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Flat currents and Mumford-Shah functionals in codimension higher than one

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Contents

List of Notation

$\mathbf{S}(T)$	Size of T	2.6
$[\![Q]\!]$	Integration current on the oriented polyhedron Q	2.6
$\mathbf{B}_0(E)$	Flat 0-dimensional boundaries of finite mass	2.7
$\mathcal{G}(Q,Q')$	Hybrid distance between flat boundaries	2.7
$W_1(\mu,\nu)$	Wasserstein 1-distance between probability measures μ,ν	2.7
$Geo(E)$	Space of constant speed geodesics parametrized on $[0,1]$	2.7
$MBV(\mathbb{R}^k,M)$	Space of functions $u:\mathbb{R}^k\to M$ of metric bounded variation	2.8
$\lvert Du\rvert$	Total variation measure of an MBV function	2.8
$\tilde{\mathbf{F}}$	Functional flat norm	2.8
$M\mu$	Maximal function of a measure μ	2.8
wd_xf	w^*-differential of f at x	2.10
ω_k	Lebesgue measure of the unit ball of $\mathbb{R}^k$	2.10
$\mathbf{J}_k(L)$	Jacobian of a linear map L betwween Banach spaces	2.10
$\mathrm{Tan}^{(k)}(S,x)$	Approximate tangent space of S at x	2.10
$d_x^S\pi$	Tangential differential	2.10
λ_V	Area factor of a Banach space V	2.10
$\Pi_k(Y)$	w^*-continuous and linear maps $\pi:Y\to\mathbb{R}^k$ with $\mathrm{Lip}(\pi)\le 1$	2.10
$W^{s,p}(\Omega,\mathbb{R}^n)$	Sobolev spaces of exponents s and p	3.1
$\dot W^{1,p}(\Omega,\mathbb{R}^n)$	Equivalence classes of distributions with weak derivative in L^p	3.1
$j(u), Ju$	prejacobian and distributional jacobian of u	3.1
deg	Brouwer's degree function	3.1
$BV(\Omega), SBV(\Omega)$	Space of real valued classical and special functions of bounded variation	3.2
$B_nV(\Omega,\mathbb{R}^n)$	Jerrard and Soner's space of functions of bounded n-variation	3.2
$[Ju]^a, [Ju]^s$	Absolutely continuous and singular part of Ju	3.2
$\mathfrak{h}^1, \mathfrak{bmo}$	Local Hardy and BMO spaces	3.2
$SB_nV(\Omega)$	Special functions of bounded n-variation	3.2
$GSB_nV(\Omega)$	Generalized special functions of bounded n-variation	3.3
$GSBV(\Omega)$	Generalized special functions of bounded variation	3.3
R_u, T_u	Regular and singular part of Ju for $u\in GSB_nV(\Omega)$	3.3
S_u	Singular set of u	3.3
$M\nabla u$	vector of minors of ∇u of every rank	4.1
$MS(u,\Omega)$	Mumford-Shah functional in higher codimension	4.2
$\mathcal{H}^1, BMO$	Classical Hardy and BMO spaces	4.3
$\mathcal{M}^{*k}_{\Omega}(S), \mathcal{M}^{k}_{*\Omega}(S)$	Upper and lower Minkowski content	5.1
$\underline{F}$, $\Gamma-\liminf_h F_h$	Lower Γ-limit of (F_h)	5.2
$\overline{F}$, $\Gamma-\limsup_h F_h$	Upper Γ-limit of (F_h)	5.2
$B(U)$	Bounded Borel functions ranging in $[0,1]$	5.2
$d(v,v')$	Convergence in measure distance	5.2
$R_n(\Omega)$	Space of maps with absolutely continuous jacobian	5.2
$X(\Omega), Y(\Omega), Z(\Omega)$	Spaces of approximating functions	5.2

CHAPTER 1

Introduction

The theory of currents was developed during the 50's in response to the difficulty of solving via the classical parametric methods [**Cou50**] the Plateau problem for surfaces of dimension higher than two in the Euclidean space. The first contributions to weak notions of surfaces came from Caccioppoli [**Cac52**] and De Giorgi [**DG54, DG55**], who developed the concepts of sets of finite perimeter and BV function and provided existence of solutions to the minimal surface problem in codimension one. In their 1960 fundamental paper [**FF60**] Federer and Fleming presented the general theory of normal and integral currents, solving the minimal surface problem in arbitrary dimension and codimension.

Following an intuition of De Giorgi [**DG95**] the theory of currents has been extended to nonsmooth spaces by Ambrosio and Kirchheim in [**AK00a**], where the duality with smooth differential forms available in the Riemannian setting is replaced by the duality with Lipschitz functions (see also [**Lan11**] for a friendly exposition and a local variant of the theory). This framework, available in a fairly general class of metric spaces, allows to prove again existence of solutions to Plateau's problem for integral currents, and more generally the existence of mass-minimizing currents in a fixed homology class (see [**Wen07**]). A general summary of these works is proposed in the first sections of chapter 2.

If we move from normal currents to flat currents with finite mass, other remarkable extensions of the classical theory have been obtained among others by Fleming in [**Fle66**], and White in [**Whi99a, Whi99b**], dealing with Euclidean spaces and general group coefficients, and by De Pauw and Hardt [**DPH12**], dealing with general spaces and general group coefficients at the same time. In particular the theory of flat G-chains presented in [**DPH12**] relaxes the requirements on the metric space by isometrically embedding E in larger Banach spaces. In this connection see also [**AW11, AK11**], where coefficients in $\mathbb{Z}_p$ are dealt with also in metric spaces, using the idea of taking the quotients of integral currents.

In [**DL02**], De Lellis proved in the metric framework the rectifiability of the "lowest dimensional part" of a flat chain with finite mass and real coefficients. As an example, one might consider the distributional derivative Du of a BV function in $\mathbb{R}^n$, that can canonically be viewed as a flat $(n-1)$-dimensional current with finite mass. In this case only the restriction of Du to the so-called jump set of u provides a $(n-1)$-rectifiable measure, while the remaining part of Du might be diffuse.

In chapter 2 we further generalize the aforementioned results with the introduction of a new quantity called size, motivated by the definition of a Mumford-Shah energy in higher codimension, introduced in chapter 4. This size quantity, which essentially is the Hausdorff measure of the set where the lower dimensional part of the current is concentrated, needs to be defined for general currents with possibly infinite mass. The space of flat k-currents $\mathbf{F}_k(E)$ in the metric context, thoroughly presented in section 2.4, proves to be the right class to treat such general objects: in particular we show how many useful properties enjoyed by normal currents extend to this larger space.

Section 2.6 is devoted to the definition of the concentration measure μ_T for flat currents and the size functional $\mathbf{S}(T) = \mu_T(E)$, the main objects of our investigation. They are defined through an integral-geometric approach that involves only the 0-dimensional slices of the current which are required, in the case of finite size, to have a finite support (and, as a consequence, finite mass, according to Theorem 2.6.3). Then, we prove lower semicontinuity of size with respect to flat convergence, obtaining in particular a closure property for sequences of currents with equibounded size, Theorem 2.6.4. In the Euclidean setting different notions of size, constructed for instance by relaxation or implementing the techniques of multiple valued functions, have been studied by several authors, and can be found in [**Alm86, DPH03, Mor89, DLS11**] together with the properties of currents satisfying suitable size bounds.

In Section 2.7 we introduce a quantity $\mathcal{G}(T, T')$, called hybrid distance, in the class $\mathbf{B}_0(E)$ of flat boundaries with finite mass: it takes into account all representations $T - T' = \partial(X + R)$, with X having finite mass and R having finite size. This results in a smaller distance, compared to the classical one where no R term is present, which allows to extend the BV estimates for the slice operator from currents with finite mass to currents with finite size. Here we use the flexibility of these BV estimates, namely the possibility to adapt them to several classes of "geometric" distances (see for instance [**PR11**]). The distance $\mathcal{G}$, though weaker than the classical flat distance, will be proved to be still sufficiently strong to control the oscillations of the atoms of the slices. In order to show the separability of $(\mathbf{B}_0(E), \mathcal{G})$, we will use some results from the theory of optimal transportation in geodesic spaces, see for instance [**AGS08**].

Since we aim to prove a rectifiability result, we recall in Section 2.8 the concept of rectifiable set and the main features of the theory of functions of bounded variation taking values in metric spaces introduced in [**Amb90b**]. In particular we will extensively use the concept of approximate upper limit of the difference quotient as a tool to measure the slope of a function: along the lines of [**AW11, Whi99b**] we can turn pointwise control of this slope into Lipschitz estimates on a family of sets which exhaust almost all the domain (see Theorem 2.3.3 for the precise statement).

The new main result regarding flat chains with finite size is described in the following rectifiability Theorem, proved in Section 2.9 of chapter 2:

Theorem 2.9.1 (Rectifiability of currents of finite size). *For every flat current $T \in \mathbf{F}_k(E)$ with finite size the measure μ_T is concentrated on a countably $\mathscr{H}^k$-rectifiable set. The least one, up to $\mathscr{H}^k$-null sets, is given by*

$$\mathit{set}(T) := \left\{ x \in E : \limsup_{r \downarrow 0} \frac{\mu_T(B_r(x))}{r^k} > 0 \right\}.$$

This result is established first for 1-dimensional currents, and then extended to the general case via an iterated slicing procedure, along the lines of [**AW11, Whi99b**] but using the distance $\mathcal{G}$ adapted to our problem. This rectifiability property is accompanied, in the Section 2.10, by a comparison between μ_T and $\mathscr{H}^k \llcorner set(T)$. Similarly to [**AK00a, AK00b**], we are able to describe the density $\lambda(x)$ of μ_T with respect to $\mathscr{H}^k \llcorner set(T)$ in terms of the geometry of the approximate tangent space $\mathrm{Tan}^{(k)}(set(T), x)$. In the Euclidean case, the factor λ is equal to 1.

The results obtained in the first chapter allow to study the fine properties of properly integrable Sobolev maps $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$, via the concept of distributional jacobian, which can be interpreted as a flat current of dimension $m-n > 1$. The space BV of functions of bounded variation, consisting of real-valued functions u defined in a domain of $\mathbb{R}^m$ whose distributional derivative Du is a finite Radon measure, may contain discontinuous functions and, precisely for this reason, can be used to model a variety of phenomena, while on the PDE side it plays an important role in the theory of conservation laws [**Daf10, Bre00**]. In more recent times, Ambrosio and De Giorgi introduced the distinguished subspace SBV of special functions of bounded variation, whose distributional derivative consists of an absolutely continuous part and a singular part concentrated on a $(m-1)$-dimensional set, called (approximate) discontinuity set S_u. See [**AFP00**] for a full account of the theory, whose applications include the minimization of the classical Mumford-Shah functional [**MS89**] and variational models in fracture mechanics. In a vector-valued setting, also the spaces BD and SBD play an important role, in connection with problems involving linearized elasticity and fracture (see also the recent work by Dal Maso on the space $GSBD$ [**DM11**]).

It is well know that $|Du|$ vanishes on $\mathscr{H}^{n-1}$-negligible sets, hence BV and all related spaces can't be used to describe singularities of higher codimension. For this reason, having in mind application to the Ginzburg-Landau theory (where typically singularities, e.g. line vortices in $\mathbb{R}^3$ have codimension 2, [**SS07**]) Jerrard and Soner introduced in [**JS02**] the space B_nV of functions of bounded higher variation, where n stands for the target dimension: roughly speaking it consists of Sobolev maps $u : \Omega \to \mathbb{R}^n$ whose distributional Jacobian Ju (well defined, at least as a distribution, under appropriate integrability assumptions) is representable by a vector-valued measure: in this case the natural vector space is the space $\Lambda_{m-n}\mathbb{R}^m$ of $(m-n)$-vectors. Remarkable extensions of the BV theory have been discovered in [**JS02**], as the counterpart of the coarea formula and of De Giorgi's rectifiability theorem for sets of finite perimeter. Even before [**JS02**], the distributional jacobian has been studied in many fundamental works as [**Mor66, Bal77, GMS98, Šve88, MS95**] in connection with variational problems in nonlinear elasticity (where typically $m = n$ and u represents a deformation map), e.g. to model cavitation effects.

As a matter of fact, since Ju can be equivalently described as a flat $(m-n)$-dimensional current, an important tool in the study of Ju is the well-developed machinery of currents, both in the Euclidean and in metric spaces, see [**Fed69, Fle66, AK00a, Whi99b**]. The fine structure of the measure Ju has been investigates in subsequent papers: using precisely tools from the theory of metric currents [**AK00a**], De Lellis in [**DL02**] characterized Ju in terms of slicing and proved rectifiability of the (measure theoretic) support S_u of the $(m-n)$-dimensional part of Ju, while in [**DLG10**] De Lellis and the author characterized the absolutely continuous part of Ju with respect to $\mathscr{L}^m$ in terms of the Sobolev gradient ∇u. Also, in [**DL02**] the analog of the space SBV has been introduced, denoted SB_nV: it consists of all functions $u \in B_nV$ such that $Ju = R + T$, with $\|R\| \ll \mathscr{L}^m$ and $\|T\|$ concentrated on a $(m-n)$-dimensional set.

The main goal of chapter 3 is to study the compactness properties of SB_nV. Even in the standard SBV theory, a uniform control on the energy of Mumford-Shah type

$$\int |u_h|^s + |\nabla u_h|^p \, d\mathscr{L}^m + \mathscr{H}^{m-n}(S_{u_h})$$

(with $s > 0$, $p > 1$) along a sequence (u_h) does not provide a control on Du_h. Indeed, only the $\mathscr{H}^{m-1}$-dimensional measure of S_{u_h} does not provide a control on the width of the jump. This difficulty leads **[DGA89]** to the space $GSBV$ of *generalized* special functions of bounded variation, i.e. the space of all real-valued maps u whose truncates $(-N)\vee u \wedge N$ are all SBV. Since both the approximate gradient ∇u and the approximate discontinuity set S_u behave well under truncation, it turns out that also the energy of $u_h^N := (-N)\vee u_h \wedge N$ is uniformly controlled, and now also $|Du_h^N|$; this is the very first step in the proof of the compactness-lower semicontinuity theorem in $GSBV$, which shows that the sequence (u_h) has limit points with respect to local convergence in measure, that any limit point u belongs to $GSBV$, and that

$$\int |u|^s + |\nabla u|^p \, d\mathscr{L}^m \le \liminf_h \int |u_h|^s + |\nabla u_h|^p \, d\mathscr{L}^m,$$
$$\mathscr{H}^{m-1}(S_u) \le \liminf_h \mathscr{H}^{m-1}(S_{u_h}).$$

In the higher codimension case, if we look for energies of the form

$$\int |u|^s + |\nabla u|^p + |M\nabla u|^\gamma \, d\mathscr{L}^m + \mathscr{H}^{m-n}(S_u) \tag{1}$$

(with $1/s + (n-1)/p < 1$, $\gamma > 1$) now involving also the minors $M(\nabla u)$ of ∇u, the same difficulty exists, but the truncation argument, even with smooth maps, does not work anymore. Indeed, the absence of S_u, namely the absolute continuity of Ju, may be due to very precise cancellation effects that tend to be destroyed by a left composition, thus causing the appearance of new singular points (see Example 3.1.3 and the subsequent observation). Also, unlike the codimension 1 theory, no "pointwise" description of S_u is presently available.

For these reasons, when looking for compactness properties in SB_nV, we have been led to define the space GSB_nV of *generalized special functions of bounded higher variation* as the space of functions u such that Ju is representable in the form $R + T$, with R absolutely continuous with respect to $\mathscr{L}^m$ and T having finite *size*, according to the theorems contained in chapter 2 (in the same vein, one can also define GB_nV, but our main object of investigation will be GSB_nV). In particular, for $u \in GSB_nV$ the distribution Ju is not necessarily representable by a measure. The similarity between $GSBV(\Omega)$ and $GSB_nV(\Omega)$ is not coincidental, and in fact we prove that in the scalar case $n = 1$ these two spaces are essentially the same; on the contrary for $n \ge 2$ their properties are substantially different. In order to study the T part of Ju, which might possibly have infinite mass, we use the rectifiability Theorem 2.9.1.

The chapter is organized as follows: after posing the definition of distributional jacobian, and present some key examples of singular maps, we briefly review in section 3.2 the space B_nV studied in **[JS02, DL02, DL03]** and present the pointwise description of the absolutely continuous part of Ju obtained in [**Mül90, DLG10**]. We proceed to the analysis of the crucial slicing Theorem 3.2.9 and it application to the SB_nV closure Theorem 3.2.11. In section 3.3 we define our new space of functions GSB_nV and show its simplest properties. The main result of the chapter is presented in section 3.4, where with the help of the slicing theorem we will generalize to our setting the compactness theorem of $GSBV$, as well as the closure theorem in SB_nV due to De Lellis in [**DL02, DL03**].

We apply the compactness theorem for GSB_nV functions to show the existence of minimizers for a general class of energies that feature both a volume and a size term. More precisely we look at energies involving an unknown function as well as a set:

$$\mathcal{A}(u, K; \Omega) = \int_{\Omega \setminus K} f(x, u, M\nabla u)\, dx + \int_{\Omega \cap K} g\, d\mathscr{H}^{m-n}.$$

Here $\Omega \subset \mathbb{R}^m$ is a bounded open set of class C^1, $u \in C^1(\Omega \setminus K, \mathbb{R}^n)$, $M\nabla u$ is the vector of minors of ∇u of every rank and K is a sufficiently regular closed set. The main novelty in this type of energies with respect to the classical Mumford-Shah energies [**MS89, AFP00, DGCL89, Amb90a, Dav05**] is the presence of a "free discontinuity" set of codimension higher than one.

The model problem is a new functional of Mumford-Shah type (1) that we introduce in section 4.1 of chapter 4, in the spirit of [**DGCL89**]. The study of this family of energies stems from some problems in image restoration and denoising, whose study culminated in the groundbreaking paper [**MS89**] where the authors introduced a new functional which became universally known as the Mumford-Shah functional. A rigorous variational treatment of the minimization problem was undertaken in [**DGCL89**], where the existence of minimizers and their regularity is addressed.

The main idea, in the special case $m = n$, is that u is a vector-valued map regular outside a finite number of points where the map covers a set of positive measure, thus imposing a singularity to its jacobian. The functional penalizes maps with an excessively large area factor $M_n\nabla u = \det \nabla u$ as well as the creation of too large singular sets S_u. Note that the p-th power of the gradient helps smoothing possible wild oscillations of u, however if $p < n$ the map might still have a singular jacobian.

As a motivation for the study of such energies involving weak notions of area deformation we recall the many results obtained in nonlinear elasticity, where the deformation u of a material is driven by the energy minimization of a functional depending on the minors of ∇u. The groundbreaking work [**Bal77**] has been followed by a rich literature, where several theories treating possible formation of fractures and cavitations are described, see [**MS95, GMS98, ADM94, Šve88, FH95**].

We analyse the minimization of (1), provided by Theorem 3.4.1, together with suitable Dirichlet boundary conditions, in the first section of the chapter: we also address the problem both in the interior and in the closure of Ω. In particular in section 4.2 we show that minimizers must be nontrivial (i.e.: $S_u \neq \emptyset$), at least for suitable boundary data; we also compare our choice of the energy with the classical p-energy of sphere-valued maps, see [**GMS98, HLW98, BCL86**]. We present also some calculations related to this specific non-triviality property, implying a density lower bound for the singular set S_u:

$$\frac{\mathscr{H}^{m-n}(S_u \cap B_r(x_0))}{\omega_{m-n} r^{m-n}} \geq \theta > 0 \qquad \text{for every } r \leq r_0 \text{ and } x \in \overline{S_u}.$$

This property, far from being straightforward, whenever proved would be the first step in the analysis of the regularity of minimizers, compare [**DMMS92, DGCL89**]. Unfortunately we are not able to prove such important result for general boundary data. Regarding the problem in $\overline{\Omega}$ the higher codimension of the singular set allows concentration of the jacobian at the boundary, providing some interesting examples that we briefly include in section 4.3. The dependence of the jacobian Ju on the boundary values is analysed, and under weak summability assumptions on the trace we are able to prove a generalization of Stokes' Theorem for jacobians. Similar variational problems in the framework of cartesian currents [**GMS98**] have been considered in [**Muc10**], where the author proves existence of minimizers in the set of maps whose graph is a normal current. The boundaries of these graphs are assumed to have equibounded mass and enjoy a decomposition into vertical parts of integer dimension, inherited from general properties of integral currents, which relates to the space B_nV. Our result avoids however such stringent finiteness hypothesis which does not seem to be a consequence of the finiteness of the energy.

In the final chapter 5 we show how the Mumford-Shah energy (1) can be approximated, in the sense of Γ-convergence, by a family of functionals defined on maps with absolutely continuous jacobian, a result presented in [**Ghi13**] following the literature [**AT90, AT92, ABO05, MM77a**]. We are concerned with the simpler energy

$$E(u,\Omega) = \int_\Omega |\nabla u|^p + |M_n \nabla u|^\gamma dx + \sigma \mathscr{H}^{m-n}(\Omega \cap S_u)$$

defined for maps $u \in GSB_nV(\Omega)$. We are motivated by the centrality of the distributional jacobian in the literature of Ginzburg-Landau problems, where the defects of constrained Sobolev maps are detected via the appearance of a singularity in Ju, and where approximation results similar to ours have been obtained, see [**BBH94, ABO03, ABO05, SS07, DF06, GMS98**].

The variational approximation of E will take place via Γ-convergence by (asymptotically degenerate) elliptic functionals E_ε, in the spirit of [**AT90, AT92**]. These densities, being absolutely continuous, are easier to handle from the numerical viewpoint. Similarly to the scalar Mumford-Shah functional, we are able to approximate the defect measure, which is singular, via a family of bulk functionals (although not uniformly elliptic), a phenomenon already outlined in the pioneering papers by Modica and Mortola [**MM77a, MM77b**]. Our choice of approximating functionals is

$$E_\varepsilon(u,v,\Omega) = \int_\Omega |\nabla u|^p + (v+k_\varepsilon)|M_n \nabla u|^\gamma dx + \int_\Omega \varepsilon^{q-n}|\nabla v|^q + \frac{W(1-v)}{\varepsilon^n} dx, \qquad (2)$$

and the limit takes place for $\varepsilon \to 0$. Since we want to approximate the maps $u \in GSB_nV$ with functions u_ε possessing "better regularity", (2) is defined for maps u having absolutely continuous jacobian. Moreover v plays the role of a control function for the pointwise determinant $M_n\nabla u$, ranging in the interval $[0,1]$, and depends on the singular set S_u; k_ε is an infinitesimal number apt to guarantee coercivity of E_ε. The second integral, referred to as the Modica-Mortola term because of the similarity with the phase transition energies contained in [**AT90**], contains a nonnegative convex potential W vanishing in 0. The variable v dims the concentration of $M(\nabla u)$: the price of the transition between 0 and 1 is captured by the Modica-Mortola term which detects $(m-n)$-dimensional sets.

After a brief analysis on the existence of minimizers for E_ε, we summarize in section 5.2 main properties of Γ-convergence. In particular the fundamental Theorem for such convergence yields:

$$(u_\varepsilon, v_\varepsilon) \text{ minimizes } E_\varepsilon, \ (u_\varepsilon, v_\varepsilon) \to (u, v) \quad \Rightarrow \quad (u, v) \text{ minimizes } E.$$

In section 5.3 we analyse the asymptotic of the sequence (v_ε), in the case $m - n = 0$, which is shown to converge towards a precise profile w_0 there studied. In particular using a slicing argument the Modica-Mortola term concentrates around the singular set S_u proportionally to its $\mathscr{H}^{m-n}$-measure.

The proof of the approximation will be carried out in two steps: first we show

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon, \Omega) \geq E(u, \Omega)$$

whenever $(u_\varepsilon, v_\varepsilon) \to (u, 1)$. This step is achieved first in codimension $m - n = 0$, where S_u is a discrete set, and then generalized to every codimension with the help of the slicing Theorem. The second part of the proof concerns the upper limit: here we construct (u_ε) truncating the function u around the singularity S_u and we use the optimal profile w_0 to build functions v_ε such that $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ and

$$\limsup_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon, \Omega) \leq E(u, \Omega).$$

In order to make this construction we will assume a mild regularity assumption on the singular set, namely

$$\limsup_{r \downarrow 0} \frac{\mathscr{L}^m(\{x \in \Omega : \operatorname{dist}(x, S_u) \leq r\})}{\mathscr{L}^n(B_1^n) r^n} = \mathscr{H}^{m-n}(S_u). \tag{3}$$

In order to conclude the proof of the Γ-convergence of E_ε to E we would need to know the density in energy of the set of GSB_nV maps satisfying (3). In the codimension 1 case this property was deduced by the regularity of minimizers of the Mumford-Shah energy, for which a lower bound on the $(m-1)$-dimensional density of the singular set is available. The analogous density property as well as a regularity result for minimizers of E is still under investigation.

In section 5.6 we prove an analog approximation result where we impose a fixed boundary condition to both u and the approximating sequence (u_ε). In the case $S_u \cap \partial\Omega \neq \emptyset$ then the transition made by v takes place partially outside the domain Ω, which translates in a loss of mass in the limit energy.

Finally in the last section we discuss a possible generalization to general Lagrangians, featuring a polyconvex integrand for the bulk part and where the size term is weighted by a continuous density. Growth and convexity assumptions will be crucial to extend the results of the previous sections to this broader class of energies.

CHAPTER 2

Metric currents and rectifiable sets

In this chapter we present the general theory of currents in metric spaces. After recalling in the first five sections the fundamental results obtained by Ambrosio and Kirchheim in [**AK00a**] and their relation with the Riemannian construction of currents [**Fed69**], we will develop a rectifiability theory for chains of possibly infinite mass via the concept of size, as in [**AG13b**]. The use of purely metric tools will not prevent us from relating the main theorems to the Euclidean setting, as in the last section where the size measure is characterized.

2.1. Metric currents

In this chapter (E, d) will be a complete metric space, and starting from section 2.2 we will make the further assumption that E is a locally compact and length space, namely that for every $x, y \in E$ it holds

$$d(x,y) = \inf \big\{ L(\gamma) : \gamma \in \mathrm{Lip}([0,1], E), \gamma(0) = x, \gamma(1) = y \big\}. \tag{4}$$

Recall that the metric version of the Hopf-Rinow Theorem [**BBI01**, 2.5.28] implies that a complete and locally compact length space is boundedly compact, that is bounded closed subsets of E are compact. As a consequence E is separable; moreover the infimum in (4) is a minimum: for every pair of points in E there exists a Lipschitz curve of minimal length connecting them. The symbol $\mathcal{B}(E)$ will wenote the σ-algebra of Borel subsets of E, and $\mathcal{B}^\infty(E)$ will be the space of real-valued bounded Borel functions. $\mathcal{M}(E)$ will be the space of nonnegative Borel measures on E. In a general metric space without differentiable structure the maximum regularity we can ask for a function is to be Lipschitz. We set $\mathrm{Lip}_c(E)$, $\mathrm{Lip}_b(E)$ and $\mathrm{Lip}_{\mathrm{loc}}(E)$ to be respectively the spaces of compactly supported, bounded and locally Lipschitz functions; moreover $\mathrm{Lip}_1(E)$ will be the space of 1-Lipschitz functions.

Recall the notions of k-dimensional Hausdorff (outer) measure: given $B \subset E$ we let

$$\mathscr{H}^k(B) := \lim_{\delta \downarrow 0} \inf \left\{ \frac{\omega_k}{2^k} \sum_i [\mathrm{diam}(B_i)]^k : B \subset \bigcup_i B_i,\ \mathrm{diam}(B_i) < \delta \right\}.$$

The upper and lower k-dimensional densities of a finite Borel measure μ at x are respectively defined by

$$\Theta^{*k}(\mu, x) := \limsup_{\rho \downarrow 0} \frac{\mu(B_r(x))}{\omega_k r^k}, \qquad \Theta^k_*(\mu, x) := \liminf_{\rho \downarrow 0} \frac{\mu(B_r(x))}{\omega_k r^k}.$$

Recall pointwise bounds on these densities imply a comparison between μ and $\mathscr{H}^k$: given $B \in \mathcal{B}(E)$

$$\Theta^{*k}(\mu, x) > t \quad \forall x \in B \quad \Rightarrow \quad \mu \geq t\mathscr{H}^k \llcorner B,$$
$$\Theta^{*k}(\mu, x) < t \quad \forall x \in B \quad \Rightarrow \quad \mu \llcorner B \leq 2^k t\mathscr{H}^k \llcorner B.$$

Our test forms are be defined as follows:

Definition 2.1.1. *We let $\mathcal{D}^k(E)$ denote the space of Lipschitz k-forms*

$$(f, \pi^1, \dots, \pi^k) \in \mathcal{D}^k(E) := \mathrm{Lip}_b(E) \times [\mathrm{Lip}(E)]^k.$$

A metric current is a functional on k-forms: our definition slightly differs from the original one in [**AK00a**, 3.1], since we do not require T to have finite mass yet.

Definition 2.1.2 (Metric currents $\mathcal{D}_k(E)$). *A metric k-current T in E is a map*

$$T : \mathcal{D}^k(E) \to \mathbb{R}$$

satisfying the following properties of multilinearity, continuity and locality:

(i) *T is multilinear in $(f, \pi^1, \dots, \pi^k)$,*
(ii) *$\lim_i T(f, \pi^1_i, \dots, \pi^k_i) = T(f, \pi^1, \dots, \pi^k)$ whenever $\pi^j_i \to \pi^j$ pointwise in E with $\mathrm{Lip}(\pi^j_i) \leq C$,*
(iii) *$T(f, \pi^1, \dots, \pi^k) = 0$ if for some $i \in \{1, \dots, k\}$ the function π_i is constant in a neighborhood of $\{f \neq 0\}$.*

The action of T on a form ω will be interchangeably denoted by $T(\omega)$ or $\langle T, \omega\rangle$.

We could replace the space $\mathcal{D}^k(E)$ by $\mathcal{D}^k_c(E) := \mathrm{Lip}_c(E) \times [\mathrm{Lip}_{\mathrm{loc}}(E)]^k$, giving rise to the space of local currents $\mathcal{D}_{k,\mathrm{loc}}(E)$: according to [**Lan11**, 2.1] the continuity assumption (ii) needs to be changed in:

(ii') $\lim_i T(f_i, \pi^1_i, \dots, \pi^k_i) = T(f, \pi^1, \dots, \pi^k)$ whenever $f_i \to f$ and $\pi^j_i \to \pi^j$ pointwise in E with $\mathrm{Lip}(f_i), \mathrm{Lip}(\pi^j_i) \leq C$ and $\mathrm{spt}(f_i) \subset K$ for some $K \Subset E$.

We will not use this generalization in the sequel. The exterior differentiation in $\mathcal{D}^k(E)$ is defined as

$$d(f, \pi^1, \dots, \pi^k) := (1, f, \pi^1, \dots, \pi^k) \in \mathcal{D}^{k+1}(E).$$

Since $\mathrm{Lip}_b(E)$ and $\mathrm{Lip}_c(E)$ are both algebras the exterior product between forms

$$(f, \pi^1, \dots, \pi^k) \wedge (g, \eta^1, \dots, \eta^l) := (fg, \pi^1, \dots, \pi^k, \eta^1, \dots, \eta^l)$$

is well defined in $\mathcal{D}^k(E)$ and $\mathcal{D}^k_c(E)$.

Definition 2.1.3 (Boundary). *Let $k \geq 1$. The boundary of a current $T \in \mathcal{D}_k(E)$ is a $(k-1)$-current defined as the adjoint of the exterior differentiation: $\partial T(\omega) := T(d\omega)$ for every $\omega \in \mathcal{D}^{k-1}(E)$.*

Proposition 2.1.4. *If T is a current, then ∂T is a current.*

Proof. The locality property (iii) is the only non-obvious one. We need to show that, if $f \in \mathrm{Lip}_b(E)$ and $(\pi^1, \pi') \in \mathrm{Lip}(E) \times \mathrm{Lip}(E)^{k-1}$ and π^1 is constant in a neighborhood of $\mathrm{spt}(f)$ then

$$T(1, f, \pi^1, \pi') = 0.$$

Assume that π^1 is constant in a ε-neighborhood of $\mathrm{spt}(f)$: let c be this constant value. It is easy to use the distance function to construct $\psi \in \mathrm{Lip}_b(E)$ such that

- $\psi = 1$ in a $\frac{\varepsilon}{3}$-neighborhood of $\mathrm{spt}(f)$;
- $\psi = 0$ in a $\frac{\varepsilon}{3}$-neighborhood of $\{\pi^1 \neq c\}$.

Then by locality $T(1, f, \pi^1, \pi') = T(\psi, f, \pi^1, \pi')$ because f is constant in a neighborhood of $1-\psi$, and $T(\psi, f, \pi^1, \pi') = 0$ because π^1 is constant in a neighborhood of $\mathrm{spt}(\psi)$. The general case can be obtained by the continuity assumption (ii), approximating π^1 with the compositions $\pi^1_h := \theta_h \circ \pi^1$, where

$$\theta_h(t) := \min\left\{t + \frac{1}{h}, \max\left\{c, t - \frac{1}{h}\right\}\right\}:$$

$\pi^1_h \to \pi^1$ uniformly and with equibounded Lipschitz constants. □

Given two metric spaces E, F, $\phi \in \mathrm{Lip}(E, F)$ and $\omega = (f, \pi^1, \ldots, \pi^k)$ we let

$$\phi^\# \omega = (f \circ \phi, \pi^1 \circ \phi, \ldots, \pi^k \circ \phi).$$

Definition 2.1.5 (Push forward). *The push forward of $T \in \mathcal{D}_k(E)$ via a Lipschitz map $\phi \in \mathrm{Lip}(E, F)$ is defined as $(\phi_\# T)(\omega) := T(\phi^\# \omega)$ for every $\omega \in \mathcal{D}^k(F)$.*

Notice that the push-forward and the boundary operator commute.

Definition 2.1.6 (Restriction). *Let $T \in \mathcal{D}_k(E)$ and $\omega = (g, \tau^1, \ldots, \tau^m)$ for some $m \leq k$. The restriction of T to ω is defined as*

$$(T \llcorner \omega)(f, \pi^1, \ldots, \pi^{k-m}) := T(fg, \tau^1, \ldots, \tau^m, \pi^1, \ldots, \pi^{k-m}).$$

for every $(f, \pi^1, \ldots, \pi^{k-m}) \in \mathcal{D}^{k-m}(E)$.

Example 2.1.7. The functional

$$T : C^1_c(\mathbb{R}) \times C^1_c(\mathbb{R}) \to \mathbb{R}, \qquad T(f, \pi) = \pi'(0)$$

cannot be extended to a metric current in $\mathbb{R}$, since the continuity axiom fails.

We now introduce our first norm on the space of current: the mass.

Definition 2.1.8 (Mass). *We say that $T \in \mathcal{D}_k(E)$ is a current of finite mass if there exists a finite Borel measure μ in E such that*

$$|T(f, \pi^1, \ldots, \pi^k)| \leq \prod_{i=1}^k \mathrm{Lip}\left(\pi^i\right) \int_E |f| d\mu, \tag{5}$$

for every $(f, \pi^1, \ldots, \pi^k) \in \mathrm{Lip}_c(E) \times [\mathrm{Lip}(E)]^k$. The total variation of T is the minimal μ satisfying (5) and is denoted by $\|T\|$. As customary we let $\mathbf{M}_k(E)$ be the Banach space of finite mass k-dimensional currents endowed with the norm $\mathbf{M}(T) := \|T\|(E)$.

By the density of $\mathrm{Lip}_b(E)$ in $L^1(E, \|T\|)$, which contains the space $\mathcal{B}^\infty(E)$ of bounded Borel functions, every $T \in \mathbf{M}_k(E)$ can be uniquely extended to k-forms in $\mathcal{B}^\infty(E) \times [\mathrm{Lip}(E)]^k$. Recall the following characterization of the mass measure as a supremum among test forms: this property is essential to establish lower semicontinuity.

Proposition 2.1.9 (Characterization of Mass, [**AK00a**, 2.7]). *A current $T \in \mathcal{D}_k(E)$ ha finite mass if and only if*

(a) *there exists a constant M such that*

$$\sum_{i=0}^{\infty} |T(f_i, \pi^1, \dots, \pi^k)| \le M$$

whenever $\sum_i |f_i| \le 1$ *and* $\mathrm{Lip}(\pi^i) \le 1$;

(b) $f \mapsto T(f, \pi^1, \dots, \pi^k)$ *is continuous along equibounded monotone sequences* (f_h), *i.e. sequences* $(f_h) \subset \mathrm{Lip}_b(E)$ *such that* $(f_h(x))$ *is monotone for every* $x \in E$,

$$\sup\{|f_h(x)|, x \in E, h \in \mathbb{N}\} < \infty$$

and the pointwise limit $x \mapsto \lim_h f_h(x)$ *belongs to* $\mathrm{Lip}_b(E)$.

If $T \in \mathbf{M}_k(E)$ *then for every Borel set* B

$$\|T\|(B) = \sup\left\{\sum_{i=0}^{\infty} |T(\chi_{B_i}, \pi_i^1, \dots, \pi_i^k)|\right\}, \tag{6}$$

where the supremum runs among all Borel partitions (B_i) *of* B *and all* 1*-Lipschitz maps* π_i^j

The proof of the following Theorem is contained of [**AK00a**, 3.5] (see Proposition 2.4 of [**Lan11**] for the analogous statement for local currents). We remark that the proof shows that every $T \in \mathcal{D}_k(E)$ is alternating in the k-tuple $(\pi^1, \dots, \pi^k)$, even if it has not finite mass. We therefore adopt the exterior algebra notations $f d\pi^1 \wedge \dots \wedge d\pi^k$ and $f d\pi$ in place of $(f, \pi^1, \dots, \pi^k)$.

Theorem 2.1.10 (Product, chain rule and locality in $\mathbf{M}_k(E)$). *Let* $T \in \mathbf{M}_k(E)$ *be extended to* $\mathcal{B}^\infty(E) \times [\mathrm{Lip}(E)]^k$. *Then*

- *(product rule)* $T\big(f d\pi^1 \wedge \dots \wedge d\pi^k\big) + T\big(\pi^1 df \wedge \dots \wedge d\pi^k\big) = T\big(1 d(f\pi^1) \wedge \dots \wedge d\pi^k\big)$ *whenever* $f, \pi^1 \in \mathrm{Lip}_b(E)$;
- *(chain rule)* $T\big(f d\psi^1(\pi) \wedge \dots \wedge d\psi^k(\pi)\big) = T\big(f \det \nabla\psi(\pi) d\pi^1 \wedge \dots \wedge d\pi^k\big)$ *whenever* $\psi \in C^1(\mathbb{R}^k, \mathbb{R}^k) \cap \mathrm{Lip}(\mathbb{R}^k, \mathbb{R}^k)$;
- *(locality)* $T\big(f d\pi^1 \wedge \dots \wedge d\pi^k\big) = 0$ *if* $\{f \neq 0\} = \cup_i B_i$ *with* B_i *Borel and* π^i *constant on* B_i.

The strengthened locality property allows to extend the restriction operator to forms $g d\tau$ with $g \in \mathcal{B}^\infty(E)$ and $\tau \in [\mathrm{Lip}(E)]^k$: in particular given a Borel set B we let $T \llcorner B := T \llcorner \chi_B$.

Example 2.1.11. Among the simplest finite mass currents we find the following: given $a \in E$ and $\gamma \in \mathrm{Lip}([0,1], E)$ we let

$$[\![a]\!] \in \mathcal{D}_0(E), \qquad [\![a]\!](f) = f(a),$$

$$\gamma_\#[\![0,1]\!] \in \mathcal{D}_1(E), \qquad \gamma_\#[\![0,1]\!](f d\pi) = \int_0^1 f(\gamma(t))(\pi \circ \gamma)'(t)\, dt.$$

An important example of finite mass metric current in the Euclidean space is the following:

EXAMPLE 2.1.12. Any function $g \in L^1(\mathbb{R}^m)$ induces a top dimensional current $[g] \in \mathbf{M}_m(\mathbb{R}^m)$ given by

$$(\mathbf{E}^m \llcorner g)(f d\pi^1 \wedge \cdots \wedge d\pi^k) := \int_{\mathbb{R}^m} g f d\pi^1, \wedge \cdots \wedge d\pi^m = \int_{\mathbb{R}^m} g f \det \nabla \pi \, dx.$$

By Hadamard's inequality $|\det(v_1, \ldots, v_m)| \leq \prod_{i=1}^m |v_i|$ we have that $\mathbf{E}^m \llcorner g \in \mathbf{M}_m(\mathbb{R}^m)$ and $\|\mathbf{E}^m \llcorner g\| = |g| \mathscr{L}^m$; moreover the continuity property is ensured by the w^*-continuity of determinants of maps in $W^{1,\infty}$ (Reshetnyak's theorem, see [**Dac08, GMS98**]). More generally given $k \leq m$, $g \in L^1(\mathbb{R}^m)$ and $\tau^1, \ldots, \tau^{m-k} \in \mathrm{Lip}(\mathbb{R}^m)$ we can construct

$$(\mathbf{E}^m \llcorner g \, d\tau^1 \wedge \cdots \wedge d\tau^{m-k})(f d\pi^1 \wedge \cdots \wedge d\pi^k) := \int_{\mathbb{R}^m} f g d\tau^1 \wedge \cdots \wedge d\tau^{m-k} \wedge d\pi^1 \wedge \cdots \wedge d\pi^k.$$

In dimension $m = 2$ the previous example is optimal: for $f \in \mathcal{B}^\infty(\mathbb{R}^2)$ and $\pi^1, \pi^2 \in \mathrm{Lip}(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)$ the functional

$$T(f, \pi^1, \pi^2) := \int_{\mathbb{R}^2} f \det \nabla \pi \, d\mu$$

satisfies the continuity axiom (ii) only if $\mu \ll \mathscr{L}^2$. This follows from a deep result on null sets in the plane by Preiss:

THEOREM 2.1.13 (Preiss). *If μ is a finite Borel measure not absolutely continuous with respect to $\mathscr{L}^2$, then there exists a sequence of maps $g_h \in \mathrm{Lip}_1(\mathbb{R}^2, \mathbb{R}^2) \cap C^1(\mathbb{R}^2, \mathbb{R}^2)$ such that $\mathrm{Lip}(g_h) \leq C$, $g_h \to g$ pointwise where $g(x, y) = (x, y)$ and*

$$\lim_h \int_{\mathbb{R}^2} \det \nabla g_h \, d\mu < \mu(\mathbb{R}^2).$$

It is still unknown if this theorem can be generalized to higher dimensions. We remark that for $m = 1$ there is an easy proof: is μ is singular with respect to $\mathscr{L}^1$, let

$$g_h(t) = t - \mathscr{L}^1(A_h \cap (-\infty, t)),$$

where A_h is a sequence of open sets such that $\mathscr{L}^1(A_h) \to 0$ and A_h contain a Lebesgue negligible set where μ is concentrated.

Conversely we have the following description of top dimensional finite mass currents in the Euclidean space:

PROPOSITION 2.1.14 ([**AK00a**, 3.8]). *A current $T \in \mathbf{M}_m(\mathbb{R}^m)$ is representable as $\mathbf{E}^m \llcorner g$ for some $g \in L^1(\mathbb{R}^m)$ if and only if $\|T\| \ll \mathscr{L}^m$.*

In light of the Preiss' Theorem and the subsequent example if $k = 1, 2$ then every finite mass current T has an absolutely continuous total variation $\|T\|$.

