geq 0\}\,; \end{cases}$$

$\widetilde{\mathcal{F}'}_\varepsilon$ is a coherent sheaf on $B^+_c \cap \{y_n > \varepsilon\}$ extending $\mathcal{F}$. $\square$

Corollary 5.17 *Let* $X^+ = \big(B_1 \smallsetminus \overline{B}_0\big) \cap \{h > 0\}$ *and* Y *be an analytic subset of* X^+ *such that* $\operatorname{depth}(\mathcal{O}_{Y,z}) \geq 3$ *for* z *near* $\{\varphi = 0\}$*. Then* Y *extends on* $B_1 \cap \{h \geq 0\}$ *by an analytic subset.*

Proof. We apply Theorem 5.16 to $X^+ \cap \{h \geq \varepsilon\}$, where $\varepsilon \sim 0$ is positive. Then, for $\nu \in \mathbb{N}$ there exists a coherent sheaf $\widetilde{\mathcal{O}}_Y^{(\nu)}$ on $B_1 \cap \{h \geq 0\}$ which extends $\mathcal{O}_Y$; $\widetilde{Y}^{(\nu)} = \operatorname{supp} \widetilde{\mathcal{O}}_Y^{(\nu)}$ is an analytic subset extending $Y \cap \big(B_1 \smallsetminus B_{1/\nu}\big) \cap \{h \geq \varepsilon\}$. In view of the strong pseudoconvexity of $\mathrm{b}B_{1/\nu}$, the subset $F_\nu = \widetilde{Y}^{(\nu)} \smallsetminus \widetilde{Y}^{(\nu+1)}$ is a finite set of points which is contained in $B_{1/\nu}$. Start by $\nu = 2$ and consider the first extension $\widetilde{Y}^{(2)}$. Then $\widetilde{Y}^{(2)} \smallsetminus F_2 \cap \big(B_{1/2} \smallsetminus B_{1/3}\big)$ coincide with Y on $\big(B_1 \smallsetminus B_{1/3}\big)$ and so on. To handle different extensions depending on ε we argue in the same way. □

5.5 Some generalizations

We conclude the capter with a few remarks that simply outline some possible generalizations of the above results.

5.5.1 Bump lemma for semi q-coronae

One can also deal with a bump lemma for semi q-coronae when $q > 1$.

A bump lemma for semi q-coronae would be similar to Lemma 5.6, but with the cohomology vanishing result in (iv) only for $q \leq r \leq p(\overline{D}; \mathcal{F}) - q - 1$. The proof follows the very same lines. Then it follows, analoguously to Prposition 5.7, that the homomorphism

$$H^r(\overline{C}^+_{\varepsilon'}, \mathcal{F}) \to H^r(\overline{C}^+_{\varepsilon}, \mathcal{F}) \tag{5.7}$$

is onto for $q \leq r \leq p(\overline{C}^+_{\varepsilon}; \mathcal{F}) - q - 1$. The finiteness result on cohomology (Theorem 5.9) this time cannot be achieved.

In order to conclude and get an isomorphism theorem, one must show also that the maps of $\mathcal{LF}$-spaces

$$\mathsf{Z}^{q-1}(\overline{C}^+_{\varepsilon'}, \mathcal{F}) \to \mathsf{Z}^{q-1}(\overline{C}^+_{\varepsilon}, \mathcal{F}) \tag{5.8}$$

is dense image. From (5.7) and (5.8) then the homomorphism

$$H^q(C^+_{\varepsilon'}, \mathcal{F}) \to H^q(C^+_{\varepsilon}, \mathcal{F})$$

turns out to be an isomorphism if the depth of $\mathcal{F}$ is less then $2q+1$. Anyhow this result is far more less interesting then its corresponding Theorem 5.4, since no corollary of extension for analytic sets follows from it.

5.5.2 Semi q-coronae in Stein spaces

To treat the cohomology problem for a semi q-corona $C_+ \subset A$, where A is an analytic 1-complete (i.e. Stein) space, a few observations are needed.

First of all for each point $p \in A$, there is a neighborhood V where $A \cap V$ can be embedded into $U \subset \mathbb{C}^n$. Then one extends the sheaf $\mathcal{O}_{A \cap V}$ to a sheaf $\mathcal{O}_{A,U}$ defined on the whole U by setting it equal to 0 outside the image of $A \cap V$. The depth of the sheaf $\mathcal{O}_{A,U}$ coincides with the depth of $\mathcal{O}_{A \cap V}$.

Then one concludes by using this sheaf in all the steps for proving the bump lemma as in the previous sections.

Chapter 6

The boundary problem

6.1 The boundary problem

Let M be a smooth and oriented real $(2m+1)$-submanifold of some n-complex manifold X. A natural question arises, whether M is the boundary of an $(m+1)$-complex analytic subvariety of X. This problem, the so called *boundary problem*, has been widely treated over the past fifty years when M is compact and X is $\mathbb{C}^n$ or $\mathbb{CP}^n$.

The case when M is a compact, connected curve in $X = \mathbb{C}^n$ $(m = 0)$, has been first solved by Wermer [103] in 1958. Later on, in 1975, Harvey and Lawson in [38, 39] solved the boundary problem in $\mathbb{C}^n$ and then in $\mathbb{CP}^n \setminus \mathbb{CP}^r$, in terms of holomorphic chains. The boundary problem in $\mathbb{CP}^n$ was studied by Dolbeault and Henkin, in [27] for $m = 0$ and in [28] for any m. Moreover, in these two papers the boundary problem is dealt with also for closed submanifolds (with negligible singularities) contained in q-concave (i.e. union of $\mathbb{CP}^q$'s) open subsets of $\mathbb{CP}^n$. This allows M to be non-compact. The results in [27, 28] were extended by Dinh in [25].

In the very last few years (2004–2006) Harvey, Lawson and Wermer (see [41–47, 104]) took a new look at the problem in order to try solve it in $\mathbb{CP}^n$. The point of view is that of generalizing Wermer's classical approach [103] that used polynomial hulls, by introducing the concept of projective hull. In $\mathbb{CP}^n$ the solution to the boundary problem is no longer unique, since to any solution can be added closed manifold without boundary. Hence there are two main interests: one is finding the minimal solution, the other is finding a solution with a prescribed number of sheets over a fixed point.

In this chapter an overview of the problem and of the estabilished theorems is given, without totally complete proofs, but sketching some of the basic ideas. Also the new areas of interest of this rich and interesting problem are described in here.

The final two chapters of this thesis will be devoted to some results on the boundary problem in $\mathbb{C}^n$ in the non-compact case, joint work with Giuseppe Della Sala [15, 16].

6.2 The boundary problem for compact curves

Let $S \Subset \mathbb{C}^n$ be a compact set, and $\mathcal{P}_n$ be the algebra of polynomials in $\mathbb{C}^n$. Let us consider the polynomial hull $\widehat{S}_{\mathcal{P}}$ of S, as defined in Subsection 1.1.3. Šilov [86] proved that, if S is connected, then also $\widehat{S}_{\mathcal{P}}$ is connected.

John Wermer, in [103], found a nice link between the boundary problem and the polynomial hull of curves. More precisely, let $\Gamma \Subset \mathbb{C}^n$ be a curve given parametrically by the equations

$$z_j = \varphi_j(u), \quad j = 1, \ldots, k, \quad u \in \mathbb{S}^1, \tag{6.1}$$

such that each φ_j is analytic in an annulus around $\mathbb{S}^1$, the functions φ_j separate points of $\mathbb{S}^1$, and $\varphi_j'(u) \neq 0$ for every $u \in \mathbb{S}^1$. In particular, Γ is a simple closed curve in $\mathbb{C}^n$.

Theorem 6.1 *Let $\Gamma \Subset \mathbb{C}^n$ be a curve given as in* (6.1)*. Then either $\widehat{\Gamma}_{\mathcal{P}} = \Gamma$ or $\widehat{\Gamma}_{\mathcal{P}}$ is an analytic surface, with at most finitely many branching points, with boundary Γ.*

Moreover, a moments' condition is shown to be a necessary and sufficient condition for Γ to be the boundary of an analytic surface:

Theorem 6.2 *Let $\Gamma \Subset \mathbb{C}^n$ be a curve given as in* (6.1)*. Γ is the boundary of an analytic surface ($\widehat{\Gamma}_{\mathcal{P}}$) if and only if for every multiindex $\alpha \in \mathbb{N}^n$*

$$\int_{u \in \mathbb{S}^1} \varphi^{\alpha}(u) \cdot \varphi_1'(u)\, du \;=\; 0, \tag{6.2}$$

where $\varphi = (\varphi_1, \ldots, \varphi_n)$.

6.2.1 Sketches of the proofs

Let $g_1, \ldots, g_k \in \mathcal{C}^0(\mathbb{S}^1)$. By $[g_1, \ldots, g_k] \subset \mathcal{C}^0(\mathbb{S}^1)$ we will denote the algebra of complex polynomials in the variables $g_1, \ldots, g_k$, and by $K[g_1, \ldots, g_k]$ its closure in $\mathcal{C}^0(\mathbb{S}^1)$.

The following extension result is at the core of the proof. Its proof can be found in [101, 103].

Theorem 6.3 *Let $f_1, f_2 \in \mathcal{C}^0(\mathbb{S}^1)$. Assume that f_1, f_2 are analytic in an annulus around $\mathbb{S}^1$, they separate points of $\mathbb{S}^1$, f_1 takes only finitely values more than once on $\mathbb{S}^1$, and $K[f_1, f_2] \neq \mathcal{C}^0(\mathbb{S}^1)$.*

Then there exist a relatively compact domain $D \Subset \mathcal{R}$ of a Riemann surface, with boundary $bD = \gamma$ a simple closed analytic curve, and a bijective conformal map χ of a neighborhood of γ on a neighborhood of $\mathbb{S}^1$ such that, defining

$$F_1(p) = f_1(\chi(p)), \quad F_2(\chi(p)), \quad p \in \gamma,$$

F_1 and F_2 have extensions in $\mathcal{A}^0(D)$.

We will also need the following theorems. Their proofs can be found respectively in [102] and in [48, 49].

Theorem 6.4 (Wermer) *Let $D \Subset \mathcal{R}$ be a relatively compact domain of a Riemann surface, with boundary $bD = \gamma$ a simple closed analytic curve. Let F_1 and F_2 be two analytic functions on $D \cup \gamma$ that separate points on γ and $dF_1 \neq 0$ on γ.*

Then there exist only finitely many pairs of points $p, q \in D \cup \gamma$ such that $F_1(p) = F_1(q)$ and $F_2(p) = F_2(q)$.

Theorem 6.5 (Helson-Quigley) *Let $S \subset \mathbb{C}^n$ be a differentiable curve such that $\widehat{S}_{\mathcal{P}} = S$. Let $z_1, \ldots, z_n$ be the coordinate functions in $\mathbb{C}^n$ restricted to S. Then $K[z_1, \ldots, z_n] = \mathcal{C}^0(S)$.*

A few lemmata are necessary before entering the proof of Theorem 6.1 and Theorem 6.2.

Lemma 6.6 *Let A be a set, $\mathcal{A}$ an algebra of functions on A which separates points on A. If $p_1, \ldots, p_N \in A$ are distinct, there is $g \in \mathcal{A}$ s.t. $g(p_i) \neq g(p_j)$, for all $i \neq j$.*

Proof. For each pair $i \neq j$, consider a function $g_{ij} \in \mathcal{A}$ s.t. $g_{ij}(p_i) \neq g_{ij}(p_j)$. Setting

$$h_i = \prod_{j \neq i} (g_{ij} - g_{ij}(p_j)),$$

the function

$$g = \sum_{k=1}^{N} k \cdot \frac{h_k}{h_k(p_k)}$$

satisfies $g(p_k) = k$, for $k = 1, \ldots, N$. $\square$

Lemma 6.7 *There exist functions $f_1, f_2 \in [\varphi_1, \dots, \varphi_n]$ that separate points on $\mathbb{S}^1$ and $f_1'(u) \neq 0$ for $u \in \mathbb{S}^1$ and f_1 takes only finitely values more than once on $\mathbb{S}^1$.*

Proof. Fix $u_0 \in \mathbb{S}^1$. Let $\{u_0, \dots, u_n\} = \{u \in \mathbb{S}^1 \ : \ \varphi_1(u) = \varphi_1(u_0)\}$ (the set is indeed finite because of the hypotesis on φ_1). Lemma 6.6 implies the existence of $g \in [\varphi_1, \dots, \varphi_k]$ s.t. $g(u_i) \neq g(u_j)$, for $i \neq j$. By the hypothesis on φ_1 there is $r > 0$ such that for all $|\varepsilon| < r$, $f_\varepsilon = \varphi_1 + \varepsilon g$ has non-zero derivative on $\mathbb{S}^1$. Setting

$$\zeta(u) \ = \ \frac{\varphi_1(u) - \varphi_1(u_0)}{g(u) - g(u_0)},$$

we obtain a meromorphic function on $\mathbb{S}^1$. Hence there is a $\overline{\varepsilon} \neq 0$ s.t. $|\overline{\varepsilon}| < r$ and $\zeta(u) \neq -\overline{\varepsilon}$ on $\mathbb{S}^1$.

We set $f_1 = f_{\overline{\varepsilon}}$. f_1 assumes the value $f_1(u_0)$ at no other point of $\mathbb{S}^1$. Indeed, suppose by contradiction that $f_1(\overline{u}) = f_1(u_0)$, $u_0 \neq \overline{u} \in \mathbb{S}^1$. Then

$$\varphi_1(\overline{u}) - \varphi_1(u_0) \ = \ -\varepsilon(g(\overline{u}) - g(u_0)).$$

If $\varphi_1(\overline{u}) - \varphi_1(u_0) = 0$, then $\overline{u} = u_i$, $i \neq 0$, hence the righthand side is not zero. Thus $\varphi_1(\overline{u}) - \varphi_1(u_0) \neq 0$, $g(\overline{u}) - g(u_0) \neq 0$, we can divide and get $\zeta(\overline{u}) = -\varepsilon$, contradiction.

Moreover $f_1' \neq 0$ on $\mathbb{S}^1$. An analytic function on $\mathbb{S}^1$ which takes one value only once and has non-vanishing derivative can take only a finite number of values more than once. Let $M \subset \mathbb{C}$ be the set of values taken more than once by f_1. M and $f^{-1}(M) \subset \mathbb{S}^1$ are finite. By Lemma 6.6, there is $f_2 \in [\varphi_1, \dots, \varphi_n]$ that separates points in $f_1^{-1}(M)$. Together f_1 and f_2 separate points of $\mathbb{S}^1$. □

Lemma 6.8 *Suppose $\widehat{\Gamma}_{\mathcal{P}} \neq \Gamma$. Then there exists a complex measure $d\mu \not\equiv 0$ on $\mathbb{S}^1$ such that*

$$\int_{\mathbb{S}^1} g(u) d\mu(u) \ = \ 0, \tag{6.3}$$

for all $g \in [\varphi_1, \dots, \varphi_n]$.

Proof. Choose $x \in \widehat{\Gamma}_{\mathcal{P}} \setminus \Gamma$. For every polynomial P, set

$$L_x\{P(\varphi_1, \dots, \varphi_n)\} \ = \ P(x).$$

The linear functional L_x on $[\varphi_1, \ldots, \varphi_n]$ is well-defined thanks to Remark 1.2. Moreover, since $x \in \widehat{\Gamma}_{\mathcal{P}}$,

$$|P(x)| \;\le\; \max_{y\in\Gamma} |P(y)| \;=\; \max_{u\in\mathbb{S}^1} |P(\varphi_1(u), \ldots, \varphi_n(u)|.$$

Hence, Riesz representation theorem implies that there exists some measure $d\nu$ on $\mathbb{S}^1$ such that

$$P(x) \;=\; \int_{\mathbb{S}^1} P(\varphi_1(u), \ldots, \varphi_n(u))\, d\nu(u), \tag{6.4}$$

for all polynomials P. Let $u_0 \in \operatorname{supp} d\nu$. Since $x \notin \Gamma$, $x \neq (\varphi_1(u_0, \ldots, \varphi_n(u_0))$. Hence there is a polynomial P_0 such that $P_0(x) = 0$ and

$$P_0(\varphi_1(u_0, \ldots, \varphi_n(u_0)) \neq 0.$$

Then the measure $d\mu = P_0(\varphi_1(u_0, \ldots, \varphi_n(u_0)) \cdot d\nu$ does not vanish identically on $\mathbb{S}^1$. From (6.4) we get

$$0 \;=\; P(x)P_0(x) \;=\; \int_{\mathbb{S}^1} P(\varphi_1(u), \ldots, \varphi_n(u))\, d\mu,$$

for all polynomials P. $\square$

Lemma 6.9 *Suppose $\widehat{\Gamma}_{\mathcal{P}} \neq \Gamma$. Then there exist a relatively compact domain $D \Subset \mathcal{R}$ of a Riemann surface, with boundary $bD = \gamma$ a simple closed analytic curve, and a homomorphism χ of γ on $\mathbb{S}^1$ such that, defining*

$$\Phi_k(p) = \varphi_k(\chi(p)), \quad p \in \gamma, \quad i = 1, \ldots, n \tag{6.5}$$

then each Φ_k has an extension in $\mathcal{O}^0(D \cup \gamma)$. We will denote the extensions again by Φ_k.

