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On the First Order Theory of Real Exponentiation

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Contents

Introduction

The first order theory of real exponentiation has been studied by many mathematicians, among which are L. van den Dries, A. Gabrielov, A. Khovanskii, A. Macintyre, J.-P. Ressayre, D. Richardson, N. Vorobjov, A. Wilkie, H. Wolter, to recall only those who are most connected to this work.

The methods of inquiry are manifold and come from model theory as well as from analytic geometry and computational algebra. One of the main motivations for this inquiry is the decidability problem posed by Tarski. In the 1930's, Alfred Tarski proved that the *elementary theory* of the real numbers is decidable. This means that there is an algorithm which, on input an elementary sentence θ, decides whether θ is true or false $-$ an elementary sentence being an expression built up using the elementary operations $+, -, \cdot$, the elementary relations $=, <$ and the boolean connectives, and quantifying only over variables which denote real numbers (and not, for example sets of numbers or functions). Tarski asked, furthermore, if his decidability result could be generalized to the case of real exponentiation, namely, if it were possible to decide algorithmically the truth of sentences in which, together with the elementary operations and relations, also the exponential function appears. This latter question turned out to be extremely difficult and remains unanswered to this day, although partial results have been achieved in the meanwhile.

The aim of this work is to give a presentation of some of the results obtained so far in this area, and to improve and refine them when possible. This will be achieved mainly by looking for a suitable axiomatization of the complete theory of real exponentiation, or of some of its fragments: it is well known that the decidability of a complete theory is equivalent to its recursive axiomatizability, and this second problem is often more accessible than the problem of finding a decision algorithm (Tarski himself used this approach to prove the decidability of the elementary theory of the real numbers). A. Macintyre and A. Wilkie proposed in [21] a certain recursive set of axioms as candidate for a complete axiomatization of the real exponential field. This candidate is complete, provided that Schanuel's Conjecture (see 4.3.7) holds.

We follow the same approach and finally propose a simplified candidate, which is again complete assuming Schanuel's Conjecture.

Some of the results contained in this thesis were obtained in collaboration with A. Berarducci, and will be objects of future publication.

0.1 Outline of this work and main results

The approach to the decidability problem described above, i.e. looking for a recursive complete axiomatization, gives us the possibility to go beyond the real exponential case, and analyse the elementary model theory and geometry of a broad class of functions over real closed fields. In the first three chapters we make a point of using elementary methods: firstly, we avoid using non first order properties like Dedekind Completeness or the compactness of a closed and bounded interval; these two properties are frequently used, implicitely or explicitely, in most of the classical proofs of the basic results of calculus and differential geometry. Secondly, we always try to keep our assumptions to the minimum; in particular, we do not have access to the powerful tools of o-minimal geometry.

In Chapter 1 we fix the general context in which we are working: we take into consideration any expansion of an ordered field such that every definable subset of the domain, which is bounded from above, has a least upper bound. We call these structures *definably complete structures* (see Chapter 1, Definition 1.1.1). They form a recursively axiomatized class which includes, in addition to the real exponential field, all the following: any expansion of the real field (for example the real field with the sine function); any o-minimal expansion of a real closed field (for example any elementary superstructure of the real field with all analytic functions restricted to a closed ball); any model of a suitable recursive fragment of the complete theory of real exponentiation (or, of $\mathbb{R}$ with a pfaffian chain of functions, or even of $\mathbb{R}$ with the sine function). We develop the theory of basic calculus in this setting. We prove, then, a uniqueness result for the definable solutions of linear differential equations (see Theorem 1.5.1), with an elementary proof which does not use o-minimality. We prove an effective version of Newton's method for the existence of nonsingular zeroes of C^2 definable maps (see Theorem 1.4.1), which will be used in Chapter 4.

In Chapter 2 we consider the basic differential topology of C^∞ definable functions in definably complete structures. We concentrate, then, on those definable functions which form a noetherian ring closed under differentiation (see Definition 2.3.2). We prove that the zero-set of a function belonging to such a ring can be decomposed into a finite union of smooth manifolds, defined via functions from the same ring (see Theorem 2.4.7). We leave as an open question to establish whether the union is disjoint. Here we do not assume Khovanskii-type finiteness properties, hence this decomposition theorem holds for a wide class of functions, which includes non-tame examples like $\sin(x)$, and may include some C^∞ but not analytic examples. Results of a similar flavour have been obtained by A. Gabrielov in a different setting, namely, in the context of real analytic functions restricted to a compact ball (see [15]). We apply our decomposition theorem to prove a Khovanskii-type finiteness theorem (see Theorem 2.6.6): given a noetherian differential ring M of functions, if every zero-dimensional zero-set of functions in M consists of finitely many points, then the zero-set of any function in M has finitely many connected components.

In Chapter 3 we take a further step and try to find out which conditions guarantee that a certain definably complete structure is o-minimal. The results of this investigation, illustrated in the chapter, have already been published in a paper together with A. Berarducci (see the introduction to the chapter for a description of the main results). We follow the approach of A. Wilkie in [37] and we give an *effective* version of his results. As a consequence, we obtain that there exists a recursively axiomatized *o-minimal* subtheory of the complete theory of real exponentiation. This result will play an important role in Chapter 4, where it will contribute to build up our candidate for a recursive axiomatization.

Despite the generality of the setting, the first three chapters should be read bearing in mind that every result appearing therein holds true in particular for the real exponential field.

In Chapter 4 we concentrate only on the real exponential field and we give an overview of the decidability problem for this structure (see the introduction to the chapter for a detailed explanation of the problem). We start by introducing the concepts of *effectively continuous function* and *effectively C^2 function* (see Definitions 4.2.1 and 4.2.4); we then prove, using the effective version of Newton's method proved in Chapter 1, that it is possible to enumerate recursively the set of tuples of effective C^2 functions which have a nonsingular common zero. In particular, since the exponential function is effectively continuous, there is an algorithm which, on input an n-tuple of exponential terms in n variables, stops if and only if the terms have a nonsingular common zero in $\mathbb{R}^n$. The relevance of this result to the decidability problem lies in the fact that, if we had an analogous result also for *singular* zeroes, then, by results of A. Macintyre and A. Wilkie, the theory of real exponentiation would be decidable.

Subsequently, we consider the universal fragment of the theory and we show that there also lie delicate problems, such as establishing when two closed terms are equal. However, we notice that, if this latter problem were decidable, then so would be the problem of establishing when two terms containing variables are equal. Finally, we apply the results of the previous chapters to the exponential case and build up a candidate for a recursive axiomatization of the theory of real exponentiation. We conclude the dissertation with a list of open problems which we intend to investigate in the future.

0.2 Notation and Prerequisites

Let n be a natural number. If $\{\overline{x}_1, \ldots, x_n\}$ is a set of variables, we denote by $\overline{x}$ the ordered n-tuple $(x_1, \ldots, x_n)$. If A is a set and $a_1, \ldots, a_n$ are elements of A, we denote by $\overline{a}$ the ordered n-tuple $(a_1, \ldots, a_n)$. If $I = (i_1, \ldots, i_n)$ is a multi-index, then we denote by $|I|$ the number $i_1 + \ldots + i_n$.

0.2.1 (First order languages and structures)**.** In this work we will consider first order languages which are expansions of the *language of ordered fields*

$$L_{\mathrm{of}} = \{+, -, \cdot, <, 0, 1\}.$$

We refer to [1] for a review of the basic concepts of first order logic.

We will consider first order structures which are expansions of some ordered ring or field. We will use the same symbol to denote a structure and its domain, whenever this is not confusing. A systematic exception will be made for the structures based on the set of real numbers $\mathbb{R}$. This latter symbol will always denote the *set*, while the structures will be denoted with subscripts, for example: $\mathbb{R}_{\mathrm{of}} = \langle \mathbb{R}, 0, 1, +, -, \cdot, < \rangle$ will denote the real ordered field, $\mathbb{R}_{\exp} = \langle \mathbb{R}, 0, 1, +, -, \cdot, <, \exp \rangle$ will denote the real ordered exponential field, $\mathbb{R}_{\sin} = \langle \mathbb{R}, 0, 1, +, -, \cdot, <, \sin \rangle$ the real ordered field with the sine function, $\mathbb{R}_{\tan} = \langle \mathbb{R}, 0, 1, +, -, \cdot, <, \tan \rangle$ the real ordered field with the tangent function, etc.

Given a language L, an L-structure $\mathcal{A}$, an L-formula $\varphi(\overline{x})$ and a tuple $\overline{a}$ of elements of the domain of $\mathcal{A}$, we write $\mathcal{A} \models \varphi(\overline{a})$ if the interpretation in $\mathcal{A}$ of $\varphi(\overline{x})$, with $\overline{x} = \overline{a}$, is *true* in $\mathcal{A}$ (see [1] for a detailed explanation of the concept of truth for first order structures).

Given two L-formulas $\varphi(\overline{x})$ and $\psi(\overline{x})$, we say that they are *equivalent* in the L-structure $\mathcal{A}$ if $\mathcal{A} \models \forall \overline{x}\ (\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$.

Fix a language L and two L-structures $\mathcal{A}$ and $\mathcal{B}$. We write $\mathcal{A} \equiv \mathcal{B}$ if the two structures $\mathcal{A}$ and $\mathcal{B}$ are *elementary equivalent*, i.e. if for every L-sentence φ,

$$\mathcal{A} \models \varphi \Leftrightarrow \mathcal{B} \models \varphi.$$

We write $\mathcal{A} \subseteq \mathcal{B}$ if $\mathcal{A}$ is a *substructure* of $\mathcal{B}$, i.e. if the domain of $\mathcal{A}$ is contained in the domain of $\mathcal{B}$ and for every *quantifier free* formula $\varphi(\overline{x})$ and every tuple $\overline{a}$ of elements of the domain of $\mathcal{A}$,

$$\mathcal{A} \models \varphi(\overline{a}) \Leftrightarrow \mathcal{B} \models \varphi(\overline{a}).$$

We write $\mathcal{A} \preceq \mathcal{B}$ if $\mathcal{A}$ is an *elementary substructure* of $\mathcal{B}$, i.e. if the domain of $\mathcal{A}$ is contained in the domain of $\mathcal{B}$ and for every formula $\varphi(\overline{x})$ and every tuple $\overline{a}$ of elements of the domain of $\mathcal{A}$,

$$\mathcal{A} \models \varphi(\overline{a}) \Leftrightarrow \mathcal{B} \models \varphi(\overline{a}).$$

0.2.2 (Theories)**.** Given a language L, a consistent set T of L-sentences is called a *theory*. A *model* of a theory T is an L-structure $\mathcal{A}$ such that every sentence in the set T is true in $\mathcal{A}$. We say that a theory T *proves* an L-sentence ψ if for every model $\mathcal{A}$ of T the sentence ψ is true in $\mathcal{A}$. Sometimes we call the elements of T *axioms*, while we call the sentences which T proves *theorems*. However, most of the times we will be interested in the *deductive closure* of T, i.e. the set of theorems of T, rather than in T itself. An L-theory T *axiomatizes* another L-theory T' if the deductive closure of T' coincides with the deductive closure of T.

A theory is *complete* if for every L-sentence ψ either ψ or $\neg\psi$ is a theorem of T. Notice that all models of a complete theory are elementary equivalent. If $\mathcal{A}$ is an L structure, we denote by $Th(\mathcal{A})$ the *complete theory of* $\mathcal{A}$, i.e. the set of all L-sentences which are true in $\mathcal{A}$. Notice that $\mathcal{A} \equiv \mathcal{B} \Leftrightarrow Th(\mathcal{A}) = Th(\mathcal{B})$.

A theory T is *model complete* if for every pair of models $\mathcal{A}, \mathcal{B}$ of T such that $\mathcal{A}$ is a substructure of $\mathcal{B}$, it is true that $\mathcal{A}$ is actually an elementary substructure of $\mathcal{B}$. A theory T is model complete if and only if for every L-formula $\varphi(\overline{x})$ there is an *existential* L-formula $\psi(\overline{x})$ such that $\varphi(\overline{x})$ is *T-equivalent* to $\psi(\overline{x})$, i.e. T proves $\forall\overline{x}\ (\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$ (see [4] for a proof of this statement).

0.2.3 (Computability)**.** In this work, we assume some familiarity with the notion of *algorithm* or *recursive procedure* (see, for example, [30]). In particular, we assume Church's Thesis, which states that every function which can be computed by some algorithm, can be also computed by some Turing

Machine. If L is a finite language, we denote by L^* the (countable) set of all *words* on L, i.e. the set of all finite strings of symbols from L.

A function $f : L^* \to \mathbb{N}$ is *computable* or *recursive* if there is an algorithm which on input $\sigma \in L^*$ stops after finitely many steps, giving $f(\sigma) \in \mathbb{N}$ as output. Analogous definitions can be given for functions with domain $\mathbb{N}, \mathbb{N}^k, (L^*)^k$.

A set $A \subseteq L^*$ is *decidable* or *recursive* if there is an algorithm which on input $\sigma \in L^*$ *always* stops, with output 1 if $\sigma \in A$ and output 0 if $\sigma \notin A$.

A set $A \subseteq L^*$ is *semi-decidable* or *recursively enumerable* if there is an algorithm which on input $\sigma \in L^*$ stops *if and only if* $\sigma \in A$. Notice that every recursive set is recursively enumerable. Moreover, a set A is recursive if and only if both A and $L^* \setminus A$ are recursively enumerable.

A set of L-sentences A is a subset of L^*, hence it makes sense to ask whether A is recursive. Notice that the set of all L-sentences is recursive.

A theory T is *decidable* if the set of the theorems of T is recursive. A theory T is a *recursive axiomatization* of a theory T' if T is a recursive set and T axiomatizes T'.

One can prove that the set of all theorems of a recursive or recursively enumerable theory T is recursively enumerable. If moreover T is complete, then the set of all theorems of T is even recursive.

0.2.4 (Definability)**.** Given an L-structure $\mathcal{A}$ and subsets $B \subseteq \mathcal{A},\ D \subseteq \mathcal{A}^n$, we say that D is *definable in $\mathcal{A}$, with parameters from* B if there is an L-formula $\varphi(\overline{x}, \overline{y})$ and a tuple $\overline{b}$ of elements of B such that

$$D = \{\overline{a} \in \mathcal{A}^n |\ \mathcal{A} \models \varphi(\overline{a}, \overline{b})\}.$$

Unless otherwise specified, we will use the adjective *definable* to mean 'definable in the structure, with parameters from the structure'. In general, if $\mathcal{A}$ is a structure, we will write 'D is a definable subset of $\mathcal{A}$ (or of $\mathcal{A}^n$)' as a shorthand for 'D is a subset of the domain of $\mathcal{A}$ (or of the n-th power of the domain of $\mathcal{A}$), which is definable in $\mathcal{A}$ with parameters from $\mathcal{A}$'.

We recall that a map $f : \mathcal{A}^n \to \mathcal{A}^m$ is *definable* if its graph $\{(\overline{x}, \overline{y}) \in \mathcal{A}^{n+m} |\ f(\overline{x}) = \overline{y}\}$ is a definable set.

Chapter 1

Definably complete structures

1.1 Introduction

In this chapter we introduce the class of definably complete structures (which has also been considered by Miller in [23]), and discuss its basic properties. This class expands the class of real closed fields, and we are able to prove most of the classical results of the basic calculus for definable functions. In the last three sections, we prove some less trivial results, which will be useful in the following chapters.

The class of definably complete structures includes o-minimal expansions of ordered fields, as a main example. This inclusion is however proper, and we can not, in general, use the powerful tools of o-minimality to prove our results, as in [8]. We will need to develop different techniques, applicable in our environment.

1.1.1 Definition (Definably complete structures)**.** Fix a language $L = \{+, -, \cdot, <, 0, \dots\}$ which is an expansion of the language of ordered rings. A *definably complete structure* $\mathbb{K}$ (in the language L) is an L-expansion of an ordered field, such that every definable subset of the domain of $\mathbb{K}$ which is bounded from above, has a least upper bound. As one can see from the axiomatization below, this is a first order weak version of Dedekind completeness for the real numbers.

1.1.2 Remark. The class of all definably complete structures in a given language L is recursively axiomatizable, with the following axiomatization:

1. [ORDERED FIELD]:

$$\begin{array}{ll}
\forall xyz(x+(y+z)=(x+y)+z) & \\
\forall xy(x+y=y+x) & \\
\forall x(x+0=x) & \\
\forall x(x+(-x)=0) & \\
\forall xyz(x\cdot(y\cdot z)=(x\cdot y)\cdot z) & (\mathit{Field\ Axioms}) \\
\forall xy(x\cdot y=y\cdot x) & \\
\forall x(x\cdot 1=x) & \\
\forall x\exists y(\neg(x=0)\to x\cdot y=1) & \\
\forall xyz(x\cdot(y+z)=x\cdot y+x\cdot z) &
\end{array}$$

$$\begin{array}{ll}
\forall x(x\leq x) & \\
\forall xy(x\leq y\wedge y\leq x\to x=y) & (\mathit{Linear} \\
\forall xyz(x\leq y\wedge y\leq z\to x\leq z) & \mathit{order\ Axioms}) \\
\forall xy(x\leq y\vee y\leq x) &
\end{array}$$

$$\begin{array}{ll}
\forall xyz(x\leq y\to x+z\leq y+z) & (\mathit{Compatibility}) \\
\forall xy(0\leq x\wedge 0\leq y\to 0\leq x\cdot y) &
\end{array}$$

2. [DEFINABLE COMPLETENESS]: for every L-formula $\varphi(\overline{x},y)$ in $n+1$ variables,

$$\forall\overline{x}\ (\exists z\,\forall y\,(\varphi(\overline{x},y)\ \to\ y\leq z)\ \to\ \exists z\,(\forall y\,(\varphi(\overline{x},y)\ \to\ y\leq z)\ \wedge\ \forall t\,\forall y\,(\varphi(\overline{x},y)\ \to\ \to\ y\leq t)\ \to\ z\leq t))$$

1.1.3 Example. Examples of definably complete structures in a given language L are:

1. Every L-expansion $\mathcal{R}$ of the real ordered field $\mathbb{R}$ (by Dedekind completeness); for example $\mathbb{R}_{\sin}$.
2. Any o-minimal L-expansion of a real closed field (see [27]).
3. Any elementary extension $\mathcal{M}$ of a structure $\mathcal{R}$ as in 1.; for example $\mathcal{M}\succeq\mathbb{R}_{\sin}$.

1.1.4 Remark. A substructure of $\mathbb{K}$ is, in general, an ordered subring (without zero-divisors), but not necessarily a field or a definably complete structure. In fact, the axioms of ordered ring are universal, whereas the existence of a multiplicative inverse is a $\forall\exists$-axiom and [DEFINABLE COMPLETENESS] is a scheme of even more complex sentences.

1.2 Basic calculus in definably complete structures

We fix a language L and a definable complete L-structure $\mathbb{K}$.

1.2.1 Definition (Topology on $\mathbb{K}$)**.** We equip $\mathbb{K}$ with the *order topology* or *interval topology*: the basic open sets are given by the (definable) sets of the form $\{x \in \mathbb{K} | \; a < x < b\}$, where $a \neq b$ are elements of $\mathbb{K}$. Such a set will be called an *open interval* and will be denoted, as usual, (a, b). Analogously, we define the *closed interval* $[a, b]$.

We equip any power $\mathbb{K}^n$ with the product topology. It is a Hausdorff topology, hence limits are well defined.

1.2.2 Remark (Limits and continuity)**.** Let $f : \mathbb{K} \to \mathbb{K}$ be a definable function. Then, f is continuous if and only if the following sentence is true in $\mathbb{K}$:

$$\forall x \; \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall y \;\; (|x - y| < \delta \to |f(x) - f(y)| < \varepsilon).$$

In this case, for a general x, we write $\lim\limits_{y \to x} f(y) = f(x)$. Notice that ε and δ are elements of $\mathbb{K}$.

An analogous ε, δ-definition can be given for the continuity of a definable map $f : \mathbb{K}^n \to \mathbb{K}^n$, where the absolute value $|x|$ of an element $x \in \mathbb{K}$ is replaced by the *pseudonorm* $|\overline{x}| := \max\{|x_i| : \;\; i = 1, \dots, n\}$ of a vector $\overline{x} = (x_1, \dots, x_n) \in \mathbb{K}^n$.

1.2.3 Theorem (Intermediate value)**.** *Let $a, b \in \mathbb{K}$ and $f : [a, b] \to \mathbb{K}$ be a continuous definable function such that $f(a) < 0$ and $f(b) > 0$. Then there exists $c \in (a, b)$ such that $f(c) = 0$.*

Proof. Consider the definable set $A = \{x \in (a, b) : \;\; f(x) \leq 0\}$. Then $A \neq \emptyset$, for $a \in A$, and A is bounded from above by b. Hence, by definable completeness, there exists $x_0 = \sup A$. Suppose $f(x_0) > 0$. Then, by continuity of f, there is an open neighbourhood U of x_0 where f is strictly positive; but this is impossible since $x_0 = \sup A$ (by left-continuity, the same argument shows that $x_0 \neq b$). For the same reason $f(x_0)$ cannot be strictly negative. Hence $f(x_0) = 0$. $\square$

In particular, every definably complete structure is a real closed field.

1.2.4 Corollary (Intermediate Value Property)**.** *Let $a, b \in \mathbb{K}$ and $f : [a, b] \to \mathbb{K}$ be a continuous definable function. Then f takes all values in $\mathbb{K}$ between* $\inf f$ *and* $\sup f$ *(which exist, possibly $\pm\infty$, by definable completeness).*

1.2.5 Remark. Miller has shown in [23] that Corollary 1.2.4 implies definable completeness in densely ordered fields. Hence the so called IVP (Intermediate Value Property) is equivalent to definable completeness. Another equivalent condition is that $\mathbb{K}$ is *definably connected*, i.e. it is not the disjoint union of two open nonempty definable sets.

1.2.6 Theorem (Weierstrass Property)**.** *Let $a, b \in \mathbb{K}$ and $f : [a, b] \to \mathbb{K}$ be a continuous definable function. Then f is achieves maximum and minimum on $[a, b]$.*

Proof. We may assume, without loss of generality, that $\forall x\ f(x) \geq 0$. We prove that f assumes a maximum value (the proof is similar for the minimum). Let us consider the set

$$B = \{x \in [a, b] |\ \forall y \exists x'\ x' \geq x \wedge f(x') > y\},$$

and suppose for a contradiction that f is not bounded from above. Hence $B \neq \emptyset$ and, by definable completeness, $\exists\ c = \sup B$. It follows that $\forall \delta\, \exists x' \in B$ such that $c - \delta < x' < c$.

Suppose $c \neq a, b$. Hence, $\forall x \in [a, b]\ (x > c \to x \notin B)$, so f is bounded on the subinterval $(c, b]$. Let $D = \sup f \restriction (c, b]$. Fix $\varepsilon > 0$. Then, by continuity of f in c, there exists $\delta > 0$ such that $\forall x\ (|x - c| < \delta \to |f(x) - f(c)| < \varepsilon)$. If $E = \max\{f(c) + \varepsilon, D\}$, then E bounds $f \restriction (c - \delta, b]$ from above, which is absurd by the definition of c. The argument works even more clearly if $c = b$ or $c = a$. $\square$

1.3 Calculus for C^∞ definable maps

1.3.1 Definition (Differentiable functions)**.** A definable function $f : \mathbb{K} \to \mathbb{K}$ is *differentiable* if for all $x \in \mathbb{K}$ the limit $\lim_{y \to x} \frac{f(y)-f(x)}{|y-x|}$ exists (where the limit is expressed, as before, via the ε, δ-definition). Notice that the derivative of a definable function (if it exists) is again a definable function. Analogously, if $f : \mathbb{K}^n \to \mathbb{K}$, we define the partial derivatives. As usual, we say that a definable function f is C^1 if it is differentiable, with continuous derivative. C^n and C^∞ are similarly defined.

1.3.2 Remark (Derivatives)**.** Let $F : \mathbb{K}^n \to \mathbb{K}$ be a definable C^∞ function and let $\overline{x}_0 \in \mathbb{K}^n$. Note that $\mathbb{K}^n$ is a topological $\mathbb{K}$-vector space, endowed with the pseudonorm $|\overline{x}| = \max_i |x_i|$.

Consider the gradient of F in $\overline{x}_0$, i.e. the vector $(\frac{\partial F}{\partial x_1}(\overline{x}_0), \ldots, \frac{\partial F}{\partial x_n}(\overline{x}_0))$. This vector represents, in the coordinates $x_1, \ldots, x_n$, that unique linear function $F'(\overline{x}_0) : \mathbb{K}^n \to \mathbb{K}$ such that

$$F(\overline{x}) = F(\overline{x}_0) + F'(\overline{x}_0)[\overline{x} - \overline{x}_0] + R(\overline{x} - \overline{x}_0),$$

for some map R such that $|R(\overline{x} - \overline{x}_0)| \leq \text{constant} \cdot |\overline{x} - \overline{x}_0|^2$.

Now, the map $F' : \mathbb{K}^n \to \mathrm{Lin}(\mathbb{K}^n, \mathbb{K})$, which sends $\overline{x}_0$ to $F'(\overline{x}_0)$, is still C^∞, so we can consider the linear map $(F')'(\overline{x}_0) \in \mathrm{Lin}(\mathbb{K}^n, \mathrm{Lin}(\mathbb{K}^n, \mathbb{K})) = \mathrm{BiLin}(\mathbb{K}^n, \mathbb{K})$.

Inductively, for all $N \in \mathbb{N}$ we have a definable map

$$F^{(N)}(\overline{x}_0) : (\mathbb{K}^n)^N \to \mathbb{K}$$

which is multilinear. Given $\overline{x}_0 = (x_{01}, \ldots, x_{0n}) \in \mathbb{K}^n$, the function $F^{(N)}(\overline{x}_0)$ applied to N copies of the vector $\overline{v} = (v_1, \ldots, v_n) \in \mathbb{K}^n$ gives the number

$$\sum_{i_1+\ldots+i_n=N} \frac{\partial^N F}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}(\overline{x}_0)(v_1 - x_{01})^{i_1} \ldots (v_n - x_{0n})^{i_n}.$$

Now, if $F = (F_1, \ldots, F_m) : \mathbb{K}^n \to \mathbb{K}^m$ is C^∞ and definable, we define $F^{(N)}(\overline{x}_0) := (F_1^{(N)}(\overline{x}_0), \ldots, F_m^{(N)}(\overline{x}_0)) : (\mathbb{K}^n)^N \to \mathbb{K}^m$. Notice that the map $F^{(N)}(\overline{x}_0)$ is N-linear.

Let $n = m$. We consider the *operator norm* on the vector space $N{-}\mathrm{Lin}(\mathbb{K}^n, \mathbb{K}^n)$, so that

$$|F^{(N)}(\overline{x}_0)| = \sup_{|\overline{x}|=1} |F^{(N)}(\overline{x}_0)[\overline{x}]|$$

The most important property of the norm is the following:

$$|F^{(N)}(\overline{x}_0)[\overline{\eta_1}, \ldots, \overline{\eta_N}]| \leq n^N |F^{(N)}(\overline{x}_0)||\overline{\eta_1}| \cdot \ldots \cdot |\overline{\eta_N}|$$

Notice that the norm is a continuous function.

1.3.3 Theorem (Taylor's Theorem).

- *Let $F : \mathbb{K}^n \to \mathbb{K}$ be a definable C^N map and let $\overline{x}_0, \overline{x} \in \mathbb{K}^n$. Then there exists $\overline{\xi}$, lying on the segment joining $\overline{x}_0$ and $\overline{x}$, such that*

$$F(\overline{x}) = F(\overline{x}_0) + F'(\overline{x}_0)[\overline{x} - \overline{x}_0] + \tfrac{F''(\overline{x}_0)}{2}[\overline{x} - \overline{x}_0, \overline{x} - \overline{x}_0] + \ldots + \tfrac{F^{(N)}(\overline{\xi})}{N!}[\overline{x} - \overline{x}_0, \ldots, \overline{x} - \overline{x}_0]$$

- *Let $F : \mathbb{K}^n \to \mathbb{K}^n$ be a definable C^{N+1} map and let $\overline{x}_0, \overline{x} \in \mathbb{K}^n$ such that $|\overline{x} - \overline{x}_0| < r$. Then*

$$|F(\overline{x}) - F(\overline{x}_0) - F'(\overline{x}_0)[\overline{x} - \overline{x}_0] - \tfrac{F''(\overline{x}_0)}{2}[\overline{x} - \overline{x}_0, \overline{x} - \overline{x}_0] - \ldots - \tfrac{F^{(N)}(\overline{x}_0)}{N!}[\overline{x} - \overline{x}_0, \ldots, \overline{x} - \overline{x}_0]| \le \sup_{\overline{y} \in B(\overline{x}_0, r)} |\tfrac{F^{(N+1)}(\overline{y})}{(N+1)!}[\overline{x} - \overline{x}_0, \ldots, \overline{x} - \overline{x}_0]|$$

Proof. First consider the case $n = 1$. The case $N = 1$ follows, as in the classical case (see for example [31]), from our version 1.2.6 of Weierstrass' Theorem (existence of extrema of a continuous definable function on closed and bounded definable sets) and from the fact that the derivative vanishes on the local extrema; the case of a general N follows inductively.

For a general n, notice that the function $f(t) := F(t\overline{x} + (1-t)\overline{x}_0)$ is C^N and

$$f(0) = F(\overline{x}_0) \wedge f(1) = F(\overline{x}) \wedge f^{(n)}(t) = F^{(n)}(t\overline{x} + (1-t)\overline{x}_0)[\overline{x} - \overline{x}_0, \ldots, \overline{x} - \overline{x}_0].$$

Hence, by what we have already proved,

$$f(1) = f(0) + f'(0)(1-0) + \ldots + \frac{f^{(N)}(\nu)}{N!}(1-0)^N, \ \forall i = 1, \ldots, n$$

for some ν, with $0 \le \nu \le 1$; this yields the conclusion of the first statement with $\overline{\xi} = \nu\overline{x} + (1-\nu)x_0$. The second statement follows immediately. □

1.3.4. Corollary (Increasing functions and the sign of the derivative). *Let $a, b \in \mathbb{K}$ and $f : (a, b) \to \mathbb{K}$ be a C^1 definable function. If for all $x \in (a, b)$ we have $f'(x) > 0$, then f is strictly increasing on (a, b).*

Proof. Let $a < x_1 < x_2 < b$. Then, by Taylor's expansion of degree 1, there is $\eta \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = f'(\eta)(x_2 - x_1) > 0$. Hence $f(x_2) > f(x_1)$. □

1.4 Newton's method

Now we will show an effective method for establishing, given a C^2 definable map, if it has nonsingular zeroes.

Let $F : \mathbb{K}^n \to \mathbb{K}^n$ be a C^2 definable map. Suppose we are given some point $\overline{x}_0 \in \mathbb{K}^n$ such that $|F(\overline{x}_0)|$ is small, $|F'(\overline{x}_0)|$ is bounded away from zero, and $|F'(\overline{x})|$ and $|F''(\overline{x})|$ are not too large on a suitable neighbourhood of $\overline{x}_0$. Then F has a zero lying near to $\overline{x}_0$. More precisely,

1.4.1 Theorem (Newton's Method). *Let $a_0, a_1, a_2 \geq 1$. Then there exist $m, r \in \mathbb{K}^+$ (which can be written as rational functions of n, a_0, a_1, a_2) such that, $\forall \overline{x}_0 \in \mathbb{K}^n$,*

If $|F(\overline{x}_0)| < m$ *and*

$\forall \overline{y} \in B(\overline{x}_0, r)$ $|F'(\overline{y})^{\dot{-}1}| < a_0$ *and* $|F'(\overline{y})| < a_1$ *and* $|F''(\overline{y})| < a_2$,

Then $\exists \overline{x}$ $F(\overline{x}) = 0$ *and* $\overline{x} \in B(\overline{x}_0, r)$.

The proof in based on a repeated use of Taylor's Theorem.

Proof. Let $r = (2n^3 a_0^2 a_1 a_2)^{\dot{-}1}$ and $m = (4n^3 a_0^3 a_1 a_2)^{\dot{-}1}$.

Let $\overline{x} \in B(\overline{x}_0, r)$ be such that $|F(\overline{x})| = \min\{|F(\overline{u})| : \overline{u} \in B(\overline{x}_0, r)\}$ (the existence of such a point $\overline{x}$ follows from the fact that the function $\overline{u} \mapsto |F(\overline{u})|$ is continuous definable). We claim $F(\overline{x}) = 0$. Let

$$\overline{y} = \overline{x} - F'(\overline{x})^{\dot{-}1} \cdot F(\overline{x}). \tag{1.1}$$

Equivalently, $F(\overline{x}) = F'(\overline{x})[\overline{x} - \overline{y}]$. It is sufficient to show that:
(i) $\overline{y} \in B(\overline{x}_0, r)$;
(ii) $|F(\overline{y})| \leq \frac{1}{2}|F(\overline{x})|$.

Proof of (i): By Taylor's formula,

$$|F(\overline{x}_0) - F(\overline{x}) - F'(\overline{x})[\overline{x}_0 - \overline{x}]| \leq \sup |\frac{F''(\overline{z})}{2}[\overline{x}_0 - \overline{x}, \overline{x}_0 - \overline{x}]|, \tag{1.2}$$

where $\overline{y} \in B(\overline{x}_0, r)$. Hence,

$$|F(\overline{x}_0) - F(\overline{x}) - F'(\overline{x})[\overline{x}_0 - \overline{x}]| \leq \frac{a_2}{2} n^2 |\overline{x}_0 - \overline{x}|^2. \tag{1.3}$$

Now, using (1.1),

$$|F(\overline{x}_0) - F'(\overline{x})[\overline{x}_0 - \overline{y}]| \leq \frac{a_2}{2} n^2 |\overline{x}_0 - \overline{x}|^2. \tag{1.4}$$

Hence $|\overline{x}_0 - \overline{y}| \leq |F'(\overline{x})^{\dot{-}1}|(|F(\overline{x}_0)| + \frac{a_2}{2} n^2 |\overline{x}_0 - \overline{x}|^2) \leq a_0(m + \frac{a_2}{2} n^2 r^2) \leq r$ (the last inequality can be easily checked by substituting the values of r, m). Therefore $\overline{y} \in B(\overline{x}_0, r)$.

Proof of (ii): By Taylor's formula and using (1.1), we get

$$|F(\overline{y})| \leq |F(\overline{x}) + F'(\overline{x})[\overline{y} - \overline{x}]| + \frac{a_2}{2} n^2 |\overline{y} - \overline{x}|^2 = 0 + \frac{a_2}{2} n^2 |\overline{y} - \overline{x}|^2 \tag{1.5}$$

Another use of Taylor's Theorem yields

$$|F(\overline{x})| \leq |F(\overline{x}_0)| + a_1 n |\overline{x} - \overline{x}_0|. \tag{1.6}$$

Hence, by (1.1),

$$|\overline{y}-\overline{x}|^2 \leq |F'(\overline{x})^{-1}|^2|F(\overline{x})|^2 \leq a_0^2(m+a_1nr)|F(\overline{x})|. \tag{1.7}$$

Putting all together, $|F(\overline{y})| \leq \frac{a_2}{2}n^2a_0^2(m+a_1nr)|F(\overline{x})| \leq \frac{1}{2}|F(\overline{x})|$. □

1.4.2 Remark. Notice also that, since $\overline{x} \in B(\overline{x}_0, r)$, then $\overline{x}$ is a *nonsingular* zero of F.

1.5 Uniqueness of solutions of linear differential equations

1.5.1. Theorem (Uniqueness Theorem for systems of linear differential equations). *Let $a, b \in \mathbb{K} \cup \{\pm\infty\}$ and $F = (f_1, \dots, f_n) : (a, b) \to \mathbb{K}^n$ a C^∞ definable map. Let $A(t) = (a_{ij}(t))$ be an $n \times n$ matrix of C^∞ definable functions from (a, b) to $\mathbb{K}$; suppose that*

$$\forall t \in (a,b) \quad F'(t) = A(t)F(t).$$

Then, either F is identically zero or else it never vanishes on (a, b).

Proof. For convenience, suppose $0 \in (a, b)$.
First, consider the case $n = 1$ and $A(t) = A \in \mathbb{K}$. We first prove that if F satisfies

$$|F'(t)| \leq |AF(t)|, \ F(0) = 0 \quad (*),$$

then $F \equiv 0$.

Let $\varepsilon > 0$. The set $C_\varepsilon = \{t \in [0, b) : \ |F(t)| \leq \varepsilon t\}$ is definable and not empty (since $0 \in C_\varepsilon$), hence it has a supremum $c_\varepsilon \leq b$. We claim that $\forall \varepsilon > 0$, either $c_\varepsilon = b$ (hence $C_\varepsilon = [0, b)$) or $c_\varepsilon \geq 1/|A|$. Suppose for a contradiction that $b \neq c_\varepsilon < 1/|A|$; then c_ε is actually a maximum (for C_ε is closed in $[0, b)$) and $|F(c_\varepsilon)| = \varepsilon c_\varepsilon < \varepsilon/|A|$ and $|F'(c_\varepsilon)| \leq |AF(c_\varepsilon)| < \varepsilon$, which is inconsistent with the definition of c_ε (because for $t < c_\varepsilon$, $F(t)$ lies between the lines $y = \varepsilon t$ and $y = -\varepsilon t$, and for $t > c_\varepsilon$, $F(t)$ lies outside that cone, so in $t = c_\varepsilon$ the magnitude of the derivative must be greater that ε).
A similar argument holds for the set $D_\varepsilon = \{t \in (a, 0] : \ |F(t)| \leq \varepsilon t\}$.
Hence, there's an open neighbourhood (d, c) (where $d = \max\{a, -1/|A|\}$ and $c = \min\{1/|A|, b\}$) of zero such that

$$\forall \varepsilon > 0, \ \forall t \in (d, c) \ \ |F(t)| \leq \varepsilon |t|,$$

i.e. F is identically zero on (d, c).
The above discussion could be carried out in a neighbourhood of any point $t_0 \in (a, b)$ such that $F(t_0) = 0$; so the set

$$\{t \in (a, b) : \; F(t) = 0\}$$

is both open and closed, hence it coincides with the whole interval (a, b). This concludes the proof of the first case.

Next, suppose $A(t)$ is any function from (a, b) to $\mathbb{K}$, and that $F'(t) = A(t)F(t)$ and $F(0) = 0$. Then, on each closed and bounded subinterval I of (a, b), $A(t)$ takes its maximum and minimum values (by Theorem 1.2.6), i.e. there exists $A_I \in \mathbb{K}^+$ such that $|A(t)| \leq A_I$ on I. Therefore, $|F'(t)| \leq A_I|F(t)|$ on I and we can argue as before to find, once again, that F vanishes on a set which is both open and closed in (a, b).

Finally, consider the general case $n \in \mathbb{N}$ and suppose $A(t) = (a_{ij}(t))$.
If $F = (f_1, \ldots, f_n)$ is a definable solution of the differential equation in the statement of the theorem, then we consider the definable function $g(t) = \sum_{i=1}^n f_i^2(t)$. Its derivative satisfies $g' = \sum_{i=1}^n 2f_i f_i' = 2\sum_{i,j=1}^n a_{ij} f_i f_j$. For each closed and bounded subinterval I of (a, b), let $A_I \in \mathbb{K}^+$ be such that $|a_{ij}(t)| \leq A_I$ on I. Then, $|g'| \leq 2A_I \sum_{i,j=1}^n |f_i||f_j| = 2A_I(\sum_{i=1}^n |f_i|)^2 \leq 2A_I n^2 \sum_{i=1}^n f_i^2 = 2A_I n^2 |g|$ on I and $g(0) = 0$, so that we can argue as before. This completes the proof of the theorem.

$\square$

1.6 Continuous definable maps on definably compact sets

In this section we quote from [23] some results of C. Miller, which we will need later on. We adopt the notation of [23] and use the abbreviation CBD for "closed, bounded and definable ".

1.6.1 Lemma (Miller,[23])**.**

1. *Let $A \subseteq \mathbb{K}^n$, $f : A \to \mathbb{K}$ be definable and b be in the closure of $A \setminus \{b\}$. Then both* $\liminf_{a \to b} f(a)$ *and* $\limsup_{a \to b} f(a)$ *exist in* $\mathbb{K} \cup \{+\infty, -\infty\}$.
2. *If $A \subseteq \mathbb{K}^{n+m}$ is CBD, then πA (the projection onto the first n coordinates of A) is also CBD.*

3. *(Weak Definable Choice) Let* $A \subseteq \mathbb{K}^{n+m}$ *be definable and such that* $\forall \overline{x} \in \mathbb{K}^n$*, the fiber* $A_{\overline{x}} := \{\overline{y} \in \mathbb{K}^m | \ (\overline{x}, \overline{y}) \in A\}$ *is CBD. Then there is a definable map* $f : \pi A \to \mathbb{K}^m$ *such that* $\mathrm{Graph}(f) \subseteq A$.

Proof. The choice function in 3) sends $\overline{x}$ to the *lexicographic minimum* of $A_{\overline{x}}$. See [23] for the details. □

1.6.2 Theorem (Miller,[23])**.** *Let* $A \subseteq \mathbb{K}^n$ *be CBD and* $f : A \to \mathbb{K}^m$ *be a continuous definable map. Then* $f(A)$ *is CBD.*

Proof. Note that $f(A)$ is definable and closed , since $\mathrm{Graph}(f)$ is closed, so it suffices to show that f is bounded, in the case $m = 1$. For all $r \in \mathbb{K}$, let $Y_r := f^{-1}([r, +\infty)) \subseteq A$. Then Y_r is CBD. We claim that there exists $M \in \mathbb{K}$ such that $Y_M = \emptyset$. Suppose not; then, by Weak Definable Choice, there is a definable function $g : \mathbb{K} \to \mathbb{K}^n$ such that $g(r) \in Y_r$. This implies that $\limsup_{t\to+\infty} g(t) \in \bigcap_{r\in\mathbb{K}} Y_r$, which is absurd because $\bigcap_{r\in\mathbb{K}} Y_r = \emptyset$. This shows that f is bounded from above, and a similar argument shows the boundedness from below. □

Chapter 2

Noetherian differential rings of functions

2.1 Introduction

The direction of this chapter is suggested by the work of A. Wilkie, in particular [35]. We first study the basic differential topology of zero-sets of definable C^∞ functions in a definably complete structure. Then, we restrict our attention to the functions which form a *Noetherian differential ring*, and derive a decomposition of the zero-sets of these functions, into finitely many smooth sets of a certain form. In the last section, we show an application of the Decomposition Theorem 2.4.7 to a more specific situation.