The space $\mathcal{D}_k(E)$ can be endowed with a weak* topology:

DEFINITION 2.1.15 (Weak* convergence of currents). *We say that a sequence $(T_h) \subset \mathcal{D}_k(E)$ converges to $T \in \mathcal{D}_k(E)$ if*

$$\lim_h T_h(f d\pi^1 \wedge \cdots \wedge d\pi^k) = T(f d\pi^1 \wedge \cdots \wedge d\pi^k)$$

for every $f d\pi^1 \wedge \cdots \wedge d\pi^k \in \mathcal{D}^k(E)$. Weak convergence is denoted by $T_h \stackrel{*}{\rightharpoonup} T$.*

Given an open set A the map $T \mapsto \|T\|(A)$ is lower semicontinuous with respect to the weak* convergence, since by Proposition 2.1.9 applied to $T \llcorner A$ we can write

$$\|T\|(A) = \sup \left\{ \sum_{i=0}^{\infty} |T(f_i d\pi_i)|, \sum_{i=0}^{\infty} |f_i| \leq \chi_A, \sup_{i,j} \mathrm{Lip}(\pi_i^j) \leq 1 \right\}.$$

Recall that the existence of the pointwise limit does not automatically give a metric current, since the continuity property (ii) in Definition 2.1.2 would require the sequence $(T_h \llcorner f)$ to be equicontinuous on bounded subsets of $[\mathrm{Lip}(E)]^k$ for every $f \in \mathrm{Lip}_b(E)$, while the second condition in Proposition 2.1.9 would require for $(T_h \llcorner d\pi)$ the same property on bounded subsets of $\mathrm{Lip}_b(E)$ for any fixed $\pi \in \mathrm{Lip}(E)^k$. We remark that in the notations of [**AK00a**] this convergence is called weak, while when E is a w^*-separable Banach space weak* convergence denotes pointwise convergence against k-forms $f d\pi \in \mathcal{D}^k(E)$ with f, π w^*-continuous. For metric currents induced by $L^1(\mathbb{R}^m)$ functions in $\mathbb{R}^m$ like in Example 2.1.12 we underline the following:

$$f_k \rightharpoonup f \text{ weakly in } L^1 \quad \Rightarrow \quad \mathbf{E}^m \llcorner f_h \overset{*}{\rightharpoonup} \mathbf{E}^m \llcorner f.$$

Let us recall the definition of supremum of a family of measures.

Definition 2.1.16. *Let $\{\mu_i\}_{i\in I}$ be a family of Borel positive measures on E. Then, for every Borel subset of E, we define*

$$\bigvee_{i\in I} \mu_i(B) = \sup \left\{ \sum_{i\in J} \mu_i(B_i) : B_i \textit{ pairwise disjoint and Borel, } \bigcup_{i\in J} B_i = B \right\}, \qquad (7)$$

where J runs through all countable subsets of I.

The set function $\bigvee_{i\in I} \mu_i$ is a Borel measure, and it is finite if and only if there exists a finite Borel measure $\sigma \geq \mu_i$ for any i. Notice that in (7) it would be equivalent to consider finite partitions of B into Borel sets $B_1, \ldots, B_N$.

Finally we present a technical lemma, whose proof can be found in [**AK00a**, 5.4]: recall that E begin boundedly compact, the closed balls $\overline{B_N(x_0)}$ are compact, hence every finite mass current is concentrated on a σ-compact set.

Lemma 2.1.17. *Let $T \in \mathbf{M}_k(E)$: there exists a countable set $D \subset \mathrm{Lip}_1(E) \cap \mathrm{Lip}_b(E)$ such that*

$$\|T\| = \bigvee_{\pi\in D} \|T \llcorner d\pi\|.$$

2.2. Normal currents

In this section we outline the main property of the class of normal currents, which together with the theory rectifiable sets and currents exposed in the following sections provide the main compactness Theorem in the solution of Plateau's problem.

Definition 2.2.1 (Normal currents). *We let $\mathbf{N}_k(E)$ be the subspace of normal currents:*

$$\mathbf{N}_k(E) = \mathbf{M}_k(E) \cap \{T : \partial T \in \mathbf{M}_k(E)\}.$$

$\mathbf{N}_k(E)$, endowed with te norm $\mathbf{M}(T) + \mathbf{M}(\partial T)$ is a Banach space.

Similarly to Example 2.1.12 and Proposition 2.1.14 we have the following characterization of top dimensional normal currents in $\mathbb{R}^m$:

PROPOSITION 2.2.2 ([**AK00a**, 3.7]). *For any $T \in \mathbf{N}_m(E)$ there exists a unique $g \in BV(\mathbb{R}^m)$ such that $T = \mathbf{E}^m \llcorner g$. Moreover $\|\partial T\| = |Dg|$.*

Moreover the total variation of a normal k-current does not charge $\mathscr{H}^k$-null sets:

THEOREM 2.2.3 ([**AK00a**, 3.9]). *Let $T \in \mathbf{N}_k(E)$ and let $N \in \mathcal{B}(\mathbb{R}^k)$ be a $\mathscr{L}^k$-negligible Borel set. Then*

$$\|T \llcorner d\pi\|(\pi^{-1}(N)) = 0 \qquad \forall \pi \in \mathrm{Lip}(E, \mathbb{R}^k).$$

Moreover $\|T\|$ vanishes on Borel $\mathscr{H}^k$-null subsets of E.

PROOF. Let $L := \pi^{-1}(N)$ and $f \in \mathrm{Lip}_b(E)$: since

$$(T \llcorner d\pi)(f\chi_L) = T \llcorner (f d\pi)(\chi_L) = \pi_{\#}(T \llcorner f)(\chi_N dx^1 \wedge \cdots \wedge dx^k)$$

and $\pi_{\#}(T \llcorner f) \in \mathbf{N}_k(\mathbb{R}^k)$, we obtain by Proposition 2.2.2 $(T \llcorner d\pi)(f\chi_L) = 0$. If $L \in \mathcal{B}(E)$ s $\mathscr{H}^k$-null and $\pi \in \mathrm{Lip}(E, \mathbb{R}^k)$ then $\pi(L)$ is contained in a Borel $\mathscr{L}^k$-negligible set $N \subset \mathbb{R}^k$, hence $\|T \llcorner d\pi\|(L) \leq \|T \llcorner d\pi\|(\pi^{-1}(N)) = 0$. Taking into account (6) we have the thesis. □

The following Proposition shows the equicontinuity property of normal currents with equibounded norm:

PROPOSITION 2.2.4 ([**AK00a**, 5.1]). *Let $T \in \mathbf{N}_k(E)$: the following estimate*

$$|T(f d\pi) - T(f, d\pi')| \leq \sum_{i=1}^{k} \int_E |f||\pi^i - \pi'^i| d\|\partial T\| + \mathrm{Lip}(f) \int_E |\pi^i - \pi'^i| d\|T\|$$

holds whenever $f, \pi^i, \pi'^i \in \mathrm{Lip}(E)$ and $\mathrm{Lip}(\pi^i) \leq 1$, $\mathrm{Lip}(\pi'^i) \leq 1$.

PROOF. Assume f, π, π' are bounded and set $d\pi_0 := d\pi^2 \wedge \cdots \wedge d\pi^k$. Then

$$\begin{aligned} &T(f\pi^1 \wedge d\pi_0) - T(f d\pi'^1 \wedge d\pi_0) \\ &\quad = T(1 d(f\pi^1) \wedge d\pi_0) - T(1 d(f\pi'^1) \wedge d\pi_0) - T(\pi^1 df \wedge d\pi_0) + T(\pi'^1 df \wedge d\pi_0) \\ &\qquad = \partial T(f\pi^1 d\pi_0) - \partial T(f\pi'^1 d\pi_0) - T(\pi^1 df \wedge d\pi_0) + T(\pi'^1 df \wedge d\pi_0), \end{aligned}$$

hence using the locality property $|T(f\pi^1 \wedge d\pi_0) - T(f d\pi'^1 \wedge d\pi_0)|$ can be estimated with

$$\int_E |f||\pi^1 - \pi'^1| d\|\partial T\| + \mathrm{Lip}(f) \int_E |\pi^1 - \pi'^1| d\|T\|.$$

Repeating the same argument for the other $k-1$ components we obtain the desired estimate. In general the boundedness hypothesis can be discharged by the continuity property of T and ∂T. □

The equicontinuity property just stated yields the main compactness Theorem for normal currents:

THEOREM 2.2.5 (Compactness, [**AK00a**, 5.2]). *Let $(T_h) \subset \mathbf{N}_k(E)$ be a bounded sequence and assume that for every $p \geq 1$ there exists a compact set $K_p \subset E$ such that*

$$\|T_h\|(E \setminus K_p) + \|\partial T_h\|(E \setminus K_p) < \frac{1}{p} \qquad \forall h \in \mathbb{N}. \tag{8}$$

Then there exists a subsequence $(T_{h(n)})$ weak converging to some $T \in \mathbf{N}_k(E)$ satisfying*

$$\|T\|(E \setminus \bigcup_{p=1}^{\infty} K_p) + \|\partial T\|(E \setminus \bigcup_{p=1}^{\infty} K_p) = 0.$$

PROOF. The proof follows a classical argument of extracting via a diagonal process a subsequence pointwise converging on a dense set, and then using equicontinuity to deduce the full convergence. Up to the extraction of a subsequence we can assume that there exist finite measures $\mu, \nu \in \mathcal{M}(E)$ such that

$$\|T_h\| \overset{*}{\rightharpoonup} \mu, \qquad \|\partial T_h\| \overset{*}{\rightharpoonup} \nu$$

weakly* in the sense of measures. It is not difficult to see that $(\mu + \nu)(E \setminus \bigcup_p K_p) = 0$. We first show that (T_h) has a pointwise converging subsequence $(T_{h(n)})$: to this aim by a diagonal argument we only need to check that for any integer $q \geq 1$ there exists a subsequence $(T_{h(n)})$ such that

$$\limsup_{m,n\to\infty} |T_{h(n)}(f d\pi) - T_{h(m)}(f d\pi)| \leq \frac{3}{q}$$

whenever $f d\pi \in \mathcal{D}^k(E)$, $|f| \leq q$, $\mathrm{Lip}(f) \leq 1$ and $\mathrm{Lip}(\pi^i) \leq 1$. To this extent by the equi-tightness expressed by (8) we can truncate with a compactly supported Lipschitz function g so that

$$\sup_n \mathbf{M}(T_{h(n)} - T_{h(n)} \llcorner g) + \mathbf{M}(\partial(T_{h(n)} - T_{h(n)} \llcorner g)) < \frac{1}{q^2}.$$

and prove the asserted convergence for $(T_{h(n)} \llcorner g)(f d\pi)$. Let $D \subset \mathrm{Lip}_1(\bigcup_p K_p))$ be a countable set dense for the uniform convergence: then there exists a subsequence $T_{h(n)}$ such that $T_{h(n)} \llcorner g(f d\pi)$ converges whenever $f, \pi^1, \ldots, \pi^k \in D$. We claim that $T_{h(n)} \llcorner g(f d\pi)$ converges whenever $f, \pi^1, \ldots, \pi^k \in \mathrm{Lip}_1(E)$: in fact for any $\tilde{f}, \tilde{\pi}^1, \ldots, \tilde{\pi}^k$ we can use Proposition 2.2.4 to obtain

$$\begin{aligned}
&\limsup_{m,n\to\infty} |T_{h(n)} \llcorner g(f d\pi) - T_{h(m)} \llcorner g(f d\pi)| \leq 2 \limsup_{h\to\infty} |T_h \llcorner g(f d\pi) - T_h \llcorner g(\tilde{f} d\tilde{\pi})| \\
&\leq \limsup_{h\to\infty} \sum_{i=1}^{k} \int_E (|f| + 1)|\pi^i - \tilde{\pi}^i| \, d[\|T \llcorner g\| + \|\partial(T_h \llcorner g)\|] + \int_E |f - \tilde{f}| \, d\|T_h \llcorner g\| \\
&\leq \sum_{i=1}^{k} \int_{\mathrm{spt} g} (|f| + 1)|\pi^i - \tilde{\pi}^i| \, d\mu + \int_E (|f| + 1)|g||\pi^i - \tilde{\pi}^i| \, d\nu + + \int_E |f - \tilde{f}||g| \, d\mu.
\end{aligned}$$

Since $\tilde{f}, \tilde{\pi}^i$ are arbitrary, this proves the convergence of $T_{h(n)} \llcorner g(f d\pi)$.

Since $T_{h(n)}(\omega)$ converge to $T(\omega)$ for any $T \in \mathcal{D}^k(E)$, T satisfies conditions (i) and (iii) in Definition 2.1.2. Passing to the limit $n \to \infty$ in the definition of mass we obtain that both T and ∂T have finite mass, and that $\|T\| \leq \mu$, $\|\partial T\| \leq \nu$. In order to check the continuity property (ii), by the finiteness of mass we can suppose that $\mathrm{spt}(f)$ is compact: passing to the limit as $h \to \infty$ in Proposition 2.2.4 we get

$$|T(f d\pi) - T(f d\pi')| \leq \sum_{i=1}^{k} \int_E |f||\pi^i - \pi'^i|\, d\mu + \mathrm{Lip}(f) \int_{\mathrm{spt}(f)} |\pi^i - \pi'^i|\, d\nu,$$

whenever $\mathrm{Lip}(\pi^i) \leq 1$, $\mathrm{Lip}(\pi'^i) \leq 1$. This estimate yields the continuity property. □

The following slicing Theorem will be crucial in the definition of currents of finite size:

THEOREM 2.2.6 (Slicing, [**AK00a**, 5.6],[**Fed69**, 4.2.1]). *Let $T \in \mathbf{N}_k(E)$, $\ell \leq k$ and $\pi \in \mathrm{Lip}(E, \mathbb{R}^\ell)$.*

- *There exist a weak*-measurable map $\mathbb{R}^\ell \ni x \mapsto \langle T, \pi, x\rangle \in \mathbf{N}_{k-\ell}(E)$ such that*

$$\langle T, \pi, x\rangle \text{ and } \partial\langle T, \pi, x\rangle \quad \text{are concentrated on } \pi^{-1}(x); \tag{9}$$

$$\int_{\mathbb{R}^\ell} \langle T, \pi, x\rangle \psi(x)\, dx = T \llcorner (\psi \circ \pi)\, d\pi \qquad \forall \psi \in \mathrm{Lip}_b(E); \tag{10}$$

$$\int_{\mathbb{R}^\ell} \|\langle T, \pi, x\rangle\| dx = \|T \llcorner d\pi\|; \tag{11}$$

- *if $\ell = 1$ then for $\mathscr{L}^1$-almost every t it holds:*

$$\langle T, \pi, t\rangle = \partial(T \llcorner \{\pi < t\}) - (\partial T) \llcorner \{\pi < t\}.$$

We will sometimes write $T_x = \langle T, \pi, x\rangle$ to shorten the writing and to emphasize the dependence of the slice on the variable x. Several properties of the slices of a normal current will be outlined in section 2.4, directly for flat currents. Remark finally that if $\mathscr{L}^m \wedge \rho \in \mathbf{M}_k(\mathbb{R}^m)$ is a finite mass current for some $\rho \in L^1(\mathbb{R}^m, \Lambda_k \mathbb{R}^m)$, and $\pi : \mathbb{R}^m \to \mathbb{R}^k$ is an orthogonal projection, then by Fubini's theorem the slices are simply the restrictions of the current to vectorfields tangent to the fiber, i.e.:

$$\langle \mathscr{L}^m \wedge \rho, \pi, x\rangle = \mathscr{H}^{m-k} \llcorner \pi^{-1}(x) \wedge (\rho \llcorner d\pi) \tag{12}$$

for $\mathscr{L}^k$-almost every $x \in \mathbb{R}^k$.

2.3. Rectifiable sets and currents

This section is devoted to the presentation of rectifiable sets and currents in the metric context. Such class of currents is the closest one to the classical concept of submanifold, because its elements are indeed concentrated on rectifiable sets of dimension equal to the dimension of the current. Key compactness properties connected with the slicing theorem 2.2.6 allow to prove the existence of minimal area (or more generally minimizers of parametric functionals).

2.3.1. Rectifiable sets. We begin by recalling the definition of countably $\mathscr{H}^k$-rectifiable set:

Definition 2.3.1 ([**Fed69**]). *A $\mathscr{H}^k$-measurable set $\Sigma \subset E$ is called countably $\mathscr{H}^k$-rectifiable if there exist countably many sets $A_j \subset \mathbb{R}^k$ and Lipschitz maps $f_j : A_j \to E$ such that*

$$\mathscr{H}^k\left(\Sigma \setminus \bigcup_j f_j(A_j)\right) = 0. \tag{13}$$

For $k = 0$ we define a countably $\mathscr{H}^0$-rectifiable set to be a finite or countable set.

We recall that since E is complete and boundedly compact, the sets A_i can be assumed to be closed or compact; moreover one can suppose that the images $f_j(A_j)$ are pairwise disjoint (see [**Kir94**, Lemma 4]).

In order to prove a rectifiability result it is often necessary to prove that a certain parameterization function is Lipschitz. Among the many ways to measure the slope of a function, the following notion is quite flexible, since it is local and behaves well under slicing:

Definition 2.3.2. *Let $A \subset \mathbb{R}^k$ Borel and $f : A \to E$ a Borel map. For $x \in A$ we define $\delta_x f$ as the smallest $N \geq 0$ such that*

$$\lim_{r\downarrow 0} \frac{1}{r^k}\mathscr{L}^k\big(\{y \in A \cap B_r(x) : \ \frac{d(f(y), f(x))}{r} > N\}\big) = 0.$$

This definition is a simplified version of Federer's definition of approximate upper limit of the difference quotients, where $|y - x|$ is replaced by r in the denominator. The next theorem, proved in [**AW11**, Theorem 5.1] (actually a simplified version of [**Fed69**, Theorem 3.1.4]), is the weak version of the total differential theorem that implements the local slope $\delta_x f$ defined above instead of the classical differential:

Theorem 2.3.3. *Let $A \subset \mathbb{R}^k$ Borel and $f : \mathbb{R}^k \to E$ be Borel.*

(i) *Let $k = n + m$, $x = (z, y)$, and assume that there exist Borel subsets A_1, A_2 of A such that $\delta_z(f(\cdot, y)) < \infty$ for all $(z, y) \in A_1$ and $\delta_y(f(z, \cdot)) < \infty$ for all $(z, y) \in A_2$. Then $\delta_x f < \infty$ for $\mathscr{L}^k$-a.e. $x \in A_1 \cap A_2$;*

(ii) *if $\delta_x f < \infty$ for $\mathscr{L}^k$-a.e. $x \in A$ there exists a sequence of Borel sets $B_n \subset A$ such that $\mathscr{L}^k(A \setminus \cup_n B_n) = 0$ and the restriction of f to B_n is Lipschitz for all n.*

Proof. For real-valued maps this result is basically contained in [**Fed69**, 3.1.4], with slightly different definitions: here, to simplify matters as much as possible, we avoid to mention any differentiability result.

(i) By an exhaustion argument we can assume with no loss of generality that, for some constant N, $\delta_z f < N$ in A_1 and $\delta_y f < N$ in A_2. Moreover, by Egorov theorem (which allows to transform pointwise limits, in our case as $r \downarrow 0$, into uniform ones, at the expense of passing to a slightly smaller domain in measure), we can also assume that

$$\lim_{r\downarrow 0} \frac{1}{r^m}\mathscr{L}^m\big(\{y' \in B^m_r(y) : \ \frac{d(f(z, y'), f(z, y))}{r} > N\}\big) = 0 \quad \text{uniformly for } (z, y) \in A_2. \tag{14}$$

We are going to show that $\delta_x f \leq 2N$ $\mathscr{L}^k$-a.e. in $A_1 \cap A_2$. By the triangle inequality, it suffices to show that

$$\lim_{r\downarrow 0} \frac{1}{r^k} \mathscr{L}^k\big(\{(z',y') \in B_r((z,y)) : \ \frac{d(f(z',y), f(z,y))}{r} > N\}\big) = 0 \tag{15}$$

and

$$\lim_{r\downarrow 0} \frac{1}{r^k} \mathscr{L}^k\big(\{(z',y') \in B_r((z,y)) : \ \frac{d(f(z',y'), f(z',y))}{r} > N\}\big) = 0 \tag{16}$$

for $\mathscr{L}^k$-a.e. $(z,y) \in A_1 \cap A_2$. The first property is clearly satisfied at all $(z,y) \in A_1$, because the sets in (15) are contained in

$$\{z' \in B_r(z) : \ \frac{d(f(z',y), f(z,y))}{r} > N\} \times B_r(y).$$

In order to show the second property (16) we can estimate the quantity therein by

$$\frac{1}{r^n} \int_{B_r^1(z)} \frac{1}{r^m} \mathscr{L}^m\big(\{y' \in B_r(y) : \ \frac{d(f(z',y'), f(z',y))}{r} > M\}\big)\, dz' + \frac{\mathscr{L}^m(B_r(y))\mathscr{L}^n(B_r^2(z))}{r^{m+n}},$$

where $B_r^1(z) = \{z' \in B_r(z) : \ (z',y) \in A_2\}$ and $B_r^2(z) := B_r(z) \setminus B_r^1(z)$. If we let $r \downarrow 0$, the first term gives no contribution thanks to (14); the second one gives no contribution as well provided that z is a density point in $\mathbb{R}^n$ for the slice $(A_2)_y := \{z' : \ (y,z') \in A_2\}$. Since, for all y, $\mathscr{L}^n$-a.e. point of $(A_2)_y$ is a density point $(A_2)_y$, by Fubini's theorem we get that $\mathscr{L}^k$-a.e. $(y,z) \in A_2$ has this property.

(ii) Let $e_0 \in E$ be fixed. Denote by C_N the subset of A where both δ_x and $d(f, e_0)$ do not exceed N. Since the union of C_N covers $\mathscr{L}^k$-almost all of A, it suffices to find a family (B_n) with the required properties covering $\mathscr{L}^k$-almost all of C_N. Let χ_k be a geometric constant defined by the property

$$\mathscr{L}^k(B_{|x_1-x_2|}(x_1) \cap B_{|x_1-x_2|}(x_2)) = \chi_k \mathscr{L}^k(B_{|x_1-x_2|}(0)).$$

We choose $B_n \subset C_N$ and $r_n > 0$ in such a way that $\mathscr{L}^k(C_N \setminus \cup_n B_n) = 0$ and, for all $x \in B_n$ and $r \in (0, r_n)$, we have

$$\mathscr{L}^k\big(\{y \in B_r(x) : \ \frac{d(f(y), f(x))}{r} > N+1\}\big) \leq \frac{\chi_k}{2} \mathscr{L}^k(B_r(x)). \tag{17}$$

The existence of B_n is again ensured by Egorov theorem.

We now claim that the restriction of f to C_n is Lipschitz. Indeed, take x_1, $x_2 \in B_n$: if $|x_1 - x_2| \geq r_n$ we estimate $d(f(x_1), f(x_2))$ simply with $4r_n^{-1} \sup_{B_n} d(f, e_0)|x_1 - x_2|$. If not, by (17) at $x = x_i$ with $r = |x_1 - x_2|$ and our choice of χ_k we can find $y \in B_r(x_1) \cap B_r(x_2)$ where

$$\frac{d(f(y), f(x_1))}{r} \leq N+1 \quad \text{and} \quad \frac{d(f(y), f(x_2))}{r} \leq N+1.$$

It follows that $d(f(x_1), f(x_2)) \leq 2(N+1)|x_1 - x_2|$. $\square$

2.3.2. Rectifiable Currents.

DEFINITION 2.3.4 (Rectifiable currents, [**AK00a**, 4.2]). *A current $T \in \mathbf{M}_k(E)$ is said to be rectifiable if*

(a) *$\|T\|$ is concentrated on a countable $\mathscr{H}^k$-rectifiable set;*
(b) *$\|T\|$ vanishes on $\mathscr{H}^k$-negligible Borel sets.*

The space of (real) rectifiable currents is denoted by $\mathcal{R}_k(E)$.

In particular for $k = 0$ a rectifiable current is a finite or countable sum of Dirac's masses, with finite total mass. Thanks to the locality property of metric currents every rectifiable k-current enjoy a parametric representation as a sum Lipschitz images of Euclidean metric currents:

PROPOSITION 2.3.5. *A metric current T belongs to $\mathcal{R}_k(E)$ if and only if there exist a sequence of compact sets $K_i \subset \mathbb{R}^k$, functions $\theta_i \in L^1(\mathbb{R}^k)$ with $\mathrm{spt}(\theta_i) \subset K_i$, and bi-Lipschitz maps $f_i : K_i \to E$ such that*

$$T = \sum_i f_{i\#}[\![\theta_i]\!], \qquad \mathbf{M}(T) = \sum_i \mathbf{M}(f_{i\#}[\![\theta_i]\!]).$$

Moreover if E is a Banach space T can be approximated in mass by normal currents.

Note how the previous Proposition implicitly attaches an orientation to the concentration set of T, via the parametrizing maps f_i. Observe also that the locality property excludes from $\mathcal{R}_k(E)$, and even from $\mathcal{D}_k(E)$ a current like

$$T(f\,d\pi) = \int_0^1 f(x,0)\frac{\partial \pi}{\partial y}(x,0)\,dx$$

in $\mathbb{R}^2$: the orientation of $\mathscr{H}^1 \llcorner ([0,1] \times \{0\})$ must be e_1. According to [**Fed86**] in the Euclidean context the mass closure of images of finite mass k-currents in $\mathbb{R}^k$ consists of flat currents with positive densities, see section 2.4. Finally the space of currents representable as $\sum_i f_{i\#}[\![\theta_i]\!]$ with $\theta_i \in L^1(\mathbb{R}^k, \mathbb{Z})$ is denoted by $\mathcal{I}_k(E)$; the intersection

$$\mathbf{I}_k(E) := \mathcal{I}_k(E) \cap \mathbf{N}_k(E)$$

is the space if integral current, which possesses crucial compactness properties.

Given $T \in \mathcal{R}_k(E)$ there exists a canonical set where T is concentrated:

THEOREM 2.3.6. *Let $T \in \mathcal{R}_k(E)$ and let*

$$set(T) := \{x \in E : \Theta^{*k}(\|T\|, x) > 0\}.$$

Then $set(T)$ is countably $\mathscr{H}^k$-rectifiable and $\|T\|$ is concentrated on $set(T)$. Moreover $set(T)$ is unique up to $\mathscr{H}^k$-negligible subsets of E.

We let

$$\mathbf{S}(T) := \mathscr{H}^k(set(T))$$

be the size of $T \in \mathcal{R}_k(E)$. Note in particular that if $T \in \mathcal{I}_k(E)$ then $\mathscr{H}^k(set(T)) \leq \mathbf{M}(T)$.

THEOREM 2.3.7 (Slices of rectifiable current,[**AK00a**, 5.7]). *Let* $T \in \mathcal{R}_k(E)$ *(resp.* $\in \mathcal{I}_k(E)$*), an let* $\pi \in \mathrm{Lip}(E, \mathbb{R}^\ell)$*, with* $1 \le \ell \le k$*. Then there exist currents* $\langle T, \pi, x\rangle \in \mathcal{R}_{k-\ell}(E)$ *(resp* $\langle T, \pi, x\rangle \in \mathcal{I}_{k-\ell}(E)$*), concentrated on set*$(T) \cap \pi^{-1}(x)$ *and satisfying* (10),(11),

$$\langle T \llcorner A, \pi, x\rangle = \langle T, \pi, x\rangle \llcorner A \qquad \forall A \in \mathcal{B}(E)$$

for $\mathscr{L}^\ell$*-a.e.* $x \in \mathbb{R}^\ell$ *and*

$$\int_{\mathbb{R}^\ell} \mathbf{S}(\langle T, \pi, x\rangle)\, dx \le c(k, m) \prod_{i=1}^{\ell} \mathrm{Lip}(\pi^i)\mathbf{S}(T).$$

For the sake of completeness we report the fundamental compactness and boundary rectifiability Theorems for rectifiable and integral currents. The proofs of these statements require some tools developed in the next sections, but we present them here for continuity with the argument of the section.

THEOREM 2.3.8 (Closure Theorem, [**AK00a**, 8.5]). *Let* $(T_h) \subset \mathbf{N}_k(E)$ *be a sequence weakly converging to* $T \in \mathbf{N}_k(E)$ *such that*

$$T_h \in \mathcal{R}_k(E), \qquad \sup_h \mathbf{M}(T_h) + \mathbf{M}(\partial T_h) + \mathbf{S}(T_h) < \infty.$$

Then $T \in \mathcal{R}_k(E)$*.*

THEOREM 2.3.9 (Boundary rectifiability). *Let* $T \in \mathbf{I}_k(E)$*. Then* $\partial T \in \mathbf{I}_k(E)$*.*

These two theorems allow to solve the Plateau's problem in w^*-separable dual Banach spaces (see section 2.10) satisfying the following isoperimetric inequality: there exists a constant $\gamma(k, Y) > 0$ such that for any $S \in \mathbf{I}_k(Y)$ with $\partial S = 0$ there exists $T \in \mathbf{I}_{k+1}(Y)$ such that $\partial T = S$ and

$$\mathbf{M}(T) \le \gamma(k, Y)[\mathbf{M}(S)]^{\frac{k+1}{k}}.$$

This assumption on the ambient space is necessary to show the equicompactness of the support of every minimizing sequence, with the help of a lower density bound on their masses provided by the cone construction.

THEOREM 2.3.10 (Plateau's problem, [**AK00a**, Theorem 10.6]). *Suppose* Y *is a* w^**-separable dual satisfying the isoperimetric inequality: then for every* $S \in \mathbf{I}_k(Y)$ *with compact support the problem*

$$\min\{\mathbf{M}(T) : T \in \mathbf{I}_{k+1}(Y), \partial T = S\}$$

has a solution.

The isoperimetric inequality is clearly not needed if one assumes that Y is a Hilbert space, since the projection on the convex hull co(spt(S)), which is compact [**Dei77**, Lemma 2.2], is 1-Lipschitz, hence the minimizing sequences can be chosen with support in such compact set.

2.4. Flat currents

The flat norm plays an important role in the contest of metric currents, as it gives rise to a topology much more flexible than the mass convergence, but stronger than the weak* convergence. Recall also that the space of normal currents $\mathbf{N}_k(E)$ is complete, but hardly separable: the push forwards of a normal current via two neighboring maps can have very large mass distance. To this extent the following definition proves advantageous:

DEFINITION 2.4.1 (Flat norm). *The flat norm of a current $T \in \mathcal{D}_k(E)$ is defined as*

$$\mathbf{F}(T) = \inf\{\mathbf{M}(T - \partial Y) + \mathbf{M}(Y) : \, Y \in \mathbf{M}_{k+1}(E)\}. \tag{18}$$

It is a straightforward calculation to show that $\mathbf{F}$ is a norm on $\mathbf{M}_k(E)$, and that

$$\mathbf{F}(\partial T) \leq \mathbf{F}(T) \leq \mathbf{M}(T). \tag{19}$$

Our primary space of currents is the following:

DEFINITION 2.4.2. *We define the space of flat currents $\mathbf{F}_k(E)$ as the $\mathbf{F}$-completion of the space of normal currents:*

$$\mathbf{F}_k(E) = \widehat{\mathbf{N}_k(E)}^{\mathbf{F}}.$$

The first inequality in (19) immediately implies that if $T \in \mathbf{F}_k(E)$ then $\partial T \in \mathbf{F}_{k-1}(E)$. Recall also that any flat current T of finite mass can be approximated by a sequence (Z_h) of normal currents in mass norm. In fact, by definition there exist currents $(T_h) \subset \mathbf{N}_k(E)$ and $(Y_h) \subset \mathbf{M}_{k+1}(E)$ such that

$$\mathbf{M}(T - T_h - \partial Y_h) + \mathbf{M}(Y_h) \to 0.$$

The hypothesis $\mathbf{M}(T) < \infty$ yields $\mathbf{M}(\partial Y_h) < \infty$, hence the currents $Z_h = T_h + \partial Y_h$ are normal and clearly $\mathbf{M}(T - Z_h) \to 0$. As we will see later on, many properties of the space of normal currents behave nicely under convergence in the flat norm (18) and therefore can be extended to the completion. On the other hand, every definition involving a completion procedure somehow hides the true nature of the objects under consideration. The following proposition partially overcomes this inconvenience:

PROPOSITION 2.4.3 ([**Fed69**, 4.1.24]). *The space of flat k-currents can be characterized as*

$$\mathbf{F}_k(E) = \{X + \partial Y : X \in \mathbf{F}_k(E), Y \in \mathbf{F}_{k+1}(E), \, \mathbf{M}(X) + \mathbf{M}(Y) < \infty\}. \tag{20}$$

PROOF. We need only to show that $\mathbf{F}_k(E)$ is contained in the right hand side, as the opposite inclusion follows by additivity and stability of flat currents under the boundary operator. Let $(T_h) \subset \mathbf{N}_k(E)$ be a sequence of normal currents rapidly converging towards $T \in \mathbf{F}_k(E)$: $\sum_h \mathbf{F}(T_{h+1} - T_h) < \infty$. There exist normal currents X_h and Y_h such that

$$T_{h+1} - T_h = X_h + \partial Y_h \;\text{ and }\; \mathbf{M}(X_h) + \mathbf{M}(Y_h) < 2\mathbf{F}(T_{h+1} - T_h).$$

Since $\mathbf{M}_k(E)$ is Banach, the $\mathbf{M}$-converging series $\sum_h X_h$ and $\sum_h Y_h$ define two flat currents, respectively $X \in \mathbf{F}_k(E)$ and $Y \in \mathbf{F}_{k+1}(E)$, of finite mass such that $T - T_0 = X + \partial Y$. □

Thanks to the representation (20), given $\omega = f d\pi \in \mathcal{D}^k(E)$ it holds

$$|T(\omega)| \le \max\{\sup|f|, \mathrm{Lip}(f)\} \prod_i \mathrm{Lip}(\pi^i)\mathbf{F}(T). \tag{21}$$

In particular convergence in the flat norm is stronger than weak* convergence. A typical example of flat current of infinite mass is

$$\partial \sum_{k=1}^{\infty} [\![2^{-2k}, 2^{1-2k}]\!] \in \mathbf{F}_0(\mathbb{R}).$$

2.4.1. Restriction and slicing. In [**AK11, AK00b**] (and [**Fed69**, 4.2.1] for the classical case in Euclidean space), it is proved that the family of slices $\langle T, u, x\rangle$ and restrictions $T \llcorner \{u < r\}$ of a normal current $T \in \mathbf{N}_k(E)$ via $u \in \mathrm{Lip}(E)$, besides the properties expressed in Theorem 2.2.6, enjoy the following estimates:

$$\int_a^{*b} \mathbf{F}(T \llcorner \{u < r\})\, dr \le (b - a + \mathrm{Lip}(u))\mathbf{F}(T), \tag{22}$$

$$\int_a^{*b} \mathbf{F}(\langle T, u, r\rangle)\, dr \le \mathrm{Lip}(u)\mathbf{F}(T), \tag{23}$$

where $\int^*$ denotes the outer integral.

Proposition 2.4.4. *The operations of restriction and slicing via a Lipschitz map can be extended to the space of flat currents in such a way that $\sum_h \mathbf{F}(T_h - T) < \infty$ implies*

$\mathbf{F}(T_h \llcorner \{u < r\} - T \llcorner \{u < r\}) \to 0$ *and* $\mathbf{F}(\langle T_h, u, r\rangle - \langle T, u, r\rangle) \to 0$ *for $\mathscr{L}^1$-a.e. $r \in \mathbb{R}$. Moreover, inequalities* (22) *and* (23) *hold for a generic $T \in \mathbf{F}_k(E)$.*

Proof. Let $T \in \mathbf{F}_k(E)$ and let (T_h) be a sequence of normal currents rapidly converging to T: $\sum_h \mathbf{F}(T_h - T) < \infty$. Thanks to the subadditivity of the outer integral it is fairly easy to show that for $\mathscr{L}^1$-a.e. r both sequences $(T_h \llcorner \{u < r\})$ and $(\langle T_h, u, r\rangle)$ are $\mathbf{F}$-Cauchy, hence they admit a limit. Note that these limits do not depend on the particular (T_h) we choose: if (T'_h) were another sequence rapidly converging to T, we could merge it with (T_h) setting $T''_{2h} = T_h$, $T''_{2h+1} = T'_h$. Then (T''_h) would have converging restrictions and slices for almost every r. Therefore the limits

$$\lim_h T_h \llcorner \{u < r\} \quad \text{and} \quad \lim_h T'_h \llcorner \{u < r\}$$

must agree for a set of values r of full measure; similarly for the sequence of slices $(\langle T_h, u, r\rangle)$. Finally we write T as an $\mathbf{F}$-convergent sum of normal currents

$$T = T_N + \sum_{h=N}^{\infty} (T_{h+1} - T_h) \text{ with } \sum_{h=N}^{\infty} \mathbf{F}(T_{h+1} - T_h) < \varepsilon.$$

Hence, since $\mathbf{F}(T_N) \le \mathbf{F}(T) + \varepsilon$, applying (22) and the subadditivity of the upper integral

$$\begin{aligned}\int_a^{*b} \mathbf{F}(T \llcorner \{u < r\})\, dr &\le \int_a^{*b} \mathbf{F}(T_N \llcorner \{u < r\})\, dr + \sum_{h=N}^{\infty} \int_a^{*b} \mathbf{F}((T_{h+1} - T_h) \llcorner \{u < r\})\, dr \\ &\overset{(22)}{\le} (b - a + \mathrm{Lip}(u))(\mathbf{F}(T) + 2\varepsilon)\end{aligned}$$

we prove the thesis. The statement for (23) can be proved in the same way. □

Proposition 2.4.4 allows us to extend many properties of slicing and restriction from normal currents to flat currents by density. First of all, given $\ell \leq k$ the slicing of a current $T \in \mathbf{F}_k(E)$ by a vector-valued map $\pi = (\pi^1, \ldots, \pi^\ell) \in \mathrm{Lip}(E, \mathbb{R}^\ell)$ can be defined inductively:

$$\langle T, \pi, x\rangle = \langle\langle T, (\pi^1, \ldots, \pi^{\ell-1}), (x_1, \ldots, x_{l-1})\rangle, \pi^\ell, x_\ell\rangle.$$

Fubini's theorem ensures that these iterated slices are meaningful for $\mathscr{L}^\ell$-a.e. $x \in \mathbb{R}^\ell$, and it is easy to show by induction that $\partial\langle T, u, r\rangle = (-1)^\ell \langle \partial T, u, r\rangle$. In particular, for every $u \in \mathrm{Lip}(E)$ slicing and boundary operator commute via the relation

$$\partial\langle T, u, r\rangle = -\langle \partial T, u, r\rangle. \qquad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbb{R}. \tag{24}$$

Lemma 2.4.5 (Slice and restriction commute). *Let $T \in \mathbf{F}_k(E)$, π, $u \in \mathrm{Lip}(E)$. Then*

$$\langle T, \pi, r\rangle \llcorner \{u < s\} = \langle T \llcorner \{u < s\}, \pi, r\rangle \qquad \textit{for } \mathscr{L}^2\textit{-a.e. } (r, s) \in \mathbb{R}^2. \tag{25}$$

Proof. We start with $T \in \mathbf{N}_k(E)$. It is immediate to check that, for s fixed, the currents in the left hand side of (25) fulfil (9) and (10) relative to $T \llcorner \{u < s\}$, therefore they coincide with $\langle T \llcorner \{u < s\}, \pi, r\rangle$ for $\mathscr{L}^1$-a.e. $r \in \mathbb{R}$. Let now T be flat and let $(T_h) \subset \mathbf{N}_k(E)$ with $\sum_h \mathbf{F}(T_h - T) < \infty$: we want to pass to the limit in the identity

$$\langle T_h, \pi, r\rangle \llcorner \{u < s\} = \langle T_h \llcorner \{u < s\}, \pi, r\rangle \qquad \text{for } \mathscr{L}^2\text{-a.e. } (r, s) \in \mathbb{R}^2. \tag{26}$$

We know that $\sum_h \mathbf{F}(T_h \llcorner \{u < s\} - T \llcorner \{u < s\}) < \infty$ for $\mathscr{L}^1$-a.e. $s \in \mathbb{R}$; for any such s by Proposition 2.4.4 we can plug $(T_h - T) \llcorner \{u < r\}$ into inequality (23) and infer that the right hand sides in (26) converge to $\langle T \llcorner \{u < s\}, \pi, r\rangle$ with respect to $\mathbf{F}$ for $\mathscr{L}^1$-a.e. $r \in \mathbb{R}$. On the other hand, we also know that $\sum_h \mathbf{F}(\langle T_h, \pi, r\rangle - \langle T, \pi, r\rangle) < \infty$ for $\mathscr{L}^1$-a.e. $r \in \mathbb{R}$; for any r for which this property holds the left hand sides in (26) converge with respect to $\mathbf{F}$ to $\langle T_h, \pi, r\rangle \llcorner \{u < s\}$ for $\mathscr{L}^1$-a.e. $s \in \mathbb{R}$, again by Proposition 2.4.4 and equation (22). Therefore, passing to the limit as $h \to \infty$ in (26), using Fubini's theorem, we conclude. □

Lemma 2.4.6 (Set additivity of restrictions). *Let $T \in \mathbf{F}_k(E)$, π_1, $\pi_2 \in \mathrm{Lip}(E)$ and $\bar{t} \in \mathbb{R}$ such that the sets $\{\pi_1 < \bar{t}\}$ and $\{\pi_2 < \bar{t}\}$ have positive distance. Let $\pi := \min\{\pi_1, \pi_2\}$. Then*

$$T \llcorner \{\pi < t\} = T \llcorner \{\pi_1 < t\} + T \llcorner \{\pi_2 < t\} \qquad \textit{for a.e. } t < \bar{t}. \tag{27}$$

Proof. Let $t < \bar{t}$. Since $\{\pi_1 < t\}$ and $\{\pi_2 < t\}$ are distant the function

$$\psi(x) = \frac{d(x, \{\pi_1 < t\})}{d(x, \{\pi_1 < t\}) + d(x, \{\pi_2 < t\})}$$

is Lipschitz and equals 0 in $\{\pi_1 < t\}$ and 1 in $\{\pi_2 < t\}$. Let (T_h) be a sequence of normal currents rapidly converging to T such that

$$\sum_h \mathbf{F}\left(T_{h+1} \llcorner \{\pi < t\} - T_h \llcorner \{\pi < t\}\right) < \infty.$$

Then the sequence $S_h = \psi T_h \llcorner \{\pi < t\} = T_h \llcorner \{\pi_2 < t\}$ satisfies

$$\mathbf{F}(S_{h+1} - S_h) \leq \max\{\sup|\psi|, \mathrm{Lip}(\psi)\} \mathbf{F}(T_{h+1} \llcorner \{\pi < t\} - T_h \llcorner \{\pi < t\}),$$

hence S_h converge to $T\llcorner\{\pi_2 < t\}$ in the flat norm. Similarly for $T\llcorner\{\pi_1 < t\}$. Equation (27) holds for normal currents, and since the same sequence (T_h) is used to define the three restrictions, set additivity is straightforward by passing to the limit. $\square$

2.4.2. Support and push forward. We adopt (see also [**Ada08**]) as definition of support of a flat current T the set:

$$\mathrm{spt}(T) = \{x \in E : T\llcorner B_r(x) \neq 0 \text{ for } \mathscr{L}^1\text{-a.e. } r > 0\}. \tag{28}$$

Observe that the a.e. in the definition is motivated by the fact that slices exist only up to $\mathscr{L}^1$-negligible sets, and that $\mathrm{spt}(T) = \mathrm{spt}\|T\|$ whenever $T \in \mathbf{M}_k(E)$.

PROPOSITION 2.4.7. $\mathrm{spt}(T)$ *is a closed set and* $x \notin \mathrm{spt}(T)$ *implies* $T\llcorner B_r(x) = 0$ *for a.e.* $r \in (0, \mathrm{dist}(x, \mathrm{spt}(T)))$.