We do not enter the detail of the proof, which is based on results of Sakakihara and of Wermer himself (see [81, 100, 103]).

Lemma 6.10 *Suppose $\widehat{\Gamma}_{\mathcal{P}} \neq \Gamma$. Let $\Phi = (\Phi_1, \ldots, \Phi_n) : D \cup \gamma \to \mathbb{C}^n$. Then $\Phi(\gamma) = \Gamma$ and $\Phi(D \cup \gamma) = \widehat{\Gamma}_{\mathcal{P}}$.*

Proof. From the definitions of Γ and of the functions Φ_k (6.5) we get that $\Phi(\gamma) = \Gamma$.

Here we will only prove the inclusion $\Phi(D \cup \gamma) \subset \widehat{\Gamma}_{\mathcal{P}}$. Fix $p \in D$ and a polynomial P. Define $P^*(p) = P(\Phi(q))$. Since Φ is analytic on $D \cup \gamma$, the same is true for P^*. By the maximum principle, then

$$|P^*(p)| \;\le\; \max_{y\in\gamma} |P^*(y)|,$$

i.e.

$$|P(\Phi(p)| \leq \max_{x\in\Gamma} |P(x)|.$$

Then $\Phi(q) \in \widehat{\Gamma}_{\mathcal{P}}$, hence $\Phi(D \cup \gamma) \subset \widehat{\Gamma}_{\mathcal{P}}$. $\square$

Proof of Theorem 6.1. Assume $\widehat{\Gamma}_{\mathcal{P}} \neq \Gamma$. Let $S \subset D \cup \gamma$, the set of points p such that there is $q \in D \cup \gamma$, $q \neq p$ and $\Phi(q) = \Phi(p)$. This happens only when $\Phi_k(p) = \Phi_k(q)$ for all k, thus Theorem 6.4 applied to the functions $F_1, F_2 \in [\Phi_1, \dots, \Phi_n]$ which correspond to the functions $f_1, f_2 \in [\varphi_1, \dots, \varphi_n]$ of Lemma 6.7 implies that S is a finite set.

Choose a point $z^0 = (z_1^0, \dots, z_n^0) \in \widehat{\Gamma}_{\mathcal{P}} \setminus \Gamma$. Lemma 6.10 implies that $z^0 \in \Phi(D \cup \gamma)$. Since Φ is analytic on $D \cup U$, only finitely many points $q_1, \dots, q_m \in D \cup \gamma$ are mapped into x_0 by Φ. Since $z^0 \notin \Gamma$, $q_i \in D$ for all $i = 1, \dots, m$.

S being a discrete set, each q_i has an open relatively compact neighborhood $q_i \in U_i \Subset D$ such that if $p \in \overline{U}_i \setminus \{q_i\}$ then $p \notin S$. In particular, Φ is bijective in $\overline{U}_i$. We can choose U_i simply connected and a disc in local coordinates λ at q_i. Since $\Phi|_{\overline{U}_i}$ is bijective and continuous, Φ maps $\overline{U}_i$ homeomorphically on its image. Hence Φ maps U_i homeomorphically on its image. Moreover each coordinate Φ_k is analytic.

Fix $y_0 \in \cup_1^m \Phi(U_i)$, which means there is i and $\lambda_0 \in U_i$ such that $\Phi(\lambda_0) = y_0$. Then we claim that there is an open neighborhood $U(y_0) \subset \mathbb{C}^n$ s.t.

$$(\widehat{\Gamma}_{\mathcal{P}} \setminus \Gamma) \cap U(y_0) \subset \bigcup_{i=1}^{m} \Phi(U_i). \tag{6.6}$$

Indeed, suppose the inclusion does not hold for any neighborhood. Then there is a sequence $\{y_l\} \subset \widehat{\Gamma}_{\mathcal{P}} \setminus \Gamma$ converging to y_0 with $y_l \notin \cup_1^m \Phi(U_i)$. Since $y_l \in \widehat{\Gamma}_{\mathcal{P}} \setminus \Gamma$, there is $p_l \in D \cup \gamma$ with $\Phi(p_l) = y_l$, hence $p_l \notin \cup_1^m U_i$. Let $\overline{p}$ be a limit point of the sequence $\{p_l\}$. $\overline{p} \notin \cup_1^m U_i$, but $\Phi(\overline{p}) = y_0 = \Phi(\lambda_0)$. Hence $\overline{p} = \lambda_0$, contradiction. Hence (6.6) holds for some neighborhood $U(y_0)$.

Consider the open set

$$U = \bigcup_{y_0 \in \cup_1^m \Phi(U_i)} U(y_0) \subset \mathbb{C}^n.$$

$z^0 \in U$ and $U \cap (\widehat{\Gamma}_{\mathcal{P}} \setminus \Gamma) \subset \cup_1^m \Phi(U_i)$, due to (6.6). The reverse inclusion is trivial, hence

$$U \cap (\widehat{\Gamma}_{\mathcal{P}} \setminus \Gamma) = \cup_1^m \Phi(U_i).$$

Hence z^0 has an open neighborhood in $\mathbb{C}^n$ which intersect $\widehat{\Gamma}_{\mathcal{P}} \setminus \Gamma$ in a number m of discs, m being the number of preimages of z^0. Hence $m = 1$ except for a finite set. Thus $\widehat{\Gamma}_{\mathcal{P}} \setminus \Gamma$ is an analytic surface with finitely many branching points. $\square$

Proof of Theorem 6.2. Assume (6.2). Since $\varphi_1'(u) \neq 0$ for $u \in \mathbb{S}^1$, the measure $\varphi_1'(u)\, du$ does not vanish identically on $\mathbb{S}^1$. Hence $K[\varphi_1, \ldots, \varphi_n] \subsetneq \mathcal{C}^0(\mathbb{S}^1)$. Theorem 6.5 then implies that $\widehat{\Gamma}_{\mathcal{P}} \neq \Gamma$.

Conversely, suppose $\widehat{\Gamma}_{\mathcal{P}} \neq \Gamma$. We can apply Lemma 6.9 and by a homeomorphism χ transform the functions $\varphi_1, \ldots, \varphi_n$ into functions $\Phi_1, \ldots, \Phi_n$ on the curve γ. These functions extend holomorphically on $D \cup \gamma$, due to Lemma 6.9. Fix a multiindex $\alpha \in \mathbb{N}^n$. $\Phi^\alpha\, d\Phi_1$ is an analytic differential on $D \cup \gamma$, hence Cauchy's theorem implies

$$\int_\gamma \Phi^\alpha \, d\Phi_1 \;=\; 0. \tag{6.7}$$

A change of variable $u = \chi(\rho)$, transforms (6.7) into (6.2). $\square$

6.2.2 Generalization to several curves

In 1966, Stolzenberg [91] generalized Wermer's result to the union of more smooth curves. More precisely, consider l smooth curves $\varphi_j : [0,1] \to \mathbb{C}^n$. Consider their union

$$K = \bigcup_{j=1}^{l} \varphi_j([0,1])$$

and a polinomially convex compact set $X = \widehat{X}_{\mathcal{P}} \Subset \mathbb{C}^n$ (which may be empty).

Theorem 6.11 (Stolzenberg [91]**)** *Setting $Y = X \cup K$, $\widehat{Y}_{\mathcal{P}} \setminus Y$ is a (possibly empty) one-dimensional analytic subset of $\mathbb{C}^n \setminus Y$. If K is simply connected and disjoint from X, then $\widehat{Y}_{\mathcal{P}} = Y$. If $\widehat{Y}_{\mathcal{P}} = Y$, then every continuous function on Y which is uniformily approximable on X by polynomials, is uniformily approximable by polynomials also on Y.*

This theorem in particular gives a complex one-dimansional subset whose boundary is a union of smooth closed curves K if $\widehat{K}_{\mathcal{P}} \neq K$, thus solving the boundary problem for unions of compact curves.

6.3 The boundary problem for compact manifolds

6.3.1 The boundary problem in terms of holomorphic chains

In the Seventies Harvey and Lawson [38, 39] showed that maximal complexity was indeed a necessary and sufficient condition for the boundary problem, once this was stated in terms of holomorphic chains.

By a *positive holomorphic p-chain* with boundary $\Gamma \subset X$ we mean a finite sum $T = \sum_k n_k[V_k]$ where $n_k \in \mathbb{N}$ and $V_k \subset X \setminus \Gamma$ is an irreducible subvariety of dimension p, so that $dT = \Gamma$ as currents on X.

By the *mass* of such a chain we mean the finite sum $T = \sum_k n_k \mathcal{H}^{2p}(V_k)$, where $\mathcal{H}^{2p}$ is the Hausdorff $2p$-dimensional measure.

Theorem 6.12 (Harvey-Lawson, [38]**)** *Let M be a compact, oriented submanifold of dimension $2m+1$ and of class $\mathcal{C}^2$ in $\mathbb{C}^n$. Suppose that M is maximally complex, or if $m = 0$, suppose that M satisfies the moments condition. Then there exists a unique holomorphic $(m+1)$-chain T in $\mathbb{C}^n \setminus M$, with $\operatorname{supp} T \Subset \mathbb{C}^n$, and with finite mass, such that $dT = [M]$ (in the sense of currents) in $\mathbb{C}^n$. Furthermore, there is a compact subset $A \subset M$ of Hausdorff $(2m+1)$-measure zero such that each point of $M \setminus A$, near which M is of class $\mathcal{C}^k$, has a neighborhood in which $\mathsf{supp}\, T \cup M$ is a regular $\mathcal{C}^k$ submanifold with boundary.*

In particular, if M is connected, then there exists a unique irreducible complex $(m+1)$-dimensional subvariety V of $\mathbb{C}^n \setminus M$ such that $d[V] = \pm[M]$, with boundary regularity as before.

The indeterminacy of the sign above is due to the fact that the orientation of M and that of V are independent one of the other.

When $m = 0$ Harvey-Lawson's Theorem follows from the results by Wermer [101] (see Theorems 6.1, 6.2) or from the one by Stolzenberg [91] (see Theorem 6.11) of the previous section.

The proof of the theorem is rather long and involving. Here we just state some key points and basic ingredients which enter the proof. For a full exposition, see [14, 38].

Hypothesis for currents

One of the fundamental facts in the proof is that the two appearently very different conditions (maximal complexity and moments condition) can indeed be unified if we consider the more general setting of currents.

Let X be a complex manifold. Recall that the space of *k-dimensional currents* in $U \subset X$ is the dual of the space of compactly supported k-differential forms in U; the space of *compactly supported k-dimensional currents* is the dual of the space of k-differential forms. Since X is a complex manifold, a k-dimensional current T_k can be split in its holomorphic and antiholomorphic part:

$$T_k = \sum_{j=0}^{k} T_{j,k-j},$$

where $T_{j,l}$ is a linear operator on the (j,l)-differential forms.

We can now extend the definition of maximal complexity and the moments condition to currents. A $(2m+1)$-current T_{2m+1} is said to be *maximally complex* if

$$T_{2m+1} = T_{m+,m} + T_{m,m+1},$$

i.e. if all the components of bidegree (r,s) with $|r-s| > 1$ vanish identically.

A $(2m+1)$-current T_{2m+1} is said to satisfy the *moments condition* if $T_{2m+1}(\alpha) = 0$ for all $\overline{\partial}$-closed forms α of bidegree $(m+1,m)$.

If $T = [M]$ is the current of integration on a manifold M, the notions given for T and for M coincide. Moreover, if T is a $(2m+1)$-dimensional maximally complex current and $m > 0$, then T satisfies the moments condition.

Plemelj formulae

In every extension result, at the core of the proof, lies some deep analytical representation formula. In this case the Plemelj formulae.

Theorem 6.13 *Let $U \subset \mathbb{C}^n$ be an open subset, and $M \subset U$ an oriented submanifold of class $\mathcal{C}^1$, dividing $U \setminus M$ in two components U^+ and U^- ($dU^+ = M$, $dU^- = -M$). Let $f \in \mathcal{C}^1_0(M)$, and let*

$$F(z) = \int_M \omega_{BM}(z,\zeta) f(\zeta)$$

denote the Bochner-Martinelli transform of f. $F|_{U^\pm}$ has a continuous extension $F^\pm$ on $U^\pm \cup M$. On M the Plemelj formulae are satisfied:

$$F^+ - F^- = f, \tag{6.8}$$

$$F^+ + F^- = 2P.V. \int_M \omega_{BM}(z,\zeta) f(\zeta). \tag{6.9}$$

The Cauchy principal value being defined as

$$P.V. \int_M \omega_{BM}(z,\zeta)f(\zeta) = \lim_{\varepsilon\to 0^+} \int_{M\setminus B_\varepsilon(z)} \omega_{BM}(z,\zeta)f(\zeta).$$

The Plemelj formulae basically enable to write a $\mathcal{C}^1$-smooth function on M as a difference of two functions on the open sets $U^\pm$. If f is a CR-function, then the functions $F^\pm$ are holomorphic.

6.3.2 The boundary problem in strictly pseudoconvex domains

The previous result of Harvey-Lawson in terms of holomorphic chains, can be improved —in terms of regularity— when the odd-dimensional manifold M is either pseudoconvex or contained in the boundary of a strictly pseudoconvex domain (see [38, 40]). More precisely,

Theorem 6.14 (Harvey-Lawson) *Let $M \Subset \mathbb{C}^n$ be a compact connected $\mathcal{C}^k$-smooth manifold of real dimension $2m+1$. If $m = 0$ suppose M satisfies the moments condition, if $m \geq 1$ suppose M is maximally complex. If M is pseudoconvex, then there exists an irreducible $(m+1)$-complex dimensional analytic subvariety $V \subset \mathbb{C}^n \setminus M$ with $\overline{V}$ having at most finitely many intrinsic singularities (i.e. not arising from self-intersection), such that $M = bV$ with $\mathcal{C}^k$ boundary regularity of V near M.*

Theorem 6.15 (Harvey-Lawson) *Let $M \subset b\Omega \Subset \mathbb{C}^n$ be a compact connected $\mathcal{C}^k$-smooth manifold of real dimension $2m+1$, contained in the boundary of a strictly pseudoconvex domain Ω. If $m = 0$ suppose M satisfies the moments condition, if $m \geq 1$ suppose M is maximally complex. Then there exists an irreducible $(m+1)$-complex dimensional analytic subvariety $V \subset \Omega$ with $\overline{V}$ having at most finitely many intrinsic singularities, such that $M = bV$ with $\mathcal{C}^k$ boundary regularity of V near M.*

Theorem 6.15 will be (partially) generalized to non-compact M in the following chapter.

6.3.3 The boundary problem and the linking number

In 2000, Alexander and Wermer [2] found a characterization of the compact manifolds for which the boundary problem can be solved via a topological condition: the positivity of the linking number.

Theorem 6.16 (Alexander-Wermer, [2]**)** *Let $\Gamma \Subset \mathbb{C}^n$ be a compact oriented smooth submanifold of real dimension $2p-1$. Then Γ bounds a positive holomorphic p-chain if and only if the linking numer*

$$\mathrm{Link}(\Gamma, Z) \geq 0,$$

for all canonically oriented algebraic subvarieties $Z \subset \mathbb{C}^n \setminus \Gamma$ of codimension p.

The *linking number* is an integer-valued topological invariant defined by the number of intersections $\mathrm{Link}(\Gamma, Z) = N \bullet Z$ of Z with any $2p$-chain $N \subset \mathbb{C}^n \setminus \Gamma$ having $bN = \Gamma$.