Throughout this chapter, L will be a fixed language and $\mathbb{K}$ a fixed definably complete L-structure, unless otherwise stated.

2.2 Varieties of C^∞ definable functions

2.2.1 Definition (Varieties)**.** If $n, m \in \mathbb{N}$, let $C^\infty(\mathbb{K}^n, \mathbb{K}^m)$ be the ring of C^∞ definable maps from $\mathbb{K}^n$ to $\mathbb{K}^m$.
If $G \in C^\infty(\mathbb{K}^n, \mathbb{K}^m)$, we define the *variety of* G as the zero-set of the map G:

$$V(G) = \{\overline{a} \in \mathbb{K}^n : \; G(\overline{a}) = \overline{0}\}.$$

Let $g_1, \dots, g_m \in C^\infty(\mathbb{K}^n, \mathbb{K})$ be the *components* of G, i.e.

$$\forall \overline{x} \;\; (G(\overline{x}) = (g_1(\overline{x}), \dots, g_m(\overline{x}))).$$

Then $V(G) = V(g_1) \cap \ldots \cap V(g_m)$; we will often write $V(g_1, \dots, g_m)$ instead of $V(G)$.

2.2.2 Remark. The variety of G is a closed subset of $\mathbb{K}^n$, for it is the preimage of a point under a continuous map: if $\overline{y}$ belongs to the closure of $V(G)$ in $\mathbb{K}^n$, then by definition $\forall\delta > 0\ \exists\overline{x} \in V(G)$ such that $|\overline{y} - \overline{x}| < \delta$; now, by continuity of $g := \sum_{i=1}^{m} g_i$ in $\overline{y}$, for every $\varepsilon > 0\ |f(\overline{y})| < \varepsilon$.

2.2.3 Remark. Let $g \in C^\infty(\mathbb{K}^n, \mathbb{K})$. The definition of the variety of g depends on the choice of the function g. As a set, however, $V(g)$ could be represented as the zero-set of another function: for example, as the zero-set of the function g^2, or as the zero-set of the function $g \cdot f$, if f is always nonzero. In the sequel we will often implicitely consider a variety together with a particular *presentation* instead of just as a set. The reason why we will do that, will be clear in Chapters 3 and 4: throughout this work, we are always interested in the *effectiveness* of the arguments; in particular, we want to deal with mathematical objects which can be finitely described (for example first order formulas). Now, while a definable function is a finite object, a subset of $\mathbb{K}^n$ is not, unless equipped with a finite presentation.

On the other hand, we will not distinguish between the set and its presentation if it is not necessary, and if it does not lead to misinterpretation.

2.2.4 Notation. Let $n \in \mathbb{N}$, $g : \mathbb{K}^n \to \mathbb{K}$ be a C^∞ definable function and $\overline{a} \in \mathbb{K}^n$. We have defined in Remark 1.3.2 the linear function $g'(\overline{a})$, which, if we fix the coordinates $\overline{x} = (x_1, \ldots, x_n)$, is represented by the vector $\nabla g(\overline{a}) = (\frac{\partial g}{\partial x_1}(\overline{a}), \ldots, \frac{\partial g}{\partial x_n}(\overline{a}))$.

If $n \geq m \in \mathbb{N}$, $G = (g_1, \ldots, g_m) : \mathbb{K}^n \to \mathbb{K}^m$ is a C^∞ definable map and $\overline{a} \in \mathbb{K}^n$, we denote by $DG(\overline{a})$ the $m \times n$ matrix whose rows are the vectors $\nabla g_1(\overline{a}), \ldots, \nabla g_m(\overline{a})$ (so $DG(\overline{a})$ is the matrix corresponding to the linear map $G'(a)$, with respect to the standard basis). If $\overline{y} \subset \overline{x}$ is a subtuple of coordinates, then we denote by $D_{\overline{y}}G(\overline{a})$ the matrix of the partial derivatives $\frac{\partial g_i}{\partial y_j}(\overline{a})$ with respect to the variables in the tuple $\overline{y}$.

2.2.1 The Implicit Function Theorem

We will use many times in this work, some version of the Implicit Function Theorem. The statement is standard, but technical and we will find it useful to fix here a notation and to refer to this subsection whenever we use the theorem.

2.2.5 Definition (Regular sets)**.** Let $n, m \in \mathbb{N}$ and $G = (g_1, \ldots, g_m) \in C^\infty(\mathbb{K}^n, \mathbb{K}^m)$. Let $\overline{a} \in V(G)$ be a point such that the linear map $G'(\overline{a})$ is surjective. Then we say that $\overline{a}$ is a *regular point* of G. The set of regular zeroes of G (the *regular set of* G, for short) is denoted by $V^{\text{reg}}(G)$. So,

$$V^{\text{reg}}(G) := \{\overline{a} \in \mathbb{K}^n : \ \overline{a} \in V(G) \text{ and } G'(\overline{a}) \text{ is onto}\}.$$

In other words, $V^{\text{reg}}(G)$ is the set of those $\overline{a} \in V(g_1, \ldots, g_m)$ such that the vectors $\nabla g_1(\overline{a}), \ldots, \nabla g_m(\overline{a})$ are $\mathbb{K}$-linearly independent (We denote by $\text{lin. span}\langle \nabla g_1(\overline{a}), \ldots, \nabla g_m(\overline{a})\rangle$ the $\mathbb{K}$-vector space generated by these vectors).

Notice, in reference to Remark 2.2.3, that the definition of regular set depends crucially on the choice of G: we have noticed that, as a set, $V(g_1) = V(g_1^2)$; but $V^{\text{reg}}(g_1) \neq V^{\text{reg}}(g_1^2)$. In fact $V^{\text{reg}}(g_1^2)$ is always empty.

If $\overline{a} \in V^{\text{reg}}(G)$, then the matrix of the linear map $DG(\overline{a})$ has a nonsingular $m \times m$ minor of the form $D_{\overline{y}}G(\overline{a})$, where $\overline{y}$ is some m-tuple of variables, that is, the *jacobian determinant* $J_{\overline{y}}G(\overline{a}) := \det D_{\overline{y}}G(\overline{a})$ is nonzero. If $n = m$, we simply write $JG(\overline{a})$.

The function $\overline{a} \mapsto J_{\overline{y}}G(\overline{a})$ is continuous, hence, if $J_{\overline{y}}G(\overline{a}) \neq 0$, then there is a whole neighbourhood of $\overline{a}$ where $J_{\overline{y}}G$ is nonzero.

If $V(G) = V^{\text{reg}}(G)$, we say that $V(G)$ is a *regular variety.*

2.2.6 Remark. Using the result 2.2.8 proved below, we will see that $V^{\text{reg}}(G)$ is locally definably diffeomorphic to an open subset of $\mathbb{K}^{n-m}$. Hence, $V^{\text{reg}}(G)$ is a *differentiable* $\mathbb{K}$*-manifold*, of dimension $n - m$.

We give now the notation which we will use for the Implicit Function Theorem and its Corollaries.

2.2.7 Notation. Let $n \geq m \in \mathbb{N}$. We write $n = k + m$ and we fix the following set of coordinates:

$$\begin{array}{rcl} \mathbb{K}^n & = & \mathbb{K}^k \times \mathbb{K}^m \\ \overline{x} & = & (\overline{u}, \overline{v}) \end{array}$$

We fix $G = (g_1, \ldots, g_m) \in C^\infty(\mathbb{K}^{k+m}, \mathbb{K}^m)$ and $\overline{x}_0 = (\overline{u}_0, \overline{v}_0) \in V(G)$ such that $D_{\overline{v}}G(\overline{x}_0)$ is non-singular.

2.2.8 Theorem (Implicit function Theorem)**.** *There exist*

1. *open definable subsets $O \subseteq \mathbb{K}^k$ and $W \subseteq \mathbb{K}^m$ such that $\overline{x}_0 \in O \times W$, and*

2. *a definable C^∞ map*
$$Y : O \to W$$
such that $Y(\overline{u}_0) = \overline{v}_0$ and
$$\forall \overline{u} \in O \forall \overline{v} \in W \quad G(\overline{u}, \overline{v}) = \overline{0} \Leftrightarrow \overline{v} = Y(\overline{u}).$$

Moreover, $D_{\overline{u}}Y(\overline{u})$ is everywhere non-singular and, if $J_{vv}G(\overline{x}) = \det D_{\overline{v}}G(\overline{x})$,

$$\forall \overline{x} \in U \quad D_{\overline{u}}Y(\overline{u}) = -J_{vv}G^{-1}(\overline{u}, Y(\overline{u})) \cdot D_{\overline{u}}G(\overline{u}, Y(\overline{u})).$$

2.2.9 Definition. The map

$$\begin{array}{rccc} \phi: & \mathbb{K}^k & \to & V(G) \cap O \times W \\ & \overline{u} & \mapsto & (\overline{u}, Y(\overline{u})) \end{array}$$

is called a *local rectangular parametrization of* $V(G)$ *around* $\overline{x}_0$, and is a definable diffeomorphism, whose inverse is the restriction to $V(G) \cap (O \times W)$ of the projection $\pi : \mathbb{K}^n \to \mathbb{K}^k$ onto the first k coordinates.

For the proof of 2.2.8, we refer you to the proof provided by L. van den Dries in [8] for the o-minimal case. The only nontrivial fact used there is our Theorem 1.6.2, hence the proof is applicable in the definably complete case as well.

We give now a list of the usual consequences of the Implicit Function Theorem.

2.2.10 Corollary. *There is a ring homomorphism (the* restriction homomorphism*)*

$$\begin{array}{rccc} \widehat{\ }: & C^\infty(O \times W, \mathbb{K}) & \to & C^\infty(O, \mathbb{K}) \\ & h & \mapsto & \widehat{h}(\overline{u}) = h(\overline{u}, Y(\overline{u})) \end{array}$$

The kernel of $\widehat{\ }$ *is the set* $\{h \in C^\infty(O \times W, \mathbb{K}) : \ h \restriction V(G) \cap (O \times W) \equiv 0\}$, *hence*

$$\widehat{C^\infty}(O \times W, \mathbb{K}) \cong C^\infty(V(G) \cap (O \times W), \mathbb{K}).$$

2.2.11 Corollary (Lagrange's Multipliers Rule)**.** *Let* $h \in C^\infty(O \times W, \mathbb{K})$. *A point* $\overline{x} = (\overline{u}, Y(\overline{u})) \in V(G) \cap (O \times W)$ *is a local extremum (maximum or minimum) of* h *on* $V(G)$ *if and only if* $\nabla\widehat{h}(\overline{u}) = 0$. *Moreover,*

$$\nabla\widehat{h}(\overline{u}) = 0 \ \Leftrightarrow \ \nabla h(\overline{u}, Y(\overline{u})) \in \operatorname{lin.span}\langle \nabla g_1(\overline{u}, Y(\overline{u})), \ldots, \nabla g_m(\overline{u}, Y(\overline{u}))\rangle.$$

2.2.12 Corollary. *Suppose* $M \subset C^\infty(U \times W, \mathbb{K})$ *is a noetherian ring closed under differentiation. Then so is* $\widehat{M}[\widehat{J_{vv}G^{-1}}]$.

Proof. Notice that $\widehat{M}[\widehat{J_{vv}G^{-1}}]$ is a finite extension of a homomorphic image of a noetherian ring, hence it is noetherian; moreover, an easy calculation and Corollary 2.2.10 show that $\widehat{M}[\widehat{J_{vv}G^{-1}}]$ is also closed under differentiation. □

2.3 Noetherian differential rings

2.3.1 Remark. Sometimes it is more useful to work with zero-sets of ideals of functions, rather than zero-sets of functions. If $g_1, \ldots, g_m \in C^\infty(\mathbb{K}^n, \mathbb{K})$ and I is the ideal generated by $\{g_1, \ldots, g_m\}$, then $V(g_1, \ldots, g_m)$ coincides with the set $V(I) := \{\overline{x} \in \mathbb{K}^n | \; f(\overline{x}) = 0 \; \forall f \in I\}$; but if J is *any* ideal of the ring $C^\infty(\mathbb{K}^n, \mathbb{K})$, then J needs not to be finitely generated; in particular, $V(J)$ could be not definable. Hence, we restrict our attention to some rings of functions which only have finitely generated ideals.

2.3.2 Definition (Noetherian differential rings)**.** Let $n \in \mathbb{N}$. Let M be a ring with the following properties:

- $M \subseteq C^\infty(\mathbb{K}^n, \mathbb{K})$;
- M is noetherian;
- M is closed under partial differentiation;
- $M \supseteq \mathbb{Z}[x_1, \dots, x_n]$.

We call M a *noetherian differential ring.*

2.3.3 Examples. The following are examples of noetherian differential rings in definably complete structures.

1. Let $\langle \mathbb{K}, +, -, \cdot, <, \exp, 0, 1, \dots \rangle$ be a definably complete structure such that exp satisfies the usual differential equation $\exp'(x) = \exp(x) \wedge \exp(0) = 1$ (for example, $\mathbb{K} = \mathbb{R}_{\exp}$), and let $\mathbb{F}$ be a subfield of $\mathbb{K}$. Then,
$$\mathbb{F}[x_1, \dots, x_n, \exp(x_1), \dots, \exp(x_n)]$$
and
$$\mathbb{F}[x_1, \dots, x_n, \exp(x_1), \dots, \exp(x_n), \exp(\exp(x_1)), \dots, \exp(\exp(x_n))]$$
are noetherian differential subrings of $C^\infty(\mathbb{K}^n, \mathbb{K})$. The same clearly holds if we consider up to k iterations of exp, for any natural number k.

2. Let $\langle \mathbb{K}, +, -, \cdot, <, g_1, .., g_l, 0, 1, \dots \rangle$ be a definably complete structure such that $g_1, .., g_l \in C^\infty(\mathbb{K}^n, \mathbb{K})$ form a *pfaffian chain*, i.e. they satisfy a *triangular* system of differential equations, with polynomial coefficients (see 2.6.2 for the precise definition. Examples are $\mathbb{R}_{\exp}$ and $\mathbb{R}_{\tan}$). Then,
$$\mathbb{F}[x_1, \dots, x_n, g_1, \dots, g_l],$$
where $\mathbb{F}$ is a subfield of $\mathbb{K}$, is a noetherian differential subring of $C^\infty(\mathbb{K}^n, \mathbb{K})$.

3. Let $\langle \mathbb{K}, +, -, \cdot, <, \sin, 0, 1, \ldots \rangle$ be a definably complete structure such that sin satisfies the usual differential equation (for example, $\mathbb{K} = \mathbb{R}_{\sin}$) and let $\mathbb{F}$ be a subfield of $\mathbb{K}$. Then,

$$\mathbb{F}[x_1, \ldots, x_n, \sin(x_1), \ldots, \sin(x_n), \cos(x_1), \ldots, \cos(x_n)],$$

where $\cos(x) := \sin'(x)$, is a noetherian differential subring of $C^\infty(\mathbb{K}^n, \mathbb{K})$ (Notice that this is not a pfaffian example).

4. Let $\langle \mathbb{K}, +, -, \cdot, <, g_1, .., g_l, 0, 1, \ldots \rangle$ be a definably complete structure such that $g_1, .., g_l \in C^\infty(\mathbb{K}^n, \mathbb{K})$ satisfy a (not necessarily triangular) system of differential equations, with polynomial coefficients (Examples are $\mathbb{R}_{\sin}$ and $\mathbb{R}_{\sinh}$). Then,

$$\mathbb{F}[x_1, \ldots, x_n, g_1, \ldots, g_l],$$

where $\mathbb{F}$ is a subfield of $\mathbb{K}$, is a noetherian differential subring of $C^\infty(\mathbb{K}^n, \mathbb{K})$.

2.3.4 Remark. In the last example, if $\mathbb{K}$ is a structure based on $\mathbb{R}$, then the functions $g_1, \ldots, g_l$ are not only C^∞, but even analytic (by Cauchy-Kowalesky Theorem, see for example [16]). On the other hand, if M is a noetherian differential ring which *is not* a finitely generated algebra, then it does not necessarily follow that M consists of real analytic functions. Unfortunately, we do not know any such example.

We now fix a noetherian differential ring $M \subseteq C^\infty(\mathbb{K}^n, \mathbb{K})$ in our definably complete structure $\mathbb{K}$, and we study the properties of the zero-sets of functions in M. The following result shows that the functions in M have a "quasi-analytic" behaviour.

2.3.5 Lemma (Lack of flat functions)**.** *Let $I \subseteq M$ be an ideal closed under differentiation; then either $V(I) = \emptyset$ or $V(I) = \mathbb{K}^n$.*

Proof. Since M is noetherian, I is finitely generated, say $I = \langle g_1, \ldots, g_s \rangle$, and hence $V(I) = V(g_1, .., g_s)$ is a closed definable subset. If $V(I) \neq \emptyset$, since $\mathbb{K}^n$ is definably connected, all we need to show is that $V(I)$ is open.

Suppose for a contradiction that this is not the case. Then there exists $\overline{x} \in V(I)$ which is not an interior point, i.e. given an arbitrary open box neighbourhood B of $\overline{x}_0$, there exists a point $\overline{y}_0 \in B$ which is not in $V(I)$. Without loss of generality, we may assume that $\overline{x}_0, \overline{y}_0$ differ in exactly one coordinate, say, the first one: $\overline{x}_0 = (s, p_2, \ldots, p_n)$, $\overline{y}_0 = (t, p_2, \ldots, p_n)$ and $s \neq t$.

Recall that $\{g_1, \ldots, g_s\}$ is a set of generators for I. Since I is closed under differentiation, it follows in particular that the derivatives with respect to the first coordinate $\partial g_1/\partial x_1(\overline{x}), \ldots, \partial g_s/\partial x_1(\overline{x})$ all belong to I, hence there exist functions $a_{ij}(\overline{x}) \in M$ such that

$$\forall \overline{x},\ \forall i = 1, \ldots, s \quad \frac{\partial g_i}{\partial x_1}(\overline{x}) = \sum_{j=1}^{s} a_{ij}(\overline{x}) g_j(\overline{x}).$$

Now, consider the restrictions $f_i(x_1) = g_i(x_1, p_2, \ldots, p_n)$ of the functions $g_1, \ldots, g_s$ to the line $L = \{\overline{x} \in \mathbb{K}^n :\ x_2 = p_2 \wedge \ldots \wedge x_n = p_n\}$, and define $F(x_1) = (f_1(x_1), \ldots, f_s(x_1))$. We have

$$F'(x_1) = A(x_1)F(x_1),$$

where $A(x_1)$ is the $s \times s$ matrix whose entries are the functions $a_{ij}(x_1, p_2, \ldots, p_n)$.

It follows from the Uniqueness Theorem for Linear Differential Equations 1.5.1 that either $F \equiv 0$ or else has no zeros. But this leads to a contradiction, since $F(t) \neq 0$ and $F(s) = 0$. □

2.3.6 Remark. An analogous result holds if $M \subseteq C^\infty(U, \mathbb{K})$, where U is a definably connected definable open subset of $\mathbb{K}^n$.

2.3.7 Corollary. *Let $G = (g_1, \ldots, g_m) \in M^m$ and $\overline{x}_0 \in V^{\mathrm{reg}}(G)$. Then either there exists $h \in M$ such that $\overline{x}_0 \in V^{\mathrm{reg}}(G, h)$ or for all $h \in M$, if $h(\overline{x}_0) = 0$, then h vanishes on a definable neighbourhood of $\overline{x}_0$ in $V^{\mathrm{reg}}(G)$.*

Proof. We refer to the notation of the Implicit Function Theorem 2.2.8, so $\overline{x}_0 = (\overline{u}_0, \overline{v}_0) \in \mathbb{K}^k \times \mathbb{K}^m$. Up to some rearrangement of the variables, we may assume that $D_{\overline{v}}G(\overline{x}_0)$ is non-singular and apply the Implicit Function Theorem in a suitable neighbourhood $O \times W$ of $\overline{x}_0$. Suppose that there is no $h \in M$ such that $\overline{x}_0 \in V^{\mathrm{reg}}(G, h)$ and let $h \in M$ be such that $h(\overline{x}_0) = 0$. Then $\nabla h(\overline{x}_0)$ belongs to $\mathrm{lin.span}\langle \nabla g_1(\overline{x}_0), \ldots, \nabla g_m(\overline{x}_0)\rangle$. This implies, by Lagrange's Multiplier Rule 2.2.11, that $\widehat{\nabla h}(\overline{u}_0) = 0$.

Consider the ideal $\widehat{I} = \{\widehat{g} \in \widehat{M[J_{\overline{v}}G^{-1}]} :\ \widehat{g}(\overline{u}) = 0\}$; what we have shown is that if $\widehat{h} \in \widehat{I}$, then its first derivatives $\partial\widehat{h}/\partial u_i$ belong to $\widehat{I}$; thus $\widehat{I}$ is closed under differentiation. Since $V(\widehat{I}) \neq \emptyset$, it follows from Lemma 2.3.5 and the subsequent Remark, that $V(\widehat{I}) = O$. This means that h vanishes on $V^{\mathrm{reg}}(G) \cap (O \times W)$. □

2.4 Decomposition of noetherian varieties

We fix, for the rest of the chapter, a noetherian differential ring M and we show further properties of *M-varieties*, i.e. zero-sets of functions belonging to M. In this section we prove that every M-variety can be decomposed into finitely many differentiable $\mathbb{K}$-manifolds of a certain form.

2.4.1 Definition (Clopen subsets)**.** Let A be a definable set; we say that S is a *definable clopen of* A if $S \subseteq A$ is a definable subset which is both open and closed in A. Clearly, the collection of all definable clopen of A is a boolean algebra $\mathcal{B}(A)$ of sets.

2.4.2 Definition (Regular components)**.** If $G \in M^m$ and S is a clopen definable subset of $V^{\text{reg}}(G)$, then S is called a *regular component*. The *dimension* of S is the $\mathbb{K}$-manifold dimension of $V^{\text{reg}}(G)$, which is $n-m$.

2.4.3 Lemma. *Let $0 \neq f \in M$ and $V(f) \subset \mathbb{K}^n$ be a nonempty M-variety; then for all $\overline{x} \in V(f)$ there exists $g \in M$ such that $\overline{x} \in V^{\text{reg}}(g)$, i.e. $g(\overline{x}) = 0 \wedge \nabla g(\overline{x}) \neq \overline{0}$.*

Proof. Take $\overline{x} \in V(f)$ and consider f together with all its partial derivatives, evaluated in $\overline{x}$. We claim that there exist a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $i_0 \in \{1, \ldots, n\}$ such that, if we put $\partial^\alpha f := \frac{\partial^{\alpha_1+\ldots+\alpha_n} f}{\partial x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}}$, then $\partial^\alpha f(\overline{x}) = 0$ and $\frac{\partial \partial^\alpha f}{\partial x_{i_0}}(\overline{x}) \neq 0$, so that we can define $g := \partial^\alpha f$. Suppose, on the contrary, that f as well as all its derivatives $\partial^\alpha f$ vanishes in $\overline{x}$ and let I be the ideal of M generated by f and all its derivatives. Notice that $V(I) \neq \emptyset$, since $\overline{x} \in V(I)$; M is noetherian, so I is finitely generated. Moreover, I is closed under differentiation, since each member of I can be written as a linear combination (with coefficients in M) of a finite number of derivatives of f. Then, Lemma 2.3.5 implies that $V(I)$ (and hence $V(f)$) coincides with $\mathbb{K}^n$, which is impossible. $\square$

2.4.4 Remark. The above argument shows also that, if $0 \neq f \in M$, then $V(f)$ has empty interior. In fact, if $V(f)$ has interior around a point $\overline{x}$, then $\overline{x}$ is necessarily not a regular zero of f (otherwise, by the Implicit Function Theorem 2.2.8, $V(f)$ would be locally diffeomorphic to $\mathbb{K}^{n-1}$ around $\overline{x}$). For the same reason, $\overline{x}$ is not a regular zero of any of the derivatives of f, hence all the derivatives of any order of f vanish in $\overline{x}$. But then, as in the proof above, $V(f)$ must be $\mathbb{K}^n$.

2.4.5 Definition. For every $\overline{x} \in \mathbb{K}^n$, we define the *$M$-degree of* $\overline{x}$, $\deg_M(\overline{x})$, as the minimal dimension of a regular component containing $\overline{x}$. Equivalently,

$$\deg_M(\overline{x}) = \min\{k |\ \exists G \in M^{n-k} \text{ such that } \overline{x} \in V^{\text{reg}}(G)\}.$$

Lemma 2.4.3 shows that every point belonging to a proper M-variety has M-degree at most $n-1$. Moreover, we can show that every proper M-variety has a point of M-degree zero (this result will be proven in the next section and will be used in Chapter 4).

2.4.6 Theorem. *Let $f \in M$ and $V(f)$ be a proper M-variety. Then, for every point $\overline{x}$ in $V(f)$, there exist $k < n$ and $G \in M^{n-k}$ and a regular component S of $V^{\text{reg}}(G)$ such that $\overline{x} \in S \subseteq V(f)$. Moreover, S is explicitely definable from G and f.*

Proof. Let $k = \deg_M(\overline{x})$ and $G \in M^{n-k}$ such that $\overline{x} \in V^{\text{reg}}(G)$. We define

$$S := \text{the interior of the set } V^{\text{reg}}(G) \cap V(f) \text{ in } V^{\text{reg}}(G).$$

We claim that $\overline{x} \in S$. In fact, by the choice of $V^{\text{reg}}(G)$ as a regular set of minimal dimension, by Corollary 2.3.7 it follows that every function $h \in M$ which vanishes in $\overline{x}$, also vanishes on an open definable neighbourhood B of $\overline{x}$ in $V^{\text{reg}}(G)$. In particular, f vanishes on some B (depending on f). Hence $\overline{x}$ has an open neighbourhood B contained in $V(f) \cap V^{\text{reg}}(G)$, i.e. $\overline{x}$ is an interior point.

We now claim that S is a regular component. S is definable, nonempty and open in $V^{\text{reg}}(G)$, by definition. We must show that S is also closed in $V^{\text{reg}}(G)$. Take a boundary point $\overline{x}_0$ of S in $V^{\text{reg}}(G)$ and consider (after permuting the variables, if necessary) the local parametrization given by the Implicit Function Theorem 2.2.8

$$\begin{array}{rccc} \phi: & \mathbb{K}^k & \to & V(G) \cap (O \times W) \\ & \overline{u} & \mapsto & (\overline{u}, Y(\overline{u})). \end{array}$$

Setting, as usual, $\widehat{f} = f \circ \phi$, we observe that $\phi^{-1}(S)$ is open in O and $\widehat{f}(\phi^{-1}(S)) = 0$. Hence, all derivatives of any order of $\widehat{f}$ vanish on $\phi^{-1}(S)$. Since $\overline{u}_0 = \phi^{-1}(\overline{x}_0)$ belongs to the closure of $\phi^{-1}(S)$, it is also true, by continuity, that $\widehat{f}$, and all its derivatives of any order, vanish in $\overline{u}_0$. By Lemma 2.3.5 and the usual argument, $V(\widehat{f}) = \mathbb{K}^k$. Hence, the open neighbourhood $O \times W$ of $\overline{x}_0$ is contained in $V(f) \cap V^{\text{reg}}(G)$, that implies $\overline{x}_0 \in S$. $\square$

2.4.7. Theorem (Decomposition of an M-variety into regular components). *Let $f \in M$ and $V(f)$ be a proper M-variety. Then $V(f)$ can be written as a finite union of regular components:*

$$\exists k \in \mathbb{N},\ \exists G_1, \ldots, G_k \in \bigcup_{l=1}^{n} M^l,\ \exists S_i \in \mathcal{B}(V^{\text{reg}}(G_i)) \text{ so that } V(f) = S_1 \cup \ldots \cup S_k.$$

Proof. By compactness. More precisely, let $\mathbb{F}$ be a $|\mathbb{K}|^+$-saturated elementary superstructure of $\mathbb{K}$ (see [32] for the existence of such an $\mathbb{F}$), so that $\mathbb{F}$ realizes all types over $\mathbb{K}$. Let $\tilde{M}$ be the set of those definable functions $\tilde{g}$ such that $g \in M$ and $\tilde{g}$ is the interpretation of g in $\mathbb{F}$ (note that $\tilde{g}$ is still a C^∞ function). Then $\tilde{M}$ is still a noetherian differential ring, hence Theorem 2.4.6 holds for $\tilde{M}$-varieties. Consider the function $\tilde{f}$ and the following set of formulas:

$$\Phi = \{\phi_{\tilde{G}} := \overline{x} \in V(\tilde{f}) \wedge \tilde{S} = \mathrm{int}_{V^{\mathrm{reg}}(\tilde{G})}(V^{\mathrm{reg}}(\tilde{G} \cap V(\tilde{f})) \wedge (\overline{x} \in \tilde{S} \to \tilde{S} \not\subset V(\tilde{f}))|\, \tilde{G} \in \bigcup_{i=1}^{n} \tilde{M}^i\}.$$

If Φ were a consistent type, then it would be realized $\mathbb{F}$. This means that there would exists an $\overline{x} \in \mathbb{F}$ such that for all $\tilde{G} \in \bigcup_{i=1}^n \tilde{M}^i$, $\overline{x} \in V(\tilde{f}) \wedge \tilde{S} = \mathrm{int}_{V^{\mathrm{reg}}(\tilde{G})}(V^{\mathrm{reg}}(\tilde{G} \cap V(\tilde{f})) \wedge (\overline{x} \in \tilde{S} \to \tilde{S} \not\subset V(\tilde{f}))$, which would contradict Theorem 2.4.6. Hence there exist $k \in \mathbb{N}$, $\tilde{G}_1, \ldots, \tilde{G}_k \in \bigcup_{i=1}^n \tilde{M}^i$, such that the conjunction $\phi_{\tilde{G}_1} \wedge \ldots \wedge \phi_{\tilde{G}_k}$ is not satisfiable in $\mathbb{F}$; in other words the following holds in $\mathbb{F}$:

$$\forall \overline{x}\ (\overline{x} \in \tilde{S}_1 \cup \ldots \cup \tilde{S}_k \wedge \tilde{S}_1 \cup \ldots \cup \tilde{S}_k \subseteq V(\tilde{f})).$$

Therefore $V(\tilde{f}) = \tilde{S}_1 \cup \ldots \cup \tilde{S}_k$.

Now, in $\mathbb{K}$ the following holds: $V(f) = S_1 \cup \ldots \cup S_k$, where $S_i := \tilde{S}_i \cap \mathbb{K}^n$ $(i = 1, \ldots, k)$ are clearly regular components in $\mathbb{K}$, hence the theorem is proved. □

2.5 Dimension of M-varieties

2.5.1 Remark. The decomposition which appears in Theorem 2.4.7 is clearly not unique, nor are unique the dimensions of the regular components appearing in two different decompositions of the same variety. For example, the algebraic variety $V(x^2 - y^2) \subset \mathbb{R}^2$ can be decomposed as $V^{\mathrm{reg}}(x^2 - y^2) \cup V^{\mathrm{reg}}(x, y)$ or as $V^{\mathrm{reg}}(x - y) \cup V^{\mathrm{reg}}(x + y)$. In the first decomposition the first regular component has dimension 1 and the second has dimension 0, while in the second decomposition both regular components have dimension 1. Moreover, in the first case the union is disjoint, and in the second case it is not. Natural questions arise, to which the answer is at the moment not known: Is it always possible to obtain a disjoint union? Is it always possible to find a decomposition where the components are of the form $V^{\mathrm{reg}}(g_1, \ldots, g_k)$, rather than just definable clopen subsets of sets of that form?

On the other hand, we obtain, as a consequence of the Decomposition Theorem, that we are able to assign a *dimension* to (not necessarily regular) M-varieties.

2.5.2 Lemma. *Let $f \in M$. Then there exists a unique natural number m such that for every decomposition of $V(f)$ into regular components, as in Theorem 2.4.7, the maximal dimension of the regular components appearing in the decomposition is m.*

Proof. Let $V(f) = S_1 \cup \ldots \cup S_k$ be a decomposition of $V(f)$ into regular components and suppose $\dim S_i \leq \dim S_1 = m$, for all $i = 2, \ldots, k$. Clearly $V(f)$ does not contain an open subset which is diffeomorphic to $\mathbb{K}^l$, for $l > m$, because otherwise such a subset would be obtained as a finite union of manifolds of dimension $\leq m$, which is clearly impossible. On the other hand, $V(f)$ does contain an open subset which is diffeomorphic to $\mathbb{K}^m$, because so does S_1. Hence, every decomposition of $V(f)$ must contain a component of dimension m, and can not contain components of bigger dimension. $\square$

2.5.3 Definition. The previous lemma allows us to define the dimension of an M-variety $V(f)$ as

$$\dim V(f) := \max\{\dim S_i |\ i = 1, \ldots, k \ \text{ and } V(f) = S_1 \cup \ldots \cup S_k\},$$

where $V(f) = S_1 \cup \ldots \cup S_k$ is any decomposition given by Theorem 2.4.7.

We now compare the dimension of a variety with the M-degree of its points (see Definition 2.4.5).

2.5.4 Lemma. *Let $g_1, \ldots, g_m \in M$ and $\overline{x} \in V^{\text{reg}}(g_1, \ldots, g_m) \subset \mathbb{K}^n$. If $\deg_M(\overline{x}) < n - m$, then there exists $f \in M$ such that $\overline{x} \in V^{\text{reg}}(g_1, \ldots, g_m, f)$.*

Proof. Since $\deg_M(\overline{x}) < n - m$, there exist $f_1, \ldots, f_{m+1} \in M$ so that $\overline{x} \in V^{\text{reg}}(f_1, \ldots, f_{m+1})$. We claim that there exists $i \in \{1, \ldots, m+1\}$ so that $\overline{x} \in V^{\text{reg}}(g_1, \ldots, g_m, f_i)$, because otherwise the (linearly independent) vectors $\nabla f_1(\overline{x}), \ldots, \nabla f_{m+1}(\overline{x})$ would all lie in the m-dimensional vector space generated by $\nabla g_1(\overline{x}), \ldots, \nabla g_m(\overline{x})$, which is impossible. $\square$

2.5.5 Proposition. *Let $V(f) \subset \mathbb{K}^n$ be an M-variety. Let $\mathbb{F}$ be any $|\mathbb{K}|^+$-saturated elementary superstructure of $\mathbb{K}$ and let $\tilde{f}$ be the interpretation of f in $\mathbb{F}$ (as in the proof of 2.4.7). Then,*

$$\dim V(f) = \max\{\deg_M(\overline{x})|\ \overline{x} \in V(\tilde{f})\}.$$

Proof. Let $V(\tilde{f}) = \tilde{S}_1 \cup \ldots \cup \tilde{S}_l$ be a decomposition of $V(\tilde{f})$ into regular components, and let $S_i = \mathbb{K}^n \cap \tilde{S}_i$. Then $V(f) = S_1 \cup \ldots \cup S_l$, hence $\dim V(\tilde{f}) = \dim V(f)$. Let $\overline{x} \in V(\tilde{f})$. Then $\overline{x} \in \tilde{S}_i$ for some i, hence $\deg_M(\overline{x}) \leq \dim \tilde{S}_i \leq \dim V(f)$. So $\dim V(f) \geq \max\{\deg_M(\overline{x})|\ \overline{x} \in V(\tilde{f})\}$.

Now we prove that there exists $\overline{x} \in V(\tilde{f})$ with $\deg_M(\overline{x}) = \dim V(f)$. Let $\tilde{S}_1 \in \mathcal{B}(V^{\mathrm{reg}}(\tilde{g_1}, \ldots, \tilde{g_k}))$ be a component of maximal dimension. Consider the set of formulas

$$\Phi = \{\overline{x} \in \tilde{S}_1 \wedge \overline{x} \notin V^{\mathrm{reg}}(\tilde{g_1}, \ldots, \tilde{g_k}, \tilde{h})|\ h \in M\}.$$

Φ is clearly finitely satisfiable in $\mathbb{F}$, because no finite union of regular sets of dimension $n-k-1$ can cover the whole of $\tilde{S}_1$, which has dimension $n-k$. By saturation, there exists $\overline{x} \in \mathbb{F}^n$ which satisfies all formulas in Φ. By Lemma 2.5.4, then, $\deg_M(\overline{x}) = \dim V(f)$. $\square$

On the other hand, we prove that every variety contains a point with M-degree equal to zero.

2.5.6 Proposition. *Let $g \in M$ and $\emptyset \neq S \in \mathcal{B}(V(g))$. Then there exists $\overline{x}_0 \in S$ such that* $\deg_M(\overline{x}_0) = 0$.

Proof. We define for every $\overline{x} \in S$ the ideal $I_{\overline{x}} = \{h \in M|\ h(\overline{x}) = 0\}$. Since M is noetherian, the collection $\{I_{\overline{x}}|\ \overline{x} \in S\}$ has a maximal element $I_0 = I_{\overline{x}_0}$. We claim that $\deg_M(\overline{x}_0) = 0$.

Let $\deg_M(\overline{x}_0) = k$ and $F = (f_1, \ldots, f_{n-k}) \in M^{n-k}$ so that $\overline{x}_0 \in V^{\mathrm{reg}}(F)$. Define, moreover, $V_S(I_0) = \{\overline{x} \in S|\ \forall h \in I_0\ h(\overline{x}) = 0\}$. Notice that $V_S(I_0)$ is a closed set.

We first prove that $V_S(I_0)$ is a definable clopen subset of $V^{\mathrm{reg}}(F)$.

Notice that

$$\forall \overline{x}\ (\overline{x} \in V_S(I_0) \to I_{\overline{x}} = I_0),$$

for $I_0 \subseteq I_{\overline{x}}$ as a consequence of the definition of $V_S(I_0)$, while the other inclusion follows by maximality of I_0.

This implies at once that $V_S(I_0) \subseteq V^{\mathrm{reg}}(F)$, since $F \in I_0^{n-k} = I_{\overline{x}}^{n-k}$ and $\det E \notin I_0 = I_{\overline{x}}$ (where E is some maximal rank minor of DF, which is nonsingular in $\overline{x}_0$), for all $\overline{x} \in V_S(I_0)$.

By the same token, if again $\overline{x} \in V_S(I_0)$, then $\overline{x}$ and $\overline{x}_0$ belong to the same regular sets, hence $\deg_M(\overline{x}) = \deg_M(\overline{x}_0)$. By Corollary 2.3.7, $\overline{x}$ has a definable open neighbourhood $U_{\overline{x}}$ such that

$$\forall h \in M \quad (h(\overline{x}) = 0 \to h_{|_{U_{\overline{x}} \cap V^{\mathrm{reg}}(F)}} \equiv 0).$$

In particular, if we set $U := S \cap \bigcup\{U_{\overline{x}}|\ \overline{x} \in V_S(I_0)\}$, then

$$\forall h \in I_0 \quad h(\overline{x}) = 0 \to h_{|_{U \cap V^{\mathrm{reg}}(F)}} \equiv 0,$$

that implies $U \cap V^{\mathrm{reg}}(F) \subseteq V_S(I_0)$. Hence $V_S(I_0)$ is a definable clopen subset of $V^{\mathrm{reg}}(F)$.

Next, we prove that $V_S(I_0)$ has dimension zero (and hence so does $V^{\text{reg}}(F)$). Since $V_S(I_0)$ is closed, for all $\overline{\eta} \in \mathbb{Z}^n$ there exists a point $\overline{x} \in V_S(I_0)$ whose "distance" from $\overline{\eta}$ is minimal, i.e. the function $h_{\overline{\eta}}(\overline{x}) = \sum_{i=1}^n (x_i - \eta_i)^2 \in M$ has a minimum in $\overline{x}$. Then, Lagrange's Multiplier Rule tells us that $2(\overline{x}-\overline{\eta}) = \nabla h_{\overline{\eta}}(\overline{x}) \in \operatorname{lin.span}\langle \nabla f_1(\overline{x}), \dots, \nabla f_{n-k}(\overline{x})\rangle$. The point is that this linear dependence condition remains true if evaluating the vectors in $\overline{x}_0$ (for this condition is equivalent to the vanishing in $\overline{x}$ of some functions of M), so that we infer

$$\forall \overline{\eta} \in \mathbb{Z}^n \ (\overline{x}_0 - \overline{\eta}) \in \operatorname{lin.span}\langle \nabla f_1(\overline{x}_0), \dots, \nabla f_{n-k}(\overline{x}_0)\rangle;$$

hence $k = 0$. □

2.5.7 Remark (Noetherian topology)**.** We notice that M-varieties form a basis of closed sets for a Noetherian topology. We first observe that a closed set in this topology is itself an M-variety: the union of two M-varieties $V(f)$ and $V(g)$ is the M-variety $V(fg)$, and an arbitrary intersection of M-varieties $\bigcap_{f\in\mathcal{F}} V(f)$, where $\mathcal{F} \subset M$ can be written as the common zero-set of all the functions in the family $\mathcal{F}$, which in turn equals the common zero-set of all functions in the ideal I generated by the family $\mathcal{F}$; since M is noetherian, the ideal I is finitely generated and, if $g_1, \dots, g_s$ are generators, then

$$\bigcap_{f\in\mathcal{F}} V(f) = V(\{f \mid f \in \mathcal{F}\}) = V(I) = V(g_1, \dots, g_s).$$

Secondly, let $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$ be a descending chain of closed sets in this topology (hence, every V_i is an M-variety $V(f_i)$). Let I_i be the ideal of all the functions in M which vanish on V_i. Then we obtain an increasing chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$, which, by noetherianity of M must eventually stabilize: there exists N such that $I_N = I_j$ for all $j > N$. This implies that $V(I_N) = V(I_j)$ for all $j > N$. We observe now that for all $j \in \mathbb{N}$, $V(I_j) = V_j$: in fact, from $f_j \in I_j$ follows the inclusion $V(I_j) \subseteq V_j$, while the other inclusion follows by definition of I_j. Hence,

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots V_N = V_{N+1} = \dots,$$

that is, every descending chain of closed sets stabilizes, as required by the definition of noetherian topology.