PROOF. Let $x \notin \mathrm{spt}(T)$: there must be a set A of radii of positive $\mathscr{L}^1$-measure such that $T\llcorner B_r(x) = 0$ for $\mathscr{L}^1$-a.e. $r \in A$. If $(T_h) \subset \mathbf{N}_k(E)$ is a sequence of normal currents rapidly converging to T, by (22) we obtain that for almost every $r \in A$

$$T_h\llcorner B_r(x) \to T\llcorner B_r(x) = 0. \tag{29}$$

rapidly. Fix now $r > 0$ with this property, $y \in B_r(x)$ and $\rho < r - d(x, y)$: we want to prove that $T\llcorner B_\rho(y) = 0$ for a.e. $\rho \in (0, r - d(x, y))$. Since T_h has finite mass we have $(T_h\llcorner B_r(x))\llcorner B_\rho(y) = T_h\llcorner B_\rho(y)$, and since the convergence in (29) is rapid, again for almost every ρ in $(0, r - d(x, y))$

$$T_h\llcorner B_\rho(y) = (T_h\llcorner B_r(x))\llcorner B_\rho(y) \to (T\llcorner B_r(x))\llcorner B_\rho(y) = 0\llcorner B_\rho(y) = 0.$$

Hence $B_r(x) \cap \mathrm{spt}(T) = \emptyset$. $\square$

PROPOSITION 2.4.8. *For all* $T \in \mathbf{F}_k(E)$ *the following properties hold:*

(i) *If* $\mathrm{spt}(f)$ *is compact then*

$$\mathrm{spt}(f) \cap \mathrm{spt}(T) = \emptyset \quad \Longrightarrow \quad T(f\, d\pi) = 0 \quad \forall \pi \in \mathrm{Lip}(E, \mathbb{R}^k). \tag{30}$$

(ii) *For all* $u \in \mathrm{Lip}(E)$

$$\mathrm{spt}(T\llcorner\{u < t\}) \subset \mathrm{spt}(T) \cap \{u \leq t\} \qquad \textit{for } \mathscr{L}^1\textit{-a.e. } t \in \mathbb{R}. \tag{31}$$

PROOF. (i) In this proof only, let us conventionally say that $T\llcorner B_r(x_0) = 0$ if there exist normal currents T_n such that $\mathbf{F}(T_n - T) \to 0$ and $\mathbf{F}(T_n\llcorner B_r(x_0)) \to 0$. In the proof of (30), we assume first that $\mathrm{spt}(f)$ is contained in a ball $B_r(x)$ such that $T\llcorner B_r(x) = 0$. By assumption $T_h\llcorner B_r(x_0)(f) \to 0$ for suitable approximating normal currents T_h; on the other hand, $T_h\llcorner B_r(x_0)(f) = T_h(f) \to T(f)$.

Now, let us consider the general case. Since the space is boundedly compact, we can find an open bounded neighborhood U of $\mathrm{spt}(f)$ such that $\overline{U} \cap \mathrm{spt}(T) = \emptyset$. By Proposition 2.4.7, any $x \in \overline{U}$ is the center of a ball B_x such that $T\llcorner B_x = 0$: we can therefore extract a finite subcover $\{B_j\}$ and build a partition of unity $\{\chi_j\}$ made of nonnegative Lipschitz functions such that $\sum_j \chi_j = 1$ in $\mathrm{spt}(f)$ and $\mathrm{spt}(\chi_j) \subset B_j$. Hence $f = \sum_j f\chi_j$ and the previous step yields

$$T(f\, d\pi) = \sum_j T(f\chi_j\, d\pi) = 0.$$

(ii) There exist $(T_h) \subset \mathbf{N}_k(E)$ such that for almost every t

$$\sum_h \mathbf{F}(T_h \llcorner \{u < t\} - T \llcorner \{u < t\}) < \infty.$$

We fix t with these properties and $x \notin \mathrm{spt}(T) \cap \{u \leq t\}$, so that $\mathrm{spt}(T) \cap \{u \leq t\} \cap B_r(x) = \emptyset$ for $r \in (0, \bar{r})$ for some $\bar{r} > 0$. We obtain that

$$0 = (T_h \llcorner \{u < t\}) \llcorner B_r(x) \to (T \llcorner \{u < t\}) \llcorner B_r(x)$$

for a.e. $r \in (0, \bar{r})$, hence $x \notin \mathrm{spt}(T \llcorner \{u < t\})$. □

Compare (28) with the classical definition of support

$$\mathrm{cl} - \mathrm{spt}(T) := E \setminus \bigcup \{A \text{ open}, \, T(f d\pi) = 0 \text{ whenever } \mathrm{spt}(f) \subset A\} :$$

since closed balls of E are compact, implication (30) easily implies that $E \setminus \mathrm{spt}(T) \subset E \setminus \mathrm{cl} - \mathrm{spt}(T)$. On the other hand if $B_r \Subset E \setminus \mathrm{cl} - \mathrm{spt}(T)$ and $T \llcorner B_r$ exist, then by (31) and the locality property given $f d\pi \in \mathcal{D}^k(E)$ it is easy to build (simply multiplying by a cutoff function) a test form $\tilde{f} d\pi$ such that $\mathrm{spt}(\tilde{f}) \subset A$ and $T(f d\pi) = T(\tilde{f} d\pi) = 0$. Hence the two definitions agree.

Given a Lipschitz map $\Phi : E \to F$ between two metric spaces and given $T \in \mathbf{F}_k(E)$ the flat norm behaves according to the next proposition:

Proposition 2.4.9. *For every $T \in \mathbf{F}_k(E)$ it holds $\Phi_\# T \in \mathbf{F}_k(F)$ and*

$$\mathbf{F}_F(\Phi_\# T) \leq (\mathrm{Lip}(\Phi))^k \mathbf{F}_E(T).$$

In particular, $\Phi_\# T \in \mathbf{F}(F)$. Furthermore, the push forward and the boundary operator commute.

Proof. Since $\mathbf{F}_F(\Phi_\# S) \leq (\mathrm{Lip}(\Phi))^k \mathbf{F}_E(S)$ holds for S normal, the current $\Phi_\# T$ is flat and the estimate holds also for flat currents. The relation $\partial \Phi_\# T = \Phi_\# \partial T$ simply comes from the definition. □

Suppose moreover that $\Phi : E \to F$ is an isometric embedding: then by the Hahn-Banach Theorem every composition $\pi \circ \Phi$ has an extension preserving the Lipschitz constant, by locality we have that $\mathbf{M}(\Phi_\# T) = \mathbf{M}(T)$ and $\|\Phi_\# T\| = \Phi_\# \|T\|$.

2.5. Comparison between metric and Euclidean currents

The theory outlined in the previous section must be related to the Euclidean one developed by Federer and Fleming during the 60s. In particular in the next chapters we will treat the notion of distributional jacobian of a Sobolev map $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ interpreting it as a flat current. Clearly Ω would be the natural ambient metric space, but unfortunately it is not complete; moreover we will need several results in the interior of Ω. We therefore now describe the main similarities and differences between the two approaches, and explain how to localize a flat current to a compact set, in order to use the theorems on rectifiability and size of the following sections.

2.5.1. Exterior algebra and projections. In $\mathbb{R}^m$ we will denote the standard basis $e_1, \dots, e_m$ and its dual $e^1, \dots, e^m$. For every $1 \le k \le m$ we let

$$\mathbf{O}_k = \left\{ \pi : \mathbb{R}^m \to \mathbb{R}^k : \pi \circ \pi^* = I_k \right\}$$

be the space of orthogonal projections of rank k. We will also need to fix coordinates according to some projection $\pi \in \mathbf{O}_k$: we agree that $\mathbb{R}^m \ni z = (x, y) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$ are orthogonal coordinates with positive orientation such that $\pi(z) = x$. In particular we let $A^x = A \cap \pi^{-1}(x)$ be the restriction of any $A \subset \mathbb{R}^m$ to the fiber $\pi^{-1}(x)$ and $i^x = \mathbb{R}^{m-k} \to \mathbb{R}^m$ be the isometric injection $i^x(y) = (x, y)$.

As customary the symbols $\Lambda_k \mathbb{R}^m$ and $\Lambda^k \mathbb{R}^m$ will respectively denote the spaces of k-vectors and k-covectors in $\mathbb{R}^m$. The contraction operation $\llcorner : \Lambda_q \mathbb{R}^m \times \Lambda^p \mathbb{R}^m \to \Lambda_{q-p} \mathbb{R}^m$ between a q-vector ζ and a p-covector α, with $q \ge p$, is defined as:

$$\langle \zeta \llcorner \alpha, \beta \rangle = \langle \zeta, \alpha \wedge \beta \rangle \qquad \text{whenever } \beta \in \Lambda^{q-p} \mathbb{R}^m. \tag{32}$$

If $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear then

$$M_n L := e_1 \wedge \cdots \wedge e_m \llcorner L^1 \wedge \cdots \wedge L^n \in \Lambda_{m-n} \mathbb{R}^m \tag{33}$$

represents the collection of determinants of $n \times n$ minors of L. In fact, if $\underline{i} : \{1, \dots, m - n\} \to \{1, \dots, m\}$ is an increasing selection of indexes, and if $\bar{i} : \{1, \dots, n\} \to \{1, \dots, m\}$ is the complementary increasing selection, then the $e_{\underline{i}}$ component of $M_n L$ is

$$\langle M_n L, e^{\underline{i}} \rangle = \langle e_1 \wedge \cdots \wedge e_m, L^1 \wedge \cdots \wedge L^n \wedge e^{\underline{i}} \rangle = (-1)^\sigma \det([L]_{\bar{i}}), \tag{34}$$

where $[L]_{\bar{i}}$ is the $n \times n$ submatrix L_j^ℓ with $j = \bar{i}(1), \dots, \bar{i}(n)$, $\ell = 1, \dots, n$ and σ is the sign of the permutation

$$(1, \dots, m) \mapsto (\bar{i}(1), \dots, \bar{i}(n), \underline{i}(1), \dots, \underline{i}(m-n)).$$

When $\mathrm{rk}(L) = n$, choosing an orthonormal frame (e_i) so that $\ker(L) = < e_{n+1}, \dots, e_m >$ we have $L = (A, \mathbf{0})$ and by (34) $M_n L = \det(A) e_{n+1} \wedge \cdots \wedge e_m$. In particular $M_n L$ is a simple $(m-n)$-vector.

Recall that the spaces $\Lambda_k \mathbb{R}^m$ and $\Lambda^k \mathbb{R}^m$ can be endowed with two different pairs of dual norms. The first one is called norm, it is denoted by $|\cdot|$ and it comes from the scalar product where the multivectors

$$\{e_{i(1)} \wedge \cdots \wedge e_{i(k)}\}_i \quad \text{and} \quad \{e^{i(1)} \wedge \cdots \wedge e^{i(k)}\}_i \tag{35}$$

indexed by increasing maps $i : \{1, \dots, k\} \to \{1, \dots, m\}$ form a pair of dual orthonormal bases. The second one is called mass, comass for the space of covectors, and it is defined as follows: the comass of $\phi \in \Lambda^k \mathbb{R}^m$ is

$$\|\phi\| := \sup \left\{ \langle \phi, v_1 \wedge \cdots \wedge v_k \rangle : v_i \in \mathbb{R}^m, |v_i| \le 1 \right\}; \tag{36}$$

and the mass of $\xi \in \Lambda_k \mathbb{R}^m$ is defined, by duality, by

$$\|\xi\| := \sup \left\{ \langle \xi, \phi \rangle : \|\phi\| \le 1 \right\}.$$

As described in [**Fed69**, 1.8.1], in general $\|\xi\| \le |\xi|$ and equality holds if and only if ξ is simple.

Therefore $|M_nL| = \|M_nL\|$. Moreover using the Pitagora's Theorem for the norm and Binet's formula we have the following relation:

$$\begin{aligned}\sup_{\pi\in\mathbf{O}_{m-n}} |M_nL \llcorner d\pi| &= \sup_{\pi\in\mathbf{O}_{m-n}} |d\pi(M_nL)| = \sup_{\pi\in\mathbf{O}_{m-n}} \Big|\sum_{\underline{i}} M_nL^{\underline{i}}\, d\pi(e_{\underline{i}})\Big| \\ &\leq \sup_{\pi\in\mathbf{O}_{m-n}} |M_nL| \Big(\sum_{\underline{i}} |d\pi(e_{\underline{i}})|^2\Big)^{\frac{1}{2}} = |M_nL| \sup_{\pi\in\mathbf{O}_{m-n}} |\det(\pi\circ\pi^*)| \\ &= |M_nL| \end{aligned} \tag{37}$$

where $d\pi$ stands for $d\pi^1 \wedge \cdots \wedge d\pi^{m-n}$, and the equality is realized by the orthogonal projection onto $\ker(L)$. We adopt the convention of choosing the mass and comass norms to measure the length of k-vectors and k-covectors respectively. Finally if $\{x^i\}$ are coordinates relative to the orthonormal base e_i, we adopt the classical notation $dx^{\underline{i}} = dx^{i(1)} \wedge \cdots \wedge dx^{i(k)}$ for the base (35) of the space $\Lambda^k\mathbb{R}^m$ of k-covectors.

2.5.2. Euclidean currents. We briefly recall the basic definitions and properties of classical currents in $\Omega \subset \mathbb{R}^m$. This theory was introduced by De Rham in [**dR55**], along the lines of the previous work on distributions by Schwartz [**Sch66**], and the subsequently put forward by Federer and Fleming in [**FF60**]; we refer to [**Fed69**] for a complete account of it. We give for granted the concepts of derivative, exterior differentiation, pull-back and support of a smooth and compactly supported differential form: they can all be defined by expressing the form in the coordinates given by the frame (35), see [**Fed69**, 4.1.6]. We begin by defining the space of smooth, compactly supported test forms:

Definition 2.5.1 (Smooth test forms). *Let $\Omega \subset \mathbb{R}^m$ be an open set. We let $\mathscr{D}^k(\Omega)$ be the space of smooth, compactly supported k-differential forms:*

$$\mathscr{D}^k(\Omega) = \bigcup_{K\Subset\Omega} \mathscr{D}^k_K(\Omega), \qquad \mathscr{D}^k_K(\Omega) = \big\{\omega \in C^\infty(\Omega, \Lambda^k\mathbb{R}^m),\ \mathrm{spt}(\omega) \subset K\big\}. \tag{38}$$

Each space $\mathscr{D}^k_K(\Omega)$ is endowed with the topology given by the seminorms

$$p_{K,j}(\omega) = \sup\{\|D^\alpha\omega(x)\|, x \in K, |\alpha| \leq j\},$$

and $\mathscr{D}^k(\Omega)$ is endowed with the finest topology making the inclusions $\mathscr{D}^k_K(\Omega) \hookrightarrow \mathscr{D}^k(\Omega)$ are continuous.

This topology is locally convex, translation invariant and Hausdorff; moreover a sequence $\omega_j \to \omega$ in $\mathscr{D}^k(\Omega)$ if and only if there exists $K \Subset \Omega$ such that $\mathrm{spt}(\omega_j) \subset K$ and $p_{K,j}(\omega_j - \omega) \to 0$ for every $j \geq 0$.

The interior and alternating multiplications in the Grassmann algebra of $\mathbb{R}^m$ yield dual operations on currents: if $T \in \mathscr{D}_k(\Omega)$ and $\xi \in C^\infty(\Omega, \Lambda_h\mathbb{R}^m)$ and $\psi \in C^\infty(\Omega, \Lambda^\ell\mathbb{R}^m)$ with $k \geq \ell$, then

$$\begin{aligned} T\wedge\xi &\in \mathcal{D}_{h+k}(\Omega), \qquad (T\wedge\xi)(\phi) = T(\xi \lrcorner \phi) \quad \forall\phi \in \mathscr{D}^k(\Omega),\\ T\llcorner\psi &\in \mathcal{D}_{k-\ell}(\Omega), \qquad (T\llcorner\psi)(\phi) = T(\psi\wedge\phi) \quad \forall\phi \in \mathscr{D}^{k-\ell}(\Omega). \end{aligned}$$

Definition 2.5.2 (Classical currents and weak* convergence). *A current T is a continuous linear functional on $\mathscr{D}^k(\Omega)$. The space of k-currents is denoted by $\mathscr{D}_k(\Omega)$.*

We say that a sequence (T_h) *weak* converges to* T, $T_h \overset{*}{\rightharpoonup} T$, *whenever*

$$T_h(\omega) \to T(\omega) \qquad \forall \omega \in \mathscr{D}^k(\Omega). \tag{39}$$

The comass norm on the space of k-covector brings to define the mass of an Euclidean current as:

$$\mathbf{M}_{\mathrm{Eucl}}(T) := \sup\{|T(\omega)| : \sup_{x\in\Omega} \|\omega(x)\| \leq 1\}. \tag{40}$$

Finiteness of (40) defines membership to the space of finite mass currents $\mathbf{M}_{\mathrm{Eucl},k}(\Omega)$; similarly for normal currents. Notice the relation with condition (a) in Proposition 2.1.9. Similarly to Riesz's representation Theorem, characterizing linear bounded functionals on $C^0_0(X)$ as integration against Radon measures when the space X is locally compact and Hausdorff ([**Fed69**, 2.5.13] and [**Bog07**, 7.11.3]), in the classical theory of currents the finiteness of (40) ensures T is representable by integration against a finite measure, compare [**Fed69**, 4.1.7]. When the domain of the functional contains forms not infinitesimal "at infinity", like the space $\mathcal{D}^k(E)$ of metric currents, then Daniell's Theorem [**Fed69**, 2.5.5], [**Bog07**, 7.8.1] (applied to $T \llcorner d\pi$, for $\mathrm{Lip}(\pi^i) \leq 1$) provides the analogous result, under the additional hypothesis of continuity along monotone sequence as condition (b) in Proposition 2.1.9. Furthermore the linear structure of $\mathbb{R}^m$ allows to see the k-tuples $d\pi^1 \wedge \cdots \wedge d\pi^k$ as a unique map taking values into $\Lambda^k\mathbb{R}^m$, endowed with the comass norm (36). This in turn allows the representation by integration $T = \vec{T} \wedge \|T\|_{\mathrm{Eucl}}$:

$$T(\phi) = \int_\Omega \langle \vec{T}(x), \phi(x)\rangle \, d\|T\|(x) \qquad \forall \phi \in \mathcal{D}^k(\Omega).$$

for some $\|T\|_{\mathrm{Eucl}}$-measurable function $\vec{T}$ satisfying $\|\vec{T}(x)\| = 1$ at $\|T\|_{\mathrm{Eucl}}$-almost every point $x \in E$, compare [**Fed69**, 2.5.12]. Such representation is presently not available in the metric context, besides the special case when T is rectifiable. Finally recall that in Federer's classical monograph the currents in the main spaces $\mathbf{M}_k, \mathbf{N}_k, \mathbf{F}_k$ are all assumed to have compact support: in this section we will explicitly mention such a requirement, when needed.

The first bridge between the classical and the metric theory is the following proposition, establishing the equivalence in $E = \mathbb{R}^m$ of metric currents of finite mass with compact support and Euclidean flat currents with finite mass and compact support.

THEOREM 2.5.3 ([**AK00a**, 11.1]). *Any* $T \in \mathbf{M}_k(\mathbb{R}^m)$ *with compact support induces a Euclidean current* $T \in \mathbf{M}_{\mathrm{Eucl},k}(\mathbb{R}^m)$

$$\tilde{T}(\omega) = \sum_{|\underline{i}|=k} T(\omega_{\underline{i}} dx^{\underline{i}})$$

for any $\omega \in \mathscr{D}^k(\mathbb{R}^m)$. *Moreover* $\mathbf{M}_{\mathrm{Eucl}}(\tilde{T}) \leq c(k,m)\mathbf{M}(T)$. *Conversely any classical finite mass current* $T \in \mathbf{M}_{\mathrm{Eucl},k}(\mathbb{R}^m)$ *with compact support induces a metric current* $\hat{T}$ *with* $\mathbf{M}(\hat{T}) \leq \mathbf{M}_{\mathrm{Eucl}}(T)$. *Furthermore the maps* $T \mapsto \tilde{T}$ *and* $T \mapsto \hat{T}$ *are one the inverse of the other when restricted to normal currents.*

Observe in particular that when we will consider classical currents and we will apply some result from the metric theory, the metric mass (5) enjoys the same bounds from above holding for the Euclidean one. Notice moreover the discrepancy between the values of $\mathbf{M}(T)$: this is due to the fact that metric currents are defined only for simple forms $f d\pi^1 \wedge \cdots \wedge d\pi^k$, while Euclidean currents are tested also on linear combinations of them.

Euclidean flat currents in $\mathbb{R}^m$ are defined as the completion of Euclidean normal currents via the metric (18). Suppose that (T_h) are normal currents $\mathbf{F}$-converging to $T \in \mathbf{F}_k(\mathbb{R}^m)$ and consider the family of 1-Lipschitz cutoff functions $\rho_n := \max\{0, \min\{n - |x|, 1\}\}$. Then for $n \to \infty$

$$\mathbf{M}(T - T \llcorner \rho_n) = \|T\|(\{|x| \geq n-1\}) \to 0,$$
$$\mathbf{M}(\partial T - \partial(T \llcorner \rho_n)) = \mathbf{M}(\partial T \llcorner (1-\rho_n) + T \llcorner d\rho_n) \leq (\|T\| + \|\partial T\|)(\{|x| \geq n-1\}) \to 0.$$

Hence $\mathrm{spt}(T_h \llcorner \rho_n) \Subset \mathbb{R}^m$; moreover for every h there exists $n(h)$ sufficiently large such that $\mathbf{F}(T \llcorner \rho_{n(h)} - T) \leq \mathbf{F}(T_h - T) + \frac{1}{h} \to 0$. Therefore by Theorem 2.5.3 the spaces of classical and metric flat currents in $\mathbb{R}^m$ coincide; moreover $\mathbf{F}(T) \leq \mathbf{F}_{\mathrm{Eucl}}(T) \leq c(k,m)\mathbf{F}(T)$. Recall also the important duality result available for Euclidean flat currents [**Fed69**, 4.1.12]:

$$\mathbf{F}_{\mathrm{Eucl}}(T) = \sup\big\{T(\omega) : \omega \in \mathscr{D}^k(\mathbb{R}^m),\ \max\{\sup_x \|\omega(x)\|, \sup_x \|d\omega(x)\|\} \leq 1\big\}. \tag{41}$$

The quantity $\max\{\sup_x \|\omega(x)\|, \sup_x \|d\omega(x)\|\}$ is customarily denoted by $\mathbf{F}(\omega)$.

Definition 2.5.4 (Local flat currents). *We say that $T \in \mathcal{D}_k(\mathbb{R}^m)$ is a local flat current if for every $\varrho \in \mathrm{Lip}_b(\mathbb{R}^m)$ with compact support $T \llcorner \varrho \in \mathbf{F}_k(\mathbb{R}^m)$. The space of local flat currents will be denoted by $\mathbf{F}_k^{\mathrm{loc}}(\mathbb{R}^m)$.*

This definition is best suited for applying the metric theory to obtain interior result for Euclidean currents in bounded open subsets Ω of $\mathbb{R}^m$. Indeed the localized current $T \llcorner \varrho$ depends only on the action on $\mathrm{spt}(\varrho)$ and at the same time is a metric flat current in the whole Euclidean space $\mathbb{R}^m$, which is a geodesic space.

Since $\Omega \subset \mathbb{R}^m$ can be written as a countable union of nested compact subsets

$$K_i = \overline{\{x \in \Omega : d(x, \mathbb{R}^m \setminus \Omega) \geq \tfrac{1}{i}\} \cap B_n(0)},$$

or alternatively there exists Lipschitz functions ϱ_i such that $0 \leq \varrho_i \leq 1$, $\mathrm{Lip}(\varrho_i) \leq 1$, $\varrho_i|_{K_i} = 1$, $\varrho_i|_{\Omega \setminus K_{i+1}} = 0$, we can define the local flat convergence in Ω of $T_h \in \mathbf{F}_k(\mathbb{R}^m)$ to $T \in \mathbf{F}_k(\mathbb{R}^m)$ as

$$\mathbf{F}_\Omega^{\mathrm{loc}}(T_h, T) \to 0 \qquad \Longleftrightarrow \qquad \mathbf{F}((T_h - T) \llcorner \varrho_i) \to 0 \quad \forall i \in \mathbb{N}.$$

Thanks to (41) to prove local flat convergence of T_h to T in Ω it is then sufficient to prove that for every i

$$\lim_h \sup\{(T_h - T)(\psi) : \psi \in \mathscr{D}^k(\mathbb{R}^m), \mathbf{F}(\psi) \leq 1, \mathrm{spt}(\psi) \subset \mathrm{spt}(\varrho_i)\} = 0. \tag{42}$$

Indeed

$$\begin{aligned}\mathbf{F}((T_h - T) \llcorner \varrho_i) &= \sup\{\langle (T_h - T), \varrho_i \omega\rangle, \omega \in \mathscr{D}^k(\mathbb{R}^m), \mathbf{F}(\omega) \leq 1\} \\ &\leq \sup\{\langle (T_h - T), \psi\rangle, \omega \in \mathscr{D}^k(\mathbb{R}^m), \mathbf{F}(\psi) \leq 1 + \mathrm{Lip}(\varrho_i), \mathrm{spt}(\psi) \subset \mathrm{spt}(\varrho_i)\},\end{aligned}$$

since $\mathbf{F}(\varrho_i\omega) \le 1 + \mathrm{Lip}(\varrho_i)$, which is the stated limit besides a scaling factor. We remark that local flat convergence in Ω is metrizable by

$$\mathbf{F}^{\mathrm{loc}}_\Omega(S,T) := \sum_{i\in\mathbf{N}} 2^{-i}\frac{\mathbf{F}((S-T)\llcorner\varrho_i)}{1+\mathbf{F}((S-T)\llcorner\varrho_i)}.$$

(We will also use the notation $\mathbf{F}^{\mathrm{loc}}_\Omega(S-T)$ to denote the invariance under translation of this distance). Note furthermore that if $T \in \mathbf{F}_k(\mathbb{R}^m)$ and $\mathrm{spt}(T) \subset K \Subset \mathbb{R}^m$, then the representation $T = X + \partial Y$ in (20) can be chosen su that $\mathrm{spt}(X) \cup \mathrm{spt}(Y) \subset K^\delta$, where K^δ is the δ-neighborhood of K, for any $\delta > 0$.

Finally it is obvious that $\mathbf{F}^{\mathrm{loc}}_\Omega$ convergence implies weak* convergence in Ω, namely convergence against fixed Lipschitz functions with compact support comtained in Ω.

2.6. Size measure of a flat current

In this section we introduce the notion of concentration measure for a flat current, possibly with infinite mass.

Definition 2.6.1 (Concentration measure). *We say that a positive Borel measure μ is a concentration measure for $T \in \mathbf{F}_k(E)$ if $\mathscr{H}^0\llcorner\mathrm{spt}(T) \le \mu$ in the case $k = 0$, and if*

$$\mu_{T,\pi} := \int_{\mathbb{R}^k} \mathscr{H}^0\llcorner\mathrm{spt}\langle T,\pi,x\rangle\, d\mathscr{L}^k(x) \le \mu \qquad \forall\pi \in \mathrm{Lip}_1(E,\mathbb{R}^k) \tag{43}$$

for $k \ge 1$. The choice of μ can be optimized by choosing the least upper bound of the family $\{\mu_{T,\pi}\}$ in the lattice of nonnegative measures:

$$\mu_T := \bigvee_{\pi\in\mathrm{Lip}_1(E,\mathbb{R}^k)} \mu_{T,\pi}. \tag{44}$$

We shall call μ_T the concentration measure of μ.

Definition 2.6.2 (Size). *We say that $T \in \mathbf{F}_k(E)$ has finite size if μ_T has finite mass. In this case we define*

$$\mathbf{S}(T) := \mu_T(E).$$

For example a polyhedral chain in the Euclidean space $\mathbb{R}^m$

$$P = \sum_{i=1}^n a_i[\![Q_i]\!]$$

where $a_i \in \mathbb{R}$ and $[\![Q_i]\!]$ are the integration currents over some pairwise disjoint k-polygons $Q_i \subset \mathbb{R}^m$, has mass $\mathbf{M}(P) = \sum_i |a_i|\mathscr{H}^k(Q_i)$ and size $\mathbf{S}(P) = \sum_i \mathscr{H}^k(Q_i)$. Notice that this definition has been given in term of the supports of the slices of T, rather than the whole support of T. This choice is motivated by the special behavior of 0-dimensional flat chains illustrated in Section 2.6.1.

2.6.1. Zero dimensional flat currents. For zero dimensional flat currents some special properties hold: it is a well-known result in the theory of distributions that any distribution supported in a finite set is a finite sum of derivatives of Dirac masses. Here we present an analogous result for flat currents of finite size, which is also similar to the representation theorem for zero dimensional flat G-chains of finite mass obtained in [**Whi99b**] and to the result on integer-valued currents in [**Sme02**].

THEOREM 2.6.3. *Every* $T \in \mathbf{F}_0(E)$ *with finite size can be represented as*

$$T = \sum_{i=1}^{\mathbf{S}(T)} a_i [\![x_i]\!] \tag{45}$$

where $\mathrm{spt}(T) = \{x_i : i = 1, \ldots, \mathbf{S}(T)\}$ *and* $a_i \in \mathbb{R}$. *In particular* T *has finite mass.*

PROOF. We will prove the theorem through several steps.
Step 1. First of all we claim that it is sufficient to prove the representation formula $T(f) = \sum_i a_i f(x_i)$ for functions $f \in \mathrm{Lip}_b(E)$ such that

(1) f has compact support,
(2) f is locally constant in a neighborhood of $\mathrm{spt}(T)$.

In fact, since bounded closed sets of E are compact and $\mathrm{spt}(T)$ is finite, we can easily approximate any $f \in \mathrm{Lip}_b(E)$ by functions f_n with uniformly bounded Lipschitz functions satisfying (a), (b) and pointwise convergent to f. We can then use the continuity axiom of currents to pass to the limit.
Step 2. Let us fix $f \in \mathrm{Lip}_b(E)$ satisfying conditions (1) and (2) above. Set $\gamma(x) = \min\{d(x, x_i),\, x_i \in \mathrm{spt}(T)\}$: for almost every $r < r_0 := \frac{1}{2}\min_{i \neq j}\{d(x_i, x_j)\}$ the current $T \llcorner \{\gamma < r\}$ is well-defined, and by Lemma 2.4.6 it equals the sum of the $T \llcorner B_r(x_i)$'s:

$$T = T \llcorner \{\gamma \geq r\} + \sum_{i=1}^{N} T \llcorner B_r(x_i).$$

The first term is null on f: in fact, by equation (31)

$$\mathrm{spt}(T \llcorner \{\gamma \geq r\}) \subset \mathrm{spt}(T) \cap \{\gamma \geq r\} = \emptyset,$$

hence by (30) $T \llcorner \{\gamma \geq r\}(f) = 0$. As a result $T(f) = \sum_i T \llcorner B_r(x_i)(f)$, independently of r.
Step 3. We reduced our problem to the characterization of $T \llcorner B_r(x_i)(f)$, whose support is $\{x_i\}$ by (31). For each j we let g_j be a Lipschitz function equal to 1 on $B_{r_0/2}(x_i)$ and equal to 0 on $E \setminus B_{r_0}(x_i)$: if $0 < s < r$ are radii such that the restrictions of T to $B_r(x_i)$ and $B_s(x_i)$ exist, the difference $T \llcorner B_r(x_i) - T \llcorner B_s(x_i)$ satisfies

$$\mathrm{spt}(T \llcorner B_r(x_i) - T \llcorner B_s(x_i)) \overset{(31)}{\subset} \overline{B_r(x_i) \backslash B_s(x_i)} \cap \{x_i\} = \emptyset.$$

Therefore again by (30) $T \llcorner B_r(x_i)(h)$ is (essentially) constant in r for any bounded Lipschitz function h. In particular (30) implies that:

- $T \llcorner B_r(x_i)(g_j)$ does not depend on $r < r_0$, and actually (30) gives $T \llcorner B_r(x_i)(g_j) = 0$ for $i \neq j$;
- since $f = \sum_j f(x_j) g_j$ in a neighborhood of $\mathrm{spt}(T)$,

$$T \llcorner B_r(x_i)(f) = f(x_i)\, T \llcorner B_r(x_i)(g_i).$$

Letting $a_i = T \llcorner B_r(x_i)(g_i)$ we obtain the thesis. $\square$

As a consequence of Proposition 2.6.3 we obtain a closure theorem for sequences of flat currents with equibounded sizes:

THEOREM 2.6.4 (Lower semicontinuity of size,[**AG13b**, Theorem 3.4]). *Let $(T_h) \subset \mathbf{F}_k(E)$ be a sequence of currents with equibounded sizes and converging to T in the flat norm:*

$$\mathbf{S}(T_h) \le C < \infty, \qquad \lim_h \mathbf{F}(T_h - T) = 0.$$

Then T has finite size and

$$\mathbf{S}(T) \le \liminf_h \mathbf{S}(T_h). \tag{46}$$

PROOF. Possibly extracting a subsequence we can assume that

$$\sum_h \mathbf{F}(T_h - T) < \infty \quad \text{and} \quad \lim_h \mathbf{S}(T_h) = \liminf_h \mathbf{S}(T_h). \tag{47}$$

If $k = 0$ we prove a slightly more general implication: for any open set $A \subset E$, $\mathbf{F}(T_h - T) \to 0$ and $\liminf_h \mathscr{H}^0(A \cap \mathrm{spt} T_h) < \infty$ implies

$$\mathscr{H}^0(A \cap \mathrm{spt}(T)) \le \liminf_h \mathscr{H}^0(A \cap \mathrm{spt}(T_h)). \tag{48}$$

Indeed, consider $x \in \mathrm{spt}(T) \cap A$. Then by Definition 28 and inequality (22) for every $\varepsilon > 0$ there exists $r < \varepsilon$ such that

(i) $B(x,r) \subset A$,
(ii) $\lim_h \mathbf{F}(T_h \llcorner B(x,r) - T \llcorner B(x,r)) = 0$,
(iii) $T \llcorner B(x,r) \neq 0$.

Point (ii) implies that for $h \ge h(x,\varepsilon)$ $T_h \llcorner B(x,r) \neq 0$, and since by Proposition 2.6.3 T_h is a finite sum of Dirac deltas

$$T_h = \sum_{j=1}^{\mathbf{S}(T_h)} a_{j,h} [\![y_{j,h}]\!]$$

at least one of the points $y_{j,h}$ must belong to $B(x,r)$. If $A \cap \mathrm{spt}(T)$ contains m distinct points $\{x_1, \dots, x_m\}$, we can take ε sufficiently small such that the family of balls $\{B(x_i, \varepsilon) : i = 1, \dots, m\}$ is disjoint. Therefore there exist radii r_i as above such that for every $h \ge \max_i h(x_i, \varepsilon)$ each ball $B(x_i, r_i)$ contains at least one point $y_{j,h'}$. Hence $m \le \mu_{T_h}(A)$ and (48) follows.

In the case $k \ge 1$ we fix a projection $\pi \in \mathrm{Lip}_1(E, \mathbb{R}^k)$. Thanks to (47) we know that for $\mathscr{L}^k$-almost every $x \in \mathbb{R}^k$ the slices $T_{h,x}$ converge to T_x; moreover Fatou's lemma implies that

$$\int_{\mathbb{R}^k} \liminf_h \mathbf{S}(T_{h,x})\, dx \le \liminf_h \int_{\mathbb{R}^k} \mathbf{S}(T_{h,x})\, dx = \liminf_h \mu_{T_h,\pi}(E) \le \lim_h \mathbf{S}(T_h) \le C < \infty.$$

The same argument can be applied for an open set $A \subset E$ and using (48) we get

$$\begin{aligned} \mu_{T,\pi}(A) = \int_{\mathbb{R}^k} \mathscr{H}^0(A \cap \mathrm{spt} T_x)\, dx &\le \int_{\mathbb{R}^k} \liminf_h \mathscr{H}^0(A \cap \mathrm{spt} T_{h,x})\, dx \\ &\le \liminf_h \int_{\mathbb{R}^k} \mathscr{H}^0(A \cap \mathrm{spt} T_{h,x})\, dx = \liminf_h \mu_{T_h,\pi}(A). \end{aligned} \tag{49}$$

The map $A \mapsto \nu(A) = \liminf_h \mu_{T_h}(A)$ is a finitely superadditive set-function, with $\nu(E)$ bounded from above by $\liminf_h \mathbf{S}(T_h)$; the chain of inequalities (49) simply expresses that $\mu_{T,\pi}(A) \le \nu(A)$ on open sets A.

If $B_1, \dots, B_N$ are pairwise disjoint Borel sets and $K_i \subset B_i$ are compact, we can find pairwise disjoint open sets A_i containing K_i and apply the superadditivity to get

$$\sum_{i=1}^{N} \mu_{T,\pi_i}(K_i) \le \sum_{i=1}^{N} \nu(A_i) \le \nu(E).$$

Since K_i are arbitrary, the same inequality holds with B_i in place of K_i. Since B_i, π_i and N are arbitrary, it follows that μ_T is a finite Borel measure and $\mu_T(E) \le \nu(E)$. □

2.7. A hybrid distance on zero dimensional flat boundaries

We let $\mathbf{B}_0(E) = \mathbf{M}_0(E) \cap \partial \mathbf{F}_1(E)$ be the space of finite mass boundaries of flat chains. We endow $\mathbf{B}_0(E)$ with the following distance of interpolation type:

DEFINITION 2.7.1 (Hybrid distance). *For every Q, $Q' \in \mathbf{B}_0(E)$ we set*

$$\mathcal{G}(Q, Q') = \inf\{\mathbf{S}(R) + \mathbf{M}(S) \,:\, Q - Q' = \partial(R + S),\ R,\ S \in \mathbf{F}_1(E)\}.$$

It is plain that the triangle inequality holds, by the subadditivity of mass and size. It is also immediate to check that $\mathcal{G}$ is finite: indeed, since $Q = \partial T$ with $T \in \mathbf{F}_1(E)$, we may write $T = X + \partial Y$ with X, Y flat and $\mathbf{M}(X) + \mathbf{M}(Y) < \infty$. Therefore $Q = \partial X$ and so $\mathcal{G}(0, Q) \le \mathbf{M}(X) < \infty$. Occasionally we shall abbreviate $\mathcal{G}(Q) = \mathcal{G}(0, Q)$.
The proof of nondegeneracy of $\mathcal{G}$, is based on a "elimination argument".

PROPOSITION 2.7.2. *Let $Q \in \mathbf{B}_0(E)$ satisfy $\mathcal{G}(Q) = 0$. Then $Q = 0$.*

PROOF. Suppose Q is not null and take an open set A such that $|Q(A)| > 0$. Since Q is a finite measure, by monotone approximation from the interior we can guarantee that the open set satisfies $\|Q\|(\partial A) = 0$. Therefore we can choose a small $\delta > 0$ such that

$$|Q(A)| > 4\|Q\|(A^\delta \setminus A), \tag{50}$$

where A^δ is the δ-neighborhood of A. Let $\varepsilon > 0$: by hypothesis we can find flat currents R, S with $Q = \partial(R + S)$ satisfying

$$\mathbf{S}(R) < \frac{\delta}{6} \quad \text{and} \quad \mathbf{M}(S) < \frac{\varepsilon\delta}{6}.$$

If ρ_1 and ρ_2 are two positive numbers in $(0, \delta)$ with $\rho_1 < \rho_2$, we let $\pi(x) = dist(x, A)$ and $\Sigma_{\rho_1,\rho_2} = \{\rho_1 \le dist(\cdot, A) < \rho_2\}$. Using π we can formally set the currents R and S to be zero within the ring Σ_{ρ_1,ρ_2} through the following relation:

$$\begin{aligned} Q \llcorner (E \setminus \Sigma_{\rho_1,\rho_2}) &= (\partial R) \llcorner (E \setminus \Sigma_{\rho_1,\rho_2}) + (\partial S) \llcorner (E \setminus \Sigma_{\rho_1,\rho_2}) \\ &= \partial(R \llcorner (E \setminus \Sigma_{\rho_1,\rho_2})) + \partial(S \llcorner (E \setminus \Sigma_{\rho_1,\rho_2})) + S_{\rho_2} - S_{\rho_1} + R_{\rho_2} - R_{\rho_1}. \end{aligned} \tag{51}$$

Note that (51) actually holds if ρ_1 and ρ_2 belong to a subset of $(0, \delta)$ of full measure, since slices and restrictions of the currents R and S exist only almost everywhere. Inequality

(2.6.1) gives

$$\int_0^{\frac{\delta}{3}} \mathscr{H}^0(\mathrm{spt} R_\rho)\, d\rho \leq \mu_{R,\pi}(E) \leq \mathbf{S}(R) < \frac{\delta}{6}$$

and since $\mathscr{H}^0(\mathrm{spt} R_\rho)$ is an integer, there must be a set of radii ρ in $(0, \frac{\delta}{3})$ of length strictly greater than $\frac{\delta}{6}$ such that $R_\rho = 0$. Moreover

$$\int_0^{\frac{\delta}{3}} \mathbf{M}(S_\rho)\, d\rho \leq \mathbf{M}(S) < \frac{\varepsilon\delta}{6}$$

and therefore $\mathbf{M}(S_\rho) < \varepsilon$ in a subset of $(0, \frac{\delta}{3})$ of measure strictly greater than $\frac{\delta}{6}$. For the same reason we can find another set of positive measure contained in $(\frac{2\delta}{3}, \delta)$ where the same requirements hold. Putting together these two results we can pick two radii $\rho_1 \in (0, \frac{\delta}{3})$ and $\rho_2 \in (\frac{2\delta}{3}, \delta)$ such that equation (51) holds, $R_{\rho_1} = R_{\rho_2} = 0$ and $\mathbf{M}(S_{\rho_1}) + \mathbf{M}(S_{\rho_2}) < 2\varepsilon$. Take now a Lipschitz function ψ such that:

- $0 \leq \psi \leq 1$,
- $\psi = 1$ in $A^{\frac{\delta}{3}}$ and $\psi = 0$ outside $A^{2\delta/3}$,
- $\mathrm{Lip}(\psi) \leq \frac{3}{\delta}$

and test it on the current $Q' = Q + S_{\rho_1} - S_{\rho_2}$. Since (50) gives $Q' = Q \llcorner \Sigma_{\rho_1,\rho_2} + \partial Y$, with Y supported in the complement of Σ_{ρ_1,ρ_2}, and since ψ is constant inside the ring Σ_{ρ_1,ρ_2}, by (50) we have $|Q'(\psi)| < |Q(A)|/4$. Hence

$$|Q(\psi)| \leq \frac{1}{4}|Q(A)| + \mathbf{M}(S_{\rho_1}) + \mathbf{M}(S_{\rho_2}) < \frac{1}{4}|Q(A)| + 2\varepsilon.$$

On the other hand equation (50) yields

$$|Q(A)| - |Q(\psi)| \leq |Q(A) - Q(\psi)| \leq \|Q\|(A^\delta \setminus A) < \frac{|Q(A)|}{4}$$

and so $|Q(A)| < \frac{4}{3}|Q(\psi)| \leq \frac{1}{3}|Q(A)| + \frac{8}{3}\varepsilon$. Since $|Q(A)| > 0$, by choosing ε sufficiently small we have a contradiction. So $Q^+(A) = Q^-(A)$ on every open set A. Since the family of open sets is stable by intersection and generates the σ-algebra of Borel sets, we get $Q^+ \equiv Q^-$, hence $Q = 0$. $\square$

In order to apply the theory of functions of metric bounded variation developed in [**Amb90b, AK00a**], we need to ensure that the space $(\mathbf{B}_0(E), \mathcal{G})$ is separable. Let us first relate the space of 0-currents to the theory of Optimal Transportation. Recall that a finite nonnegative Borel measure μ has finite first moments if $d(\cdot, x_0)$ belongs to $L^1(\mu)$ for some, and thus all, $x_0 \in X$. Given two such measures μ and ν with finite first moments and equal total mass ($\mu(E) = \nu(E)$) we let

$$W_1(\mu,\nu) = \inf\left\{\int_{E\times E} d(x,y)d\sigma(x,y) : \sigma \in \mathcal{M}_+(E\times E),\ \pi_{1\#}\sigma = \mu, \pi_{2\#}\sigma = \nu\right\} \quad (52)$$

be their 1-Wasserstein distance, where π_1 is the projection on the first coordinate and π_2 is the projection on the second one. For the many properties and applications of this distance we refer to the monograph [**Vil09**]. Among them, we recall that the infimum (52) is attained by at least one nonnegative Borel measure σ, which we call optimal plan. Since E is a geodesic space the Wasserstein distance can be lifted to the space of geodesics $Geo(E)$ of constant speed geodesics parametrized on $[0,1]$:

$$W_1(\mu,\nu) = \inf\left\{\int_{Geo(E)} d(\gamma(0),\gamma(1))\,d\lambda(\gamma),\ \lambda\in\mathcal{M}_+(Geo(E)),\ (e_0,e_1)_\#\lambda = (\mu,\nu)\right\}. \tag{53}$$

Here $e_t(\gamma)=\gamma(t)$ denoted the evaluation map at time t. This allows us to make the following observation:

LEMMA 2.7.3. *Let $Q\in\mathbf{M}_0(E)$ be such that $Q(1)=0$ and the total variation measure $\|Q\|$ has finite first moment. Then Q is representable as ∂Y for some $Y\in\mathbf{F}_1(E)$ with $\mathbf{M}(Y)\le W_1(Q^+,Q^-)$. In particular $\mathcal{G}(Q)\le W_1(Q^+,Q^-)$.*

PROOF. The two measures Q^+ and Q^- given by Hahn decomposition theorem have finite first moments and have the same mass. We let $\lambda\in\mathcal{M}_+(Geo(E))$ be an optimal measure in problem (53) and we build

$$Y = \int_{Geo(E)} \gamma_\#[\![0,1]\!]\,d\lambda(\gamma).$$

Since $\partial\gamma_\#[\![0,1]\!] = \delta_{\gamma(1)}-\delta_{\gamma(0)}$, it is easily proved that Y is actually a normal current with $Q=\partial Y$ and that

$$\mathbf{M}(Y)\le\int_{Geo(E)}\mathbf{M}(\gamma_\#[\![0,1]\!])\,d\lambda(\gamma) = \int_{Geo(E)} d(x,y)\,d\lambda(x,y) = W_1(Q^+,Q^-).$$

□

PROPOSITION 2.7.4. *The metric space $(\mathbf{B}_0(E),\mathcal{G})$ is separable.*

PROOF. We first show that the class of currents with bounded support is dense. In fact, let us fix a basepoint $x_0\in E$ and $Q=\partial(R+S)$ with $\mathbf{S}(R)<\infty$ and $\mathbf{M}(S)<\infty$: as in Proposition 2.7.2, there are arbitrarily big radii r such that $R_r=0$, $\mathbf{M}(S_r)$ is finite and $\mathrm{spt}(S_r)\subset\partial B_r(x_0)$. As in (51), for a.e. $r>0$ we obtain

$$Q\llcorner(E\setminus B_r(x_0)) = \partial(R\llcorner(E\setminus B_r(x_0))) + \partial(S\llcorner(E\setminus B_r(x_0))) + S_r,$$

so that $Q\llcorner B_r(x_0)+S_r$ belongs to $\mathbf{B}_0(E)$. Clearly $Q\llcorner B_r(x_0)+S_r$ is supported in $\overline{B}_r(x_0)$, and its $\mathcal{G}$-distance from Q can be estimated by

$$\mathcal{G}(Q,Q\llcorner B_r(x_0)+S_r) = \mathcal{G}(Q\llcorner(E\setminus B_r(x_0))-S_r)\le\|S\|(E\setminus B_r(x_0))+\mathbf{S}(E\setminus B_r(x_0))$$

which is arbitrarily small provided we take r sufficiently large.