The notion of boundary here is in the sense of currents (i.e. Stokes' Theorem holds), however for Γ smooth one has boundary regularity almost everywhere, and for Γ real analytic, one has complete boundary regularity (see [38]).

This result has been re-proved later on, using geometric measure theory methods, by Harvey and Lawson [46].

6.4 The boundary problem in q-concave domains

After estabilishing the theorem for compact manifolds in $\mathbb{C}^n$, the next obvious step is to look for the result in $\mathbb{CP}^n$. Unluckily, things are a lot more difficult in the projective setting. Here we will recall the results obtained by Harvey-Lawson [39] and Dolbeault-Henkin [27, 28] in the q-concave setting, leaving to the next section the recent results toward a resolution of the boundary problem in $\mathbb{CP}^n$.

So let us define q-concavity in $\mathbb{CP}^n$. Let $q \leq n$. Consider the Grassmannian $G_{\mathbb{C}}(q+1, n+1)$ of the $(q+1)$-dimensional subspaces of $\mathbb{C}^{n+1}$. To every point $\nu \in G_{\mathbb{C}}(q+1, n+1)$ naturally corresponds a q-dimensional projective subspace of $\mathbb{CP}^n$ that we will denote by P_ν. A domain $X \subset \mathbb{CP}^n$ is said to be *q-concave* if there exists a non-empty domain $V \subset G_{\mathbb{C}}(q+1, n+1)$ such that

$$X = \bigcup_{\nu \in V} P_\nu.$$

Note that if X is q-concave, then it is r-concave for all $r \leq q$.

Theorem 6.17 (Dolbeault-Henkin [28]**)** *Let $X \subset \mathbb{CP}^n$ be q-concave, and $M \subset X$ be a $(2m+1)$-dimensional oriented closed submanifold of class $\mathcal{C}^2$, with $1 \leq n-m \leq q$. The following conditions are equivalent:*

1. *M is the boundary of a holomorphic $(m+1)$-chain in X of locally finite mass;*

2. *M is maximally complex and there is a point $\nu_0 \in G_{\mathbb{C}}(q+1, n+1)$ such that for all points ν in a neighborhood of ν_0, $P_\nu \subset X$ and $M \cap P_\nu$ is a curve γ_ν boundary of a holomorphic 1-chain of P_ν of finite mass.*

In $\mathbb{CP}^n$ maximal complexity is no longer a sufficient condition by itself. Indeed, a real hypersurface in an algebraic subvariety $Y \subset \mathbb{CP}^n$ is maximally complex, but it cannot be the boundary of a holomorphic chain in $\mathbb{CP}^n$ if it is not homologous to 0 in Y.

Condition 2. of Theorem 6.17 basically asks for a local condition (maximal complexity) and a global one (moments condition on the curves γ_ν).

Dolbeault and Henkin proved this result using the system of choc-wave equations.

The result of Dolbeault and Henkin is a generalization of the result obtained by Harvey and Lawson in 1977 [39] when the q-concave domain of $\mathbb{CP}^n$ is $X = \mathbb{CP}^n \setminus \mathbb{CP}^{n-q}$.

Dihn [25] lowered the regularity requirements in the hypothesis of Theorem 6.17.

6.5 The boundary problem in $\mathbb{CP}^n$

Let as always $M \subset \mathbb{CP}^n$ be a real manifold of real dimension $2m+1$.

In the last few years the boundary problem has been studied in $\mathbb{CP}^n$. The main difference with the boundary problem in $\mathbb{C}^n$ is the non-uniqueness of the solution. Indeed, in $\mathbb{CP}^n$ there are closed manifolds, i.e. without boundary, of any dimension, e.g. $\mathbb{CP}^{m+1} \subset \mathbb{CP}^n$. Hence, if W is a solution of the boundary problem, i.e. $bW = M$, then also $W \cup \mathbb{CP}^{m+1}$ is a solution.

The non-uniqueness of the solution, as a consequence, gives birth to some very natural different problems, other than that of finding a solution.

The first problem is that of finding the minimal solution, i.e. a solution W_0 such that if W_1 is also a solution, then $W_1 = W_0 \cup T_1$, where $bT_1 = \emptyset$.

A second problem is that of finding (necessary and sufficient) conditions for the existence of a positive chain solution with mass less then a fixed amount. Of course this problem is strictly related to the previous. The minimal solution is the one of least mass.

A third problem is the following. Fix a projection $\pi : \mathbb{CP}^n \to \mathbb{CP}^{m+1}$, proper on M, a base point $a \in \mathbb{CP}^{m+1} \setminus \pi(M)$ and $l \in \mathbb{N}$. Find (necessary and sufficient) conditions for M to bound a positive $(m+1)$-chain l-sheeted over a.

In this section we state the main results obtained since 2004 in the direction of solving these problems, and outline the ideas lying under the methods used. A huge variety of methods was used and still is under exploration.

6.5.1 The projective hull

In order to generalize Wermer's result (see Theorem 6.1, and [103]), Harvey and Lawson [43] defined a notion in $\mathbb{CP}^n$ that generalizes the notion of polynomial hull in $\mathbb{C}^n$.

Let $K \subset \mathbb{CP}^n$ be a compact set. The projective hull of K, $\widehat{K}_{\mathbb{CP}^n}$ is defined as the set of all points $x \in \mathbb{CP}^n$ such that there exists a constant C_x such that for all $d > 0$ and all sections $P \in H^0(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(d))$, where $\mathcal{O}_{\mathbb{CP}^n}(d)$ denotes the d^{th} power of the hyperplane bundle with the standard metric (i.e. for all P homogenous polynomials in of degree d in affine coordinates), such that

$$\| P(x) \| \leq C_x^d \sup_K \| P \| . \tag{6.10}$$

The projective hull is indeed a generalization of the polynomial hull in the following sense. Suppose $K \Subset \Omega \subset \mathbb{CP}^n$, Ω being an affine open subset of $\mathbb{CP}^n$. Then the definition immediately implies that

$$\widehat{K}_{\mathcal{P}(\Omega)} \subset \widehat{K}_{\mathbb{CP}^n}. \tag{6.11}$$

Moreover,

$$\widehat{K}_{\mathbb{CP}^n} \Subset \Omega \implies \widehat{K}_{\mathcal{P}(\Omega)} = \widehat{K}_{\mathbb{CP}^n}. \tag{6.12}$$

This result is non-trivial and proved in [43, Section 12]. Moreover the projective hull is subordinate to the Zariski hull. If $K \subset Z \subset \mathbb{CP}^n$, where Z is an algebraic subvariety, then $\widehat{K}_{\mathbb{CP}^n} \subset Z$, and if γ is a real curve, $\gamma \subset Z \subset \mathbb{CP}^n$, Z being an irreducible algebraic curve, then $\widehat{\gamma}_{\mathbb{CP}^n} = Z$.

For $x \in \widehat{K}_{\mathbb{CP}^n}$, the infimum of C_x for which inequality (6.10) holds is a minimum. This *best constant function* $C_K : \widehat{K}_{\mathbb{CP}^n} \to \mathbb{R}^+$ has a central rôle in the study of projective hulls. It is usually extended outside of $\widehat{K}_{\mathbb{CP}^n}$ as $+\infty$, thus having $C_K : \mathbb{CP}^n \to \mathbb{R}^+ \cup \{+\infty\}$. The best constant function C_K is related to the notion of quasi-plurisubharmonic function and to pluripotential theory.

Theorem 6.18 (Harvey-Lawson [43]**)** *Let $\gamma \subset \mathbb{CP}^n$ be a finite union of real analytic curves. Then $\widehat{\gamma}_{\mathbb{CP}^n}$ has Hausdorff dimension 2. If the Hausdorff 2-measure of $\overline{\widehat{\gamma}_{\mathbb{CP}^n}}$ is finite in a neighborhood of some algebraic hypersurface, then $\widehat{\gamma}_{\mathbb{CP}^n} \setminus \gamma$ is a 1-dimensional complex analytic subvariety of $\mathbb{CP}^n \setminus \gamma$.*

The same conclusion holds for any pluripolar curve $\gamma \subset \mathbb{CP}^2$.

One would like to get rid of the finite-measure condition, i.e. to prove the following

Conjecture 6.19 *Let $\gamma \subset \mathbb{CP}^n$ be a finite union of simple closed real analytic curves. Then $\widehat{\gamma}_{\mathbb{CP}^n} \setminus \gamma$ is a 1-dimensional complex analytic subvariety of $\mathbb{CP}^n \setminus \gamma$.*

The conjecture is true, under the additional assumption that γ is *stable*, i.e. that the best constant function C_γ is bounded on $\widehat{\gamma}_{\mathbb{CP}^n}$.

Theorem 6.20 (Harvey-Lawson-Wermer [47]) *Let $\gamma \subset \mathbb{CP}^n$ be a finite union of simple closed real analytic curves and assume γ is stable. Then $\widehat{\gamma}_{\mathbb{CP}^n} \setminus \gamma$ is a 1-dimensional complex analytic subvariety of $\mathbb{CP}^n \setminus \gamma$.*

Then $\widehat{\gamma}_{\mathbb{CP}^n}$ is the image of a compact Riemann surface with analytic boundary under a holomorphic map to $\mathbb{CP}^n$.

The notion of projective hulls is further analyzed by Wermer in [104].

6.5.2 The projective linking number

The projective linking number was introduced by Harvey and Lawson [42, 44] in order to generalize Alexander-Wermer's result (see Theorem 6.16) connecting linking numbers and the boundary problem in $\mathbb{C}^n$.

Suppose $\Gamma \subset \mathbb{CP}^n$ is a compact oriented smooth curve, and let $Z \subset \mathbb{CP}^n \setminus \Gamma$ be an algebraic variety of codimension 1. The *projective linking number* of Γ with Z is given by

$$\mathrm{Link}_{\mathbb{P}}(\Gamma, Z) = N \bullet Z - \deg(Z) \int_N \omega$$

where ω is the standard Kähler form on $\mathbb{CP}^n$ and N is any integral 2-chain with $bN = \Gamma$ (in the sense of currents) in $\mathbb{CP}^n$. Z has the canonical orientation and

$$\bullet : H_2(\mathbb{CP}^n, \Gamma) \times H_{2n-2}(\mathbb{CP}^n \setminus \Gamma) \to \mathbb{Z}$$

is the topologically defined intersection pairing. This definition is independent of the choice of N. The *associated reduced linking number* is defined as

$$\widetilde{\mathrm{Link}}_{\mathbb{P}}(\Gamma, Z) = \frac{1}{\deg(Z)} \mathrm{Link}_{\mathbb{P}}(\Gamma, Z).$$

Theorem 6.21 (Harvey-Lawson, [42]) *Let Γ be an oriented stable real analytic curve in $\mathbb{CP}^n$ with possible positive integer multiplicities on each component. Then the following are equivalent* *(i)* Γ *is the boundary of a positive holomorphic* 1*-chain of mass at most* Λ *in* $\mathbb{CP}^n$*,*

(ii) $\widetilde{\mathrm{Link}}_{\mathbb{P}}(\Gamma, Z) \geq -\Lambda$ *for all algebraic hypersurfaces* Z *in* $\mathbb{CP}^n \setminus \Gamma$*.*

Moreover, if Γ bounds any positive holomorphic 1-chain, then there is a unique such chain T_0 of least mass (all the others being obtained by adding algebraic 1-cycles to T_0):

Corollary 6.22 *Let* Γ *be as in Theorem 6.21 and suppose* T *is a positive holomorphic* 1*-chain with* $dT = \Gamma$*. Then* T *is the unique holomorphic chain of least mass with* $dT = \Gamma$ *if and only if*

$$\inf_Z \left\{ \frac{T \bullet Z}{\deg(Z)} \right\} = 0$$

where the infimum ranges over all algebraic hypersurfaces in the complement $\mathbb{CP}^n \setminus \Gamma$*.*

It is worth saying that condition (ii) in Theorem 6.21 has several equivalent formulations, in terms of projective winding number and of ω-quasi-plurisubharmonic functions.

The bigger dimensional version of Theorem 6.21 has been reduced to the following conjecture (of which Harvey and Lawson say it is likely to be true).

Conjecture 6.23 *Let* $\gamma \subset \mathbb{CP}^2$ *be a compact embedded real analytic curve such that for some choice of orientation and positive integer multiplicity condition (ii) of Theorem 6.21 is satisfied. Then* γ *is stable.*

Suppose $\Gamma \subset \mathbb{CP}^n$ is a compact oriented submanifold of real dimension $2m+1$, and let $Z \subset \mathbb{CP}^n \setminus \Gamma$ be an algebraic variety of codimension $m+1$. The *projective linking number* of Γ with Z is given by

$$\mathrm{Link}_{\mathbb{P}}(\Gamma, Z) = N \bullet Z - \deg(Z) \int_N \omega^{m+1}$$

where ω is the standard Kähler form on $\mathbb{CP}^n$ and N is any integral 2-chain with $bN = \Gamma$ (in the sense of currents) in $\mathbb{CP}^n$. This definition is independent of the choice of N. The *associated reduced linking number* is defined as

$$\widetilde{\mathrm{Link}}_{\mathbb{P}}(\Gamma, Z) = \frac{1}{(m+1)!\deg(Z)} \mathrm{Link}_{\mathbb{P}}(\Gamma, Z).$$

Theorem 6.24 (Harvey-Lawson, [44]**)** *Let* $\Gamma \subset \mathbb{CP}^n$ *be a compact oriented real analytic submanifold of dimension* $2m+1$ *with possible integer multiplicities on each component. If Conjecture 6.23 holds, then the following are equivalent* *(i)* Γ *is the boundary of a positive holomorphic* $m+1$*-chain of mass at most* Λ *in* $\mathbb{CP}^n$*,*

(ii) $\widetilde{\text{Link}}_{\mathbb{P}}(\Gamma, Z) \geq -\Lambda$ *for all algebraic subvarieties* Z *of codimension* $m+1$ *in* $\mathbb{CP}^n \setminus \Gamma$*.*

Again, if the cycle Γ bounds a positive chain T, then there is a unique chain T_0 of least mass with $dT_0 = \Gamma$ (all others being obtained by adding positive algebraic $(m+1)$-cycles to T_0):

Corollary 6.25 *Let* Γ *be as in Theorem 6.24 and suppose* T *is a positive holomorphic* $(m+1)$*-chain with* $dT = \Gamma$*. Then* T *is the unique holomorphic chain of least mass with* $dT = \Gamma$ *if and only if*

$$\inf_Z \left\{ \frac{T \bullet Z}{\deg(Z)} \right\} = 0$$

where the infimum ranges over all positive algebraic $(n-m-1)$*-cycles in the complement* $\mathbb{CP}^n \setminus \Gamma$*.*

6.5.3 l-sheeted solutions

Consider, as always, a $(2m+1)$-real maximally complex submanifold $M \subset \mathbb{CP}^n$. Fix a projection $\pi : \mathbb{CP}^n \to \mathbb{CP}^{m+1}$, proper on M, and a base point $a \in \mathbb{CP}^{m+1} \setminus \pi(M)$. Harvey and Lawson, in [41], have proved that necessary and sufficient condition for M to bound a positive holomorphic $(m+1)$-chain which is l-sheeted over a is satisfying a sequence of (explicit and algorithmically computable) non-linear moments conditions which depend on l.

The precise statement of the theorem goes beyond the purpose of this survey chapter. Anyhow it is worth noticing that the moments conditions in this case are non-linear and highly more complicated that their equivalent for curves in $\mathbb{C}^n$.

6.6 The boundary problem in an arbitrary complex manifold X

The boundary problem, as stated, can be considered in much more generality, e.g. in a Stein manifold, a complex manifold or even a complex space. If X is a Stein manifold, the boundary problem was solved by Harvey and Lawson in their classical paper of 1975 [38]. The statement of the theorem is exactly the same we stated just for $\mathbb{C}^n$ (see Theorem 6.12). Also the proof goes along the same lines.

For an arbitrary complex manifold X, Harvey and Lawson have recently proved (2006, [45]) a result which generalizes their own results stated before (the characterizations via the linking number). In order to understand the statement, we need some additional definitions.