2.5.8 Definition. A closed set is called *irreducible* if it can not be written as the (not necessarily disjoint) union of two closed proper subsets.

Recall that

2.5.9 Proposition. *Every closed set in a noetherian topology can be written as a finite union of irreducible closed sets (called* irreducible components*).*

2.5.10 Lemma. *Let $V = V_1 \cup \ldots \cup V_l$ be a decomposition of the M-variety V into irreducible components. Then there exists i such that $\dim V = \dim V_i$.*

Proof. Trivial from the definition of dim. $\square$

2.5.11 Definition (Krull dimension)**.**

- Let V be an irreducible M-variety. Then, define $\mathrm{krull}(V)$ as the maximal length of a chain of irreducible proper subvarieties of V:

$$\mathrm{krull}(V) = \max\{k \in \mathbb{N} \mid \exists V_0, \ldots, V_k \text{ irreducible} \quad V_0 \subset \ldots \subset V_k \subset V\}.$$

- Let V be any variety. Then define $\mathrm{krull}(V)$ as the maximal Krull dimension of its irreducible components.

In this section we try to understand the following:

2.5.12 Problem. *Let V be an M-variety. Then, what is the relationship between* $\dim V$ *and* $\mathrm{krull}\, V$*?*

If M is a polynomial ring, the answer to this question is that $\dim V = \mathrm{krull}\, V$. We are not able to answer the question in the general case completely, but we will give partial answers, assuming that M satisfies further conditions.

Suppose we were to try and prove that also in the general case the two definitions of dimension coincide. In the polynomial case, it is enough to prove that the following two conditions hold:

2.5.13 (Condition 1)**.** Let V be an M-variety of dimension $\dim V = k$. Then there exists an M-variety $W \subset V$ of dimension $\dim W = k - 1$.

2.5.14 (Condition 2)**.** Let V and W be two irreducible varieties such that $W \subset V$. Then, $\dim W < \dim V$.

Now we prove, for any noetherian differential ring M,

2.5.15 Proposition. *Suppose Conditions 1 and 2 hold. Then, for all M-varieties V, it is true that* $\dim V = \mathrm{krull}\, V$.

Proof. Let V be an M-variety of dimension $\dim V = k$. Let V_1 be an irreducible component of dimension k. By Condition 1, we find $W \subset V_1$ such that $\dim W = k - 1$, and we take one of its irreducible components of maximal dimension W_1. With this procedure, we construct a chain of irreducible varieties of length k, so $\operatorname{krull} V \geq \dim V$. On the other hand, Condition 2 implies that such a chain is maximal, hence $\dim V = \operatorname{krull} V$.

We will discuss now Condition 1.

2.5.16 Lemma. *Let $V \subset \mathbb{K}^{k+m}$ be an M-variety of dimension $\dim V = k$. Then there exists $I = (i_1, \ldots, i_m)$, with $1 \leq i_1 < \ldots < i_m \leq k + m$, so that the projection $\pi_I(V)$ onto the k coordinates which are not in I, has nonempty interior.*

Proof. Let $V = S_1 \cup \ldots \cup S_l$ be a decomposition as in Theorem 2.4.7, with $\dim S_1 = k$. Suppose $S_1 \in \mathcal{B}(V^{\mathrm{reg}}(g_1, \ldots, g_m))$ and take $\overline{a} \in S_1$. Then, by definition of $V^{\mathrm{reg}}(g_1, \ldots, g_m)$, there exists $I = (i_1, \ldots, i_m)$, so that the minor, with respect to this set of coordinates, of the matrix of the linear map $D(g_1, \ldots, g_m)(\overline{a})$ is nonsingular. Hence, by the Implicit Function Theorem 2.2.8, there are open neighbourhoods U of $\overline{a}$ and O of $\pi_I(\overline{a})$, such that the projection π_I induces a diffeomorphism of $V \cap U$ and O. In particular, $\pi_I(V)$ has nonempty interior. □

2.5.17 Proposition. *Suppose $M \subset C^\infty(\mathbb{K}^n, \mathbb{K})$ is a noetherian differential ring which satisfies furthermore:*

- *If $\overline{a} = (a_1, \ldots, a_n)$ is a point in $\mathbb{K}^n$ such that $\deg_M(\overline{a}) = 0$, then the constant functions $a_1, \ldots, a_n$ belong to M.*

Then Condition 1 holds for M-varieties.

Proof. Let $n = k + m$ and $V = V(f) \subset \mathbb{K}^{k+m}$ be an M-variety of dimension $\dim V = k$. Let $V = S_1 \cup \ldots \cup S_l$ be a decomposition as in Theorem 2.4.7, with $\dim S_1 = k$. Suppose $S_1 \in \mathcal{B}(V^{\mathrm{reg}}(g_1, \ldots, g_m))$ and take $\overline{a} = (a_1, \ldots, a_n) \in S_1$ such that $\deg_M(\overline{a}) = 0$. Suppose, to simplify the notation, that the projection $\pi(V)$ onto the first k coordinates has nonempty interior around $\pi(\overline{a})$. Now, define the M-variety

$$W := V(f, x_k - a_k, x_{k+1} - a_{k+1}, \ldots, x_n - a_n).$$

Now clearly $\dim W = k - 1$, as required. □

2.5.18 Remark. It seems plausible that the further condition imposed on M could be superfluous. In particular, we could try to avoid using coordinate functions and find $g \in M$ such that $\overline{a} \in V^{\mathrm{reg}}(g_1, \ldots, g_m, g)$ and moreover $V^{\mathrm{reg}}(g_1, \ldots, g_m, g) = V(g_1, \ldots, g_m, g)$, to ensure that the dimension decreases. The main problem with this approach is that we do not have an answer to the following question: let S be a regular component of dimension k; is there a variety of dimension k which contains S?

Condition 2 is unfortunately much more difficult to deal with, because it involves an understanding of *irreducible* varieties, which we do not seem to have, at least in the general case. What we can say is that in the case when M is a polynomial ring in n variables over a field K, then Condition 2 holds, and the proof uses the following consequence of Noether´s Normalization Lemma:

2.5.19 Theorem. *Let K be a field and $V_p \subset \ldots \subset V_1 \subseteq K^n$ be irreducible algebraic varieties. Then there is a finite-to-one polynomial map $F : K^n \to K^n$ and a sequence of integers $1 \leq k_1 < \ldots < k_p \leq n$ such that*

- $\forall i \in \{1, \ldots p\} \quad F(V_i) \subseteq \{0\}^{k_i} \times K^{n-k_i}$ *and*
- $\forall i \in \{1, \ldots p\} \quad \pi_{k_i}(F(V_i))$ *is Zariski-dense in* K^{n-k_i}.

The theorem is used in the following way: it is easy to see in general that a proper variety $W \subset K^n$ must have dimension strictly smaller than n (this is true in general for M-varieties), and we use the theorem to reduce the situation "$W \subset V$ irreducible varieties" to the simpler situation "$W \subset K^n$ ".

2.5.20 Remark. It might be possible to mimic the proof of Condition 2 for the polynomial case, for the case when M is a finitely generated algebra, but the general case seems unlikely to be true.

2.6 Khovanskii rings

In this section, we investigate a class of noetherian differential rings, called *Khovanskii rings*, which satisfy some further condition, i.e. the finiteness of the number of regular common zeroes of n functions in n variables. The classical example of such a ring is $M = \mathbb{R}[\overline{x}, f_1, \ldots, f_k]$, where the functions f_i for a Pfaffian chain (see 2.6.2, below), as it was proved by Khovanskii in [18]. In the same paper, Khovanskii proved that all M-varieties, where M is as above, have finitely many connected components (see 4.7.14 for the precise statement of Khovanskii's Theorem in the case of real exponentiation). Here we prove, with a method which differs from the approach in [18], that if M is a Khovanskii ring in a definably complete structure, then all M-varieties have finitely many *definably connected components.* The natural question is: under which further hypotheses can we prove that every definable set, not only M-varieties, has finitely many definably connected components (in other words, $\mathbb{K}$ is o-minimal)? An answer to this question will be given in the next chapter.

2.6.1 Definition (Khovanskii rings)**.** Let $n \in \mathbb{N}$. Let M be a ring with the following properties:

- $M \subseteq C^\infty(\mathbb{K}^n, \mathbb{K})$;
- M is a noetherian differential ring;
- $\forall g_1, \ldots, g_n \in M \quad |V^{\text{reg}}(g_1, \ldots, g_n)| < \infty$.

Then we say that M is a *Khovanskii ring.*

Let $\{M_n |\ n \in \mathbb{N}\}$ be a collection of rings such that:

- M_n is a ring of definable C^∞ functions from $\mathbb{K}^n$ to $\mathbb{K}$;
- M_n is a Khovanskii ring;
- $M_n \subset M_{n+1}$ (in the obvious sense);
- M_n is closed under permutation of the variables.

Then we say that $\{M_n |\ n \in \mathbb{N}\}$ is a *collection of Khovanskii rings.*

A similar definition appears in [10].

2.6.2 Examples. Examples of Khovanskii rings over the real numbers are:

- The ring generated by a *Pfaffian chain of functions* (see [19]), that is, a finite sequence of differentiable functions $g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$, such that there exist polynomials q_{ij} ($i = 1, \ldots, m;\ j = 1, \ldots, n$) over $\mathbb{R}$, with:

$$\frac{\partial g_i}{\partial x_j}(\overline{x}) = q_{ij}(\overline{x}, g_1, \ldots, g_i) \qquad (i = 1, \ldots, m;\ j = 1, \ldots, n).$$

- Any noetherian differential ring of functions definable in an o-minimal expansion of the real field;
- The ring generated by the restrictions to a bounded interval of the real functions $\exp(x), \sin(x)$ and $\cos(x)$ (see [19]).

2.6.3 Remark. Fix $n, m \in \mathbb{N},\ m \leq n$. Let $M \subseteq C^\infty(\mathbb{K}^n, \mathbb{K})$ be a noetherian differential ring (not necessarily a Khovanskii ring) and let $F \in M^m$. Then the set of regular zeroes of F can be expressed as the projection of a finite union of *regular* varieties of dimension $\dim V^{\text{reg}}(F)$. To see this, let $E_1(\overline{x}), \dots, E_l(\overline{x})$ be the maximum rank minors of the matrix $DF(\overline{x})$. Now consider $V_i := V(F(\overline{x}), x_{n+1} \det E_i(\overline{x}) - 1)$. Then V_i is a regular sub-variety of $\mathbb{K}^{n+1}$ and $\pi_{n+1}(\bigcup_{i=1}^{l} V_i) = V^{\text{reg}}(F)$ (where π_{n+1} is the projection onto the first n coordinates).

Notice that $\dim V_i = n + 1 - (m + 1) = n - m = \dim V^{\text{reg}}(F)$. Moreover, if $M = M_n$ belongs to a collection of Khovanskii rings, then the map $(F(\overline{x}), x_{n+1} \det E_i(\overline{x}) - 1)$ belongs to M_{n+1}^{n-k+1}.

2.6.4 Proposition. *Fix $n, m \in \mathbb{N},\ m \leq n - 1$. Let $M \subseteq C^\infty(\mathbb{K}^n, \mathbb{K})$ be a Khovanskii ring and $f_1, \dots, f_m \in M$ be such that $V(f_1, \dots, f_m) = V^{\text{reg}}(f_1, \dots, f_m)$, i.e. $V(f_1, \dots, f_m)$ is a regular variety. Then there exists a definable set G such that:*

- *$\emptyset \neq G \subset V(f_1, \dots, f_m)$;*
- *For every clopen definable subset S of $V(f_1, \dots, f_m)$, the intersection $S \cap G$ is not empty;*
- *$\forall \overline{x} \in G\ \exists h \in M\quad (\overline{x} \in V^{\text{reg}}(f_1, \dots, f_m, h))$.*

Proof. For all $h \in M$, consider the matrix of partial derivatives $D(f_1, \dots, f_m, h)$. Let $\overline{x} \in V(f_1, \dots, f_m)$. Then, this matrix, if we evaluate all the entries in $\overline{x}$, has rank at least m, because the common zeroes of the functions $f_1, \dots, f_m$ are all regular zeroes, by hypothesis. Let $H_i(\overline{x})$ $(i = 1, \dots, \binom{n}{m+1})$ be the minors of rank $m+1$ of the matrix $D(f_1, \dots, f_m, h)$ evaluated in $\overline{x}$ and define $h^*(\overline{x}) = \sum_{i=1}^{l} (\det H_i)(\overline{x})^2 \in M$. Then $\overline{x}$ is a critical point of h on $V(f_1, \dots, f_m)$ if and only if $h^*(\overline{x}) = 0$. And (see 2.2.11),

$$h^*(\overline{x}) = 0 \Leftrightarrow \nabla h(\overline{x}) \in \text{lin. span}(\nabla f_1(\overline{x}), \dots, \nabla f_m(\overline{x})).$$

We take $n+1$ points $P_0, \dots, P_n$ in $\mathbb{Z}^n$ such that the vectors $\overrightarrow{P_0P_1}, \dots, \overrightarrow{P_0P_n}$ are linearly independent over $\mathbb{K}$. For example, let us take $P_0 = \overline{0}$ and P_i to be the tuple with the i-th coordinate equal to 1 and the other coordinates equal to 0 (for $i = 1, \dots, n$). Now consider the following "distance " functions:

$$d_0(\overline{x}) = \sum_{j=1}^{n} x_j^2, \quad d_i(\overline{x}) = (x_i - 1)^2 + \sum_{j=1, j\neq i}^{n} x_j^2 \quad i = 1, \dots, n.$$

Clearly these functions belong to M.

For every $S \in \mathcal{B}(V(f_1, \ldots, f_m))$, for every $i = 0, \ldots, n$, consider the set $V_S(d_i^*) = S \cap V(d_i^*)$ of the critical points of the function d_i on S and let $\operatorname{bd}_S V_S(d_i^*) = V_S(d_i^*) \setminus \operatorname{int}_S(V_S(d_i^*))$ be the set of boundary points of $V_S(d_i^*)$ in S. Now define

$$G := \bigcup_{S \in \mathcal{B}(V(f_1,\ldots,f_m))} \bigcup_{i=0,\ldots,n} \operatorname{bd}_S V_S(d_i^*).$$

Step 1. We first observe that G is definable and $G \subseteq V(f_1, \ldots, f_m)$.

Step 2. Next, we note that for all $S \in \mathcal{B}(V(f_1, \ldots, f_m))$, for every $i = 0, \ldots, n$, the set S contains a point whose distance from P_i is minimal, i.e. $V_S(d_i^*)$ is nonempty. This follows from the fact that d_i increases on balls centered in P_i and of increasing radius, so Theorem 1.2.6 applies.

Step 3. Now we show that G meets every nonempty definable clopen of $V(f_1, \ldots, f_m)$ (in particular, G is not empty). Equivalently, we show that for all $S \in \mathcal{B}(V(f_1, \ldots, f_m)) \setminus \{\emptyset\}$, there exists $i \in \{0, \ldots, n\}$ such that the set $V_S(d_i^*)$ is not open in S. Suppose for a contradiction that this is not the case. Then for all $i = 0, \ldots, n$ the set $V_S(d_i^*)$, which is clearly closed and definable, in also open in $V(f_1, \ldots, f_m)$, and hence it belongs to $\mathcal{B}(V(f_1, \ldots, f_m))$. Now consider the boolean subalgebra $\mathcal{A}$ of $\mathcal{B}(V(f_1, \ldots, f_m))$ generated by $V_S(d_0^*), \ldots, V_S(d_n^*)$. Since $\mathcal{A}$ is finite, there is an atom, say, $C \in \mathcal{A}$. Let $C_i = C \cap V_S(d_i^*)$; by Step 1, C_i is nonempty for all $i = 0, \ldots, n$, and hence $C_i = C$. But this implies that $\emptyset \neq C \subseteq V(d_0^*, \ldots, d_n^*)$. But this is not possible, because the vectors $\nabla d_i(\overline{x})$ span $\mathbb{K}^n$ at all points $\overline{x}$. If $\overline{x} \in V(d_0^*, \ldots, d_n^*)$, then $\forall i = 0, \ldots, n$, $\nabla d_i(\overline{x}) \in \operatorname{lin.span}(\nabla f_1(\overline{x}), \ldots, \nabla f_m(\overline{x}))$, which is absurd.

Step 4. We now show that $\forall \overline{x} \in G \ \exists h \in M \quad \overline{x} \in V^{\text{reg}}(f_1, \ldots, f_m, h)$. Suppose for a contradiction that there exists $\overline{x} \in G$ such that it is not possible to cut transversally $V(f_1, \ldots, f_m)$ at $\overline{x}$ by any $h \in M$. Now arguing as in the last paragraph of the proof of Theorem 2.4.6, we show that every $h \in M$ must vanish on a suitable neighbourhood of $\overline{x}$ in $V(f_1, \ldots, f_m)$. But by definition of G, every point $\overline{x}$ of G is a boundary point of some $V_S(d_i^*)$, i.e.

$$\begin{array}{l} \forall \overline{x} \in G \quad \exists S \in \mathcal{B}(V(f_1, \ldots, f_m)) \quad \exists i \in \{0, \ldots, s\} \\ d_i^*(\overline{x}) = 0 \quad \wedge \ \forall r > 0 \exists \overline{y} \in S \cap B(\overline{x}, r) \ d_i^*(\overline{y}) \neq 0, \end{array} \tag{2.1}$$

and this leads to a contradiction. □

2.6.5. Theorem (Cutting transversally the clopen subsets of $V(f_1, \ldots, f_m)$). *Fix $n, m \in \mathbb{N}$, $m \leq n - 1$. Let $M \subseteq C^\infty(\mathbb{K}^n, \mathbb{K})$ be a Khovanskii ring and $f_1, \ldots, f_m \in M$ be such that $V(f_1, \ldots, f_m) = V^{\text{reg}}(f_1, \ldots, f_m)$, i.e. $V(f_1, \ldots, f_m)$ is a regular variety. Then there exists a definable set G such that:*

- *$\emptyset \neq G \subset V(f_1, \ldots, f_m)$;*

- *For every clopen definable subset S of $V(f_1,\dots,f_m)$, the intersection $S \cap G$ is not empty;*

- $\exists l \in \mathbb{N},\ \exists h_1,\dots,h_l \in M \quad G \subset V^{\mathrm{reg}}(f_1,\dots,f_m,h_1) \cup \dots \cup V^{\mathrm{reg}}(f_1,\dots,f_m,h_l)$.

Proof. By compactness, using an argument similar to the one used in the proof of 2.4.7. More precisely, let $\mathbb{F}$ be a $|\mathbb{K}|^+$-saturated elementary superstructure of $\mathbb{K}$ (see [32] for the existence of such an $\mathbb{F}$), so that $\mathbb{F}$ realizes all types over $\mathbb{K}$. Let $\widetilde{M}$ be the set of those definable functions $\tilde{g}$ such that $g \in M$ and $\tilde{g}$ is the interpretation of g in $\mathbb{F}$ (note that $\tilde{g}$ is still a C^∞ function). Then $\widetilde{M}$ is still a Khovanskii ring, hence Proposition 2.6.4 holds for $\widetilde{M}$-varieties. Consider the map $\widetilde{F} = (\widetilde{f}_1,\dots,\widetilde{f}_m)$ and the following set of formulas:

$$\Phi = \{\phi_{\widetilde{h}} := (\overline{x} \in G \wedge \overline{x} \notin V^{\mathrm{reg}}(\widetilde{F},\widetilde{h}))|\ h \in M\}.$$

If Φ were a consistent type, then it would be realized in $\mathbb{F}$. This means that there would exist $\overline{x} \in G$ such that for all $h \in M$, $\overline{x} \notin V^{\mathrm{reg}}(\widetilde{F},\widetilde{h})$, which is not possible by Proposition 2.6.4. Hence there exist $h_1,\dots,h_l \in M$ such that the conjunction $\phi_{\widetilde{h}_1} \wedge \dots \wedge \phi_{\widetilde{h}_l}$ is not satisfiable; in other words the following holds in $\mathbb{F}$:

$$\forall \overline{x}\ \overline{x} \in G \rightarrow \overline{x} \in V^{\mathrm{reg}}(\widetilde{F},\widetilde{h}_1) \cup \dots \cup V^{\mathrm{reg}}(\widetilde{F},\widetilde{h}_l).$$

Pulled back to $\mathbb{K}$, this proves the theorem. $\square$

2.6.6 Theorem (Finiteness of $\mathcal{B}(V(F))$). *Let $\{M_n|\ n \in \mathbb{N}\}$ be a collection of Khovanskii rings. Then, for all $n,m \in \mathbb{N}$ and $F \in (M_n)^m$, the boolean algebra $\mathcal{B}(V(F))$ is finite.*

Proof. By induction, using Propositions 2.6.5, 2.4.7 and Remark 2.6.3.

More precisely, we first prove by induction on $k = n - m$ that $\forall n \in \mathbb{N}$, $\mathcal{B}(V^{\mathrm{reg}}(F))$ is finite.

The case $k = 0$ follows from the fact that M_n is a Khovanskii ring. Next, suppose the statement true for $n - m < k$ and consider $F \in M_n^{n-k}$. If $V(F) = V^{\mathrm{reg}}(F)$, then by Theorem 2.6.5, there exist a definable set G and functions $h_1,\dots,h_l \in M_n$ such that

- $G \subset V(F)$;

- G meets every clopen subset of $V(F)$;

- $\exists h_1,\dots,h_l \in M \quad G \subset V^{\mathrm{reg}}(F,h_1) \cup \dots \cup V^{\mathrm{reg}}(F,h_l)$.

By inductive hypothesis, $\mathcal{B}(V^{\mathrm{reg}}(F,h_i))$ is finite, and hence so is $\mathcal{B}(V^{\mathrm{reg}}(F))$, assuming $V(F) = V^{\mathrm{reg}}(F)$.

If $V(F) \neq V^{\mathrm{reg}}(F)$, then, by remark 2.6.3, $V^{\mathrm{reg}}(F)$ is the projection of a finite union of regular varieties V_i still of dimension k, hence it follows from what we have just proved that $\mathcal{B}(V_i)$ is finite, and hence so is $\mathcal{B}(\pi(\bigcup V_i)) = \mathcal{B}(V^{\mathrm{reg}}(F))$.

Finally, if $V(F)$ is any variety, not necessarily regular, then by Theorem 2.4.7 it follows that $V(F)$ is a finite union of clopen subsets of regular sets, hence, by what we have just proved, $\mathcal{B}(V(F))$ is finite.

$\square$

2.6.7 Remark (Definably connected components)**.** Since the boolean algebra $\mathcal{B}(V(F))$ is finite, then there is an atom. If A is an atom, then it is clearly a maximal definably connected subset, i.e. a definably connected component. Hence we have proved that $V(F)$ has a finite number of definably connected components.

Chapter 3

Effective o-minimality

3.1 Introduction

The results of this chapter are due to A.Beraducci and myself, and appear in the paper [2].

We recall the following definition:

3.1.1 Definition. Let $\mathbb{K}$ be an expansion of an ordered field. Then $\mathbb{K}$ is *o-minimal* if every definable subset of the domain of $\mathbb{K}$ is a finite union of intervals and points.

We have already noticed, at the beginning of Chapter 1, that every o-minimal expansion of a field is a definably complete structure; we have also shown that the converse is not true, since for example $\mathbb{R}_{\sin}$ is a non o-minimal definably complete structure. In this chapter we try to answer the following question, which was anticipated at the beginning of last section: under which hypotheses is a definably complete structure o-minimal? The answer we find has the following form: under certain assumptions (which we will discuss below, but which are for example satisfied by $\mathbb{R}_{\exp}$), we can find a recursive scheme of axioms which, added to the axioms of definably complete structure, ensures the o-minimality of all models.

In [37] Wilkie proved, using the notion of Charbonnel closure introduced in [5], a general "theorem of the complement" which in particular implies that in order to establish the o-minimality of an expansion of $\mathbb{R}$ with C^∞ functions it suffices to prove uniform (in the parameters) bounds on the number of connected components of quantifier free definable sets.

Here we prove an effective version of Wilkie's theorem of the complement. In particular we prove that, given an expansion of $\mathbb{R}$ with finitely many C^∞ functions, if there are uniform and computable upper bounds on the number of connected components of quantifier free definable sets, then there are such uniform and computable bounds for all definable sets. In such a case the theory of the structure is effectively o-minimal: there is a recursively axiomatized subtheory such that all the models are o-minimal. The hypotheses of our theorem hold in the case of an expansion of $\mathbb{R}$ with Pfaffian functions by [18], so in particular we obtain a proof of the effective o-minimality of any expansion of $\mathbb{R}$ by finitely many Pfaffian functions.

The main result of this chapter (Theorem 3.8.4) applies to the sets definable in the language associated to an "o-minimal effective W-structure" (Definition 3.4.1) which is "effectively determined by its smooth functions" (Definition 3.5.1). This is an effective analogue of the setting of [37]. Since the definitions involved are rather technical, we state in this outline a particular case of the theorem which is easier to formulate. We then derive some corollaries.

3.1.2 Definition. For $X \subseteq \mathbb{R}^n$ let $cc(X)$ be the number of connected components of X and let $\gamma(X)$ be the least $n \in \mathbb{N}$ such that for every affine set $L \subseteq \mathbb{R}^n$ (i.e. a set defined by a system of linear equations over $\mathbb{R}$) we have $cc(X \cap L) \leq n$, with the convention that $\gamma(X) = \infty$ if n does not exist.

Clearly $cc(X) \leq \gamma(X)$. It is well known that a first order structure with domain $\mathbb{R}$ is o-minimal if and only if for every definable set in the structure one has $cc(X) < \infty$. If a structure is o-minimal it then follows that one actually has $\gamma(X) < \infty$. We can now state the particular case of Theorem 3.8.4:

3.1.3 Theorem. *Let $\mathcal{R}$ be an L-structure which is an expansion of $(\mathbb{R}; +, \cdot, 0, 1)$ by finitely many C^∞ functions. Assume that there is a recursive function Γ_0 which, given a quantifier free L-formula $\phi(\overline{x})$ computes a finite upper bound $\Gamma_0(\phi) \in \mathbb{N}$ on $\gamma(X)$, where $X \subseteq \mathbb{R}^n$ is the set defined by ϕ. Then there is a recursive function Γ which, given an arbitrary L-formula $\theta(\overline{x})$ computes a finite upper bound $\Gamma(\theta) \in \mathbb{N}$ on $\gamma(Y)$, where $Y \subseteq \mathbb{R}^n$ is the set defined by θ.*

The corresponding result, dropping the word "recursive", is due to Wilkie [37]. The formulas involved in the above theorem are without parameters, so it makes sense to speak of recursive functions taking such formulas as inputs (an L-formula is just a string of symbols from some finite alphabet). We have not attempted a complexity analysis, but it should be clear by the analysis of the proof that if Γ_0 is primitive recursive, then Γ can also be found primitive recursive. For technical reasons we did not include the order relation in the language. However the order can be defined as usual from $+, \cdot$ using existential quantifiers.

3.1.4 Remark. Theorem 3.1.3 refers to formulas without parameters. The following easy observation allows us to obtain bounds on γ also in the presence of parameters: if $X \subseteq \mathbb{R}^n$ is defined by a formula $\phi(\overline{x}, \overline{b})$ with n free variables $\overline{x}$ and k parameters $\overline{b} \in \mathbb{R}$, then $\gamma(X) \leq \gamma(Y)$, where $Y \subseteq \mathbb{R}^{n+k}$ is defined by the formula without parameters $\phi(\overline{x}, \overline{y})$.

Let $\mathbb{R}_{\mathrm{Pfaff}}$ be an expansion of $(\mathbb{R}, +, \cdot, 0, 1)$ by finitely many Pfaffian functions (for instance by the exponential function e^x). Then the hypothesis of Theorem 3.1.3 are verified by [18] (reasoning as in [37, Theorem 1.9]). We thus obtain:

3.1.5 Corollary. *If $X \subseteq \mathbb{R}^n$ is defined by a formula $\phi = \phi(x_1, \ldots, x_n)$ in the language of $\mathbb{R}_{\mathrm{Pfaff}}$, then $\gamma(X) < \Gamma(\phi)$, where Γ: Formulas $\to \mathbb{N}$ is a computable (even primitive recursive) function.*

3.1.6 Corollary. *Let T_{Pfaff} be the complete theory of $\mathbb{R}_{\mathrm{Pfaff}}$. There is a recursively axiomatized subtheory T_{omin} of T_{Pfaff} such that all the models of T_{omin} are o-minimal.*

Proof. A structure M is o-minimal if and only if every definable subset of M, possibly with parameters, is a finite union of open intervals and points. So it suffices to define T_{omin} as the theory which contains, for each formula $\phi(x_1, x_2, \ldots, x_n)$ in the language of T_{Pfaff}, an axiom stating that $\forall x_2, \ldots, x_n$ the set $\{x_1 \mid \phi(x_1, x_2, \ldots, x_n)\}$ is the union of at most $\Gamma(\phi)$ open intervals and points. $\square$

3.2 Outline of the proof of the main theorem

The proof of Theorem 3.8.4 and its consequences, as stated in the previous section, is based on suitable effective versions of the results in [37], and in particular of the cell decomposition theorem contained in that paper. We cannot go as far as to claim that there is an algorithm to perform the cell decomposition theorem, since, in the case of $\mathbb{R}_{\exp}$, this would be equivalent to the decidability of $T_{\exp}$: in fact a sentence φ in the language of $\mathbb{R}_{\exp}$ is true in $\mathbb{R}_{\exp}$ if and only if the subset of $\mathbb{R}$ defined by $(\varphi \wedge x = x)$ is non-empty (and one would expect that a reasonable notion of algorithmic cell decomposition should be able to tell if a set is empty). However we will see that, despite this obstacle, we can extract from [37] some "non-deterministic" or "multivalued" algorithms which are good enough for Theorem 3.8.4 and its corollaries. In order to carry out this program we begin by presenting the results in [37] in a form that suits our purposes. The main idea in [37] is to give a new characterization of the definable sets, under suitable assumptions. The new characterization is based on the notion of *Charbonnel closure*, introduced by Charbonnel in [5], which we now describe.

Let $\mathcal{S}_n$ be a collection of subsets of $\mathbb{R}^n$ and let $\mathcal{S} = \langle \mathcal{S}_n \mid n \in \mathbb{N} \rangle$. The definable sets in the structure $\mathcal{S}$ form the smallest collection of sets stable under the boolean operations (inside each $\mathbb{R}^n$) and the operation of taking the image of a set under a linear projection $\Pi_n^{n+k} \colon \mathbb{R}^{n+k} \to \mathbb{R}^n$ (projection onto the first n coordinates). Let $\mathrm{Def}_n(\mathcal{S}) \supseteq \mathcal{S}_n$ be the collection of all definable subsets of $\mathbb{R}^n$ in the sense just described. We call $\mathrm{Def}(\mathcal{S}) = \langle \mathrm{Def}_n(\mathcal{S}) \mid n \in \mathbb{N} \rangle$ the definable closure of $\mathcal{S}$.

The Charbonnel closure $\mathrm{Ch}(\mathcal{S}) = \langle \mathrm{Ch}_n(\mathcal{S}) \mid n \in \mathbb{N} \rangle$ of $\mathcal{S}$ is defined similarly, but instead of the boolean operation of taking the complement one has the operation of taking the topological closure. More precisely one considers the operations of binary unions, projections, and the operation sending a sequence $A, B_1, \ldots, B_k$ of subsets of $\mathbb{R}^n$ into $A \cap \overline{B_1} \cap \ldots \cap \overline{B_k}$, where $\overline{B_i}$ is the topological closure of B_i. We will work in this paper with an equivalent definition, where we replace the latter with the simpler operation of taking the topological closure $B \mapsto \overline{B}$ and we add a rather limited form of intersection with linear sets (see Definition 3.3.4 below).

Clearly $\mathrm{Def}_n(\mathcal{S}) \supseteq \mathrm{Ch}_n(\mathcal{S})$ (since the topological closure is a definable operation). In [37] Wilkie proves the following two results under suitable assumptions on $\mathcal{S}$. First, for every $X \in \mathrm{Ch}_n(\mathcal{S})$ we have $\gamma(X) < \infty$ (see Definition 3.1.2 for the definition of γ). This essentially amounts to proving that the operations in the definition of $\mathrm{Ch}(\mathcal{S})$ preserve the finiteness of γ, at least if they are applied to sets already in $\mathrm{Ch}(\mathcal{S})$. Second, and more difficult, under some additional "smoothness assumptions" on $\mathcal{S}$ it is shown that the complement of a set in $\mathrm{Ch}_n(\mathcal{S})$ is also in $\mathrm{Ch}_n(\mathcal{S})$. From this it clearly follows that the equality $\mathrm{Def}(\mathcal{S}) = \mathrm{Ch}(\mathcal{S})$ holds. The needed assumptions on $\mathcal{S}$ are verified if, for instance, $\mathcal{S}_n$ is the collection of all the exponential varieties included in $\mathbb{R}^n$. In this case the sets in $\bigcup_n \mathrm{Def}_n(\mathcal{S})$ coincide with the definable sets in the structure $\mathbb{R}_{\exp}$ and the o-minimality of $\mathbb{R}_{\exp}$ follows.

Our goal is to prove effective versions of these results. In order to do so it is technically convenient to weaken the assumptions on $\mathcal{S}$ (with respect to [37]), so as to allow for instance the possibility that $\mathcal{S}_n$ is the collection of all those exponential varieties included in $\mathbb{R}^n$ which are defined as the zero-sets of exponential polynomials with coefficients in $\mathbb{Z}$ (so in particular we do not require that all the semialgebraic subsets of $\mathbb{R}^n$ are in $\mathcal{S}_n$: actually we do not even require that all the singletons $\{a\}$ with $a \in \mathbb{R}$ are in $\mathcal{S}_1$). Assuming that $\mathcal{S}$ is an "effective W-structure" (Definition 3.4.1), the sets in $\mathrm{Ch}_n(\mathcal{S})$ can be naturally coded by "Ch-formulas" (Definition 3.4.3), which correspond to a subset of the first order formulas of the language associated to $\mathcal{S}$ (Definition 3.8.2). Roughly our Ch-formulas correspond to the "Charbonnel formulas" in [22], although our definition is different (recall that we work with different assumptions on $\mathcal{S}$ and with a different definition of $\mathrm{Ch}(\mathcal{S})$). If $\mathcal{S}_n$ consists of the exponential varieties included in $\mathbb{R}^n$ which are defined as the zero-sets of exponential polynomials with coefficients in $\mathbb{Z}$, then the Ch-formulas correspond to a subset of the first order formulas of $T_{\exp}$.

Our first result (Lemma 3.3.10, Theorem 3.4.4) is that if $A \subseteq \mathbb{R}^n$ is defined by a Ch-formula A, then $\gamma(A) < \Gamma(\mathsf{A})$, where $\Gamma\colon$ Ch-Formulas $\to \mathbb{N}$ is a computable function.

Using this fact we prove a "non-deterministic" effective version of Wilkie's theorem of the complement. More precisely we show (Theorem 3.8.1) that there is a recursive function which, given a Ch-formula for a set $A \in \mathrm{Ch}_n(\mathcal{S})$, returns a finite set of Ch-formulas, one of which defines the complement of A in $\mathbb{R}^n$, although we are not able to tell which one.

Granted this the results stated in the introduction follow easily. First we deduce that there is a recursive function which, given a first order formula ϕ (in the language associated to $\mathcal{S}$), returns a finite set of Ch-formulas, one of which defines the subset of $\mathbb{R}^n$ defined by ϕ. In other words we have an effective non-deterministic translation from first order formulas to Ch-formulas. This is the effective version of the result $\mathrm{Def}(\mathcal{S}) = \mathrm{Ch}(\mathcal{S})$. Finally we deduce Theorem 3.8.4, namely we obtain a recursive function which, given a first order formula $\phi(x_1, \dots, x_n)$ in the language associated to $\mathcal{S}$ returns an upper bound on $\gamma(A)$, where $A \subseteq \mathbb{R}^n$ is the set defined by ϕ (it suffices to take the maximum of the bounds for the Ch-formulas associated to ϕ).

So it remains to prove Theorem 3.8.1, the non-deterministic effective version of Wilkie's theorem of the complement. The latter depends on a preliminary (and difficult) "boundary theorem" asserting that for every closed set $X \in \mathrm{Ch}(\mathcal{S})$ there is a closed set with empty interior $Y \in \mathrm{Ch}(\mathcal{S})$ such that Y contains the boundary ∂X of X (a posteriori it will follow that ∂X itself is in $\mathrm{Ch}(\mathcal{S})$). By an analysis of Wilkie's proof it is not difficult to obtain an effective version of this result (Theorem 3.5.12), namely it can be shown that from a Ch-formula for X we can effectively find a Ch-formula for Y. Note that we are not claiming that Wilkie's <u>proof</u> is constructive. What we claim is only that Wilkie's <u>definition</u> of Y (implicit in the proof) is constructive, although the proof that Y has the desired properties may not be so.

Granted the boundary theorem, the complement theorem in [37] follows from a cell-decomposition argument. The latter is non-constructive because it is a proof by cases and the task of distinguishing the cases by a computable function seems hopeless (we have already seen that even telling if a set is empty or not is connected with the decidability of $T_{\exp}$). To prove an effective version we do not distinguish the cases. We simply try them all and in at least one case we will obtain the correct result.

To give an idea of how this works, let us prove the non-deterministic effective version of the complement theorem (Theorem 3.8.1) in the basic case of subsets of $\mathbb{R}$. Note that in [37] this case is obvious and does not even requires the boundary theorem. The effective version is instead nontrivial even in the basic case so it is worth sketching a proof (in the official proof we will give a more complicated argument which is more suitable for the generalization to $\mathbb{R}^n$). So let $A \subseteq \mathbb{R}$ be a set in $\mathrm{Ch}_1(\mathcal{S})$ which we assume to be closed for simplicity. Then $\gamma(A) < \infty$ and therefore A is a finite union of closed intervals. The complement A^c of A is then a finite union of open intervals. We want to prove first of all that A^c is in $\mathrm{Ch}_1(\mathcal{S})$ (this is trivial in [37] since in that paper all the semialgebraic sets are in $\mathcal{S}$) and also that, given a Ch-formula for A, we can effectively find a finite set of Ch-formulas one of which defines A^c. The algorithm is the following. First, using the boundary theorem we find a Ch-formula for a set $B \in \mathrm{Ch}_1(\mathcal{S})$ with empty interior containing the boundary ∂A of A. Note that B is then a finite set of cardinality $\gamma(B)$. Since Ch-formulas do not have negations, it is not clear a priori whether one can effectively find, knowing the Ch-formula for B, a Ch-definition of the least element of the finite set B, or of the other elements of B. This however would be possible if we knew the cardinality of B. So we proceed as follows. First, using the Ch-formula for B, we compute an upper bound N on $\gamma(B)$. Then we choose non-deterministically a number $k \leq N$. At least one choice will give us the cardinality of B. Now given k we consider, for each $i \leq k \leq N$, the set $P_i^k = \{x \in \mathbb{R} \mid \exists x_1 \dots x_k \in B(x_1 < \dots < x_k \wedge x = x_i)\}$, which can be defined by a Ch-formula as a projections on $\mathbb{R}$ of a Ch-definable set lying in $\mathbb{R}^{k+1}$. If k was the cardinality of B, as we temporarily assume, then these sets are singletons and B is the union of these singletons. Moreover the boundary of A is the union of a subset of these singletons. We now guess non-deterministically which of the above singletons and which of the open intervals determined by the such singletons are disjoint from A, and we take their union. This is the complement of A. If we were unlucky and k was not the cardinality of B, we can still make sense of the rest of the algorithm (e.g. if $P, Q \subseteq \mathbb{R}$ are not singletons we can still define the pseudo-interval $(P, Q) := \{x \mid \exists y \in P \exists z \in Q(y < x < z)\}$). At least one of the non-deterministic choices will lead to a Ch-formula for A^c.

In the general case (in $\mathbb{R}^n$) the proof of Theorem 3.8.1 will require a rather complex effective non-deterministic version of Wilkie's cell decomposition theorem. At a crucial point of the cell decomposition theorem we must define a certain number of functions (the functions which bound the cells) where the i-th functions picks the i-th point of a certain finite set $A \in \mathrm{Ch}_1(\mathcal{S})$. The problem is that we do not know the cardinality of A, but we can compute an upper bound on it (since we can compute an upper bound on $\gamma(A)$ given a Ch-formula for A). We have thus only finitely many possibilities and we can non-deterministically guess the exact cardinality and proceed with the construction.

We remark on a difference between our approach and Wilkie's in the structure of the induction. We perform induction on the notion of "rank" introduced in Definition 3.3.5. Moreover we make the assumption (not present in [37]) that all sets in $\mathcal{S}$ are closed (other differences on $\mathcal{S}$ and $\mathrm{Ch}(\mathcal{S})$ have already been explained). This assumption is inessential and can be dropped (assuming our EDSF condition, Definition 3.5.1), however since this would not produce any essential gain of generality in the main result, we decided to keep the assumption to simplify some arguments.