Now, if $Q\in\mathbf{B}_0(E)$ has bounded support we may represent $Q=\partial Y$ for some normal current Y, so that

$$\mathcal{G}(Q,aQ)\le|1-a|\mathbf{M}(Y).$$

This inequality can be used to show that the class of Q's with bounded support such that $c(Q)=Q^+(E)=Q^-(E)$ is a rational number is dense.

Now, recall that the space of Borel probability measures in E endowed with the W_1 distance is separable (see for instance [**AGS08**, Proposition 7.1.5]) and let us denote by $\mathcal{D}$ a countable dense subset. If $Q\in\mathbf{B}_0(E)$ and $c=Q^+(E)=Q^-(E)\in\mathbb{Q}$, we may consider families ν_n, μ_n contained in $\mathcal{D}$ converging respectively to Q^+/c and Q^-/c in Wasserstein distance and use the inequality

$$\mathcal{G}(Q, c\mu_n - c\nu_n) \leq \mathcal{G}(Q^+, c\mu_n) + \mathcal{G}(Q^-, c\nu_n) \leq W_1(Q^+, c\mu_n) + W_1(Q^-, c\nu_n)$$

to get $\mathcal{G}(Q, c\mu_n - c\nu_n) \to 0$. This proves the separability of $(\mathbf{B}_0(E), \mathcal{G})$. □

2.8. Functions of metric bounded variation

It is a well-known fact that, in absence of a linear structure, as in a generic metric space (M, d_M), Lipschitz functions play the role of coordinates. Bearing in mind this idea we begin with a definition:

DEFINITION 2.8.1. *A metric space (M, d_M) is called weakly separable if there exists a countable family $(\varphi_h)_{h\in\mathbf{N}} \subset \mathrm{Lip}_1(M) \cap \mathrm{Lip}_b(M)$ such that*

$$d_M(x, y) = \sup_h |\varphi_h(x) - \varphi_h(y)| \qquad \forall x,\, y \in M. \tag{WS}$$

Notice that separable metric spaces are particular cases of the class defined above, as it is sufficient to take as φ_h truncations of the functions $d_M(\cdot, x_h)$ where x_h run in a dense subset of M; similarly w^*-separable dual Banach spaces, namely spaces $Y = G^*$ with G Banach and separable, fulfil the same property by simply taking a countable dense set $\{g_n\}$ of the unit ball of G and letting $\varphi_n(x) := \langle x, g_n\rangle$.

Weakly separable metric spaces can be isometrically embedded into ℓ^∞ via the Kuratowski map $j : M \to \ell^\infty$ defined by

$$j(x) = (\varphi_0(x) - \varphi_0(x_0), \varphi_1(x) - \varphi_1(x_0), \dots).$$

Note also that since every subset of ℓ^∞ is weakly separable ($\phi_h(x) = x_h$), condition (WS) is a characterization of metric spaces which can be isometrically embedded in ℓ^∞. In general any metric space M can be isometrically embedded into a Banach space, for instance $C^0_b(M)$ endowed with the sup norm, via the map

$$\iota(x) = d_M(\cdot, x) - d_M(\cdot, x_0).$$

Proposition 2.7.4 ensures that $(\mathbf{B}_0(E), \mathcal{G})$ is a weakly separable metric space. Observe also that given a Borel function $u : \mathbb{R}^k \to M$ we have that $\mathscr{L}^k$-a.e. $x \in \mathbb{R}^k$ is an approximate continuity point, namely

$$\{y :\ d_M(u(y), z) > \varepsilon\}$$

has 0-density at x for all $\varepsilon > 0$, for some $z \in M$:

$$\lim_{r\downarrow 0} \frac{1}{r^k} \mathscr{L}^k(\{y :\ d_M(u(y), z) > \varepsilon\}) = 0.$$

The point z is unique and we will denote it by $\tilde{u}(x)$. We shall denote, as in [**AFP00, Amb90b**], by S_u the set of approximate discontinuity points: it is a Lebesgue negligible Borel set and $\tilde{u} = u$ $\mathscr{L}^n$-a.e. in $\mathbb{R}^k$.

The oscillations of a function $u : \mathbb{R}^k \to M$ are detected through the composition with each φ_h. In analogy with the case where $M = \mathbb{R}^N$ is a Euclidean space, a natural definition of metric space valued BV_{loc} function would require that

$$\text{the distribution } D(\varphi_h \circ u) \text{ is a locally finite measure for every } h. \tag{D}$$

Although this conditions easily characterizes the space $BV_{\rm loc}(\mathbb{R}^k,\mathbb{R}^N)$ if we take among the functions φ_h (truncates of) coordinate projections, in the general context of metric spaces a uniformity among the measures $\{|D(\varphi_h\circ u)|\}_h$ is not a byproduct of condition (D). Therefore, as in [**Amb90b**, Definition 2.1], we define:

DEFINITION 2.8.2 (Metric bounded variation). *Let (M,d_M) be a weakly separable metric space and let $u:\mathbb{R}^k\to M$ be a Borel function. We say that u has metric bounded variation if there exists a finite measure σ in $\mathbb{R}^k$ such that*

$$|D(\varphi_h\circ u)|\le\sigma \quad \text{for every } h. \tag{54}$$

where the set (φ_h) satisfies (WS). *We denote by with $MBV(\mathbb{R}^k,M)$ the space of such functions and by $|Du|$ the least possible measure σ in* (54).

For our purposes, it is also necessary to work with the classical definition of function of bounded variation defined on intervals of the real line (see [**Fed69**, 4.5.10]): if $u:(a,b)\to M$ is a Borel function we let

$$\text{ess-Var}_a^b u=\sup\left\{\sum_{k=1}^N d_M(\tilde u(x_{k-1}),\tilde u(x_k)),\ a<x_0<\ldots<x_N<b,\ x_k\in D\right\} \tag{55}$$

where D is any countable dense set in $(a,b)\setminus S_u$. As it is proved in [**Fed69**, 4.5.10] and in [**Amb90b**, Remark 2.2], $u\in BV((a,b))$ if and only if $\text{ess-Var}_a^b u$ is finite and $\text{ess-Var}_a^b u=|Du|((a,b))$. Hypothesis (WS) comes into play when dealing with many measure theoretic properties of the space MBV. For instance:

LEMMA 2.8.3. *If $u\in MBV((a,b),M)$ then the approximate upper limit of incremental quotient $\delta_x u$ is finite $\mathscr{L}^1$-almost everywhere in (a,b).*

PROOF. We can assume with no loss of generality that $b-a<\infty$. Then, the composition $u_h:=\varphi_h\circ u$ belongs to $BV((a,b))$: hence there exists a $\mathscr{L}^1$-negligible set $N_h\subset(a,b)$ such that

$$u_h(x)-u_h(y)=Du_h((x,y))\quad\forall\, x,\,y\notin N_h.$$

Moreover by Vitali's covering theorem the set

$$N'=\Big\{x\in(a,b):\ \limsup_{r\downarrow 0}\frac{|Du|(B_r(x))}{2r}=\infty\Big\}$$

where the upper density $\Theta_1^*(|Du|,\cdot)$ is infinite is Lebesgue negligible; since

$$|u(x)-u(y)|\le|Du|((x,y))\qquad\forall x,\,y\in(a,b)\setminus\bigcup_h N_h,$$

therefore $\delta_x u\le\Theta_1^*(|Du|,x)<\infty$ for $x\notin N'\cup\bigcup_h N_h$. □

In particular, by Theorem 2.3.3(ii), for all $u\in MBV(\mathbb{R}^k,(\mathbf{B}_0(E),\mathcal{G}))$ there exist Borel sets B_n and constants L_n such that

$$\mathcal{G}(u(x_1),u(x_2))\le L_n|x_1-x_2|\quad\forall\, x_1,\,x_2\in B_n\qquad\text{and}\qquad\mathscr{L}^k\big(\mathbb{R}^k\setminus\bigcup_n B_n\big)=0. \tag{56}$$

The theory of MBV functions allows to deduce interesting rectifiability properties of currents by looking at their slices. For instance when $M = \mathbf{M}_0(E)$ is endowed with the flat-type norm

$$\tilde{\mathbf{F}}(T) := \sup\big\{T(\phi), \mathrm{spt}(\phi) \Subset E, \max\{\sup|\phi|, \mathrm{Lip}(\phi)\} \leq 1\big\}, \tag{57}$$

it becomes a weakly separable metric space. In fact it is easy to show that $\tilde{\mathbf{F}}$ is a norm; moreover the valuation map

$$\mathbf{M}_0(E) \ni T \mapsto T(\phi) \in \mathbb{R}$$

is 1-Lipschitz when $\sup|\phi| \leq 1$ and $\mathrm{Lip}(\phi) \leq 1$ by definition of the metric $\tilde{\mathbf{F}}$. Since the family of balls $\{\overline{B(x_0,n)}\}$ is countable and made of compact sets, it is sufficient to recall the Stone-Weierstrass Theorem to deduce the separability of each space

$$\{\phi \in \mathrm{Lip}(E), \mathrm{spt}(\phi) \subset \overline{B(x_0,n)}, \max\{\sup|\phi|, \mathrm{Lip}(\phi)\} \leq 1\}.$$

We can prove the following criterion:

THEOREM 2.8.4. *Let E we a weakly separable metric space and let $M := \mathbf{M}_0(E)$ endowed with the norm* (57). *Then for every $T \in MBV(\mathbb{R}^k, \mathbf{M}_0(E))$ there exists a $\mathscr{L}^k$-negligible set $N \subset \mathbb{R}^k$ such that*

$$\bigcup_{z \in \mathbb{R}^k \setminus N} \{x \in \mathbb{R}^k : \|T(z)\|(\{x\}) > 0\}$$

is countably $\mathscr{H}^k$-rectifiable.

PROOF. Recall the notion of maximal function of a nonnegative finite measure μ:

$$M\mu(x) := \sup_{\rho>0} \frac{\mu(B_\rho(x))}{\omega_k \rho^k}.$$

By Besicovitch convering theorem, $\mathscr{L}^k(\{M|Du| > \lambda\})$ can be estimated from above by a constant times $|Du|(\mathbb{R}^k)/\lambda$, hence the function $M\mu(\cdot)$ is finite $\mathscr{L}^k$-almost everywhere, see for instance [**Ste93**]. Given (φ_h) as in definition 2.8.1 we observe that $|D(\varphi_h \circ T)| \leq |DT|$, which implies that

$$M|D(\varphi_h \circ T)| \leq M|DT| \quad \mathscr{L}^k\text{-a.e.}$$

Thanks to the classical Morrey estimate [**AFP00**, 5.34] there exists a constant C_k and a set $L_h \subset \mathbb{R}^k$ of full measure (the Lebesgue points of $\phi_h \circ T$) such that for every $z, z' \in L_h$

$$\tilde{\mathbf{F}}\left(\varphi_h(T(z)), \varphi_h(T(z'))\right) \leq C_k \left[M|DT|(z) + M|DT|(z')\right] |z - z'|. \tag{58}$$

Therefore whenever z and z' belong to the set $\bigcap_h L_{\varphi_h \circ T} \cap \{M|DT| < \infty\}$, which has full measure by assumption 2.8.1, we can take the supremum on h of (59) and obtain

$$\tilde{\mathbf{F}}(T(z), T(z')) \leq C_k[M|DT|(z) + M|DT|(z')]|z - z'|. \tag{59}$$

Set $N := \mathbb{R}^k \setminus \bigcap_h L_{\varphi_h \circ T} \cap \{M|DT| < \infty\}$: T_z will be a short notation for $T(z)$. Let $Z_{\varepsilon,\delta}$ be the set of point $z \in \mathbb{R}^k \setminus N$ such that $MDT(z) < \frac{1}{2\varepsilon}$ and

$$\|T_z\|(\{x\}) \geq \varepsilon \qquad \Rightarrow \qquad \|T_z\|(B_{3\delta}(x)) \leq \frac{\varepsilon}{3}.$$

Setting $R_{\varepsilon,\delta} := \{x \in \mathbb{R}^k : \|T_z\|(\{x\}) \geq \varepsilon$ for some $z \in \mathbb{R}^k \setminus Z_{\varepsilon,\delta}\}$ we note that it suffices to prove the rectifiability of $R_{\varepsilon,\delta}$.

We claim that for every set $B \subset E$ with diameter less than δ it holds

$$d(x,x') \leq \frac{3c(k)(\delta+1)}{\varepsilon^2}|z-z'|, \tag{60}$$

whenever $x, x' \in B$, $\|T_z\|(\{x\}) \geq \varepsilon$ and $\|T_z'\|(\{x'\}) \geq \varepsilon$ for some $z, z' \in Z_{\varepsilon,\delta}$. In fact setting $d := d(x,x') \leq \delta$ we can define a Lipschitz function $\phi(y)$ equal to $d(y,x)$ in $B_d(x)$, equal ro 0 in $E \setminus B_{2\delta}(x)$ with $\sup|\phi| \leq d$ and $\mathrm{Lip}(\phi) \leq 1$. Since

$$|T_z(\phi)| \leq \frac{\varepsilon d}{3}, \qquad |T_{z'}(\phi)| \geq \varepsilon d - \frac{\varepsilon d}{3}$$

we get

$$\frac{\varepsilon}{3} d(x,x') \leq |T_z(\phi) - T_{z'}(\phi)| \leq \frac{c(k)(\delta+1)}{\varepsilon}|z-z'|.$$

By (60) it follows that for any $z \in Z_{\varepsilon,\delta}$ there exists at most one $x = f(z) \in B$ such that $\|T_z\|(\{x\}) \geq \varepsilon$; moreover denoting D the domain of f the map $f : D \mapsto B$ is Lipschitz and onto, hence B is contained in the countably $\mathscr{H}^k$-rectifiable set $f(\overline{D})$. A covering argument proves that $R_{\varepsilon,\delta}$ is contained in a countably $\mathscr{H}^k$-rectifiable set. □

The following Theorem is based on the observation of Jerrard and Soner in the Euclidean context, [**JS02**], that if $T \in \mathbf{N}_k(E)$ and $\pi \in \mathrm{Lip}(E,\mathbb{R}^k)$ then the slicing map $x \mapsto \langle T, \pi, x\rangle$ is MBV:

THEOREM 2.8.5 (Rectifiability and rectifiability of slices, [**AK00a**, 8.1]). *Let $T \in \mathbf{N}_k(E)$. Then $T \in \mathcal{R}_k(E)$ if and only if for any $\pi \in \mathrm{Lip}(E,\mathbb{R}^k)$ it holds*

$$\langle T, \pi, x\rangle \in \mathcal{R}_0(E) \qquad \text{for } \mathscr{L}^k\text{-a.e. } x \in \mathbb{R}^k.$$

PROOF. Let $\pi \in \mathrm{Lip}(E,\mathbb{R}^k)$ with $\mathrm{Lip}(\pi^i) \leq 1$ and let $T \in \mathbf{N}_k(E)$. We claim that the map

$$\mathbb{R}^k \ni x \mapsto \langle T, \pi, x\rangle \in \mathbf{M}_0(E)$$

has metric bounded variation, when we endow $\mathbf{M}_0(E)$ with the flat norm (57). In fact let $\psi \in C^1_0(\mathbb{R}^k)$ and $\phi \in \mathrm{Lip}_b(E)$ with $\sup|\phi| \leq 1$ and $\mathrm{Lip}(\phi) \leq 1$. Using Proposition 2.2.4 we have

$$\begin{aligned}
(-1)^{i-1}\int_{\mathbb{R}^k} \langle T, \pi, x\rangle(\phi)\frac{\partial\psi}{\partial x^i}(x)\,dx &= (-1)^{i-1} T \llcorner d\pi\big(\phi\frac{\partial\psi}{\partial x^i}\circ\pi\big)\\
&= T(\phi\, d(\psi\circ\pi)\wedge d\hat{\pi}^i) =\\
&= \partial T(\phi(\psi\circ\pi)d\hat{\pi}^i) - T(\psi\circ\pi\, d\phi\wedge d\hat{\pi}^i)\\
&\leq \|\partial T\|(\psi\circ\pi) + \|T\|(\psi\circ\pi),
\end{aligned}$$

where $d\hat{\pi}^i = d\pi^1 \wedge \cdots \wedge d\pi^{i-1} \wedge d\pi^{i+1} \wedge \cdots \wedge d\pi^k$. Since ψ is arbitrary $x \mapsto \langle T, \pi, x\rangle$ belongs to $BV(\mathbb{R}^k)$ and

$$|D\langle T, \pi, x\rangle| \leq k\pi_\#\|T\| + k\pi_\#\|\partial T\|.$$

Since ϕ is arbitrary $x \mapsto \langle T, \pi, x\rangle$ belongs to $MBV(\mathbb{R}^k, \mathbf{M}_0(E))$.

Now if T is rectifiable, Theorem 2.3.7 yields generic rectifiability of the slices. On the other hand if $\|T\|$ is concentrated on a σ-compact set L, then Theorem 2.8.4 gives an $\mathscr{L}^k$-negligible set $N \subset \mathbb{R}^k$ such that

$$\bigcup_{x \in \mathbb{R}^k \setminus N} \{y \in L : \|\langle T, \pi, x\rangle\|(\{y\}) > 0\}$$

is contained in a $\mathscr{H}^k$-countably rectifiable set R_π. If $\langle T, \pi, x\rangle \in \mathcal{R}_0(E)$ for almost every x then

$$\|T \llcorner d\pi\|(E \setminus R_\pi) \leq \int_{\mathbb{R}^k} \|\langle T, \pi, x\rangle\|(L \setminus R_\pi)\, dx = 0.$$

Hence $T \llcorner d\pi$ os concentrated on a countably $\mathscr{H}^k$-rectifiable subset of E for any $\pi \in \mathrm{Lip}(E, \mathbb{R}^k)$. By Lemma 2.1.17 the same holds for T. □

2.9. Rectifiability of flat currents with finite size

This section contains the main rectifiability result for currents of finite size.

Theorem 2.9.1 (Rectifiability of currents of finite size). *For every flat current $T \in \mathbf{F}_k(E)$ with finite size the measure μ_T is concentrated on a countably $\mathscr{H}^k$-rectifiable set. The least one, up to $\mathscr{H}^k$-null sets, is given by*

$$set(T) := \left\{ x \in E : \limsup_{r \downarrow 0} \frac{\mu_T(B_r(x))}{r^k} > 0 \right\}. \tag{61}$$

Notice that, even for flat chains for finite mass, the theorem provides no information on the rectifiability of the measure $\|T\|$, which fails to be true in general. So, our goal is to find a countably $\mathscr{H}^k$-rectifiable set Σ such that $\mu_T(E \setminus \Sigma) = 0$. We start by proving the existence of a countably $\mathscr{H}^k$-rectifiable set $\Sigma = \Sigma_\pi$ satisfying

$$\mu_{T,\pi}(E \setminus \Sigma_\pi) = 0 \tag{62}$$

for a fixed $\pi \in \mathrm{Lip}_1(E, \mathbb{R}^k)$: since $\mathbf{S}(T)$ is finite, for $\mathscr{L}^k$-almost every $x \in \mathbb{R}^k$ the slice $T_x = \langle T, \pi, x\rangle$ has finite size, hence by Theorem 2.6.3 it is a finite sum of Dirac's masses. Therefore $\mathrm{spt}(T_x) = \{y \in E : \|T_x\|(\{y\}) > 0\}$, moreover $T_x = X_x + (\partial Y)_x$ which entails $\|T_x\| \leq \|X_x\| + \|(\partial Y)_x\|$ again almost everywhere. This implies that

$$\mathrm{spt}(T_x) \subset \{y \in E : \|X_x\|(\{y\}) > 0\} \cup \{y \in E : \|(\partial Y)_x\|(\{y\}) > 0\}. \tag{63}$$

So, in order to investigate the rectifiability of the measure $\mu_{T,\pi} = \int_{\mathbb{R}^k} \mathscr{H}^0 \llcorner \mathrm{spt}(T_x) dx$ we will prove that there are countably many Lipschitz graphs that contain the right hand side of (63), for $\mathscr{L}^k$-almost every x. Since $X \in \mathbf{F}_k(E)$ is a flat current with finite mass, the statement regarding its atoms has already been obtained in the proof of [**DL02**, Theorem 3.2]. The result reads:

Theorem 2.9.2. *Let $X \in \mathbf{F}_k(E)$ be a flat current of finite mass. Then, for all $\pi \in \mathrm{Lip}(E, \mathbb{R}^k)$ there exists a countably $\mathscr{H}^k$-rectifiable set $\Sigma_{X,\pi}$ such that, for $\mathscr{L}^k$-a.e. $x \in \mathbb{R}^k$, the atoms of the measure $\langle X, \pi, x\rangle$ are contained in $\Sigma_{X,\pi}$.*

Actually one can even prove, arguing as in Section 2.9.3, that a countably $\mathscr{H}^k$-rectifiable set can be chosen independently of π, but we shall not need this fact.

Proof. Recall X is a finite mass flat current, therefore by the observation after Definition 2.4.2 there exists a sequence $(X_h) \subset \mathbf{N}_k(E)$ of normal currents converging in mass to X. This implies in particular that given $\pi \in \mathrm{Lip}(E, \mathbb{R}^k)$ each atom in $\{y \in E : \|X_x\|(\{y\}) > 0\}$ belongs to one of the currents X_h:

$$\{y \in E : \|X_x\|(\{y\}) > 0\} \subset \bigcup_h \{y \in E : \|X_{h,x}\|(\{y\}) > 0\}.$$

The argument of the proof of Theorem 2.8.5 shows how for a normal current these sets are actually countably $\mathscr{H}^k$-rectifiable. □

Observe that the proof uses the dual construction of the flat norm $\tilde{\mathbf{F}}$ (57) to show that the slicing map is MBV. Unfortunately our hybrid distance $\mathcal{G}$ does not seem to have a similar duality property. Instead, we consider the classical definition of function of bounded variation recalled in Section 2.8 to prove the theorem in dimension $k = 1$. Then, the total differential Theorem 2.3.3 and Proposition 2.9.8 will allow us to pass to the general dimension.

2.9.1. The 1-dimensional case. First of all we fix a map $\pi \in \mathrm{Lip}_1(E)$.

Proposition 2.9.3. *Let $T \in \mathbf{F}_1(E)$ be a flat 1-current with finite size, let us write $T = X + \partial Y$ with $\mathbf{M}(X) + \mathbf{M}(Y) < \infty$ and denote by Q_x the slicing map*

$$Q_x : \mathbb{R} \to \mathbf{B}_0(E), \quad Q_x = \langle T - X, \pi, x\rangle = \langle \partial Y, \pi, x\rangle.$$

Then $Q_x \in MBV(\mathbb{R}, (\mathbf{B}_0, \mathcal{G}))$ and $|DQ_x|(\mathbb{R}) \le \mathbf{M}(X) + \mathbf{S}(T)$.

Proof. Since μ_π is a finite measure, for almost every x the support of $\langle T, \pi, x\rangle$ is finite. By Proposition 2.6.3 we know that $\langle T, \pi, x\rangle$ must have finite mass. Therefore $Q_x = \langle T, \pi, x\rangle - \langle X, \pi, x\rangle$ belongs to $\mathbf{M}_0(E)$. Moreover Q_x is a boundary

$$Q_x = \langle T - X, \pi, x\rangle = \partial(\partial Y \llcorner \{\pi < x\}) - \partial^2 Y \llcorner \{\pi < x\} = \partial((T - X)\llcorner\{\pi < x\}),$$

which proves that the map Q takes values in $\mathbf{B}_0(E)$. These properties hold whenever the slices exist and restrictions can be made: as explained in Section 2.4.1 these operations are meaningful in a set of full measure. Therefore for every $x_1 < x_2$ both outside a set of measure zero we can perform the following computation:

$$Q_{x_2} - Q_{x_1} = \langle T - X, \pi, x_2\rangle - \langle T - X, \pi, x_1\rangle = \partial((T - X)\llcorner\{x_1 \le \pi < x_2\}),$$

hence

$$\mathcal{G}(Q_{x_2}, Q_{x_1}) \le \mathbf{M}(X\llcorner\{x_1 \le \pi < x_2\}) + \mathbf{S}(T\llcorner\{x_1 \le \pi < x_2\}).$$

Therefore choosing $x_0 < x_1 < \ldots < x_N$, from (55), we obtain that $|DQ_x|(\mathbb{R}) = \text{ess-Var}_{-\infty}^{+\infty} Q_x \le \mathbf{M}(X) + \mathbf{S}(T)$, which is the thesis. □

Theorem 2.9.4. *Let $Q \in MBV(\mathbb{R}, (\mathbf{B}_0, \mathcal{G}))$. There exists a $\mathscr{L}^1$-negligible set $\Lambda \subset \mathbb{R}$ such that the set of atoms*

$$\Sigma_Q = \{y \in E : \text{ there exists } x \in \mathbb{R} \setminus \Lambda \text{ such that } \|Q_x\|(\{y\}) > 0\}$$

is countably $\mathscr{H}^1$-rectifiable. In particular, for all $T \in \mathbf{F}_1(E)$ with finite size and all $\pi \in \mathrm{Lip}(E)$ this property holds for the map $Q_x = \langle T, \pi, x\rangle$.

Proof. Fix ε, $\delta > 0$ and let $\Lambda = \mathbb{R} \setminus \bigcup_n B_n$ be the Lebesgue negligible set, where B_n are the Borel sets given by Theorem 2.3.3(ii) in which the estimate (56) holds:

$$\mathcal{G}(Q_{x_1}, Q_{x_2}) \leq L_n |x_1 - x_2| \qquad \forall\, x_1,\, x_2 \in B_n, \tag{64}$$

for suitable constants L_n. We then take the set $\Sigma_{\varepsilon,\delta,n}$ of points $y \in E$ such that for some $x \in B_n$:

(a) $\|Q_x\|(\{y\}) \geq \varepsilon$,
(b) $\|Q_x\|(B_{2\delta}(y) \setminus \{y\}) \leq \frac{\varepsilon}{8}$.

It is easy to notice that with this choice of Λ the set Σ_{Q_x} is the union of $\Sigma_{\varepsilon,\delta,n}$ for a countable set of parameters ε and δ, therefore it is sufficient to our purpose to prove the rectifiability of the latter sets. In addition the hypothesis of separability allows us to write E as a countable union of disjoint Borel sets E^δ_k of diameter at most δ, and again it is sufficient to prove the rectifiability of $\Sigma_{\varepsilon,\delta,n,k} := \Sigma_{\varepsilon,\delta,n} \cap E^\delta_k$. Let us take two points y_1 and y_2 in $\Sigma_{\varepsilon,\delta,n,k}$ and let $x_1 \leq x_2$ be their basepoints in B_n: we know that $d = d(y_1, y_2) \leq \delta$. Take T, $X \in \mathbf{F}_1(E)$ such that

$$Q_{x_1} - Q_{x_2} = \partial(X + T) \quad \text{and} \quad \mathbf{M}(X) + \mathbf{S}(T) \leq 2\mathcal{G}(Q_{x_1}, Q_{x_2}). \tag{65}$$

Then either $\mathbf{S}(T) \geq \frac{d}{9}$ or not. In the first case

$$d(y_1, y_2) \leq 9\mathbf{S}(T) \leq 18\mathcal{G}(Q_{x_1}, Q_{x_2}),$$

and since x_1, $x_2 \in B_n$, we obtain by (64)

$$d(y_1, y_2) \leq 18 L_n |x_1 - x_2|. \tag{66}$$

In the latter case $\mathbf{S}(T) < \frac{d}{9}$, hence by definition of size, slicing T with the distance function from y_1, we infer that

$$T_r = \langle T, d(y_1, \cdot), r\rangle = 0$$

for radii r in at least $\frac{8}{9}$ of the segment $[0, d]$. Therefore we can find radii $\rho_1 \in (0, d/3)$, $\rho_2 \in (2d/3, d)$ satisfying

$$T_{\rho_1} = T_{\rho_2} = 0 \quad \text{and} \quad \mathbf{M}(X_{\rho_1}) + \mathbf{M}(X_{\rho_2}) \leq \frac{9}{d}\mathbf{M}(X). \tag{67}$$

In order to remove the ring $\mathcal{R} = \{\rho_1 \leq d(y_1, \cdot) < \rho_2\}$ from T and X we set $T' = T \llcorner (E \backslash \mathcal{R})$ and $X' = X \llcorner (E \backslash \mathcal{R})$. We obtain, as in (51),

$$\partial(T' + X') = [\partial(T + X)] \llcorner (E \backslash \mathcal{R}) + X_{\rho_1} - X_{\rho_2} = (Q_{x_1} - Q_{x_2}) \llcorner (E \backslash \mathcal{R}) + X_{\rho_1} - X_{\rho_2}. \tag{68}$$

Take now a Lipschitz function ϕ such that $0 \leq \phi \leq 1$, $\phi = 1$ in $B_{d/3}(y_1)$, $\phi = 0$ in $E \setminus B_{2d/3}(y_1)$, and $\mathrm{Lip}(\phi) \leq 3/d$. By hypothesis (b) above

$$\begin{aligned} |(Q_{x_1} - Q_{x_2}) \llcorner \mathcal{R}(\phi)| &\leq \|Q_{x_1}\|(\chi_{\mathcal{R}}\phi) + \|Q_{x_2}\|(\chi_{\mathcal{R}}\phi) \\ &\leq \|Q_{x_1}\|(B_{2\delta}(y_1) \backslash \{y_1\}) + \|Q_{x_2}\|(B_{2\delta}(y_2) \backslash \{y_2\}) \leq \frac{\varepsilon}{4}, \end{aligned} \tag{69}$$

since $\mathcal{R} \subset B_{2\delta}(y_i) \backslash \{y_i\}$. The first two assumptions on ϕ imply that

$$|(Q_{x_1} - Q_{x_2})(\phi) - Q_{x_1}(\{y_1\})| \leq \|Q_{x_1}\|(B_{\frac{2d}{3}}(y_1) \backslash \{y_1\}) + \|Q_{x_2}\|(B_{\frac{2d}{3}}(y_1)) \leq \frac{\varepsilon}{4},$$

which gives

$$|(Q_{x_1} - Q_{x_2})(\phi)| \geq |Q_{x_1}(\{y_1\})| - |(Q_{x_1} - Q_{x_2})(\phi) - Q_{x_1}(\{y_1\})| \geq \frac{3}{4}\varepsilon. \tag{70}$$

Putting together (69) and (70) we obtain

$$|(Q_{x_1} - Q_{x_2})\llcorner(E\backslash\mathcal{R})(\phi)| \geq |(Q_{x_1} - Q_{x_2})(\phi)| - |(Q_{x_1} - Q_{x_2})\llcorner\mathcal{R}(\phi)| \geq \frac{\varepsilon}{2}. \tag{71}$$

We can now test equation (68) with ϕ:

$$\begin{aligned}\frac{\varepsilon}{2} &\overset{(71)}{\leq} |(Q_{x_1} - Q_{x_2})\llcorner(E\backslash\mathcal{R})(\phi)| = |(T' + X')(d\phi) + (X_{\rho_2} - X_{\rho_1})(\phi)| \\ &\leq |(T' + X')(d\phi)| + \mathbf{M}(X_{\rho_1}) + \mathbf{M}(X_{\rho_2}) \overset{(67)}{\leq} |(T' + X')(d\phi)| + \frac{9}{d}\mathbf{M}(X). \end{aligned} \tag{72}$$

Since ϕ is constant in a neighborhood of $B_{\rho_1}(y_1)$ and in a neighbourhood of $E \setminus B_{\rho_2}(y_1)$, we deduce from Lemma 2.9.5 (splitting $T' + X'$ in two parts) that $(T' + X')(d\phi) = 0$. Hence, estimates (72) and (65) yield

$$\frac{\varepsilon}{2} \leq \frac{18}{d}\mathcal{G}(Q_{x_1}, Q_{x_2}) \leq \frac{18L_n}{d}|x_1 - x_2|, \tag{73}$$

since we took $x_i \in B_n$. Hence putting together the two cases (66) and (73) we obtain

$$d(y_1, y_2) \leq \max\{18L_n, \frac{36L_n}{\varepsilon}\}|x_1 - x_2|. \tag{74}$$

In particular for every $x \in \mathbb{R}\setminus\Lambda$ there exists at most one atom y of Q_x in the set $\Sigma_{\varepsilon,\delta,n,k}$, denoted by $f(x)$: let $D_{\varepsilon,\delta,n,k} \subset \mathbb{R} \setminus \Lambda$ denote the set of points x where this atom exists. The estimate (74) implies that the map $f : D_{\varepsilon,\delta,n,k} \to E$ has a global Lipschitz extension.

Finally, the last part of the statement follows by Proposition 2.9.3. □

LEMMA 2.9.5. *Let $T \in \mathbf{F}_k(E)$ and $u \in \mathrm{Lip}(E)$. For $\mathscr{L}^1$-almost every $t \in \mathbb{R}$ the following property holds:*

$$\partial(T\llcorner\{u < t\})(\phi) = 0$$

for every $\phi \in \mathrm{Lip}_b(E)$ constant in a neighborhood of $\{u < t\}$.

PROOF. By definition there exists a sequence of normal currents (T_h) satisfying $\sum_h \mathbf{F}(T_h - T) < \infty$, so that that for almost every t it holds $\mathbf{F}(\partial(T_h\llcorner\{u < t\}) - \partial(T\llcorner\{u < t\})) \to 0$. Since T_h has finite mass, we can write $\partial(T_h\llcorner\{u < t\})(\phi) = T_h(\chi_{\{u<t\}}d\phi)$ and we can use the locality property of finite mass metric currents ([**AK00a**, Theorem 3.5]) to get $T_h(\chi_{\{u<t\}}d\phi) = 0$. Passing to the limit in h the statement follows. □

2.9.2. The general k-dimensional case. In this section we analyse the general case $k \geq 1$.

We shall need three technical statements. The first one provides a useful commutativity property of the iterated slice operator, the second one provides a measurable selection, see for instance [**CV77**, III.6, III.11]. The third, proved in [**AW11**, 5.2], relates 1-dimensional rectifiable sets and projections.

LEMMA 2.9.6 (Commutativity of slices). *Let $T \in \mathbf{F}_k(E)$ and let $\pi = (p, q)$ satisfy $p \in \mathrm{Lip}(E, \mathbb{R}^{m_1})$, $q \in \mathrm{Lip}(E, \mathbb{R}^{m_2})$, $m_i \geq 1$ and $m_1 + m_2 \leq k$. Then*

$$\langle\langle T, p, z\rangle, q, y\rangle = (-1)^{m_1 m_2}\langle\langle T, q, y\rangle, p, z\rangle \qquad \text{for } \mathscr{L}^{m_1+m_2}\text{-a.e. } (z, y) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}. \tag{75}$$

PROOF. If $T \in \mathbf{N}_k(E)$ it is immediate to check that $\langle\langle T, q, y\rangle, p, z\rangle$ satisfy condition(a) of Section 2.4.1 and

$$\int \psi(y, z)\langle\langle T, q, y\rangle, p, z\rangle \, dydz = T \llcorner \psi(p, q) dq \wedge dp = (-1)^{m_1+m_2} T \llcorner \psi(p, q) dp \wedge dq,$$

hence (75) holds. The general case can be achieved choosing a sequence $(T_h) \subset \mathbf{N}_k(E)$ with $\sum_h \mathbf{F}(T_h - T) < \infty$. □

LEMMA 2.9.7. *Let us assign for all $x \in \mathbb{R}^k$ a finite set $A(x) \subset E$, and let us assume that $\{x : \; A(x) \cap C \neq \emptyset\}$ is Lebesgue measurable for all closed sets $C \subset E$. Then the sets*

$$B_n := \left\{x \in \mathbb{R}^k : \; \mathrm{card}\, A(x) = n\right\} \qquad n \geq 1$$

are Lebesgue measurable and there exist Lebesgue measurable maps $f_1, \ldots, f_n : B_n \to E$ such that

$$A(x) = \{f_1(x), \ldots, f_n(x)\} \qquad \text{for } \mathscr{L}^k\text{-a.e. } x \in B_n. \tag{76}$$

PROPOSITION 2.9.8. *Let $K \subset \Gamma \subset E$, with K countably $\mathscr{H}^1$-rectifiable, and let $\pi \in \mathrm{Lip}(E)$ be injective on Γ. Then $\delta(\pi|_\Gamma)^{-1}$ is finite $\mathscr{L}^1$-a.e. on $\pi(K)$.*

PROOF. Assume first that $K = f(C)$ with $C \subset \mathbb{R}$ closed and $f : C \to K$ Lipschitz and invertible. The condition $\delta(\pi|_\Gamma)^{-1} < \infty$ clearly holds at all points $t = \pi(x)$ of density 1 for $\pi(K)$, with $x \in K$ satisfying

$$\liminf_{y \in K \to x} \frac{|\pi(y) - \pi(x)|}{d(y, x)} > 0.$$

Indeed, at these points $t = \pi(x)$ we have $x = (\pi|_\Gamma)^{-1}(t)$ and

$$\liminf_{s \in \pi(K) \to t} \frac{|(\pi|_\Gamma)^{-1}(s) - x|}{|s - t|} < \infty.$$

If $N \subset K$ is the set where the condition above fails, the Lipschitz function $p = \pi \circ f$ has null derivative at all points in $f^{-1}(N)$ where it is differentiable, hence $\mathscr{L}^1(p(f^{-1}(N))) = 0$. It follows that $\mathscr{L}^1(\pi(N)) = 0$.

In the general case, write $K = N \cup \cup_i K_i$ with $\mathscr{H}^1(N) = 0$ and $K_i = f_i(C_i)$ pairwise disjoint, with $C_i \subset \mathbb{R}$ closed and $f_i : C_i \to K_i$ Lipschitz and invertible. Let $B_i \subset \pi(K_i)$ be Borel sets such that the inverse g_i of $\pi|_{K_i}$ satisfies $\delta g_i < \infty$ on B_i and $\mathscr{L}^1(\pi(K_i) \setminus B_i) = 0$. Since $\mathscr{H}^1(\pi(N)) = 0$, the union $\cup_i \pi(K_i)$ covers $\mathscr{L}^1$-almost all of $\pi(K)$. Hence, it suffices to show that $\delta(\pi|_\Gamma)^{-1} < \infty$ at all points of density 1 for one of the sets B_i. This property easily follows from the definition of δ and from the fact that $(\pi|_\Gamma)^{-1}$ and g_i coincide on B_i. □

We are ready prove the rectifiability of the atoms of $\langle \partial Y, \pi, x\rangle$ for general $k \geq 1$ and, as a consequence, the rectifiability of $\mu_{T,\pi}$.

THEOREM 2.9.9. *Let $\pi \in \mathrm{Lip}(E, \mathbb{R}^k)$ and suppose $T = X + \partial Y \in \mathbf{F}_k(E)$ has finite size, with $\mathbf{M}(X) + \mathbf{M}(Y) < \infty$. Then there exists a Lebesgue negligible set $\Lambda \subset \mathbb{R}^k$ such that the set of atoms*

$$\Sigma_{\partial Y,\pi} = \Big\{ y \in E : \text{ there exists } x \in \mathbb{R}^k \setminus \Lambda \text{ such that } \|(\partial Y)_x\|(\{y\}) > 0 \Big\}$$

is a countably $\mathscr{H}^k$-rectifiable set. In particular

$$\mu^*_{\partial Y,\pi} = \int_{\mathbb{R}^k} \mathscr{H}^0 \llcorner \mathrm{Atoms}(\langle \partial Y, \pi, x\rangle)\, dx \tag{77}$$

is concentrated on a countably $\mathscr{H}^k$-rectifiable set.

PROOF. First of all notice that the statement of the theorem allows us to ignore sets of atoms whose projection under π is Lebesgue negligible. We will split the family of atoms in countably many subfamilies (indexed by m and n), according to their weight and the cardinality in each fiber.

Since T has finite size and X has finite mass, by Proposition 2.6.3 for almost every $x \in \mathbb{R}^k$ the equality

$$Q_x = \langle \partial Y, \pi, x\rangle = \langle T - X, \pi, x\rangle$$

implies that Q_x has finite mass, so for every $m \geq 1$ the set of points $y \in E$ such that $\|Q_x\|(\{y\}) \geq 1/m$ is finite almost everywhere. We fix a representative Q_x of the slicing map and denote by $\mathcal{N}$ the Lebesgue negligible set of points where Q_x has infinite mass.
Step 1. In this step we view the set of atoms with weight larger than $1/m$ as images of suitable maps defined on subsets of $\mathbb{R}^k$. To this aim, consider the set-valued function

$$A_m(x) := \begin{cases} \{y \in \pi^{-1}(x) : \|Q_x\|(\{y\}) \geq \frac{1}{m}\} & \text{if } x \in \mathbb{R}^k \setminus \mathcal{N}, \\ \emptyset & \text{if } x \in \mathcal{N} \end{cases}$$

and notice that it fulfils the measurability assumption of Lemma 2.9.7. Indeed, let $C \subset E$ be compact and let $\{y_q\}$ be dense in E, We claim that all $x \notin \mathcal{N}$ it holds

$$\exists y \in C : \|Q_x\|(\{y\}) \geq \frac{1}{m} \quad \Longleftrightarrow \quad \forall \ell\, \exists q : \|Q_x\|(B_{\frac{1}{\ell}}(y_q) \cap C) \geq \frac{1}{m}.$$

The implication $\Rightarrow$ is trivial by density; if on the other hand there is a sequence $(y_{q(\ell)})$ such that $\|Q_x\|(B_{\frac{1}{\ell}}(y_{q(\ell)}) \cap C) \geq \frac{1}{m}$, any limit point $\bar{y}$ must belong to C and satisfies $\|Q_x\|(B_{\frac{1}{n}}(\bar{y}) \cap C) \geq \frac{1}{m}$ for any given n, so that $y \in A_m(x)$. Hence $\{x : A_m(x) \cap C \neq \emptyset\}$ can be written as

$$\bigcap_{\ell} \bigcup_{q} \left\{ x \in \mathbb{R}^k \backslash \mathcal{N} : \|Q_x\|(B_{\frac{1}{\ell}}(y_q) \cap C) \geq \frac{1}{m} \right\}. \tag{78}$$

The map $x \mapsto \|Q_x\|(B)$ is measurable for every Borel set B and for every $T \in \mathbf{F}_k(E)$ (see [**AW11**, Section 3] for the proof of this result), hence the set in (78) is Lebesgue measurable. Since any closed set C is a countable union of compact sets we obtain that A_m satisfies the measurability assumption of Lemma 2.9.7. As a consequence, for all $n \geq 1$ we obtain disjoint measurable sets $B_n = \{x : \mathscr{H}^0(A_m(x)) = n\}$ and measurable maps $f_1, \ldots, f_n$ satisfying (76).
Step 2. In order to show that the collection of atoms is countably $\mathscr{H}^k$-rectifiable, modulo sets with Lebesgue negligible projection on $\mathbb{R}^k$, we can use Lusin's theorem and

the inner regularity of the Lebesgue measure to restrict the domain of the functions $f_1, \dots, f_n$ to a compact set $C \subset B_n$ and assume that these restrictions are continuous. Notice that since $f_i(x) \neq f_j(x)$ whenever $x \in B_n$ and $i \neq j$ we can also assume that the sets $K_i := f_i(C)$, $i = 1, \dots, n$, are pairwise disjoint, by a further decomposition of C in countably many pieces. Observe also that $\pi : K_i \to C$ is injective and its inverse is f_i. In order to prove the theorem it suffices to show that the sets $K_i \setminus \pi^{-1}(V_i)$ for suitable Lebesgue negligible sets $V_i \subset \mathbb{R}^k$, are countably $\mathscr{H}^k$-rectifiable: we fix an index i once and for all.