Let X be a hermitian manifold (non necessarily compact). Suppose there exists a closed subset $\Sigma_\Gamma \subset X$ of zero Hausdorff $(2m+1)$-measure, and an oriented, properly embedded, $(2m+1)$-dimensional $\mathcal{C}^1$ submanifold of $X \setminus \Sigma_\Gamma$ with connected components Γ_k $(k \in \mathbb{N})$. If for $n_k \in \mathbb{Z}$,

$$\Gamma = \sum_{k=0}^{\infty} n_k \Gamma_k$$

defines a d-closed current of locally finite mass in X, then Γ will be called a *scarred* $(2m+1)$*-cycle of class* $\mathcal{C}^1$ in X.

Suppose Γ is an embedded $(2m+1)$-dimensional oriented submanifold of a complex manifold. We say that Γ can be *pushed out at* $z \in \Gamma$ if there exists a complex p-dimensional submanifold with boundary $(W, -\Gamma)$[1] containing the point z.

Theorem 6.26 (Harvey-Lawson, [45]**)** *Let* Γ *be a scarred* $(2m+1)$*-cycle of class* $\mathcal{C}^1$ *in* X *such that each component* Γ_k *can be pushed out at some point. Then* $\Gamma = dV$ *(in the sense of currents) where* V *is a positive holomorphic* $(m+1)$*-chain with mass* $M(V) \leq \Lambda$ *if and only if the* $(m+1, m+1)$*-linking numbers of* Γ *are bounded below by* $-\Lambda$*.*

[1] i.e. $bW = \Gamma$ with opposite orientation

Chapter 7

Non-compact boundaries of complex analytic varieties

This chapter is based on [15]. The last section, containing an example due to both authors of that paper, was not present in there.

7.1 Introduction

The main theorem proved by Harvey and Lawson in [38] is that if $M \subset \mathbb{C}^n$ is compact and maximally complex then M is the boundary of a unique holomorphic chain of finite mass (see Theorem 6.12). Moreover, if M is contained in the boundary $b\Omega$ of a strictly pseudoconvex domain Ω then M is the boundary of a complex analytic subvariety of Ω, with isolated singularities [34,38,40] (see Theorem 6.15). The aim of this chapter is that to generalize this last result to a non compact, connected closed and maximally complex submanifold M of the connected boundary $b\Omega$ of an unbounded weakly pseudoconvex domain $\Omega \subset \mathbb{C}^n$. The pseudoconvexity of Ω is needed both for the local result and to prove that the singularities are isolated.

Maximal complexity of M and Lewy extension theorem for CR-functions [61] (see Theorem 3.2) allow us to prove the following local result (see Corollary 7.4):

> *Assume that $n \geq 3$, $m \geq 1$ and the Levi-form $\mathcal{L}(b\Omega)$ of $b\Omega$ has at least $n - m$ positive eigenvalues at every point $p \in M$. Then there exist a tubular open neighborhood I of $b\Omega$ and a complex submanifold W_0 of $\overline{\Omega} \cap I$ with boundary, such that $bW_0 \cap b\Omega = M$ i.e. a complex manifold $W_0 \subset I \cap \Omega$ such that the closure $\overline{W}_0$ of W_0 in I is a smooth submanifold with boundary M.*

A very simple example (see Example 7.1) shows that in general the local result fails to be true for $m = 0$.

In order to prove that W_0 extends to a complex analytic subvariety W of Ω with boundary M we first treat the case when Ω is convex and does not contain straight lines. This is the crucial step. For technical reasons we divide the proof in two cases: $m \geq 2$ and $m = 1$. We cut $\overline{\Omega}$ by a family of real hyperplanes H_λ which intersect M along smooth compact submanifolds. Then the natural foliation on each H_λ by complex hyperplanes induces on $M \cap H_\lambda$ a foliation by compact maximally complex $(2m-1)$-real manifolds M'. Thus a natural way to proceed is that to apply Harvey-Lawson's theorem to each M' and to show that the family $\{W'\}$ of the corresponding Harvey-Lawson solutions actually organizes in a complex analytic subvariety W, giving the desired extension (see Theorem 7.5). This is done by following Zaitsev's idea (see Lemma 7.7).

The same method of proof is used in the last section in order to treat the problem when Ω is pseudoconvex. In this case, M is requested to fulfill an additional condition. Precisely,

($\star$) if $\overline{M}^\infty$ denotes the closure of $M \subset \mathbb{C}^n \subset \mathbb{CP}^n$ in $\mathbb{CP}^n$, then there exists an algebraic hypersurface V such that $V \cap \overline{M}^\infty = \emptyset$.

Equivalently

($\star'$) there exists a polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ such that

$$M \subset \left\{ z \in \mathbb{C}^n : |P(z)|^2 > (1 + |z|^2)^{\deg P} \right\}.$$

We recall that a similar condition was first pointed out by Lupacciolu [64] in studying the extension problem for CR-functions in unbounded domains (see Theorem 3.13). Lupacciolu's ($\star$) condition is also used by Simioniuc and Tomassini in [87] to obtain an unbounded version of Bremermann-Welsh lemma.

In here ($\star$) allows us to build a nice family of hypersurfaces, which play the role of the hyperplanes in the convex case, and so to prove the main theorem of this chapter:

Theorem 7.1 *Let Ω be an unbounded domain in $\mathbb{C}^n$ ($n \geq 3$) with smooth boundary $b\Omega$ and M be a maximally complex closed $(2m+1)$-real submanifold ($m \geq 1$) of $b\Omega$. Assume that*

(i) $b\Omega$ is weakly pseudoconvex and the Levi-form $\mathcal{L}(b\Omega)$ has at least $n - m$ positive eigenvalues for every point of M

(ii) M satisfies condition $(\star)$

Then there exists a unique $(m+1)$*-complex analytic subvariety* W *of* Ω*, such that* $bW = M$*. Moreover the singular locus of* W *is discrete and the closure of* W *in* $\overline{\Omega} \setminus \mathsf{Sing}\ W$ *is a smooth submanifold with boundary* M*.*

We do not deal with the 1-dimensional case. There are two different kinds of difficulty. First of all a local strip as in Corollary 7.4 could not exist (see Example 7.1). Secondly, even though it does exist, it could be non extendable to whole Ω (see Example 4.1) and it is not clear at all how it is possible to generalize the *moments condition*.

Finally we end the chapter with an example (see Example 7.2) in which the $(\star)$ condition is not satisfied but our methods of proof can be nevertheless used in order to show that the extension holds. We do not have an example where our boundary problem in a strictly pseudoconvex domain (not satisfying $(\star)$, of course) is not solvable.

Another similar approach can be followed to treat the *semi-local* boundary problem i.e. given an open subset U of the boundary of Ω, find an open subset $\Omega' \subset \Omega$ such that, for any maximally complex submanifold $M \subset U$, there exists a complex subvariety W of Ω' whose boundary is M. We deal with this problem in the following chapter.

7.2 The local result

The aim of this section is to prove the local result. Given a real smooth hypersurface S in $\mathbb{C}^n$, we denote by $\mathcal{L}_p(S)$ the Levi-form S at the point p. Let 0 be a point of M. We have the following inclusions of tangent spaces:

$$\mathbb{C}^n \supset T_0(S) \supset H_0(S) \supset H_0(M);$$

$$T_0(S) \supset T_0(M) \supset H_0(M).$$

Lemma 7.2 *Let* M *be a maximally complex submanifold of a hypersurface* S*,* $\dim_{\mathbb{R}} M = 2m+1$*,* $m \geq 1$*,* $0 \in M$*. Suppose that* $\mathcal{L}_0(S)$ *has at least* $n-m$ *eigenvalues of the same sign. Then*

$$H_0(S) \not\supset T_0(M).$$

Proof. Should the thesis fail we would have the following chain of inclusions

$$\mathbb{C}^n \supset T_0(S) \supset H_0(S) \supset T \supset T_0(M) \supset H_0(M)$$

where T is the smallest complex space containing $T_0(M)$ (since M is maximally complex, $\dim_{\mathbb{C}} T = m+1$). Hence, if $m+2 \leq n-1$ we may choose local complex coordinates $z_k = x_k + iy_k$, $k = 1, \ldots, m+1$, $w_l = u_l + iv_l$, $l = m+2, \ldots, n$, in a neighborhood of 0 in such a way that:

- $H_0(M) = \mathsf{span}\ (\partial/\partial x_k, \partial/\partial y_k)$, $k = 1, \ldots, m$
- $T_0(M) = \mathsf{span}\ (\partial/\partial x_k, \partial/\partial y_k, \partial/\partial x_{m+1})$, $k = 1, \ldots, m$
- $T = \mathsf{span}\ (\partial/\partial x_k, \partial/\partial y_k)$, $k = 1, \ldots, m+1$
- $H_0(S) = \mathsf{span}\ (\partial/\partial x_k, \partial/\partial y_k, \partial/\partial u_l, \partial/\partial v_l)$, $k = 1, \ldots, m+1$, $l = m+2, \ldots, n-1$, if $m+2 \leq n-1$
  or
- $H_0(S) = T$, if $m = n-2$;
- $T_0(S) = \mathsf{span}\ (\partial/\partial x_k, \partial/\partial y_k, \partial/\partial u_l, \partial/\partial v_l, \partial/\partial u_n)$, $k = 1, \ldots, m+1$, $l = m+2, \ldots, n-1$, if $m+2 \leq n-1$
  or
- $T_0(S) = \mathsf{span}\ (\partial/\partial x_k, \partial/\partial y_k, \partial/\partial u_n)$ $k = 1, \ldots, m+1$, if $m = n-2$.

We denote by z the first $m+1$ coordinates, by $\hat{z}$ the first m, and by π the projection on T; π is obviously a local embedding of M near 0, and we set $M_0 = \pi(M)$.
Locally at 0, S is a graph over its tangent space:

$$S = \{v_n = h(u_n, u_j, v_j, x_i, y_i)\}.$$

Observe that the Levi-form of h has $n-m+1$ positive eigenvalues. In order to obtain a similar description of M, we proceed as follows. First, we have

$$M_0 = \{(\hat{z}, z_{m+1}) : y_{m+1} = \varphi(\hat{z}, x_{m+1})\}.$$

Then, we choose $f_j(\hat{z}, x_{m+1}) = f_j^1(\hat{z}, x_{m+1}) + i f_j^2(\hat{z}, x_{m+1})$ (where f_j^1 and f_j^2 are real-valued) defined in a neighborhood of M_0 in T in such a way that

$$M = \{w_{m+2} = f_{m+2}(\hat{z}, x_{m+1}), \ldots, w_n = f_n(\hat{z}, x_{m+1})\}.$$

Observe that the function $(f_{m+2}(\hat{z}, x_{m+1}), \ldots, f_n(\hat{z}, x_{m+1}))$ is just $\pi^{-1}|_{M_0}$, and since M is maximally complex it has to be a CR map.

By hypothesis, the following equation holds in a neighborhood of 0:

$$f_n^2(\hat{z}, x_{m+1}) = h\left(f_n^1(\hat{z}, x_{m+1}), f_j^k(\hat{z}, x_{m+1}), \hat{z}, x_{m+1}\right)$$

After a computation of the second derivatives, taking into account that all first derivatives of f_j^k, of h and of φ vanish in the origin, we obtain

$$\frac{\partial^2 f_n^2}{\partial z_j \partial \overline{z}_k}(0) \;=\; \frac{\partial^2 h}{\partial z_j \partial \overline{z}_k}(0),$$

i.e. the Levi-form of h and f_n coincide in $H_0(M)$. By hypothesis $\mathcal{L}_0(h)$ is strictly positive definite on a non-zero subspace of $H_0(M)$. We shall obtain a contradiction by showing that $\mathcal{L}_0(f_n)$ vanishes on $H_0(M)$. Let $\xi \in H_0(M)$. We may assume (up to unitary linear transformation of coordinates of $H_0(M)$) that $\xi = \partial/\partial z_1$.

Set $f = f_n$. Then, since f is a CR-function on M_0, we have:

$$\frac{\partial}{\partial \overline{z}_k} f(\hat{z}, x_{m+1}) = -\alpha(\hat{z}, x_{m+1}) \frac{\partial}{\partial \overline{z}_k} \varphi(\hat{z}, x_{m+1}), \quad k = 1, \ldots, m$$

and

$$\frac{\partial}{\partial \overline{z}_{m+1}} f(\hat{z}, x_{m+1}) = -i\alpha(\hat{z}, x_{m+1}) + \frac{\alpha(\hat{z}, x_{m+1})}{2} \frac{\partial}{\partial x_{m+1}} \varphi(\hat{z}, x_{m+1})$$

where $\alpha(\hat{z}, x_{m+1}) = \alpha^1(\hat{z}, x_{m+1}) + i\alpha^2(\hat{z}, x_{m+1})$ is a complex valued function. Differentiating and calculating in 0 we obtain

$$\frac{\partial^2 f}{\partial z_1 \partial \overline{z}_1}(0) = \alpha(0) \frac{\partial^2 \varphi}{\partial z_1 \partial \overline{z}_1}(0) \tag{7.1}$$

$$0 = \frac{\partial f}{\partial x_{m+1}}(0) = i\alpha(0). \tag{7.2}$$

i.e. $\alpha(0) = 0$. From equation (7.1) we deduce that $\partial^2 f / \partial z_1 \partial \overline{z}_1(0) = 0$. Contradiction. □

Lemma 7.3 *Under the hypothesis of Lemma 7.2, assume that S is the boundary of an unbounded domain $\Omega \subset \mathbb{C}^n$, $0 \in M$ and that the Levi-form of S has at least $n - m$ positive eigenvalues. Then*

1. *there exists an open neighborhood U of 0 and an $(m+1)$-complex submanifold $W_0 \subset U$ with boundary, such that $bW_0 = M \cap U$;*
2. *$W_0 \subset \Omega \cap U$.*

Proof. The first assertion is proved by observing that the Levi form of M is the restriction of the Levi-form of S, in the following sense: to compute

$\mathcal{L}_0^M(\zeta_0, \overline{\zeta}_0)$ it suffices to choose a smooth local section ζ of $H_0(M)$ (or $H_0(S)$) such that $\zeta(0) = \zeta_0$ and compute the projection of the bracket $[\zeta, \overline{\zeta}](0)$ on the real part of $T_0(M)$. By Lemma 7.2 we know that if this bracket does not vanish then it has a non-zero component along the real part of $T_0(S)$. Moreover, by hypothesis the intersection of the space where $\mathcal{L}_0(S)$ is positive with $H_0(S)$ is non empty. We project (generically) M over a $\mathbb{C}^{m+1}$ in such a way that the projection π is a local embedding around 0: since the restriction of π to M is a CR-function, and since the Levi form of M has —by the arguments stated above— at least one positive eigenvalue, it follows that the Levi-form of $\pi(M)$ has at least one positive eigenvalue. Thus, in order to obtain W_0, it is sufficient to apply the Lewy extension theorem (see Theorem 3.2, [61]) to the CR-function $\pi^{-1}|_M$.

As for the second statement, we observe that the projection by π of the normal vector of S pointing towards Ω lies into the domain of $\mathbb{C}^{m+1}$ where the above extension W_0 is defined. Indeed, Theorem 3.2 gives an holomorphic function in the connected component of (a neighborhood of 0 in) $\mathbb{C}^n \smallsetminus \pi(M)$ for which $\mathcal{L}_0(\pi(M))$ has a positive eigenvalue when $\pi(M)$ is oriented as the boundary of this component. This is precisely the component towards which the projection of the normal vector of S points when the orientations of S and M are chosen accordingly. This fact, combined with Lemma 7.2 (which states that any extension of M must be transverse to S) implies that locally $W_0 \subset \Omega \cap U$. $\Box$

Corollary 7.4 (Semi global existence of W) *Under the same hypothesis as in Lemma 7.3, there exist an open tubular neighborhood I of S in $\overline{\Omega}$ and an $(m+1)$-complex submanifold W_0 of $\overline{\Omega} \cap I$, with boundary, such that $S \cap bW_0 = M$.*

Proof. By Lemma 7.3, for each point $p \in M$, there exist a neighborhood U_p of p and a complex manifold $W_p \subset \overline{\Omega} \cap U_p$ bounded by M. We cover M with countable many such open sets U_i, and consider the union $W_0 = \cup_i W_i$. W_0 is contained in the union of the U_i's, hence we may restrict it to a tubular neighborhood I_M of M. It is easy to extend I_M to a tubular neighborhood I of S.