3.3 W-structures and Charbonnel closure

The following definition of W-structure is a modification of the notion of weak-structure in [37]. The difference is that we do not require that all the semi-algebraic sets are in the structure.

3.3.1 Definition. Let $\mathcal{S} = \langle \mathcal{S}_n : n \in \mathbb{N}^+ \rangle$, where $\mathcal{S}_n$ is a collection of subsets of $\mathbb{R}^n$. We say that $\mathcal{S}$ is a *W-structure* if for all $n \in \mathbb{N}$,

- W(pol): $\mathcal{S}_n$ contains every subset of $\mathbb{R}^n$ defined as the zero-set of a system of finitely many polynomials with coefficients in $\mathbb{Z}$;
- W(perm): if $A \in \mathcal{S}_n$, then $\Sigma[A] \in \mathcal{S}_n$, where $\Sigma \colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear bijection induced by a permutation of the variables;
- W($\cap$): if $A, B \in \mathcal{S}_n$, then $A \cap B \in \mathcal{S}_n$;
- W($\times$): if $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_m$, then $A \times B \in \mathcal{S}_{n+m}$.

3.3.2 Definition. We say that a W-structure $\mathcal{S}$ is *closed* if for every n and $A \in \mathcal{S}_n$, A is a closed subset of $\mathbb{R}^n$; $\mathcal{S}$ is *semi-closed* if for every n and $A \in \mathcal{S}_n$, A can be obtained as the projection onto the first n coordinates of some closed set $B \in \mathcal{S}_{n+k}$, for some suitable k . We say that a W-structure is *o-minimal* if for every n and $A \in \mathcal{S}_n$ we have $\gamma(A) < \infty$.

3.3.3 Example. Let $\mathcal{S}_n$ be the collection of all zero- sets $X \subseteq \mathbb{R}^n$ of polynomials with coefficients in $\mathbb{Z}$, then $\mathcal{S} = \langle \mathcal{S}_n \mid n \in \mathbb{N} \rangle$ is a W-structure, indeed the minimal one.

The following definition is different from the corresponding one in [37] but equivalent to it.

3.3.4 Definition. Let $\mathcal{S}$ be an o-minimal W-structure. The *Charbonnel closure* $\widetilde{\mathcal{S}} = \mathrm{Ch}(\mathcal{S}) = \langle \widetilde{\mathcal{S}}_n : n \in \mathbb{N}^+ \rangle$ of $\mathcal{S}$ is defined as follows:

Ch(base): $\widetilde{\mathcal{S}}_n$ is a collection of subsets of $\mathbb{R}^n$ and $\mathcal{S}_n \subseteq \widetilde{\mathcal{S}}_n$.

Ch($\cup$): If $A, B \in \widetilde{\mathcal{S}}_n$, then $A \cup B \in \widetilde{\mathcal{S}}_n$.

Ch($\cap_\ell$): If $A \in \widetilde{\mathcal{S}}_n$ and $L \subseteq \mathbb{R}^n$ is the zero-set of a system of linear polynomials with coefficients in $\mathbb{Z}$, then $A \cap L \in \widetilde{\mathcal{S}}_n$ (the "ℓ" in the label stands for "linear"). We call such an L a $\mathbb{Z}$-*affine set*.

Ch(π): if $A \in \widetilde{\mathcal{S}}_{n+k}$ and $\Pi^{n+k}_n : \mathbb{R}^{n+k} \to \mathbb{R}^n$ is the projection onto the first n coordinates, then $\Pi^{n+k}_n[A] \in \widetilde{\mathcal{S}}_n$;

Ch($\overline{x}$): if $A \in \widetilde{\mathcal{S}}_n$, then $\overline{A} \in \widetilde{\mathcal{S}}_n$, where $\overline{A}$ is the topological closure of A.

$\bigcup_n \widetilde{\mathcal{S}}_n$ is minimal with these properties.

Our aim is to prove that if $\mathcal{S}$ is a closed o-minimal W-structure, then $\widetilde{\mathcal{S}}$ is a semi-closed o-minimal W-structure. The same conclusion would be valid if $\mathcal{S}$ were only assumed to be semi-closed, but we do not need this fact.

Sometimes we write $A \in \widetilde{\mathcal{S}}$ instead of $A \in \widetilde{\mathcal{S}}_n$ if n is implicit or irrelevant. Similar conventions apply to $\mathcal{S}$.

3.3.5 Definition. A *Ch-description* of $A \in \widetilde{\mathcal{S}}$ is an expression which illustrates one of the possible ways to obtain A from sets in $\mathcal{S}$ using the Ch-operations Ch($\cup$), Ch($\cap_\ell$), Ch(π) and Ch($\overline{x}$). More precisely, we fix a set Σ of symbols (called *labels*) of the same cardinality as $\bigcup_n \mathcal{S}_n$ and a surjection from Σ to $\bigcup_n \mathcal{S}_n$, so that every set $A \in \mathcal{S}$ has a label $\mathsf{A} \in \Sigma$ (possibly not unique). If A is a label for the set $A \in \mathcal{S}$, then A is a Ch-description of A. Inductively, if B, C are Ch-descriptions of the sets $B, C \in \widetilde{\mathcal{S}}$, and if L is a label for a $\mathbb{Z}$-affine set L, then the strings of symbols $(\mathsf{B} \cup \mathsf{C}), (\mathsf{B} \cap \mathsf{L}), \overline{\mathsf{B}}$, and $\Pi^{n+k}_n \mathsf{B}$ (the last one includes the strings needed to define the integers n, k) are Ch-descriptions of the sets $(B \cup C), (B \cap L), \overline{B}$ and $\Pi^{n+k}_n B$ respectively. So for instance the expressions $(\mathsf{B} \cup \mathsf{C}) \cap \mathsf{L}$ and $(\mathsf{B} \cap \mathsf{L}) \cup (\mathsf{C} \cap \mathsf{L})$ (where we have omitted the external parenthesis), are two different Ch-descriptions for the same set of $\widetilde{\mathcal{S}}$.

The *rank* ρ of a Ch-description of A is defined as follows:

- If $\mathsf{A} \in \Sigma$ is a label, then $\rho(\mathsf{A}) = 0$;
- $\rho(\mathsf{B} \cup \mathsf{C}) = 1 + \max\{\rho(\mathsf{B}), \rho(\mathsf{C})\}$;
- $\rho(\mathsf{B} \cap \mathsf{L}) = 1 + \rho(\mathsf{B})$;

- $\rho(\Pi_n^{n+k}\mathsf{B}) = 1 + \rho(\mathsf{B})$;
- $\rho(\overline{\mathsf{B}}) = 4 + \rho(\mathsf{B})$.

Finally we define the rank $\rho(A)$ of a set $A \in \widetilde{\mathcal{S}}$ as the least possible rank of a Ch-description of A.

Thus the sets of rank zero are exactly the sets in $\mathcal{S}$, but there are Ch-descriptions of sets in $\mathcal{S}$ of arbitrarily high rank. Note that the equalities in the definition of the rank of a Ch-description become inequalities if we refer to the sets rather than their descriptions. For instance $\rho(A \cup B) \leq 1 + \max\{\rho(A), \rho(B)\}$ and the inequality can be strict since the *set* $A \cup B$ can admit simpler Ch-descriptions besides the one which presents it as the union of A and B. The reason why we need to let the operation $\mathrm{Ch}(\overline{x})$ raise the rank so much will be clear in the proof of Lemma 3.3.10.

3.3.6 Remark. Since $\mathcal{S}_n \subseteq \widetilde{\mathcal{S}}_n$, $\widetilde{\mathcal{S}}$ satisfies W(pol). Notice also that, since linear bijections (induced by a permutation of the variables) commute with union, intersection, projection and closure, $\widetilde{\mathcal{S}}$ satisfies W(perm). Moreover, by an application of W(perm) the rank of a Ch-description does not increase.

Given a closed o-minimal W-structure $\mathcal{S}$, to prove that $\widetilde{\mathcal{S}}$ is a W-structure it remains to show that it verifies W($\times$) and W($\cap$). This will be done by induction on the rank.

3.3.7 Lemma. *If $X \in \widetilde{\mathcal{S}}_m$ and $Y \in \widetilde{\mathcal{S}}_n$, then $X \times Y \in \widetilde{\mathcal{S}}_{m+n}$. Moreover $\rho(X \times Y) \leq \rho(X) + \rho(Y)$.*

Proof. We prove by induction on $\rho(\mathsf{X}) + \rho(\mathsf{Y})$ the following stronger result: if X, Y are Ch-descriptions of X, Y, then there is a Ch-description $\mathsf{X} \times \mathsf{Y}$ of $X \times Y$ such that $\rho(\mathsf{X} \times \mathsf{Y}) = \rho(\mathsf{X}) + \rho(\mathsf{Y})$. We use the following facts:

- $\mathcal{S}$ is closed under $\times$. This handles the case $\rho(\mathsf{X}) + \rho(\mathsf{Y}) = 0$.
- $$(A \cup B) \times Z = (A \times Z) \cup (B \times Z).$$

  This settles the case when one of the two Ch-descriptions X, Y is obtained from descriptions of smaller rank by the operation Ch($\cup$), say $\mathsf{X} = \mathsf{A} \cup \mathsf{B}$ and $\mathsf{Y} = \mathsf{Z}$ (the symmetric case is handled, both here and below, by permuting the variables). In fact, $\rho(\mathsf{A}) + \rho(\mathsf{Z})$ and $\rho(\mathsf{B}) + \rho(\mathsf{Z})$ are strictly smaller than $\rho(\mathsf{A} \cup \mathsf{B}) + \rho(\mathsf{Z})$, hence by induction $(A \times Z)$ and $(B \times Z)$ are in $\widetilde{\mathcal{S}}$ and have Ch-descriptions of the prescribed rank. An application of Ch($\cup$) puts $(A \times Z) \cup (B \times Z)$ in $\widetilde{\mathcal{S}}$. The correct evaluation of the ranks follows from an easy computation: $\rho(\mathsf{X} \times \mathsf{Y}) =$

$1 + \max\{\rho(\mathsf{A}) + \rho(\mathsf{Y}), \rho(\mathsf{B}) + \rho(\mathsf{Y})\} = 1 + \max\{\rho(\mathsf{A}), \rho(\mathsf{B})\} + \rho(\mathsf{Y}) = \rho(\mathsf{X}) + \rho(\mathsf{Y})$.

- $$(A \cap L) \times Z = (A \times Z) \cap (L \times \mathbb{R}^n),$$
  where $Z \subseteq \mathbb{R}^n$. This handles the case when one of the two descriptions is obtained from a description of smaller rank by the operation $\mathrm{Ch}(\cap_\ell)$. It is important to note that, if L is $\mathbb{Z}$-affine, so is $L \times \mathbb{R}^n$.

- $$Z \times \Pi_n^{n+k} A = \Pi_{m+n}^{m+n+k}(Z \times A),$$
  where $Z \subseteq \mathbb{R}^m$. This handles the case when one of the two descriptions is obtained from a description of smaller rank by the operation $\mathrm{Ch}(\pi)$.

- It remains to show that if $A \in \widetilde{\mathcal{S}}_m$ and $Z \in \widetilde{\mathcal{S}}_n$, then $\overline{A} \times Z \in \widetilde{\mathcal{S}}_{m+n}$. If Z has a description Z obtained from descriptions of smaller rank by one of the operations considered above, then we are in one of the preceding cases. In the remaining cases Z is either a description for a set in $\mathcal{S}$, or for a set of the form $\overline{B}$. In any case Z is a closed set, so we can write
  $$\overline{A} \times Z = \overline{A \times Z}.$$
  Note that $\rho(\mathsf{A}) + \rho(\mathsf{Z}) < \rho(\overline{\mathsf{A}}) + \rho(\mathsf{Z})$, hence by induction $A \times Z$ is in $\widetilde{\mathcal{S}}$ and we conclude by an application of $\mathrm{Ch}(\overline{x})$.

□

3.3.8 Lemma. *If $A \in \widetilde{\mathcal{S}}_n$ and $B \in \widetilde{\mathcal{S}}_n$ then $A \cap B \in \widetilde{\mathcal{S}}_n$. Moreover $\rho(A \cap B) \leq 2 + \rho(A) + \rho(B)$.*

Proof. $A \cap B = \Pi_n^{2n}[(A \times B) \cap \Delta]$ where $\Delta \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is the diagonal $\{(\overline{x}, \overline{x}) \mid \overline{x} \in \mathbb{R}^n\}$. The estimate on the rank follows from Lemma 3.3.7. □

3.3.9 Lemma. *$\widetilde{\mathcal{S}}$ is semi-closed, i.e. if $A \in \widetilde{\mathcal{S}}_n$, then there exist $k \in \mathbb{N}$ and a closed set $B \in \widetilde{\mathcal{S}}_{n+k}$ such that $A = \Pi_n^{n+k}[B]$.*

Proof. By induction on the rank of a Ch-description of A, using the following facts:

- If $A \in \mathcal{S}$ or A is obtained by an application of $\mathrm{Ch}(\overline{x})$, there is nothing to prove, since A is already closed.

- If $X = \Pi_n^{n+k}[B]$ and $Y = \Pi_n^{n+h}[C]$, then $X \cup Y = \Pi_n^{n+k+h}[(B \times \mathbb{R}^h) \cup (C \times \mathbb{R}^k)]$.

  This handles the case when $A = X \cup Y$ is obtained by an application of $\mathrm{Ch}(\cup)$.

- If $X = \Pi_n^{n+k}[B]$ and L is $\mathbb{Z}$-affine, then $X \cap L = \Pi_n^{n+k}[B \cap (L \times \mathbb{R}^k)]$.

  This handles the case when $A = X \cap L$ is obtained by an application of $\mathrm{Ch}(\cap_\ell)$.

- $\Pi_n^{n+h} \circ \Pi_{n+h}^{n+h+k}[B] = \Pi_n^{n+h+k}[B]$.

  This handles the case when A is obtained by an application of $\mathrm{Ch}(\pi)$.

□

The fact that $\widetilde{\mathcal{S}}$ is semi-closed will be useful in Section 3.8 to prove Theorem 3.8.1. Let us prove that $\widetilde{\mathcal{S}}$ is o-minimal.

3.3.10 Lemma. *If $A \in \widetilde{\mathcal{S}}_n$, then $\gamma(A) < \infty$.*

Proof. By induction on the rank of a Ch-description of A, using the following facts:

- $\gamma(B \cup C) \leq \gamma(B) + \gamma(C)$.
- If L is $\mathbb{Z}$-affine, $\gamma(B \cap L) \leq \gamma(B)$.
- $\gamma(\Pi_n^{n+k} B) \leq \gamma(B)$.
- $\gamma(\overline{B}) \leq \gamma((B \times \mathbb{R}^{m+2}) \cap E)$, where $m = n^2 + n$, E is the semi-algebraic set $\{(\overline{x}, \overline{y}, R, \varepsilon) \in \mathbb{R}^{n+m+2} \colon |p(\overline{x}, \overline{y})| < \varepsilon^2 \wedge \sum_{i=1}^n x_i^2 < R^2\}$, and p is a polynomial with coefficients in $\mathbb{Z}$ with the property that every subset of $\mathbb{R}^n$ defined by a system of linear polynomials over $\mathbb{R}$ is of the form $\{\overline{x} \mid p(\overline{x}, \overline{y}) = 0\}$ for a suitable $\overline{y}$.

The existence of p and the proof that $\gamma(\overline{B}) \leq \gamma((B \times \mathbb{R}^{m+2}) \cap E)$ is in [22, Claim 1.9]. Since E is semi-algebraic, it is the projection of an algebraic set, and moreover in our case it is the projection of the zero-set of a polynomial with coefficients in $\mathbb{Z}$. It thus follows that E is a set in $\widetilde{\mathcal{S}}$ of rank at most 1. So by Lemma 3.3.7 and Lemma 3.3.8, the rank of the Ch-description $(\mathsf{B} \times \mathsf{R}^{m+2}) \cap \mathsf{E}$ of the set $(B \times \mathbb{R}^{m+2}) \cap E$ is strictly smaller than $\rho(\overline{\mathsf{B}})$. It is now clear how to complete the proof by induction. □

We have thus proved:

3.3.11 Theorem. *If $\mathcal{S}$ is a closed o-minimal W-structure, then its Charbonnel closure $\widetilde{\mathcal{S}}$ is a semi-closed o-minimal W-structure.*

3.4 Effective W-structures

Let $\widetilde{\mathcal{S}}$ be the Charbonnel closure of a closed o-minimal W-structure $\mathcal{S}$. We have seen that each set $A \in \widetilde{\mathcal{S}}_n$ admits a Ch-description which shows how to obtain it from sets in $\mathcal{S}$. If we now assume that each set in $\mathcal{S}$ admits a description as a string of symbols from a finite alphabet (this implies in particular that each $\mathcal{S}_n$ is countable), then a Ch-description of a set $A \in \widetilde{\mathcal{S}}$ becomes itself a string of symbols from a finite alphabet, and it makes sense to ask whether an upper bound on $\gamma(A)$ can be effectively found from the description of A. We will see that the answer is positive if we make some natural assumptions on how the sets in the $\mathcal{S}$ are described.

3.4.1 Definition. Let $\mathcal{S}$ be a W-structure such that each $\mathcal{S}_n$ is countable. A *coding* of $\mathcal{S}$ is a surjective map $\mathcal{I}\colon \mathsf{Expr} \to \bigcup_n \mathcal{S}_n$, where Expr is a recursive set of strings of symbols from some finite alphabet. If $\mathcal{I}(\mathsf{A}) = A \in \mathcal{S}_n$, we say that A is a *W-formula* for A. We say that $(\mathcal{S}, \mathcal{I})$ is an *effective W-structure* if the following conditions hold:

EW(sort): There is a recursive function which, given a W-formula for $A \in \bigcup_n \mathcal{S}_n$, computes the unique integer n (called the *sort* of A) such that $A \in \mathcal{S}_n$.

EW(pol): There is a recursive function which, given the coefficients of a system of polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$, compute a W-formula for the zero-set of the system.

EW(perm) There is a recursive function which, given a W-formula for $A \in \mathcal{S}_n$ and a permutation σ of $\{1, \ldots, n\}$, computes a W-formula for the set $\Sigma[A]$, where $\Sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ is the linear bijection induced by the permutation σ on the coordinates.

EW($\cap$): There is a recursive function which, given W-formulas for the sets $A, B \in \mathcal{S}_n$, computes a W-formula for the set $A \cap B$.

EW($\times$): There is a recursive function which, given W-formulas for $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_m$ computes a W-formula for the set $A \times B$.

An effective *o-minimal* W-structure satisfies furthermore:

EW(o-min): There is a recursive function such that, given a W-formula for $A \in \mathcal{S}_n$, computes an upper bound for $\gamma(A)$.

3.4.2 Example. An example of an effective o-minimal W-structure $\mathcal{S}$ is the following: let $\mathcal{S}_n$ be the collection of all the subsets X of $\mathbb{R}^n$ such that X is the zero-set of a system of exponential polynomials with coefficients in $\mathbb{Z}$. A W-formula for X is any of the systems defining it. By the results of [18] this coding turns $\mathcal{S}$ into an effective o-minimal W-structure.

Our aim is to show that the Charbonnel closure of a closed effective o-minimal W-structure is an effective o-minimal W-structure with respect to an induced coding which we now describe.

3.4.3 Definition. Let $(\mathcal{S}, \mathcal{I})$ be an effective W-structure. The notion of *Ch-formula* for a set in $\widetilde{\mathcal{S}}$ is defined exactly as the notion of Ch-description (Definition 3.3.5) with the further requirement that the sets of $\mathcal{S}$ are labeled by their W-formulas (so a Ch-formula is a string of symbols from a finite alphabet). So for instance if B is a W-formula for $B \in \mathcal{S}_n$ and C is a W-formula for $C \in \mathcal{S}_n$, then B and C are Ch-formulas for B and C respectively, and the expression $(\mathsf{B} \cup \mathsf{C})$ is a Ch-formula for the set $B \cup C \in \widetilde{\mathcal{S}}_n$.

We define a surjective map $\widetilde{\mathcal{I}}$ from the set of all Ch-formulas to $\bigcup_n \widetilde{\mathcal{S}}_n$ as follows: $\widetilde{\mathcal{I}}(\mathsf{A}) = A$ if A is a Ch-formula for the set A. We call $\widetilde{\mathcal{I}}$ the coding induced by $\mathcal{I}$.

We define the notion of rank of a Ch-formula exactly as the rank of a Ch-description and we use the same notation.

3.4.4 Theorem. *If $(\mathcal{S}, \mathcal{I})$ is a closed effective o-minimal W-structure, then $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{I}})$ is an effective o-minimal W-structure which is semi-closed.*

Proof. It suffices to follow the proof of Theorem 3.3.11 and notice that one can extract from it the additional information required: for instance the proof of Lemma 3.3.7 actually shows that there is a recursive function which, given Ch-formulas for $X \in \widetilde{\mathcal{S}}_n$ and $Y \in \widetilde{\mathcal{S}}_m$ yields a Ch-formula for $X \times Y$; the proof of Lemma 3.3.10 gives a recursive function which, given a Ch-formula for $A \in \widetilde{\mathcal{S}}_n$, computes an upper bound for $\gamma(A)$. $\square$

3.4.5 Remark. Following the proof of Lemma 3.3.9 it can also be shown that $\widetilde{\mathcal{S}}$ is *effectively semi-closed*, i.e. given a Ch-formula for $A \in \widetilde{\mathcal{S}}_n$, we can effectively find $k \in \mathbb{N}$ and the Ch-formula for a closed set $B \in \widetilde{\mathcal{S}}_{n+k}$ such that $A = \Pi_n^{n+k}[B]$.

3.5 Smooth approximation of the boundary

The results of this section correspond to the ones in [37, Section 3]. We include the proofs because we work with a slightly different definition of W-structure (see Remark 3.5.8). Moreover we find it convenient to give a definition of approximation (our Definition 3.5.4, corresponding to [37, Def.

3.2]) and a proof of the approximation theorem (Theorem 3.5.11) which does not make explicit use of Wilkie's notion of "moduli". We replace the moduli by a systematic use of the quantifier "for all sufficiently small" (the moduli are essentially the Skolem functions associated to the quantifiers).

Let $\mathcal{S}$ be a closed o-minimal W-structure.

3.5.1 Definition.

- We say that $\mathcal{S}$ is *determined by its smooth functions* (DSF) if, given a set $A \in \mathcal{S}_n$, there exist $k \in \mathbb{N}$ and a C^∞ function $f_A : \mathbb{R}^{n+k} \to \mathbb{R}$ whose graph lies in $\mathcal{S}$, such that A is the projection onto the first n coordinates of the zero-set of f_A.
- Moreover, $\mathcal{S}$ is *effectively determined by its smooth functions* (EDSF) if $\mathcal{S}$ is an effective W-structure and there is an algorithm that, given a Ch-formula A for A, yields k and a Ch-formula for the graph of f_A.

3.5.2 Example. Let $\mathcal{S}_n$ be the collection of all zero-sets $X \subseteq \mathbb{R}^n$ of exponential polynomials with coefficients in $\mathbb{Z}$. Then $\mathcal{S} = \langle \mathcal{S}_n \mid n \in \mathbb{N} \rangle$ is an effective W-structure, which is EDSF. More generally, let $\mathcal{S}_n$ be the collection of all quantifier free definable sets of the structure $\mathcal{R}$, where $\mathcal{R}$ is an expansion of $(\mathbb{R}, +, \cdot)$ with finitely many C^∞ functions satisfying a Khovanskii- type result, namely for which there are recursive bounds on the number of connected components (actually on γ) of quantifier free definable sets. Then $\mathcal{S} = \langle \mathcal{S}_n \mid n \in \mathbb{N} \rangle$ is an effective W-structure, which is EDSF (we argue as in [37, Theorem 1.9] eliminating negations and compositions by introducing existential quantifiers).

Recall that the aim is to prove that the Charbonnel closure of $\mathcal{S}$ coincides with the definable closure of $\mathcal{S}$. Since $\widetilde{\mathcal{S}}$ is closed under finite unions and projections, it remains to show that $\widetilde{\mathcal{S}}$ is closed under complementation (Theorem 3.6.11). We still have not proved that if $A \in \widetilde{\mathcal{S}}$ then the boundary $\partial A = \overline{A} \setminus \mathrm{int}(A)$ is in $\widetilde{\mathcal{S}}$. Anyway, by assuming the DSF condition we are able to confine the boundary of a closed set $A \in \widetilde{\mathcal{S}}_n$, into a closed set $B \in \widetilde{\mathcal{S}}_n$ with empty interior, and this will suffice to prove the stability of $\widetilde{\mathcal{S}}$ under complementation (this will be clear in section 6). The set B will be obtained as the projection of a sort of "limit of smooth manifolds", by a procedure described in [37]. Moreover, if $\mathcal{S}$ is EDSF, the Ch-description of B can be effectively found from a Ch-description of A. The main difficulty lies in the attempt to confine the boundary of the projection of a set (and this is the reason why we need a *smooth* approximation). For this we make use of Lemma 3.9.6, which is a variant of the fact that, for a smooth function f having zero as a regular value with compact preimage, the boundary in $\mathbb{R}^n$ of a set of the form $\{\overline{x} \mid \mathbb{R}^n : \exists x_{n+1}(f(\overline{x}, x_{n+1}) = 0)\}$ is contained in the set with empty interior $\{\overline{x} \mid \mathbb{R}^n : \exists x_{n+1}(f(\overline{x}, x_{n+1}) = 0 \wedge \frac{\partial f}{\partial x_{n+1}}(\overline{x}, x_{n+1}) = 0)\}$.

To give the precise notion of limit, we need some definitions and lemmas.

3.5.3 Definition. Let $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$.

- Given $A \subseteq \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}_+$, define the *$\varepsilon$-neighborhood* A^ε of A as the set $\{x \in \mathbb{R}^n \mid \exists y \in A\, |x - y| < \varepsilon\}$.

- The *Hausdorff distance* $d(A, B)$ between two subsets A, B of $\mathbb{R}^n$ is the infimum of all the $\varepsilon \in \mathbb{R}_+$ such that the ε-neighborhood of each set contains the other.

- (The quantifier "for all sufficiently small") We write $\forall^s \varepsilon \phi$ as a shorthand for $\exists \mu \forall \varepsilon < \mu \phi$, where μ, ε are always assumed to range in $\mathbb{R}_+$. These quantifiers can be iterated: so $\forall^s \varepsilon_1 \forall^s \varepsilon_2 \phi$ abbreviates $\exists \mu_1 (\forall \varepsilon_1 < \mu_1) \exists \mu_2 (\forall \varepsilon_2 < \mu_2) \phi$, which is not the same as $\forall^s \varepsilon_2 \forall^s \varepsilon_1 \phi$. The expression $\forall^s \varepsilon_1, \ \ldots, \forall^s \varepsilon_k \phi$ can be read as: ϕ holds for all sufficiently small $\varepsilon_1, \ldots, \varepsilon_k$ provided each ε_i with $i > 1$ is also sufficiently small with respect to the preceding ones.

- (Sections) Given $S \subseteq \mathbb{R}^n \times \mathbb{R}^k_+$ and given $\varepsilon_1, \ldots, \varepsilon_k \in \mathbb{R}^+$, we define $S_{\varepsilon_1, \ldots, \varepsilon_k}$ as the set $\{x \in \mathbb{R}^n \mid (x, \varepsilon_1, \ldots, \varepsilon_k) \in S\}$.

The Hausdorff distance is a metric if we restrict to compact subsets of $\mathbb{R}^n$. In this case $\lim_{t \to 0} A_t = B$ if $\forall^s \varepsilon \forall^s t (B \subseteq A_t^\varepsilon \wedge A_t \subseteq B^\varepsilon)$. This is equivalent to $\forall^s \varepsilon \forall^s t (B \subseteq A_t^\varepsilon) \wedge \forall^s \varepsilon \forall^s t (A_t \subseteq B^\varepsilon)$.

3.5.4 Definition. ([37, Def. 3.2]) Let $A \subseteq \mathbb{R}^n, S \subseteq \mathbb{R}^n \times \mathbb{R}^k_+$.

1. S approximates A from below if

$$\forall^s \varepsilon_0 \forall^s \varepsilon_1 \ldots \forall^s \varepsilon_k (S_{\varepsilon_1, \ldots, \varepsilon_k} \subseteq A^{\varepsilon_0}).$$

2. S approximates A from above on bounded sets if

$$\forall^s \varepsilon_0 \forall^s \varepsilon_1 \ldots \forall^s \varepsilon_k (A \cap B(0, 1/\varepsilon_0) \subseteq S^{\varepsilon_0}_{\varepsilon_1, \ldots, \varepsilon_k})$$

   where $B(0, 1/\varepsilon_0) \subseteq \mathbb{R}^n$ is the compact ball of radius $1/\varepsilon_0$ centered at the origin.

Note that if A is bounded, we can omit in the above definition the intersection with the compact ball, and we recover in the special case $k = 1$ the limit in the Hausdorff distance.

3.5.5 Definition. Let $M(\mathcal{S}) = \bigcup_n M_n(\mathcal{S})$, where $M_n(\mathcal{S})$ is the smallest ring of functions from $\mathbb{R}^n$ to $\mathbb{R}$ closed under partial differentiation and containing:

- all polynomials $p \in \mathbb{Z}[x_1, \ldots, x_n]$;
- all functions f_A, for $A \in \mathcal{S}$, which provide the DSF condition for $\mathcal{S}$ (see Definition 3.5.1);
- the functions $(x_1, \ldots, x_n) \mapsto (1 + x_{i_1}^2 + \ldots + x_{i_s}^2)^{-1}$, with $s \leq n$ and $1 \leq i_1 \ldots i_s \leq n$.

Note that every function in $M(\mathcal{S})$ is C^∞ and we have $M(\mathcal{S}) \subseteq \widetilde{\mathcal{S}}$, in the sense that if $f \in M_n(\mathcal{S})$, then the graph of f is in $\widetilde{\mathcal{S}}_{n+1}$. In fact in [22, Lemma 4.11] it is proved that if $f \in \widetilde{\mathcal{S}}_{n+1}$ is a C^1 function, then all partial derivatives $\partial f/\partial x_i$ belong to $\widetilde{\mathcal{S}}_{n+1}$. The idea is to simulate the limit of the differential quotient using sections and the topological closure:

$$\mathrm{Graph}(\partial f/\partial x_i) = (\overline{\{(\overline{x}, y, \varepsilon)|\ y\varepsilon = f(\overline{x} + \overline{\varepsilon_i}) - f(\overline{x})\}})_0,$$

where $\overline{\varepsilon_i} = (0, \ldots, 0, \varepsilon, 0, \ldots, 0)$, with ε in the i^{th} coordinate and we have used the notation $X_0 = \{\overline{u} \mid (\overline{u}, 0) \in X\}$. This also shows that, given a Ch-description for f, we can effectively find a Ch-description for $\partial f/\partial x_i$.

3.5.6 Definition. An $M(\mathcal{S})$-*constituent* is a set of the form

$$\{(\overline{x}, \overline{\varepsilon}) \in \mathbb{R}^n \times \mathbb{R}^k_+ \mid \exists \overline{y} \in \mathbb{R}^{k-1}\ F(\overline{x}, \overline{y}) = \overline{\varepsilon}\}$$

where $F\colon \mathbb{R}^{n+k-1} \to \mathbb{R}^k$ belongs to $M(\mathcal{S})^k$. An $M(\mathcal{S})$-*set* $S \subseteq \mathbb{R}^n \times \mathbb{R}^k_+$ is a finite union of $M(\mathcal{S})$-constituents (with the same k).

Given a set $A \in \widetilde{\mathcal{S}}_n$ and an $M(\mathcal{S})$-set $S \subseteq \mathbb{R}^{n+k}$, we say that S is an $M(\mathcal{S})$-*approximant* for A if S both approximates $\partial\overline{A}$ from above on bounded sets and approximates $\overline{A}$ from below.

3.5.7 Lemma. *Every $M(\mathcal{S})$-set $S \subseteq \mathbb{R}^{n+k}$ is in $\widetilde{\mathcal{S}}_{n+k}$ and has empty interior.*

Proof. The fact that $S \in \widetilde{\mathcal{S}}_{n+k}$ depends on the inclusion $M(\mathcal{S}) \subseteq \widetilde{\mathcal{S}}$ and the closure properties of $\widetilde{\mathcal{S}}$. To show that every such set S has empty interior, first recall that, as a consequence of Sard's Theorem, the image of a C^∞ function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ with $m > n$ has empty interior. Now, let $T = \{(\overline{x}, \overline{\varepsilon})|\ \exists \overline{y} \in \mathbb{R}^{k-1}\ F(\overline{x}, \overline{y}) = \overline{\varepsilon}\}$ be an $M(\mathcal{S})$-constituent of S and, for each fixed $\overline{x}$, consider the fiber $T_{\overline{x}} = \{\overline{\varepsilon} \mid (\overline{x}, \overline{\varepsilon}) \in T\}$ over $\overline{x}$. Note that $T_{\overline{x}}$ is the (positive part of the) image of the C^∞ function $h : \mathbb{R}^{k-1} \to \mathbb{R}^k$ which sends $\overline{y}$ to $F(\overline{x}, \overline{y})$, hence for every $\overline{x}$, $T_{\overline{x}}$ has empty interior by the remark above. It follows that T (and hence S) has empty interior. □

3.5.8 Remark. Our definition of W-structure is different from the corresponding definitions given in [5, 22, 37], where it is required that $\mathcal{S}$ contains all real semi-algebraic sets. Nevertheless we can apply to our $\widetilde{\mathcal{S}} = \mathrm{Ch}(\mathcal{S})$ all the results of these authors concerning the regularity properties of the sets in (their) $\widetilde{\mathcal{S}}$ (e.g. the fact that if a set $A \in \widetilde{\mathcal{S}}$ has empty interior, then so does its closure $\overline{A}$). The reason is the following. Let $\mathcal{S}_n^*$ be the collection of all sets of the form $A \cap L$ where $A \in \mathcal{S}_n$ and $L \subseteq \mathbb{R}^n$ is defined by a system of linear equations with coefficients in $\mathbb{R}$. We call $\mathcal{S}^* = \langle \mathcal{S}_n^* \mid n \in \mathbb{N} \rangle$ the enlargement of $\mathcal{S}$ with parameters from $\mathbb{R}$. Next, define $\widehat{\mathcal{S}^*}$ as the closure of $\mathcal{S}^*$ under the Ch-operation $\mathrm{Ch}(\pi)$. It can be readily verified that, if $\mathcal{S}$ is a closed o-minimal W-structure with DSF, then $\langle \widehat{\mathcal{S}_n^*} \mid n \in \mathbb{N} \rangle$ is a semi-closed o-minimal W-structure with DSF, and since $\mathcal{S}^*$ contains all real semi-algebraic sets, $\widehat{\mathcal{S}^*}$ is also a weak structure in Wilkie's sense. Moreover $\widetilde{\mathcal{S}} \subseteq \mathrm{Ch}(\widehat{\mathcal{S}^*})$, so we can apply to our $\widetilde{\mathcal{S}}$ the regularity results of $\mathrm{Ch}(\widehat{\mathcal{S}^*})$. To prove the DSF condition for $\widehat{\mathcal{S}^*}$ note that a generic set in $\widehat{\mathcal{S}^*}$ is of the form $\Pi_n^{n+k}[A \cap L]$, where $A \in \mathcal{S}$ and L is the zero-set of a system of linear polynomial $p_1, \dots, p_r$ over $\mathbb{R}$; the DSF condition for $\mathcal{S}$ provides a C^∞ function f_A with graph in $\mathcal{S}$ such that $A = \Pi_{n+k}^{n+k+h}[V(f_A)]$ (we recall that $V(f)$ is the zero-set of f); then the function $g = f_A^2 + \Sigma_i p_i^2$ is C^∞ with graph in $\widehat{\mathcal{S}^*}$ (note that the graph of the square of a function is existentially definable) and $\Pi_n^{n+k}[A \cap L] = \Pi_n^{n+k+h}[V(g)]$.

We need the following result of Charbonnel [5].

3.5.9 Lemma.

- *If $A \in \widetilde{\mathcal{S}}_n$ has empty interior, then so does $\overline{A}$.*
- *If $A \in \widetilde{\mathcal{S}}_{n+1}$ and $A \subseteq \mathbb{R}^n \times \mathbb{R}_+$ then $\overline{A}_0 = \{\overline{x} \in \mathbb{R}^n \mid (\overline{x}, 0) \in A\} \in \widetilde{\mathcal{S}}_n$, and if A has no interior points nor does $\overline{A}_0$.*

Proof. See [5] or also [22, Lemma 2.7] for the first statement and [37, Lemma 2.2] for the second. The proof depends on the o-minimality condition for $\widetilde{\mathcal{S}}$. □

From Lemma 3.5.7 and Lemma 3.5.9 we obtain:

3.5.10 Lemma. *Suppose $S \subseteq \mathbb{R}^n \times \mathbb{R}_+^k$ is an $M(\mathcal{S})$-set. Then the section $\overline{S}_{\overline{0}} = \{\overline{x} \in \mathbb{R}^n \mid (\overline{x}, \overline{0}) \in \overline{S}\}$ is closed, lies in $\widetilde{\mathcal{S}}_n$, and has empty interior.*

3.5.11 Theorem.

- *Suppose $\mathcal{S}$ is DSF; then, every set $A \in \widetilde{\mathcal{S}}_n$, has an $M(\mathcal{S})$- approximant $S \subseteq \mathbb{R}^n \times \mathbb{R}_+^k$ for some $k \geq 0$.*

- *Moreover, if $\mathcal{S}$ is EDSF, then there is an algorithm which, given a Ch-description for A, produces a Ch-description for S.*

The first part is in [37, Theorem 3.13], except that we are working with a slightly different definition of $\mathcal{S}$ and $\widetilde{\mathcal{S}}$. From the analysis of the proof it is easy to obtain the second part. We include a proof in the last section.

A weaker form of Theorem 3.5.11 - i.e. given a set in $A \in \widetilde{\mathcal{S}}_n$ we can find an $M(\mathcal{S})$-set $S \subseteq \mathbb{R}^{n+k}$ (for some k) such that S approximates $\partial\overline{A}$ from above on bounded sets - would be enough to our purposes, but we are not able to prove the weaker statement without proving the statement of 3.5.11 first.

Let us prove the main theorem of this section (corresponding to [37, Theorem 3.1]).

3.5.12 Theorem.

- *Let $\mathcal{S}$ be a closed o-minimal W-structure which is DSF. Then, given a closed set $A \subseteq \mathbb{R}^n$ in $\widetilde{\mathcal{S}}$, there exists a closed set $B \subseteq \mathbb{R}^n$ in $\widetilde{\mathcal{S}}$ such that B has empty interior and $\partial A \subseteq B$.*

- *Furthermore, if $\mathcal{S}$ is EDSF, then there is an effective procedure which, given a Ch-description for A, produces a Ch-description for B.*

Proof. Given a set $A \in \widetilde{\mathcal{S}}_n$, we can find an $M(\mathcal{S})$-approximant $S \subseteq \mathbb{R}^n \times \mathbb{R}^k_+$ for A as in Theorem 3.5.11. So in particular S approximates $\partial\overline{A} \in \mathbb{R}^n$ from above on bounded sets. But then so does the section $\overline{S}_{\overline{0}} = \{\overline{x} \in \mathbb{R}^n | \ (\overline{x}, \overline{0}) \in \overline{S}\}$ (see the proof of Lemma 3.3 in [37]). Moreover, the set $\overline{S}_{\overline{0}}$ is closed, lies in $\widetilde{\mathcal{S}}_n$ and has empty interior, by Lemma 3.5.10. Hence we can set $B = \overline{S}_{\overline{0}}$.

As to the effectiveness of this procedure, in case $\mathcal{S}$ is EDSF, notice that given a Ch-description for A, we can effectively find a Ch-description for the set S, from which we can easily compute a Ch-description for the set $\overline{S}_{\overline{0}}$. $\square$

3.6 Cell decomposition

We give a presentation of Wilkie's cell decomposition omitting some details of the proofs but emphasizing the definitions implicit in the proofs. We will refer to such definitions in the next section, where we will give an effective non-deterministic version of these results.

Fix a W-structure $\mathcal{S}$ which is DSF and let $\widetilde{\mathcal{S}}$ be the Ch-closure of $\mathcal{S}$.

3.6.1 Definition. Given $A \in \widetilde{\mathcal{S}}_n$, consider the set with empty interior $B \in \widetilde{\mathcal{S}}_n$ with $\partial A \subseteq B$ given by Theorem 3.5.12 and define $A^* \in \widetilde{\mathcal{S}}_n$ as $B \cap \overline{A}$. So $\partial A \subseteq A^* \subseteq \overline{A}$ and A^* has empty interior.

So A^* may contain, besides ∂A, some points in the interior of A.

3.6.2 Definition. Given $C \in \widetilde{\mathcal{S}}_n$ and given two functions $f\colon C \to \mathbb{R}$ and $g\colon C \to \mathbb{R}$, both in $\widetilde{\mathcal{S}}_{n+1}$, we denote by $(f)_C$ the graph of f and by $(f,g)_C$ the set $\{(\overline{x}, y) \in C \times \mathbb{R} \mid f(\overline{x}) < y < g(\overline{x})\}$.

In the sequel we identify a function with its graph, so a function $f : \mathbb{R}^n \to \mathbb{R}$ is a subset of $\mathbb{R}^{n+1}$.