Writing $x = (z,t)$ with $z \in \mathbb{R}^{k-1}$ and $t \in \mathbb{R}$, let us consider the sets

$$C_z := \{t \in \mathbb{R} : \ (z,t) \in C\}, \qquad K_{iz} := \{x \in K_i : \ (\pi_1, \dots, \pi_{k-1})(x) = z\}$$

and the maps $g_{iz}(t) := f_i(z,t) : C_z \to K_{iz}$. We claim that, for $\mathscr{L}^{k-1}$-a.e. z, $\delta_t g_{iz} < \infty$ $\mathscr{L}^1$-a.e. in C_z. Indeed,

$$Q_x = \langle S_z, \pi_k, t\rangle \qquad \text{with} \qquad S_z := \langle T - X, (\pi_1, \dots, \pi_{k-1}), z\rangle$$

we know that for $\mathscr{L}^{k-1}$-a.e. z the flat chain $S_z \in \mathbf{F}_1(E)$ is the sum of a flat current with finite size and of a flat current with finite mass and (thanks to Lemma 2.9.6) $\langle S_z, \pi_k, t\rangle = Q_x$ for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$. It follows that $\langle S_z, \pi_k, t\rangle \llcorner K_i = Q_x \llcorner K_i$ is a Dirac mass concentrated on $g_{iz}(t)$ for $\mathscr{L}^1$-a.e. $t \in C_z$.

We fix now a point z with these properties: combining Theorem 2.9.4 (applied to the part with fine size of S_z) and Theorem 2.9.2 (applied to the part with finite mass of S_z) we get a countably $\mathscr{H}^1$-rectifiable set $G_z \subset E$ and a $\mathscr{L}^1$-negligible set $N_z \subset \mathbb{R}$ such that the atoms of $\langle S_z, \pi_k, t\rangle$ lying in K_i are contained in G_z for all $t \in \mathbb{R} \setminus N_z$. We denote by $\tilde{K}_{iz} \subset G_z$ the set

$$\tilde{K}_{iz} := \{g_{iz}(t) : \ t \in C_z \setminus N_z\}$$

which is countably $\mathscr{H}^1$-rectifiable as well and contained in K_{iz}. Also $\mathscr{L}^1(\pi_k(K_{iz} \setminus \tilde{K}_{iz})) = 0$ because this set is contained in N_z. Since $\pi_k|_{K_{iz}}$ is injective, we can now apply Proposition 2.9.8 with $K = \tilde{K}_{iz}$ and $\Gamma = K_{iz}$ to obtain that $\delta_t((\pi_k)|_{K_{iz}})^{-1} < \infty$ $\mathscr{L}^1$-a.e. on $\pi_k(\tilde{K}_{iz})$ and therefore $\mathscr{L}^1$-a.e. on $\pi_k(K_{iz})$. But, since the inverse of $\pi|_{K_i}$ is f_{j_i}, the inverse of $(\pi_k)|_{K_{iz}}$ is g_{iz}. It follows that $\delta_t g_{iz} < \infty$ $\mathscr{L}^1$-a.e. on C_z. This proves the claim.

Using the commutativity of the slice operator, we see that a similar property is fulfilled by f_i with respect to the other $(k-1)$ variables, hence Theorem 2.3.3(i) ensures that $\delta_x f_i < \infty$ $\mathscr{L}^k$-a.e. on C. This ensures that Theorem 2.3.3(ii) is applicable to f_i, so that we can cover $\mathscr{L}^k$-almost all of C with Borel sets C_k such that the restriction of f to C_k is Lipschitz. Since $f(\cup_k C_k)$ is countably $\mathscr{H}^k$-rectifiable, we can can choose $V_i = C \setminus \cup_k C_k$ to conclude the proof. □

2.9.3. Proof of the main result. In this section we prove Theorem 2.9.1. Let $T = X + \partial Y$. For a given $\pi \in \mathrm{Lip}_1(E, \mathbb{R}^k)$, Theorem 2.9.2 and Theorem 2.9.9 provide us two countably $\mathscr{H}^k$-rectifiable sets $\Sigma_{X,\pi}$ and $\Sigma_{\partial Y,\pi}$ such that $\mu_{X,\pi}$ is concentrated on $\Sigma_{X,\pi}$ and $\mu_{\partial Y,\pi}$ is concentrated on $\Sigma_{\partial Y,\pi}$. In particular $\mu_{T,\pi}$ is concentrated on the countably $\mathscr{H}^k$-rectifiable set $\Sigma_{T,\pi} = \Sigma_{X,\pi} \cup \Sigma_{\partial Y,\pi}$. Consider for any $n \in \mathbf{N}$ a finite set $J_n \subset \mathrm{Lip}_1(E; \mathbb{R}^k)$ of projections such that

$$\mu_T(E) \leq \left(\bigvee_{\pi \in J_n} \mu_{T,\pi}\right)(E) + 2^{-n}$$

(its existence is a direct consequence of (7)). Then, denoting by J the union of the sets J_n, the measure

$$\sigma := \bigvee_{\pi \in J} \mu_{T,\pi}$$

is smaller than μ_T and with the same total mass, hence it coincides with μ_T. Since J is countable, a countably $\mathscr{H}^k$-rectifiable concentration set Σ for μ_T can be obtained by taking the union $\cup_{\pi \in J} \Sigma_{T,\pi}$.

Finally, defining $set(T)$ as in (61), since μ_T is concentrated on Σ the spherical differentiation theory gives $\Theta^*_k(\mu_T, x) = 0$ for $\mathscr{H}^k$-a.e. $x \in E \setminus \Sigma$, hence $set(T) \subset \Sigma$ up to $\mathscr{H}^k$-negligible sets.

2.10. Characterization of the size measure

In this section we improve the result of Theorem 2.9.1, showing a formula for the density of μ_T with respect to $\mathscr{H}^k \llcorner set(T)$ that involves only the local geometry of $set(T)$. We start by stating some differentiability properties of Lipschitz maps and rectifiable sets contained in [**AK00b**].

2.10.1. Dual of separable Banach spaces. Recall that by w^*-separable dual Banach space we mean $Y = G^*$ with G Banach and separable, and that if $\{g_n\}$ is a dense subset of $B_1(0) \subset G$ then the distance

$$d_w(x, y) = \sum_n 2^{-n} |\langle x - y, g_n \rangle|$$

induces the weak* topology on bounded subsets of Y and makes (Y, d_w) separable. The sequence space ℓ^∞ is an example of such spaces. Throughout the rest of the section Y will indicate a w^*-separable dual Banach space.

It is helpful for our purposes to consider w^*-separable dual spaces as an ambient space. There are mainly two reasons for this. First, since E is separable it is also weakly separable, and we gain some linear structure by embedding E into the vector space ℓ^∞. The second reason is related to the following Rademacher-type Theorem:

THEOREM 2.10.1 ([**AK00b**, 3.5]). *Let $f \in \mathrm{Lip}(\mathbb{R}^k, Y)$: for $\mathscr{L}^k$-a.e. $x \in \mathbb{R}^k$ there exists a linear map $wd_x f : \mathbb{R}^k \to Y$ such that*

$$w^* - \lim_{y \to x} \frac{f(y) - f(x) - wd_x f(y - x)}{|y - x|} = 0 \quad \textit{and} \tag{79}$$

$$\lim_{y \to x} \frac{\|f(y) - f(x)\| - \|wd_x f(y - x)\|}{|y - x|} = 0. \tag{80}$$

The map $wd_x f$ is called the w^*-differential of f at x.

PROOF. Let $D \subset G$ be a countable dense vector space over $\mathbb{Q}$: the classical Rademacher Theorem yields the existence of a $\mathscr{L}^k$-negligible $N \subset \mathbb{R}^k$ such that $f_g(x) = \langle f(x), g \rangle$ is differentiable at any $x \in \mathbb{R}^k \setminus N$ for any $g \in D$. By continuity, we can find for any $x \in \mathbb{R}^k \setminus N$ a linear function $\nabla f(x) : \mathbb{R}^k \to Y$ such that $\langle \nabla f(x), g \rangle = \nabla f_g(x)$ for all $g \in D$. By a density argument it is not difficult to check that f is w^*-differentiable at any $x \in \mathbb{R}^k \setminus N$ and $wd_x f = \nabla f(x)$, namely (79).

Using the lower w^*-semicontinuity of the norm we infer

$$\|wd_x f(v)\| \le \liminf_{t\downarrow 0} \frac{\|f(x+tv)-f(x)\|}{t} \qquad \forall v \in \mathbb{R}^k.$$

If we let $D' \subset S^{k-1}$ be countable and dense, and set $\nabla f_g = 0$ and $\nabla f = 0$ in N, then for any $x \in \mathbb{R}^k$ and any $v \in D'$ we let $\nabla_v f(x)$ to be the unique $y \in Y$ such that $\langle y, g\rangle = \nabla_v f_g$ for any $g \in D$. By a classical result on differentiation of Sobolev functions [**Zie89**, 2.1.4] there exists a $\mathscr{L}^k$-negligible $N' \subset \mathbb{R}^k$ such that

$$\langle f(x+tv) - f(x)\rangle = \int_0^t \nabla_v f_g(x+sv)\, ds$$

$$\lim_{\rho\downarrow 0} \frac{1}{\rho}\int_0^\rho \|\nabla_v f(x+sv)\|\, ds = \|\nabla_v f(x)\|$$

for any $t>0$, $v \in D'$, $g \in D$ and $x \in \mathbb{R}^k \setminus N'$. By density this gives

$$|\langle f(x+tv) - f(x)\rangle| \le \int_0^t \|\nabla_v f(x+sv)\|\, ds.$$

If $x \in \mathbb{R}^k \setminus (N \cup N')$ and $v \in D'$ we can divide both sides by t and let $t \downarrow 0$ to obtain

$$\limsup_{t\downarrow 0} \frac{\|f(x+tv)-f(x)\|}{t} \le \|wd_x f(v)\|.$$

By density again the last inequality holds for any $v \in S^{k-1}$ and gives (80). □

Note that if Y is uniformly convex, the two limits (79) and (80) imply that f is Fréchet differentiable at x. An example of non w^*-differentiability is given by the isometric map

$$[0,1] \ni t \mapsto \chi_{[0,t]} \in Y = L^1([0,1])$$

which shows the necessity to deal with dual spaces. If we take $Y = (C^0[0,1])^*$ the space of Radon measures on $[0,1]$ endowed with the total variation norm, then (79) holds with $wd_t f = \delta_t$, nevertheless (80) fails.

2.10.2. Jacobians and the area formula. Aiming to generalize the area formula for maps between metric spaces we need to adapt the concept of jacobian of a linear map in the metric context. We recall that the k-dimensional Hausdorff measure of the unit ball in $\mathbb{R}^k$ $\omega_k := \mathscr{L}^k(B_1^k)$ is actually the Hausdorff measure of the unit ball in any k-dimensional Banach space, see [**Kir94**, Lemma 6].

DEFINITION 2.10.2. *Given a linear map $L : V \to W$ between two Banach spaces V and W, with* $\dim(V) = k$, *we let*

$$\mathbf{J}_k(L) = \frac{\omega_k}{\mathscr{H}^k_V(\{x : \|L(x)\| \le 1\})}, \tag{81}$$

If V, W are Hilbert spaces than by the polar representation theorem it is known that $\mathbf{J}_k(L) = (\det L^* \circ L)^{1/2}$. We will use the following form of the classical Binet theorem:

LEMMA 2.10.3. *If* $\dim U = \dim V = k \le \dim W$ *and* $K : U \to V$, $L : V \to W$ *are linear maps, then*

$$\mathbf{J}_k(L \circ K) = \mathbf{J}_k(L)\mathbf{J}_k(K).$$

The key theorem related to $\mathbf{J}_k(L)$ is the area formula: for every $f \in \mathrm{Lip}(\mathbb{R}^k, Y)$, every $A \in \mathcal{B}(\mathbb{R}^k)$ and every Borel function $\theta : Y \to [0, \infty]$ it holds

$$\int_A \theta(f(x))\mathbf{J}_k(wd_x f)\, dx = \int_Y \theta(y)\mathscr{H}^0(A \cap f^{-1}(y))\, d\mathscr{H}^k(y). \tag{82}$$

In particular taking $L : V \to W$ and $\phi : \mathbb{R}^k \to V$ linear, and setting $\theta = 1$ and $A := \phi^{-1}(B_1^V(0))$ in (82) we obtain

$$\mathbf{J}_k(L) = \frac{\mathbf{J}_k(L \circ \phi)}{\mathbf{J}_k(\phi)} = \frac{\mathscr{H}^k_W(\{L(v) : v \in B_1^V(0)\})}{\mathscr{H}^k_V(B_1^V(0))} = \frac{\mathscr{H}^k_W(\{L(v) : v \in B_1^V(0)\})}{\omega_k}. \tag{83}$$

2.10.3. Approximate tangent space and orientation.

Definition 2.10.4 (Approximate tangent space). *Let $S \in \mathcal{B}(Y)$ and assume $S = f(B)$ for some Lipschitz $f : \mathbb{R}^k \to Y$, one to one in $B \in \mathcal{B}(\mathbb{R}^k)$. For any $x \in S$ such that f is w^*-differentiable at $y = f^{-1}(x)$ and $\mathbf{J}_k(wd_y f) > 0$ we let*

$$\mathrm{Tan}^{(k)}(S, x) := wd_y f(\mathbb{R}^k).$$

If $S \subset Y$ is a countably $\mathscr{H}^k$-rectifiable set and $S \subset \bigcup_i S_i$ with $S_i = f_i(B_i)$ as above we let

$$\mathrm{Tan}^{(k)}(S, x) := \mathrm{Tan}^{(k)}(S_i, x) \qquad \text{for } \mathscr{H}^k - \text{a.e. } x \in S \cap S_i.$$

The definition is well posed thanks to the area formula and the following locality property: if $S_i = f_i(B_i)$ with $f_i \in \mathrm{Lip}(\mathbb{R}^k, Y)$ and one to one in B_i then

$$\mathrm{Tan}^{(k)}(S_i, x) = \mathrm{Tan}^{(k)}(S_j, x) \qquad \text{for } \mathscr{H}^k - \text{a.e. } x \in S_i \cap S_j.$$

Note moreover that $\dim \mathrm{Tan}^{(k)}(S, x) = k$, and that $\mathrm{Tan}^{(k)}(S, x)$ inherits the norm of Y by restriction.

If $S \subset E$ is countably $\mathscr{H}^k$-rectifiable and E is a separable metric space then we can pose

$$\mathrm{Tan}^{(k)}(S, x) := \mathrm{Tan}^{(k)}(j(S), j(x))$$

for some isometric embedding $j : S \to Y$ with Y w^*-separable dual space. Different choices of embedding produce different tangent spaces: however all such spaces are isometric, and their intrinsic norm is called local norm [**Kir94**, Theorem 9]. This fact stems from the characterization of $\mathrm{Tan}^{(k)}(S, x)$ as w^*-limits of secant vectors and the existence of a good projection map: for $\mathscr{H}^k$-a.e. $x \in S$ there exists $S_x \in \mathcal{B}(Y)$ and $\pi_x : Y \to \mathrm{Tan}^{(k)}(S, x)$ w^*-continuous linear map such that $\Theta^{*k}(S \setminus S_x, x) = 0$,

$$\mathrm{Tan}^{(k)}(S, x) \cap \partial B_1(x) = \left\{ p : p = w^* - \lim_{y \in S_x \to x} \frac{y - x}{\|y - x\|} \right\}$$

and

$$\limsup_{\rho \downarrow 0} \left\{ \left| \frac{\|\pi_x(y) - \pi_x(z)\|}{\|y - z\|} - 1 \right| : y, z \in S_x \cap B_\rho(x) \right\} = 0.$$

The metric jacobian (81) allows to measure the length of simple k-vectors $\tau = \tau_1 \wedge \cdots \wedge \tau_k \in \Lambda_k Y$ simply taking

$$\mathbf{J}_k(L_\tau), \qquad \text{where } L_\tau : \mathbb{R}^k \to Y, \qquad L_\tau(x) := \sum_{i=1}^k x^i \tau_i.$$

(right composition with $GL(k)$ maps and the Binet's formula 2.10.3 guarantees the passage from $\bigotimes_k Y$ to the quotient $\Lambda_k Y$). An orientation of a countably $\mathscr{H}^k$-rectifiable set S is a unit simple k-vector $\tau = \tau_1 \wedge \cdots \wedge \tau_k$ such τ_i are Borel functions spanning the approximate tangent space to S almost everywhere.

2.10.4. Tangential differentiability and coarea formula. Let Z be another w^*-separable dual space, let $S \subset Y$ be countably $\mathscr{H}^k$-rectifiable and let $\pi \in \mathrm{Lip}(Y, Z)$. Then for $\mathscr{H}^k$-a.e. $x \in S$ there exists a linear map

$$d_x^S \pi : \mathrm{Tan}^{(k)}(S, x) \to Z$$

called the tangential differential of π at x. Such map can be characterized by requiring that

$$wd_y(\pi \circ f) = d^S_{f(y)}\pi \circ wd_y f$$

for any parametrization f as in (13). In the case $Z = \mathbb{R}^p$ the tangential differentials of each coordinate of π give rise to a k-covector $\Lambda^k d_x^S \pi := d_x^S \pi^1 \wedge \cdots \wedge d_x^S \pi^k$. In particular if $p = k$ and $x = f(y)$ by Lemma 2.10.3 we have

$$\det \nabla(\pi \circ f)(y) = \langle \Lambda^k d_x^S \pi, \tau(y) \rangle,$$

with $\tau(y) = wd_y f(e_1) \wedge \cdots \wedge wd_y f(e_k)$. Therefore

$$\mathbf{J}_k(d_x^S \pi) = \frac{|\det \nabla(\pi \circ f)(y)|}{\mathbf{J}_k(L_{\tau(y)})} = |\langle \Lambda^k d_x^S \pi, \frac{\tau(y)}{\mathbf{J}_k(L_{\tau(y)})} \rangle|$$

and by the arbitrariness of f we deduce

$$\mathbf{J}_k(d_x^S \pi) = |\langle \Lambda^k d_x^S \pi, \sigma(y) \rangle|$$

where σ is any orientation of S. The importance of $\mathbf{J}_k$ relies on the following general coarea formula [**AK00b**, 9.4]:

$$\int_S \theta(x) \mathbf{J}_k(d_x^S \pi) \, d\mathscr{H}^k(x) = \int_{\mathbb{R}^k} \int_{S \cap \pi^{-1}(y)} \theta(x) \, d\mathscr{H}^0(x) \, d\mathscr{H}^k(y). \tag{84}$$

We restrict our attention to the Euclidean case $Z = \mathbb{R}^k$. In order to relate μ_T to $\mathscr{H}^k \llcorner set(T)$ we need to calculate the supremum of the k-Jacobians $\mathbf{J}_k(d^S \pi)$ among all possible functions π. As explained in the next two lemmas, it turns out that this quantity depends only on the norm of tangent space $\mathrm{Tan}^{(k)}(S, x)$.

Let V be a k-dimensional Banach space and denote by B_1^V its unit ball. We call ellipsoid any set $R = L(B)$, where $L : \mathbb{R}^k \to V$ is a linear map and B is a ball in the Euclidean space $\mathbb{R}^k$. The supremum

$$\lambda_V := \sup \left\{ \frac{\mathscr{H}^k(B_1^V)}{\mathscr{H}^k(R)} : B_1^V \subset R, \ R \text{ ellipsoid} \right\} \tag{85}$$

is called the area factor of V, and is clearly related to the problem of finding the best ellipsoid enclosing a convex set in $\mathbb{R}^k$. For instance if V is a Hilbert space, then the spectral theorem implies $\lambda_V = 1$. The following lemma relates λ_V to the k-jacobian of linear maps ([**AK00a**, Lemma 9.2]:

LEMMA 2.10.5. *Let V be a k-dimensional Banach space. Then*

$$\lambda_V = \sup\{\mathbf{J}_k(\zeta) : \zeta : V \to \mathbb{R}^k \text{ linear, } \mathrm{Lip}(\zeta) \le 1\}.$$

More generally if the linear maps are taken with Lipschitz constant (i.e.: operator norm) bounded by C, then the supremum is $\lambda_V C^k$.

PROOF. Without loss of generality we can assume that the map ζ is non singular. Then, the ellipsoid $\{v \in V : |\zeta(v)| \le 1\} = \zeta^{-1}(B_1)$ contains $B_1^V(0)$ if and only if $\mathrm{Lip}(\zeta) \le 1$. Hence for such maps the area formula implies that

$$\mathbf{J}_k(\zeta) = \frac{\omega_k}{\mathscr{H}^k(\{v \in V : |\zeta(v)| \le 1\})} = \frac{\mathscr{H}^k(B_1^V)}{\mathscr{H}^k(\{v \in V : |\zeta(v)| \le 1\})} \le \lambda_V.$$

On the other hand by definition any nontrivial ellipsoid $R = L(B)$ can be written as $\zeta^{-1}(B)$, for some linear map ζ and some Euclidean ball B, just setting $\zeta = L^{-1}$, By possibly rescaling one can assume that B has radius 1. At this point $R = \zeta^{-1}(B_1) = \{v \in V : |\zeta(v)| \le 1\}$ and the same inequality as above completes the proof. The general case simply follows by homothety. □

Also, we shall need the following density result [**AK00a**, Lemma 9.4]:

LEMMA 2.10.6. *Let $\Pi_k(Y)$ be the collection of all w^*-continuous linear maps $\pi : Y \to \mathbb{R}^k$, with $\mathrm{Lip}(\pi) \le 1$. There exists a countable set $\{\pi_j\} \subset \Pi_k(Y)$ such that $\mathrm{Lip}(\pi_j) \le 1$ for every $j \in \mathbf{N}$ and*

$$\sup_j \mathbf{J}_k(\pi_j|_V) = \sup\{\mathbf{J}_k(\pi|_V) : \pi \in \Pi_k(Y),\, \mathrm{Lip}(\pi) \le 1\}$$

for any k-dimensional subspace $V \subset Y$.

PROOF. In fact it is sufficient to consider the set of maps

$$\left\{\sum_{i=1}^k \langle x, g_i\rangle e_i : \, g_i \in D\right\} \subset L^*(Y, \mathbb{R}^k)$$

where D is a countable dense subset of G: it is not hard to show that this set is dense in the space $L^*(Y, \mathbb{R}^k)$ of linear and w^*-continuous maps from Y to $\mathbb{R}^k$ for the topology of uniform convergence on bounded subsets of Y. As a consequence its subspace $\Pi_k(Y)$ is also separable, and if $\pi_h \to \pi$ uniformly in $B_1^V(0)$ then

$$\mathscr{H}^k(\{\pi(v) : v \in B_1^V(0)\}) = \lim_h \mathscr{H}^k(\{\pi_h(v) : v \in B_1^V(0)\}),$$

which according to (81) and (83) makes the map $\pi \mapsto \mathbf{J}_k(\pi|_V)$ continuous along the sequence. □

THEOREM 2.10.7 (Characterization of μ_T). *For any $T \in \mathbf{F}_k(Y)$ with finite size it holds*

$$\frac{\lambda}{k^{\frac{k}{2}}} \mathscr{H}^k \llcorner set(T) \le \mu_T \le \lambda \mathscr{H}^k \llcorner set(T),$$

where $\lambda(x) = \lambda_{\mathrm{Tan}^{(k)}(set(T),x)}$ is the function defined in (85). If moreover Y is a Hilbert space then $\mu_T = \mathscr{H}^k \llcorner set(T)$.

PROOF. The area formula (84) implies that if $\pi \in \mathrm{Lip}_1(Y, \mathbb{R}^k)$ and $A \subset Y$ is a Borel set, then

$$\mu_{T,\pi}(A) = \int_{\mathbb{R}^k} \mathscr{H}^0(A \cap set(T) \cap \pi^{-1}(y))\, dy = \int_{A \cap set(T)} \mathbf{J}_k(d_x^{set(T)}\pi)\, d\mathscr{H}^k(x),$$

so that

$$\mu_T = \bigvee_{\pi \in \mathrm{Lip}_1(Y,\mathbb{R}^k)} \mathbf{J}_k(d^{set(T)}\pi)\mathscr{H}^k \llcorner set(T). \tag{86}$$

By Lemma 2.10.5 $\mu_T \leq \lambda \mathscr{H}^k \llcorner set(T)$. On the other hand, choosing π to be one element of the countable family of maps π_j provided by Lemma 2.10.6, μ_T can be bounded below by $\sup_j \mathbf{J}_k(d^{set(T)}\pi_j)\mathscr{H}^k \llcorner set(T)$. Moreover since every linear $\zeta : V \to \mathbb{R}^k$ with $\mathrm{Lip}(\zeta) \leq 1$ can be extended to a w^*-continuous linear $\tilde{\zeta}$ with $\mathrm{Lip}(\tilde{\zeta}) \leq k^{\frac{1}{2}}$, by rescaling for every k-dimensional subspace V the restrictions $\pi_j\big|_V$ are dense in

$$\{\zeta : V \to \mathbb{R}^k, \mathrm{Lip}(\zeta) \leq \frac{1}{k^{\frac{1}{2}}}\},$$

hence by Lemma 2.10.5

$$\sup_j \mathbf{J}_k(\pi_j|_{\mathrm{Tan}^{(k)}(set(T),\cdot)}) \geq \frac{\lambda_{\mathrm{Tan}^{(k)}(set(T),\cdot)}}{k^{\frac{k}{2}}}$$

and so $\mu_T \geq \frac{\lambda}{k^{\frac{k}{2}}} \mathscr{H}^k \llcorner set(T)$. Finally if Y is a Hilbert space, the extension of any ζ to the whole Y can be made by composing with the orthogonal projection onto V, thus preserving the Lipschitz constant. In that case the restrictions to V are dense in the set of 1-Lipschitz linear maps, yielding

$$\mu_T = \lambda \mathscr{H}^k \llcorner set(T);$$

furthermore every approximate tangent space has the Hilbert structure induced by Y, hence $\lambda = 1$. □

CHAPTER 3

The distributional jacobian and the space GSB_nV

In this chapter we introduce the space of Generalised special bounded higher variation GSB_nV of vector valued maps $u : \mathbb{R}^m \to \mathbb{R}^n$. The definition implements the concepts of size of the last chapter to Ju, leading to a compactness Theorem in the same spirit of the one available for SBV maps [**Amb89**]. We also present some key example that clarify the similarities as well as the differences with the scalar case $n = 1$.

3.1. Distributional jacobian

We will assume throughout all the chapter that $m \geq n$ are positive integers and that p and s are positive exponents satisfying

$$\frac{1}{s} + \frac{n-1}{p} \leq 1. \tag{87}$$

We will consider Sobolev maps $u : \Omega \to \mathbb{R}^n$ where $\Omega \subset \mathbb{R}^m$ is an open subset of the Euclidean space. Recall that if Ω is bounded and has Lipschitz boundary then the cone property holds, namely for some $a, b > 0$ and for every point $x \in \partial\Omega$ there exists an orthonormal coordinate frame such that $\Omega \supset \{y \in \mathbb{R}^m : (y^1 - x^1)^2 + \cdots + (y^{n-1} - x^{n-1})^2 \leq a(y^n - x^n)^2 \leq b\}$. The Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$ is the set of (equivalence classes) of maps $u \in L^p(\Omega, \mathbb{R}^n)$ such that the distributional derivative $\nabla u \in L^p(\Omega, \mathbb{R}^{n\times m})$, endowed with the norm

$$\|u\|_{L^p} + \|\nabla u\|_{L^p} = \|u\|_{L^p} + \left(\int_\Omega \Big(\sum_{i=1}^n |\nabla u^i|^2 \Big)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}}.$$

The pointed Sobolev space $\dot{W}^{1,p}(\Omega, \mathbb{R}^n)$ is defined as the factor space

$$\{u \in \mathscr{D}'(\Omega) : \nabla u \in L^p\}/\mathcal{N}.$$

The denominator $\mathcal{N} = \{u \in \mathscr{D}'(\Omega) : \nabla u = 0 \text{ in } \Omega\}$ coincides with the set of functions whose restriction to every connected component of Ω is constant. $\dot{W}^{1,p}$ is endowed with the norm $\|\nabla u\|_{L^p}$, which makes it a Banach space. By the Poincaré inequality [**Maz85**, 1.1.11]

$$\inf_{v\in\mathcal{N}} \|u - v\|_{L^p} \leq C\|\nabla u\|_{L^p},$$

valid if Ω is a bounded open set with Lipschitz boundary, the representatives of any equivalence class of $\dot{W}^{1,p}(\Omega, \mathbb{R}^n)$ belong to $W^{1,p}(\Omega, \mathbb{R}^n)$; more generally if Ω is arbitrary, then every representative in $\dot{W}^{1,p}(\Omega, \mathbb{R}^n)$ is actually in $L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. We will always assume Ω to be bounded and Lipschitz regular.

The definition of distributional jacobian takes advantage of the divergence structure of jacobians

$$d(u^1 du^2 \wedge \cdots \wedge du^n) = du^1 \wedge \cdots \wedge du^n \quad \forall u \in C^1(\mathbb{R}^n, \mathbb{R}^n),$$

which allows to pass the exterior derivative to the test form and hence weakens the minimal regularity assumptions on the map u.

Definition 3.1.1 (Distributional Jacobian). *Let $u \in \dot{W}^{1,p}(\Omega, \mathbb{R}^n) \cap L^s(\Omega, \mathbb{R}^n)$. We denote by $j(u)$ the $(m-n+1)$-dimensional flat current*

$$\langle j(u), \omega \rangle := (-1)^n \int_\Omega u^1 du^2 \wedge \cdots \wedge du^n \wedge \omega \qquad \forall \omega \in \mathscr{D}^{m-n+1}(\mathbb{R}^m); \tag{88}$$

we define the distributional Jacobian of u as the $(m-n)$-dimensional flat current

$$Ju := \partial j(u) \in \mathbf{F}_{m-n}(\mathbb{R}^m). \tag{89}$$

If $m = n$, $[Ju]$ is a distribution and a simple calculation gives that $Ju = \mathrm{div}[\mathrm{Cof}(\nabla u) u]$, where $\mathrm{Cof}(\nabla u)$ is the matrix of cofactors of ∇u. This case of Definition 3.1.1 was first introduced by Ball in [**Bal77**], in the context of nonlinear elasticity, the extension of the distributional Jacobian to the case $m > n$ is due to Jerrard and Soner in [**JS02**].

A few observations around Definition 3.1.1 are in order: first of all the integrability assumption $u \in \dot{W}^{1,p} \cap L^s$ and the exponent bound (87) ensure that $j(u)$ is a well-defined flat current of finite mass, since it acts on test forms as the integration against an $L^1(\mathbb{R}^m, \Lambda_{m-n+1}\mathbb{R}^m)$ function:

$$j(u) = (-1)^n \mathbf{E}^m \llcorner \chi_\Omega u^1 du^2 \wedge \cdots \wedge du^n.$$

As a consequence $Ju \in \mathbf{F}_{m-n}(\mathbb{R}^m)$ as declared in (89). Furthermore for $p \geq \frac{mn}{m+1}$ the constraint (87) is satisfied with the Sobolev exponent p^* in place of s, hence definition 3.1.1 makes sense for $u \in W^{1,p}$ in this range of summability. In [**BN11**] the authors showed that Ju can be defined in the space $W^{1-\frac{1}{m},m}(\mathbb{R}^m)$, which contains $L^s \cap \dot{W}^{1,p}$ for every s, p as in (87). This extension exploits the trace space nature of $W^{1-\frac{1}{m},m}$, expressing Ju as a boundary integral in $\mathbb{R}^m_+$.

Finally in the special situation $n = 1$ the minimal requirement to give meaning to (88) is $u \in L^1(\Omega)$, and the Jacobian in Ω reduces to the distributional derivative $Ju = -\partial(\mathbf{E}^m \llcorner u)$: for any $\omega \in \mathscr{D}^{m-n}(\mathbb{R}^m)$ with $\mathrm{spt}(\omega) \subset \Omega$

$$\Big\langle Ju, \sum_i (-1)^{i-1} \omega_i \widehat{dx^i} \Big\rangle = -\sum_i \int_\Omega u \frac{\partial \omega_i}{\partial x^i} dx = \sum_i \langle D_i u, \omega_i \rangle. \tag{90}$$

Regarding the convergence properties of these currents, we note the following:

Proposition 3.1.2. *Let $u_h, u \in \dot{W}^{1,p}(\Omega, \mathbb{R}^n) \cap L^s(\Omega, \mathbb{R}^n)$ satisfy*

- *$u_h \to u$ in $L^s(\Omega, \mathbb{R}^n)$,*
- *$\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega, \mathbb{R}^{n\times m})$.*

Then $\mathbf{F}^{\mathrm{loc}}_\Omega(Ju_h - Ju) \to 0$.

PROOF. Recall (ϱ_i) is the sequence of Lipschitz cutoff functions as in (42). Let us rewrite the difference $u_h^1 du_h^2 \wedge \cdots \wedge du_h^n - u^1 du^2 \wedge \cdots \wedge du^n$ in the following way:

$$\begin{aligned} & u_h^1 du_h^2 \wedge \cdots \wedge du_h^n - u^1 du^2 \wedge \cdots \wedge du^n = \\ &= (u_h^1 - u^1) du_h^2 \wedge \cdots \wedge du_h^n + u^1 \sum_{k=2}^n du_h^2 \wedge \cdots \wedge du_h^{k-1} \wedge d(u_h^k - u^k) \wedge du^{k+1} \wedge \cdots \wedge du^n. \end{aligned}$$

We can actually write each addendum in the last summation as

$$-(u_h^k - u^k) du_h^2 \wedge \cdots \wedge du_h^{k-1} \wedge du^1 \wedge du^{k+1} \wedge \cdots \wedge du^n + d\zeta_h^k,$$

where we set

$$\zeta_h^k = (-1)^{k-2} u^1 (u_h^k - u^k) du_h^2 \wedge \cdots \wedge du_h^{k-1} \wedge du^{k+1} \wedge \cdots \wedge du^n \in L^1(\Omega, \Lambda^{n-2}\mathbb{R}^m). \quad (91)$$

Notice that we can always assume $s \geq p$, hence $\zeta_h^k \in L^1$. To show (91) it is sufficient to approximate both u and u_h in the strong topology with regular functions and apply the Leibniz rule; the same approximation shows that $d\zeta_h^k \in L^1$ and hence $\int_\Omega d\zeta_h^k \wedge d\omega = 0$ for each $\omega \in \mathscr{D}^{m-n}(\Omega)$. In particular if $\mathrm{spt}(\omega) \subset \mathrm{spt}(\varrho_i)$ by the calculations above we can estimate

$$\begin{aligned} |\langle Ju_h - Ju, \omega\rangle| &= |\langle j(u_h) - j(u), d\omega\rangle| \\ &\leq \|d\omega\|_{L^\infty} \sum_{k=1}^n \|u_h^k - u^k\|_{L^s} \|du_h^1\|_{L^p} \cdots \|du_h^{k-1}\|_{L^p} \|du_h^{k+1}\|_{L^p} \cdots \|du_h^n\|_{L^p} \\ &\leq C\, \mathbf{F}(\omega) \left(\sup_h \|\nabla u_h\|_{L^p}\right)^{n-1} \|u_h - u\|_{L^s}. \end{aligned}$$

Taking the supremum on test functions ω with $\mathbf{F}(\omega) \leq 1 + \mathrm{Lip}(\varrho_i)$ we immediately obtain the asserted convergence by (42). □

A natural question is the relation between the summability exponent p and the regularity of the distribution Ju. There is a main difference between $p \geq n$ and $p < n$: if the gradient ∇u has a sufficiently high summability, then Ju is an absolutely continuous measure. In fact let $u_h = u * \rho_h$, where ρ_h is a standard approximation of the identity: since $p \geq n$ the continuous embedding $W^{1,p} \hookrightarrow W^{1,n}_{\mathrm{loc}}$ implies that $u_h \to u$ both in $W^{1,p} \cap L^s$ and $W^{1,n}_{\mathrm{loc}}$. Taking a test form ψ with compact support we can use Proposition 3.1.2 to pass to the limit in the integration by parts formula

$$\langle Ju_h, \psi\rangle = (-1)^n \int_\Omega u_h^1 du_h^2 \wedge \cdots \wedge du_h^n \wedge d\psi = \int_\Omega du_h^1 \wedge du_h^2 \wedge \cdots \wedge du_h^n \wedge \psi,$$

yielding $Ju = \mathbf{E}^m \llcorner du^1 \wedge \cdots \wedge du^n$.

On the other hand when $p < n$ there are several examples of functions whose jacobian is not in L^1: for instance when $m = n$ the "monopole" function $u(x) := \frac{x}{|x|}$ satisfies $Ju = \mathscr{L}^n(B_1)[\![0]\!]$, where $[\![0]\!]$ is the Dirac's mass in the origin. More generally:

EXAMPLE 3.1.3 (Zero homogeneous functions, $m = n$, [**JS02**, 3.2]). Let $\gamma : S^{n-1} \to \mathbb{R}^n$ be smooth and let $u(x) := \gamma(\frac{x}{|x|})$. Then

$$Ju = Area(\gamma)[\![0]\!] \quad (92)$$

where $Area(\gamma)$ is the signed area enclosed by γ.

PROOF. Outside the origin u is smooth and takes values into the $(n-1)$-dimensional submanifold $\gamma(S^{n-1})$, hence $\mathrm{spt}(Ju) \subset \{0\}$. Set $t = |x|$ and $y = \frac{x}{|x|}$: then $d\gamma = \sum_i \frac{\partial u}{\partial x^k} t dy^k$, so

$$t^{n-1} du^2 \wedge \cdots \wedge du^n = d\gamma^2 \wedge \cdots \wedge d\gamma^n \in \Lambda^{n-1} \mathrm{Tan}\, S^{n-1}.$$

Hence the only term of $d\omega$ surviving in the wedge product is $\frac{\partial \omega}{\partial t} dt$. Therefore

$$\begin{aligned}
(-1)^n \int_{\mathbb{R}^n} u^1 du^2 \wedge \cdots \wedge du^n \wedge d\omega &= (-1)^n \int_{\mathbb{R}^n} \frac{\partial \omega}{\partial t} \gamma^1(y) du^2 \wedge \cdots \wedge du^n \wedge dt \\
&= - \int_{\mathbb{R}^n} \frac{\partial \omega}{\partial t} u^1(y) dt \wedge du^2 \wedge \cdots \wedge du^n \\
&= - \int_{\partial B_t} \left(\int_0^{+\infty} \frac{\partial \omega}{\partial t} dt \right) u^1(x) du^2 \wedge \cdots \wedge du^n \\
&= - \int_{\partial B_1} \left(\int_0^{+\infty} \frac{\partial \omega}{\partial t} dt \right) \gamma^1(y) d\gamma^2 \wedge \cdots \wedge d\gamma^n \\
&= \omega(0) \int_{S^{n-1}} \gamma^1(y) d\gamma^2 \wedge \cdots \wedge d\gamma^n. \qquad (93)
\end{aligned}$$

Setting $\Upsilon(t, y) := t\gamma(y)$ the Lipschitz extension to the unit ball $B_1 \subset \mathbb{R}^n$, by Stokes' Theorem (93) equals to

$$\begin{aligned}
\omega(0) \int_{\partial B_1} \Upsilon^1(1, y) d\Upsilon^2 \wedge \cdots \wedge d\Upsilon^n &= \omega(0) \int_{B_1} d\Upsilon^1 \wedge \cdots \wedge d\Upsilon^n \\
&= \omega(0) \int_{B_1} \det(\nabla \Upsilon) dx = \omega(0) \int_{\mathbb{R}^n} \deg(\Upsilon, w, B_1) dw. \qquad (94)
\end{aligned}$$

It is well known that (94) represents the signed area enclosed by the surface $\gamma(S^{n-1})$. $\square$

This example immediately outlines one of the biggest differences with the scalar case. Consider as in [**JS02**] the "eight-shaped" loop in $\mathbb{R}^2$:

$$\gamma(\theta) = \begin{cases} (\cos(2\theta) - 1, \sin(2\theta)) & \text{for } \theta \in [0, \pi], \\ (1 - \cos(2\theta), \sin(2\theta)) & \text{for } \theta \in [\pi, 2\pi]. \end{cases} \qquad (95)$$

and let u be the zero homogeneous extension. γ encloses the union $B_1(-e_1) \cup B_1(e_1)$ with degree $+1$ and -1 respectively: in light of (92) $Ju = 0$. However a left composition, even with a smooth map $F : \mathbb{R}^2 \to \mathbb{R}^2$, generically destroys the cancellation, causing the appearance of a Dirac's mass in 0. Hence the estimate

$$\|J(F \circ u)\| \leq \mathrm{Lip}(F)^2 \|Ju\| \qquad (96)$$

doesn't hold anymore if u is not regular. Note that this phenomenon does not appear for $n = 1$ and $u \in BV(\Omega)$, as Vol'pert chain rule provides exactly the estimate (96) (see [**AFP00**, Theorem 3.96]).

The failure of (96) is related to the validity of a strong coarea formula for jacobians of vector valued maps, namely equation (1.7) in [**JS02**]. The (weak) coarea formula amounts to decompose the current Ju into the superposition of integral currents corresponding to the level sets of u: letting $u_y(x) := \frac{u(x)-y}{|u(x)-y|}$, it is proved in [**JS02**, Theorem 1.2] that if $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ then for $\mathscr{L}^n$-a.e. $y \in \mathbb{R}^n$ $u_y \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty$ and

$$\langle Ju, \omega \rangle = \frac{1}{\mathscr{L}^n(B_1)} \int_{\mathbb{R}^n} \langle Ju_y, \omega \rangle \, dy$$

for every $\omega \in \mathscr{D}^{m-n}(\Omega)$; moreover for every $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ it holds

$$\langle J(F \circ u), \omega \rangle = \frac{1}{\mathscr{L}^n(B_1)} \int_{\mathbb{R}^n} \det \nabla F(y) \langle Ju_y, \omega \rangle \, dy. \tag{97}$$

This last equation can be interpreted as an extension of the chain rule for jacobians of smooth functions, which amounts to Binet's Lemma 2.10.3 thanks to their pointwise nature. Clearly (97) entails

$$\|J(F \circ u)\| \leq \frac{[\mathrm{Lip}(F)]^n}{\mathscr{L}^n(B_1)} \int_{\mathbb{R}^n} \|Ju_y\| \, dy :$$

however, because of some cancellation phenomena like in (95), (96), the strong version of the coarea formula

$$\|Ju\| = \frac{1}{\mathscr{L}^n(B_1)} \int_{\mathbb{R}^n} \|Ju_y\| \, dy \tag{98}$$

might well fail. As proved in [**DL03**, Theorems 13, 14] if $u \in W^{1,p}(\Omega, \mathbb{R}^n) \cap C^0$ for some $p > n-1$ and $\int_{\mathbb{R}^n} \|Ju_y\|(\Omega) \, dy < \infty$ then $F \circ u \in B_nV(\Omega)$ and $J(F \circ u) = \det \nabla F(u) Ju$, which in turn implies (98). The strong coarea formula is also valid if $u \in W^{1,n-1} \cap B_nV$ takes values in S^{n-1}, as shown in [**JS02**, Lemma 4.8]. Once again observe that for $n = 1$ the equality (98) has been proved by Fleming and Rishel to holds for every $u \in BV$, see [**AFP00**, Theorem 3.40]. For a more detailed analysis we refer to [**JS02, DL03, MS95, DP12**].