The fact that $W_{i|U_{ij}} = W_{j|U_{ij}}$ if $U_i \cap U_j = U_{ij} \neq \emptyset$ immediately follows from the construction made in Lemma 7.3, in view of unicity of the holomorphic extension of CR-functions. $\Box$

Example 7.1 Corollary 7.4 could be restated by saying that if a submanifold $M \subset S$ ($dim_{\mathbb{R}} M \geq 3$) is locally extendable at each point as a complex manifold, then (one side of) the extension lies in Ω. This is no longer true, in general, for curves as shown in $\mathbb{C}^{n+1}_{(w,z_1,\ldots,z_n)}$ by the following case:

$$S = \left\{ v = u^2 + \sum_k |z_k|^2 \right\}, \ \Omega = \left\{ v > u^2 + \sum_k |z_k|^2 \right\},$$

$$M = \left\{ y_1 = 0, \ v = x_1^2, \ u = 0, \ z_2 = \cdots = z_{n-1} = 0 \right\}$$

and

$$W = \left\{ w = iz_1^2, \ z_2 = \cdots = z_{n-1} = 0 \right\};$$

we have that $S \cap W = M$ and $W \subset \mathbb{C}^n \smallsetminus \Omega$.

Remark 7.1 Suppose that S is strongly pseudoconvex and choose, in $\mathbb{C}^n$, with coordinates $z_1, \ldots, z_n$, a local plurisubharmonic equation ρ for S: $S = \{\rho = 0\}$. Consider the curve

$$\gamma = \{z_j = \gamma_j(t), \ j = 1, \ldots, n, \ t \in (-\varepsilon, \varepsilon)\},$$

$\gamma \subset S$. Assume that γ is real analytic, so that locally there exists a complex extension $\widetilde{\gamma} \supset \gamma$. Then one side of $\widetilde{\gamma}$ lies in Ω if and only if

$$\sum_j Re \frac{\partial \rho}{\partial z_j} \frac{\partial \gamma_j}{\partial t} \neq 0. \tag{7.3}$$

Observe that condition (7.3), which depends only on γ (when S is given), is not satisfied in Example 7.1. Sufficiency of (7.3) is true when S is *any* real hypersurface: indeed, from a geometric point of view, the condition is equivalent to the transversality of $T(\widetilde{\gamma})$ and $H(S)$ (and hence $T(S)$). Pseudoconvexity is required to establish the necessity.

7.3 The global result

In order to make the proof more transparent we first treat the case when Ω is an unbounded convex domain with smooth boundary $b\Omega$. In the next section we will prove the main theorem in all its generality.

Theorem 7.5 *Let M be a maximally complex (connected) $(2m+1)$-real submanifold $(m \geq 1)$ of $b\Omega$. Assume that Ω does not contain straight lines and $b\Omega = S$ satisfies the conditions of Lemma 7.2. Then there exists an $(m+1)$-complex subvariety W of Ω, with isolated singularities, such that $bW = M$.*

We observe that under the hypothesis of Theorem 7.5, there exists a complex strip in a tubular neighborhood with boundary M (see Corollary 7.4). Moreover, since Ω does not contain straight lines, we can approximate uniformly from both sides $b\Omega$ by strictly convex domains, cfr. [70]. It follows that we can found a real hyperplane L such that, for any translation L' of L, $L' \cap \overline{\Omega}$ is a compact set. We choose an exhaustive sequence L_k of such hyperplanes, and we set Ω_k as the bounded connected component of $\Omega \setminus L_k$. Then, approximating from inside, we can choose a strictly convex open subset $\Omega'_k \subset \Omega$ such that $b\Omega'_k \cap \Omega_k \subset I$, where I is the tubular neighborhood of Corollary 7.4. It is easily seen, then, that we are in the situation of Proposition 7.6.

We treat the cases $m \geq 2$ and $m = 1$ separately. Indeed all the main ideas of the proof lie in the case $m \geq 2$, while the case $m = 1$ simply adds technical difficulties.

7.3.1 M is of dimension at least 5 ($m \geq 2$).

Theorem 7.5 follows from

Proposition 7.6 *Let $D \Subset B \Subset \mathbb{C}^n$ ($n \geq 4$) be two strictly convex sets. Let $D_+ = D \cap \{\mathsf{Re}\ z_n > 0\}$, $B_+ = B \cap \{\mathsf{Re}\ z_n > 0\}$. Then every $(m+1)$-complex subvariety ($m \geq 1$) with isolated singularities, $A \subset B_+ \setminus \overline{D_+} \doteqdot C_+$, is the restriction of a complex subvariety $\widetilde{A}$ of B_+ with isolated singularities.*

Before proving the proposition, we make some considerations and we prove two lemmata that will be useful.

Let φ be a strictly convex function[1] defined in a neighborhood of B such that $B = \{\varphi < 0\}$. Fixing $\varepsilon > 0$ small enough $\varphi < -\varepsilon$ is a strictly convex domain $B' \subset B$ whose boundary H intersects A in a smooth maximally complex submanifold N. A natural way to proceed is to slice N with complex hyperplanes, in order to apply Harvey-Lawson's theorem. Each slice of B' is strictly convex, hence strongly pseudoconvex, and so the holomorphic chain we obtain is contained in B'. Thus the set made up by collecting the chains is contained in B'. Analyticity of this set is the hard part of the proof.

Because of Sard's lemma, for all $z \in D_+$, there exist a vector v arbitrarily near to $\partial/\partial z_n$, and $k \in \mathbb{C}$ such that $z \in v_k \doteqdot v^\perp + k$ and $A_k \doteqdot v_k \cap N$ is transversal and compact, and thus smooth.

In a neighborhood of each fixed $z_0 \in D_+$, the same vector v realizes the transversality condition. Hence we should now fix our attention to a

[1] In the general case φ will be a strongly plurisubharmonic function

neighborhood of the form

$$\widehat{U} \doteqdot \bigcup_{k\in U} v_k \cap B_+.$$

Let $k_0 \in \mathbb{C}$ correspond to z_0. Let $\pi : \widehat{U} \to \mathbb{C}^m$ be a generic projection: we use (w', w) as holomorphic coordinates on $v_{k_0} = \mathbb{C}^m \times \mathbb{C}^{n-m-1}$ (and also for k near to k_0). Let $V_k = \mathbb{C}^m \setminus \pi(A_k)$, and $V = \cap_k V_k$.

Since A_{k_0} has a local extension (given by $v_{k_0} \cap A$), it is maximally complex and so, by Harvey-Lawson's theorem, there is an holomorphic chain $\widetilde{A}_{k_0}$ with $b\widetilde{A}_{k_0} = A_{k_0}$, which extends holomorphically A_{k_0}.

Our goal is to show that $\widetilde{A}_U = \cup_k \widetilde{A}_k$ is analytic in $\pi^{-1}(V)$. From this, it will follow that $\widetilde{A}_U$ is an analytic subvariety of $\widehat{U}$, π being a generic projection.

Following an idea of Zaitsev, for $k \in U$, $w' \in \mathbb{C}^m \setminus \pi(A_k)$ and $\alpha \in \mathbb{N}^{n-m-1}$, we define

$$I^\alpha(w', k) \ \doteqdot \ \int_{(\eta',\eta)\in A_k} \eta^\alpha \omega_{BM}(\eta' - w'),$$

ω_{BM} being the Bochner-Martinelli kernel.

Lemma 7.7 (Zaitsev) *Let $F(w', k)$ be the multiple-valued function which represents $\widetilde{A}_k$ on $\mathbb{C}^m \setminus \pi(A_k)$; then, if we denote by $P^\alpha(F(w', k))$ the sum of the α^{th} powers of the values of $F(w', k)$, the following holds:*

$$P^\alpha(F(w', k)) = I^\alpha(w', k).$$

In particular, $F(w', k)$ is finite.

Proof. Let V_0 denote the unbounded component of V_k (where, of course, $P^\alpha(F(w', k)) = 0$). It is easy to show, following [38], that on V_0 also $I^\alpha(F(w', k)) = 0$: in fact, if w' is far enough from $\pi(A_k)$, then $\beta = \eta^\alpha \omega_{BM}(\eta' - w')$ is a regular $(m, m-1)$-form on some Stein neighborhood O of A_k. So, since in O there exists γ such that $\overline{\partial}\gamma = \beta$, we may write in the language of currents

$$[A_k](\beta) = [A_k]_{m,m-1}(\overline{\partial}\gamma) = \overline{\partial}[A_k]_{m,m-1}(\gamma) = 0.$$

In fact, since A_k is maximally complex, $[A_k] = [A_k]_{m,m-1} + [A_k]_{m-1,m}$ and $\overline{\partial}[A_k]_{m,m-1} = 0$, see [38]. Moreover, since $[A_k](\beta)$ is analytic in the variable w', $[A_k](\beta) = 0$ for all $w' \in V_0$.

To conclude our proof, we just need to show that the "jumps" of the functions $P^\alpha(F(w', k))$ and $I^\alpha(w', k)$ across the regular part of the common boundary of two components of V_k are the same.

So, let $z' \in \pi(A_k)$ be a regular point in the common boundary of V_1 and V_2. Locally in a neighborhood of z', we can write $\widetilde{A}_k$ as a finite union of graphs of holomorphic functions, whose boundaries A_k^i are either in A_k or empty. In the first case, the A_k^i are CR graphs over $\pi(A_k)$ in the neighborhood of z'. We may thus consider the jump j_i of $P^\alpha(F(w',k))$ due to a single function. We remark that the jump for a function f is $j_i = f(z')^\alpha$. The total jump will be the sum of them.

To deal with the jump of $I^\alpha(w',k)$ across z', we split the integration set in the sets A_k^i (thus obtaining the integrals I_i^α) and $A_k \setminus \cup_i A_k^i$ (I_0^α). Thanks to Plemelj formulae (see Theorem 6.13, and [38]) the jumps of I_i^α are precisely j_i. Moreover, since the form $\eta^\alpha \omega_{BM}(\eta' - z')$ is C^∞ in a neighborhood of $A_k \setminus \cup_i A_k^i$, the jump of I_0^α is 0. So $P^\alpha(F(w',k)) = I^\alpha(w',k)$. $\square$

Remark 7.2 Lemma 7.7 implies, in particular, that the $P^\alpha(F(w',k))$ are continuous in k. Indeed, they are represented as integrals of a fixed form over submanifolds A_k which vary continuously with the parameter k.

The functions $P^\alpha(F(w',k))$ and the holomorphic chain $\widetilde{A}_{k_0}$ uniquely determine each other and so, proving that the union over k of the $\widetilde{A}_k$ is an analytic set is equivalent to proving that the functions $P^\alpha(F(w',k))$ are holomorphic in the variable $k \in U \subset \mathbb{C}$.

Lemma 7.8 *$P^\alpha(F(w',k))$ is holomorphic in the variable $k \in U \subset \mathbb{C}$, for each $\alpha \in \mathbb{N}^{n-m-1}$.*

Proof. The proof is very similar to the one of Lewy's main lemma in [61]. Let us fix a point $(w', \underline{k})$ such that $w' \notin A_{\underline{k}}$ (this condition remains true for $k \in B_\epsilon(\underline{k})$). Consider as domain of $P^\alpha(F)$ the set $\{w'\} \times B_\epsilon(\underline{k})$. In view of Morera's theorem, we need to prove that for any simple curve $\gamma \subset B_\epsilon(\underline{k})$,

$$\int_\gamma P^\alpha(F(w',k))dk \;=\; 0.$$

Let $\Gamma \subset B_\epsilon(\underline{k})$ be an open set such that $b\Gamma = \gamma$. By $\gamma * A_k$ ($\Gamma * A_k$) we mean the union of A_k along γ (along Γ). Note that these sets are submanifolds of N ($\Gamma * A_k$ is an open subset) and $b(\Gamma * A_k) = \gamma * A_k$. By Lemma 7.7 and

Stoke's theorem

$$\begin{aligned}\int_\gamma P^\alpha(F(w',k))dk &= \int_\gamma I^\alpha(w',k)dk = \\ &= \int_\gamma \left(\int_{(\eta',\eta)\in A_k} \eta^\alpha \omega_{BM}(\eta'-w')\right) dk = \\ &= \iint_{\gamma * A_k} \eta^\alpha \omega_{BM}(\eta'-w') \wedge dk = \\ &= \iint_{\Gamma * A_k} d\left(\eta^\alpha \omega_{BM}(\eta'-w') \wedge dk\right) = \\ &= \iint_{\Gamma * A_k} d\eta^\alpha \wedge \omega_{BM}(\eta'-w') \wedge dk = \\ &= 0.\end{aligned}$$

The last equality follows from the fact that since η^α is holomorphic, in $d\eta^\alpha$ appear only holomorphic differentials. Since all the holomorphic differentials supported by $\Gamma * A_k$ already appear in $\omega_{BM}(\eta'-w')\wedge dk$ the integral is zero. □

We may now prove Proposition 7.6, when $m \geq 2$.

Proof of Proposition 7.6 ($m \geq 2$). Up to this point we have extended the complex manifold A to an analytic set

$$\widetilde{A}_U \doteqdot A \cup \bigcup_{k\in U} \widetilde{A}_k \subset V_U \doteqdot C_+ \cup \bigcup_{k\in U} (v_k \cap B_+)\,.$$

The open sets V_U are an open covering of B_+.

Moreover the open sets

$$\omega_U \doteqdot \bigcup_{k\in U} (v_k \cap B_+)$$

are an open covering of each compact set $K_\delta \doteqdot \overline{B}_{\varepsilon+\epsilon} \cap \{z_n \geq \delta\}$. Hence there exist $\omega_1, \ldots, \omega_l$ which cover K_δ and such that $\omega_i \cap \omega_{i+1} \cap C_+ \neq \emptyset$, for $i = 1, \ldots, l-1$ and therefore there exists a countable open cover $\{\omega_i\}_{i\in\mathbb{N}}$ of $\overline{B}_{\varepsilon+\epsilon} \cap B_+$ such that, for all $i \in \mathbb{N}$, $\omega_i \cap \omega_{i+1} \cap C_+ \neq \emptyset$.
So we may extend A to $C_+ \cup \omega_1$ by proceeding as above.
Suppose now that we have extended A to $C^i \doteqdot C_+ \cup \bigcup_{j=1}^i \omega_j$ with an analytic set A_i. On the non-empty intersection $C^i \cap \omega_{i+1} \cap C_+$ A_i and the extension $\widetilde{A}_{i+1}$ of A to $C_+ \cup \omega_{i+1}$ coincide (as they both coincide with A), hence by

analicity they coincide everywhere. Consequently we may extend A to C^{i+1} by $A_{i+1} \doteqdot A_i \cup \widetilde{A}_{i+1}$. It follows that

$$\widetilde{A} \doteqdot A \cup \bigcup_{j\in\mathbb{N}} A_j.$$

$\widetilde{A}$ is the desired extension of A to B_+. In order to conclude the proof we have to show that $\widetilde{A}$ has isolated singularities. Let $\mathsf{Sing}\,\widetilde{A} \subset B'_+$ be the singular locus of $\widetilde{A}$.

Recall that φ is a strongly plurisubharmonic defining function for B. Let us consider the family

$$(\phi_\lambda = \lambda(\varphi) + (1-\lambda)(z_n))_{\lambda\in[0,1]}$$

of strongly plurisubharmonic functions. For λ near to 1, $\{\phi_\lambda = 0\}$ does not intersect the singular locus $\mathsf{Sing}\,\widetilde{A}$. Let $\overline{\lambda}$ be the biggest value of λ for which $\{\phi_\lambda = 0\} \cap \mathsf{Sing}\,\widetilde{A} \neq \emptyset$. Then

$$\{\phi_{\overline{\lambda}} < 0\} \cap B_+ \subset B_+$$

is a Stein domain in whose closure the analytic set $\mathsf{Sing}\,\widetilde{A}$ is contained, touching the boundary in a point of strict pseudoconvexity. So, by Kontinuitätsatz, $\{\phi_{\overline{\lambda}} = 0\} \cap \mathsf{Sing}\,\widetilde{A}$ is a set of isolated points in $\mathsf{Sing}\,\widetilde{A}$. By repeating the argument, we conclude that $\mathsf{Sing}\,\widetilde{A}$ is made up by isolated points. □

Proof of Theorem 7.5 ($m \geq 2$). Thanks to Corollary 7.4, we have a regular submanifold W_1 of a tubular neighborhood I, with boundary M.