3.6.3 Definition.

1. A cell in $\mathbb{R}$ is either a singleton $\{a\}$ belonging to $\widetilde{\mathcal{S}}_1$ or an interval $(a,b) \in \widetilde{\mathcal{S}}_1$.
2. A cell in $\mathbb{R}^{n+1}$ is either a set of the form $(f)_C$, where $f\colon C \to \mathbb{R}$ is a continuous function in $\widetilde{\mathcal{S}}_{n+1}$ and C is a cell in $\mathbb{R}^n$, or else a set of the form $(f,g)_C$ where C is a cell in $\mathbb{R}^n$ and $f,g\colon C \to \mathbb{R}$ are continuous bounded functions in $\widetilde{\mathcal{S}}_{n+1}$ satisfying $f(\overline{x}) < g(\overline{x})$ for all $\overline{x} \in C$.

The definition of cell depends on $\widetilde{\mathcal{S}}$, so our cells are $\widetilde{\mathcal{S}}$-cells. According to our definition, which departs from the usual one, a singleton $\{a\} \subseteq \mathbb{R}$ is not necessarily a cell, unless it belongs to $\widetilde{\mathcal{S}}_1$ (recall that we did not put in $\mathcal{S}_1$ all the singletons). Similarly an interval (a,b) is a cell only if it belongs to $\widetilde{\mathcal{S}}_1$. Note moreover that every cell is bounded, as in [37].

3.6.4 Definition. Let $D \in \widetilde{\mathcal{S}}_n$ be a cell. A cell decomposition $\mathcal{D}$ of D is a partition of D into cells where we require, if $n > 1$, that the projections $\Pi^n_{n-1}E$ of the cells $E \in \mathcal{D}$ form a cell decomposition of $\Pi^n_{n-1}D$ (which is clearly a cell). We say that $\mathcal{D}$ is compatible with a set $A \subseteq \mathbb{R}^n$ if $A \cap D$ is the union of some cells of $\mathcal{D}$. We say that $\mathcal{D}$ is compatible with a finite collection of sets, if it is compatible with each of them.

3.6.5 Remark. A cell decomposition $\mathcal{D}$ of D which is compatible with A^* is also compatible with $\overline{A}$.

3.6.6 Lemma. *Let $\mathcal{D}$ and $\mathcal{F}$ be two cell decompositions of the same cell $D \in \widetilde{\mathcal{S}}_n$. If $\mathcal{D}$ is compatible with the closure of each cell of $\mathcal{F}$, then $\mathcal{D}$ is compatible with each cell of $\mathcal{F}$.*

Proof. By induction on the definition of cell one shows that given two distinct cells C_0 and C_1 of $\mathcal{F}$, the closure of C_i $(i = 0,1)$ does not intersect C_{i-1}. Granted this, if for a contradiction there is a cell E of $\mathcal{D}$ which intersects two distinct cells C_0, C_1 of $\mathcal{F}$, then by the compatibility condition E is included in $\overline{C_i}$ for $i = 1,2$. Now $E \cap C_i$ is nonempty and is included in $\overline{C_1}$ and $\overline{C_2}$, so the closure of each C_i intersects C_{i-1}, and we have a contradiction. $\square$

We can now state the cell decomposition theorem:

3.6.7 Theorem. *Let $n \geq 1$ and suppose that D is a cell in $\mathbb{R}^n$. Given a finite collection $\mathcal{A} = \{A_1, \ldots, A_m\}$ of subsets of D which are closed in D and lie in $\widetilde{\mathcal{S}}_n$, there exists a cell decomposition $\mathcal{D}$ of D compatible with each set of the collection.*

The proof is by induction on n. The key step in the induction is based on Lemma 3.6.8 below, which provides the functions needed to define the cells.

3.6.8 Lemma. *Let $D = (f,g)_C \in \widetilde{\mathcal{S}}_{n+1}$ be an open cell (i.e. a cell which is an open subset of $\mathbb{R}^{n+1}$) and let $A \in \widetilde{\mathcal{S}}_{n+1}$ be a subset of D which is closed in D. There is a finite collection $\mathcal{H} \subseteq \widetilde{\mathcal{S}}_n$ of subsets of C which are closed in C and such that, if $\mathcal{F}$ is a cell decomposition of C compatible with $\mathcal{H}$ and C' is an open cell of $\mathcal{F}$, then:*

1. *the fibers of A^* over C' (namely the sets $A^*_{\overline{x}} = \{y \in \mathbb{R} \mid (\overline{x}, y) \in A^*\}$ for $\overline{x} \in C'$), have constant finite cardinality $\kappa = \kappa(C') \leq \gamma(A^*)$;*
2. *for $1 \leq i \leq \kappa$ the function $f_i \colon C' \to \mathbb{R}$, where $f_i(\overline{x})$ is defined as the i-th point in increasing order of $A^*_{\overline{x}}$, is continuous and lies in $\widetilde{\mathcal{S}}_{n+1}$ (this is vacuous if $\kappa = 0$).*

The lemma permits us to decompose a "large" subset of $D = (f,g)_C \subseteq \mathbb{R}^{n+1}$ compatibly with A^* provided we can find a decomposition $\mathcal{F}$ of the open cell $C \subseteq \mathbb{R}^n$ as required in the lemma. Indeed, for each open cell $C' \subseteq C$ of $\mathcal{F}$, the functions $f_i \colon C' \to \mathbb{R}$, together with $f_{|C'}$ and $g_{|C'}$, allow us to define a cell decomposition of $(f,g)_{C'} = (C' \times \mathbb{R}) \cap D$ compatible with A^*. In this way we decompose the union $\bigcup_{C'} (C' \times \mathbb{R}) \cap D$, where C' varies among the open cells of C. This set is large in D in the sense that its relative complement in D has empty interior.

Proof of Lemma 3.6.8. We must define $\mathcal{H}$ and, for each C', the functions $f_i \colon C' \to \mathbb{R}$. Let $\{A^* \geq i\} \subseteq C$ be the set of points $\overline{x} \in C$ such that the fiber $A^*_{\overline{x}} \subseteq \mathbb{R}$ of A^* over x has cardinality $\geq i$. This set is in $\widetilde{\mathcal{S}}_n$ since it admits the definition

$$\{A^* \geq i\} = \{x \in C \mid \exists y_1, \ldots, y_i (y_1 < \ldots < y_i \wedge \bigwedge_{j=1}^{i} (x, y_j) \in A^*)\}$$

which presents it as a projection of a set in $\widetilde{\mathcal{S}}_{n+i}$. Note that if a fiber $A^*_{\overline{x}}$ has cardinality $> \gamma(A^*)$ then it has a nonempty interior. By The Kuratowski-Ulam Theorem (see for example [26, Theorem 15.1]), the set of those points $\overline{x} \in C$ for which this happens has empty interior, as otherwise A^* would have interior. So, by Lemma 3.5.9, for $N > \gamma(A^*)$ the set $\overline{\{A^* \geq N\}}$ has empty interior. Note that $\{A^* \geq N\} = \{A^* \geq N'\}$ for $N, N' > \gamma(A^*)$. Now consider the following sets:

$$H = \{(\overline{x}, \varepsilon) \in C \times \mathbb{R}_+ \mid \exists y_1, y_2 (y_1 < y_2,\ (\overline{x}, y_1) \in A,\ (\overline{x}, y_2) \in A^*,\ y_2 - y_1 = \varepsilon)\}$$

$$H_f = \{(\overline{x}, \varepsilon) \in C \times \mathbb{R}_+ \mid \exists y ((\overline{x}, y) \in A^* \wedge y - f(\overline{x}) = \varepsilon)\}$$
$$H_g = \{(\overline{x}, \varepsilon) \in C \times \mathbb{R}_+ \mid \exists y ((\overline{x}, y) \in A^* \wedge g(\overline{x}) - y = \varepsilon)\}$$

Let $\widetilde{H} = \{\overline{x} \in C \mid (\overline{x}, 0) \in \overline{H}\}$, $\widetilde{H}_f = \{\overline{x} \in C \mid (\overline{x}, 0) \in \overline{H_f}\}$ and define $\widetilde{H}_g$ similarly. Finally define:

$$\mathcal{H} := \{\overline{\{A^* \geq 1\}}, \ldots, \overline{\{A^* \geq N\}}, \widetilde{H}, \widetilde{H}_f, \widetilde{H}_g\}.$$

Using the fact that A^* is a closed set with empty interior, it is not difficult to prove (see [37, Theorem 4.5]) that given a cell decomposition $\mathcal{F}$ of C compatible with $\mathcal{H}$, and given an open cell C' of $\mathcal{F}$, there is an integer $\kappa = \kappa(C') \leq \gamma(A^*)$ such that the set $(C' \times \mathbb{R}) \cap A^*$ is the union of the graphs of κ continuous functions $f_i \colon C' \to \mathbb{R}$ $(1 \leq i \leq \kappa)$ with $f_1 < \ldots < f_\kappa$ on C'. Granted this, it remains to prove that $f_i \in \widetilde{\mathcal{S}}_{n+1}$. This follows from the following definition

$$f_i = \{(\overline{x}, y) \in C' \times \mathbb{R} \mid \exists y_1, \ldots, y_\kappa (y_1 < \ldots < y_\kappa \wedge \bigwedge_{j=1}^{\kappa} (\overline{x}, y_j) \in A^* \wedge y = y_i)\}$$

which presents f_i as the projection of a set in $\widetilde{\mathcal{S}}_{n+1+\kappa}$. □

3.6.9 Remark. The definition of f_i given above depends on κ and so it is nonconstructive inasmuch as we do not know how to compute $\kappa = \kappa(C')$ given a description of C'. Using negations we could give an alternative definition of f_i which makes no reference to κ: $(\overline{x}, y) \in f_i$ if there are at least i points in the fiber $A^*_{\overline{x}}$ below y, and it is not the case that there are at least $i+1$ points in $A^*_{\overline{x}}$ below y. Unfortunately we cannot give this definition until we prove that $\widetilde{\mathcal{S}}$ is stable under complementation.

We are now ready to prove the cell decomposition theorem.

Proof of Theorem 3.6.7: case $n = 1$. Suppose $n = 1$, namely D is a cell of $\mathbb{R}$. If D is a singleton the result to be proved is obvious, so assume that D is an interval $(a, b) \in \widetilde{\mathcal{S}}_1$. Assume first that $m = 1$, namely the collection $\{A_1, \ldots, A_m\}$ contains only one set A. The set $A^* \subseteq D$ is finite since it has empty interior and belongs to $\widetilde{\mathcal{S}}_1$. Let κ be the cardinality of A^*. The singleton of the i-th point of A^* belongs to $\widetilde{\mathcal{S}}_1$ since it admits the definition

$$P_{i,\kappa} = \{y \in \mathbb{R} \mid \exists y_1 \ldots y_\kappa (y_1 < \ldots < y_\kappa \wedge \bigwedge_{j=1}^{\kappa} y_j \in A^* \wedge y = y_i)\}$$

which presents $P_{i,\kappa}$ as a projection of a set in $\widetilde{\mathcal{S}}_{\kappa+1}$ $(1 \leq i \leq \kappa)$. Define $P_{0,\kappa} = \{a\}$ and $P_{\kappa+1,\kappa} = \{b\}$. A cell decomposition of D is obtained by considering the singletons $P_{i,\kappa}$ $(1 \leq i \leq \kappa)$ and the intervals $(P_{i,\kappa}, P_{i,\kappa+1})$ $(0 \leq i < i+1 \leq \kappa+1)$. This decomposition is compatible with A^*, hence with A. The case $m > 1$ is similar. $\square$

To prove the general case we need:

3.6.10 Lemma. *For each cell C in $\mathbb{R}^n$ there exists a unique sequence of integers $1 \leq i_1 < \ldots < i_d \leq n$ such that if we let $\pi\colon \mathbb{R}^n \to \mathbb{R}^d$ be the projection $\pi(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_d})$ we have that the restriction of π to C is an homeomorphism onto an open cell of $\mathbb{R}^d$.*

Proof. The lemma is well known but we give a proof for future reference. A cell of the form $(f)_C$ is homeomorphic to its base C through the projection onto the first coordinates. So if $i_1, \ldots, i_d$ is the sequence associated to C, then the same sequence is associated to $(f)_C$, while the sequence associated to a cell of the form $(f, g)_C$ is $i_1, \ldots, i_d, i_n$. $\square$

Proof of Theorem 3.6.7: general case. Assume that the theorem holds in dimension $< n+1$. We prove it for $n+1$, dealing first with the case in which $m = 1$, namely the collection $\{A_1, \ldots, A_m\}$ contains only one set A. There are two cases two distinguish.

Case 1: suppose D is a cell of $\mathbb{R}^{n+1}$ which is not open. Then there is $d < n+1$ and integers $1 \leq i_1 < \ldots < i_d \leq n$ such that the projection $\pi\colon \mathbb{R}^n \to \mathbb{R}^d$, $(x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_d})$, maps D homeomorphically onto an open cell of $\mathbb{R}^d$. By induction we can decompose the image of D under π compatibly with the image of A. The preimages give us the desired decomposition of D.

Case 2: suppose $D = (f, g)_C$ is an open cell of $\mathbb{R}^{n+1}$. Consider the finite collection $\mathcal{H} \subseteq \widetilde{\mathcal{S}}_n$ of Lemma 3.6.8. By induction there is a cell decomposition $\mathcal{F}$ of C compatible with each set in $\mathcal{H}$. If C' is an open cell of $\mathcal{F}$, then we decompose $(f_{|C'}, g_{|C'})_{C'} = (C' \times \mathbb{R}) \cap D$ into cells bounded by the functions $f_{|C'}, g_{|C'}$ and the functions $f_i\colon C' \to \mathbb{R}$ of Lemma 3.6.8 $(1 \leq i \leq \kappa(C'))$. On the other hand if C' is a cell of $\mathcal{F}$ which is not open in $\mathbb{R}^n$, then the cell $(f_{|C'}, g_{|C'})_{C'}$ is not open in $\mathbb{R}^{n+1}$ and we argue as in case 1.

It remains to consider the case in which $m > 1$, namely we want a decomposition of a cell $D \in \widetilde{\mathcal{S}}_{n+1}$ compatible with a finite collection $\mathcal{A} \subseteq \widetilde{\mathcal{S}}_{n+1}$ of subsets of D which are closed in D. To begin with we apply the construction of case 1 and 2 to each $A \in \mathcal{A}$ separately. We obtain in this way, for each $A \in \mathcal{A}$, a cell decomposition $\mathcal{D}_A$ of D compatible with A. Projecting from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ we obtain, for each $A \in \mathcal{A}$, a decomposition $\mathcal{D}_{A,C}$ of $C = \Pi^{n+1}_n D$ compatible with $\Pi^{n+1}_n A$. By induction there is a cell decomposition $\mathcal{D}_C$ of C compatible with the following closed sets: (i) the closures of the cells of the various decompositions $\mathcal{D}_{A,C}$ of C described above; (ii) the sets of the form $\{x \in \mathbb{R}^n \mid h_i(x) = h_j(x)\}$, where h_i, h_j are functions bounding some cell in some of the given decompositions $\mathcal{D}_A$ of D (the nontrivial case is when h_i, h_j belong to different decompositions relative to distinct choices of $A \in \mathcal{A}$). By Lemma 3.6.6, $\mathcal{D}_C$ is then compatible with the cells of $\mathcal{D}_{A,C}$, not only with their closures. It is now obvious how to lift $\mathcal{D}_C$ to a cell decomposition of D. It suffices to consider cells which are bounded by the same functions $h_i, h_j, \ldots$ as before, but with domain restricted to the appropriate cells of $\mathcal{D}_C$ (this guarantees that the graphs of two different functions do not intersect). $\square$

The cell decomposition theorem is only proved for closed sets. However using the fact that $\widetilde{\mathcal{S}}$ is semi-closed we can reduce to closed sets and conclude as in [37, Theorem 1.8]:

3.6.11 Theorem. *$\widetilde{\mathcal{S}}$ is closed under complementation.*

A nonconstructive aspect of the above proof is that it makes use of Lemma 3.6.8 and therefore it requires the knowledge of the number $\kappa = \kappa(C')$. We observe that $\kappa \leq \gamma(A^*)$, so if $\mathcal{S}$ is an effective W-structure we can compute an upper bound on κ. This will suffice to turn the above proof into a "multivalued" algorithm, namely an algorithm which tries systematically all the possible values of κ (inductively) so as to yield a finite list of "objects" among which there is the description of a cell-decomposition compatible with A. To make this precise it turns out that the main difficulty is to give the correct definitions of what kind of objects our algorithm is going to manipulate. The problem here is that the notion of cell itself is not very constructive: from the Ch-description of a set we do not know how to recognize if the set is empty, or a singleton, a function, a continuous function, a cell of the form $(f, g)_C$, etc. Moreover, until we prove that $\widetilde{\mathcal{S}}$ coincides with the family of sets which are first order definable from sets in $\mathcal{S}$, from the Ch-description of a cell of the form $(f, g)_C$ it is not clear how to obtain a Ch-description of f and g. To handle these problems we will define in the sequel the notion of good representation of a cell.

3.7 Effective non-deterministic cell decomposition

Fix a closed o-minimal effective W-structure $\mathcal{S}$ which is EDSF.

3.7.1 Definition. *good representation of a cell* is given by the Ch-formulas for all the functions which are needed to define the cell. More precisely:

- A good representation of a singleton $P \in \widetilde{\mathcal{S}}_1$ is the sequence of length one whose only element is a Ch-formula P for P. Such a sequence is denoted by (P).

- A good representation of a cell of the form $(a, b) \in \widetilde{\mathcal{S}}_1$ is a pair whose first element is a Ch-formula P for the singleton $P = \{a\}$ and whose second element is a Ch-formula Q for the singleton $Q = \{b\}$. Such a pair is denoted by (P, Q).

- A good representation of a cell of the form $(f, g)_C$ is a triple whose first two elements are Ch-formulas f and g for f and g respectively and whose third element is a good representation C of C. Such a triple is denoted by $(\mathsf{f}, \mathsf{g})_{\mathsf{C}}$.

- A good representation of a cell of the form $(f)_C$ is a pair whose first element is a Ch-formula f for f and whose second element is a good representation C of C. Such a pair is denoted by $(\mathsf{f})_{\mathsf{C}}$.

We denote by Cell_n the set of all good representations of cells in $\widetilde{\mathcal{S}}_n$.

So Cell_n is a hereditary finite sequence of Ch-formulas, namely a finite sequence whose elements are Ch-formulas or other hereditary finite sequences. A hereditary sequence of Ch-formulas can be considered as a syntactic expression, namely a finite sequence of symbols from some finite alphabet. So it makes sense to ask whether Cell_n is a recursive set of syntactic expressions. A priori there is no reason to believe so, since we are not able to determine if a Ch-formula is the Ch-formula for a function. This is the reason to consider a larger recursive set $\mathrm{PCell}_n \supseteq \mathrm{Cell}_n$ which is defined exactly as Cell_n but without the requirement that the various Ch-formulas involved are Ch-formulas for functions. The precise definition follows.

3.7.2 Definition. The set PCell_n is defined by induction on n as follows:

- If P is the Ch-formula for a set $P \in \widetilde{\mathcal{S}}_1$ (not necessarily a singleton), then the sequence of length 1 whose only element is P belongs to PCell_1. Such a sequence is denoted by (P) and represents the set P.

- If P and Q are Ch-formulas for two sets P and Q in $\widetilde{\mathcal{S}}_1$ (not necessarily singletons), then the pair whose first element is P and whose second element is Q belongs to PCell_1. Such a pair is denoted by (P, Q) and represents the set

$$(P, Q) := \{y \in \mathbb{R} \mid \exists u \in P \exists v \in Q . u < y < v\}.$$

- If f and g are Ch-formulas for two sets $f \in \widetilde{\mathcal{S}}_{n+1}$ and $g \in \widetilde{\mathcal{S}}_{n+1}$ (not necessarily functions), and $\mathsf{C} \in \mathrm{PCell}_n$, then the triple whose first element is f, whose second element is g, and whose third element is C, belongs to PCell_{n+1}. Such a triple is denoted $(\mathsf{f}, \mathsf{g})_{\mathsf{C}}$ and represents the set

  $$(f, g)_C := \{(\overline{x}, y) \in C \times \mathbb{R} \mid \exists u, v \in \mathbb{R}.(\overline{x}, u) \in f \wedge (\overline{x}, v) \in g \wedge u < y < v\}$$

  where $C \in \widetilde{\mathcal{S}}_n$ is the set represented by C.

- If f is a Ch-formula for a set $f \in \widetilde{\mathcal{S}}_{n+1}$, and $\mathsf{C} \in \mathrm{PCell}_n$, then the pair whose first element is f and whose second element is C belongs to PCell_{n+1}. Such a triple is denoted $(\mathsf{f})_{\mathsf{C}}$ and represents the set

  $$(f)_C := \{(\overline{x}, y) \in C \times \mathbb{R} \mid (\overline{x}, y) \in f\}$$

  where $C \in \widetilde{\mathcal{S}}_n$ is the set represented by C.

Unlike Cell_n, the set PCell_n can be considered as a recursive set of syntactic expressions. The expressions in PCell_n will be called *representations of pseudo cells*, and the sets they represent will be called *pseudo cells*. Clearly $\mathrm{Cell}_n \subseteq \mathrm{PCell}_n$.

A pseudo cell $D \in \widetilde{\mathcal{S}}_n$ admits two kinds of representations. We can represent D by a Ch-formula, or we can represent it as a pseudo cell, namely by a syntactic expression in PCell_n. The advantage of this second representation is that it allows us to compute a representation for the boundary of the cells. For instance from the representation of a cell of the form $(f, g)_C$ as a pseudo cell, we can extract the representations of f and g (which are part of its boundary). The next lemma shows that there is an algorithm to pass from the pseudo cell representation to the Ch-formula. We denote by Ch_n the set of Ch-formulas for sets in $\widetilde{\mathcal{S}}_n$.

3.7.3 Lemma. *For each $n > 0$ there is a recursive function $\psi_n \colon \mathrm{PCell}_n \to \mathrm{Ch}_n$ (uniform in n) such that if $\mathsf{D} \in \mathrm{PCell}_n$ represents the pseudo cell $D \in \widetilde{\mathcal{S}}_n$ (according to Definition 3.7.2), then $\psi_n(\mathsf{D})$ is a Ch-formula for D. So in particular from a good representation of a cell we can compute its Ch-formula.*

Here and below, "uniform in n" means that the function is recursive even as a function of n.

Proof. Given $\mathsf{D} \in \mathrm{PCell}_n$, let D be the set represented by D according to Definition 3.7.2. That definition is inductive and can be naturally turned into an algorithm to compute the Ch-formula for D. □

3.7.4 Definition. We represent a cell decomposition D by the set of the good representations of its cells according to definition 3.7.2. So the representation of D belongs to the set $\mathrm{Dec}_n := \wp_{<\omega}(\mathrm{Cell}_n)$ (the family of all finite subsets of Cell_n). Anyway it is convenient to work with $\mathrm{PDec}_n := \wp_{<\omega}(\mathrm{PCell}_n) \supseteq \mathrm{Dec}_n$, since PDec_n can be naturally identified with a recursive set of syntactic expressions.

3.7.5 Definition. A non-deterministic function f from A to B is a function $f\colon A \to \wp_{<\omega}(B)$, namely a function from A to the finite subsets of B. We write $f\colon A \Rightarrow B$ as a shorthand for $f\colon A \to \wp_{<\omega}(B)$. So if $f\colon A \Rightarrow B$ and $b \in f(a)$ ($a \in A, b \in B$) we can consider b as one of the possible non-deterministic outputs of $f(a)$. We say that a non-deterministic function from A to B is recursive if it is recursive as a function from A to $\wp_{<\omega}(B)$ (this makes sense if A, B are recursive sets of strings of symbols).

We can now give our effective version of Wilkie's cell decomposition theorem.

3.7.6 Theorem. *For each $n > 0$ there is a recursive non-deterministic function*

$$F_n\colon \mathrm{PCell}_n \times \wp_{<\omega}(\mathrm{Ch}_n) \Rightarrow \mathrm{PDec}_n$$

(uniform in n) such that if $\mathsf{D} \in \mathrm{PCell}_n$ is a good representation of a cell $D \in \widetilde{\mathcal{S}}_n$ and $\mathsf{A}_1, \ldots, \mathsf{A}_m \in \mathrm{Ch}_n$ are Ch-formulas for subsets $A_1, \ldots, A_m$ of D which are closed in D, then there is a decomposition $\mathcal{D}$ of D compatible with $A_1, \ldots, A_m$ which admits a representation $\{\mathsf{D}_1, \ldots, \mathsf{D}_s\} \in F_n(\mathsf{D}, \{\mathsf{A}_1, \ldots, \mathsf{A}_m\})$.

The theorem says that from the expressions representing $D, A_1, \ldots, A_m$ we can effectively find a finite set $Y = F_n(\mathsf{D}, \{\mathsf{A}_1, \ldots, \mathsf{A}_m\})$ of candidates, one of which is a representation of a cell decomposition of D compatible with each A_i.

Proof of Theorem 3.7.6: case $n = 1$. We describe in the sequel the algorithm to compute $F_1(\mathsf{D}, \{\mathsf{A}_1, \ldots, \mathsf{A}_m\})$ where D is a representation of a pseudo cell D of $\mathbb{R}$ and A_i is a Ch-formula for $A_i \in \widetilde{\mathcal{S}}_n$. Assume first that $m = 1$, namely $\{\mathsf{A}_1, \ldots, \mathsf{A}_m\} = \{\mathsf{A}\}$. If D has the form (P) where $\mathsf{P} \in \mathrm{Ch}_1$, then F_1 outputs the finite set Y whose only element is (P). This does what is required, since the unique possible decomposition of a singleton is the singleton itself. Consider now the case when D has the form (P, Q) and represents the pseudo cell (P, Q) (with P, Q not necessarily singletons). The set $A^* \subseteq \mathbb{R}$ (see Definition 3.6.1) is finite since it has empty interior and belongs to $\widetilde{\mathcal{S}}_1$. Given A we can compute a Ch-formula A^* for A^* and an upper bound $N = \Gamma(\mathsf{A}^*)$ on $\gamma(A^*)$. Choose non-deterministically a non-negative integer $\kappa \leq N$ (this means that we try all the possible values of κ and we proceed with the construction for each possible choice, putting in the final output all the outcomes of the various computation paths). For $1 \leq i \leq \kappa$ compute the Ch-formula $\mathsf{P}_{i,\kappa}$ for the set

$$P_{i,\kappa} = \{y \in \mathbb{R} \mid \exists y_1 \ldots y_\kappa (y_1 < \ldots < y_\kappa \wedge y_i \in A^* \cap (P, Q) \wedge y = y_i)\}.$$

(If $\kappa = 0$ we skip this step.) Define $P_{0,\kappa} = P$ and $P_{\kappa+1,\kappa} = Q$. The output corresponding to these non-deterministic choices is the element of PDec_1 consisting of the following set of expressions: $\mathsf{P}_{i,\kappa}$ (for $1 \leq i \leq \kappa$) and $(\mathsf{P}_{i,\kappa}, \mathsf{P}_{i+1,\kappa})$ (for $0 \leq i < i+1 \leq \kappa+1$). To verify that the algorithm does what is required to do, note that, if the input $\mathsf{D} = (\mathsf{P}, \mathsf{Q}) \in \mathrm{PCell}_n$ were a good representation of a cell (i.e. if P, Q are singletons) and if κ was non-deterministically chosen as the cardinality of $A^* \cap (P, Q)$, then all the $\mathsf{P}_{i,\kappa}$ represent singletons and the output represents a cell decomposition of (P, Q) compatible with A^*, hence with A (if A was closed in D: see Remark 3.6.5). The case $m > 1$ is similar. $\square$

3.7.7 Definition. In the proof of Lemma 3.6.10 we have defined, for each cell C in $\mathbb{R}^n$, a sequence of integers $1 \leq i_1 < \ldots < i_d \leq n$ and the corresponding projection $\pi \colon \mathbb{R}^n \to \mathbb{R}^d$ (if $d = 0$ the sequence is empty and we project onto $\mathbb{R}^0$). The sequence can be computed by an algorithm which takes as input the representation C of C as a pseudo cell, in the sense of Definition 3.7.2. If C represents a pseudo cell C which is not an actual cell, the algorithm will still return a projection $\pi \colon \mathbb{R}^n \to \mathbb{R}^d$, but in this case $\pi_{|C}$ may not be an homeomorphism onto its image (for instance $(f)_E$ need not be homeomorphic to E if f is not a function). In any case we call d the *pseudo dimension* of C and π the *associated projection* (it may depend on the representation C, not just on the set C, in case C is only a pseudo cell).

Proof of Theorem 3.7.6: general case. We assume that $F_1, \ldots, F_n$ have already been defined with the desired properties and we describe the algorithm to compute $F_{n+1}(\mathsf{D}, \{\mathsf{A}_1, \ldots, \mathsf{A}_m\})$, dealing first with the case in which $\{\mathsf{A}_1, \ldots, \mathsf{A}_m\} = \{\mathsf{A}\}$. First we compute the pseudo dimension d of D and the associated projection $\pi \colon \mathbb{R}^{n+1} \to \mathbb{R}^d$. Let D be the pseudo cell represented by D. We distinguish two cases.

Case $d < n+1$. Note that the image $D' = \pi D$ of the pseudo cell D is a pseudo cell, and moreover we can compute its representation $\mathsf{D}' \in \mathrm{PCell}_d$. Now we compute the Ch-formula A' for the image of A under π and we choose non-deterministically an element $\mathsf{E}' \in F_d(\mathsf{D}', \{\mathsf{A}'\})$. Then $\mathsf{E}' \in \mathrm{PDec}_d$ and we can find an element $\mathsf{E} \in \mathrm{PDec}_n$ which represents the set of those pseudo cells which are the preimages under $\pi_{|D}\colon D \to D'$ of the pseudo cells of E'. The output corresponding to this non-deterministic computation is $\mathsf{E} \in F_{n+1}(\mathsf{D}, \{\mathsf{A}\})$. We must verify that this does the required job in case D was an actual cell and A was closed in D. Indeed in this case, by our inductive assumption on F_d, at least one of the non-deterministic choices of E' is a cell decomposition of πD compatible with $A' = \pi A$. Corresponding to this choice, E represents a cell decomposition of D compatible with A.

Case $d = n+1$. In this case D has necessarily the form $(\mathsf{f}, \mathsf{g})_{\mathsf{C}}$ where $\mathsf{C} \in \mathrm{PCell}_n$ has pseudo-dimension n, and represents a pseudo cell $D = (f,g)_C \in \widetilde{\mathcal{S}}_{n+1}$ (f, g are not necessarily functions). We can consider the sets $\widetilde{H}, \widetilde{H}_f$t and $\widetilde{H}_g$ defined exactly as in Lemma 3.6.8, except that in the definition of H_f we replace "$y - f(x) = \varepsilon$" with "$(x, y-\varepsilon) \in f$" (which makes sense even if f is not a function) and similarly with g in the role of f. From the available data we can compute Ch-formulas for these sets. Now from A we can compute a Ch-formula A^* for the set A^* and an upper bound $N = \Gamma(\mathsf{A}^*) + 1$ on $\gamma(A^*)$. Define

$$\mathcal{H} := \{\overline{\{A^* \geq 1\}}, \dots, \overline{\{A^* \geq N\}}, \widetilde{H}, \widetilde{H}_f, \widetilde{H}_g\}$$

as in Lemma 3.6.8 and compute Ch-formulas for all the elements of $\mathcal{H}$. Let $\mathsf{H} \in \wp_{<\omega}(\mathrm{Ch}_n)$ be the set of these Ch-formulas. Choose non deterministically an element $\mathsf{F} \in F_n(\mathsf{C}, \mathsf{H})$ and let $\mathcal{F} \subseteq \widetilde{\mathcal{S}}_n$ be the corresponding family of sets. So $\mathcal{F}$ is a candidate for a cell decomposition of C. From A and D we can compute an upper bound N on $\gamma(A^* \cap D)$. For each $\mathsf{C}' \in \mathsf{F}$ of pseudo dimension n, choose non-deterministically a nonnegative integer $\kappa(C') \leq N$ and define

$$f_i = \{(\overline{x}, y) \in C' \times \mathbb{R} \mid \exists y_1, \dots, y_\kappa (y_1 < \dots < y_\kappa \wedge \bigwedge_{j=1}^{\kappa} (\overline{x}, y_j) \in A^* \cap D \wedge y = y_i)\}.$$

Although f_i may not be a function, we can proceed as in the proof of Theorem 3.6.7 to define some pseudo cells over C' "bounded" by the various f_i, together with $(f)_{C'} = \{(\overline{x}, y) \in C' \times \mathbb{R} \mid (\overline{x}, y) \in f\}$ and $g_{|C'}$ (defined similarly). Clearly we can compute representations for all these pseudo cells. For $\mathsf{C}' \in \mathsf{F}$ of pseudo dimension $< n$ we can proceed as in Theorem 3.6.7 and compute non-deterministically, with the help of the functions $F_1, \dots, F_{n-1}$, the appropriate pseudo cell decompositions. Putting everything together we have obtained, non-deterministically, an element of PDec_n. If D was an actual cell, and A was a subset of D closed in D, at least one of these non-deterministic computations gives the correct result, namely a representation of a cell decomposition of D compatible with A.

We leave to the reader the definition of $F_{n+1}(\mathsf{D}, \{\mathsf{A}_1, \dots, \mathsf{A}_m\})$ in the case in which $m > 1$. $\square$

3.8 Non-deterministic computation of the complement

Fix as above a closed o-minimal effective W-structure $\mathcal{S}$ which is EDSF and let $\widetilde{\mathcal{S}}$ be its Charbonnel closure.

3.8.1 Theorem. *For each $n > 0$ there is a recursive non-deterministic function $G_n \colon \mathrm{Ch}_n \Rightarrow \mathrm{Ch}_n$ (uniform in n) which, given a Ch-formula for a set $A \in \widetilde{\mathcal{S}}_n$, returns a finite set of Ch-formulas, one of which defines the complement of A in $\mathbb{R}^n$.*

Proof. Since $\widetilde{\mathcal{S}}$ is effectively semi-closed (Remark 3.4.5), given a Ch-formula for $A \in \widetilde{\mathcal{S}}_n$, we can compute a Ch-formula which defines a closed set $B \in \widetilde{\mathcal{S}}_{n+k}$ such that $A = \Pi^{n+k}_n[B]$. We can easily find a semi-algebraic homeomorphism $f \colon \mathbb{R}^{n+k} \to D$ in $\widetilde{\mathcal{S}}$, where $D \in \widetilde{\mathcal{S}}_{n+k}$ is a cell, such that f commutes with the projection Π^{n+k}_n. Compute the Ch-formula B' of $B' = f(B)$. Apply Theorem 3.7.6 to compute a finite set of candidates for a good representation of a cell decomposition of D compatible with B'. Choose non-deterministically a candidate $\mathcal{D}$. Take the preimages under f of the sets of $\mathcal{D}$ (and find their Ch-formulas). If $\mathcal{D}$ was the correct candidate, we obtain a partition of $\mathbb{R}^{n+k}$ (technically it is not a cell decomposition since cells must be bounded) such that B is a finite union of classes of the partition. Project all these preimages down to $\mathbb{R}^n$ using Π^{n+k}_n: we obtain a finite collection of subsets of $\mathbb{R}^n$ (together with their Ch-formulas) which is a candidate for a partition of $\mathbb{R}^n$ such that A is the union of some classes of the partition. Select non-deterministically a sub-collection (a candidate for the sets of the collection which do not meet A), consider their union, and return its Ch-formula as the output corresponding to these non-deterministic choices. At least one of the possible outputs is a Ch-formula of the complement of A. $\square$

3.8.2 Definition. The first order language associated to $\mathcal{S}$ consists of an n-ary predicate symbol P_{A} for every Ch-formula A, which is interpreted as the set $A \subseteq \mathbb{R}^n$ associated to A (we identify a predicate with the set of elements which satisfy it).

3.8.3 Corollary. *There is a recursive function which, given a first order formula $\phi(x_1, \ldots, x_n)$ in the language associated to $\mathcal{S}$, returns a finite set of Ch-formulas, one of which denotes the subset of $\mathbb{R}^n$ defined by ϕ.*

Proof. We can assume that the only logical connectives of ϕ are existential quantifiers, disjunctions, and negations. The first two can be simulated by the Ch-operations of projections and unions, while negations can be non-deterministically simulated with complements using Theorem 3.8.1. $\square$

Finally we can prove our main result:

3.8.4 Theorem. *There is a recursive function which, given a first order formula $\phi(x_1, \ldots, x_n)$ in the language associated to $\mathcal{S}$, returns an upper bound on $\gamma(A)$, where $A \subseteq \mathbb{R}^n$ is the set defined by ϕ.*

Proof. Given ϕ we compute a finite set of Ch-formulas, one of which denotes the set A defined by ϕ. Using the effective o-minimality of $\widetilde{\mathcal{S}}$ (see Theorem 3.4.4), for each such Ch-formula, we compute an upper bound on γ of the corresponding set. Taking the greatest of these upper bounds, we also get an upper bound on $\gamma(A)$. $\square$

3.9 Proof of Theorem 3.5.11

In this section we will go through the proof of Theorem 3.5.11. Most of the proofs of the following Lemmas can be found in [37], we simply give a presentation suitable to our purposes. The Lemmas 3.9.3, 3.9.6, 3.9.8 below are of the form: "given certain sets in $\widetilde{\mathcal{S}}$, we can find other sets in $\widetilde{\mathcal{S}}$ with some required properties". The proofs show that, if $\mathcal{S}$ is EDSF, then the procedure is effective.

3.9.1 Remark. $|(1, x_1, \ldots, x_n)|^2 \leq 1/\varepsilon$ iff $\exists y.(1 + x_1^2 + \ldots + x_n^2 + y^2)^{-1} = \varepsilon$. Recall that the function $(x_1, \ldots, x_n, y) \mapsto (1 + x_1^2 + \ldots + x_n^2 + y^2)^{-1}$ belongs to $M(\mathcal{S})$.

The first task is to find an $M(\mathcal{S})$-approximant for the zero-set of a smooth function.

3.9.2 Lemma. *Let $g \colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function with $g \geq 0$ and let $S = \{(\overline{x}, t) \in \mathbb{R}^n \times \mathbb{R}_+ |\ g(\overline{x}) = t\}$. Then S approximates $\partial[g^{-1}(0)]$ from above on bounded sets, i.e. $\forall^s \varepsilon \forall^s t (g^{-1}(t)^\varepsilon \supseteq \partial[g^{-1}(0)] \cap B(0, 1/\varepsilon))$.*

Proof. Fix $\varepsilon > 0$ and suppose for a contradiction that there is a sequence of positive real numbers t_n converging to zero such that the inclusion fails for t_n, so that we can choose $\overline{x}_n \in \partial[g^{-1}(0)] \cap B(0, 1/\varepsilon)$ with $\overline{x}_n \notin g^{-1}(t_n)^\varepsilon$. By compactness of $B(0, 1/\varepsilon)$, choosing a subsequence we can assume that $\overline{x}_n \to \overline{x} \in \partial[g^{-1}(0)] \cap B(0, 1/\varepsilon)$. Let O be the $\varepsilon/2$-neighborhood of $\overline{x}$. Since $\overline{x} \in \partial[g^{-1}(0)]$, g assumes some positive value γ on O, and since O is connected g assumes all values in the interval $[0, \gamma]$ on O. Now choose n so big that $\overline{x}_n \in O$ and $t_n < \gamma$. Then O intersects $g^{-1}(t_n)$ and therefore it is contained in $g^{-1}(t_n)^\varepsilon$. So $\overline{x}_n \in g^{-1}(t_n)^\varepsilon$, contrary to the choice of $\overline{x}_n$. $\square$

3.9.3 Lemma. (See [37, Lemma 3.8]) *If $f \colon \mathbb{R}^n \to \mathbb{R}$ is in $M(\mathcal{S})$, then its zero-set $V(f)$ has an $M(\mathcal{S})$-approximant $S \in \widetilde{\mathcal{S}}_{n+2}$.*

Proof. let $S = \{(x_1, \ldots, x_n, \varepsilon_1, \varepsilon_2) \in \mathbb{R}^n \times \mathbb{R}^2_+ \mid |(1, x_1, \ldots, x_n)|^2 \leq 1/\varepsilon_1 \wedge f^2(\overline{x}) = \varepsilon_2\}$. By Remark 3.9.1, S is an $M(\mathcal{S})$-set. We prove that S approximates $V(f)$ from below, namely $\forall^s \varepsilon_0 \forall^s \varepsilon_1 \forall^s \varepsilon_2 S_{\varepsilon_1,\varepsilon_2} \subseteq V(f)^{\varepsilon_0}$. To see this note first that for fixed $\varepsilon_0, \varepsilon_1$, $S_{\varepsilon_0,\varepsilon_1}$ is contained in the compact set $K = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid |(1, x_1, \ldots, x_n)|^2 \leq 1/\varepsilon_1\}$. If ε_2 is smaller than the minimum of f on $K - V(f)^{\varepsilon_0}$ (or ε_2 is arbitrary if this set is empty), then $S_{\varepsilon_1,\varepsilon_2} \subseteq V(f)^{\varepsilon_0}$.

We prove that S approximates $\partial V(f)$ from above on bounded sets, namely $\forall^s \varepsilon_0 \forall^s \varepsilon_1 \forall^s \varepsilon_2 . \partial V(f) \cap B(0, 1/\varepsilon_0) \subseteq S^{\varepsilon_0}_{\varepsilon_1,\varepsilon_2}$. Fix ε_0 and choose ε_1 so that the set K considered above contains $B(0, 1/\varepsilon_0)$. By Lemma 3.9.2 for all sufficiently small ε_2, setting $g = f^2$, we have $g^{-1}(\varepsilon_2)^{\varepsilon_0} \supseteq \partial V(f) \cap B(0, 1/\varepsilon_0)$. Thus $\partial V(f) \cap B(0, 1/\varepsilon_0) \subseteq S^{\varepsilon_0}_{\varepsilon_1,\varepsilon_2}$. □

The smoothness assumptions are used in the following key lemma, which give us some information on the boundary of the projection of a set.