For later purposes we report the dipole construction, introduced by Brezis, Coron and Lieb in [**BCL86**]: it consists of a map taking values into a sphere which is constant outside a prescribed compact set, its jacobian is the difference of two Dirac's masses and satisfies suitable $W^{1,p}$ estimate. We write $(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and denote by $\mathcal{N} = (0, 1) \in S^{n-1}$ the north pole.

Example 3.1.4 (Dipole, [**BN11**, 2.2]). Let $n \geq 2$, $\nu \in \mathbb{Z}$, $\rho > 0$: there exists a map $f_{\nu,\rho} : \mathbb{R}^n \to S^{n-1}$ with the following properties:

- $f_{\nu,\rho} \equiv \mathcal{N}$ outside $\{|y| + |z| < \rho\}$;
- $f_{\nu,\rho} - \mathcal{N} \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ for every $p < n$ with estimates

$$\|\nabla f_{\nu,\rho}\|_{L^p}^p \leq C_p \, \nu^{\frac{p}{n-1}} \rho^{n-p};$$

- $J f_{\nu,\rho} = \nu \mathscr{L}^n(B_1^n)\big([\![(0, -\rho)]\!] - [\![(0, \rho)]\!]\big)$.

The construction starts from $f_{\nu,\rho}(\cdot, 0) : \mathbb{R}^{n-1} \to S^{n-1}$, which is a smooth map equal to $\mathcal{N}$ outside $B_\rho^{n-1}(0)$ and such that $\deg(f_{\nu,\rho}(\cdot, 0)) = \nu$ (here S^{n-1} is given the orientation $\tau_{S^{n-1}}$ such that the n-vector $\tau_{S^{n-1}} \wedge \frac{x}{|x|}$ is positive). When $\nu = \pm 1$ such map can be assumed to satisfy $\|\nabla_y f_{1,\rho}\|_{L^\infty(\mathbb{R}^{n-1})} \leq C\rho^{-1}$; for general $\nu \in \mathbb{Z}$ by rescaling and translating $f_{\nu,\rho}$ can be constructed by gluing ν copies of $f_{1,r}$ each being constant outside a ball of radius r: comparing the volumes of these ν disjoint balls of radius r to $B_\rho(0)$ we deduce the condition $\rho^{n-1} \sim \nu r^{n-1}$, yielding the pointwise bound

$$\|\nabla_y f_{\nu,\rho}\|_{L^\infty(\mathbb{R}^{n-1})} \leq C\nu^{\frac{1}{n-1}}\rho^{-1}.$$

For $|z| < \rho$ we extend by $f_{\nu,\rho}(y,z) = f_{\nu,\rho}(\frac{\rho y}{\rho-|z|}, 0)$ and we set $f_{\nu,\rho} \equiv \mathcal{N}$ at points $|z| \geq \rho$. Clearly $\operatorname{spt}(Jf_{\nu,\rho}) \subset \{(0,-\rho),(0,+\rho)\}$; moreover since $f_{\nu,\rho}$ is 0-homogeneous in suitable small neighborhoods of $(0,\pm\rho)$, by Example 3.1.3 we get the asserted formula for $Jf_{\nu,\rho}$.

The locality of the dipole construction allows to glue several copies of dipoles to produce interesting examples.

Example 3.1.5 (Finiteness of $\mathbf{F}(Jg)$ does not imply finiteness of $\mathbf{M}(Jg)$). With the help of the family of maps $\{f_{\nu,\rho}\}$ of the previous example we build a map g such that $\mathbf{F}(Jg)$ is finite and Jg has an infinite mass. Choosing a sequence of positive radii (ρ_k) we can glue an infinite number of dipoles along the z axis:

$$g(y,z) = f_{1,\rho_k}(y, z - z_k) \qquad \text{for} \qquad |z - z_k| \leq 2\rho_k,$$

where $z_0 = 0$ and $z_k = 2\sum_{j=0}^k \rho_j$. The function g belongs to $L^\infty \cap W^{1,p}$ provided $\sum_k \rho_k^{n-p} < \infty$: in this case

$$Jg = \mathscr{L}^n(B_1)\sum_k [\![(0, z_k - \rho_k)]\!] - [\![(0, z_k + \rho_k)]\!],$$

hence $\mathbf{F}(Jg) \leq 2\mathscr{L}^n(B_1)\sum_k \rho_k < \infty$ but $\mathbf{M}(Jg) = +\infty$.

More complicated examples, including maps such that Ju is not even a Radon measure, are presented in [**JS02, MS95, ABO05**].

3.2. The space B_nV of Jerrard and Soner

The space of functions of bounded n-variation has been introduced by Jerrard and Soner in the fundamental paper [**JS02**]. In light of the results of chapter 1 we chose to consider our currents in the ambient space $\mathbb{R}^m$, which is a geodesic space, we adopt the following definition, consistent with the local theory of current:

Definition 3.2.1. *$B_nV(\Omega)$ is the space of functions $u \in W^{1,p}(\Omega,\mathbb{R}^n) \cap L^s(\Omega,\mathbb{R}^n)$ such that Ju has current finite mass in Ω, namely there exists a constant C such that for every $i \in \mathbf{N}$*

$$\mathbf{M}(Ju \llcorner \varrho_i) \leq C.$$

Equivalently one can require

$$\sup\{\langle Ju, \omega\rangle : \omega \in \mathscr{D}^{m-n}(\Omega), \sup_{x\in\Omega}\|\omega(x)\| \leq 1\} \leq C.$$

Notice that in this case the action of Ju against test form compactly supported in Ω can be represented as the integration against a $\Lambda_{m-n}\mathbb{R}^m$-valued Radon measure. In turn thanks to Theorem 2.1.10 we can restrict Ju to Ω and obtain a finite mass current in $\mathbb{R}^m$:

$$Ju \llcorner \Omega \in \mathbf{M}_{m-n}(\mathbb{R}^m).$$

To simplify the notations, we will assume that every BnV map belongs to $L^s \cap W^{1,p}$, with p and s as in (87). Among the literature related to this class of maps we underline the works in elasticity by Šverak [**Šve88**] and Müller and Spector [**MS95**], devoted to analyze the regularity properties of B_nV maps, such as the existence of a precise representative, the structure of the singular set and their invertibility; in particular the second paper gives an existence theory for deformations of a material body that allows for cavitation. A powerful variational theory to problems in elasticity has been developed by Giaquinta, Modica and Souček (see [**GMS98**] for a detailed presentation) exploiting techniques from geometric measure theory. In some relevant situations, this latter approach and the one with the distributional Jacobian are equivalent, as shown in [**CDL03**] (see also [**Hen09**] for further developments in this direction).

The following local w^*-convergence statement is an easy improvement of 3.1.2, since every continuous function with compact support can be uniformly approximated by a Lipschitz function with the same L^∞ bound.

Corollary 3.2.2 (Local w^*-convergence in the sense of measures). *Assume the same hypotheses of Proposition 3.1.2. If in addition*

$$(u_h) \subset B_nV(\Omega) \qquad \text{and} \qquad \|Ju_h \llcorner \Omega\|(\mathbb{R}^m) \le C < \infty$$

then $u \in B_nV(\Omega)$ and $Ju_h \llcorner \Omega \overset{*}{\rightharpoonup} Ju \llcorner \Omega$ in the sense of measures in Ω, that is in the dual space $(C^0_c(\Omega, \Lambda^{m-n}\mathbb{R}^m))^*$.

Furthermore by Theorem 2.2.3 we know that if $u \in B_nV(\Omega, \mathbb{R}^n)$ then $Ju \llcorner \varrho_i$ is a normal current, hence $\|Ju \llcorner \varrho_i\| \ll \mathscr{H}^k$. In light of the Example 3.2.3 below (with a trivial extension in case $m > n$) this absolute continuity property is the only possible bound on the Hausdorff dimension of Ju.

3.2.1. Pointwise description of Ju. Regarding the properties of a B_nV function inside its domain Ω, as in the theory of BV functions Ju satisfies a canonical decomposition in three mutually singular parts according to the dimensions (see [**DL02, AFP00, JS02**]):

$$Ju \llcorner \Omega = \nu \cdot \mathscr{L}^m \llcorner \Omega + J^c u + \theta \cdot \mathscr{H}^{m-n} \llcorner (S_u \cap \Omega) \tag{99}$$

where the decomposition is uniquely determined by these three properties:

- $\nu = \frac{dJu}{d\mathscr{L}^m} \in L^1(\Omega, \Lambda_{m-n}\mathbb{R}^m)$ is the Radon-Nikodym derivative of $Ju \llcorner \Omega$ with respect to $\mathscr{L}^m$;
- $\|J^c u\|(F) = 0$ whenever $\mathscr{H}^{m-n}(F) < \infty$;
- $\theta \in L^1(\Omega, \Lambda_{m-n}\mathbb{R}^m, \mathscr{H}^{m-n})$ is a $\mathscr{H}^{m-n}$-measurable function and $S_u \subset \Omega$ is σ-finite w.r.t. $\mathscr{H}^{m-n}$.

The intermediate measure $J^c u$ is known as the Cantor part of Ju.

Example 3.2.3 (Summability exponent p versus $\dim_{\mathscr{H}} \operatorname{spt}(Ju)$, [**Mül93**, Theorem 5.1]). For every $\alpha \in [0, n]$ there exists a continuous B_nV map

$$u_\alpha \in C^0(\mathbb{R}^n, \mathbb{R}^n) \cap \bigcap_{p<n} W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$$

such that Ju_α is a nonnegative Cantor measure satisfying

$$c\mathscr{H}^\alpha \llcorner \operatorname{spt}(Ju_\alpha) \le Ju_\alpha \le C\mathscr{H}^\alpha \llcorner \operatorname{spt}(Ju_\alpha)$$

for some $c, C > 0$. In particular $\operatorname{spt}(Ju)$ has Hausdorff dimension α.

Hence no bound on p is sufficient to constrain the singularity of Ju. Adding $m-n$ dummy variables to the domain the same examples show α can range in the interval $[m-n, m]$ regardless how close p is to n.

The set S_u is unique up to $\mathscr{H}^{m-n}$-negligible sets, and can be characterized by

$$S_u := \left\{ x \in \Omega : \limsup_{\rho \downarrow 0} \frac{\|Ju\|(B_\rho(x))}{\rho^{m-n}} > 0 \right\}.$$

Moreover thanks to Theorem 2.8.5 it has been shown in [**DL02**] that S_u is countably $\mathscr{H}^{m-n}$-rectifiable and that for $\mathscr{H}^{m-n}$-a.e. $x \in S_u$ the multivector $\theta(x)$ is simple and it orients the approximate tangent space $\operatorname{Tan}^{(m-n)}(S_u, x)$.

We now prove a result contained in [**Mül90**] and [**DLG10**]:

THEOREM 3.2.4. *Let $u \in L^s \cap W^{1,p}(\Omega, \mathbb{R}^m)$ be a B_nV map and let ν be the density of the absolutely continuous part of the distributional Jacobian Ju with respect to the Lebesgue measure:*

$$Ju \llcorner \Omega = \nu \mathscr{L}^m \llcorner \Omega + [Ju]^s \;=\; [Ju]^a + [Ju]^s. \tag{100}$$

Then $\nu(x) = \det \nabla u(x)$ for $\mathscr{L}^m$-almost every $x \in \Omega$.

The theorem can be generalized to the case $m > n$.

THEOREM 3.2.5. *If $u \in L^s \cap W^{1,p}(\Omega, \mathbb{R}^n)$ is a B_nV map, then $\nu(x) = (e_1 \wedge \cdots \wedge e_m) \llcorner du^1(x) \wedge \cdots \wedge du^n(x)$ for $\mathscr{L}^m$-a.e. $x \in \Omega$.*

Theorem 3.2.4 was originally proved by Müller in [**Mül90**] assuming $u \in W^{1,p} \cap B_nV$ with

$$p \geq \frac{n^2}{n+1} \tag{101}$$

Note that, by Sobolev's embedding, (101) implies that $u \in L^s$ for some s satisfying (87). We here present the proof contained in [**DLG10**], which is valid in the full range of exponents (87). Similarly to [**Mül90**], Theorem 3.2.4 will be proved using a blow up procedure. In order to perform it we will need two preliminary lemmas.

LEMMA 3.2.6. *If $u \in B_nV(B_R, \mathbb{R}^n)$ then for $\mathscr{L}^1$-a.e. $\rho \in (0, R)$:*

$$Ju(B_\rho) = \int_{\partial B_\rho} u^1 du^2 \wedge \cdots \wedge du^n = \int_{\partial B_\rho} \langle u^1 du^2 \wedge \cdots \wedge du^n, \tau \rangle d\mathcal{H}^{n-1}, \tag{102}$$

where τ is the simple $(n-1)$-vector orienting ∂B_ρ as the boundary of B_ρ.

PROOF. Let

$$\varphi_{\delta,r}(x) = \begin{cases} 1 & \text{for} \quad |x| \leq r - \delta, \\ \frac{r-x}{\delta} & \text{for} \quad r - \delta \leq |x| \leq r, \\ 0 & \text{elsewhere.} \end{cases}$$

Let $f(r) := \int_{\partial B_r} u^1 du^2 \wedge \cdots \wedge du^n$: then $f \in L^1([0,1])$ because of (87) and Fubini's theorem. This implies that $\mathscr{L}^1$-a.e. r is a Lebesgue point, that is:

$$\frac{1}{2\delta} \int_{r-\delta}^{r+\delta} |f(s) - f(r)| ds \to 0 \text{ as } \delta \to 0.$$

Note also that

$$\langle Ju, \varphi_{\delta,r}\rangle = \langle j(u), d\varphi_{\delta,r}\rangle = \int -u^1 d\varphi_{\delta,r}(x) \wedge du^2 \wedge \cdots \wedge du^n =$$

$$= \frac{1}{\delta}\int_{r-\delta}^{r} dt \wedge \int_{\partial B_t} u^1 du^2 \wedge \cdots \wedge du^n = \frac{1}{\delta}\int_{r-\delta}^{r} \left(\int_{\partial B_t} u^1 du^2 \wedge \cdots \wedge du^n\right) d\mathscr{L}^1(t),$$

hence at every Lebesgue point

$$\langle Ju, \varphi_{\delta,r}\rangle \to \int_{\partial B_r} u^1 du^2 \wedge \cdots \wedge du^n;$$

on the other hand by Lebesgue dominated convergence theorem $\langle Ju, \varphi_{\delta,r}\rangle \to Ju(B_r)$, that proves the proposition. □

DEFINITION 3.2.7. *Let $u \in B_nV(\Omega, \mathbb{R}^n)$ and let $x_0 \in B_R \subset \Omega$. We define*

$$u_\varepsilon(y) := \frac{u(x_0 + \varepsilon y) - u(x_0)}{\varepsilon}.$$

LEMMA 3.2.8. *Let u be as above and set $\delta_a(x) := a(x - x_0)$. Then*

$$Ju_\varepsilon = \frac{1}{\varepsilon^n}\delta_{\frac{1}{\varepsilon}\#}Ju.$$

PROOF. Let $\phi \in C_c^\infty(B_1)$ be a test function.

$$\langle Ju_\varepsilon, \phi\rangle = \langle j(u_\varepsilon), d\phi\rangle =$$

$$= (-1)^n \int_{B_1} \frac{u^1(x_0+\varepsilon y) - u^1(x_0)}{\varepsilon} \det(\nabla u^2(x_0+\varepsilon y), \ldots, \nabla u^n(x_0+\varepsilon y), \nabla\phi(y))dy$$

$$= (-1)^n \int_\Omega \frac{u^1(x) - u^1(0)}{\varepsilon^{n+1}} \det\left(\nabla u(x), \nabla\phi\left(\frac{x-x_0}{\varepsilon}\right)\right) dx$$

$$= \frac{1}{\varepsilon^n}\left\langle j(u), d\left[\phi\left(\frac{x-x_0}{\varepsilon}\right)\right]\right\rangle \frac{1}{\varepsilon^n}\left\langle Ju, \phi\left(\frac{x-x_0}{\varepsilon}\right)\right\rangle.$$

□

In particular, taking the supremum over $\{\phi \in C_c^\infty(B_1) : \|\phi\|_\infty \le 1\}$ we have

$$Ju_\varepsilon = \frac{1}{\varepsilon^n}\delta_{\frac{1}{\varepsilon}\#}Ju, \quad \|Ju_\varepsilon\| = \frac{1}{\varepsilon^n}\delta_{\frac{1}{\varepsilon}\#}\|Ju\|.$$

Note that the Radon-Nikodym decomposition commutes with the push forward:

$$[Ju_\varepsilon]^a = \frac{1}{\varepsilon^n}\delta_{\frac{1}{\varepsilon}\#}[Ju]^a, \quad [Ju_\varepsilon]^s = \frac{1}{\varepsilon^n}\delta_{\frac{1}{\varepsilon}\#}[Ju]^s.$$

This property, together with the previous observation, allows to conclude that $\forall r > 0$

$$\|[Ju_\varepsilon]^s\|(B_r(x_0)) = \frac{\|[Ju]^s\|(B_{\varepsilon r}(x_0))}{\varepsilon^n}. \tag{103}$$

We are now ready to prove Theorem 3.2.4. To simplify the notation we denote by (u_h) the sequence $(u_{\varepsilon_h}), \varepsilon_h = \frac{1}{h}$. We wish to apply formula (102) to the blow-up sequence (u_h) around a "good" point x_0

$$Ju_h(B_\rho(x_0)) = \int_{\partial B_\rho(x_0)} u_h^1 du_h^2 \wedge \cdots \wedge du_h^n \tag{104}$$

and let h tend to ∞ to obtain

$$\nu(x_0)|B_\rho| = \int_{\partial B_\rho(x_0)} (L{\cdot}x)^1 L^2 \wedge \cdots \wedge L^n = \int_{\partial B_\rho(x_0)} (L{\cdot}x)^1 \mathrm{cof}(L)^1_k {\cdot} \eta^k = \det(L)|B_\rho|, \tag{105}$$

where $L := \nabla u(0)$ and η is the exterior unit normal to ∂B_ρ.

Step 1: by standard properties of Sobolev functions (see [**AFP00**], [**Fed69**], [**EG92**]), $\mathscr{L}^n$-almost every point $x_0 \in \Omega$ satisfies the following properties:

(a)

$$\lim_{r \downarrow 0} \frac{1}{r^n} \left\{ \|[Ju]^s\|(B_r(x_0)) + \int_{B_r(x_0)} |\nu(x) - \nu(x_0)|\, dx \right\} \;=\; 0\,;$$

(b) ∇u is approximately continuous at x_0 and in particular

$$\lim_{r \downarrow 0} \frac{1}{r^n} \int_{B_r(x_0)} |\nabla u(x) - \nabla u(x_0)|^p dx \;=\; 0\,.$$

From now on we fix x_0 satisfying (a) and (b) and, without loss of generality, we assume $x_0 = 0$. Observe first of all that condition (a) and equation (103) imply

$$Ju_h(B_r(0)) = h^n Ju(B_{\frac{r}{h}}(0)) = o(1) + h^n \int_{B_{\frac{r}{h}}(0)} \nu(y)dy \to \nu(0)|B_r| \qquad \forall r > 0\,. \tag{106}$$

Step 2: We observe that, being (u_h) a sequence, there is a set of radii $\rho \in (0,1)$ of full measure such that (102) holds for every h. Moreover by (b), using Fubini's and Fatou's Theorems, for almost every ρ there exists a subsequence (not relabeled and possibly depending on ρ) such that $\nabla u_h \to L := \nabla u(0)$ in $L^p(\partial B_\rho)$. We fix now a radius ρ such that all the properties above hold and we do not relabel the relevant subsequence. Hence

$$du_h^2 \wedge \cdots \wedge du_h^n \to L^2 \wedge \cdots \wedge L^n \tag{107}$$

in $L^{\frac{p}{n-1}}(\partial B_\rho)$, since it is sufficient to rewrite the difference as

$$du_h^2 \wedge \cdots \wedge du_h^n - L^2 \wedge \cdots \wedge L^n = \sum_i L^2 \wedge \cdots \wedge (du_k^i - L^i) \wedge \cdots \wedge du_k^n\,. \tag{108}$$

Suppose first of all that $p > n-1$. Then by the Poincaré's inequality and the Sobolev embedding theorem, the sequence (u_h) is equicontinuous, with the estimate

$$\|u_h - L \cdot x - C_h\|_{C^{0,\alpha}(\partial B_\rho)} \le C\|\nabla u_h - L\|_{L^p(\partial B_\rho)} \to 0.$$

Here C_h is the average of u_h on ∂B_ρ. Since $\int_{\partial B_\rho} du_h^2 \wedge \ldots \wedge du_h^n = 0$, we conclude

$$Ju_h(B_\rho) = \int_{\partial B_\rho} u_h^1 du_h^2 \wedge \cdots \wedge du_h^n = \int_{\partial B_\rho} \left(u_h^1 - C_h^1\right) du_h^2 \wedge \cdots \wedge du_h^n$$

$$\to \int_{\partial B_\rho} (L\cdot x)^1 L^2 \wedge \cdots \wedge L^n = \det(L)|B_\rho|.$$

In the borderline case $p = n-1$, the convergence (107) is improved to the local Hardy space $\mathfrak{h}^1(\partial B_\rho)$ because of the Coifman-Lions-Meyer-Semmes estimate (see [**CLMS93**]):

$$\|\langle dv^2 \wedge \cdots \wedge dv^n, \tau\rangle\|_{\mathfrak{h}^1(\partial B_\rho)} \le C\|dv^2\|_{L^{n-1}(\partial B_\rho)} \cdots \|dv^n\|_{L^{n-1}(\partial B_\rho)}\,.$$

Indeed recall the definition of local Hardy space is given via a compactly supported test function of class C^1 ϕ: letting $\phi_t(x) = \frac{1}{t^n}\phi(\frac{x}{t})$ we have

$$\mathfrak{h}^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : \sup_{0<t<1} |(\phi_t * f)(x)| \in L^1\}.$$

The local Hardy space is endowed with the norm $\|f\|_{\mathfrak{h}^1(\mathbb{R}^n)} := \|\phi_t * f\|_{L^1(\mathbb{R}^n)}$. Since this class is closed under truncation by a compactly supported test function, as well as composition via diffeomorphisms, $\mathfrak{h}^1$ can be defined also for functions defined on compact manifolds via a partition of unity, see [**Gol79**]. Moreover the classical duality $\mathcal{H}^1(\mathbb{R}^n)^* = BMO(\mathbb{R}^n)$ established by Fefferman (see [**Ste93**], chapter IV, and [**Gol79**]) has a local counterpart $\mathfrak{h}^1(\mathbb{R}^n)^* = \mathfrak{bmo}(\mathbb{R}^n)$ where

$$\mathfrak{bmo}(\mathbb{R}^n) = \left\{ g \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n,\, r<1} \fint_{B_r(x)} \big|g - \fint_{B_r(x)} g\big| < \infty,\ \sup_{x\in\mathbb{R}^n,\, r\ge 1} \fint_{B_r(x)} |g| < \infty \right\}.$$

(The norm in $\mathfrak{bmo}$ is the largest of the two suprema). In particular we have that

$$\left|\int fg\right| \le C\|f\|_{\mathfrak{h}^1}\|g\|_{\mathfrak{bmo}}$$

whenever $fg \in L^1$.

Therefore if $p = n-1$ we use the John-Nirenberg embedding and Poincaré's inequality for the sequence $v_h := u_h - C_h - L\cdot x$: if $r < 1$ then $\mathscr{H}^{n-1}(B^{n-1}_r(x)) \ge cr^n$ for some constant c, thus we can bound the $\mathfrak{bmo}$ norm

$$\fint_{B^{n-1}_r} |v_h - \fint_{B^{n-1}_r} v_h| \le \left(\fint_{B^{n-1}_r} |v_h - \fint_{B^{n-1}_r} v_h|^n\right)^{\frac{1}{n}} \le C\left(\int_{B^{n-1}_r} |\nabla v_h|^n\right)^{\frac{1}{n}},$$

and if $r \ge 1$ since $\fint_{\partial B_\rho} v_h = 0$ we have

$$\fint_{B^{n-1}_r} |v_h| \le C\fint_{\partial B_\rho} |v_h| = C\fint_{\partial B_\rho} |v_h - \fint_{\partial B_\rho} v_h| \le C\left(\int_{\partial B_\rho} |\nabla v_h|^n\right)^{\frac{1}{n}}.$$

Thus $\|(u_h - C_h) - L\cdot x\|_{\mathfrak{bmo}(\partial B_\rho)} \to 0$. On the other hand the Coifman-Lions-Meyer-Semmes estimates holds even more in the local setting, since the maximal function needs to be estimated only for averages at scales bounded above by 1, hence

$$\|du^2_h \wedge \cdots \wedge du^n_h - L^2 \wedge \cdots \wedge L^n\|_{\mathfrak{h}^1(\partial B_\rho)} \to 0.$$

We thus infer that

$$\int_{\partial B_\rho} (u^1_h - C^1_h)\, du^2_h \wedge \cdots \wedge du^n_h \to \int_{\partial B_\rho} (L\cdot x)^1 L^2 \wedge \cdots \wedge L^n = \det(L)|B_\rho|.$$

□

3.2.2. Slicing Theorem and SB_nV functions. We aim to apply the slicing operation to $Ju \in \mathbf{F}_{m-n}(\Omega)$ in the special case $\ell = m - n$, thus reducing ourselves to 0-dimensional slices; moreover we want to relate these slices to the Jacobian of the restriction $J(u|_{\pi^{-1}(x)})$. In [**DL02**], the author extended a classical result on restriction of BV functions (see [**AFP00**, Section 3.11]) to Jacobians:

THEOREM 3.2.9 (Slicing). *Let $u \in W^{1,p} \cap L^s(\Omega, \mathbb{R}^n)$ and let $\pi \in \mathbf{O}_{m-n}$. Then for $\mathscr{L}^{m-n}$-almost every $x \in \mathbb{R}^{m-n}$*

$$\langle Ju, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(Ju^x), \tag{109}$$

where $u^x = u \circ i^x$. Moreover $u \in B_nV(\Omega)$ if and only if for every $\pi \in \mathbf{O}_{m-n}$ the following two conditions hold:

$$\text{(i)} \qquad u^x \in B_nV(\Omega^x) \quad \textit{for} \quad \mathscr{L}^{m-n}\textit{-almost every } x \in \mathbb{R}^{m-n},$$

$$\text{(ii)} \qquad \int_{\pi(\Omega)} \|Ju^x\|(\Omega^x)\, d\mathscr{L}^{m-n}(x) < \infty.$$

If $u \in B_nV(\Omega, \mathbb{R}^n)$ the slicing property (109) *holds separately for the absolutely continuous part, the Cantor part and the Jump part of Ju, namely:*

- $\langle J^a u, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(J^a u^x)$,
- $\langle J^c u, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(J^c u^x)$,
- $\langle J^s u, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(J^s u^x)$.

Thanks to Theorem 3.2.9 we can now prove Corollary 3.2.5:

PROOF OF THEOREM 3.2.5. Set $\pi(x) = (x^1, \dots, x^{m-n})$, and $y = (x^{m-n+1}, \dots, x^n)$, then, by Theorem 3.2.4 and Theorem 3.2.9,

$$\langle [Ju]^a, f d\pi\rangle = \langle [Ju]^a \llcorner d\pi, f\rangle = \int_{\mathbb{R}^{m-n}} \langle [Ju]^a, \pi, x\rangle(f) d\mathscr{L}^{m-n}(x) =$$

$$= \int_{\mathbb{R}^{m-n}} \left(\int_{\mathbb{R}^n} (-1)^{(m-n)n} \det(\nabla_y u(x,y)) f(x,y) d\mathscr{L}^n(y) \right) d\mathscr{L}^{m-n}(x) =$$

$$= \int_{\mathbb{R}^m} \det(\nabla_y u(x,y)) f(x,y) dy \wedge d\pi = \int_{\mathbb{R}^m} f\langle e_1 \wedge \dots \wedge e_m \llcorner du^1 \wedge \dots \wedge du^n, d\pi\rangle d\mathscr{L}^m.$$

It's then sufficient to write a generic form as $\omega = \sum_I f_I dx^I$. It is easy to show that for every $A \in GL(n, \mathbb{R})$ it holds

$$[J(u \circ A)] = \deg(A) \cdot (A^{-1}_\#)[Ju]$$

where $\deg(A)$ is the sign of the determinant of A. If then I is a multiindex of length $m-n$, and $\pi^I(x) = (x^{i_1}, \dots, x^{i_{m-n}})$, we let A be a permutation matrix satisfying $\pi = \pi^I \circ A$. Then

$$\langle [Ju]^a, f_I d\pi^I\rangle = \deg(A)\langle [J(u \circ A)]^a \llcorner d\pi, f_I \circ A\rangle =$$

$$= \deg(A) \int_{\mathbb{R}^m} f_I \circ A \langle e_1 \wedge \dots \wedge e_m \llcorner d(u^1 \circ A) \wedge \dots \wedge d(u^n \circ A), d(\pi^I \circ A)\rangle d\mathscr{L}^m =$$

$$= \deg(A) \int_{\mathbb{R}^m} A^*(f_I du^1 \wedge \dots \wedge du^n \wedge d\pi^I) = \int_{\mathbb{R}^m} f_I du^1 \wedge \dots \wedge du^n \wedge d\pi^I .$$

This concludes the proof. $\square$

In analogy with the SBV theory we pose the following definition:

DEFINITION 3.2.10. *We denote by* $SB_nV(\Omega)$ *the set of* $B_nV(\Omega)$ *functions such that* $J^c u = 0$.

Thus the jacobian of a SB_nV map consists of an absolutely continuous part plus a lower dimensional part concentrated on a countably $\mathscr{H}^{m-n}$-rectifiable set S_u. The space SB_nV enjoys a closure property proved in [**DL02**]:

THEOREM 3.2.11 (Closure Theorem for SB_nV). *Let us consider* u, $u_h \in B_nV(\Omega)$ *and suppose that*

(a) $u_h \to u$ *strongly in* $L^s(\Omega, \mathbb{R}^n)$ *and* $\nabla u_h \rightharpoonup \nabla u$ *weakly in* $L^p(\Omega, \mathbb{R}^{n\times m})$,

(b) *if we write*

$$Ju_h \llcorner \Omega = \nu_h \cdot \mathscr{L}^m + \theta \cdot \mathscr{H}^{m-n} \llcorner S_{u_h}$$

then $|\nu_h|$ *are equiintegrable in* Ω *and* $\mathscr{H}^{m-n}(S_{u_h}) \le C < \infty$.

Then $u \in SB_nV(\Omega, \mathbb{R}^n)$ *and*

$$\nu_h \rightharpoonup \nu \text{ weakly in } L^1(\Omega, \Lambda_{m-n}\mathbb{R}^m), \quad \mathscr{H}^{m-n}(S_u) \le \liminf_h \mathscr{H}^{m-n}(S_{u_h}).$$

3.3. A new space of functions: GSB_nV

We are now interested in broadening the class B_nV to include vector valued maps satisfying a weaker control than the mass bound: this lack of control on $\mathbf{M}(Ju)$ already appears in Theorem 3.2.11 when we require *a priori* the limit u to be in B_nV. We relax our energy by considering a mixed control of Ju, where we bound part of the current Ju with its size. The idea is to apply Definition 2.6.1 of size to the current Ju, which we recall is available also for currents with infinite mass, borrowing some ideas already used by Hardt and Rivière in [**HR03**], Almgren [**Alm86**], Federer [**Fed86**].

We observe that the definition of size in the metric space context of chapter 2 can be slightly modified in the Euclidean setting, replacing the family of slicing maps $\pi \in \mathrm{Lip}_1(\Omega, \mathbb{R}^k)$ in the supremum (44) with the subfamily of orthogonal projections. When the ambient space is Euclidean, the rectifiability and lower semicontinuity results obtained there, as well as the characterization of μ_T in terms of $set(T)$ can be readily proved using only the subset of orthogonal projections. Similarly the proof of the compactness Theorem 3.4.1 below as well as the Γ-convergence Theorem 5.2.8 can be adapted to the broad definition of size with the help of the coarea formula. The question of the equality between the two definitions is however interesting and seems non trivial to address even in $\mathbb{R}^m$. Clearly it holds $\mu_{T,\mathbf{O}_k} \le \mu_T$, hence by the representation Theorem 2.10.7 if T has finite size then $\mathscr{H}^k(set(T)_{\mathbf{O}_k} \setminus set(T)) = 0$. The equivalence would then be established if we knew that $\mathbf{S}_{\mathbf{O}_k}(T) < \infty$ implies $\mathbf{S}(T) < \infty$ and $\mathscr{H}^k(set(T) \setminus set(T)_{\mathbf{O}_k}) = 0$. Regarding the second condition if T had finite mass then the restriction $T \llcorner (set(T) \setminus set(T)_{\mathbf{O}_k})$ would be a rectifiable current and equation (11) applied with $\pi \in \mathbf{O}_k$ would imply that the current is zero. Unfortunately we are not able to prove such statements for general flat currents with infinite mass.

The space of generalized functions of bounded higher variation is described in terms of the decomposition (99): we relax the requirement on the addendum of lower dimension and require only a size bound, retaining the mass bound on the diffuse part. Following the previous definitions we consider the Sobolev functions u whose jacobian can be split in the sum of two parts, R and T, such that:

- R has finite mass and $\|R\|(F) = 0$ whenever $\mathscr{H}^{m-n}(F) < \infty$;
- T is a flat chain of finite size.

We can also require $\|R\|$ to be absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^m$: these two possible choices lead to the following definitions:

DEFINITION 3.3.1 (Special functions of bounded higher variation). *The space of generalized functions of bounded higher variation is defined by*

$$\begin{aligned} GB_nV(\Omega) = \big\{u \in W^{1,p}(\Omega, \mathbb{R}^n) \cap L^s(\Omega, \mathbb{R}^n) : \exists R, T \in \mathbf{F}_{m-n}(\mathbb{R}^m),& \\ \operatorname{spt}(R) \cup \operatorname{spt}(T) \subset \overline{\Omega},\ Ju \llcorner \Omega = R + T,& \\ \mathbf{M}(R) + \mathbf{S}(T) < \infty,\ \|R\|(F) = 0\, \forall F : \mathscr{H}^{m-n}(F) < \infty\big\}.& \quad (110) \end{aligned}$$

Analogously, the space of generalized special functions of bounded higher variation is defined by

$$\begin{aligned} GSB_nV(\Omega) = \big\{u \in W^{1,p}(\Omega, \mathbb{R}^n) \cap L^s(\Omega, \mathbb{R}^n) : \exists R, T \in \mathbf{F}_{m-n}(\mathbb{R}^m),& \\ \operatorname{spt}(R) \cup \operatorname{spt}(T) \subset \overline{\Omega},\ Ju \llcorner \Omega = R + T,& \\ \mathbf{M}(R) + \mathbf{S}(T) < \infty,\ \|R\| \ll \mathscr{L}^m\big\}.& \quad (111) \end{aligned}$$

In accordance with the classical BV theory we denote $S_u := set(T_u)$.

This space is clearly meant to mimic the aforementioned SB_nV class. In particular, thanks to the slicing properties of flat currents and the definition of size, the slicing theorem for $GSB_nV(\Omega)$ can be stated in the following way:

$$u \in GSB_nV(\Omega) \Longleftrightarrow \forall \pi \in \mathbf{O}_{m-n} \quad \text{and for } \mathscr{L}^{m-n}\text{-a.e. } x \in \mathbb{R}^{m-n}$$

$$\begin{cases} u^x \in GSB_nV(\Omega^x), \\ \int_{\pi(\Omega)} \mathbf{M}(R_{u^x}) + \mathbf{S}(T_{u^x})\, d\mathscr{L}^{m-n}(x) < \infty. \end{cases} \quad (112)$$

In the following propositions we describe some useful properties of the class $GSB_nV(\Omega)$.

LEMMA 3.3.2. *If $m = n$ then $GSB_nV(\Omega) = SB_nV(\Omega)$.*

PROOF. The statement relies on Theorem 2.6.3, whose statement is reminiscent of Schwartz lemma for distributions, namely the fact that a flat 0-current of finite size coincides with a finite sum of Dirac masses, and in particular it has finite mass. Since

$$T = Ju \llcorner \Omega - R$$

has finite mass, hence $\mathbf{M}(Ju \llcorner \Omega) \le \mathbf{M}(R) + \mathbf{M}(T) < \infty$ which means $u \in B_nV(\Omega)$. □

Since the Radon-Nikodym decomposition of a measure into the sum of an absolutely continuous and a singular part is unique, by slicing also R and T are uniquely determined in the decomposition. Therefore we can write $Ju \llcorner \Omega = R_u + T_u$, so that S_u is a well defined set.

A very well known space of functions implemented in the calculus of variations is $GSBV$. The main idea behind this space, introduced in [**DGA89**] (see also [**AFP00**, Section 4.5]), is to consider functions u whose derivative Du loses any kind of local integrability, but nevertheless retains some of the structure of SBV functions. Setting $u^N := (-N) \vee u \wedge N$ for every $N > 0$ we define

$$GSBV(\Omega) = \{u : \Omega \to \mathbb{R} \text{ Borel} : u^N \in SBV(\Omega) \text{ for all integers } N > 0\}.$$

The countable set of truncation given by $N \in \mathbf{N}$ is enough to provide the existence of an approximate differential $\nabla^* u$ and of a countably $\mathscr{H}^{m-1}$-rectifiable singular set S_u^* such that for every N

$$\|Du^N\| \le |\nabla^* u|\chi_{\{|u|\le N\}}\mathscr{L}^m + 2N\mathscr{H}^{m-1} \llcorner S_u^*.$$

Moreover an analog of the slicing theorem for BV function is available also in $GSBV$, see [**AFP00**, Proposition 4.35].

Proposition 3.3.3 (Comparison between GSB_1V and $GSBV$). *A function u belongs to $GSB_1V(\Omega)$ if and only if $u \in GSBV(\Omega)$, $u \in L^1(\Omega)$, $\nabla^* u \in L^1(\Omega, \mathbb{R}^n)$ and $\mathscr{H}^{m-1}(S_u^*) < \infty$.*

Proof. With abuse of notation, motivated by (90) we identify for scalar functions the action of Ju on $\mathscr{D}^{m-1}(\Omega)$ with the action of the distributional derivative Du on $C_c^\infty(\Omega, \mathbb{R}^n)$ (see the map $\mathbf{D}^{m-1}$ in [**Fed69**, 1.5.2]). Consider first the case $m = 1$. Let $u \in GSB_1V(\Omega)$: writing $R_u = \rho\mathscr{L}^1$ and $T_u = \sum_{k=1}^{\mathbf{S}(T_u)} a_k [\![x_k]\!]$, thanks to (90) we know that for $\omega \in \mathscr{D}^0(\Omega)$

$$\langle Du, \omega\rangle = \int_\Omega \rho\omega\, dx + \sum_{k=1}^{\mathbf{S}(T_u)} a_k \omega(x_k).$$

This proves that $u \in SBV(\Omega)$, $S_u \subset set(T_u)$ and $u'(x) = \rho(x)$ almost everywhere. In particular for $N > 0$ fixed

$$\|Du^N\| \le |\rho|\mathscr{L}^1 + 2N\mathscr{H}^{m-1} \llcorner set(T_u). \tag{113}$$

For $m \ge 2$ the slicing Theorem 3.2.9 applied to a coordinate projection onto a hyperspace implies that almost every slice u^x is in $GSB_1V(\Omega^x)$, hence for every $N > 0$ the estimate (113) holds for u^x. Integrating back we have $\|Du^N\|(\Omega) < \infty$, hence $u \in GSBV(\Omega)$.

On the other if $u \in GSBV(\Omega) \cap L^1(\Omega)$ we know that $Du^N \overset{*}{\rightharpoonup} Du$ in the sense of distributions, and also in the flat norm, since the weak derivative is a distribution of order 1. Moreover $\nabla u^N \to \nabla^* u$ strongly in L^1, hence also in the flat norm. Therefore the jump parts also converge to some flat T_u:

$$D^j u^N \overset{\mathbf{F}_\Omega^{\mathrm{loc}}}{\to} T_u \in \mathbf{F}_{m-1}(\mathbb{R}^m).$$

Recall that for $v \in BV$ the jump part of the derivative Dv can be expressed in terms of the approximate upper and lower limits $v_\pm$ and of the approximate tangent $(m-1)$-vector τ in the following way:

$$D^j v = (v_+ - v_-)\tau\, \mathscr{H}^{m-1} \llcorner S_v. \tag{114}$$

Hence if $m = 1$ then $\mathscr{H}^0(\mathrm{spt} T_u) \le \lim_N \mathscr{H}^0(S_{u^N}) \le \mathscr{H}^0(S_u^*)$; in the general case can be achieved using the slicing Theorem 3.2.9 and Proposition [**AFP00**, 4.35]. $\square$

3.3.1. Some examples. The following observation shows that when $n \geq 2$ it is hopeless to rely on truncation to get mass bounds for Ju.

EXAMPLE 3.3.4 (L^∞ bound for $n \geq 2$). For $n \geq 2$ let $\gamma_k : S^{n-1} \to S^{n-1}$ be a smooth map with degree k, and call u_k its zero homogeneous extension to $\mathbb{R}^n$. Then $\|u_k\|_{L^\infty} \leq 1$ but by Example 3.1.3 $Ju = k\mathscr{L}^n(B_1)[\![0]\!]$.

On the contrary for $n = 1$ and $u \in BV(\Omega)$ the approximate upper and lower limits $u_\pm$ of u characterize the singular set: $S_u = \{x \in \Omega : u_+(x) > u_-(x)\}$. Equation (114) implies that an L^∞ bound on u together with a size bound $\mathscr{H}^{m-1}(S_u) < \infty$ gives a mass bound on Du.

We now adapt the construction in 3.1.5, building a map whose jacobian has infinite mass but finite size:

EXAMPLE 3.3.5 ($u \in GSB_nV(\mathbb{R}^m)$, $\mathbf{S}(Ju) < \infty$ but $\mathbf{M}(Ju) = \infty$). Set $m = n+1$ and let us write (x, y, z) for the coordinates of $\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$. Besides $\nu \in \mathbb{Z}$ and $\rho > 0$ fix an extra parameter $R \geq \rho$. We extend the function $f_{\nu,\rho}(y, z)$ of Example 3.1.4 to $\mathbb{R}^{n+1}$ by

$$h_{\nu,\rho,R}(x, y, z) = \begin{cases} f_{\nu,\rho}\left(\frac{Ry}{R-|x|}, \frac{Rz}{R-|x|}\right) & \text{for} \quad |x| < R, \\ \mathcal{N} & \text{for} \quad |x| \geq R. \end{cases}$$

Clearly $h_{\nu,\rho,R} \neq \mathcal{N}$ in the set $\{|x|/R + |y|/\rho + |z|/\rho < 1\}$; by simmetry we can do the computations in $\{x < 0\}$. Let us estimate the partial derivatives:

$$\begin{aligned} &\left|\frac{\partial h_{\nu,\rho,R}}{\partial x}(x, y, z)\right| \leq \frac{R(|y|+|z|)}{(R+x)^2}|\nabla f_{\nu,\rho}|\left(\frac{Ry}{x+R}, \frac{Rz}{x+R}\right) \leq \frac{\rho}{x+R}|\nabla f_{\nu,\rho}|\left(\frac{Ry}{x+R}, \frac{Rz}{x+R}\right), \\ &|\nabla_{y,z} h_{\nu,\rho,R}| \leq \frac{R}{x+R}|\nabla f_{\nu,\rho}|\left(\frac{Ry}{x+R}, \frac{Rz}{x+R}\right). \end{aligned}$$

Since $\rho \leq R$

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}} |\nabla h_{\nu,\rho,R}|^p \, dxdydz \\ &\quad \leq 2(n+1) \int_{-R}^0 \int_{\{|y|/\rho+|z|/\rho<(x+R)/R\}} \left(\frac{R}{x+R}\right)^p |\nabla f_{\nu,\rho}|^p \left(\frac{Ry}{x+R}, \frac{Rz}{x+R}\right) dydz \, dx \\ &\quad \leq 2(n+1) \int_0^R \left(\frac{R}{x}\right)^{p-n} \int_{\{|y|+|z|<\rho\}} |\nabla f_{\nu,\rho}(y, z)|^p \, dydz \, dx \\ &\quad \leq C_p \, \nu^{\frac{p}{n-1}} \rho^{n-p} R. \end{aligned} \tag{115}$$

Moreover $Jh_{\nu,\rho,R}$ is the integral cycle $\nu \cdot \zeta_\# [\![0, 1]\!]$, where $\zeta : [0, 1] \to \mathbb{R}^{n+1}$ is the following closed curve:

$$\zeta(t) = \begin{cases} (4Rt - R, 0, -4\rho t) & \text{for} \quad t \in [0, \frac{1}{4}], \\ (4Rt - R, 0, 4\rho t - 2\rho) & \text{for} \quad t \in [\frac{1}{4}, \frac{1}{2}], \\ (3R - 4Rt, 0, 4\rho t - 2\rho) & \text{for} \quad t \in [\frac{1}{2}, \frac{3}{4}], \\ (3R - 4Rt, 0, 4\rho - 4\rho t) & \text{for} \quad t \in [\frac{3}{4}, 1]. \end{cases}$$

Since $\mathrm{Lip}(\zeta) \leq CR$ we have $\mathbf{M}(Jh_{\nu,\rho,R}) \leq C\nu R$ and $\mathbf{S}(Jh_{\nu,\rho,R}) \leq CR$. Like in 3.1.4 we glue infinite copies of h_{ν_k,ρ_k,R_k} along the z axis and obtain a map g: the Sobolev norm of g can be estimated by (115):

$$\|\nabla g\|_{L^p}^p \leq C \sum_k \nu_k^{\frac{p}{n-1}} \rho_k^{n-p} R_k$$

and

$$\mathbf{M}(Jg) \leq C \textstyle\sum_k \nu_k R_k,$$
$$\mathbf{S}(Jg) \leq C \textstyle\sum_k R_k.$$

Choosing $\nu_k = k$, $R_k = \frac{1}{k^2}$ and $\rho_k = e^{-k}$ we obtain a S^{n-1}-valued $W^{1,p}$ function constant outside a compact set and whose Jacobian has infinite mass but finite size.