Suppose $0 \in M$. The real hyperplanes $H_k \doteqdot T_0(S) + k$, $k \in \mathbb{R}$, intesect S in a compact set. If the intersection is non-empty, $\overline{\Omega}$ is divided in two sets. Let Ω_k be the compact one. We can choose a sequence H_{k_n} such that Ω_{k_n} is an exaustive sequence for $\overline{\Omega}$.

We apply proposition 7.6 with $B_+ = \Omega_{k_n}$, $C_+ = I \cap \Omega_{k_n}$, and $A = W_1 \cap \Omega_{k_n}$, to obtain an extension of W_1 in Ω_{k_n}. Since, by the identity principle, two such extensions coincide in $\Omega_{k_{\min\{n,m\}}}$, their union is the desired submanifold W. □

7.3.2 M is of dimension 3 ($m = 1$)

We prove now the statement of Proposition 7.6 for $m = 1$.

Our first step is to show that when we slice transversally N with complex hyperplanes, we obtain 1-real submanifolds which satisfy the moments condition.

Again, we fix our attention to a neighborhood of the form

$$\widehat{U} \doteqdot \bigcup_{k \in U} v_k \cap B_+.$$

In $\widehat{U}$, with coordinates $w_1, \dots, w_{n-1}, k$, we choose an arbitrary holomorphic $(1,0)$-form which is constant with respect to k.

Lemma 7.9 *The function*

$$\Phi_\omega(k) \; = \; \int_{A_k} \omega$$

is holomorphic in U.

Proof. We use again Morera's theorem. We need to prove that for any simple curve $\gamma \subset U$, $\gamma = b\Gamma$,

$$\int_\gamma \Phi_\omega(k) dk \; = \; 0.$$

Applying Stoke's theorem, we have

$$\begin{aligned} \int_\gamma \Phi_\omega(k) dk \; &= \; \int_\gamma \left(\int_{A_k} \omega \right) dk \; = \\ &= \; \iint_{\gamma * A_k} \omega \wedge dk \; = \\ &= \; \iint_{\Gamma * A_k} d(\omega \wedge dk) \; = \\ &= \; \iint_{\Gamma * A_k} \partial\omega \wedge dk \; = \\ &= \; 0. \end{aligned}$$

The last equality is due to the fact that $\Gamma * A_k \subset N$ is maximally complex and thus supports only $(2,1)$ and $(1,2)$-forms, while $\partial\omega \wedge dk$ is a $(3,0)$-form. □

Now we can prove Proposition 7.6 and Theorem 7.5 also when $m = 1$.

We can find a countable covering of B_+ made of open subsets $\omega_i = \widehat{U}_i \cap B_+$ in such a way that:

1. $\omega_0 \subset C_+$;

2. if

$$B_l = \bigcup_{i=1}^{l} \omega_i$$

then $\omega_{l+1} \cap B_l \supset v_{l+1} \cap B_+$, where v_{l+1} is a complex hyperplane in $\widehat{U}_{l+1}$.

Now, suppose we have already found $\widetilde{A}_l$ that extends A on B_l (observe that in $B_0 = \omega_0$ $\widetilde{A}_0 = A$). To conclude the proof we have to find $\widetilde{A}_{l+1}$ extending A on B_{l+1}.

Each slice of N in B_l is maximally complex, and so are $v_{l+1} \cap N$ and $v_\epsilon \cap N$, for $v_\epsilon \subset \omega_{l+1}$ sufficiently near to v_{l+1} (because they are in B_l as well).

Now we use Lemma 7.9 with $\widehat{U} = \widehat{U}_{l+1}$. What we have just observed implies that, for all holomorphic $(1,0)$-form η, $\Phi_\eta(k)$ vanishes in an open subset of U and so is identically zero on U. This implies that all slices in ω_{l+1} are maximally complex. Again we may apply Harvey-Lawson's theorem slice by slice and conclude by the methods of the proof of Proposition 7.6 in the case $m \geq 2$.

7.3.3 M is of dimension 1 $(m = 0)$

We have already observed that if M is one-dimensional the local extension inside Ω may not exist (see Example 7.1). Even though there is a local strip in which we have an extension, the methods used to prove Proposition 7.6 do not work, since the transversal slices M are either empty or isolated points. Indeed, as Example 4.1 shows (just let $B = B_c$, $D = B_\varepsilon$), that extension result does not hold for $m = 0$.

7.4 Extension to pseudoconvex domains.

We may now prove

Theorem 7.10 *Let Ω be an unbounded domain in $\mathbb{C}^n$ $(n \geq 3)$ with smooth boundary $b\Omega$ and M be a maximally complex closed $(2m+1)$-real submanifold $(m \geq 1)$ of $b\Omega$. Assume that*

(i) $b\Omega$ is weakly pseudoconvex and the Levi-form $\mathcal{L}(b\Omega)$ has at least $n-m$ positive eigenvalues for every point of M;

(ii) M satisfies condition $(\star)$.

Then there exists a unique $(m+1)$-complex analytic subvariety W of Ω, such that $bW = M$. Moreover the singular locus of W is discrete and the closure of W in $\overline{\Omega} \smallsetminus \mathsf{Sing}\ W$ is a smooth submanifold with boundary M.

Proof. Assume, for the moment that condition $(\star)$ is replaced by the stronger condition

$(\star_\Omega)$ $\overline{\Omega}^\infty \cap \Sigma_0 = \emptyset$ where $\overline{\Omega}^\infty$ denotes the projective closure of Ω.

The only thing we have to show in order to conclude the proof (by using the methods of the previous section) is that, up to a holomorphic change of coordinates and a holomorphic embedding $V : \mathbb{C}^n \to \mathbb{C}^N$, we can choose a sequence of real hyperplanes $H_k \subset \mathbb{C}^N$, $k \in \mathbb{N}$, which are exhaustive in the following sense:

1. $H_k \cap V(S)$ is compact, for all $k \in \mathbb{N}$;
2. one of the two halfspaces in which H_k divides $\mathbb{C}^N$, say H_k^+, intersects $V(\Omega)$ in a relatively compact set;
3. $\cup_k (H_k^+ \cap V(\Omega)) = V(\Omega)$.

The arguments of Proposition 7.6, indeed - excluded the proof that the singularities are isolated - depend only on the fact that we can cut M by complex hyperplanes, obtaining compact maximally complex submanifolds. Once we have found $W' \subset V(\mathbb{C}^n)$ (W' is in fact contained in $V(\mathbb{C}^n)$ by analytic continuation, since it has to coincide with the strip in a neighborhood of M), we set $W = V^{-1}(W')$. Observe that the hypersurfaces $V^{-1}(H_k)$ are an exhaustive sequence for Ω; let Ω_k be correspondent sequence of relatively compact subsets. Since Ω is a domain of holomorphy, for each k we can choose a strongly pseudoconvex open subset $\Omega_k' \subset \Omega$ such that $b\Omega_k' \cap \Omega_k \subset I$, where I is the tubular neighborhood found in the local section. So, in each Ω_k we can suppose that we deal with a strongly pseudoconvex open set, and thus the proof of the fact that the singularities are isolated is the same as in Proposition 7.6.

Following [64] (see proof of Theorem 3.13) we divide the proof in two steps.

Step 1. P linear. We consider $\overline{\Omega} \subset \mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}_\infty$, which is disjoint from $\Sigma_0 = \{P = 0\}$. So we can consider new coordinates of $\mathbb{CP}^n$ in such a way that Σ_0 is the $\mathbb{CP}^{n-1}$ at infinity. Now Ω is a relatively compact open set of $(\mathbb{C}^n)' = \mathbb{CP}^n \setminus \Sigma_0$, and $H_\infty = \mathbb{CP}^{n-1}_\infty \cap (\mathbb{C}^n)'$ is a complex hyperplane containing the boundary of S. Let $H^{\mathbb{R}}_\infty \supset H_\infty$ be a real hyperplane. The intersection between S and a translated of $H^{\mathbb{R}}_\infty$ is either empty or compact. For all $z \in \Omega$, there exist a real hyperplane $H^{\mathbb{R}}_\infty \not\ni z$, intersecting Ω, and a small translated H_{ε_z} such that $z \in H^+_{\varepsilon_z}$. Since $\Omega = \cup_z (H^+_{\varepsilon_z} \cap \Omega)$, and Ω is a countable union of compact sets, we may choose an exhaustive sequence H_k.

Step 2. P generic. We use the Veronese map v (see the proof of Theorem 3.13) to embed $\mathbb{CP}^n$ in a suitable $\mathbb{CP}^N$ in such a way that $v(\Sigma_0) = L_0 \cap v(\mathbb{CP}^n)$, where L_0 is a linear subspace. If $P = \sum_{|I|=d} \alpha_I z^I$, then $v(\Sigma_0) = L_0 \cap v(\mathbb{CP}^n)$, where

$$L_0 = \left\{ \sum_{|I|=d} \alpha_I w_I = 0 \right\}.$$

Again we can change the coordinates so that L_0 is the $\mathbb{CP}^{N-1}$ at infinity. We may now find the exhaustive sequence H_k as in Step 1.

This achieves the proof in the case when $\overline{\Omega}^\infty \cap \Sigma_0 = \emptyset$. The general case is now an easy consequence.

Indeed, since $\mathbb{CP}^n \setminus \Sigma_0$ is Stein, there is a strictly plursubharmonic exhaustion function ψ. The sets

$$\Omega_c \ = \ \{\psi < c\}$$

are an exhaustive strongly pseudoconvex family for $\mathbb{CP}^n \setminus \Sigma_0$. Thus in view of $(\star)$ there exists $\overline{c}$ such that $\overline{M} \subset \Omega_{\overline{c}}$. $\Omega' = \Omega \cap \Omega_{\overline{c}}$, up to a regularization of the boundary, is a strongly pseudoconvex open set verifying $(\star_\Omega)$ in whose boundary lies M, and thus M can be extended thanks to what has been already proved. $\square$

7.5 On the Lupacciolu's $(\star)$ condition

We end the chapter with an example which shows that the method used in here can be indeed applied also in some cases where the Lupacciolu's $(\star_\Omega)$ condition does not hold, hence showing that the Lupacciolu's $(\star_\Omega)$ condition is sufficient, but not necessary for the positive answer to the boundary problem.

Example 7.2 Let $z = (z_1, z')$ denote the coordinates in $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$. For every $c \in \mathbb{R}$, the domain

$$\Omega_c = \left\{ z \in \mathbb{C}^n \ \middle| \ |z|^2 + \log |z'|^2 < c \right\}$$

is a strongly pseudoconvex domain which contains the complex line $L_0' = \{z' = 0\}$. Since it contains a complex line, it does not verify the Lupacciolu's $(\star_\Omega)$ condition. Anyhow, notice that for each fixed $z_1^0 \in \mathbb{C}$, $\Omega_c \cap \{z_1 = z_1^0\}$ is relatively compact. Hence the cut and paste method described in this chapter can be used to prove the very same extension result, at least in the case $m \geq 1$. It is worth observing that anyhow the proof of the discreteness of singularities does no longer works, since Ω_c does contain analytic sets of dimension greater than zero, e.g. L_0'.

Chapter 8

Semi-local extension of maximally complex submanifolds

This chapter is based on [16].

8.1 Introduction

In this chapter we deal with the boundary problem for complex analytic varieties in a "semi-local" setting.

More precisely, let $\Omega \subset \mathbb{C}^n$ be a strongly pseudoconvex open domain in $\mathbb{C}^n$, and $b\Omega$ its boundary. Let M be a maximally complex $(2m+1)$-dimensional real closed submanifold ($m \geq 1$) of some open domain $A \subset b\Omega$, and let $K = bA$ be its boundary. We want to find a domain $\widetilde{A}$ in Ω, independent of M, and a complex subvariety W of $\widetilde{A}$ such that:

(i) $b\widetilde{A} \cap b\Omega = A$;

(ii) $bW \cap b\Omega = M$,

In this chapter we show that, if $A \Subset b\Omega$, the problem we are dealing with has a solution $(\widetilde{A}, W)$ whose $\widetilde{A}$ can be determined in terms of the envelope $\widehat{K}$ of K with respect to the algebra of functions holomorphic in a neighborhood of $\overline{\Omega}$, i.e.

> *For any maximally complex $(2m+1)$-dimensional closed real submanifold M of A, $m \geq 1$, there exists an $(m+1)$-dimensional complex variety W in $\Omega \setminus \widehat{K}$, with isolated singularities, such that $bW \cap (A \setminus \widehat{K}) = M \cap (A \setminus \widehat{K})$.*

This result echoes that of Lupacciolu on the extension of CR-functions (see Theorem 3.6, and [63])

If A is not relatively compact, this result can be restated in terms of "principal divisors hull", leading to a global result for unbounded strictly pseudoconvex domains, different from the results of the previous chapter. Indeed, this method of proof allows us to drop the Lupacciolu ($\star$) hypothesis in Theorem 7.1 and extend the maximally complex submanifold to a domain, which can anyhow not be the whole of Ω. If the Lupacciolu hypothesis holds, then the domain of extension is in fact all of Ω. So this result is actually a generalization of Theorem 7.1.

The crucial question of the maximality of the domain $\widetilde{A}$ we construct is not answered; in some simple cases the domain is indeed maximal (see Example 8.1).

In the last section, by the same methods, the extension result is proved for analytic sets (see Theorem 8.10).

Recall that in Chapter 5 (see also [83]) related results are obtained via a bump lemma and cohomological methods. That approach may be generalized to complex spaces.

8.2 Main result

Let $\Omega \subset \mathbb{C}^n$ be a strongly pseudoconvex open domain in $\mathbb{C}^n$. Let A be a subdomain of $b\Omega$, and $K = bA$. For any Stein neighborhood Ω_α of Ω we set $\widehat{K}_\alpha$ to be the hull of K with respect to the algebra of holomorphic functions of Ω_α, i.e.

$$\widehat{K}_\alpha = \{x \in \Omega_\alpha \ : \ |f(x)| \leq \|f\|_K \ \forall f \in \mathcal{O}(\Omega_\alpha)\} \, .$$

We define $\widehat{K}$ as the intersection of the $\widehat{K}_\alpha$ when Ω_α varies through the family of all Stein neighborhoods of Ω. Observe that, since Ω is strongly pseudoconvex (and thus admits a fundamental system of Stein neighborhoods, see [95]), $\widehat{K}$ coincides with the hull of K with respect to the algebra of the functions which are holomorphic in some neighborhood of $\overline{\Omega}$. We claim that the following result holds:

Theorem 8.1 *For any maximally complex $(2m+1)$-dimensional closed real submanifold M of A, $m \geq 1$, there exists an $(m+1)$-dimensional complex variety W in $\Omega \setminus \widehat{K}$, with isolated singularities, such that $bW \cap (A \setminus \widehat{K}) = M \cap (A \setminus \widehat{K})$.*

Following the same strategy as in the previous chapter we first have a semi-global extension result (see Lemma 8.2 below). In order to "globalize" the extension the main differences with respect to the previous chapter are due to the fact that we have to cut Ω with level-sets of holomorphic functions instead of hyperplanes. This creates some additional difficulties: first of all it is no longer possible to use the parameter which defines the level-sets as a coordinate; secondly the intersections between tubular domains (see Lemmas 8.6, 8.8 and 8.9) may not be connected.

With the same proof as in the previous chapter we have

Lemma 8.2 *There exist a tubular neighborhood I of A in Ω and an $(m+1)$-dimensional complex submanifold with boundary $W_I \subset \overline{\Omega} \cap I$ such that $S \cap bW_I = M$*

Now, the hypothesis on the hull of K allows us to prove the following

Lemma 8.3 *Let $z^0 \in \Omega \setminus \widehat{K}$. Then there exist an open Stein neighborhood $\Omega_\alpha \supset \Omega$ and $f \in \mathcal{O}(\Omega_\alpha)$ such that*

1) $f(z^0) = 0$;

2) $\{f = 0\}$ is a regular complex hypersurface of $\Omega_\alpha \setminus \widehat{K}$;

3) $\{f = 0\}$ intersects M transversally in a compact manifold.

Remark 8.1 If f is such a function for z^0, for any point z' sufficiently near to z^0, $f(z) - f(z')$ satisfies conditions 1), 2) and 3) for z'.