3.9.4 Lemma. (See [37, Lemma 2.9]) *Let $F \colon \mathbb{R}^{m+k} \to \mathbb{R}^k$ be a C^1-function in $\widetilde{\mathcal{S}}$, and consider the manifold $V = F^{-1}(a) \subseteq \mathbb{R}^{m+k}$, where $a \in \mathbb{R}^k$ is a regular value of F. Consider the projection $\pi \colon \mathbb{R}^{m+k} \to \mathbb{R}^m$ on the first m coordinates. Let O be an open ball in $\mathbb{R}^m$ intersecting $\partial \pi V$. Then for every sufficiently small $\varepsilon > 0$, O intersects $\pi V[\varepsilon]$, where $V[\varepsilon] \subseteq V$ is defined as the set of points $(x_1, \ldots, x_{m+k}) \in V$ such that one of the following conditions is satisfied for some $1 \leq i_1 < \ldots < i_k \leq m + k$:*

- $|(1, x_{m+1}, \ldots, x_{m+k})|^2 = 1/\varepsilon$;
- $\det \left(\frac{\partial^k F}{\partial (x_{i_1} \ldots x_{i_k})} \right)^2 = \varepsilon$.

3.9.5 Definition. We call $V[\varepsilon]$ the *ε-critical part* of V. This of course depends on the representation $V = F^{-1}(a)$.

3.9.6 Lemma. (See [37, Lemma 3.10]) *If $A \subseteq \mathbb{R}^{n+1}$ has an $M(\mathcal{S})$-approximant $S \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^k_+$, then there is a $M(\mathcal{S})$-approximant $S' \subseteq \mathbb{R}^n \times \mathbb{R}^{k+1}_+$ for $\Pi^{n+1}_n A \subseteq \mathbb{R}^n$.*

Proof. The sections $S_{\varepsilon_1,\ldots,\varepsilon_k} \subseteq \mathbb{R}^{n+1}$ of S have the form:

$$S_{\varepsilon_1,\ldots,\varepsilon_k} = \Pi^{n+1+k-1}_{n+1}\{F_1 = (\varepsilon_1, \ldots, \varepsilon_k)\} \cup \ldots \cup \Pi^{n+1+k-1}_{n+1}\{F_s = (\varepsilon_1, \ldots, \varepsilon_k)\},$$

where $F_i \colon \mathbb{R}^{n+1+k-1} \to \mathbb{R}^k$ is a C^∞ function in $M(\mathcal{S})$ and $\{F_i = (\varepsilon_1, \ldots, \varepsilon_k)\}$ is the pre-image of $(\varepsilon_1, \ldots, \varepsilon_k) \in \mathbb{R}^k$ under F_i. Define $S_{\varepsilon_1,\ldots,\varepsilon_k}[\varepsilon_k]$ as the set

$$\Pi^{n+1+k-1}_{n+1}(\{F_1 = (\varepsilon_1, \ldots, \varepsilon_k)\}[\varepsilon_{k+1}]) \cup \ldots \cup \Pi^{n+1+k-1}_{n}(\{F_s = (\varepsilon_1, \ldots, \varepsilon_k)\}[\varepsilon_{k+1}])$$

where $\{F_i = (\varepsilon_1, \ldots, \varepsilon_k)\}[\varepsilon_{k+1}]$ be the ε_{k+1}-critical part of $\{F_i = (\varepsilon_1, \ldots, \varepsilon_k)\}$ Define S' as the set whose sections $S'_{\varepsilon_1,\ldots,\varepsilon_{k+1}} \subseteq \mathbb{R}^n$ are given by:

$$S'_{\varepsilon_1,\ldots,\varepsilon_{k+1}} = \Pi_n^{n+1} S_{\varepsilon_1,\ldots,\varepsilon_k}[\varepsilon_{k+1}].$$

It is easy to see that S' is an $M(\mathcal{S})$-set. Let us verify that S' approximates $\overline{\Pi_n^{n+1} A}$ from below. From the definition of S' it follows that $S'_{\varepsilon_1,\ldots,\varepsilon_k,\varepsilon_{k+1}} \subseteq \Pi_n^{n+1} S_{\varepsilon_1,\ldots,\varepsilon_k}$. On the other hand since S approximates $\overline{A}$ from below, given $\varepsilon_0 > 0$, we have $\forall^s \varepsilon_1 \ldots \forall^s \varepsilon_k \; S_{\varepsilon_1,\ldots,\varepsilon_k} \subseteq (\overline{A})^{\varepsilon_0}$. It follows that $\forall \varepsilon_0 > 0 \forall^s \varepsilon_1 \ldots \forall^s \varepsilon_k$ we have $S'_{\varepsilon_1,\ldots,\varepsilon_{k+1}} \subseteq \Pi_n^{n+1} S_{\varepsilon_1,\ldots,\varepsilon_k} \subseteq (\overline{\Pi_n^{n+1} A})^{\varepsilon_0}$.

It remains to verify that S' approximates $\partial\overline{\Pi_n^{n+1} A}$ from above on bounded sets. Fix $\varepsilon_0 > 0$. Choose open balls $O_1, \ldots, O_m \subseteq \mathbb{R}^n$ of radius ε_0 such that $O_1 \cup \ldots \cup O_m \supseteq \partial\overline{\Pi_n^{n+1}[A]} \cap B(0, 1/\varepsilon_0)$ and each O_i intersects $\partial\overline{\Pi_n^{n+1} A}$. Then O_i intersects $\Pi_n^{n+1} \partial\overline{A}$, and since S approximates $\partial\overline{A}$ from above on bounded sets, it easily follows that $\forall^s \varepsilon_1, \ldots, \forall^s \varepsilon_k \;\; O_i$ intersects $\Pi_n^{n+1} S_{\varepsilon_1,\ldots,\varepsilon_k}$. On the other hand since S approximates $\overline{A}$ from below and $O_i \nsubseteq \overline{\Pi_n^{n+1} A}$, it is easy to see that O_i is not included in $\Pi_n^{n+1} S_{\varepsilon_1,\ldots,\varepsilon_k}$, and therefore must intersect its frontier. Thus by Lemma 3.9.4, $\forall^s \varepsilon_1, \ldots, \forall^s \varepsilon_{k+1} \;\; O_i$ intersects $\Pi_n^{n+1} S_{\varepsilon_1,\ldots,\varepsilon_k}[\varepsilon_{k+1}] = S'_{\varepsilon_1,\ldots,\varepsilon_{k+1}}$, so O_i is contained in the ε_0 neighborhood of the latter set. Since $\partial\overline{A} \cap B(0, 1/\varepsilon_0)$ is covered by the balls O_i, it is contained in $S'^{\varepsilon_0}_{\varepsilon_1,\ldots,\varepsilon_{k+1}}$. □

We give without proof the following easy lemma.

3.9.7 Lemma. *Let $A, B \subseteq \mathbb{R}^n$ be closed sets, and let $K \subseteq \mathbb{R}^n$ be compact. Then $\forall^s \varepsilon_1 \forall^s \varepsilon_2 \; A^{\varepsilon_2} \cap B^{\varepsilon_2} \cap K \subseteq (A \cap B)^{\varepsilon_1}$.*

3.9.8 Lemma. (See [37, Lemma 3.12]) *Let $A \in \widetilde{\mathcal{S}}$ have an $M(\mathcal{S})$-approximant $S \subseteq \mathbb{R}^n \times \mathbb{R}^k_+$ and suppose Y is an $n-1$ dimensional $\mathbb{Z}$-affine set; suppose further that $\overline{A} \cap Y = \partial\overline{A} \cap Y$. Then there is an $M(\mathcal{S})$-approximant $S' \subseteq \mathbb{R}^n \times \mathbb{R}^{k+2}_+$ for $\overline{A} \cap Y$.*

Proof. The requirement on the frontier is equivalent to asking that Y does not meet the interior of $\overline{A}$, hence we only need to worry about a subset of $\partial\overline{A}$. Suppose Y is the zero-set of a linear polynomial l with coefficients in $\mathbb{Z}$. The sections $S_{\varepsilon_1,\ldots,\varepsilon_k} \subseteq \mathbb{R}^n$ of S have the form:

$$S_{\varepsilon_1,\ldots,\varepsilon_k} = \Pi_n^{n+k-1}\{F_1 = (\varepsilon_1, \ldots, \varepsilon_k)\} \cup \ldots \cup \Pi_n^{n+k-1}\{F_s = (\varepsilon_1, \ldots, \varepsilon_k)\},$$

where $F_i \colon \mathbb{R}^{n+k} \to \mathbb{R}^k$ is a C^∞ function in $M(\mathcal{S})$ and $\{F_i = (\varepsilon_1, \ldots, \varepsilon_k)\}$ is the pre-image of $(\varepsilon_1, \ldots, \varepsilon_k) \in \mathbb{R}^k$ under F_i.

Define $S' \subseteq \mathbb{R}^{n+k+2}$ as the set whose sections $S'_{\varepsilon_1,\ldots,\varepsilon_{k+2}} \subseteq \mathbb{R}^n$ have the form:

$$S'_{\varepsilon_1,\ldots,\varepsilon_{k+2}} = S_{\varepsilon_3,\ldots,\varepsilon_{k+2}} \cap Y(\varepsilon_2) \cap K_{\varepsilon_1},$$

where $K_{\varepsilon_1} = \{\overline{x} \in \mathbb{R}^n | \ |(1, x_1, \dots, x_n)|^2 \leq 1/\varepsilon_1\} = \{\overline{x}| \ \exists x_{n+k} \ (1 + \sum_{i=1}^n x_i^2 + x_{n+k}^2)^{-1} = \varepsilon_1\}$ and $Y(\varepsilon_2) = \{\overline{x}| \ \exists x_{n+k+1} \ l(x_1, \dots, x_n)^2 + x_{n+k+1}^2 = \varepsilon_2\}$, so that S' is an $M(\mathcal{S})$-set.

Let us prove that S' approximates $\overline{A} \cap Y$ from below. By Lemma 3.9.7

$$\forall^s \varepsilon_1 \forall^s \varepsilon_2 \ \overline{A}^{\varepsilon_2} \cap B(0, \varepsilon_1^{-1}) \cap Y(\varepsilon_2) \subseteq (\overline{A} \cap Y)^{\varepsilon_1}.$$

Since S approximates $\overline{A}$ from below we have:

$$\forall^s \varepsilon_0 \forall^s \varepsilon_1 \dots \forall^s \varepsilon_{k+2} \quad S'_{\varepsilon_1, \dots, \varepsilon_{k+2}} \subseteq S_{\varepsilon_3, \dots, \varepsilon_{k+2}} \subseteq \overline{A}^{\varepsilon_2}.$$

From the definition it follows that $S'_{\varepsilon_1, \dots, \varepsilon_{k+2}} \subseteq K_{\varepsilon_1} \cap Y(\varepsilon_2)$, hence combining all these equations we get

$$\forall^s \varepsilon_0 > 0 \forall^s \varepsilon_1 \dots \forall^s \varepsilon_{k+2} \quad S'_{\varepsilon_1, \dots, \varepsilon_{k+2}} \subseteq (\overline{A} \cap Y)^{\varepsilon_0}.$$

It remains to prove that S' approximates $\overline{A} \cap Y$ from above on bounded sets. Since S approximates $\overline{A}$ from above on bounded sets we have:

$$\forall^s \varepsilon_2 \dots \forall^s \varepsilon_{k+2} \quad \partial \overline{A} \cap B(0, \varepsilon_2^{-1}) \subseteq S^{\varepsilon_2}_{\varepsilon_3 \dots \varepsilon_{k+2}}.$$

Since $\forall^s \varepsilon_0 \forall^s \varepsilon_1 \ B(0, \varepsilon_2^{-1}) \subseteq K_{\varepsilon_1}$ and by our hypothesis $\partial A \cap Y = \overline{A} \cap Y$, we obtain, using again Lemma 3.9.7:

$$\forall^s \varepsilon_0 \forall^s \varepsilon_1 \dots \forall^s \varepsilon_{k+2} \quad \overline{A} \cap Y \subseteq (S_{\varepsilon_3, \dots, \varepsilon_{k+2}} \cap Y(\varepsilon_2) \cap K_{\varepsilon_1})^{\varepsilon_0}.$$

This concludes the proof of the lemma. □

Proof of theorem 3.5.11. We prove the first part, the second part follows by the analysis of the proof of the first. We proceed by induction on the rank (see definition 3.3.5) of a Ch-description of A of A. Assume that $\mathcal{S}$ has DSF (EDSF for the second part).

If A is described as a set in $\mathcal{S}$, then Lemma 3.9.3 and Lemma 3.9.6, combined with the DSF condition, provide the result; this is the only reason why we had to assume DSF.

If A is described as $A_1 \cup A_2$, then an $M(\mathcal{S})$-approximant for A is given by the union of the $M(\mathcal{S})$-approximants for A_1 and A_2, respectively. The reason why this arguments works is that topological closure commutes with union. The same is not true with intersection instead of union, and this is the reason why we will need a more complicated argument for the intersection.

If A is described as $\Pi_n^{n+h}[A_1]$, then an iterated use of Lemma 3.9.8 tells us what to do.

If A is described as $\overline{B}$, then it is trivial since by definition an $M(\mathcal{S})$-approximant for B is an $M(\mathcal{S})$-approximant for A.

So, the only case which requires more care is the case when A is described as $A_1 \cap L$, where L is $\mathbb{Z}$-affine. We need to analyze all subcases.

If A_1 is described as a set in $\mathcal{S}$, then A too can be described as a set in $\mathcal{S}$ and we already know how to deal with these sets. If A_1 is obtained by an application of $\mathrm{Ch}(\cup)$, then by the distributivity laws for $\cup, \cap$, by inductive hypothesis and by an application of the argument above on how to approximate unions, we know how to approximate A. If $A_1 = \Pi^m_n[U]$, then we use the equation

$$\Pi^m_n[U] \cap L = \Pi^m_n[(U \times L) \cap (\Delta \times \mathbb{R}^{m-n})],$$

where $\Delta \subset \mathbb{R}^{2n}$ is the diagonal, and we conclude again by an application of Lemma 3.9.6 and by inductive hypothesis (notice that $U \times L$ has a description $\mathsf{U} \times \mathsf{L}$ of the same rank as U, by Lemma 3.3.7). If A_1 is obtained by an application of $\mathrm{Ch}(\cap_\ell)$, then we conclude by the inductive hypothesis (as the intersection of two $\mathbb{Z}$-affine sets is $\mathbb{Z}$-affine). The only difficult case is when A_1 is described as $\overline{U}$. Let $L = Y_1 \cap \ldots \cap Y_m$, where Y_i is a $\mathbb{Z}$-affine set of codimension 1. Notice that

$$\overline{U} \cap Y_1 = \overline{U \cap Y_1} \cup (\overline{U \cap Y_1^+} \cap Y_1) \cup (\overline{U \cap Y_1^-} \cap Y_1),$$

where, Y_1 is the zero set of a linear polynomial l over $\mathbb{Z}$, $Y_1^+ = \{\overline{x} \in \mathbb{R}^n | \, l(\overline{x}) > 0\}$, and Y_1^- is defined similarly by $l < 0$.

The descriptions of $U \cap Y_1^\pm$ have lower rank than $\rho(\overline{U})$, hence the inductive hypothesis can be applied to them. Now, Y_1 does not meet the interior of $\overline{U \cap Y_1^\pm}$ (since it does not meet the interior of $\overline{Y_1^\pm}$), hence to approximate the sets $\overline{U \cap Y_1^\pm} \cap Y_1$ we can use Lemma 3.9.8; while by inductive hypothesis we can get an approximant for the set $U \cap Y_1$. Now notice that $\overline{U} \cap Y_1$ has empty interior, so that we can make use of Lemma 3.9.8 for $(\overline{U} \cap Y_1) \cap Y_2$, and continue this way until we complete the proof of the theorem. $\square$

Chapter 4

Remarks on the decidability problem for the real exponential field

4.1 Introduction: The decidability problem

Let L be a language and $\mathcal{M}$ be an L-structure. We recall that the structure $\mathcal{M}$ is *decidable* if there exists an algorithm which decides, given an L sentence φ, whether φ is true or not in $\mathcal{M}$. Equivalently, $\mathcal{M}$ is decidable if the complete theory $Th(\mathcal{M})$ of the structure admits a recursive (or recursively enumerable) axiomatization. To prove the decidability of a structure $\mathcal{M}$, we can thus follow two different approaches: we can look for a decision algorithm, or we can look for a recursive axiomatization of the theory $Th(\mathcal{M})$. A recursive axiomatization is a recursive list of L-sentences which are true in $\mathcal{M}$ and which build up a complete theory T. Hence we say that a set of L-sentences T axiomatizes $Th(\mathcal{M})$ if $T \subseteq Th(\mathcal{M})$ and T is complete.

As an example of decidability theorem, we will briefly discuss the following

4.1.1 Theorem (Tarski, in [33]). *Let $L_{\mathrm{of}} = \{+, -, \cdot, <, 0, 1\}$ and $\mathbb{R}_{\mathrm{of}}$ be the real ordered field, i.e. the L_{of}-structure based on $\mathbb{R}$, with the usual ring operations and the usual order relation. The structure $\mathbb{R}_{\mathrm{of}}$ is decidable.*

We extract from the proof of the Theorem a few useful remarks. The main point is that the theory $Th(\mathbb{R}_{\mathrm{of}})$ admits *effective elimination of quantifiers*: there is an algorithm which, given a L_{of}-formula φ, produces a quantifier free L_{of}-formula $\psi(\overline{x})$, which is equivalent to $\varphi(\overline{x})$ in $\mathbb{R}_{\mathrm{of}}$ (which means that $\mathbb{R}_{\mathrm{of}} \models \forall \overline{x}\ (\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})))$. This property simplifies the problem: we only need to decide the truth of quantifier free formulas, which is in this structure extremely easy to do, due to the particularly simple form of those formulas (we give more details in Section 4.3). Moreover, Tarski found an extremely elegant recursive axiomatization of $Th(\mathbb{R}_{\mathrm{of}})$, namely the theory of *Real Closed Fields.*

Recall that $L_{\exp} = \{+, -, \cdot, \exp, <, 0, 1\}$ and $\mathbb{R}_{\exp}$ is the real ordered exponential field, i.e. the $L_{\exp}$-structure based on $\mathbb{R}$, with the usual ring operations, the usual order relation and the usual exponentiation. $T_{\exp}$ is the complete theory of $\mathbb{R}_{\exp}$.

In this chapter we apply the results of the previous chapters to the structure $\mathbb{R}_{\exp}$, with the aim of understanding the decidability problem for this structure. Here the situation is much more complicated than in the case of $\mathbb{R}_{\mathrm{of}}$. For example, quantifier elimination does not hold (see [6]). There is, however, a model completeness result (due to Wilkie, in [35]): every $L_{\exp}$-formula is equivalent in $\mathbb{R}_{\exp}$ to an existential formula. Unfortunately, the equivalence is not known to be effective. Moreover, even the truth of quantifier free sentences is in this case difficult to decide. Hence we will need to develop alternative methods to the ones used by Tarski.

Most of the results of this chapter are implicit in the works of A. Macintyre and A. Wilkie on the subject, in particular [21], but we thought it useful to highlight some aspects and make the results more explicit for the sake of a better understanding.

We will start with some results about *effectively continuous functions* (see Definition 4.2.1); we will establish some results in this general setting and then use them in the exponential case (the exponential function being itself an effectively continuous function) in the subsequent sections. We will then concentrate only on the exponential function and in Section 4.3 we will treat some special instances of the decidability problem, namely the decidability of the quantifier free part of the theory. We will see that even this subproblem presents major difficulties, and we will not be able to solve it completely. Then in Section 4.4 we will treat the problem of establishing the existence of solutions of term-defined equations, and again we will only reach partial conclusions. In Section 4.5 we will give the first of the two main results of [21], that is the fact that the decidability of the whole theory of $\mathbb{R}_{\exp}$ depends only on the decidability of the existential fragment of the theory, and indeed on the problem of deciding the existence of solutions of term-defined equations, treated in Section 4.4. The second main result of [21] is that a positive answer to this latter problem can be given, provided that Schanuel's Conjecture holds. In Section 4.6, we will restate the decidability problem in four different equivalent ways, each of them highlighting a different aspect of the question, and providing arguments to understand the philosophy of the second main result of [21], without needing to go into the difficult details of its proof. In the Sections 4.3, 4.4, 4.5 and 4.6 we work only within the structure $\mathbb{R}_{\exp}$, and we only use elementary first order logic and the intuitive notion of algorithm. In Section 4.7, however, we will change our point of view: the decidability problem is restated in terms of finding a recursive axiomatization of the theory $T_{\exp}$. We will build up a recursive subtheory T of $T_{\exp}$ and prove that T is complete, provided that Schanuel's Conjecture holds. We will add schemes of axioms one at the time, at every step discussing the properties of the models. We will end up with a candidate for a recursive axiomatization T for $T_{\exp}$, which is simpler than the one proposed in [21], and we will conclude with a list of open questions about the models of this theory.

In what follows we will deal with exponential terms, so we recall some notation. A term in n variables is, as usual, an expression built up, starting from a subset of $\{0, 1, x_1, \ldots, x_n\}$, by repeated use of addition, subtraction, multiplication and exponentiation. The number of iterated operations used to obtain a term t from the set $\{0, 1, x_1, \ldots, x_n\}$ is called the *complexity* of t. A term which involves no variables is called a *closed* term. We will also use the obvious notation: $x^2 := x \cdot x$, $2 := 1 + 1$ etc.

4.1.2 Definition. A term is *special, of complexity 0* if it is a constant symbol or a variable.

If $m \in \mathbb{N}$, $p \in \mathbb{Z}[u_1, \ldots, u_m, v_1, \ldots, v_m]$ is a polynomial (in reduced normal form, i.e. a sum of distinct monomials) and $t_1, \ldots, t_m$ are special terms of complexities $\leq N - 1$, then $p(t_1, \ldots, t_m, e^{t_1}, \ldots, e^{t_m})$ is *special, of complexity* $\leq N$.

A special term of complexity ≤ 1 is called a *simple exponential term* or *simple exponential polynomial.*

4.1.3 Remark. Using Theorem 4.1.1, it is easy to see that every exponential term is effectively equivalent to a term in special form: in fact the field axioms are enough to prove the equivalence. Hence in what follows we will freely assume terms to be special whenever this is convenient.

4.2 Effectively continuous functions

4.2.1 Definition. A map $f : \mathbb{R}^n \to \mathbb{R}$ is *effectively continuous* if there exists a computable function φ with the following properties:

- to each n-tuple of open intervals $(a_1, b_1), \ldots, (a_n, b_n)$ with rational endpoints, φ associates an open interval (c, d) with rational endpoints such that
$$\forall \overline{x}\ (\overline{x} \in (a_1, b_1) \times \ldots \times (a_n, b_n) \ \Rightarrow\ f(\overline{x}) \in (c, d));$$

- $\forall M > 0\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall a_i, b_i$, if c, d is the output of φ on input a_i, b_i, then $(a_i, b_i \in [-M, M]\ \wedge\ \max\limits_{i=1,\dots,n} |a_i - b_i| < \delta\ \Rightarrow\ |c - d| < \varepsilon)$.

In this case we say that φ witnesses the effective continuity of f, or simply that φ *computes* f.

4.2.2 Remark. Notice that we do not require δ to be *computable* from ε, M (if ε, M are rational numbers). However, it is easy to see that δ is automatically computable: if $\varepsilon \in \mathbb{Q}$ and $M, k \in \mathbb{N}$, we can cover the cube $[-M, M]^n$ with finitely many boxes $(a_1, b_1) \times \ldots \times (a_n, b_n)$ with rational endpoints and diameter $< 1/k$. For each box, we compute the corresponding interval (c, d). If we fix M and let k increase, we eventually obtain that each interval (c, d) has diameter smaller than ε. Hence we have an algorithm which, on input ε, M, computes $\delta = 1/k$. An effectively continuous function is thus *effectively uniformly continuous* on every compact cube $[-M, M]^n$ (see [3]), uniformly in M.

4.2.3 Remark. We notice that from the computable function φ we can recover the effectively continuous function f computed by φ (because for every $\overline{x}$ we can compute a sequence of rational numbers approximating $f(\overline{x})$). Hence there are countably many effectively continuous functions, at most one for every computable function. In particular, not every constant function is effectively continuous.

4.2.4 Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *effectively* C^2 if it is C^2 and if f and its derivatives of order 1 and 2 are effectively continuous.

4.2.5 Example. The function $x \mapsto |x|$ is effectively continuous. If $f_1, \dots, f_n$ are effectively continuous, then the function $\max_{i=1,\dots,n} |f_i|$ is effectively continuous. All polynomials with rational coefficients in any number of variables are effectively C^2. It is easy to see that the function $\sin x$ is effectively C^2, and so is the exponential function (see 4.3.5).

4.2.6 Lemma. *The set of all effectively continuous functions is effectively closed under composition, i.e. given computable functions φ, φ' which compute two effectively continuous functions f, g respectively, we can effectively find a computable function φ'' which computes the composition $f \circ g$ (when the latter is defined). The same holds for effectively C^2 functions.*

Proof. To simplify the notation, we prove the lemma for functions of one variable. The proof of the general case is identical. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two effectively continuous functions. We rewrite the definition of effectively continuous function expliciting the relations of dependence of the variables.

A function $h : \mathbb{R} \to \mathbb{R}$ is effectively continuous if there is a recursive function φ with the following properties:

- given $a, b \in \mathbb{Q}$, φ returns $c(a,b,h), d(a,b,h) \in \mathbb{Q}$ such that
$$\forall x \ (x \in (a,b) \ \Rightarrow \ h(x) \in (c(a,b,h), d(a,b,h)));$$
- $\forall M \in \mathbb{N} \ \forall \varepsilon > 0 \ \exists \delta = \delta_h(M,\varepsilon) > 0 \ \forall a, b$, if c, d is the output of φ on input a, b, then $(|a-b| < \delta \wedge a, b \in [-M, M] \ \Rightarrow \ |c(a,b,h) - d(a,b,h)| < \varepsilon)$.

Given $a, b \in \mathbb{Q}$, we first compute $c := c(a,b,g), d := d(a,b,g)$; then we compute $c' := c(c,d,f), d' := d(c,d,f)$. It is clear that if $a < x < b$ then $c' < f(g(x)) < d'$. Moreover, given M, ε, consider $\delta := \delta_f(M,\varepsilon)$ and $\delta' := \delta_g(M,\delta)$. Then, if $a, b \in [-M, M]$ and $|a-b| < \delta'$, it also holds that $|c' - d'| < \varepsilon$.

An analogous argument works for effectively C^2 functions. □

4.2.7 Example. If $f_1, \dots, f_n$ are effectively C^2 functions, then so is every polynomial (with rational coefficients) in the f_is. In particular, the jacobian determinant of $(f_1, \dots, f_n)$ is effectively continuous.

4.2.8 Definition. Let

$$EC = \{\varphi | \ \varphi \text{ computes an effectively continuous function } f_\varphi : \mathbb{R}^n \to \mathbb{R}\}.$$

and

$$EC^2 = \{\varphi | \ \varphi \text{ computes an eff. } C^2 \text{ funct. } f_\varphi \text{ and its first and second derivatives}\}$$

4.2.9 Remark. Since the set of programs which compute recursive functions is recursively enumerable, we will identify a recursive function φ with a code for φ in the enumeration. Hence, a collection of recursive functions can be viewed as a set of natural numbers, and it make sense to ask whether such a set is recursively enumerable.

4.2.10 Proposition. *Let $\mathcal{T}$ be a subset of EC. Then the set*

$$\{(\varphi, \overline{x}_0) | \ \overline{x}_0 \in \mathbb{Q}^n, \ \varphi \in \mathcal{T} \text{ and } f_\varphi(\overline{x}_0) < 0\}$$

is recursively enumerable relatively to $\mathcal{T}$, i.e. there is an algorithm which, on input $\overline{x}_0 \in \mathbb{Q}^n$, $\varphi \in \mathcal{T}$, stops if and only if $f(\overline{x}_0) < 0$.

Proof. Let $\overline{x}_0 \in \mathbb{Q}^n$, $\varphi \in \mathcal{T}$ and $f = f_\varphi$. It follows directly from the fact that f is effectively continuous, that it is possible to compute two sequences of rational numbers $(p_N, q_N)_{N\in\mathbb{N}}$ such that $p_N \leq f(\overline{x}_0) \leq q_N$ and $q_N - p_N < 1/N$. Hence $f(\overline{x}_0)$ is simultaneously approximable with rational numbers from below and above, to any required degree of accuracy. The algorithm of recursive enumerability works as follows: given $N \in \mathbb{N}$, compute p_N and q_N. If $p_N \cdot q_N \geq 0$ and $p_N, q_N \leq 0$, then stop. Otherwise, repeat the procedure with accuracy $N + 1$.

The procedure will clearly stop if and only if $f(\overline{x}_0)$ is eventually trapped between two negative rational numbers. Notice that if $f(\overline{x}_0) = 0$, then the algorithm will produce for every N a strictly negative lower approximant p_N and a strictly positive upper approximant q_N.

□

The main result of this section is the following application of Newton's Method 1.4.1.

4.2.11 Theorem. *Let $\mathcal{T}$ be a subset of EC^2. Then the set*

$$\{(\varphi_1, \dots, \varphi_n) \in \mathcal{T}^n |\ \exists \overline{x} \in V^{\mathrm{reg}}(f_{\varphi_1}, \dots, f_{\varphi_n})\}$$

is recursively enumerable relatively to $\mathcal{T}$, i.e. there is an algorithm which, on input $(\varphi_1, \dots, \varphi_n) \in \mathcal{T}^n$, stops if and only if the functions $f_{\varphi_1}, \dots, f_{\varphi_n}$ have a common nonsingular zero.

Proof. We recall the statement of Newton's method in $\mathbb{R}$ (although we have proved it to be valid for any C^2 map definable in a definably complete structure, see Theorem 1.4.1): let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 map. Suppose we are given $a_0, a_1, a_2 \geq 1$ and a point $\overline{x}_0 \in \mathbb{R}^n$ such that $|F(\overline{x}_0)| < m = (4n^3 a_0^3 a_1 a_2)^{-1}$ and $|F'(\overline{x})^{-1}| < a_0$, $|F'(\overline{x})| < a_1$, $|F''(\overline{x})| < a_2$ for all $\overline{x}$ in the ball of center $\overline{x}_0$ and radius $r = (2n^3 a_0^2 a_1 a_2)^{-1}$; then F has a nonsingular zero in that same ball.

We observe that, by continuity of F and its derivatives, F has a nonsingular zero **if and only if** there exist $a_0, a_1, a_2 \in \mathbb{Q}$ and a point $\overline{x}_0 \in \mathbb{Q}^n$ such that the conditions of Newton's method are satisfied.

Let $(\varphi_1, \dots, \varphi_n) \in \mathcal{T}^n$, $f_i = f_{\varphi_i}$ and $F = (f_1, \dots, f_n)$. We need first to make some remarks:

1. The linear map $F'(\overline{x})$ can be represented as the $n \times n$ matrix A, whose entries are $\frac{\partial f_i}{\partial x_j}$. If A is invertible, then there is an $n \times n$ matrix $\mathrm{ad}A$ such that $A^{-1} = \frac{\mathrm{ad}A}{\det A}$. The entries of $\mathrm{ad}A$ are $(-1)^{i+j} \det M_{ji}$, where M_{ij} is the $(n-1) \times (n-1)$ minor of A obtained from A by eliminating

the i-th row and the j-th column. In particular, the entries of $\mathrm{ad}A$ are polynomials in the entries of A.

2. The functions $|F(\overline{x})|, |F'(\overline{x})|, |F''(\overline{x})|, |\mathrm{ad}F'(\overline{x})|, |JF(\overline{x})|$ are effectively continuous (computed by recursive functions $\psi_0, \psi_1, \psi_2, \psi_3, \psi_4$ respectively), because so are the functions $f_i(\overline{x})$, $\frac{\partial f_i}{\partial x_j}(\overline{x})$, $\frac{\partial^2 f_i}{\partial x_j \partial x_k}(\overline{x})$.

Let r be a positive rational number and $\overline{x}_0 = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^n$. Let $Q(\overline{x}_0, r)$ be the box whose sides are the intervals $(\alpha_i - r, \alpha_i + r)$. We fix the following notation: on input $\overline{x}_0, r$,

ψ_0 computes c_0, d_0 such that, if $\overline{x} \in Q(\overline{x}_0, r)$, then $c_0 < |F(\overline{x})| < d_0$;
ψ_1 computes c_1, d_1 such that, if $\overline{x} \in Q(\overline{x}_0, r)$, then $c_1 < |F'(\overline{x})| < d_1$;
ψ_2 computes c_2, d_2 such that, if $\overline{x} \in Q(\overline{x}_0, r)$, then $c_2 < |F''(\overline{x})| < d_2$;
ψ_3 computes c_3, d_3 such that, if $\overline{x} \in Q(\overline{x}_0, r)$, then $c_3 < |\mathrm{ad}F'(\overline{x})| < d_3$;
ψ_4 computes c_4, d_4 such that, if $\overline{x} \in Q(\overline{x}_0, r)$, then $c_4 < |JF(\overline{x})| < d_4$.

Notice that

$$\forall \overline{x} \in Q(\overline{x}_0, r),\ JF(\overline{x}) \neq 0 \ \Leftrightarrow\ c_4 > 0$$

and, in this case, $|F'(\overline{x})^{-1}| < a_0 \ \Leftrightarrow\ |\mathrm{ad}F'(\overline{x})| < a_0|JF(\overline{x})|$.

Hence, we can say that the map F has a nonsingular zero if and only if there exist $a_0, a_1, a_2 \geq 1$ and a point $\overline{x}_0 \in \mathbb{Q}^n$ such that, when $r = (2n^3 a_0^2 a_1 a_2)^{\dot{-}1}$, the following condition (which can be checked recursively) holds:

$$d_0 < m = (4n^3 a_0^3 a_1 a_2)^{\dot{-}1} \ \wedge\ d_1 < a_1 \ \wedge\ d_2 < a_2 \ \wedge\ c_4 > 0 \ \wedge\ d_3 < a_0 c_4.$$

This clearly provides a procedure which stops, on input the recursive functions computing F, if and only if F has a nonsingular zero (a predicate defined by an existential quantification preceding a recursive predicate, is recursively enumerable). □

4.3 On the sign of closed exponential terms (The Quantifier Free Theory)

We will now concentrate our attention on the theory of real exponentiation. Our first task will be an attempt to understand the structure of the Quantifier Free part of this theory.

4.3.1 Remark (The Quantifier Free theory of exp)**.** A difference between the structures $\mathbb{R}_{\mathrm{of}}$ and $\mathbb{R}_{\exp}$ is that in the latter the quantifier free part of the theory is not trivially decidable. In $\mathbb{R}_{\mathrm{of}}$ every quantifier free sentence is effectively equivalent to some finite boolean combination of formulas either of the form $p(q_1, \ldots, q_n) = 0$ or of the form $p(q_1, \ldots, q_n) < 0$, where $p \in \mathbb{Z}[x_1, \ldots, x_n]$ and $q_1, \ldots, q_n \in \mathbb{Z}$, so the truth of a quantifier free sentence is trivially established.

In $\mathbb{R}_{\exp}$, however, the atomic formulas are more complicated. We can reduce to the study of the truth value of sentences the forms $t < 0$ and $t = 0$, where t is a closed term.

Here is an example of closed term: $t = 11 - 3e^{-7} + 2e^{e^3}e^3 - 8e^{2e^{e^2}}e^5$.

Now, it is not clear, in general, how to determine the sign (positive, negative, or possibly zero) of such an expression. Let's start with a simple example.

4.3.2 Example (Simple closed exp-terms)**.** Let t be a closed term such that only positive integer powers of e appear in t, i.e. $t = p(e)$, where $p(x) \in \mathbb{Z}[x]$. Then, since e is transcendent, necessarily $t \neq 0$ (unless p is the polynomial 0). So in this case the truth of the sentence $t = 0$ is easily decided. Now, to decide the truth of the sentence $t < 0$, we can approximate simultaneously from above and below the number e with rational numbers, until eventually t is trapped between to strictly positive or strictly negative rational numbers. This can be done for example using Taylor approximations (see below for the details), and the approximating procedure eventually gives a univocal answer on the sign of t, because $t \neq 0$.

4.3.3 Remark (Taylor approximation of e)**.** We will define two sequences of rational numbers which approximate the number e from below and from above, respectively. It follows from Taylor's Theorem that $e = \sum_{n=0}^{\infty} 1/n!$, hence, for $N \geq 3$,

$$R_N = e - \sum_{n=0}^{N} \frac{1}{n!} = \sum_{n>N} \frac{1}{n!} = \sum_{k \geq 1} \frac{1}{(k+N)!} \leq \frac{1}{N!} \sum_{k \geq 1} \frac{1}{k!} \leq \frac{1}{N!}(e-1) < \frac{1}{N-2}.$$

Now define, for all $M > 2$, $p_M = \sum_{n=0}^{M+2} 1/n! \in \mathbb{Q}$ and $q_M = p_M + 1/M \in \mathbb{Q}$. Then $\forall M\ p_M < e < q_M$ and $q_M - p_M = 1/M$. This provides a simultaneous approximation of e to any required degree of accuracy.

4.3.4 Example. Now, let us consider a more complicated case, i.e. suppose iterations of e appear in t. For example, let $t = a + be + ce^e \cdot e + de^{2e^e}$, where $a, b, c, d \in \mathbb{Z}$. In this case it is not even clear if $t \neq 0$: it is not known if the numbers e and e^{e^e} are algebraically independent, so we can not infer, as we did in Example 4.3.2, that such an expression is always nonzero. If we try the method described in Example 4.3.2 to decide the sign of t, we can use again the Taylor approximation for e (and hence, e^{e^e}) to approximate simultaneously t from below and above. This time, though, if $t = 0$, such a procedure will never stop and in fact will produce, for every $M \in \mathbb{N}$, a negative approximant from below and a positive approximant from above, hence in this case the method described is not helpful.

So far, nothing suggests that the quantifier free part of the theory of $\mathbb{R}_{\exp}$ is decidable. The above remarks, however, lead us to the following result.

4.3.5 Proposition. *The function* exp *is effectively continuous, i.e. there is a computable function* φ *with the following properties:*

- *given* $a, b \in \mathbb{Q}$, φ *returns* $c, d \in \mathbb{Q}$ *such that*

$$\forall x \ (x \in (a, b) \ \Rightarrow \ \exp(x) \in (c, d));$$

- $\forall M \in \mathbb{N} \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall a, b$, *if* c, d *is the output of* φ *on input* a, b, *then* $|a - b| < \delta \ \wedge \ a, b \in [-M, M] \ \Rightarrow \ |c - d| < \varepsilon$.

It follows automatically that exp is effectively C^2.

Proof. Let $a, b \in \mathbb{Q}$ and let M be a natural number such that $|a|, |b| < M$. Let $n \in \mathbb{N}$ be the biggest natural number for which $|a - b| < 1/n$ holds. We define

$$c = \sum_{i=0}^{n} \frac{a^i}{i!} \ \wedge \ d = \sum_{i=0}^{M^{3+n}} \frac{b^i}{i!} + \frac{M^2}{(M-1)M^n}.$$

Then, c, d are clearly computable, given a, b. Moreover, by monotonicity of exp and Taylor's expansion, for every $x \in (a, b)$, it is true that $\exp(x) > \exp(a) > c$. The following computation shows that $\exp(b) < d$. We need to check that the M^{n+3}-th remainder of the Taylor series of e^{b_m} is smaller than the number $\frac{M^2}{(M-1)M^n}$:

$$\sum_{i=M^{3+n}+1}^{\infty} \frac{b^i}{i!} \leq \sum_{j=0}^{\infty} \frac{M^{M^{3+n}+1+j}}{(M^{3+n}+1+j)!} = M \frac{M^{M^{3+n}}}{M^{3+n}!} \cdot \sum_{j=0}^{\infty} \frac{M^j}{\prod_{l=0}^{j}(M^{3+n}+1+l)}.$$

Now,

$$\sum_{j=0}^{\infty} \frac{M^j}{\prod_{l=0}^{j}(M^{3+n}+1+l)} \leq \sum_{j=0}^{\infty} \frac{M^j}{(M^{3+n})^j} \leq \sum_{j=0}^{\infty} \frac{1}{M^j} = \frac{M}{M-1}.$$

Finally, we claim that $\frac{M^{M^{3+n}}}{M^{3+n}!} \leq \frac{1}{M^n}$, in fact

$$\frac{M^{M^{3+n}}}{M^{3+n}!} = \left(\frac{M}{M^{3+n}} \cdot \frac{M}{M^{3+n}-1} \cdot \ldots \cdot \frac{M}{M^{3+n}-M+1}\right) \cdot \ldots \cdot \left(\frac{M}{M} \cdot \frac{M}{M-1} \cdot \ldots \cdot \frac{M}{1}\right).$$

Each of the multiplicand in the first bracket is $\leq \frac{1}{M}$ and each of the multiplicand in the second bracket is $\leq M$, so the product of the terms explicited in the above equation is ≤ 1. As for the terms in the central dotted part of the equation, they are certainly all ≤ 1 and there is at least one of them (for example, the term $\frac{M}{M^{1+n}}$) which is $\leq \frac{1}{M^n}$, and this proves the claim.

Hence, $\forall x \ (x \in (a,b) \ \Rightarrow \ \exp(x) \in (c,d))$.

Now we observe that

$$d - c \leq |d - \exp(b)| + |\exp(b) - \exp(a)| + |\exp(a) - c|.$$

By continuity of exp, for every $\varepsilon > 0$ and every $M \in \mathbb{N}$, there is a $\delta > 0$ such that, if $b - a < \delta$ and $a, b \in [-M, M]$, then $\exp(b) - \exp(a) < \varepsilon/3$; by Taylor's Theorem on the other hand, there is a δ such that $d - \exp(b) < \varepsilon/3$ and $\exp(a) - c < \varepsilon/3$. Hence the proposition is proved.