3.3.2. ***Ju*** **and approximate differentiability.** We now extend to GSB_nV the pointwise characterization of the absolutely continuous part of Ju.

PROPOSITION 3.3.6 (*Det* = *det* in the GSB_nV class). *Let $u \in GSB_nV(\Omega)$ and write $Ju \llcorner \Omega = R + T$ as in Definition 3.3.1. Let ∇u be the approximate differential of u. Then*

$$\frac{dR}{d\mathscr{L}^m} = M_n \nabla u \quad \mathscr{L}^m\text{-almost everywhere in } \Omega. \tag{116}$$

PROOF. For the ease of notation let $\nu := \frac{dR}{d\mathscr{L}^m}$. Fix a projection $\pi \in \mathbf{O}_{m-n}$ and let us write the coordinates $z = (x, y)$ accordingly. For a fixed $x \in \mathbb{R}^{m-n}$ we note that the injection i^x and the complementary projection $\pi^\perp|_{\pi^{-1}(x)}$ are one the inverse of the other. Recall the slicing Theorem for general Sobolev functions gives

$$\langle Ju \llcorner \Omega, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(Ju^x \llcorner \Omega). \tag{117}$$

Taking Lemma 3.3.2 into account, for almost every $x \in \mathbb{R}^{m-n}$ it holds $u^x \in B_nV(\Omega^x)$ and $\mathbf{M}(\langle R, \pi, x\rangle)) + \mathbf{S}(\langle T, \pi, x\rangle) < \infty$, hence (12) gives

$$\langle Ju \llcorner \Omega, \pi, x\rangle = \nu(x, \cdot) \llcorner d\pi \mathscr{H}^n \llcorner \pi^{-1}(x) + \langle T, \pi, x\rangle. \tag{118}$$

Pushing forward (118) via $\pi^\perp$ by (117) it follows that

$$(-1)^{(m-n)n} Ju^x \llcorner \Omega = \nu(x, \cdot) \llcorner d\pi \mathscr{L}^n \llcorner \Omega + \tilde{T}^x,$$

with $\tilde{T}^x = \pi^\perp_\# \langle T, \pi, x\rangle$. But the finiteness of the size of $\langle T, \pi, x\rangle$ implies that $\tilde{T}^x$ is a sum of $\mathbf{S}(\langle T, \pi, x\rangle)$ Dirac masses. In particular by Theorem 3.2.4 in the case $m = n$ we know that

$$(-1)^{(m-n)n} \nu(x, \cdot) \llcorner d\pi = \det \nabla_y u(x, \cdot).$$

Using (34) we obtain $\nu(x, \cdot) \llcorner d\pi = M_n \nabla u(x, \cdot) \llcorner d\pi$ for almost every x. We recover the equality (116) by taking orthogonal projections π onto every $(m-n)$-dimensional coordinate subspace. $\square$

It will be useful to extend the result of Proposition 3.3.6 to the lower order determinants: let $u \in GSB_nV(\Omega)$ and $w \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$. We denote by $\Gamma(u, w)$ the sum of the jacobians of the functions obtained by replacing at least one component of u with the respective component of w, but not all of them. More precisely for every $I \subset \{1, \ldots, n\}$ such that $0 < |I| < n$ we construct the function u_I whose components are

$$u_I^k = \begin{cases} u^k \text{ if } k \notin I, \\ w^k \text{ if } k \in I. \end{cases}$$

Then we let $\Gamma(u,w) = \sum_{0<|I|<n} Ju_I$. By the multilinearity of jacobians, it is easy to check that if u is Lipschitz the identity

$$J(u+w) = Ju + \Gamma(u,w) + Jw \tag{119}$$

holds pointwise $\mathscr{L}^m$-a.e. in Ω.

Corollary 3.3.7. *Let $\Omega \subset \mathbb{R}^m$, $w \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$ and $u \in GSB_nV(\Omega)$. Then, in the sense of distributions, it holds*

$$J(u+w) = Ju + \Gamma(u,w) + Jw \quad \text{in } \Omega. \tag{120}$$

Proof. The proof uses the following observation: if $u_h \to u$ in L^s and $\nabla u_h \rightharpoonup \nabla u$ in L^p, then by Reshetnyak's Theorem and the inequality $p > n-1$ every minor of ∇u of order $k < n$ is weakly continuous in $L^{\frac{p}{k}}$. It follows that $\Gamma(u_h, w) \to \Gamma(u,w)$, so that we can pass to the limit in (119) to obtain (120). $\square$

3.4. Compactness

Theorem 3.4.1 (Compactness for the class GSB_nV). *Let $s > 0$, $p > 1$ be exponents with $\frac{1}{s} + \frac{n-1}{p} \leq 1$ and let $\Psi : [0,\infty) \to [0,\infty)$ be a convex increasing function satisfying $\lim_{t\to\infty} \Psi(t)/t = \infty$.*
Let $(u_h) \subset GSB_nV(\Omega)$ be such that $u_h \to u$ in $L^s(\Omega, \mathbb{R}^n)$ and $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega, \mathbb{R}^{n\times m})$. Assume that the Jacobians $Ju_h \llcorner \Omega = R_{u_h} + T_{u_h}$ fulfil

$$K := \sup_h \int_\Omega \Psi\left(\left|\frac{dR_{u_h}}{d\mathscr{L}^m}\right|\right) d\mathscr{L}^m + \mathbf{S}(T_{u_h}) < \infty. \tag{121}$$

Then $u \in GSB_nV(\Omega)$ and, writing $Ju \llcorner \Omega = R_u + T_u$,

$$\frac{dR_{u_h}}{d\mathscr{L}^m} \rightharpoonup \frac{dR_u}{d\mathscr{L}^m} \qquad \text{weakly in } L^1(\Omega, \Lambda_{m-n}\mathbb{R}^m), \tag{122}$$

$$\mathbf{S}(T_u) \leq \liminf_h \mathbf{S}(T_{u_h}). \tag{123}$$

Proof. Without loss of generality we can assume Ψ to have at most a polynomial growth at infinity, for otherwise it is sufficient to take $\tilde{\Psi}(t) := \min\{\Psi(t), t^2\}^{**}$, where f^{**} is the convexification of f (recall that the convexification of a superlinear function remains superlinear, since $\ell \leq f$ if and only if $\ell \leq f^{**}$, whenever ℓ is affine). In particular we will use the inequality

$$\Psi(2t) \leq C\Psi(t) \qquad \forall t > 0 \tag{124}$$

(this inequality is known as Δ_2 condition in the literature, see for instance [**AF03**, 8.6]). We shorten T_h, R_h in place of T_{u_h} and R_{u_h} respectively and denote by $\rho_h = M_n \nabla u_h$ the densities of R_h with respect to $\mathscr{L}^m$. We know from Proposition 3.1.2 that $Ju_h \llcorner \Omega \to Ju \llcorner \Omega$ in the local flat norm. Possibly extracting a subsequence we can assume with no loss of generality that:

(a) the limit $\lim_h \mathbf{S}(T_h)$ exists,
(b) $\rho_h \rightharpoonup \rho$ weakly in $L^1(\Omega, \Lambda_{m-n}\mathbb{R}^m)$,

(c) (u_h) rapidly converges to u in L^s: as a consequence of Proposition 3.1.2 we have also

$$\sum_h \mathbf{F}_\Omega^{\mathrm{loc}}(Ju_h \llcorner \Omega - Ju \llcorner \Omega) < \infty. \tag{125}$$

Indeed, if we prove the result under these additional assumptions, then we can use the weak compactness of ρ_h in L^1 provided by the Dunford-Pettis Theorem, and the fact that any subsequence admits a further subsequence satisfying (a), (b), (c) to obtain the general statement.

We shall let $R := \rho \mathscr{L}^m$ be the limit current: since the flat and weak convergences in (b) and (c) are stronger than the weak* convergence for currents, putting them together we obtain a flat current $T := Ju \llcorner \Omega - R$ such that

$$T_h \overset{*}{\rightharpoonup} T \quad \text{in } \Omega. \tag{126}$$

The proof is divided in four steps: we first address the special case $m = n$, then we use this case and the slicing Theorem (112) to show the lower semicontinuity of size along the slices in the second step. The main difficulty is in the third step, where we prove (122), because weak convergence behaves badly under the slicing operation. In the last step we conclude the lower semicontinuity of the size.

Step 1: $m = n$**.** We can apply a very particular case of Blaschke's compactness Theorem [**AT04**, 4.4.15] to the sets $\mathrm{spt}(T_h)$, which have equibounded cardinality, to obtain a finite set $N \subset \overline{\Omega}$ and a subsequence $(T_{h'})$ such that $\mathrm{spt}(T_{h'}) \longrightarrow N$ in the sense of Hausdorff convergence. By (126) we immediately obtain that $\mathrm{spt}(T) \subset N \cap \Omega$, hence $\mathbf{S}(T) < \infty$ and $u \in GSB_nV(\Omega)$. In addition, since any point in $\mathrm{spt}(T)$ is the limit of points in $\mathrm{spt}(T_{h'})$ it follows that

$$\mathbf{S}(T) \leq \liminf_{h'} \mathbf{S}(T_{h'}) = \lim_h \mathbf{S}(T_h).$$

Finally, since $Ju \llcorner \Omega = R + T$ it must be $T = T_u$, which yields (123), and $R = R_u$, which together with (b) yields (122) for the full sequence (u_h).

Step 2: $m \geq n$**.** Let us fix $A \subset \Omega$ open, $\pi \in \mathbf{O}_{m-n}$ and $\varepsilon \in (0,1)$: the bound (121), (12) and Fatou's lemma imply that

$$\begin{aligned}
+\infty > K &\geq \liminf_h \left\{ \mu_{T_h}(A) + \varepsilon \int_A \Psi(|\rho_h|) d\mathscr{L}^m \right\} && (127)\\
&\geq \liminf_h \left\{ \mu_{T_h,\pi}(A) + \varepsilon \int_A \Psi(|\rho_h \llcorner d\pi|) d\mathscr{L}^m \right\} && (128)\\
&\geq \int_{\mathbb{R}^{m-n}} \liminf_h \left[\mathscr{H}^0(A^x \cap \mathrm{spt}(\langle T_h, \pi, x\rangle)) + \varepsilon \int_{A^x} \Psi(|\rho_h \llcorner d\pi|) dy \right] dx \\
&= \int_{\mathbb{R}^{m-n}} \liminf_h \left[\mathscr{H}^0(A^x \cap \mathrm{spt}(T_{u_h^x})) + \varepsilon \int_{A^x} \Psi(|\rho_h^x|) dy \right] dx, && (129)
\end{aligned}$$

with $\rho_h^x := M \nabla u_h(x, \cdot) \llcorner d\pi$. By (127) we can choose for almost every $x \in \mathbb{R}^{m-n}$ a subsequence $h' = h'(x, A)$, possibly depending on x and on the set A, realizing the finite lower limit:

$$\liminf_h \mathscr{H}^0(A^x \cap \mathrm{spt}(T_{u_h^x})) + \varepsilon \int_{A^x} \Psi(|\rho_h^x|) dy.$$

Recall that thanks to (c) $Ju_h^x \llcorner \Omega \xrightarrow{\mathbf{F}_\Omega^{\mathrm{loc}}} Ju^x \llcorner \Omega$ for almost every x. We can therefore apply step 1 to the sequence $u_h^x \in GSB_nV(\Omega^x)$, which converges rapidly to u^x, to conclude that $u^x \in GSB_nV(\Omega^x)$ and that

$$\begin{aligned}\mathscr{H}^0(A^x \cap \operatorname{spt}(T_{u^x})) &\le \liminf_{h'} \mathscr{H}^0(A^x \cap \operatorname{spt}(T_{u^x_{h'}}))\\ &\le \liminf_{h'} \mathscr{H}^0(A^x \cap \operatorname{spt}(T_{u^x_{h'}})) + \varepsilon \int_{A^x} \Psi(|\rho^x_{h'}|)dy\\ &= \liminf_{h} \mathscr{H}^0(A^x \cap \operatorname{spt}(T_{u^x_{h}})) + \varepsilon \int_{A^x} \Psi(|\rho^x_{h}|)dy. \end{aligned} \tag{130}$$

Integrating in x and applying (128) as well as the monotonicity of Ψ we entail

$$\int_{\mathbb{R}^{m-n}} \mathscr{H}^0(A^x \cap \operatorname{spt}(T_{u^x}))\, dx \le \liminf_h \left\{ \mu_{T_h,\pi}(A) + \varepsilon \int_A \Psi(|\rho_h|) d\mathscr{L}^m \right\} =: \eta_\varepsilon(A). \tag{131}$$

Step 3: proof of (122)**.** In order to prove (122), since the space $\Lambda_{m-n}\mathbb{R}^m$ is finite dimensional, we will prove that

$$\rho_h \llcorner d\pi \rightharpoonup M_n \nabla u \llcorner d\pi \qquad \text{weakly in } L^1(\Omega, \Lambda_{m-n}\mathbb{R}^m) \tag{132}$$

for every orthogonal projection π onto a coordinate subspace. We fix an open $A \subset \Omega$ and $a \in \mathbb{R}$. From now on $w : A \to \mathbb{R}^n$ will be an affine map such that

$$\nabla_x w = 0, \qquad \det(\nabla_y w) = a.$$

Let us compute $J(u_h + w)$: thanks to Corollary 3.3.7 we get

$$J(u_h + w) = Ju_h + \Gamma(u_h, w) + a\mathbf{E}^m \llcorner d\pi \quad \text{in } \Omega.$$

We are now ready to prove the weak convergence of the regular parts. We argue as in step 2, but this time we change the form of the energy and we analyse the convergence of a perturbed sequence of maps. First of all we note that the sequence

$$\int_A \Psi\left(\left|\rho_h \llcorner d\pi + a\right|\right) d\mathscr{L}^m + \varepsilon \mu_{T_h,\pi}(A) + \varepsilon \int_A |\nabla u_h|^p d\mathscr{L}^m \tag{133}$$

is still bounded from above, because $|\alpha+\beta|^p \le 2^{p-1}(|\alpha|^p + |\beta|^p)$, (124) and the convexity of Ψ imply that

$$\begin{aligned}\int_A \Psi\left(\left|\rho_h \llcorner d\pi + a\right|\right) d\mathscr{L}^m &\le \frac{C}{2} \int_A \Psi\left(\left|\rho_h\right|\right) d\mathscr{L}^m + \frac{C}{2}\Psi(|a|)\mathscr{L}^m(A)\\ &\le \frac{C}{2}\big(K + \Psi(|a|)\mathscr{L}^m(A)\big).\end{aligned}$$

We consider the sequence $(u_h + w) \subset GSB_nV(A)$ and the perturbed energy (133): arguing as in the chain of inequalities (127)-(129) for almost every x we can find a suitable subsequence $h' = h'(x, A)$ realizing the finite lower limit of the sliced energies

$$\int_{A^x} \Psi\left(\left|\rho_h^x + a\right|\right) dy + \varepsilon \mathscr{H}^0(A^x \cap \operatorname{spt}(T_{u_h^x})) + \varepsilon \int_{A^x} |\nabla u_h^x|^p dy. \tag{134}$$

Since Ψ is superlinear at infinity, up to subsequences the densities $\rho^x_{h'} + a$ weakly converge to some function r^x in $L^1(A^x)$: in particular the associated currents weak* converge

$$(\rho^x_{h'} + a)\mathbf{E}^n \llcorner A^x \overset{*}{\rightharpoonup} r^x \mathbf{E}^n \llcorner A^x. \tag{135}$$

Thanks to the fast convergence (c) we also know that $u^x \to u$ in $L^s(A^x)$; moreover the boundedness of the Dirichlet term in (134) implies also that $\nabla_y u^x_{h'} \rightharpoonup \nabla u^x$ in $L^p(A^x, \mathbb{R}^n)$, hence by step 1 we get

$$u^x \in GSB_nV(A^x) \qquad \text{and} \qquad T_{u^x_{h'}} \overset{*}{\rightharpoonup} T_{u^x} \quad \text{in } \Omega. \tag{136}$$

The weak convergence of the gradients in L^p also allows to use the continuity property of $\Gamma(\cdot, w^x)$ along the sequence of restrictions $(u^x_{h'})$ and deduce that

$$(\rho^x_{h'} + a)\mathbf{E}^n \llcorner A^x = J(u^x_{h'} + w^x) - \Gamma(u^x_{h'}, w^x) - T_{u^x_{h'}} \overset{*}{\rightharpoonup} J(u^x + w^x) - \Gamma(u^x, w^x) - T_{u^x}$$

in the sense of distributions. By Corollary 3.3.7 and Proposition 3.3.6 we are able to identify the weak limit in (135)

$$r^x = \det \nabla_y u^x + a = M_n \nabla u(x, \cdot) \llcorner d\pi + a. \tag{137}$$

We fix a a convex increasing function with superlinear growth φ satisfying

$$\lim_{t \to +\infty} \frac{\Psi(t)}{\varphi(t)} = +\infty. \tag{138}$$

Using the previous convergence (135), (137) on almost every slice and integrating with respect to x we deduce by the convexity of φ that

$$\int_A \varphi\left(\left|M_n \nabla u \llcorner d\pi + a\right|\right) d\mathscr{L}^m \le \liminf_h \int_A \varphi(|\rho_h \llcorner d\pi + a|) d\mathscr{L}^m + \varepsilon \mu_{T_h, \pi}(A) + \varepsilon \int_A |\nabla u_h|^p d\mathscr{L}^m.$$

Adding this inequality on a finite number of disjoint open subsets A_j, with arbitrary choices of $a_i \in \mathbb{R}$, we obtain

$$\int_\Omega \varphi\left(\left|M_n \nabla u \llcorner d\pi + \xi\right|\right) d\mathscr{L}^m \le \liminf_h \int_\Omega \varphi(|\rho_h \llcorner d\pi + \xi|) d\mathscr{L}^m + \varepsilon \mathbf{S}(T_h) + \varepsilon \int_\Omega |\nabla u_h|^p d\mathscr{L}^m,$$

where $\xi := \sum_j a_j \chi_{A_j}$. Letting $\varepsilon \downarrow 0$ we can disregard the size and Dirichlet terms in the last inequality to get

$$\int_\Omega \varphi\left(\left|M_n \nabla u \llcorner d\pi + \xi\right|\right) d\mathscr{L}^m \le \liminf_h \int_\Omega \varphi(|\rho_h \llcorner d\pi + \xi|) d\mathscr{L}^m. \tag{139}$$

Taking $\varphi_n(t) := \frac{\varphi(t)}{n} \vee t$, we have that φ_n are still convex, increasing, superlinear at infinity and satisfy (138), therefore (139) is applicable with $\varphi = \varphi_n$. Given $\delta > 0$ fix C_δ such that $\varphi_1(t) \le \delta \Psi(t)$ for $t > C_\delta$; we also let $\Omega_{h,\delta} = \left\{|\rho_h \llcorner d\pi + \xi| > C_\delta\right\}$. By applying

(139) with $\varphi = \varphi_n$ we have therefore

$$\begin{aligned}
\int_\Omega \big|M_n\nabla u \llcorner d\pi + \xi\big| d\mathscr{L}^m &\le \int_\Omega \varphi_n\left(\big|M_n\nabla u \llcorner d\pi + \xi\big|\right) d\mathscr{L}^m \\
&\le \liminf_h \int_\Omega \varphi_n(|\rho_h \llcorner d\pi + \xi|) d\mathscr{L}^m \\
&\le \liminf_h \int_\Omega \big|\rho_h \llcorner d\pi + \xi\big| d\mathscr{L}^m + \limsup_h \int_{\Omega_{h,\delta}} \varphi_1\left(\big|\rho_h \llcorner d\pi + \xi\big|\right) d\mathscr{L}^m \\
&\qquad + \sup_{0\le t\le C_\delta}\{\varphi_n(t) - t\}\mathscr{L}^m(\Omega^c_{h,\delta}) \\
&\le \liminf_h \int_\Omega \big|\rho_h \llcorner d\pi + \xi\big| d\mathscr{L}^m + \limsup_h \delta \int_{\Omega_{h,\delta}} \Psi\left(\big|\rho_h \llcorner d\pi + \xi\big|\right) d\mathscr{L}^m \\
&\qquad + \sup_{0\le t\le C_\delta}\{\varphi_n(t) - t\}\mathscr{L}^m(\Omega).
\end{aligned}$$

Letting $n \to \infty$ the third term vanishes because $\varphi_n(t) \downarrow t$ uniformly on compact sets. Eventually, sending $\delta \downarrow 0$ we obtain

$$\int_\Omega \big|M_n\nabla u \llcorner d\pi + \xi\big| \le \liminf_h \int_\Omega \big|\rho_h \llcorner d\pi + \xi\big|. \tag{140}$$

Inequality (140) is actually valid for every $\xi \in L^1(\Omega)$ by approximation, since the set of functions of type $\sum_j a_j\chi_{A_j}$ is dense in L^1. We therefore address the last point (132) thanks to Lemma 3.4.2 below: the weak limit ρ must be the equal to $M_n\nabla u$. Moreover the full convergence of R_h to R implies that $\mathscr{L}^{m-n}$-almost everywhere, along a suitable subsequence depending on the point $h'(x, A)$ (for instance the one providing (134)-(136) with $a = 0$), it holds:

$$\langle Ju_{h'} - R_{h'}, \pi, x\rangle \overset{*}{\rightharpoonup} \langle Ju - R, \pi, x\rangle = \langle T, \pi, x\rangle,$$

hence $\langle T, \pi, x\rangle = (-1)^{(m-n)n}\iota_\#(T_{u^x})$: therefore $\iota(\mathrm{spt}(T_{u^x})) = \mathrm{spt}(\langle T, \pi, x\rangle)$. In particular the left hand side of (131) equals $\mu_{T,\pi}(A)$.

Step 4: conclusion. We are now ready to prove the last part of the Theorem. The map (131) $A \mapsto \eta_\varepsilon(A)$ is a finitely superadditive set-function, with $\eta_\varepsilon(\Omega) \le \liminf_h \mathbf{S}(T_{u_h}) + K\varepsilon$. Therefore if $B_1, \dots, B_N$ are pairwise disjoint Borel sets and $K_i \subset B_i$ are compact, we can find pairwise disjoint open sets A_i containing K_i and apply the superadditivity to get

$$\sum_{i=1}^N \mu_{T,\pi_i}(K_i) \le \sum_{i=1}^N \eta_\varepsilon(A_i) \le \eta_\varepsilon(\Omega).$$

Since K_i are arbitrary, the same inequality holds with B_i in place of K_i; since also B_i, π_i and N are arbitrary, it follows that $\mu_T = \bigvee_{\pi\in\mathbf{O}_{m-n}} \mu_{T,\pi}$ is a finite Borel measure and $\mu_T(\Omega) \le \eta_\varepsilon(\Omega)$. Hence $u \in GSB_nV(\Omega)$ because $Ju \llcorner \Omega = R + T$, $\mathbf{S}(T) < \infty$ and R is an absolutely continuous measure. Letting $\varepsilon \downarrow 0$ we also prove (123). For later purposes we notice that we proved

$$\mu_{T_u}(A) \le \liminf_h \mu_{T_{u_h}}(A). \tag{141}$$

□

Lemma 3.4.2. *Let $(z_h) \subset L^1(\Omega)$ be a weakly compact sequence and suppose that, for some $z \in L^1(\Omega)$, it holds*

$$\int_\Omega |z+\xi|\, d\mathscr{L}^m \le \liminf_h \int_\Omega |z_h+\xi|\, d\mathscr{L}^m \qquad \forall \xi \in L^1(\Omega).$$

Then $z_h \rightharpoonup z$ weakly in $L^1(\Omega)$.

Proof. Without loss of generality we can assume $z_h \rightharpoonup \zeta$ in L^1: let $P := \{\zeta > z\}$ and $N := \{\zeta \le z\}$. Since $(z_h - z)$ are equiintegrable for every $\varepsilon > 0$ there exists $k > 0$ such that

$$\int_{\{|z_h-z|>k\}} |z_h - z| < \varepsilon.$$

Setting $\xi = -z - k\chi_P + k\chi_N$ it is not hard to prove the following estimate:

$$|\xi + z_h| \le k - (z_h - z)\chi_P + (z_h - z)\chi_N + 2|z_h - z|\chi_{\{|z_h-z|>k\}}.$$

Integrating we obtain

$$k\mathscr{L}^m(\Omega) = \int_\Omega |\zeta + z| \le \liminf_h \int_\Omega |\xi + z_h| \le k\mathscr{L}^m(\Omega) - \int_P (\zeta - z) + \int_N (\zeta - z) + 2\varepsilon.$$

Therefore $\int_\Omega |\zeta - z| \le 2\varepsilon$ and the Lemma is proved. $\square$

CHAPTER 4

A new functional of Mumford-Shah type of codimension higher than one

In this chapter we introduce a new functional of Mumford-Shah type that features a singular set of higher codimension. As an application of Theorem 3.4.1 we will first prove a general existence result for a wide class of minimization problems. The choice of Lagrangians generalizes the classical Mumford-Shah energy [**MS89, DGCL89, AFP00, Dav05**] to vector valued maps with singular set of codimension at least 2: in this model we replace the singularities of the derivative by the singularities of the jacobian and we measure them with the size functional of section 2.6.

4.1. Existence of minimizers for the Dirichlet and Neumann problems

In this chapter we study a new functional in the calculus of variations of Mumford-Shah type introduced in [**AG13a**], where the minimization involves an unknown function as well as a set:

$$\mathcal{A}(u, K; \Omega) = \int_{\Omega \setminus K} f(x, u, M\nabla u)\, dx + \int_{\Omega \cap K} g\, d\mathscr{H}^{m-n}.$$

Here $\Omega \subset \mathbb{R}^m$ is a bounded open set of class C^1, $u \in C^1(\Omega \setminus K, \mathbb{R}^n)$, $M\nabla u$ is the vector of minors of ∇u of every rank and K, which plays the role of a "free discontinuity" set, is sufficiently regular and closed. In the next Theorem we show the existence of minimizers for the weak formulation of the problem: we let $GSB_nV(\Omega)$ be the space of competitors, and we replace K with the singular set S_u of the jacobian Ju. The simplified idea, in the special case $m = n$, is that u is a vector-valued map regular outside a finite number of points where the map covers a set of positive measure, thus imposing a singularity to its jacobian. The functional penalizes maps with an excessively large area factor $M\nabla u$ as well as the creation of too large singular sets S_u.

We first fix the notations for the dependence of a Lagrangian on the several minors of the gradient of a vector valued map. For approximately differentiable maps $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$, which include $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^m, \mathbb{R}^n)$ functions [**EG92**, 6.1.3], we let $M\nabla u$ be the vector of all minors of $k \times k$ submatrices of ∇u, with k ranging from 1 to n:

$$M\nabla u = (\nabla u, M_2\nabla u, \ldots, M_n\nabla u)$$

and we let $\kappa = \sum_{k=1}^{n} \binom{m}{k}\binom{n}{k}$ be its dimension. Its length will be measured with the norm of $\mathbb{R}^\kappa$:

$$|M\nabla u| = \left(\sum_{k=1}^{n} |M_k \nabla u|^2 \right)^{\frac{1}{2}}.$$

Given $w \in \mathbb{R}^\kappa$ we let w_ℓ the variables relative to the $\ell \times \ell$ minors. We also denote $\mathscr{L}_m$ the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^m$ and $\mathscr{B}(\mathbb{R}^{n+\kappa})$ the σ-algebra of Borel subsets of $\mathbb{R}^{n+\kappa}$. For the bulk part of the energy it is natural to treat polyconvex Lagrangians: the lower semicontinuity properties of such energies with respect to the weak $W^{1,p}$ convergence for $p < n$ has been thoroughly studied, see [**CDM94, FH95, FLM05, Mar86**].

THEOREM 4.1.1 (Existence of minimizers for polyconvex Lagrangians). *Assume r, p satisfy $r < \infty$, $\frac{1}{r} + \frac{n-1}{p} < 1$ and let $c > 0$ and $c_0 \geq 0$ be given constants. Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^\kappa \to [0, +\infty)$ satisfy the following hypotheses:*

(a) *f is $\mathscr{L}_m \times \mathscr{B}(\mathbb{R}^{n+\kappa})$-measurable;*
(b) *for $\mathscr{L}^m$-a.e. $x \in \Omega$, $(u, w) \mapsto f(x, u, w)$ is lower semicontinuous;*
(c) *for $\mathscr{L}^m$-a.e. $x \in \Omega$, $w \mapsto f(x, u, w)$ is convex in $\mathbb{R}^\kappa$ for every $u \in \mathbb{R}^n$;*
(d) *$f(x, u, w) \geq c_0 |u|^r + c\big(|w_1|^p + \Psi(|w_n|)\big)$ for some function Ψ satisfying the hypotheses of Theorem 3.4.1.*

Let also $g : \Omega \to [c, +\infty)$ be a lower semicontinuous function.
Then if $p^* = \frac{mp}{m-p} \geq r$, for every $u_0 \in W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^n)$ there exists a solution to the problem

$$\min_{u \in GSB_nV(\Omega),\, u = u_0 \text{ on } \partial\Omega} \left\{ \int_\Omega f\big(x, u(x), M\nabla u(x)\big)\, dx + \int_{\Omega \cap S_u} g(x)\, d\mathscr{H}^{m-n}(x) \right\}. \tag{P}$$

Similarly if $c_0 > 0$ the Neumann problem

$$\min_{u \in GSB_nV(\Omega)} \left\{ \int_\Omega f\big(x, u(x), M\nabla u(x)\big)\, dx + \int_{\Omega \cap S_u} g(x)\, d\mathscr{H}^{m-n}(x) \right\} \tag{N}$$

has a solution.

PROOF. Suppose the energy (P) is finite for some function in $GSB_nV(\Omega)$ with trace u_0, otherwise there is nothing to prove. Pick a minimizing sequence (u_h), and let first analyse problem (P).

Comparing with a $W^{1,p}$ extension of u_0 in the interior Ω, by the growth assumption (d), by the Poincaré inequality we have that (u_h) is bounded in $W^{1,p}$. Since $p^* > r$ we can find s satisfying (87) and a subsequence, not relabeled, such that $u_h \to u$ strongly in L^s and $\nabla u_h \rightharpoonup \nabla u$ weakly in L^p. The trace constraint $u = u_0$ on $\partial\Omega$ is convex, hence being strongly closed by the continuity of the trace map $T : W^{1,p}(\Omega, \mathbb{R}^n) \to W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^n)$, it is also closed for the weak $W^{1,p}$ topology. Hence the boundary datum is attained in the limit.

In the Neumann problem (N), since $c_0 > 0$, we can analogously find a minimizing subsequence and a function $u \in L^r \cap W^{1,p}$ such that

$$u_h \to u \text{ in } L^1 \tag{142}$$

and $\nabla u_h \rightharpoonup \nabla u$ in L^p. Since $\frac{1}{r} + \frac{n-1}{p} < 1$ we can choose $s < r$ such that $\frac{1}{s} + \frac{n-1}{p} \leq 1$: by Chebycheff's inequality we know that $u_h \to u$ in L^s. In both cases we recover the convergence of the jacobians

$$\mathbf{F}^{\text{loc}}_\Omega(Ju_h \llcorner \Omega - Ju \llcorner \Omega) \to 0.$$

Moreover we know that the absolutely continuous parts of $Ju_h \llcorner \Omega$ satisfy

$$\int_\Omega \Psi\left(\left|M_n \nabla u_h\right|\right)\, dx \le C,$$

and that by the lower bound $g \ge c$ we also have:

$$\sup_h \mathscr{H}^{m-n}(S_{u_h} \cap \Omega) < \infty.$$

Hence by the compactness Theorem 3.4.1, together with the classical Reshetnyak's Theorem for the minors of order less than n, we know that

$$M\nabla u_h \rightharpoonup M\nabla u \quad \text{weakly in } L^1. \tag{143}$$

By (142) and (143) the lower semicontinuity result of Ioffe [**Iof77a, Iof77b**] (see also [**AFP00**, Theorem 5.8]) implies

$$\liminf_h \int_\Omega f\big(x, u_h(x), M\nabla u_h(x)\big)\, dx \ge \int_\Omega f\big(x, u(x), M\nabla u(x)\big)\, dx.$$

Finally, g being lower semicontinuous, the superlevel sets $\{g > t\}$ are open, hence

$$\begin{aligned}
\liminf_h \int_{\Omega\cap S_{u_h}} g(x)\, d\mathscr{H}^{m-n}(x) &= \liminf_h \int_0^{+\infty} \mathscr{H}^{m-n}(S_{u_h} \cap \{g > t\})\, dt \\
&\ge \int_0^{+\infty} \liminf_h \mathscr{H}^{m-n}(S_{u_h} \cap \{g > t\})\, dt \\
&\ge \int_0^{+\infty} \mathscr{H}^{m-n}(S_u \cap \{g > t\})\, dt \\
&= \int_{\Omega\cap S_u} g(x)\, d\mathscr{H}^{m-n}(x),
\end{aligned}$$

because the size is lower semicontinuous on open sets, see (141). □

Recall that by Sobolev embedding we can drop the growth condition on $p^* \ge r$ in problem (P) provided $p > \frac{mn}{m+1}$. Notice also that we can formulate problem (P) and the corresponding boundary value condition in a slightly different way, in order to include in the energy the possible appearance of singularities at the boundary. Let $U \Supset \Omega$ be a bounded open subset of $\mathbb{R}^m$: we formulate the minimization problem in the following way:

$$\min_{u\in GSB_nV(U),\, u=u_0 \text{ in } U\setminus\Omega} \left\{ \int_U f\big(x, u(x), M\nabla u(x)\big)\, dx + \int_{U\cap S_u} g(x)\, d\mathscr{H}^{m-n}(x) \right\} \tag{P'}$$

Every competitor being equal to u_0 in $U \setminus \overline{\Omega}$, problem (P') accounts for variations of Ju in the closure $\overline{\Omega}$. Moreover Theorem 4.1.1 readily applies to this case, as the condition $u = u_0$ in $U \setminus \Omega$ is closed for the strong L^1 convergence. To explicit the dependence on the energy on the datum u_0 and on the domain U we adopt in the sequel of the chapter the notation

$$F(u, \Omega; u_0, U)$$

for the energy in (P').

4.2. A Mumford-Shah functional of codimension higher than one

The study of general functionals of the form (P) is modeled on the Mumford-Shah type functional

$$MS(u,\Omega) := \int_\Omega |u|^r + |\nabla u|^p + |M_n \nabla u|^\gamma \, dx + \mathscr{H}^{m-n}(S_u \cap \Omega) \tag{144}$$

defined on $GSB_nV(\Omega)$, with r, p satisfying $\frac{1}{r} + \frac{n-1}{p} < 1$, $\gamma > 1$, together with suitable boundary data. Theorem 4.1.1 shows the existence of minimizers of (144) for both Dirichlet and Neumann problems (P), (N) and (P'): it is however desirable that at least for some boundary datum u_0 the minimizer presents some singularity. In the next proposition we show that this is the case:

PROPOSITION 4.2.1 (Nontrivial minimizers for MS, formulation (P')). *Let $m = n$ and $u_0 : B_2 \to \mathbb{R}^n$ be the identity: $u_0(x) = x$. Then for ε sufficiently small every minimizer $u \in GSB_nV(B_2)$ of*

$$MS^\varepsilon(u, B_1; x, B_2) := \int_{B_2} \varepsilon\big(|u|^r + |\nabla u|^p\big) + |\det \nabla u|^\gamma \, dx + \varepsilon \mathscr{H}^0(S_u \cap B_2)$$

such that $u(x) = x$ in $B_2 \setminus B_1$ must satisfy

$$S_u \cap \overline{B_1} \neq \emptyset.$$

PROOF. We show that for every competitor v with $\|Jv\| \ll \mathscr{L}^n$ and for ε small enough it holds:

$$MS^\varepsilon(v, B_1; x, B_2) > MS^\varepsilon(w, B_1; x, B_2),$$

where

$$w(x) = \begin{cases} \frac{x}{|x|} & \text{in } B_1, \\ x & \text{in } B_2 \setminus B_1. \end{cases}$$

For the rest of the proof c will denote a generic positive constant we do not keep track of. Let us compute the energy of $\frac{x}{|x|}$: the Dirichlet and L^r parts are simply constants. Moreover

$$\det \nabla w = \chi_{B_2 \setminus B_1} \quad \text{and} \quad S_w = \{0\}.$$

Hence $MS^\varepsilon(w, B_1; x, B_2) = c\varepsilon + \mathscr{L}^n(B_2 \setminus B_1)$. On the contrary by Lemma 3.2.6 for almost every radius ρ it holds:

$$\int_{B_\rho} \det \nabla v \, dx = \int_{\partial B_\rho} v^1 dv^2 \wedge \cdots \wedge dv^n.$$

Since $u(x) = x$ outside B_1 for almost every $\rho \in (1,2)$ we have $\int_{B_\rho} \det \nabla v \, dx = \mathscr{L}^n(B_\rho)$, hence by Jensen's inequality

$$\int_{B_\rho} |\det \nabla v|^\gamma \, dx \geq \mathscr{L}^n(B_\rho).$$

Summing up:

$$MS^\varepsilon(v, B_1; x, B_2) \geq \int_{B_\rho} |\det \nabla v|^\gamma \, dx \geq \mathscr{L}^n(B_\rho) > c\varepsilon + \mathscr{L}^n(B_2 \setminus B_1) = MS^\varepsilon(w, B_1; x, B_2)$$

choosing first ρ sufficiently close to 2 and then ε sufficiently small. Therefore the minimizer u must have a nonempty singular set S_u, and since u is linear in the open set $B_2 \setminus B_1$, the singularity must be in $\overline{B_1}$. $\square$

It is easy to generalize the same proposition to the case $m \geq n$: take $u : B_1^{m-n} \times B_1^n \to \mathbb{R}^n$ the trivial extension in the extra variables. Reasoning slice per slice it is not difficult to show that every minimizer has a nontrivial singular set. The appearence of singularities might however be induced by the nonsmoothness of the boundary datum. This is not the case as we can show that the same phenomenon appears with a Lipschitz trace: consider the domain $\mathcal{C} := \{(x, y) \in \mathbb{R}^{m-n} \times \mathbb{R}^n : |x| + |y| \leq 1\}$ and let

$$w(x,y) = \begin{cases} \frac{(1-|x|)^+}{|y|} y \text{ in } \mathcal{C}, \\ y \text{ in } 2\mathcal{C} \setminus \mathcal{C}. \end{cases}$$

Note that $w|_{\partial\mathcal{C}} \in \mathrm{Lip}(\partial\mathcal{C}, \mathbb{R}^n)$. A careful calculation shows that:

$$\int_C |M_n \nabla w|^\gamma \, dxdy = \mathscr{L}^n(B_1^n) \mathscr{H}^{m-n-1}(S^{m-n-1}) \int_0^1 (1-t)^{\frac{n-1}{2}\gamma + n} t^{m-n-1} dt,$$

and that if $\|Jv\| \ll \mathscr{L}^m$:

$$\int_C |M_n \nabla v|^\gamma \, dxdy \geq \mathscr{L}^n(B_1^n) \mathscr{H}^{m-n-1}(S^{m-n-1}) \int_0^1 (1-t)^n t^{m-n-1} dt.$$

As the exponent of $1-t$ in the first integral is strictly larger than the one in the second estimate, the contribution of $M_n \nabla w$ is strictly lower that of $M_n \nabla v$: the gap is sufficient to absorb every other term of $MS_\varepsilon(w, \mathcal{C}; w, 2\mathcal{C})$ for ε sufficiently small, ruling out the minimality of v. We can actually be more quantitative and get a lower bound on the measure of the singular set for this special case. Consider a generic competitor $u \in GSB_nV(U)$ and let

$$\Sigma = \left\{ x \in B_1^{m-n}(0) : \mathbf{S}(\langle Ju, \pi, x\rangle) \neq 0 \right\}.$$

Then we can bound below the energy $MS^\varepsilon(u, \mathcal{C}; w, 2\mathcal{C})$ as follows:

$$\int_C |M_n \nabla u|^\gamma \, dxdy \geq \mathscr{L}^n(B_1^n) \int_0^1 (1-t)^n \mathscr{H}^{m-n-1}(\Sigma^c \cap \partial B_t^{m-n}) \, dt.$$

Hence subtracting to the energy of w:

$$\begin{aligned} \mathscr{L}^n(B_1^n) &\int_0^1 (1-t)^n \mathscr{H}^{m-n-1}(\Sigma^c \cap \partial B_t^{m-n}) \, dt \\ &\leq \mathscr{L}^n(B_1^n) \int_0^1 (1-t)^{\frac{n-1}{2}\gamma+n} \mathscr{H}^{m-n-1}(\partial B_t^{m-n}) \, dt \\ &\qquad + \varepsilon \left(\int_{\mathcal{C}} |\nabla w|^p - |\nabla u|^p \, dxdy + \mathscr{H}^{m-n}(\Sigma^c \cap B_1^{m-n}) \right), \end{aligned}$$

or rearranging the terms

$$\begin{aligned}
c_{n,\gamma} :=& \int_0^1 [(1-t)^n - (1-t)^{\frac{n-1}{2}\gamma+n}]\mathscr{H}^{m-n-1}(\partial B_t^{m-n})\,dt \\
\leq& \int_0^1 [(1-t)^n - \varepsilon]\mathscr{H}^{m-n-1}(\Sigma\cap\partial B_t^{m-n})\,dt \\
&\qquad\qquad + \frac{\varepsilon}{\mathscr{L}^n(B_1^n)}\left(\int_{\mathcal{C}} |\nabla w|^p - |\nabla u|^p\,dxdy + \mathscr{H}^{m-n}(B_1^{m-n})\right) \\
\leq& \mathscr{H}^{m-n}(\Sigma\cap B_1^{m-n}) + \frac{\varepsilon}{\mathscr{L}^n(B_1^n)}\left(\int_{\mathcal{C}} |\nabla w|^p\,dxdy + \mathscr{H}^{m-n}(B_1^{m-n})\right).
\end{aligned}$$

Therefore if $2\varepsilon(\int_{\mathcal{C}} |\nabla w|^p\,dxdy + \mathscr{H}^{m-n}(B_1^{m-n})) \leq c_{n,\gamma}\mathscr{L}^n(B_1^n)$ then

$$\mathscr{H}^{m-n}(\Sigma\cap B_1^{m-n}) \geq \frac{1}{2}c_{n,\gamma}.$$

In analogy with **[HLW98]**, we expect however the singularities to appear in the interior.