Proof. By definition of $\widehat{K}$, since $z^0 \in \Omega \setminus \widehat{K}$ there is a Stein neighborhood Ω_α such that $z^0 \notin \widehat{K}_\alpha$. So we can find a holomorphic function g in Ω_α such that $g(z^0) = 1$ and $\|g\|_K < 1$; $h(z) = g(z) - 1$ is a holomorphic function whose zero set does not intersect $\widehat{K}$. Since regular level sets are dense, by choosing a suitable small vector v and redefining h as $h(z+v) - h(z^0+v)$ we can safely assume that h satisfies both 1) and 2).

We remark that $\{h = 0\} \cap b\Omega \Subset A$ by Alexander's Theorem (see [1, Theorem 3]), and this shows compactness. Then, we may suppose that M is not contained in $\{z_1 = z_1^0\}$ and, for ε small enough, we consider the function $f(z) = h(z) + \varepsilon(z_1 - z_1^0)$. It's not difficult to see (by applying Sard's Lemma) that 3) holds for generic ε. $\square$

Now, we divide the proof of Theorem 8.1 in two cases: $m \geq 2$ and $m = 1$. This is due to the fact that in the latter case proving that we can apply Harvey-Lawson to $\{f = 0\} \cap M$ is not automatic.

8.2.1 Dimension of M greater than or equal to 5 ($m \geq 2$)

For any $z^0 \in \Omega \setminus \widehat{K}$, Lemma 8.3 provides a holomorphic function such that the level-set $f_0 = \{f = 0\}$ contains z^0 and intersects M transversally in a compact manifold M_0. The intersection is again maximally complex (it is the intersection of a complex manifold and a maximally complex manifold, see [38]), so we can apply Harvey-Lawson theorem (Theorem 6.15) to obtain a holomorphic chain W_0 such that $bW_0 = M_0$. For τ in a small neighborhood U of 0 in $\mathbb{C}$, the hypersurface $f_\tau = \{f - \tau = 0\}$ intersects M transversally along a compact submanifold M_τ which, again by Theorem 6.15, bounds a holomorphic chain W_τ. Observe that since $M_\tau \subset f_\tau$, $W_\tau \subset f_\tau$.

We claim the following proposition holds:

Proposition 8.4 *The union $W_U = \bigcup_{\tau \in U} W_\tau$ is a complex variety contained in the open set $\widetilde{U} = \bigcup_{\tau \in U} f_\tau$.*

We need some intermediate results. Let us consider a generic projection $\pi : \widetilde{U} \to \mathbb{C}^m$ and set $\mathbb{C}^n = \mathbb{C}^{m+1} \times \mathbb{C}^{n-m-1}$, with holomorphic coordinates (w', w), $w' \in \mathbb{C}^{m+1}$, $w = (w_1, \dots, w_{n-m-1}) \in \mathbb{C}^{n-m-1}$. Let $V_\tau = \mathbb{C}^{m+1} \setminus \pi(M_\tau)$.

For $\tau \in U$, $w' \in \mathbb{C}^{m+1} \setminus \pi(M_\tau)$ and $\alpha \in \mathbb{N}^{n-m-1}$, we define

$$I^\alpha(w', \tau) \doteqdot \int_{(\eta', \eta) \in M_\tau} \eta^\alpha \omega_{BM}(\eta' - w'),$$

where ω_{BM} is the Bochner-Martinelli kernel.

Recall Lemma 7.7 and Remark 7.2.

Lemma 8.5 *$P^\alpha(F(w', \tau))$ is holomorphic in the variable $\tau \in U \subset \mathbb{C}$, for each $\alpha \in \mathbb{N}^{n-m-1}$.*

Proof. Let us fix a point $(w', \underline{\tau})$ such that $w' \notin M_{\underline{\tau}}$ (this condition remains true for $\tau \in B_\epsilon(\underline{\tau})$). Consider as domain of $P^\alpha(F)$ the set $\{w'\} \times B_\epsilon(\underline{\tau})$. In view of Morera's Theorem, we need to prove that for any simple curve $\gamma \subset B_\epsilon(\underline{\tau})$,

$$\int_\gamma P^\alpha(F(w', \tau)) d\tau = 0.$$

Let $\Gamma \subset B_\epsilon(\underline{\tau})$ be an open set such that $b\Gamma = \gamma$. By $\gamma * M_\tau$ ($\Gamma * M_\tau$) we mean the union of M_τ along γ (along Γ). Note that these sets are submanifolds of $\mathbb{C} \times \mathbb{C}^n$. The projection $\pi : \Gamma * M_\tau \to \mathbb{C}^n$ on the second factor is injective and $\pi(\Gamma * M_\tau)$ is an open subset of M bounded by $\pi(b\Gamma * M_\tau) = \pi(\gamma * M_\tau)$. By Lemma 7.7 and Stokes' theorem

$$\begin{aligned}
\int_\gamma P^\alpha(F(w',\tau))d\tau &= \int_\gamma I^\alpha(w',\tau)d\tau = \\
&= \int_\gamma \left(\int_{(\eta',\eta)\in M_\tau} \eta^\alpha \omega_{BM}(\eta'-w')\right) d\tau = \\
&= \iint_{\gamma * M_\tau} \eta^\alpha \omega_{BM}(\eta'-w') \wedge d\tau = \\
&= \iint_{\Gamma * M_\tau} d\left(\eta^\alpha \omega_{BM}(\eta'-w') \wedge d\tau\right) = \\
&= \iint_{\Gamma * M_\tau} d\eta^\alpha \wedge \omega_{BM}(\eta'-w') \wedge d\tau = \\
&= \iint_{\pi(\Gamma * M_\tau)} d\eta^\alpha \wedge \omega_{BM}(\eta'-w') \wedge \pi_* d\tau = \\
&= 0.
\end{aligned}$$

The last equality follows from the fact that in $d\eta^\alpha$ appear only holomorphic differentials, η^α being holomorphic. But since all the holomorphic differentials supported by $\pi(\Gamma * M_\tau) \subset M$ already appear in $\omega_{BM}(\eta'-w') \wedge \pi_* d\tau$ (due to the fact that M is maximally complex and contains only $m+1$ holomorphic differentials) the integral is zero. □

Proof of Proposition 8.4. From [40] it follows that each W_τ has only isolated singularities[1]. So, let us fix a regular point $(w_0', w_0) \in f_{\tau_0} \subset \widetilde{U}$. In a neighborhood of this point $W = W_U$ is a manifold, since the construction depends continuously on the initial data. We want to show that W is indeed analytic in $\widetilde{U}$.

Let us fix $j \in \{1, \ldots, n-m-1\}$ and consider multiindexes α of the form $(0, \ldots, 0, \alpha_j, 0, \ldots, 0)$; let P_j^α be the corresponding $P^\alpha(F(w',\tau))$. Observe that for any j we can consider a finite number of P_j^α (it suffices to use $h = P_j^0(F(w',\tau))$ of them; not that h is independent of j). By a linear combination of the P_j^α with rational coefficients, we obtain the elementary symmetric functions $S_j^0(w',k), \ldots, S_j^h(w',\tau)$ in such a way that for any point $(w',w) \in W$ there exists $\tau \in U$ such that $(w',w) \in W_\tau$; thus, defining

$$Q_j(w',w,\tau) = S_j^h(w',\tau) + S_j^{h-1}(w',\tau)w_j + \cdots + S_j^0(w',\tau)w_j^h = 0,$$

[1]There could be singularities coming up from intersections of the solutions relative to different connected components of M_τ. These singularities are analytic sets and therefore should intersect the boundary. This cannot happen and so also these singularities are isolated.

we have, in other words,

$$W \subset V = \bigcup_{\tau \in U} \bigcap_{j=1}^{n-m-1} \{Q_j(w', w, \tau) = 0\}.$$

Define $\widetilde{V} \subset \mathbb{C}^n(w', w) \times \mathbb{C}(\tau)$ as

$$\widetilde{V} = \bigcap_{j=1}^{n-m-1} \{Q_j(w', w, \tau) = 0\}$$

and

$$\widetilde{W} = W_\tau * U \subset \widetilde{V}.$$

Observe that, since the functions S_j^α are holomorphic, $\widetilde{V}$ is a complex subvariety of $\mathbb{C}^n \times U$. Since $\widetilde{V}$ and $\widetilde{W}$ have the same dimension, in a neighborhood of (w_0', w_0, τ) $\widetilde{W}$ is an open subset of the regular part of $\widetilde{V}$, thus a complex submanifold. We denote by $\mathsf{Reg}\,\widetilde{W}$ the set of points $z \in \widetilde{W}$ such that $\widetilde{W} \cap \mathcal{U}$ is a complex submanifold in a neighborhood $\mathcal{U}$ of z. It is easily seen that $\mathsf{Reg}\,\widetilde{W}$ is an open and closed subset of $\mathsf{Reg}\,\widetilde{V}$, so a connected component. Observing that the closure of a connected component of the regular part of a complex variety is a complex variety we obtain the that $\widetilde{W}$ is a complex variety, $\widetilde{W}$ being the closure of $\mathsf{Reg}\,\widetilde{W}$ in $\widetilde{V}$.

Finally, since the projection $\pi : \widetilde{W} \to W$ is a homeomorphism and so proper, it follows that W is a complex subvariety as well. □

Now we prove that the varieties $\widetilde{W}_U$ that we have found — which are defined in the open subsets of type $\widetilde{U}$ (see Proposition 8.4) — patch together in such a way to define a complex variety on the whole of $\Omega \setminus \widehat{K}$.

Lemma 8.6 *Let $\widetilde{U}_f$ and $\widetilde{U}_g$ be two open subsets as in Proposition 8.4 and let W_f and W_g be the corresponding varieties. Let $z^1 \in \widetilde{U}_f \cap \widetilde{U}_g$. Then W_f and W_g coincide in a neighborhood of z^1.*

Proof. Let $\lambda = f(z^1)$ and $\tau = g(z^1)$ and consider

$$L(\lambda', \tau') = \{f = \lambda'\} \cap \{g = \tau'\} \subset \Omega$$

for (λ', τ') in a neighborhood of (λ, τ). Note that for almost every (λ', τ') $L(\lambda', \tau')$ is a complex submanifold of codimension 2 of $\widetilde{U}_f \cap \widetilde{U}_g$. Moreover, $W_f \cap L(\lambda', \tau')$ and $W_g \cap L(\lambda', \tau')$ are both solutions of the Harvey-Lawson problem for $M \cap L(\lambda', \tau')$, consequently they must coincide. Since the complex subvarieties $L(\lambda', \tau')$ which are regular form a dense subset, W_f and W_g coincide on the connected component of $\widetilde{U}_f \cap \widetilde{U}_g$ containing z^1. □

Remark 8.2 The above proof does not work in the case $m = 1$ since in that case $M \cap L(\lambda', \tau')$ is generically empty.

In order to end the proof of Theorem 8.1, we have to show that the set S of the singular points of W is a discrete subset of $\Omega \setminus \widehat{K}$. Let $z^1 \in \Omega \setminus \widehat{K}$, and choose a function h, holomorphic in a neighborhood of Ω such that $h(z^1) = 1$ and $K \subset \{|h| \leq \frac{1}{2}\}$ and consider $f = h - \frac{3}{4}$. Observe that $z^1 \in \{\mathsf{Re}\, f > 0\}$ and $K \subset \{\mathsf{Re}\, f < 0\}$. Choose a defining function φ for $b\Omega$, strongly psh in a neighborhood of Ω and let us consider the family

$$(\phi_\lambda = \lambda(\varphi) + (1-\lambda)\mathsf{Re}\, f)_{\lambda \in [0,1]}$$

of strongly plurisubharmonic functions. For λ near to 1, $\{\phi_\lambda = 0\}$ does not intersect the singular locus. Let $\overline{\lambda}$ be the biggest value of λ for which

$$\{\phi_\lambda = 0\} \cap S \neq \emptyset.$$

Then the analytic set S touches the boundary of the Stein domain

$$\{\phi_{\overline{\lambda}} < 0\} \cap \Omega \subset \Omega.$$

So $\{\phi_{\overline{\lambda}} = 0\} \cap S$ is a set of isolated points in S. By repeating the same argument, we conclude that S is made up by isolated points.

8.2.2 Dimension of M equal to 3 $(m = 1)$

The first goal is to show that when we slice transversally M with complex hypersurfaces, we obtain 1-dimensional real submanifolds which satisfy the moments condition.

Again, we fix our attention to a neighborhood of the form

$$\widetilde{U} = \bigcup_{\tau \in U} g_\tau.$$

Let us choose an arbitrary holomorphic $(1,0)$-form ω in $\mathbb{C}^n$.

Lemma 8.7 *The function*

$$\Phi_\omega(\tau) \;=\; \int_{M_\tau} \omega$$

is holomorphic in U.

Proof. Using again Morera's Theorem, we need to prove that for any simple curve $\gamma \subset U$, $\gamma = b\Gamma$,

$$\int_\gamma \Phi_\omega(\tau)d\tau \;=\; 0.$$

By Stokes Theorem, we have

$$\begin{aligned}
\int_\gamma \Phi_\omega(\tau)d\tau &= \int_\gamma \left(\int_{M_k} \omega\right) d\tau \;= \\
&= \iint_{\gamma * M_\tau} \omega \wedge d\tau \;= \\
&= \iint_{\Gamma * M_\tau} d(\omega \wedge d\tau) \;= \\
&= \iint_{\Gamma * M_\tau} \partial\omega \wedge d\tau \;= \\
&= \iint_{\pi(\Gamma * M_\tau)} \partial\omega \wedge \pi_* d\tau \;= \\
&= 0.
\end{aligned}$$

The last equality is due to the fact that $\pi(\Gamma * M_\tau) \subset M$ is maximally complex and thus supports only $(2,1)$ and $(1,2)$-forms, while $\partial\omega \wedge \pi_* d\tau$ is a $(3,0)$-form. □

Lemma 8.8 *Let g be a holomorphic function on a neighborhood of Ω, and suppose $\{|g| > 1\} \cap \widehat{K} = \emptyset$. Then there exists a variety W_g on $\Omega \cap \{|g| > 1\}$ such that $bW_g \cap b\Omega = M \cap \{|g| > 1\}$.*

Lemma 8.9 *Given two functions g_1 and g_2 as above, then W_{g_1} and W_{g_2} agree on $\{|g_1| > 1\} \cap \{|g_2| > 1\}$.*

Proof of Lemma 8.8. We are going to use several times open subsets of the type $\widetilde{U}$ as in Proposition 8.4, so we need to fix some notations. Given an open subset $U \subset \mathbb{C}$, define $\widetilde{U}$ by

$$\widetilde{U} = \bigcup_{\tau \in U} \{f = \tau\}.$$

From now on we use open subsets of the form $U = B(\overline{\tau}, \delta)$, where $B(\overline{\tau}, \delta)$ is the disc centered at $\overline{\tau}$ of radius δ. We say that $\{f = \overline{\tau}\}$ is the *core* of $\widetilde{U}$ and δ is its *amplitude*.

For a fixed $d > 1$ consider the compact set $H_d = \overline{\Omega} \cap \{|g| \geq d\}$; we show that W_g is well defined on H_d. Let us fix also a compact set $C \subset \Omega$ such that W_I (see Lemma 8.2) is a closed submanifold in $H_d \setminus C$.

Consider all the open subsets $V_\alpha = \widetilde{U}_\alpha \cap \Omega$, constructed using only the function $f = g - 1$ up to addition of the function $\varepsilon(z_j - z_j^0)$ (see Lemma 8.3). If we do not allow ε to be greater than a fixed $\overline{\varepsilon} > 0$, then by a standard argument of semicontinuity and compactness we may suppose that the amplitude of each $\widetilde{U}$ is greater than a positive δ.

We claim that it is possible to find a countable covering of H_d made by a countable sequence V_i of those V_α in such a way to have

1. $V_0 \subset H_d \setminus C$;
2. if
$$B_l = \bigcup_{i=1}^{l} V_i$$
then $V_{l+1} \cap B_l \cap \Omega \neq \emptyset$.

The only thing we have to prove is the existence of V_0, since the second statement follows by a standard compactness argument.

Set $L = \max_{H_d} \mathsf{Re}\, g$. Since $\mathsf{Re}\, g$ is a non constant pluriharmonic function, $\{\mathsf{Re}\, g = L\}$ is a compact subset of $b\Omega \cap H_d$. Then we can choose $\eta > 0$ such that $\{\mathsf{Re}\, g = L - \eta\} \cap \Omega$ is contained in $H_d \setminus C$, and this allows to define V_0.