□

4.3.6 Corollary. *The sets*

$$\{t \mid t \text{ closed term and } t < 0\}$$

and

$$\{t \mid t \text{ closed term and } t > 0\}$$

are recursively enumerable.

Proof. We notice that the set $\mathcal{T}$ of all closed $L_{\exp}$-terms is recursive. Hence, the corollary follows from Proposition 4.2.10. □

There is actually a famous conjecture from transcendental number theory which could help us to establish the decidability of the quantifier free theory of exponentiation:

4.3.7 (Schanuel's Conjecture SC)**.** Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(a_1, \ldots, a_n, e^{a_1}, \ldots, e^{a_n}) \geq n.$$

We show with an example how SC helps to establish if a closed term is nonzero.

4.3.8 Example. Define inductively $\exp^1(x) = e^x \ \wedge \ \exp^{n+1}(x) = e^{\exp^n(x)}$. Let $0 \neq p \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial and let

$$t = p(\exp(1), \exp^2(1), \exp^3(1), \ldots, \exp^n(1)).$$

Then SC implies that $t \neq 0$. In fact, arguing by induction on n: the case $n = 1$ follows from the fact that e is transcendental over $\mathbb{Q}$; assuming that the statement holds for all $k < n$, we obtain that the numbers $\alpha_1 = 1,\ \alpha_2 = \exp(1), \alpha_3 = \exp^2(1), \ldots,\ \alpha_n = \exp^{n-1}(1)$ are algebraically independent (in particular, they are linearly independent). Hence, SC implies that

$$n \leq \operatorname{trdeg}_{\mathbb{Q}}(1, \exp(1), \exp^2(1), \ldots, \exp^{n-1}(1), \exp(1), \exp^2(1), \exp^3(1), \ldots, \exp^n(1)) =$$
$$= \operatorname{trdeg}_{\mathbb{Q}}(\exp(1), \exp^2(1), \exp^3(1), \ldots, \exp^n(1)),$$

so any nontrivial polynomial relation between the numbers

$$\exp(1), \exp^2(1), \exp^3(1), \ldots, \exp^n(1)$$

is nonzero, and this proves the inductive step.

In general, the following holds:

4.3.9 Proposition. *SC implies the decidability of the quantifier free theory of real exponentiation.*

The proposition is a particular case of Theorem 4.5.11.

4.4 On the roots of exponential terms (The existential theory)

In this section we will consider the problem of determining the existence of zeroes of functions from $\mathbb{R}^n$ to $\mathbb{R}$, defined by exponential terms.

4.4.1 Problem. *Find an algorithmic procedure which, given a term $t(\overline{x})$ in n variables, establishes if there exists $\overline{\alpha} \in \mathbb{R}^n$ such that $t(\overline{\alpha}) = 0$.*

There are several special cases which have been taken under consideration: see for example [29], [38], [34].

We will now consider the following problem: given n terms $t_1(\overline{x}), \ldots, t_n(\overline{x})$ in n variables, establish whether the system $t_1 = \ldots = t_n = 0$ has a nonsingular solution, i.e. a solution $\overline{\alpha} \in \mathbb{R}^n$such that the determinant of the Jacobian matrix of the system, evaluated at $\overline{\alpha}$ (denoted from now on by $J\overline{t}(\overline{\alpha})$), is nonzero.

Using the results of the previous section, we prove the following:

4.4.2 Theorem. *The set of all $n \times n$ system of term-defined equations which have a nonsingular solution is recursively enumerable.*

More precisely, there is an algorithm that, on input $\bar{t} = (t_1(\overline{x}), \ldots, t_n(\overline{x}))$, stops if and only if

$$\exists \overline{\alpha} \in \mathbb{R}^n \ t_1(\overline{\alpha}) = \ldots = t_n(\overline{\alpha}) = 0 \ \wedge \ J\bar{t}(\overline{\alpha}) \neq 0.$$

Proof. We notice that the set $\mathcal{T}$ of all $L_{\exp}$-terms is recursive. Hence, the theorem follows from Theorem 4.2.11. □

The most important result about finding the zeroes of exponential terms is due to Macintyre and Wilkie, and concerns *simple exponential polynomials*, i.e. special terms of complexity 1. Unfortunately, the theorem relies upon Schanuel's Conjecture.

4.4.3 Theorem (See [21])**.** *Given $n \in \mathbb{N}$, assume SC holds for all n-tuples of real numbers. Then, the set of all simple exponential polynomials in n variables which have a zero is recursively enumerable.*

In particular, in [21] an algorithm A is described, with the following property: If A stops on inputs $n \in \mathbb{N}$ and a simple exponential polynomial $p(x_1, \ldots, x_n)$, then p has a root in $\mathbb{R}^n$; if SC holds, then the reverse implication is also true: if p has a root in $\mathbb{R}^n$, then the algorithm A halts on inputs n, p after finitely many steps. An explicit description of the algorithm A can be found in [36].

4.4.4 Remark. The case $n = 1$ of SC is actually a theorem (proved by Lindemann, see [20]), hence we have the following conjecture-free statement: the set of all simple exponential polynomials in 1 variable which have a zero (singular or nonsingular) is recursively enumerable.

The importance of Theorem 4.4.3 will be clarified in the next section.

We end this section with a last result about exponential terms. We recall the inductive definition $\exp^0(x) = x$ and $\exp^{n+1}(x) = e^{\exp^n(x)}$. In the proofs of the following two theorems, $\overline{x}$ is an n-tuple of variables and we use the symbols $I, J, K, I_j, \ldots$ to denote multi-indexes of length n. All sums are finite, although we do not indicate it because the number of addends is irrelevant.

4.4.5 Theorem. *There is a recursive procedure which, given*

$$f \in \mathbb{Z}[\{\exp^k(x_1), \ldots, \exp^k(x_n)\}_{k \in \mathbb{N}}],$$

decides if $\forall \overline{x} \ f(\overline{x}) = 0$.

Proof. Consider an elementary superstructure $\mathcal{M} \succeq \mathbb{R}_{\exp}$ such that there exists an *infinite* element $y \in \mathcal{M}$, i.e. such that $\forall N \in \mathbb{N}\ y > N$. Then define

$$y_1 = y, y_2 = y_1^{\log(y_1)}, y_3 = y_2^{\log(y_2)}, \dots, y_n = y_{n-1}^{\log(y_{n-1})}.$$

Firstly we show that

$$y_1 \ll y_2 \ll \dots \ll y_n \ll \exp^1(y_1) \ll \dots \ll \exp^1(y_n) \ll \exp^2(y_1) \ll \dots,$$

where $a \ll b$ stands for $\forall m \in \mathbb{N}\ |a|^m < |b|$ (we write $\mathbb{N} \ll a$ if a is an infinite element).

To prove this, we only need to put the following remarks together:

1. y_i is of the form $y^{(\log(y))^m}$, for some natural number m (as can be easily seen arguing by induction).
2. $\forall m, k \in \mathbb{N} \quad k(\log(y))^m < y$, because y is bigger than any natural number.
3. $y_n \ll \exp^1(y_1)$, because of the two previous steps.
4. $y_i \ll y_{i+1}$, because y is bigger than any natural number.
5. If $\mathbb{N} \ll a \ll b$, then $e^a \ll e^b$.

This observation allows us to define a linear order on the set of all monomials of the ring $\mathbb{Z}[\{\exp^k(x_1), \dots, \exp^k(x_n)\}_{k\in\mathbb{N}}]$: if we set $y_i^k = \exp^k(y_i)$, we see that the set of all monomials in the variables $\{y_i^k |\ i = 1, \dots, n,\ k \in \mathbb{N}\}$ is ordered anti-lexicographically.

Secondly, take $f \in \mathbb{Z}[\{\exp^k(x_1), \dots, \exp^k(x_n)\}_{k\in\mathbb{N}}]$ and suppose that in f at most k iterations of exp appear. Then f has the form

$$f(\overline{x}) = \sum_{I_0,\dots,I_k} a_{I_0 \dots I_k} \overline{x}^{I_0} \exp^1(I_1 \cdot \overline{x}) \dots \exp^k(I_k \cdot \overline{x}),$$

where $I_i = (j_{i,1}, \dots, j_{i,n})$, $|I_i| < M \in \mathbb{N}$, $a_{I_0 \dots I_k} \in \mathbb{Z}$ and all monomials are distinct. Now, if we consider $\overline{x} = (y_1, \dots, y_n)$, with y_i as above, we induce a linear ordering of the monomials of $f(\overline{x})$. Thus the following holds in $\mathcal{M}$ (and hence in $\mathbb{R}_{\exp}$),

$$\forall \overline{x}\ f(\overline{x}) = 0 \ \Leftrightarrow\ \forall i = 0, \dots, k\ \forall I_i\ a_{I_0 \dots I_k} = 0.$$

The condition on the right hand side can be clearly checked algorithmically. $\square$

4.4.6 Theorem. *Let $\mathcal{O}$ be an oracle which tells, for every closed $L_{\exp}$-term t_0, whether $t_0 = 0$. Then there is an algorithm, with oracle $\mathcal{O}$, which decides, given an $L_{\exp}$-term $t(\overline{x})$, whether $\forall \overline{x}\ t(\overline{x}) = 0$.*

Proof. We first consider terms of a special normal form. We define inductively:

- A normal term of rank 0 is a polynomial $\sum_K b_K \overline{x}^K$ in reduced normal form, with coefficients closed terms b_K.

  A normal term of rank 0 is a constant if $\forall K \neq \overline{0},\ b_K = 0$.

- A normal term of rank $N+1$ is of the form

$$\sum_{I,j} a_{Ij} \overline{x}^I e^{t_j} + \sum_K b_K \overline{x}^K,$$

  where a_{Ij}, b_K are closed terms, t_j are nonconstant terms of rank $\leq N$ and all monomials are distinct.

  A normal term of rank $N+1$ is constant if $\forall I, j,\ a_{Ij} = 0$ and $\forall K \neq \overline{0},\ b_K = 0$.

We show that we can order the monomials of a normal term. We will use again an elementary superstructure $\mathcal{M}$ of $\mathbb{R}_{\exp}$ and the tuple $(y_1, \ldots, y_n)$ defined in the proof of Theorem 4.4.5. Using this trick, it is easy to compare two monomials of the form $\overline{x}^I$ and $\overline{x}^J$. It is also clear that, if t_j is a nonconstant term, then $\overline{x}^K \ll \overline{x}^I e^{t_j}$. We now show that, if $t_j, t_{j'}$ are nonconstant terms and $t_j \ll t_{j'}$, then $\overline{x}^I e^{t_j} \ll \overline{x}^{I'} e^{t_{j'}}$. In fact,

$$|\overline{x}^I e^{t_j}| < |e^{t_j}| \cdot |e^{t_j}| \ll |e^{t_{j'}}| < |\overline{x}^{I'} e^{t_{j'}}|.$$

Thus, if $t(\overline{x}) = \sum_{I,j} a_{Ij} \overline{x}^I e^{t_j} + \sum_K b_K \overline{x}^K$ is a normal term, it is true in $\mathcal{M}$ (and hence in $\mathbb{R}_{\exp}$) that

$$\forall \overline{x}\ t(\overline{x} = 0) \ \Leftrightarrow\ \forall I, j, K\ a_{Ij} = 0\ \wedge\ b_K = 0.$$

The condition on the right hand side can be checked using the oracle $\mathcal{O}$.

We now prove, with the help of $\mathcal{O}$, that given any term $t(\overline{x})$, we can effectively find a normal term $t^N(\overline{x})$ such that

$$\mathbb{R}_{\exp} \models \forall \overline{x}\ (t(\overline{x}) = t^N(\overline{x})).$$

We argue by induction on the *complexity* of $t(\overline{x})$ (i.e. the number of iterations of the operations $+, -, \cdot, \exp$ necessary to build up $t(\overline{x})$ from $\overline{x}$).

If $t(\overline{x})$ is a variable, then it is already in normal form. If $t(\overline{x}) = e^{s(\overline{x})}$, where $s(\overline{x})$ is a term whose complexity is lower than the complexity of $t(\overline{x})$, then we consider a normal term $s^N(\overline{x})$, equivalent to $s(\overline{x})$, and we observe that $e^{s^N(\overline{x})}$ is a normal term equivalent to $t(\overline{x})$. If $t(\overline{x}) = t_1(\overline{x}) + t_2(\overline{x})$, where t_i are terms of lower complexity, and t_i^N is a normal term equivalent to t_i, then we order the monomials of t_1^N and t_2^N. If the same monomial m appears both in t_1^N and t_2^N, with coefficient a_1 and a_2 respectively, we use the oracle $\mathcal{O}$ to check whether $a_1 + a_2 = 0$, and in that case we eliminate the monomial from the resulting sum; after taking care of this detail, the sum $t_1^N + t_2^N$ can be easily represented as a normal term t^N, which is equivalent to t. If $t(\overline{x}) = t_1(\overline{x}) \cdot t_2(\overline{x})$, where t_i are terms of lower complexity, and t_i^N is a normal term equivalent to t_i, then we first multiply each monomial of t_1^N with each monomial of t_2^N. If $\overline{x}^I e^{t_i}$ and $\overline{x}^J e^{t_j}$ are monomials of t_1^N and t_2^N respectively, then $\overline{x}^I e^{t_i} \cdot \overline{x}^J e^{t_j} = \overline{x}^{I+J} e^{t_i + t_j}$. After having checked if $t_i + t_j = 0$, we proceed to sum the products of monomials as explained above, and finally we easily represent $t_1^N(\overline{x}) \cdot t_2^N(\overline{x})$ as a normal term t^N, which is equivalent to t. $\square$

4.5 Reduction to the Existential Theory

In this section we will examine the decidability problem in all its generality, and we will show that it is sufficient to restrict our attention only to formulas of a certain simple form.

To do this, we will need to consider, together with the $L_{\exp}$-structure $\mathbb{R}_{\exp}$, two other first order structures, still with $\mathbb{R}$ as underlying set.

4.5.1 Definition. Let e and ϵ be a unary function symbol. We consider the first order languages
$L_{\exp} = \{+, -, \cdot, <, 0, 1, \exp\}$,
$L_\epsilon = \{+, -, \cdot, <, 0, 1, \epsilon\}$
and
the $L_{\exp}$-structure $\mathbb{R}_{\exp}$, where exp is the ordinary exponentiation,
the L_ϵ-structure $\mathbb{R}_\epsilon$, where $\forall x\ (0 < x < 1 \to \epsilon(x) = e^x\ \wedge\ (x \leq 0 \vee x \geq 1) \to \epsilon(x) = 0)$.

The function $\epsilon(x)$ is called *restricted exponentiation* .

4.5.2 Definition. Let E be one of the symbols $\exp, \epsilon$. Let

- $\mathcal{F}_E$ be the set of all L_E-formulas;

- $\exists\mathcal{F}_E$ be the set of all *existential* L_E-formulas[1];
- T_E be the set of all L_E-sentences which are true in $\mathbb{R}_E$;
- $\exists T_E$ be the set of all existential L_E-sentences which are true in $\mathbb{R}_E$.

4.5.3 Definition (Simple L_E-terms)**.** A simple L_E-term (in n variables) is an L_E-term of the form $p(x_1, \dots, x_n, E(x_1), \dots, E(x_n))$, where $p(\overline{x}, \overline{y})$ is a polynomial in $2n$ variables with coefficients on $\mathbb{Z}$.

4.5.4 Proposition (Normal form of existential formulas)**.** *Let $\phi(\overline{x})$ be an existential L_E-formula with free variables $\overline{x} = (x_1, \dots, x_n)$. Then, we can effectively find a natural number $k \in \mathbb{N}$ and a simple L_E-term t in $n+k$ variables such that $\phi(\overline{x})$ is equivalent (in $\mathbb{R}_E$) to the formula*

$$\exists y_1, \dots, y_k \ (t(\overline{x}, y_1, \dots, y_k) = 0).$$

Proof. Let $\phi(\overline{x}) = \exists \overline{z} \theta(\overline{x}, \overline{z})$, where θ is a quantifier free formula and $\overline{z} = (z_1, \dots, z_m)$. We may assume θ to be in disjunctive normal form, and even of the form

$$\theta(\overline{x}, \overline{z}) = \bigvee_{i=1}^{N} \bigwedge_{j=1}^{M} (t_{ij}(\overline{x}, \overline{z}) * 0),$$

where $t_{ij}(\overline{x}, \overline{z})$ are terms and $* \in \{=, \neq, <, \geq\}$ (this assumption is justified by the fact that the steps required to transform a generic quantifier free formula into an equivalent formula of the above type are totally effective; note that we have used, other than logical equivalences, the basic properties of field operations).

We notice that

- $t_{ij}(\overline{x}, \overline{z}) \neq 0 \leftrightarrow \exists y(y \cdot t_{ij}(\overline{x}, \overline{z}) - 1 = 0)$;
- $t_{ij}(\overline{x}, \overline{z}) < 0 \leftrightarrow \exists y(y^2 \cdot t_{ij}(\overline{x}, \overline{z}) + 1 = 0)$;
- $t_{ij}(\overline{x}, \overline{z}) \geq 0 \leftrightarrow \exists y(t_{ij}(\overline{x}, \overline{z}) - y^2 = 0)$;

and finally,

[1] An existential formula is of the form $\exists \overline{x}\ \theta(\overline{x})$, where $\theta(\overline{x})$ is a quantifier free formula.

- if t is a term obtained by composition of other terms, i.e. $t = s(s_1, \ldots, s_h)$ (s, s_i terms), then

$$t = 0 \leftrightarrow \exists y_1, \ldots, y_h(\bigwedge_{i=1}^{h}(s_i - y_i = 0) \wedge s(y_1, \ldots, y_h) = 0).$$

Now, by taking the care of naming new variables y whenever introducing a new existential quantifier, we have that such added quantifiers commute with finite disjunctions and conjunctions, hence we may assume that θ is of the form

$$\theta(\overline{x}, \overline{z}) = \exists \overline{y} \bigvee_{i=1}^{N} \bigwedge_{j=1}^{M} (p_{ij}(\overline{x}, \overline{z}, \overline{y}) = 0),$$

where this time p_{ij} are *simple* L_E-terms.

Finally, we observe that

$$\bigvee_{i=1}^{N} \bigwedge_{j=1}^{M} (p_{ij} = 0) \Leftrightarrow \prod_{i=1}^{N} \sum_{j=1}^{M} p_{ij}^2 = 0$$

and this concludes the proof of the proposition. □

4.5.5 Remark. In the proof of the above proposition the only property of $\mathbb{R}_E$ that we have used is that $\mathbb{R}_E$ is an expansion by function symbols of a real closed field. Hence, more generally, we have proved that, if RCF is the L_E-theory of real closed fields and $\phi(\overline{x}) \in \exists\mathcal{F}_E$, then there exist $k \in \mathbb{N}$ and a simple L_E-term t such that

$$\mathrm{RCF} \vdash \forall \overline{x}(\phi(\overline{x}) \leftrightarrow \exists y_1, \ldots, y_k \ (t(\overline{x}, y_1, \ldots, y_k) = 0)).$$

The main result of this section is due to Wilkie and Macintyre (see [21]):

4.5.6 Theorem (Effective model completeness of $\mathbb{R}_\epsilon$)**.** *There is an effective procedure which, given an L_ϵ-formula $\phi(\overline{x})$, produces an existential L_ϵ-formula $\psi(\overline{x})$, which is equivalent to ϕ (in $\mathbb{R}_\epsilon$).*

The theorem is proven by exhibiting a recursively axiomatized subtheory T_r of T_ϵ, which is model complete. We will give more details in the last section of this chapter, where we will also show that the axiomatization proposed in [21] can be simplified (see Theorem 4.7.23).

The main consequence of Theorem 4.5.6 is the following:

4.5.7 Corollary. *The decidability of the theory T_ϵ relies only on the decidability of $\exists T_\epsilon$.*

In fact, if we have a method for deciding the truth of existential sentences, then it is enough, given any sentence ϕ, to find an existential formula ψ, equivalent to ϕ, and to decide the truth of ψ.

Finally, it follows from Proposition 4.5.4 that it is enough to provide a decision method for the existence of roots of simple L_ϵ-terms.

An analogous of Theorem 4.5.6 for $\mathbb{R}_{\exp}$ is not known at the moment, although Wilkie has proved in [35] that $T_{\exp}$ is model complete.

There is, nevertheless, a relationship between $\mathbb{R}_{\exp}$ and $\mathbb{R}_\epsilon$ which will be very useful to our purposes, and which we will now explain.

Firstly, notice that the function ϵ is definable in $\mathbb{R}_{\exp}$, hence the theory T_ϵ is definable in $T_{\exp}$ (but not viceversa), so that we can consider the former as a subtheory of the latter.

Secondly, the following important result holds.

4.5.8 Theorem. *If T_ϵ is decidable, then so is $T_{\exp}$.*

The theorem was proved in [21], but it can also be derived from Ressayre's work [28]. We follow this second approach: in particular, Ressayre proves

4.5.9 Theorem. *$\mathbb{R}_{\exp}$ is recursively axiomatized over $\mathbb{R}_\epsilon$, via a recursive set of axioms which we will call* Ressayre's Axioms.

As a consequence, if T_ϵ is decidable, then it admits a recursive set of axioms A, which, together with Ressayre's Axioms, provide a recursive axiomatization for $T_{\exp}$. We will give more details in Section 4.7.

Hence we have reduced the decidability problem for $T_{\exp}$ to the decidability of the existential fragment $\exists T_{\exp}$. Moreover, even the recursive enumerability of $\exists T_{\exp}$ would be enough for our purposes: such a result would imply the recursive enumerability of $T_{\exp}$ which, being a *complete* theory, would then be automatically decidable (see [11] for an explanation of this last statement). This latter statement can be further simplified, combining Corollary 4.5.7 and Theorem 4.5.8:

4.5.10 Corollary. *If $\exists T_\epsilon$ is decidable, then so is $T_{\exp}$.*

Finally, this explains the importance of Theorem 4.4.3: combined with Proposition 4.5.4, it provides a (conditional, unfortunately) answer to the decidability problem.

4.5.11 Corollary (Macintyre, Wilkie, in [21])**.** *SC implies that $T_{\exp}$ is decidable.*

A final remark about restricted exponentiation: the simple L_ϵ-terms which appear in the normal form for existential L_ϵ-sentences are not in general continuous functions. This can be sometimes inconvenient, hence we give the following Definition.

4.5.12 Definition (Restricted simple exponential polynomials)**.** Let $n, r \in \mathbb{N}$, with $r \leq n$. Define $M_{n,r}$ as the ring of those functions (called *restricted simple exponential polynomials*)

$$f : (0,1)^r \times \mathbb{R}^{n-r} \to \mathbb{R}$$

such that there exists a polynomial $p \in \mathbb{Z}[x_1, \ldots, x_r, x_{r+1}, \ldots, x_n, y_1, \ldots, y_r]$ so that

$$\forall \overline{x} \in (0,1)^r \times \mathbb{R}^{n-r} \quad f(\overline{x}) = p(x_1, \ldots, x_r, x_{r+1}, \ldots, x_n, e^{x_1}, \ldots, e^{x_r}).$$

If $f \in M_{n,r}$, then define $V(f) = \{\overline{x} \in (0,1)^r \times \mathbb{R}^{n-r} |\ f(\overline{x}) = 0\}$.

Note that restricted simple exponential polynomials are smooth functions on their domain, hence, for $\overline{f} = (f_1, \ldots, f_n) \in M^n_{n,r}$, the set

$$V^{\mathrm{reg}}(f_1, \ldots, f_n) = \{\overline{x} \in (0,1)^r \times \mathbb{R}^{n-r} |\ \overline{f}(\overline{x}) = 0 \ \wedge \ J\overline{f}(\overline{x}) \neq 0\}$$

is well defined.

4.5.13 Remark. Notice that, as a function, a restricted exponential polynomial is really the restriction of an exponential polynomial to a proper subset of $\mathbb{R}^n$, whereas an ϵ-term is defined on all $\mathbb{R}^n$, and can not be represented as a single exp-term on its whole domain.

4.5.14 Claim. *We claim that the decidability of $\exists T_\epsilon$ depends on finding a decision method for the existence of roots of restricted simple exponential polynomials.*

Let $p \in \mathbb{Z}[\overline{x}, \overline{y}]$ be a polynomial in $2n$ variables.

Let $I = (i_1, \ldots, i_n)$, with $i_j = 0, 1$, be a binary multi-index. Let $B_0 = [0,1]$ and $B_1 = \mathbb{R} \setminus [0,1]$. Define $Q_I = B_{i_1} \times \ldots \times B_{i_n}$ and $\overline{\xi_I} = (\xi_{i_1}, \ldots, \xi_{i_n})$, where $\xi_{i_j} = e^{x_j}$ if $i_j = 0$ and $\xi_{i_j} = 0$ otherwise. Then,

$\exists \overline{x}\ p(\overline{x}, \epsilon(\overline{x})) = 0 \Leftrightarrow$

$\exists \overline{x}\ \bigvee_I (\overline{x} \in Q_I \wedge p(\overline{x}, \overline{\xi_I}) = 0) \Leftrightarrow$

$\bigvee_I \exists \overline{x}\ (\overline{x} \in Q_I \wedge p(\overline{x}, \overline{\xi_I}) = 0)$.

Note that the disjunction is finite and that, up to a permutation of the variables, the functions $p(\overline{x}, \overline{\xi_I})$ are elements of some ring $M_{n,r}$. Hence the claim is proved.

4.6 Equivalent statements of decidability

In this section we give a list of statements, each of which is equivalent to the decidability of the real exponential field. The equivalences do not depend on SC or on any other unproven conjecture.

The next three conjectures appear in [21].

4.6.1 (Weak Schanuel's Conjecture WSC)**.** There exists an effective procedure which, given $n \in \mathbb{N}$ and

$$f_1, \ldots, f_n, g \in \mathbb{Z}[x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}]$$

produces $\eta = \eta(n, f_1, \ldots, f_n, g) \in \mathbb{N}^+$ such that, for all $\overline{\alpha} \in \mathbb{R}^n$, if $\overline{\alpha} \in V^{\text{reg}}(f_1, \ldots, f_n)$, then either $g(\overline{\alpha}) = 0$ or $|g(\overline{\alpha})| > \eta^{-1}$.

To illustrate the meaning of WSC, we show:

4.6.2 Proposition. *WSC implies that the set*

$$\{t |\ t \text{ closed } L_{\exp}\text{-term } \wedge\ \mathbb{R}_{\exp} \models t = 0\}$$

is recursive.

Proof. Let t be a closed term. The idea is to reduce the problem of deciding if $t = 0$ to the problem (which we have explained how to solve in Section 4.3) of deciding if $|t| < 1/N$, where N is a natural number which, thanks to WSC, can be found effectively from t. We assume all terms to be in *special form* (see 4.1.2) and by induction on the complexity of the term t, we first show that:

(*) We can effectively find $n \in \mathbb{N}$ and simple exponential polynomials $f_1, \ldots, f_n, g$ in n variables such that

$$\exists! \overline{x} \in \mathbb{R}^n\ \overline{x} \in V^{\text{reg}}(f_1, \ldots, f_n)\ \wedge\ \forall \overline{x} \in \mathbb{R}^n\ (\overline{x} \in V^{\text{reg}}(f_1, \ldots, f_n)\ \Rightarrow g(\overline{x}) = t).$$

If $t = 0$, then we take $n = 1$ and $f = g = x$, and analogously for $t = 1$. Suppose the statement true for all special terms of complexity $< m$ and suppose $t = f(t_1, \ldots, t_k)$, for some simple exponential polynomial $f \in \mathbb{Z}[\overline{x}, e^{\overline{x}}]$ and some terms t_i of complexity $< m$. By inductive hypothesis, for all $i = 1, \ldots, k$, we can find $n_i, \in \mathbb{N}$ and simple exponential polynomials $f_1^i, \ldots, f_{n_i}^i, g^i$ in the variables $\overline{x}_i = (x_1^i, \ldots, x_{n_i}^i)$, such that

$$\exists! \overline{x}_i \in \mathbb{R}^n\ \overline{x}_i \in V^{\text{reg}}(f_1^i, \ldots, f_{n_i}^i) \wedge \forall \overline{x}_i \in \mathbb{R}^n\ (\overline{x}_i \in V^{\text{reg}}(f_1^i, \ldots, f_{n_i}^i)\ \Rightarrow g^i(\overline{x}_i) = t_i).$$

Now consider, for all $i = 1, \ldots, k$, new variables u_i, together with the simple exponential polynomials $h_i(\overline{x}_i, u_i) = u_i - g^i(\overline{x}_i)$.

Now define $n = \sum_{i=1}^{k} n_i$ and the n-tuple of variables $\overline{z} = (\overline{x}_1, \ldots, \overline{x}_k, u_1, \ldots, u_k)$ and, for all $i = 1, \ldots, k$, the simple exponential maps $F_i(\overline{z}) = (f_1^i(\overline{x}_i), \ldots, f_{n_i}^i(\overline{x}_i))$ and $H_i(\overline{z}) = h_i(\overline{x}_i, u_i)$ and $G(\overline{z}) = f(u_1, \ldots, u_k)$.

Since we took the care of nominating new variables, it easy to see that

$$\exists! \ \overline{z} \quad \overline{z} \in V^{\mathrm{reg}}(F_1, \ldots, F_k, H_1, \ldots, H_k),$$

and

$$\forall \overline{z} \ \ \overline{z} \in V^{\mathrm{reg}}(F_1, \ldots, F_k, H_1, \ldots, H_k) \Rightarrow G(\overline{z}) = t.$$

It is in fact easy to see that the jacobian matrix of the above system of equations is blockwise upper triangular, with nonzero elements on the diagonal, hence nonsingular. This proves the statement (*).

Now we can use WSC and see that

$$\begin{array}{lcl} t = 0 & \Leftrightarrow & (\forall \overline{x} \in \mathbb{R}^n \ \ \overline{x} \in V^{\mathrm{reg}}(f_1, \ldots, f_n) \ \rightarrow \ g(\overline{x}) = 0) \\ & \Leftrightarrow & (\forall \overline{x} \in \mathbb{R}^n \ \ \overline{x} \in V^{\mathrm{reg}}(f_1, \ldots, f_n) \ \rightarrow \ |g(\overline{x})| < \eta^{-1}(n, f_1, \ldots, f_n, g)) \\ & \Leftrightarrow & |t| < \eta^{-1}. \end{array}$$

We know from Section 4.3 that there is an algorithm that on input (t, η) stops if and only if $|t| < \eta^{-1}$, and hence, if WSC holds, if and only if $t = 0$. Thus both the set

$$\{t| \ t \text{ closed } L_{\exp}\text{-term } \wedge \ \mathbb{R}_{\exp} \models t = 0\},$$

and its complement are recursively enumerable, which proves the statement of the proposition. □

The next two conjectures concern the problem of bounding the norm of the roots of simple exponential polynomials.

4.6.3 (Last Root Conjecture LRC)**.** There exists an effective procedure which, given $n \in \mathbb{N}$ and

$$f_1, \ldots, f_n \in \mathbb{Z}[x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}]$$

produces $\nu = \nu(n, f_1, \ldots, f_n) \in \mathbb{N}^+$ such that, for all $\overline{\alpha} \in \mathbb{R}^n$, if $\overline{\alpha} \in V^{\mathrm{reg}}(f_1, \ldots, f_n)$, then $|\overline{\alpha}| < \nu$.

4.6.4 (First Root Conjecture FRC)**.** There exists an effective procedure which, given $n \in \mathbb{N}$ and

$$f \in \mathbb{Z}[x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}]$$

produces $\mu = \mu(n, f) \in \mathbb{N}^+$ such that, if f has a zero in $\mathbb{R}^n$, then there exists $\overline{\alpha} \in \mathbb{R}^n$ such that $f(\overline{\alpha}) = 0$ and $|\overline{\alpha}| < \mu$.

We have shown in the previous section that, in order to solve the general decidability problem for the real exponential field, it is enough to focus on the restricted exponential case. We have also shown that it is enough to prove the recursive enumerability of the existential fragment of the theory of restricted exponentiation $\exists T_\epsilon$. We will hence state each conjecture for *restricted* simple exponential polynomials (see Definition 4.5.12), and show the equivalence with the recursive enumerability of $\exists T_\epsilon$.

Then, we can restate

4.6.5 (WSC for restricted exp)**.** There exists an effective procedure which, given $n, r \in \mathbb{N}$, with $r \leq n$ and $f_1, \dots, f_n, g \in M_{n,r}$ produces $\eta = \eta(n, r, f_1, \dots, f_n, g) \in \mathbb{N}^+$ such that, for all $\overline{\alpha} \in (0,1)^r \times \mathbb{R}^{n-r}$, if $\overline{\alpha} \in V^{\text{reg}}(f_1, \dots, f_n)$, then either $g(\overline{\alpha}) = 0$ or $|g(\overline{\alpha})| > \eta^{-1}$.

4.6.6 (LRC for restricted exp)**.** There exists an effective procedure which, given $n, r \in \mathbb{N}, r \leq n$ and $f_1, \dots, f_n \in M_{n,r}$ produces $\nu = \nu(n, r, f_1, \dots, f_n) \in \mathbb{N}^+$ such that, for all $\overline{\alpha} \in (0,1)^r \times \mathbb{R}^{n-r}$, if $\overline{\alpha} \in V^{\text{reg}}(f_1, \dots, f_n)$, then $|\overline{\alpha}| < \nu$.

4.6.7 (FRC for restricted exp)**.** There exists an effective procedure which, given $n, r \in \mathbb{N}, r \leq n$ and $f \in M_{n,r}$ produces $\mu = \mu(n, r, f) \in \mathbb{N}^+$ such that, if f has a zero in $(0,1)^r \times \mathbb{R}^{n-r}$, then there exists $\overline{\alpha} \in (0,1)^r \times \mathbb{R}^{n-r}$ such that $f(\overline{\alpha}) = 0$ and $|\overline{\alpha}| < \mu$.

Notice that for $r = n$, FRC and LRC are trivially true, because in this case the domain of the functions is $(0,1)^n$, so $\nu = \mu = 1$ is a good bound.

Now we are ready to prove

4.6.8 Theorem. *The following statements are equivalent:*

1. $\exists T_{\exp}$ *($\exists T_\epsilon$, respectively) is recursively enumerable;*
2. *The First Root Conjecture (FRC for restricted exp, respectively) is true;*
3. *The Last Root Conjecture (LRC for restricted exp, respectively) is true;*

4. *The Weak Schanuel's Conjecture (WSC for restricted exp, respectively) is true;*

Proof. The proof is very similar for restricted and unrestricted exp. We will prove the theorem in the restricted case.

(1⇒2).

If $\exists T_{\exp}$ is recursively enumerable, then there is an algorithm A which, on input an existential sentence ϕ, stops if and only if ϕ is true. For every $n, r, N \in \mathbb{N}, r \leq n$ and $f \in M_{n,r}$, consider the existential sentence

$$\phi(n,r,f,N) = (\exists \overline{y} \exists \overline{x} \ \ f(\overline{y}) = 0 \rightarrow \ \ \overline{x} \in (0,1)^r \times \mathbb{R}^{n-r} \ f(\overline{x}) = 0 \ \wedge \ |\overline{x}| < N).$$

Then A stops on input $\phi(n,r,f,N)\}$ if and only if the sentence $\phi(n,r,f,N)$ is true, and for all $n, r \in \mathbb{N}, r \leq n$ and $f \in M_{n,r}$, there exists $N \in \mathbb{N}$ such that A stops on input $\phi(n,r,f,N)\}$. Define the function $\mu'(n,r,f) = \min\{(N,t) :$ A stops on input $\phi(n,r,f,N)$ in less than t steps$\}$. Then the projection μ of μ' onto the first coordinate, is a recursive function, witnessing the requirement.

(2⇒3).

Let $(f_1, \ldots, f_n) \in M^n_{n,r}$. We now need to recall Khovanskii's result 4.7.14 in its full power, namely, we can compute a bound $M = M(\overline{f}) \in \mathbb{N}$ on the cardinality of $V^{\mathrm{reg}}(f_1, \ldots, f_n)$. Define $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{f} = (f_1, \ldots, f_n)$. There are $M+1$ cases to consider:
Case 0. Define $g_0 = 0$ and $\mu_0 = 0$.
Case 1. Define $g_1(\overline{x}, y) = \sum_{i=1}^n (f_i(\overline{x}))^2 + (J\overline{f}(\overline{x})y - 1)^2$. Notice that

$$\forall \overline{x} y \ \ g_1(\overline{x}, y) = 0 \ \Leftrightarrow \ \overline{x} \in V^{\mathrm{reg}}(\overline{f}) \ \wedge \ y = J\overline{f}^{-1}(\overline{x}).$$

Compute, using FRC, the number $\mu_1 = \mu(n+1, r, g_1)$.
Case 2. Define

$$g_2(\overline{x}_1, y_1, \overline{x}_2, y_2, w) = g_1(\overline{x}_1, y_1) + g_1(\overline{x}_2, y_2) + \prod_{i=1}^n [w(x_{1,i} - x_{2,i}) - 1]^2.$$

Notice that

$$\forall \overline{x}_1, y_1, \overline{x}_2, y_2, w \ \ g_2(\overline{x}_1, y_1, \overline{x}_2, y_2, w) = 0 \ \Leftrightarrow \ \overline{x}_1, \overline{x}_2 \in V^{\mathrm{reg}}(\overline{f}) \wedge \overline{x}_1 \neq \overline{x}_2.$$

Compute, using FRC, the number $\mu_2 = \mu(2n+3, r, g_2)$.
$\vdots$
Case M. Define

$$g_M(\overline{x}_1, \ldots, \overline{x}_M, y_1, \ldots, y_M, w_{1,2}, \ldots, w_{M-1,M}) =$$

$$= \sum_{i=1}^{M} g_1(\overline{x}_i, y_i) + \sum_{ij} \prod_{k=1}^{n} [w_{i,j}(x_{i,k} - x_{j,k}) - 1]^2.$$

Notice that, if $\overline{z} = (\forall \overline{x}_1, \ldots, \overline{x}_M, y_1, \ldots, y_M, w_{1,2}, \ldots, w_{M-1,M})$, then

$$\forall \overline{z}\ g_M(\overline{z}) = 0 \ \Leftrightarrow\ \bigwedge_{i=1}^{M} \overline{x}_i \in V^{\mathrm{reg}}(f_1, \ldots, f_n) \ \wedge\ \bigwedge_{i \neq j} \overline{x}_i \neq \overline{x}_j.$$

Compute, using FRC, the number $\mu_M = \mu(M(n+1) + M(M-1), r, g_M)$.

We claim that $\nu = \max\{\mu_i |\ i = 0, \ldots, M\}$ is a recursive bound on the number of regular solutions of $(f_1, \ldots, f_n)$. In fact, suppose $V^{\mathrm{reg}}(f_1, \ldots, f_n)$ consists of exactly $k \in \{0, \ldots, M\}$ elements. Then, in particular, $\exists \overline{z}\ g_k(\overline{z}) = 0$. By FRC, there exists $\overline{z}$ such that $g_k(\overline{z}) = 0$ and $|\overline{z}| < \mu_k$. This in turn implies, by definition of g_k, that

$$\exists \overline{x}_1, \ldots, \overline{x}_k\ \bigwedge_{i=1}^{k} \overline{x}_i \in V^{\mathrm{reg}}(f_1, \ldots, f_n) \ \wedge\ \bigwedge_{i \neq j} \overline{x}_i \neq \overline{x}_j \ \wedge\ \bigwedge_{i=1}^{k} |\overline{x}_i| < \mu_k.$$

Since there are exactly k regular solutions, this concludes the proof.

($3 \Rightarrow 4$).

Let $f_1, \ldots, f_n, g$ be as in the statement 4.6.5. Consider a new variable y and let $h(\overline{x}, y) := y g(\overline{x}) - 1$. Notice that $\overline{f} = (f_1, \ldots, f_n)$ does not depend on the variable y and $(\overline{x}, y)$ is a root of h iff $g(\overline{x}) \neq 0$ and $y = g^{-1}(\overline{x})$. Hence,

$$\begin{gathered} \forall \overline{x}, y \in (0,1)^r \times \mathbb{R}^{n+1-r},\ (\overline{x}, y) \in V^{\mathrm{reg}}(f_1, \ldots, f_n, h) \ \Leftrightarrow \\ \Leftrightarrow\ \overline{x} \in V^{\mathrm{reg}}(f_1, \ldots, f_n) \ \wedge\ g(\overline{x}) \neq 0 \wedge y = g^{-1}(\overline{x}). \end{gathered}$$

If we apply LRC to the system $(\overline{f}, h)$, we find $N := \nu(n+1, r, \overline{f}, h)$ such that

$$\forall \overline{x}, y \in (0,1)^r \times \mathbb{R}^{n+1-r}\ ((\overline{x}, y) \in V^{\mathrm{reg}}(f_1, \ldots, f_n, h) \ \Rightarrow\ |\overline{x}, y| < N.$$

Combining the two above equations, we find

$$\forall \overline{x} \in (0,1)^r \times \mathbb{R}^{n-r}\ \overline{x} \in V^{\mathrm{reg}}(f_1, \ldots, f_n) \wedge g(\overline{x}) \neq 0 \Rightarrow |\overline{x}| < N \wedge |g(\overline{x})| > N^{-1}.$$

Hence, if we define $\eta(n, r, \overline{f}, g) := \nu(n+1, r, \overline{f}, h)$, then η is a recursive function, witnessing the requirement.

(4⇒1).

We prove that the set

$$D = \{(n,r,g) |\ n,r \in \mathbb{N}, r \leq n,\ g \in M_{n,r}\ \wedge\ \exists \overline{x} \in (0,1)^r \times \mathbb{R}^{n-r}\ g(\overline{x}) = 0\}$$

is recursively enumerable, which is enough, by the result 4.5.4 of Section 4.5.

The following algorithm stops on input (n,r,g) if and only if $(n,r,g) \in D$.

Fix a recursive enumeration of the tuples $(\overline{x}_0, y_0, f_1, \ldots, f_n, M, m)$, where $(\overline{x}_0, y_0) \in \mathbb{Q}^{n+1},\ f_1, \ldots, f_n \in M_{n,r},\quad M, m \in \mathbb{N}$.

Step 1. Consider a tuple $(\overline{x}_0, y_0, f_1, \ldots, f_n, M, m)$.

Step 2. Compute $\eta = \eta(n, r, f_1, \ldots, f_n, g)$, as in 4.6.5.

Step 3. Run the Newton algorithm described in 4.4.2 on the tuple

$$(\overline{x}_0, y_0; f_1(\overline{x}), \ldots, f_n(\overline{x}), h(\overline{x}, y); M; m),$$

where $h(\overline{x}, y) = (1 - \eta^2 g(\overline{x})^2)y^2 - 1$ (notice that $h(\overline{x}, y) = 0$ for some y iff $|g(\overline{x})| < 1/\eta$). This means that we run m steps of Newton's algorithm and then stop, with positive/negative answer if the due inequalities are/are not satisfied by the given data.

Step 4. If Steps 3 gives a positive answer, then stop. Otherwise, go to Step 1 and consider another tuple.

Suppose the algorithm just described stops on some tuple $(\overline{x}_0, y_0, f_1, \ldots, f_n, M, m)$. Then, by Step 3, there exists $\overline{\alpha} \in \mathbb{R}^n$ such that $\overline{\alpha} \in V^{\text{reg}}(f_1, \ldots, f_n)$ and $|g(\overline{\alpha})| < \eta^{-1}$. Then, by WSC, necessarily $g(\overline{\alpha}) = 0$.

Vice versa, suppose $\exists \overline{\alpha} \in \mathbb{R}^n |\ g(\overline{\alpha}) = 0$. Then by the result 2.5.6 in Chapter 2, there exist $f_1, \ldots, f_n \in M_{n,r}$ and a point (which for simplicity we will still call $\overline{\alpha}$) in $\mathbb{R}^n$ such that $\overline{\alpha} \in V(g) \cap V^{\text{reg}}(f_1, \ldots, f_n)$.

Let $\eta = \eta(n, r, f_1, \ldots, f_n, g,)$ and $h(\overline{x}, y) = (1 - \eta^2 g(\overline{x})^2)y^2 - 1$.

Now we observe that $(\overline{\alpha}, 1) \in V^{\text{reg}}(f_1, \ldots, f_n, h)$: in fact $h(\overline{\alpha}, 1) = 0$ and $\partial h/\partial y(\overline{\alpha}, 1) = 2 \neq 0$. Hence there are $(\overline{x}_0, y_0) \in \mathbb{Q}^{n+1}$ and $M, m \in \mathbb{N}$ such that Step 3 is satisfied on input $(\overline{x}_0, y_0; f_1(\overline{x}), \ldots, f_n(\overline{x}), h(\overline{x}, y); M, m)$. Hence the algorithm stops.

□

By the results of the previous section, we have

4.6.9 Corollary. *Each of the statements in the theorem above is equivalent to the decidability of* $T_{\exp}$.

Notice that, combining Theorem 4.6.8 with the results of the previous section, and Theorem 4.5.11 in particular, we obtain a proof of "SC⇒WSC"
.

We now restate the above results in a more model-theoretic way, with a conjecture which has a very concise formulation (which will be related to the results of next section):

4.6.10 (Effective Archimedeicity Conjecture EAC)**.** There is an effective bound on the norm of 0-definable elements.

More precisely, there is a recursive function $\lambda : \{\phi(x)\} \to \mathbb{N}$ such that

$$\exists! x \phi(x) \to \forall x(\phi(x) \to |x| < \lambda(\phi)).$$

It is clear that the conjecture is true if $T_{\exp}$ is decidable. To see the reverse implication, we show that EAC implies LRC:

Let $\phi(x)$ be a formula that expresses "x is the minimum number y such that for all $\overline{\alpha} \in \mathbb{R}^n$, if $\overline{\alpha} \in V^{\text{reg}}(f_1, \ldots, f_n)$, then $|\overline{\alpha}| \leq y$ (or zero, if $V^{\text{reg}}(f_1, \ldots, f_n) = \emptyset$) ". Then define $\nu(n, f_1, \ldots, f_n)$ as $\lambda(\phi)$. $\square$

4.6.11 Remark. It is natural to investigate EAC in the case when the formula ϕ is of some special form. For example, the answer is known when $\phi(x)$ is $f(x) = 0$ (f simple exponential term). For this reason, we concentrate now our attention on the case of restricted exponentiation, where we know (by effective model completeness) that every formula $\phi(x)$ is effectively equivalent to a formula of the type $\exists \overline{y} f(x, \overline{y}) = 0$ (where f is a simple exponential term and $\overline{y}$ is a tuple of variables $(y_1, \ldots, y_n)$). Hence, EAC can be expressed in the following equivalent way: let $p(x, \overline{y})$ be a *restricted* simple exponential polynomial such that $\exists! x \exists \overline{y} p(x, \overline{y}) = 0$; then it is possible to find an effective bound on the norm of such an x.

Let a be the unique x such that $\exists \overline{y} p(x, \overline{y}) = 0$. We know that $\forall \overline{y}$ such that $p(a, \overline{y}) = 0$, all the derivatives $\frac{\partial p}{\partial y_i}(a, \overline{y})$ are zero, otherwise the Implicit Function Theorem 2.2.8 would be violated (there is no open neighbourhood of a where we can apply the theorem). On the other hand, we know by noetherianity (applied to the noetherian differential ring $\mathbb{Z}[a, e^a][\overline{y}_0, e^{\overline{y}_0}]$) that there is an N such that $\frac{\partial^i p}{\partial x^i}(a, \overline{y}) = 0$ (for $i < N$) and $\frac{\partial^N p}{\partial x^N}(a, \overline{y}) \neq 0$ (but unfortunately we have no means to bound EFFECTIVELY such an N).

Proceeding in our investigation, we note that, by o-minimality, the real (restricted) exponential field has definable Skolem functions: in particular, there are $q_1, \ldots, q_n$ definable functions such that $p(a, q_1(a), \ldots, q_n(a)) = 0$. By classical results, the functions q_i are continuous, differentiable, smooth, analytic out of some compact set. Our result about effective o-minimality, though, suggests that there is no intelligible way of recovering from the formula ϕ the compact set out of which f has the above mentioned properties.

4.7 Axiomatizations

In this section we start with a different approach to the decidability problem. If we could provide a complete recursive axiomatization for $\mathbb{R}_{\exp}$, then the theory of this structure would be decidable, since every sentence would be algorithmically proved or disproved from the axioms (see [11] for a detailed proof of the latter statement).

Macintyre and Wilkie have given a result in this direction, namely, they have given in [21] a recursive list of axioms, which is complete, provided that Schanuel's Conjecture holds. Here we simplify this list of axioms in the following way: we prove that some of the axioms given in [21] are superfluous, i.e. they can be proved from the remaining axioms; moreover, using the effective o-minimality result of Chapter 3, we propose to substitute a subset of the axiomatization proposed in [21], with another (stronger and more elegant) set of axioms (the details can be found in Theorem 4.7.25). As a result we obtain a list of four recursive schemes of axioms, which is complete if Schanuel's Conjecture holds, and hence constitutes a candidate for a complete axiomatization of $\mathbb{R}_{\exp}$. We will build up such a candidate by adding schemes of axioms one by one, and discussing at every step the properties of the models.

4.7.1 Definition (Definably complete exponential fields)**.** Consider $L_{\exp} = \{+, -, \cdot, <, 0, 1, \exp\}$. A *definably complete exponential field* is a definably complete $L_{\exp}$-structure (see Chapter 1, Definition 1.1.1) such that exp is a C^1 function and $\forall x \ \exp'(x) = \exp(x) \ \wedge \ \exp(0) = 1$.

4.7.2 Remark. The class of definably complete exponential fields is recursively axiomatized, by the following axiom schemes:

4.7.3 Axioms.

1. [ORDERED FIELD] (see 1.1.2)
2. [DEFINABLE COMPLETENESS] (see 1.1.2)
3. [EXP′=EXP]: $\exp(0) = 1 \wedge$

$$\forall x \ \forall \varepsilon \ \exists \delta \ \forall y \neq x \ (|x-y| < \delta \ \rightarrow \left| \frac{\exp(x) - \exp(y)}{|x-y|} - \exp(x) \right| < \varepsilon).$$

We have discussed in Chapter 1 the basic properties of definably complete structure, and here we recall some of them, applied to the particular case of exponentiation.

We fix a definably complete exponential field $\mathbb{K}$.

4.7.4 Proposition. *For every $N \in \mathbb{N}$, for every $x \in \mathbb{K}$, we define $p_N(x) = \sum_{i=0}^{N} \frac{x^i}{i!}$ (the N^{th}* Taylor Polynomial of exp*). Then,*

1. $\forall N \in \mathbb{N}, \forall x \in \mathbb{K} \exists \xi \in \mathbb{K} \quad |\xi| < |x| \ \wedge \ \exp(x) - p_N(x) = \exp(\xi) \frac{x^{N+1}}{(N+1)!}$.

2. $\forall M \in \mathbb{N}, \forall x \in \mathbb{K} \ \exists N \in \mathbb{N} \quad |\exp(x) - p_N(x)| < 1/M^2$.

3. *The following sentences (*Ressayre's Axioms*) are true in $\mathbb{K}$:*
   (HOM) $\forall x, y \quad \exp(x+y) = \exp(x) \cdot \exp(y)$
   (INCR) $\forall x, y \quad x < y \rightarrow \exp(x) < \exp(y)$
   (GA_n) ($n \in \mathbb{N}$) $\forall x \quad x > (n+1)! \rightarrow \exp(x) > x^n$
   (SURJ) $\forall y \quad y > 0 \rightarrow \exists x \quad \exp(x) = y$.

Proof. Statement (1) follows from Taylor's Theorem 1.3.3, [EXP′=EXP] and the fact, proved below, that the function exp is increasing. Statement (2) follows immediately from (1).

Proof of Statement (3): Let $y \in \mathbb{K}$ and consider the definable C^1 function $\begin{array}{rccc} h: & \mathbb{K} & \rightarrow & \mathbb{K} \\ & x & \mapsto & \exp(x+y) - \exp(x)\exp(y) \end{array}$ Then h satisfies the linear differential equation

$$h'(x) = \exp'(x+y) - \exp'(x)\exp(y) = \exp(x+y) - \exp(x)\exp(y) = h(x),$$

and $h(0) = 0$. Hence, by the uniqueness result 1.5.1, h is identically zero, which proves (HOM).

To prove (INCR), by Corollary 1.3.4, we only need to prove that for all $x \in \mathbb{K}$ we have $\exp'(x) > 0$. Since $\exp'(0) = \exp(0) = 1 > 0$, if $\exp'(x)$ were negative in some point x, then by the Intermediate value Theorem 1.2.3, $\exp'$ (and hence exp) would vanish somewhere, which is impossible by Uniqueness result 1.5.1. Hence exp is increasing over $\mathbb{K}$.

Now let $b \in \mathbb{K}$. Consider the $(n+2)$-th Taylor expansion of exp in the interval $(0, b)$:

$$\exp(b) = 1 + b + \frac{b^2}{2} + \ldots + \frac{b^{n+1}}{(n+1)!} + \frac{\exp(c) b^{n+2}}{(n+2)!},$$

for some $c \in (0, b)$. Then it is clear that for $b > (n+1)!$, we have $\exp(b) > b^n$, which proves (GA_n).

[2]Notice that we can not infer that the limit of the partial sums of Taylor's series is exp, as such a limit may not exist or may be only infinitesimally close to exp

From (GA_n) it follows that $\sup \exp = +\infty$. We also know that $\forall x \ \exp(x) > 0$ and, by (HOM), that $\exp(-1)\exp(1) = 1$. Hence, for $(-x)$ big enough, $\exp(-x) > (-x)^n$; in particular, for n even, $0 < \exp(x) < \frac{1}{x^n}$, hence $\inf \exp = 0$. So, by Corollary 1.2.4, (SURJ) holds.

□

4.7.5 Remark. We can now go back to Section 4.4 and notice that the proofs of Theorems 4.4.5 and 4.4.6 only use Ressayre's Axioms, hence both results hold not only in $\mathbb{R}_{\exp}$ but also in every definably complete exponential field.

4.7.6 Proposition (On the Axiom [EXP′=EXP])**.** *The Axiom [EXP′=EXP] can be replaced with a statement ψ in* universal form *(i.e. a statement consisting of a sequence of universal quantifiers followed by a quantifier free formula), such that [EXP′=EXP] and ψ are equivalent in the theory of definably complete ordered fields. Hence, we will rename [EXP′=EXP]:=ψ and observe that the Axiom [EXP′=EXP] holds in every substructure of $\mathbb{K}$.*

Proof. We start by observing that the fact that exp is a continuous function can be expressed as follows:

$$\forall x \forall \varepsilon > 0\, \exists \delta\, \forall y \ \ |x-y| < \delta \to |\exp(x) - \exp(y)| < \varepsilon.$$

To transform this into a universal sentence it is enough to show that we can choose a δ that can be expressed as a quantifier free definable function of x, ε. More precisely, we show that the following holds:

$$\forall x \forall \varepsilon \forall y \ \ 0 < \varepsilon < \exp(x) \wedge |y-x| < \varepsilon \exp(-x) - \frac{\varepsilon^2 \exp(-2x)}{2} \ \to \ |\exp(x) - \exp(y)| < \varepsilon.$$

We observe that, putting $z = x - y$, we have $|\exp(x) - \exp(y)| = \exp(x)|1 - \exp(z)|$; hence, it is enough to find a suitable function $\eta(\varepsilon)$ such that

$$\forall \varepsilon \, \forall z \text{ if } 0 < \varepsilon < 1 \text{ and } |z| < \eta(\varepsilon) \text{ then } |1 - \exp(z)| < \varepsilon,$$

and then consider $\delta(x, \varepsilon) = \eta(\exp(-x)\varepsilon)$ for the general case, whenever $\exp(-x)\varepsilon < 1$.

We show now that the function $\eta(\varepsilon) = \varepsilon - \varepsilon^2/2$ is suitable for our need. It is enough to consider the case when $z > 0$ (because when $z < 0$ the first derivative of $\exp(z)$ is smaller than when z is positive, and hence, all the more so, the same η is suitable in this case too). Then, $|1 - \exp(z)| < \varepsilon$ if and only if $\exp(z) - 1 < \varepsilon$, if and only if $z < \log(1+\varepsilon)$, where log is the inverse function of exp defined on $\mathbb{K}^+$. The existence of log follows from Proposition 4.7.4, and it is easy to see that the Taylor polynomials of $\log(1+x)$ are $q_N(x) =$

$\sum_{i=1}^{N}(-1)i+1\frac{x^i}{i}$ ($N \in \mathbb{N}$) and that $p_{2k}(x) < \log(1+x) < p_{2k+1}(x)$ $\forall k \in \mathbb{N}$ and $\forall x$ such that $0 < x < 1$. In particular, $\varepsilon - \varepsilon^2/2 < \log(1+\varepsilon)$, so $\forall \varepsilon$ such that $0 < \varepsilon < 1$ and $\forall z$ one has $|z| < \varepsilon - \varepsilon^2/2 \ \rightarrow\ |1-\exp(z)| < \varepsilon$.

A similar argument provides a universal sentence ensuring that exp is C^1 and that $\exp'(x) = \exp(x)$:

$$\forall x \forall \varepsilon \forall t \ \ 0 < \varepsilon < \exp(x) \wedge |t| < \delta(x,\varepsilon) \ \rightarrow \left| \ \frac{\exp(x+t)-\exp(t)}{t} - \exp(x) \ \right| < \varepsilon,$$

where $\delta(x,\varepsilon) = \exp(-x)\varepsilon$. To see the details, we use the same trick and work around zero, with $t > 0$. We note that $\forall t < 1$, $\exp(t) < 1 + t + t^2$. In fact, by Taylor's Theorem, $\exists \xi$ such that $0 < \xi < t$ and $\exp(t) - 1 - t - t^2/2 = \exp(\xi)t^3/6 < 3t^3/6 = t^3/2 < t^2$. Hence, $\frac{\exp(t)-1-t}{t} < t$, so for $t < \varepsilon$, the quotient $\frac{\exp(t)-1-t}{t}$ is smaller than ε, as required.

□

4.7.7 Proposition. *Let $\mathbb{K}$ be a definably complete exponential field. Suppose that $\mathbb{K}$ has a definably complete substructure $\mathbb{F}$ which is an archimedean field. Then, for every term $t(\overline{x})$, if $\mathbb{R}_{\exp} \models \exists \overline{x}\, t(\overline{x}) = 0$, then $\mathbb{K} \models \exists \overline{x}\, t(\overline{x}) = 0$.*

Proof. If L is an ordered field, we define the cut of an element $a \in L$ over $\mathbb{Q}$ as the set $\mathrm{Cut}^L_{\mathbb{Q}}(a) := \{(q_1, q_1) \in \mathbb{Q}^2 |\ L \models q_1 < a < q_2\}$. $\mathbb{F}$ being archimedean, to different elements of $\mathbb{F}$ correspond different cuts. Then, it is well known that the function ϕ, sending an element $a \in \mathbb{F}$ to the unique real number r such that $\mathrm{Cut}^{\mathbb{F}}_{\mathbb{Q}}(a) = \mathrm{Cut}^{\mathbb{R}}_{\mathbb{Q}}(r)$, is an injective ordered field homomorphism of $\mathbb{F}$ into $\mathbb{R}$. On $\mathbb{F}$ two exponential functions are given: the exponential $\exp^{\mathbb{K}}$ inherited from $\mathbb{K}$ and the function $\exp^{\mathbb{R}}(x) = \phi^{-1}(e^{\phi(x)})$.

We first note that, by Proposition 4.7.6, $\mathbb{F}$ is a definably complete exponential field, i.e. the axiom [EXP′=EXP] holds in $\mathbb{F}$ for $\exp^{\mathbb{K}}$.

In particular, Taylor's Theorem holds, so

$$\forall x \in \mathbb{F}\, \forall \varepsilon \in \mathbb{Q}^+\, \exists N \in \mathbb{N}, \left| \ \exp^{\mathbb{K}}(x) - \sum_{n=0}^{N} \frac{x^n}{n!} \ \right| < \varepsilon.$$

As for $\exp^{\mathbb{R}}$, though not being a definable function in the structure $\mathbb{F}$, it still satisfies Taylor's Theorem, by pull-back from $\mathbb{R}$ (ϕ commutes with sums and products, and fixes rational numbers):

$$\forall x \in \mathbb{F}\, \forall \varepsilon \in \mathbb{Q}^+\, \exists N \in \mathbb{N}, \left| \ \exp^{\mathbb{R}}(x) - \sum_{n=0}^{N} \frac{x^n}{n!} \ \right| < \varepsilon.$$

Hence,

$$\forall x \in \mathbb{F}\ \forall \varepsilon \in \mathbb{Q}^+ \ \ |\exp^{\mathbb{K}}(x) - \exp^{\mathbb{R}}(x)| < \left| \exp^{\mathbb{K}}(x) - \sum_{i \le N} \frac{x^n}{n!} \right| + \left| \sum_{i \le N} \frac{x^n}{n!} - \exp^{\mathbb{R}}(x) \right| < 2\varepsilon.$$

This means that the difference between the two exponentials is infinitesimal in $\mathbb{F}$, hence it is zero.

Having observed this, we proceed with the proof. Let $t(\overline{x})$ be a term and suppose that t has a root $\overline{a} \in \mathbb{R}^n$. Let B be a closed box (i.e. a product of closed intervals) with rational vertices, containing $\overline{a}$. We consider in $\mathbb{F}$ the minimum value of $|t(\overline{x})|$ over B (it exists by definable completeness). We aim to prove that such a minimum is zero. Now, if $t^{\mathbb{R}}(\overline{x})$ is the function defined by the term t when exp is interpreted as $\exp^{\mathbb{R}}$, then $|t^{\mathbb{R}}(\overline{x})|$ attains arbitrarily small values on rational points which are close enough to $\overline{a}$. More precisely, for every $\varepsilon \in \mathbb{Q}^+$, there exists $\overline{q} \in \mathbb{Q}^n \cap B$ such that $|t^{\mathbb{R}}(\overline{q})| < \varepsilon$. But, as we noticed at the beginning of the proof, $t^{\mathbb{R}}(\overline{q}) = t^{\mathbb{K}}(\overline{q})$, where $t^{\mathbb{K}}(\overline{x})$ is the function defined by the term t when exp is interpreted as $\exp^{\mathbb{K}}$. Hence the minimum of $|t^{\mathbb{K}}(\overline{x})|$ in $B \cap \mathbb{F}$ is zero. □

4.7.8 Remark. The last result holds more generally for all existential sentences: we recall that by 4.5.4 every existential sentence is effectively equivalent to a sentence of the form $\exists \overline{x}\ t(\overline{x}) = 0$, where t is a term (actually, a simple term).

4.7.9 Remark. Let $\mathbb{K}$ be a definably complete exponential field. Going through the results of Chapter 2, we now observe that, given $n \in \mathbb{N}$, the ring $M_n = \mathbb{Z}[x_1, \dots, x_n, e^{\overline{x}_1}, \dots, e^{\overline{x}_n}]$, whose elements are interpreted as functions from $\mathbb{K}^n$ to $\mathbb{K}$, is a (finitely generated) noetherian differential ring. Hence, all the results of Chapter 2 for noetherian differential rings, hold for M_n.

4.7.10 Definition (0-definable points and exp-algebraic points)**.** Let $\mathbb{K}$ be a definably complete exponential field. For all $n \in \mathbb{N}$, we put $M_n = \mathbb{Z}[x_1, \dots, x_n, e^{\overline{x}_1}, \dots, e^{\overline{x}_n}]$. Recall the definition of M-degree 2.4.5. We define:

$$D_{\mathbb{K}} = \{a \in \mathbb{K} |\ \exists \text{ a formula } \varphi \text{ so that } \mathbb{K} \models \exists! x \varphi(x) \wedge \varphi(a)\}$$

and

$$E_{\mathbb{K}} = \{a \in \mathbb{K} |\ \exists n \in \mathbb{N}, \exists x_2, \dots, x_n\ \deg_{M_n}(a, x_2, \dots, x_n) = 0\}.$$

4.7.11 Remark. Notice that $D_{\mathbb{K}}$ is a substructure of $\mathbb{K}$ and a field: both the field operations and exponentiation are 0-definable. $E_{\mathbb{K}}$ is also a substructure of $\mathbb{K}$ and a field. To see this, let us show for example that, if $a \in E_{\mathbb{K}}$, then $e^a \in E_{\mathbb{K}}$ (an analogous argument works for $+, -, \cdot, /$): suppose there exist $n \in \mathbb{N}$ and $\overline{f} = (f_1, \ldots, f_n)$ an n-tuple of simple exponential polynomials such that $\mathbb{K} \models \exists x_2, \ldots, x_n \; (a, x_2, \ldots, x_n) \in V^{\text{reg}}(\overline{f})$. Consider a new variable y and the simple exponential polynomial $g(y, x_1) = y - e^{x_1}$. Then $g(e^a, a) = 0$. We claim that $\exists x_2, \ldots, x_n \; (e^a, a, x_2, \ldots, x_n) \in V^{\text{reg}}(g, \overline{f})$: in fact the Jacobian matrix of this system is blockwise upper triangular (since $\overline{f}$ does not depend on the variable y), with nonzero elements on the diagonal (since $\frac{\partial g}{\partial y} = 1$). In particular, $e^a \in E_{\mathbb{K}}$.

$D_{\mathbb{K}}$ and $E_{\mathbb{K}}$ are natural candidates for a substructure of $\mathbb{K}$ with the characteristics required to apply Proposition 4.7.7 (this explains also the statement of the conjecture EAC 4.6.10). It is in general, though, not clear what the relationship between $D_{\mathbb{K}}$ and $E_{\mathbb{K}}$ is, nor if either of the two structures is definably complete.

4.7.12 Remark. $\mathbb{K}$ and $E_{\mathbb{K}}$ satisfy the same existential sentences. In fact, let ϕ be an existential sentence such that $\mathbb{K} \models \phi$. By 4.5.4, we may assume that ϕ is of the form $\exists \overline{x} g(\overline{x}) = 0$, where g is a simple exponential polynomial in n variables. Then, by Proposition 2.5.6 in Chapter 2, there exist simple exponential polynomials $f_1, \ldots, f_n$ such that

$$\mathbb{K} \models \exists \overline{x} g(\overline{x}) = 0 \wedge \overline{x} \in V^{\text{reg}}(f_1, \ldots, f_n).$$

In particular, there exist $a_1, \ldots, a_n \in E_{\mathbb{K}}$ such that $g(a_1, \ldots, a_n) = 0$. Hence, $E_{\mathbb{K}} \models \phi$.

4.7.13 Remark. If $Th(\mathbb{K})$ is model complete, then $D_{\mathbb{K}}$ is contained in $E_{\mathbb{K}}$. In fact, let $a \in D_{\mathbb{K}}$ be defined by a formula $\varphi(x)$. By model completeness, we may assume that φ is of the form $\exists \overline{y}(x, \overline{y}) = 0$, where g is a simple exponential polynomial in $1 + n$ variables. So, $\mathbb{K} \models \exists! x \exists \overline{y} g(x, \overline{y}) = 0$. Again, by result 2.5.6 of Chapter 2, there exist simple exponential polynomials $f_1, \ldots, f_{n+1}$ such that

$$\mathbb{K} \models \exists x \, \exists \overline{y} \, g(x, \overline{y}) = 0 \wedge (x, \overline{y}) \in V^{\text{reg}}(f_1, \ldots, f_{n+1}).$$

By uniqueness of such an x, it is then true that $\exists \overline{y}(a, \overline{y}) \in V^{\text{reg}}(f_1, \ldots, f_{n+1})$, so $a \in E_{\mathbb{K}}$.

It is natural to ask whether M_n is in general a Khovanskii ring (see 2.6.1). It is, of course, if $\mathbb{K} = \mathbb{R}_{\exp}$, because of Khovanskii's Theorem:

4.7.14 Theorem (Khovanskii [18]). *Let $m \leq n, N \in \mathbb{N}$ and $p_1, \ldots, p_n \in \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be polynomials of degree at most N. Then there are recursive functions*

$$\mu : \mathbb{N}^2 \to \mathbb{N} \text{ and } \nu : \mathbb{N}^3 \to \mathbb{N}$$

such that, if

$$\begin{array}{lll} F : \mathbb{R}^n \to \mathbb{R}^n & \text{and} & G : \mathbb{R}^n \to \mathbb{R}^m \\ F(\overline{x}) = (p_1(\overline{x}, e^{\overline{x}}), \ldots, p_n(\overline{x}, e^{\overline{x}})) & & G(\overline{x}) = (p_1(\overline{x}, e^{\overline{x}}), \ldots, p_m(\overline{x}, e^{\overline{x}})) \end{array}$$

then

1. *The number of regular zeroes of F is finite and bounded (uniformly in the coefficients of the p_js) by $\mu(n, N)$.*
2. *The number of connected components of $G^{-1}(0)$ is finite and bounded (uniformly in the coefficients of the p_js) by $\nu(n, m, N)$.*

For the general case it is not clear if the axioms we gave are sufficient to conclude that M_n is a Khovanskii ring. However, since Khovanskii's result is *effective*, we can add a new scheme of axioms which will ensure that $\{M_n | \, n \in \mathbb{N}\}$ is a collection of Khovanskii rings in all definably complete exponential fields which satisfy it. More precisely,

4.7.15 Axiom ([KHOVANSKII'S FINITENESS]). Let $n, N \in \mathbb{N}$ and $p_1, \ldots, p_n \in \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be polynomials of degree at most N. Set $f_i(\overline{x}) := p_i(\overline{x}, e^{\overline{x}}) \in M_n$. Then,

$$|V^{\text{reg}}(f_1, \ldots, f_n)| \leq \mu(n, N).$$

4.7.16 Remark. Let $\mathbb{K}$ be a definably complete exponential field which satisfies the axiom's scheme 4.7.15, and let M_n be as before. Notice that thanks to Theorem 2.6.6, all M_n-varieties have finitely many definably connected components. It is not clear from our proof, though, if, as in the classical case over the real numbers, there is an *effective* bound on the number of connected components.

4.7.17 Remark. Let $\mathbb{K}$ be a definably complete exponential field which satisfies the scheme [KHOVANSKII'S FINITENESS] Then, $E_{\mathbb{K}}$ is contained in $D_{\mathbb{K}}$. In fact, let $a \in E_{\mathbb{K}}$; we are looking for a formula that defines the singleton $\{a\}$. Let $n, N \in \mathbb{N}, b_2, \ldots, b_n \in \mathbb{K}$ and $f_1, \ldots, f_n$ be simple exponential polynomials such that $(a, b_2, \ldots, b_n) \in V^{\text{reg}}(f_1, \ldots, f_n)$. Now, $V^{\text{reg}}(f_1, \ldots, f_n)$ consists of finitely many points, say, at most N. We want to enumerate these points, according to a linear order: this is possible because the lexicographic order of a finite $\emptyset$-definable set is clearly $\emptyset$-definable. Suppose, without loss of generality, that $(a, b_2, \ldots, b_n)$ is the first point of $V^{\text{reg}}(f_1, \ldots, f_n)$ in this order. Then a is defined by the formula saying: "there exist $x_2, \ldots, x_n$ such that $(a, x_2, \ldots, x_n) \in V^{\text{reg}}(f_1, \ldots, f_n)$ and $(a, x_2, \ldots, x_n)$ is the first element of $V^{\text{reg}}(f_1, \ldots, f_n)$, according to the lexicographic order ".

Notice that the finiteness hypothesis given by Axiom [KHOVANSKII'S FINITENESS] is crucial in the proof of this remark. In particular, it is possible to find an elementary superstructure $\mathcal{M} \succeq \mathbb{R}_{\sin}$ such that there exists an $x \in \mathcal{M}$ with $\sin(x) = 0$, but x is not definable.

4.7.18 Remark. So far we have considered models of the schemes of axioms [ORDERED FIELD], [DEFINABLE COMPLETENESS], [EXP′=EXP] and [KHOVANSKII'S FINITENESS]. In view of the results of Chapter 3, we can replace the schemes [DEFINABLE COMPLETENESS] and [KHOVANSKII'S FINITENESS] with a single scheme of stronger axioms, denoted by [O-MINIMALITY], ensuring the o-minimality of the models (see 3.1.6). This gives us a recursive subtheory of $\mathbb{R}_{\exp}$ which is o-minimal, so that we can use all the machinery and tools of o-minimal geometry to study its models. Unfortunately, this does not give us necessarily a *model complete* recursive theory, nor is it known at the moment of the existence of a recursive model complete subtheory of $T_{\exp}$. This is somehow inconvenient, and will force us, as explained in Section 4.5, to consider the theory of restricted exponentiation.

To prove the effective model completeness of restricted exponentiation, we need to add another scheme of axioms, as it was proved in [21]. This last scheme is very technical and ensures a "polynomially bounded behaviour " of some functions defined from the function $\exp(\frac{1}{1+x^2})$. We need some notation first.

4.7.19 Definition. Define

- $D(x) := \frac{1}{1+x^2}$;
- fix $r, n, m \in \mathbb{N}$, with $r \leq n$; $\forall p \in \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$ of degree $\leq m$, set

  $$f(\overline{x}) = p(x_1, \ldots, x_n, D(x_1), \ldots, D(x_r), \exp(D(x_1)), \ldots, \exp(D(x_r))).$$

  We say that f has *polynomial degree* at most m.

4.7.20 Theorem (Macintyre, Wilkie in [21])**.** *There exists a recursive function*

$$\tau : \mathbb{N}^3 \to \wp_{<\omega}(\mathbb{Q})$$

such that, for all

- $n, r, m \in \mathbb{N}$, *with* $r \leq n$,
- $f_1, \ldots, f_n \in \mathbb{Z}[x_1, \ldots, x_n, D(x_1), \ldots, D(x_r), \exp(D(x_1)), \ldots, \exp(D(x_r))]$ *having polynomial degree at most* m,
- $\phi_2(x, y), \ldots, \phi_n(x, y)$ *formulas*,

the following sentence is true in $\mathbb{R}_{\exp}$*:*
$\forall x_0$,
IF

- φ_i *defines the graph of a function* $\phi_i : (x_0, +\infty) \to \mathbb{K} \quad (i = 2, \ldots, n)$,
- $\forall x > x_0 \ f_i(x, \phi_2(x), \ldots, \phi_n(x)) = 0 \quad (i = 2, \ldots, n)$ *and*
- $\det \frac{\partial(f_2,\ldots,f_n)}{\partial(x_2,\ldots,x_n)}(x, \phi_2(x), \ldots, \phi_n(x)) \neq 0$

THEN
there is $q \in \tau(n, r, m)$ *such that, either* $\forall x > x_0 \ f_1(x, \phi_2(x), \ldots, \phi_n(x)) = 0$, *or the function* $x^q \cdot f_1(x, \phi_2(x), \ldots, \phi_n(x))$ *tends to a finite nonzero limit, when* $x \to \infty$.

4.7.21 Definition. The collection of all sentences of the above type, with n, r, m, f_i, φ_i as above, forms a scheme of sentences, which we denote by [PUISEUX EXPANSION].

We will only try to give a rough idea of the meaning of [PUISEUX EXPANSION], and of why Theorem 4.7.20 holds. The existence of a finite nonzero limit of the function $x^q \cdot f_1(x, \phi_2(x), \ldots, \phi_n(x))$ in Theorem 4.7.20 follows from the fact that $h(x) := f_1(x, \phi_2(x), \ldots, \phi_n(x))$ is *finitely subanalytic* (see [9] for the definition) and from the following result, due to Van den Dries:

4.7.22 Theorem (Van den Dries, [9])**.** *Let* $a \in \mathbb{R}$ *and* $h : (a, +\infty) \to \mathbb{R}$ *be a finitely subanalytic function. Then there exists* $b \geq a$ *such that, on* $(b, +\infty)$, *either* h *is always zero or* h *has a convergent Puiseux expansion:*

$$\forall x \in (b, +\infty) \ \ h(x) = \sum_{i \geq p} a_i x^{-\frac{i}{d}} \ \ (p \in \mathbb{Z}, \ d \in \mathbb{N}, \ a_i \in \mathbb{R}, \ a_p \neq 0).$$

In particular, there exists a rational number q *(that is,* $q = \frac{p}{d}$*) such that the function* $x^q \cdot h$ *tends to a finite nonzero limit, when* $x \to \infty$.

The definition of the recursive function τ in Theorem 4.7.20 follows from a careful manipulation of the recursive function μ in Theorem 4.7.14. The details can be found in [21].

We are now able to compare our results with the results contained in [21]. In this latter paper the authors exhibit a recursive subtheory T_{res} of T_ϵ, which is proven to be model complete (without assuming any unproven conjecture). The models of T_{res} are definably complete ordered fields, with a function ϵ which satisfies the differential equation $\epsilon'(x) = \epsilon(x)$ if $x \in (0, 1)$ and $\epsilon(x) = 0$ if $x \notin (0, 1)$, such that Khovanskii's property, expressed in 4.7.15, holds for n-tuples of simple restricted exponential polynomials, and the axiom scheme [PUISEUX EXPANSION] holds (notice that we do not need full exponentiation, but just $\epsilon(x)$ to define the function $e^{\frac{1}{1+x^2}}$, so [PUISEUX EXPANSION] is a set of sentences in T_ϵ). These properties of T_{res} are expressed in Axioms A1,A2,A5,A6,A7 in [21]. There it is also required that the models of T_{res} satisfy the Uniqueness Theorem 1.5.1 and the Implicit Function Theorem 2.2.8, expressed in axioms A3,A4 of [21]. We have however shown in Chapters 1 and 2 that these two theorems hold in every definably complete structure, hence we can discharge axioms A3 and A4. Moreover, in view of the results of Chapter 3, we can replace the axioms ensuring definable completeness and Khovanskii's finiteness (i.e. axioms A2 and A6 in [21]), with a single scheme of axioms, ensuring the o-minimality of the models.

Hence we can conclude

4.7.23 Corollary. *The recursive L_ϵ-theory T_r axiomatized by*

- *[ORDERED FIELD],*
- *[ϵ'=ϵ]:= "$\epsilon'(x) = \epsilon(x)$ if $x \in (0, 1)$ and $\epsilon(x) = 0$ if $x \notin (0, 1)$",*
- *[O-MIN(ϵ)]:= a scheme of axioms ensuring o-minimality,*
- *[PUISEUX EXPANSION],*

is model complete.

The proof of this result, which can be found in [35] and [21], uses the axiom scheme [PUISEUX EXPANSION]: given a model $\mathcal{M}$ of T_r and a submodel $\mathcal{N}$, the submodel $\mathcal{N}$ is proven to be existentially closed in $\mathcal{M}$, by reducing the problem, via [PUISEUX EXPANSION], to the analogous result for real closed fields.

4.7.24 Corollary. *Any two o-minimal L_ϵ-structures $\mathcal{A} \subseteq \mathcal{B}$ which both satisfy [PUISEUX EXPANSION] and [ϵ'=ϵ], satisfy also $\mathcal{A} \preceq \mathcal{B}$.*

We proceed in our analysis of the paper [21]. The other result of the paper is that a recursive $L_{\exp}$-theory is exhibited, with the property that, if Schanuel's Conjecture is true, then this theory is complete, and hence provides a recursive axiomatization for $T_{\exp}$. The models of this theory are definably complete ordered fields, with a function exp which satisfies the differential equation $\exp'(x) = \exp(x)$ and $\exp(0) = 1$, such that the axiom schemes [KHOVANSKII'S FINITENESS] and [PUISEUX EXPANSION] hold. Moreover, the models satisfy, together with the Uniqueness Theorem 1.5.1 and the Implicit Function Theorem 2.2.8, two more requirements: it is possible to establish the sign of a nonzero simple closed term (this property is ensured by the axiom scheme T_H in [21]) and Newton's Theorem 1.4.1 holds (this property is axiomatized by T_{NA} in [21]). Notice that we have previously shown that all these further requirements are superfluous, since the sign of a nonzero simple closed term can be established using Taylor's expansion and Newton's Theorem 1.4.1 holds in every definably complete structure. Finally, using again the results of Chapter 3, we can replace the schemes [DEFINABLE COMPLETENESS] and [KHOVANSKII'S FINITENESS] with the single scheme [O-MINIMALITY], already defined in Remark 4.7.18, ensuring the o-minimality of the models (see 3.1.6).

Now we are able to produce a simplified version of Theorem 4.5.11:

4.7.25 Theorem. *Schanuel's Conjecture implies that the recursive subtheory T of $T_{\exp}$, axiomatized by*

- *[ORDERED FIELD],*
- *[EXP′=EXP],*
- *[O-MINIMALITY],*
- *[PUISEUX EXPANSION],*

is complete and hence provides a recursive axiomatization of $T_{\exp}$.

The proof of 4.5.11 applies to this simplified situation as well, but the axiomatization proposed here is simpler than the one proposed in [21] and, using Ressayre's work, as explained in Section 4.5, the arguments can be simplified. More precisely, we first observe that, by Ressayre's result 4.5.9, T axiomatizes $T_{\exp}$, and only if, T_r axiomatizes T_ϵ (recall that T prove Ressayre's Axioms, see 4.7.4). By model completeness, we now know that $T_r \cup \exists T_\epsilon$ axiomatizes T_ϵ, so T_r axiomatizes T_ϵ if and only if T_r axiomatizes the existential part $\exists T_\epsilon$. Finally, if SC holds, then T axiomatizes $\exists T_{\exp}$ (and hence $\exists T_\epsilon$), because the arguments used in [21] can be directly applied to out simpler axiomatization T.

Now we can state a corollary of Proposition 4.7.7

4.7.26 Corollary. *Suppose every model of the theory T has an archimedean submodel. Then T is complete.*

We are thus interested in studying the properties of the models of T. We conclude with a list of open questions about the models of the axiom schemes appearing in this theory.

4.7.27 Problem. *Let* $\mathbb{K}$ *be a definably complete exponential field, i.e. an* $L_{\exp}$*-model of the axiom schemes* [ORDERED FIELD], [DEFINABLE COMPLETENESS], [EXP′=EXP]. *For* $n \in \mathbb{N}$, *is the ring* $M_n = \mathbb{Z}[x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)] \subset C^\infty(\mathbb{K}^n, \mathbb{K})$ *necessarily a Khovanskii ring? The proof of Khovanskii's Theorem 4.7.14 uses some nonelementary facts, as for example Sard's Theorem, thus it does not automatically transfer to structures not based on the real numbers. And if it is indeed a Khovanskii ring, is the structure* $\mathbb{K}$ *even o-minimal?*

4.7.28 Problem. *Let* $(\mathbb{K}, \exp)$ *be an o-minimal exponential field, i.e. an* $L_{\exp}$*-model of the axiom schemes* [ORDERED FIELD], [EXP′=EXP], [O-MINIMALITY]. *Is the structure* $(\mathbb{K}, \exp \restriction (0,1))$ *polynomially bounded?*

4.7.29 Problem. *Let* $(\mathbb{K}, \exp)$ *be an o-minimal exponential field such that* $(\mathbb{K}, \exp \restriction (0,1))$ *is a polynomially bounded structure. Is* $(\mathbb{K}, \exp)$ *elementary equivalent to* $\mathbb{R}_{\exp}$*?*

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