Let $\widetilde{U}_1$ and $\widetilde{U}_2$ be two such open sets and $z^0 \in \widetilde{U}_1 \cap \widetilde{U}_2$. We can suppose that the cores of $\widetilde{U}_1$ and $\widetilde{U}_2$ contain z^0. They are of the form

$$f + \varepsilon_1(z_j - z_j^0) = \tau(\varepsilon_1) \text{ and } f + \varepsilon_2(z_j - z_j^0) = \tau(\varepsilon_2).$$

For $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, we consider the open sets $\widetilde{U}_\varepsilon$ whose core, passing by z^0, is $f + \varepsilon(z_j - z_j^0) = \tau(\varepsilon)$. We must show that the set

$$\Lambda = \left\{\varepsilon \in (\varepsilon_1, \varepsilon_2) : \exists W_\varepsilon \ s.t.\ W_\varepsilon \cap (\widetilde{U}_1 \cap \widetilde{U}_\varepsilon) = W_1 \cap (\widetilde{U}_1 \cap \widetilde{U}_\varepsilon)\right\}$$

is open and closed, where W_ε is a variety in $\widetilde{U}_\varepsilon$.

Λ is open. Indeed, if $\varepsilon \in \Lambda$, then for ε' in a neighborhood of ε the core of $\widetilde{U}_{\varepsilon'}$ is contained in $\widetilde{U}_\varepsilon$ and so its intersection with M is maximally complex. Because of Lemma 8.7 the condition holds also for all the level sets in $\widetilde{U}_{\varepsilon'}$ and then we can apply again the Harvey-Lawson Theorem [38] and the arguments of Proposition 8.4 in order to obtain $W_{\varepsilon'}$. Moreover, there is a connected component of $U_\varepsilon \cap U_{\varepsilon'}$ which contains z^0 and touches the boundary of Ω, where the W_ε and $W_{\varepsilon'}$ both coincide with W_I (see Lemma 8.2). By virtue of the analytic continuation principle, they must coincide in the whole connected component.

Λ is closed. Indeed, since each $\widetilde{U}$ has an amplitude of at least δ, we again have that, for $\overline{\varepsilon} \in \overline{\Lambda}$, the intersection of $\widetilde{U}_{\overline{\varepsilon}}$ and $\widetilde{U}_{\varepsilon}$ must include (for $\varepsilon \in \Lambda$, $|\varepsilon - \overline{\varepsilon}|$ sufficiently small) a connected component containing z^0 and touching the boundary. We then conclude as in the previous case. □

Proof of Lemma 8.9. Let us consider the connected components of $W_{g_1} \cap \{|g_2| > 1\}$. For each connected component W_1 two cases are possible:

1. W_1 touches the boundary of Ω: $W_1 \cap b\Omega \neq \emptyset$;

2. the boundary of W_1 is inside Ω:

$$bW_1 \Subset \{|g_1| = 1\} \cup \{|g_2| = 1\} \subset \Omega$$

In the former, the result easily follows in view of the analytic continuation principle (remember that on a strip near the boundary W_{g_1} and W_{g_2} coincide).

The latter is actually impossible. Indeed, suppose by contradiction that the component W_1 satisfies (2). Restrict g_1 and g_2 to W_1 and choose $t > 1$ such that

$$W_t \ \doteqdot \ \{|g_i| > t,\ i = 1, 2\} \ \Subset \ W_1.$$

The boundary bW_t of W_t consists of points where either $|g_1| = t$ or $|g_2| = t$. Choose a point z_0 of the boundary where $|g_1| = t$ and $|g_2| > t$, then $|g_2|$ is a plurisubharmonic function on the analytic set

$$A = \{g_1 = g_1(z_0)\} \cap \{|g_2| \geq t\}\,.$$

Since $W_t \Subset W_1$, the boundary of the connected component of A through z_0 is contained in $\{|g_2| = t\}$. This is a contradiction, because of the maximum principle for plurisubharmonic functions. □

8.3 Some remarks

8.3.1 Maximality of the solution

As stated above, we have not a complete answer to the problem of the maximality of $\widetilde{A}$. Nevertheless, here is a simple example where the constructed domain is actually maximal.

Example 8.1 Let $\Omega \subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary, $0 \in \Omega$, and let h be a pluriharmonic function defined in a neighborhood

U of $\overline{\Omega}$ such that $h(0) = 0$ and $h(z) = h(z_1, \ldots, z_{n-1}, 0)$ (i.e. h does not depend on z_n). Pose

$$H = \{z \in U : \; h(z) = 0\}$$

and let

$$A = b\Omega \cap \{z \in U : \; h(z) > 0\}.$$

Then

$$\widetilde{A} = \Omega \cap \{z \in U : \; h(z) > 0\}.$$

In order to show that $\widetilde{A}$ is maximal for our problem, it suffices to find, for any $z \in H \cap \Omega$, a complex manifold $W_z \subset \widetilde{A}$ such that $M_z = \overline{W}_z \cap A$ is smooth and W_z cannot be extended through any neighborhood of z. We may suppose $z = 0$.

So, let $f \in \mathcal{O}(\overline{\Omega})$ be such that $\mathsf{Re}\, f = h$, $f(0) = 0$. We define

$$W_0 = \{z \in \widetilde{A} : \; z_n = e^{\frac{1}{f(z)}}\};$$

W_0 extends as a closed submanifold of $U \setminus \{f = 0\}$. Moreover, observe that each point of $\{f = 0\}$ is a cluster point of W_0. Suppose by contradiction that W_0 extends through a neighborhood V of 0 by a complex manifold W_0'; then $\{f = 0\} \cap V \subset W_0'$, thus $\{f = 0\} \cap V = W_0' \cap V$. This is a contradiction.

8.3.2 The unbounded case

Let $\Omega \subset \mathbb{C}^n$ be a strictly pseudoconvex domain, and $A \subset b\Omega$ an unbounded open subset of $b\Omega$.

Consider the set

$$\mathcal{A} \;=\; \{A' \Subset b\Omega \mid A' \subset A, \; A' \; \textit{domain}\}\,.$$

For an arbitrary $A' \in \mathcal{A}$ ($bA' = K'$), let $D_{A'}$ be the compact connected component of $\Omega \setminus \widehat{K'}$. Set

$$D \;=\; \bigcup_{A' \in \mathcal{A}} D_{A'}.$$

From Theorem 8.1 it follows that for every maximally complex closed $(2m+1)$-dimensional real submanifold M of A, there is an $(m+1)$-dimensional complex closed subvariety W of D, with isolated singularities, such that $bW \cap A = M$. So the domain D is a possible solution of our extension problem.

When $A = b\Omega$, we may restate the previous result in a more elegant way. In the same situation as above, consider

$$\mathbb{C}^n \subset \mathbb{CP}^n, \ \mathbb{C}^n = \mathbb{CP}^n \setminus \mathbb{CP}^{n-1}_\infty$$

and define the *principal divisors hull* $\widehat{C}_\mathcal{D}$ of $C = \overline{\Omega} \cap \mathbb{CP}^{n-1}_\infty$ by

$$\widehat{C}_\mathcal{D} \ = \ \left\{ z \in \Omega \mid \forall f \in \mathcal{O}(\overline{\Omega}) \ \overline{L}_{f,z} \cap C \neq \emptyset \right\},$$

where $\overline{L}_{f,z}$ is the closure of the connected component (in $\overline{\Omega}$) of the level-set $\{f = f(z)\}$ passing through z. Then

$$D = \Omega \setminus \widehat{C}_\mathcal{D}.$$

Indeed, if $z \in D$, then there are an open subset $A' \subset b\Omega$ and a function $f \in \mathcal{O}(\overline{\Omega})$ such that $\overline{L}_{f,z} \cap b\Omega$ is a compact submanifold of A'. In particular $z \notin \widehat{C}_\mathcal{D}$. Vice versa, if $z \notin \widehat{C}_\mathcal{D}$ then there is a function $g \in \mathcal{O}(\Omega')$ ($\Omega' \supset \Omega$ domain) such that $N = \overline{L}_{g,z} \cap C = \emptyset$, i.e. it is a compact submanifold of $b\Omega$. By choosing a relatively compact open subset $A' \subset b\Omega$ large enough to contain N it follows that $z \in D_{A'} \subset D$.

8.4 Generalization to analytic sets

Let Ω, A and K be as before. We want now to consider the extension problem for analytic sets.

We will say that $M \subset A$ is a *k-deep trace* of an analytic subset if there are

i) an open set $U \subset \mathbb{C}^n$ ($U \cap b\Omega = A$);

ii) an $(m+1)$-dimensional irreducible analytic set W_M, whose ideal sheaf $\mathcal{I}_{W_M}$ has depth at least k at each point of U, such that $W_M \cap b\Omega = M$.

In this case, we say that the real dimension of M is $2m+1$.

Theorem 8.10 *For any $(2m+1)$-dimensional 4-deep trace of analytic subset $M \subset A$, there exists an $(m+1)$-dimensional complex variety W in $\Omega \setminus \widehat{K}$, such that $bW \cap (A \setminus \widehat{K}) = M \cap (A \setminus \widehat{K})$.*

Observe that in this situation we already have a strip U on which the set M extends. So we only need to generalize Lemma 8.3 and the results in Section 8.2.1.

Lemma 8.11 *Let $z^0 \in \Omega \setminus \widehat{K}$. Then there exist an open Stein neighborhood $\Omega_\alpha \supset \Omega$ and $f \in \mathcal{O}(\Omega_\alpha)$ such that*

1. *$f(z^0) = 0$;*
2. *$\{f = 0\}$ is a regular complex hypersurface of $\Omega_\alpha \setminus \widehat{K}$;*
3. *$\{f = 0\}$ intersects M in a compact set and W_M in an analytic subset (of depth at least 3).*

Proof. The proof of the first two conditions is exactly the same as before. So, we focus on the third one.

Again, Alexander's Theorem (see [1, Theorem 3]) implies compactness of the intersection with M. Then, we may suppose that W_M is not contained in $\{z_1 = z_1^0\}$ and, for ε small enough, let $f : \Omega_\alpha \to \mathbb{C}$ be the function $f(z) = h(z) + \varepsilon(z_1 - z_1^0)$, where Ω_α and h are as defined in Lemma 8.3. Consider the stratification of W_M in complex manifolds. By Sard's Lemma, the set of ε for which the intersection of $\{f(z) = 0\}$ with a fixed stratum is transversal is open and dense. Hence the set of ε for which the intersection of $\{f(z) = 0\}$ with each stratum is transversal is also open and dense, in particular it is non-empty. The conclusion follows. □

The previous Lemma enables us to extend each analytic subset

$$W_0 = W_M \cap \{f = 0\}$$

to an analytic set defined on the whole of

$$\Omega \cap \{f = 0\}.$$

Indeed, on a strictly pseudoconvex corona the depth of W_0 is at least 3 and thus W_0 extends in the hole (see e.g. [4, 82]). Obviously the extension lies in $\{f = 0\}$.

Observe that, up to a arbitrarily small modification of $b\Omega$ we can suppose that it intersects each stratum of the stratification of W_M transversally. In this situation M is a smooth submanifold with negligible singularities of Hausdorff codimension at least 2 (see [28]).

Again, we consider a generic projection $\pi : \widetilde{U} \to \mathbb{C}^m$ and we use holomorphic coordinates (w', w), $w = (w_1, \ldots, w_{n-m-1})$ on

$$\mathbb{C}^n = \mathbb{C}^{m+1} \times \mathbb{C}^{n-m-1}.$$

Keeping the notations used in Section 8.2.1, let $V_\tau = \mathbb{C}^{m+1} \setminus \pi(M_\tau)$.

For $\tau \in U$, $w' \in \mathbb{C}^{m+1} \setminus \pi(M_\tau)$ and $\alpha \in \mathbb{N}^{n-m-1}$, we define

$$I^\alpha(w',\tau) \doteqdot \int_{(\eta',\eta)\in \operatorname{Reg}(M_\tau)} \eta^\alpha \omega_{BM}(\eta' - w'),$$

ω_{BM} being the Bochner-Martinelli kernel.

Observe that the previous integral is well-defined and converges. In fact, $W_\tau = W_M \cap \{f = \tau\}$ is an analytic set and thus, by Lelong's Theorem, its volume is bounded near the singular locus. Hence, by Fubini's Theorem, also the regular part of $M_\tau = W_\tau \cap b\Omega$ has finite volume up to a small modification of $b\Omega$.

Lemma 8.12 *Let $F(w',\tau)$ be the multiple-valued function which represents $\widetilde{M_\tau}$ on $\mathbb{C}^m \setminus \pi(M_\tau)$; then, if we denote by $P^\alpha(F(w',\tau))$ the sum of the α^{th} powers of the values of $F(w',\tau)$, the following holds:*

$$P^\alpha(F(w',\tau)) = I^\alpha(w',\tau).$$

In particular, $F(w',\tau)$ is finite.

Proof. Let V_0 be the unbounded component of V_τ. On V_0, $P^\alpha(F(w',\tau)) = 0$. Following [38], it is easy to show that on V_0 also $I^\alpha(F(w',\tau)) = 0$. Indeed, if w' is far enough from $\pi(\mathsf{Reg}\, M_\tau)$, then $\beta = \eta^\alpha \omega_{BM}(\eta' - w')$ is a regular $(m, m-1)$-form on some ball B_R of $\mathsf{Reg}\, M_\tau$. So, since in B_R there exists γ such that $\overline{\partial}\gamma = \beta$, we may write in the sense of currents

$$[\mathsf{Reg}\, M_\tau](\beta) = [\mathsf{Reg}\, M_\tau]_{m,m-1}(\overline{\partial}\gamma) = \overline{\partial}[\mathsf{Reg}\, M_\tau]_{m,m-1}(\gamma).$$

We claim that $\overline{\partial}[\mathsf{Reg}\, M_\tau]_{m,m-1}(\gamma) = 0$ and, in order to prove this, we first show that $[\mathsf{Reg}\, M_\tau]$ is a closed current. Indeed, observe that $d[\mathsf{Reg}\, M_\tau]$ is a flat current, since it is the differential of an $\mathrm{L}^1_{\text{loc}}$ current (see [30]). Moreover

$$S = \operatorname{supp}(d[\mathsf{Reg}\, M_\tau]) \subset \mathsf{Sing}\, M_\tau,$$

hence, denoting by $\dim_{\mathcal{H}}$ the Hausdorff dimension and by $\mathcal{H}^s$ the s-Hausdorff measure, we have

$$\dim^{\mathcal{H}}(S) \le \dim_{\mathcal{H}}(\operatorname{sing}(M_\tau)) \le \dim_{\mathcal{H}}(\mathsf{Reg}\, M_\tau) - 2$$

and consequently that

$$\mathcal{H}_{\dim_{\mathcal{H}}(\mathsf{Reg}\, M_\tau)-1}(S) = 0.$$

By Federer's support theorem (see [30]), this implies that

$$d[\mathsf{Reg}\, M_\tau] = 0.$$

Now, since $\mathsf{Reg}\, M_\tau$ is maximally complex,

$$[\mathrm{Reg}(M_\tau)] = [\mathsf{Reg}\, M_\tau]_{m,m-1} + [\mathrm{Reg}(M_\tau)]_{m-1,m}.$$

Since $\overline{\partial}[\mathsf{Reg}\, M_\tau]_{m,m-1}$ is the only component of bidegree $(m, m-2)$ of $d[\mathsf{Reg}\, M_\tau]$ and $d[\mathsf{Reg}\, M_\tau] = 0$ then

$$\overline{\partial}[\mathsf{Reg}\, M_\tau]_{m,m-1} = 0.$$

Moreover, since $[\mathrm{Reg} M_\tau](\beta)$ is analytic in the variable w', $[\mathrm{Reg} M_\tau](\beta) = 0$ for all $w' \in V_0$.

The rest of the proof goes as in Lemma 7.7. $\square$

Lemma 8.13 *$P^\alpha(F(w', \tau))$ is holomorphic in the variable $\tau \in U \subset \mathbb{C}$, for each $\alpha \in \mathbb{N}^{n-m-1}$.*

Proof. The only difference with the proof for the case of manifolds is the fact that I is an integration over the regular part of $\Gamma * M_\tau$ and not all over $\Gamma * M_\tau$. It is easy to see that Stokes' theorem is valid also in this situation, so the chain of integrals in Lemma 8.5 holds in this case, too. $\square$

The rest of the proof of Theorem 8.10 goes as in the proof of Theorem 8.1 (see Subsection 8.2.1).

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