The argument in Proposition 4.2.1 essentially exploits the presence of the jacobian term: this is not coincidental, as the next proposition shows. Recall that the sum of a GSB_nV function and a C^1 function is again in GSB_nV.

PROPOSITION 4.2.2. *Every local minimizer of*

$$u \in GSB_nV(\Omega) \mapsto \int_\Omega |\nabla u|^p\,dx + \mathscr{H}^{m-n}(S_u\cap\Omega)$$

is locally of class $C^{1,\alpha}$ in Ω.

PROOF. It is sufficient to perform an outer variation of the minimizer u along a $\phi\in C_c^1(\Omega,\mathbb{R}^n)$ map: $\varepsilon\mapsto u+\varepsilon\phi$ and apply Corollary 3.3.7 to obtain that

$$S_{u+\varepsilon\phi} = S_u.$$

Hence the size term is constant and u satisfies:

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi\,dx = 0 \quad \forall\phi\in C_c^1(\Omega,\mathbb{R}^n).$$

Therefore u is a p-harmonic $W^{1,p}$ function, hence $u\in C^{1,\alpha}_{\mathrm{loc}}$ by **[Uhl77, DiB83]**. □

4.3. Traces

In the spirit of solving (P'), the nonuniqueness Example 4.3.1 below raises the problem of the dependence of the energy on the extension $u_0 : U\setminus\Omega\to\mathbb{R}^n$ to a given Sobolev trace $u|_{\partial\Omega}$. The example was communicated to us by C. De Lellis. It shows that if we want to detect the presence of singularities of Ju at the boundary of Ω, the Sobolev trace is not sufficient to characterize it.

EXAMPLE 4.3.1 (Singularity at the boundary). Let $u:\mathbb{R}^2\to S^1$ be defined by

$$u(x,y) = \left(\frac{y^2-(x-1)^2}{(x-1)^2+y^2}, \frac{2(1-x)y}{(x-1)^2+y^2}\right). \tag{145}$$

This map represents the normal unit vectorfield of the family of circles centered on the real axis and tangent to S^1 in the point $(1,0)$. If θ is the angle that the vector

$(x-1,y)$ makes with the real axis, we can write $u(x,y) = (-\cos(2\theta), -\sin(2\theta))$, hence by Example (3.1.3) $Ju = 2\pi[\![(1,0)]\!]$. Note that u is the identity map when restricted to S^1. Nonetheless we can construct another map $\tilde{u}$

$$\tilde{u}(x,y) = \begin{cases} u(x,y) & \text{for} \quad |x| < 1, \\ \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) & \text{for} \quad |x| \geq 1. \end{cases} \tag{146}$$

In this case, by Example 3.1.3, $J\tilde{u} = \pi[\![(1,0)]\!]$. Hence $u|_{B_1}$ admits two different Sobolev extensions u and $\tilde{u}$ sharing the same trace at the boundary but whose jacobians are different in $\overline{\Omega}$: the trace of a Sobolev function does not characterize the jacobian $Jv \llcorner \partial\Omega$ of all the possible extensions v.

It is interesting to know when part of the distributional jacobian can be represented as a boundary integral. Recall that the slicing Theorem 3.2.9 already provides an answer to this question, because if $u : \Omega \to \mathbb{R}^n$ then $\partial(j(u) \llcorner \{\pi > t\}) = Ju \llcorner \{\pi > t\} + \langle j(u), \pi, t\rangle$, where π is the distance from $\partial\Omega$. However, as Example 4.3.1 shows, this statement holds only for $\mathscr{L}^1$-a.e. t. The following proposition improves the general result by slicing, under additional hypotheses on the summability of u and of its trace. Denote for simplicity $g(u) := u^1 du^2 \wedge \cdots \wedge du^n$.

Proposition 4.3.2 (Stokes' Theorem). *Assume that $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ and $u|_{\partial\Omega} \in W^{1,n-1}(\partial\Omega, \mathbb{R}^n) \cap L^\infty(\partial\Omega, \mathbb{R}^n)$. Then the Stokes' theorem holds:*

$$\partial(j(u) \llcorner \Omega) = Ju \llcorner \Omega + \langle j(u), \partial\Omega\rangle$$

with the representation

$$\langle j(u), \partial\Omega\rangle(\omega) = \int_{\partial\Omega} \langle g(u), \tau_{\partial\Omega}\rangle\, \omega \, d\mathscr{H}^{n-1},$$

where $\tau_{\partial\Omega}$ orients $\partial\Omega$ as the boundary of Ω. In particular $\langle j(u), \partial\Omega\rangle$ depends only on the trace $u|_{\partial\Omega}$.

Proof. Suppose for simplicity that $\Omega = \mathbb{R}^n_+ = \mathbb{R}^n \cap \{x^n > 0\}$, $\operatorname{spt}(u) \subset B_1$ and let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a positive convolution kernel with compact support in $\mathbb{R}^{n-1}$. Set

$$u_\varepsilon(x', x^n) = \frac{1}{\varepsilon^{n-1}} \int_{\mathbb{R}^{n-1}} u(x'-y', x^n)\phi\left(\frac{x'-y'}{\varepsilon}\right) dy' :$$

since the convolution in the x' variables commutes with the trace operator we still have $u_\varepsilon|_{\mathbb{R}^{n-1}}(x') = u_\varepsilon(x',0)$; moreover $u_\varepsilon(\cdot,0) \in C^1(\mathbb{R}^{n-1}, \mathbb{R}^n)$ and the following estimates hold:

$$\|u_\varepsilon\|_{W^{1,n}(\mathbb{R}^n_+, \mathbb{R}^n)} \leq \|u\|_{W^{1,n}(\mathbb{R}^n_+, \mathbb{R}^n)}, \tag{147}$$

$$\|u_\varepsilon(\cdot,0)\|_{W^{1,n-1}(\mathbb{R}^{n-1}, \mathbb{R}^n)} \leq \|u(\cdot,0)\|_{W^{1,n-1}(\mathbb{R}^{n-1}, \mathbb{R}^n)} \tag{148}$$

and since the translations are strongly continuous in L^p,

$$\|u_\varepsilon - u\|_{W^{1,n}(\mathbb{R}^n_+, \mathbb{R}^n)} + \|u_\varepsilon(\cdot,0) - u(\cdot,0)\|_{W^{1,n-1}(\mathbb{R}^{n-1}, \mathbb{R}^n)} \to 0. \tag{149}$$

We claim that Stokes' Theorem holds for u_ε: for every $\omega \in \mathscr{D}^0(\mathbb{R}^n)$

$$\partial(j(u_\varepsilon)\llcorner\mathbb{R}^n_+)(\omega) = \int_{\mathbb{R}^n_+} \omega \det \nabla u_\varepsilon \, dx + \int_{\mathbb{R}^{n-1}\times\{0\}} \omega g(u_\varepsilon(\cdot,0)). \tag{150}$$

In fact extending $u_\varepsilon(x', x^n) := u_\varepsilon(x', 0)$ for $x^n \in [-1, 0]$ and then convolving with a smooth kernel ρ_δ supported in $B_\delta(0)$ we obtain a smooth $u_{\varepsilon,\delta} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{spt}(u_{\varepsilon,\delta}) \subset B_2 \times [-2, 2]$,

$$\begin{aligned} u_{\varepsilon,\delta}(x', x^n) &\to u_\varepsilon(x', 0) && \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n_-, \mathbb{R}^n),\\ u_{\varepsilon,\delta} &\to u_\varepsilon && \text{in } W^{1,n}_{\mathrm{loc}}(\mathbb{R}^{n-1}\times(-1,+\infty), \mathbb{R}^n). \end{aligned} \tag{151}$$

More precisely it holds: $u_{\varepsilon,\delta}(x', -\delta) \to u_\varepsilon(x', 0)$ in $C^1(\mathbb{R}^{n-1}, \mathbb{R}^n)$. Hence

$$\partial(j(u_{\varepsilon,\delta})\llcorner\{x^n > -\delta\})(\omega) = \int_{\{x^n>-\delta\}} \omega \det \nabla u_{\varepsilon,\delta} \, dx + \int_{\mathbb{R}^{n-1}\times\{-\delta\}} \omega g(u_{\varepsilon,\delta}(\cdot, -\delta)) :$$

letting $\delta \downarrow 0$ the left hand side converges to $\partial(j(u_\varepsilon)\llcorner\mathbb{R}^n_+)(\omega)$ by (151). The boundary term in right hand side tends to

$$\int_{\mathbb{R}^{n-1}\times\{0\}} \omega(\cdot, 0) g(u_\varepsilon(\cdot, 0))$$

because the convergence is C^1 and ω is smooth. Regarding the volume integral we can estimate

$$|\nabla u_{\varepsilon,\delta}(x)| = |(\rho_\delta * \nabla u_\varepsilon)(x)| \le \|u_\varepsilon(\cdot,0)\|_{C^1} + \fint_{B_\delta(x)\cap\{y^n>0\}} |\nabla u_\varepsilon(y)| \, dy$$

hence

$$\begin{aligned} \left| \int_{\{|x^n|<\delta\}} \omega \det \nabla u_{\varepsilon,\delta} \, dx \right| &\le \|\omega\|_{C^0} \int_{\{|x^n|<\delta\}} |\nabla u_{\varepsilon,\delta}|^n \, dx \\ &\le c_n \|\omega\|_{C^0} \left(\|u_\varepsilon(\cdot,0)\|^n_{C^1} \delta + \int_{\{|x^n|<\delta\}} \fint_{B_\delta(x)\cap\{y^n>0\}} |\nabla u_\varepsilon(y)|^n \, dy \, dx \right) \\ &\le c_n \|\omega\|_{C^0} \left(\|u_\varepsilon(\cdot,0)\|^n_{C^1} \delta + \int_{\{0<x^n<2\delta\}} |\nabla u_\varepsilon(x)|^n \, dx \right) \to 0. \end{aligned}$$

Clearly $\int_{\{x^n>\delta\}} \omega \det \nabla u_{\varepsilon,\delta} \, dx \to \int_{\{x^n>0\}} \omega \det \nabla u_\varepsilon \, dx$, therefore (150) is true.

We now want to pass to the limit for $\varepsilon \downarrow 0$ in (150). The left hand side goes to $\partial(j(u)\llcorner\mathbb{R}^n_+)(\omega)$ because of (149); similarly for the volume term. Regarding the boundary term the convergence of the minors on the slice needs to be improved. The estimates (147) and the classical result **[CLMS93]** gives a uniform bound of the Hardy norm **[Ste93**, Chapter IV] of the minors of order $n-1$:

$$\|du^2_\varepsilon(\cdot,0) \wedge \cdots \wedge du^n_\varepsilon(\cdot,0)\|_{\mathcal{H}^1(\mathbb{R}^{n-1},\mathbb{R}^n)} \le \|u(\cdot,0)\|^{n-1}_{W^{1,n-1}(\mathbb{R}^{n-1},\mathbb{R}^n)}.$$

We already know from Reshetnyak's Theorem that $du^2_\varepsilon(\cdot,0) \wedge \cdots \wedge du^n_\varepsilon(\cdot,0) \overset{*}{\rightharpoonup} du^2(\cdot,0) \wedge \cdots \wedge du^n(\cdot,0)$ in the sense of distributions; moreover smooth functions are dense in

$VMO(\mathbb{R}^{n-1},\mathbb{R}^n)$ and $VMO^*=\mathcal{H}^1$, so

$$du_\varepsilon^2(\cdot,0)\wedge\cdots\wedge du_\varepsilon^n(\cdot,0)\overset{*}{\rightharpoonup} du^2(\cdot,0)\wedge\cdots\wedge du^n(\cdot,0)\qquad\text{in}\quad \sigma(\mathcal{H}^1,VMO).$$

Finally the trace $u_\varepsilon(\cdot,0)$ belongs to

$$W^{1-\frac{1}{n},n}(\mathbb{R}^{n-1},\mathbb{R}^n)\subset VMO(\mathbb{R}^{n-1},\mathbb{R}^n)$$

(see [**Ada75**, Theorem 7.58], [**BN95**, Example 2] for the inclusions). Hence $\|u_\varepsilon(\cdot,0)-u(\cdot,0)\|_{VMO}\to 0$ strongly and we can pass to the limit in (150)

$$\int_{\mathbb{R}^{n-1}}\omega g(u_\varepsilon(\cdot,0))\to\int_{\mathbb{R}^{n-1}}\omega g(u(\cdot,0)).$$

By (149) also the left hand side of (150) converges to $\int_{\mathbb{R}^n_+}\omega\det\nabla u\,dx$. □

In the example above the smooth extension $\tilde u$ is certainly preferable to u, where an "extra" singularity comes from the outside. A partial answer to this problem can be given if we assume a better differentiability of the outer extension, up to $\partial\Omega$:

Proposition 4.3.3 (See [**GMS98**, 3.2.5 Theorem 1]). *Let $v,w\in L^s\cap W^{1,p}(U,\mathbb{R}^n)$ satisfy the following conditions:*

- $v|_\Omega=w|_\Omega$;
- $v|_{U\setminus\Omega}, w|_{U\setminus\Omega}\in W^{1,n}(U\setminus\Omega,\mathbb{R}^n)$;
- $v|_{\partial\Omega}=w|_{\partial\Omega}\in W^{1,n-1}(\partial\Omega,\mathbb{R}^n)$.

Then:

$$Jv-Jw=\big(\det\nabla v-\det\nabla w\big)\mathbf{E}^n\llcorner(U\setminus\Omega).$$

Proof. We can write

$$Jv=\partial j(v)=\partial(j(v)\llcorner\Omega)+\partial(j(v)\llcorner(U\setminus\Omega))=\partial(j(v)\llcorner\Omega)+Jv\llcorner(U\setminus\Omega)-\langle j(v),\partial\Omega\rangle.$$

Subtracting the analogous expression for Jw we obtain

$$Jv-Jw=(Jv-Jw)\llcorner(U\setminus\Omega)-\langle j(v)-j(w),\partial\Omega\rangle=\big(\det\nabla v-\det\nabla w\big)\mathbf{E}^n\llcorner(U\setminus\Omega)$$

because Proposition 4.3.2 applied to the open set $U\setminus\Omega$ implies that $v|_{\partial\Omega}=w|_{\partial\Omega}$, hence $\langle g(v)-g(w),\tau_{\partial\Omega}\rangle=0$. □

Therefore, if we aim at formulating problem (P') in a local way, that is depending only on the values of u in $\overline{\Omega}$, at least when the trace is sufficiently "nice", we can proceed as follows. If $u|_{\partial\Omega}$ belongs to $W^{1,n-1}$ and admits a $W^{1,n}$ extensions outside Ω, we can conventionally agree to pick one of such extensions to $U\setminus\Omega$: the result of Proposition 4.3.3 implies that the jacobian in $\overline{\Omega}$ of every competitor does not depend on the particular choice we made. Note however that the smoothness of the trace does not imply membership of the extension to $GSB_nV(U)$. In fact, it is sufficient to place the infinite dipoles of the function g in Example 3.1.4 so that the singularities lie on ∂B_1 and do not overlap. The constant extension outside the ball provides a map whose jacobian has both infinite mass and size.

In conclusion, in order to solve Problem (P') it seems necessary to impose membership of the competitors to $GSB_nV(U)$, while for a fairly broad class of boundary data the energy in $\overline{\Omega}$ shall not depend on the particular extension.

CHAPTER 5

An approximation via Γ-convergence

In this chapter we show how to approximate the Mumford-Shah energy

$$MS(u,\Omega) = \int_\Omega |u|^r + |\nabla u|^p + |M_n \nabla u|^\gamma \, dx + \mathscr{H}^{m-n}(S_u \cap \Omega) \tag{152}$$

by a sequence of (asymptotically degenerate) elliptic functionals E_ε defined for regular maps, using the tool of Γ-convergence. These densities, being absolutely continuous, are easier to handle from the numerical viewpoint. A similar result was already obtained for the classical Mumford-Shah functional by Ambrosio and Tortorelli in [**AT90, AT92**], the approximation being inspired by the classical works of Modica and Mortola for the Cahn-Hilliard energy [**MM77b, MM77a**], where the authors were able to approximate the defect measure, which is singular, via a family of bulk functionals (although not uniformly elliptic). The size term in (152) is replaced in the approximation by a phase transition energy suitably conceived to concentrate on sets of dimension $(m-n)$. The blow-up profiles are described in detail in section 5.3.

5.1. Preliminary definitions

For the sake of exposition we drop the term $|u|^r$ in (144), and introduce a constant factor in front of the size term. This choice does not modify the core of the proofs, since the removed term is of lower order in the number of derivatives. We will show the approximation for the following energy:

Definition 5.1.1. *Let $\gamma > 1$ and $\sigma > 0$. For every $u \in GSB_nV(\Omega)$ we set*

$$E(u,\Omega) = \int_\Omega |\nabla u|^p + |M_n \nabla u|^\gamma dx + \sigma \mathscr{H}^{m-n}(\Omega \cap S_u).$$

Recall the result of the previous chapter Theorem 4.1.1 entails the following result for $E(u,\Omega)$:

Theorem 5.1.2. *Let Ω be a regular open and bounded subset of $\mathbb{R}^m$ and let U be an open neighborhood of $\overline{\Omega}$. Let $\phi \in GSB_nV(U)$ be a given function and suppose $p^* = \frac{mp}{m-p} > s$. Then the minimum problem*

$$\inf \left\{ E(u,\overline{\Omega}) : \, u \in GSB_nV(U), \, u = \phi \text{ in } U \setminus \Omega \right\} \tag{153}$$

has a solution. Similarly for the Neumann problem if $r > s$ and $g \in L^r(\Omega,\mathbb{R}^n)$ is given, then

$$\inf \left\{ E(u,\Omega) + \int_\Omega |u-g|^r \, dx : \, u \in GSB_nV(\Omega) \right\} \tag{154}$$

has a solution.

5.1.1. Minkowski content. As Theorem 5.2.8 below involves the concept of Minkowski content, we here briefly review its definition and main properties.

DEFINITION 5.1.3. *Let $S \subset \mathbb{R}^m$ and let $k \in [0, m]$ be and integer. The lower and upper Minkowski contents of S in Ω are defined respectively as*

$$\mathcal{M}^k_{*\Omega}(S) = \liminf_{r\downarrow 0} \frac{\mathscr{L}^m(\{x \in \Omega : dist(x, S) \le r\})}{\mathscr{L}^{m-k}(B_1) r^{m-k}}, \tag{155}$$

$$\mathcal{M}^{*k}_{\Omega}(S) = \limsup_{r\downarrow 0} \frac{\mathscr{L}^m(\{x \in \Omega : dist(x, S) \le r\})}{\mathscr{L}^{m-k}(B_1) r^{m-k}}, \tag{156}$$

where $\mathscr{L}^{m-k}(B_1)$ is the measure of the unit ball in $\mathbb{R}^{m-k}$. We omit the subscript when $\Omega = \mathbb{R}^m$. If $\mathcal{M}^k_*(S) = \mathcal{M}^{*k}(S)$ we define the Minkowski content of S as this common value.

We must observe that neither $\mathcal{M}^k_*$ nor $\mathcal{M}^{*k}$ is a measure, and that they both give the same value to a set and its closure. It is natural to compare the upper and lower Minkowski contents with the k-dimensional Hausdorff measure: it can be proved (see [**Fed69**, 3.2.37-39], [**AFP00**, 2.101]) that for every countably $\mathscr{H}^k$-rectifiable and closed set S

$$\mathcal{M}^k_*(S) \ge \mathscr{H}^k(S).$$

By inner regularity of the Hausdorff measure the last inequality holds also relative to Ω. Various assumptions on S besides rectifiability are possible in order to have that $\mathcal{M}^k(S) = \mathscr{H}^k(S)$. One of the most general is the following:

PROPOSITION 5.1.4 ([**AFP00**, Proposition 2.104]). *Let S be a countably $\mathscr{H}^k$-rectifiable set such that*

$$\nu(B_\rho(x)) \ge c\rho^k \quad \forall x \in S \quad \forall \rho \in (0, \rho_0) \tag{157}$$

for a suitable Radon measure $\nu \ll \mathscr{H}^k$ and $c, \rho_0 > 0$. Then

$$\mathcal{M}^k(S) = \mathscr{H}^k(S).$$

Note that the equality implies that $\mathscr{H}^k(S) = \mathscr{H}^k(\overline{S})$. To ease the notation we will denote $S_r = \{x \in \Omega : 0 < \mathrm{dist}(x, S) \le r\}$ and $V(r) = \mathscr{L}^m(S_r)$. Let $S \subset \mathbb{R}^m$ be a closed set, and consider the distance function from it. Then (see [**Fed69**, 3.2.34])

$$|\nabla dist(\cdot, S)| = 1 \quad \mathscr{L}^m\text{-a.e. in } \{dist(\cdot, S) > 0\}. \tag{158}$$

Moreover the following property holds:

LEMMA 5.1.5. *The function $V(t) = \mathscr{L}^m(\{0 < dist(\cdot, S) \le t\})$ is absolutely continuous and*

$$V'(t) = \mathscr{H}^{m-1}(\Omega \cap \{dist(\cdot, S) = t\})$$

for $\mathscr{L}^1$-almost every $t > 0$.

PROOF. Recall the Coarea formula ([**Fed69**, 3.2.11-12]): if $f : \Omega \to \mathbb{R}$ is a Lipschitz function and $g : \Omega \to \mathbb{R}$ is a non-negative Borel function, then

$$\int_\Omega g(x)|\nabla f(x)|\, dx = \int_0^{+\infty} \int_{\{f=t\}} g\, d\mathscr{H}^{m-1}\, dt. \tag{159}$$

In particular taking $f(x) = \operatorname{dist}(x, S)$ and g the characteristic function of the set $\{\operatorname{dist}(\cdot, S) \leq t\}$ we obtain that for every $t > 0$

$$V(t) = \int_0^t \mathscr{H}^{m-1}(\Omega \cap \{\operatorname{dist}(\cdot, S) = s\}) ds.$$

Therefore $V(t)$ is an absolutely continuous function with

$$V'(t) = \mathscr{H}^{m-1}(\Omega \cap \{\operatorname{dist}(\cdot, S) = t\})$$

$\mathscr{L}^1$-almost everywhere. □

5.2. Variational approximation

In this section we state our main approximation theorem. We start by recalling the fundamental features of the variational convergence we will use, the Γ-convergence, and we refer to **[Bra02, DM93]** for a thorough presentation. Let X be a separable metric space and let a sequence of functions $F_h : X \to [0, \infty]$ be given. We define the upper and the lower Γ-limits as follows:

$$\underline{F}(x) = (\Gamma - \liminf_{h\to\infty} F_h)(x) = \inf\{\liminf_{h\to\infty} F_h(x_h) : x_h \to x\}, \tag{160}$$

$$\overline{F}(x) = (\Gamma - \limsup_{h\to\infty} F_h)(x) = \inf\{\limsup_{h\to\infty} F_h(x_h) : x_h \to x\}. \tag{161}$$

Both $\underline{F}$ and $\overline{F}$ are lower semicontinuous by construction, and we say that F_h Γ-converges to F if $\underline{F} = \overline{F}$. The statement $\Gamma - \lim_h F_h = F$ is equivalent to the fulfillment of the following two conditions: for every $x \in X$

$$\forall x_h \to x \text{ we have } \liminf_h F_h(x_h) \geq F(x), \tag{162}$$

$$\exists x_h \to x \text{ such that } \limsup_h F_h(x_h) \leq F(x).$$

The following Theorem describes the fundamental properties of this type of convergence, in particular the behaviour of sequences of minima:

THEOREM 5.2.1. *Assume F_h Γ-converges to F.*

(a) *Let $t_h \downarrow 0$. Then any cluster point of the sequence of sets*

$$\{x \in X : F_h(x) \leq \inf_X F_h + t_h\}$$

minimizes F.

(b) *Assume also that F_h are lower semicontinuous, and that for every $t \geq 0$ there exists a compact set $K_t \subset X$ such that*

$$\{F_h \leq t\} \subset K_t.$$

Then every function F_h has a minimizer, and any sequence of minimizers admits a subsequence converging to some minimizer of F.

(c) *Given a continuous function $G : X \to [0, \infty]$ we have*

$$\Gamma - \liminf_h (F_h + G) = (\Gamma - \liminf_h F_h) + G,$$

$$\Gamma - \limsup_h (F_h + G) = (\Gamma - \limsup_h F_h) + G.$$

The following remark recalls a useful tool in proving Γ-convergence results.

REMARK 5.2.2. Let $X' \subset X$ and $F, F_h : X \to \mathbb{R}$ as above: we say that X' is dense in energy in X if for every $x \in X$ there exists a sequence $(x'_h) \subset X'$ such that $x'_h \to x$ and $F(x'_h) \to F(x)$. A simple diagonal argument shows that in order to prove $\Gamma - \lim F_h = F$, whilst already knowing the $\Gamma - \liminf$ inequality $F \leq \underline{F}$ (namely the validity of (162)), it is enough to prove that for every $\delta > 0$ and $x \in X'$ there exists $x_h \to x$ such that $\limsup_h F_h(x_h) \leq F(x) + \delta$.

5.2.1. Main Theorem. We introduce now the function spaces involved in our approximation Theorem. Given an open set $U \subset \mathbb{R}^n$ we let $B(U)$ be the space of Borel functions ranging in $[0, 1]$:

$$B(U) = \{v : U \to [0, 1] : v \text{ is a Borel function}\},$$

endowed with a distance that induces the convergence in measure, namely:

$$d(v, v') = \int_\Omega \frac{|v - v'|}{1 + |v - v'|} dx.$$

We want to approximate the maps $u \in GSB_nV$ with functions u_ε possessing "better regularity", namely having absolutely continuous jacobian. It will be therefore handier to have a name for the space of function of bounded n-variation with absolutely continuous jacobian:

DEFINITION 5.2.3 (Regular maps). *We let*

$$R_n(\Omega) := \{u \in B_nV(\Omega) : \|Ju \llcorner \Omega\| \ll \mathscr{L}^m\}$$

be the space of regular maps.

We want to approach the energy $E(u, \Omega)$ by a sequence $E_h(u_h, v_h, \Omega)$ where the functions u_h belong to $R_n(\Omega)$, namely $Ju_h \llcorner \Omega = R_{u_h} = M_n \nabla u_h \mathscr{L}^m \llcorner \Omega$. Our function spaces will be the following:

DEFINITION 5.2.4. *We define the space $X(\Omega) := L^s(\Omega, \mathbb{R}^n) \times B(\Omega)$ with the following convergence notion:*

$$(u_h, v_h) \to (u, v) \iff u_h \to u \text{ in } L^s(\Omega, \mathbb{R}^n), \quad v_h \to v \text{ in measure.} \tag{163}$$

The subspace $Y(\Omega)$ will be:

$$Y(\Omega) := R_n(\Omega) \times B(\Omega) \subset X(\Omega),$$

endowed with the same topology.

The convergence (163) is clearly metrizable. We also introduce two subspaces of $X(\Omega)$ and $Y(\Omega)$ where the trace is fixed in a strong sense:

DEFINITION 5.2.5. *Given $U \Supset \Omega$ open and $\phi \in L^s(U)$ we let*

$$X^\phi = \{(u, v) \in X(U) : u = \phi \text{ in } U \setminus \Omega\},$$
$$Y^\phi = \{(u, v) \in Y(U) : u = \phi \text{ in } U \setminus \Omega\}.$$

Following **[AT90, AT92, MM77a]**, we introduce a Modica-Mortola type energy to approximate the size term $\mathbf{S}(T_u) = \mathscr{H}^{m-n}(S_u \cap \Omega)$. Observe that the parameter ε is present with suitable exponents in order for the energy to concentrate on $(m - n)$-dimensional sets: in particular it concentrates on points if $m = n$.

DEFINITION 5.2.6. *Let $W \in C^1(\mathbb{R})$ be a nonnegative convex potential vanishing only at* 0 *and let $q > n$ be a given exponent. If $v \in B(\Omega)$ we set*

$$MM_\varepsilon(v,\Omega) = \int_\Omega \varepsilon^{q-n}|\nabla v|^q + \frac{W(1-v)}{\varepsilon^n}dx.$$

Note in particular that W is increasing in the positive real axis. We are now ready to introduce our family of energies:

DEFINITION 5.2.7. *Let $\gamma > 1$ and $q > n$ be fixed exponents. We set, for $(u,v) \in X(\Omega)$:*

$$E(u,v,\Omega) = \begin{cases} \int_\Omega |\nabla u|^p + |M_n(\nabla u)|^\gamma dx + \sigma\mathscr{H}^{m-n}(S_u \cap \Omega) & \textit{if } u \in GSB_nV(\Omega) \textit{ and } v = 1, \\ +\infty & \textit{otherwise,} \end{cases}$$

and

$$E_\varepsilon(u,v,\Omega) = \begin{cases} \int_\Omega |\nabla u|^p + (v+k_\varepsilon)|M_n(\nabla u)|^\gamma dx + MM_\varepsilon(v,\Omega) & \textit{for } (u,v) \in Y(\Omega), \\ +\infty & \textit{otherwise,} \end{cases}$$

where the constant σ is defined by the minimum problem 5.3.1 and k_ε is an infinitesimal faster than ε^γ.

The first functional $E(u,v,\Omega)$ is clearly a trivial extension to $X(\Omega)$ of Definition 5.1.1, as $E(u,1,\Omega) = E(u,\Omega)$. We fix once and for all a sequence ε_h of positive numbers converging to zero and to simplify the notation we write E_h instead of E_{ε_h}. We will also write

$$F(u,1,\Omega) = F(u,\Omega) = \int_\Omega |\nabla u|^p + |M_n \nabla u|^\gamma\, dx,$$
$$F_\varepsilon(u,v,\Omega) = \int_\Omega |\nabla u|^p + (v+k_\varepsilon)|M_n\nabla u|^\gamma\, dx$$

for the part of the energy explicitly depending on u.

In Definition 5.2.7 v is a control function for the pointwise determinant $M_n\nabla u$, ranging in the interval $[0,1]$, and depends on the singular set S_u; k_ε is an infinitesimal number apt to guarantee coercivity of E_ε. The addeddum $MM_\varepsilon(v,\Omega)$, referred to as the Modica-Mortola term because of the similarity with the phase transition energies used in [**AT90**], contains a nonnegative convex potential W vanishing in 0. As ε goes to 0, the potential term $W(1-v)$ forces v_ε to converge to 1 in measure; on the contrary v_ε becomes closer to 0 where the jacobian of the functions u_ε tends to form a singularity, and compensates the loss of energy due to this damping with the Modica-Mortola term. Because of the scaling property of the Modica-Mortola part the transition from $v_\varepsilon \sim 0$ to $v_\varepsilon \sim 1$ happens in a set of width of order ε, and up to a rescaling v_ε converges to a precise profile w_0 analysed in section 5.3. In particular this transition energy concentrates around the singular set S_u proportionally to its $\mathscr{H}^{m-n}$-measure.

We can now state our main Theorem: we prefer to present separately the lower and upper limit part of the Γ-convergence, since it is more clear where the hypotheses are used.

THEOREM 5.2.8. *Let Ω be a bounded open subset of class C^1 of $\mathbb{R}^m$ and suppose*

$$s \geq \frac{np}{n-p}, \qquad 1 < \gamma \leq \frac{1}{\frac{n-1}{p} + \frac{1}{s}}, \qquad q > n.$$

(a) *For every sequence $\big((u_h, v_h)\big) \subset Y(\Omega)$ such that $(u_h, v_h) \to (u, v)$ in $X(\Omega)$ we have*

$$\liminf_{h\to\infty} E_h(u_h, v_h, \Omega) \geq E(u, v, \Omega);$$

moreover

$$\liminf_{h\to\infty} E_h(u_h, v_h, \Omega) < \infty \quad \Rightarrow \quad u \in GSB_nV(\Omega) \text{ and } v = 1.$$

(b) *For every $u \in GSB_nV(\Omega)$ such that $E(u, 1, \Omega) < \infty$ and $\mathcal{M}^{*m-n}(S_u) = \mathscr{H}^{m-n}(S_u)$ there exists a sequence $\big((u_h, v_h)\big) \subset Y(\Omega)$ such that $(u_h, v_h) \to (u, 1)$ in $X(\Omega)$ and*

$$\limsup_{h\to\infty} E_h(u_h, v_h, \Omega) \leq E(u, 1, \Omega).$$

Note that in particular the restrictions of E_h and E to the subspace

$$Z(\Omega) = \{u \in GSB_nV(\Omega) : \mathcal{M}^{*m-n}_{\Omega}(S_u) = \mathscr{H}^{m-n}(S_u)\} \times B(\Omega)$$

satisfy (with the convergence (163))

$$\Gamma - \lim_h E_h|_{Z(\Omega)} = E|_{Z(\Omega)}.$$

The proof of point (a) is achieved first in codimension $m - n = 0$, where S_u is a discrete set, and then generalized to every codimension with the help of the slicing Theorem. The second part of the proof concerns the upper limit: here we construct (u_ε) truncating the function u around the singularity S_u and we use the optimal profile w_0 to build functions v_ε such that $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ and fulfilling the upper limit inequality. In order to make this construction we will assume a mild regularity assumption on the singular set, namely that its Hausdorff measure agrees with its Minkowski content. In order to conclude the proof of the Γ-convergence of E_ε to E we would need to know the density in energy of the set $Z(\Omega)$. In the codimension 1 case this property was deduced by the regularity of minimizers of the Mumford-Shah energy, for which a lower bound on the $(m-1)$-dimensional density of the singular set is available. The analogous density property as well as a regularity result for minimizers of E is still subject to investigation.

We start the analysis on the whole family of energies (E_h) by proving that at a fixed positive scale ε_h the functional E_h has a minimizer in $Y(\Omega)$, once we assign suitable Dirichlet or Neumann boundary conditions.

THEOREM 5.2.9. *Let $C \geq 0$ and $h \in \mathbb{N}$ be fixed. The sets*

$$\big\{(u, v) \in Y(U) : u = \phi \text{ in } U \setminus \Omega,\ E_h(u, v, U) \leq C\big\}; \tag{164}$$

with U a neighborhood of $\overline{\Omega}$, $p^* > s$ and $\phi \in GSB_nV(\Omega)$; and

$$\big\{(u, v) \in Y(\Omega) : E_h(u, v, \Omega) + \int_\Omega |u - g|^r\, dx \leq C\big\} \tag{165}$$

with $g \in L^r(\Omega, \mathbb{R}^n)$ and $r > s$, are compact subsets of $X(\Omega)$. Moreover the unions of the sets (164) for h varying in $\mathbb{N}$, as well as the union of the sets (165), are precompact in $X(\Omega)$.

Proof. Recall that it is sufficient to check sequential compactness, since (164) and (165) are subsets of the metric space $X(\Omega)$. As the product of two precompact spaces is precompact, we can examine separately the bounds on u and v:

$$\int_U |\nabla u|^p \, dx \le C, \qquad MM_h(v,\Omega) \le C.$$

Concerning u the gradients ∇u are bounded in L^p, and since

$$\|\nabla u - \nabla \phi\|_{L^p(U,\mathbb{R}^{n\times m})} \quad \text{and} \quad \|u-\phi\|_{W^{1,p}(U,\mathbb{R}^n)}$$

are equivalent, by Sobolev embedding the set of $u-\phi$'s is precompact in L^s, and so is the set of u's since $\phi \in L^s$. Similarly in the Neumann problem the L^p gradient bound and the L^r bound on u give precompactness in every Lebesgue space of exponent strictly smaller than $\max\{r,p^*\}$, in particular in L^s. Clearly the constraint $u=\phi$ outside Ω in (164) is preserved. To get compactness for v we can apply Young's inequality $ab \le \frac{a^s}{s} + \frac{b^t}{t}$ with $s=\frac{q}{n}$ and $t=\frac{q}{q-n}$ to the two integrand addenda:

$$\begin{aligned} MM_h(v,\Omega) &= \int_\Omega \varepsilon_h^{q-n}|\nabla v|^q + \frac{W(1-v)}{\varepsilon_h^n}\, dx \ge \\ &\ge \int_\Omega \left(\frac{q}{n}\varepsilon_h^{q-n}|\nabla v|^q\right)^{\frac{n}{q}} \left(\frac{q}{q-n}\varepsilon_h^{-n}W(1-v)\right)^{\frac{q-n}{q}} dx = \\ &= c_{n,q}\int_\Omega |\nabla v|^n W(1-v)^{\frac{q-n}{q}}\, dx = c'_{n,q}\int_\Omega \big|\nabla[F(1-v)]\big|^n\, dx, \end{aligned} \tag{166}$$

with $F(t)=\int_0^t W^{\frac{q-n}{qn}}(s)ds$. Since Ω is bounded and $0\le v\le 1$ we can use the compact embedding $W^{1,n}(\Omega)\hookrightarrow L^n(\Omega)$ to deduce that the set of $F(1-v_h)$'s is precompact in L^n, hence the set of v's is precompact for the convergence in measure topology since F has a continuous inverse. Furthermore these estimates do not depend on h, hence the claimed precompactness for $h\in\mathbb{N}$ variable. It remains to prove the closedness of (164) and (165): this is equivalent to show the respective energies being lower semicontinuous. Suppose then $\big((u_i,v_i)\big)\subset X(\Omega)$ a convergent sequence and h fixed. The phase transition term MM_h is clearly lower semicontinuous (see the proof of Proposition 5.3.1); so are also $\int_\Omega |\nabla u|^p$ and $\int_\Omega |u-g|^r$. Moreover since $k_h>0$

$$\int_\Omega |M_n\nabla u_i|^\gamma\, dx \le \frac{C}{k_h} < \infty,$$

therefore up to subsequences we have $Ju_i \overset{*}{\rightharpoonup} Ju$, and by Theorem 3.4.1 $Ju \ll \mathscr{L}^m$, thus $u\in R_n(\Omega)$. Furthermore $M_n\nabla u_i \rightharpoonup M_n\nabla u$ weakly in L^1: we claim that

$$\int_\Omega (v(x)+k_h)|M_n\nabla u(x)|^\gamma\, dx \le \liminf_i \int_\Omega (v_i(x)+k_h)|M_n\nabla u_i(x)|^\gamma\, dx.$$

In fact following [**Giu03**, Theorem 4.4], since $v_i\to v$ in measure for every $\delta>0$ there exists $G\Subset\Omega$ compact such that $v_i\to v$ uniformly in G, v and $M_n\nabla u$ are continuous in

G and $\int_{\Omega\setminus G}(v+k_h)|M_n\nabla u|^\gamma dx < \delta$. Therefore

$$\liminf_i \int_\Omega (v_i(x)+k_h)|M_n\nabla u_i(x)|^\gamma dx \geq \liminf_i \int_G (v_i+k_h)|M_n\nabla u|^\gamma dx$$
$$+\liminf_i \int_G \gamma(v+k_h)|M_n\nabla u|^{\gamma-2}\langle M_n\nabla u, M_n\nabla u_h - M_n\nabla u\rangle dx$$
$$+\liminf_i \int_G \gamma(v_i-v)|M_n\nabla u|^{\gamma-2}\langle M_n\nabla u, M_n\nabla u_h - M_n\nabla u\rangle dx :$$

The first integral tends to $\int_G(v+k_h)|M_n\nabla u|^\gamma dx$ by uniform convergence; the second integral is infinitesimal by weak convergence, the term $\gamma(v+k_h)|M_n\nabla u|^{\gamma-2}M_n\nabla u$ being bounded; finally the last addendum can be bounded by

$$\gamma\|M_n\nabla u_h - M_n\nabla u\|_{L^1(\Omega)}\|M_n\nabla u\|^{\gamma-1}_{L^\infty(G)}\sup_G|v_i-v|$$

which is infinitesimal by uniform convergence. Therefore we can bound below the lower limit with $\int_\Omega(v+k_h)|M_n\nabla u|^\gamma dx - \delta$: letting $\delta\downarrow 0$ we obtain the claimed property. □

In particular the previous Theorem guarantees that the energies (E_h) are equicoercive. As a consequence the functionals satisfy condition (b) of Proposition 5.2.1, validating the choice of the topology 5.2.4 in the Γ-limit.

5.3. Optimal profile

In order to investigate the asymptotic behaviour of the functionals E_ε it is useful to understand the behaviour of the Modica-Mortola term, to single out the optimal profile and to study its properties. We consider the fixed scale $\varepsilon = 1$.

PROPOSITION 5.3.1. *We define, for $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$I(f) = \int_{\mathbb{R}^n} |\nabla f|^q + W(f)\,dx.$$

The infimum

$$\sigma = \inf\left\{I(f)\,\middle|\, I(f)<\infty,\ f(0)=1\right\} \tag{167}$$

is meaningful, positive and attained by a unique radial function $w_0 \in B(\mathbb{R}^n)\cap C^{0,\alpha}(\mathbb{R}^n)$, with $\alpha = 1-\frac{n}{q}$, satisfying:

$$\lim_{x\to\infty} w_0(x) = 0. \tag{168}$$

PROOF. First of all it is important to specify that we implicitly set $I(f)=\infty$ whenever f does not possess weak derivatives in L^1_{loc}; moreover since $q>n$ the constraint requirement $f(0)=1$ in the minimization problem is meaningful, because the Sobolev embedding Theorem (see [**AF03**], 4.12) ensures that a function f with $I(f)<\infty$ has a pointwise continuous representative. We will always consider the continuous representative, without specifying it anymore. Observe furthermore that since W is increasing in $\mathbb{R}^+$ and nonnegative, by truncation we can reduce to minimize the energy among functions in $B(\mathbb{R}^n)$ which are ranging in the interval $[0,1]$. Take a minimizing sequence (f_h): again by Sobolev embedding Theorem the functions (f_h) are uniformly Hölder continuous, and equibounded on every compact subset thanks to the constraint $f_h(0)=1$. Hence by the Ascoli-Arzelà Theorem the sequence is precompact in the topology of the local uniform convergence, and we can extract a subsequence converging to $w_0 \in C^{0,\alpha}$ locally uniformly. Hence $w_0(0) = 1$, $W(f_h) \to W(w_0)$ locally uniformly and it is not difficult to check that $\nabla f_h \rightharpoonup \nabla w_0$ in L^q_{loc}. By lower semicontinuity w_0 achieves the infimum. Moreover a radial monotone rearrangement decreases the energy (see [**Tal76, LL01**]) and by the strict convexity of the gradient part there is only one minimizer, w_0, and it is radial. Hölder continuity forces w_0 to be positive on a small ball around 0 implying that the minimum energy σ is strictly positive; for the same reason, since $\int W(w_0) < \infty$, equation (168) must be satisfied. $\square$

Observe that $I(f) = MM_1(1-f, \mathbb{R}^n)$ for $f \in B(\mathbb{R}^n)$. As our optimal function w_0 is radial it is worth investigating its one dimensional profile. Setting $w : [0,\infty) \to \mathbb{R}$, $w(|x|) = w_0(x)$ we have:

$$\sigma = \int_{\mathbb{R}^n} |\nabla w_0|^q + W(w_0)\, dx = \mathscr{H}^{n-1}(S^{n-1}) \int_0^\infty t^{n-1} \left[|w'(t)|^q + W(w(t)) \right] dt, \quad (169)$$

and the Euler-Lagrange equation in $\mathbb{R}^n \setminus \{0\}$ is

$$-q\Delta_q w_0 + W'(w_0) := -q \operatorname{div}(|\nabla w_0|^{q-2}\nabla w_0) + W'(w_0) = 0.$$

In radial coordinates it becomes

$$-\frac{q}{t^{n-1}} \left(t^{n-1}|w'(t)|^{q-2}w'(t) \right)' + W'(w) = 0 \quad (170)$$

outside the origin. We have the following Lemma:

LEMMA 5.3.2. *Let $w : [0,\infty) \to \mathbb{R}$ be the profile of the minimizer of* (167). *Then w is convex, belongs to $C^1(0,+\infty) \cap C^2(\{0 < w < 1\})$ and the following two properties hold:*

$$\lim_{t\to 0} t^n |w'(t)|^q = 0, \quad (171)$$

$$\lim_{t\to+\infty} t^n[|w'(t)|^q + W(w(t))] = 0. \quad (172)$$

PROOF. Since w is nonnegative and decreasing, and $W' \geq 0$ by convexity, the Euler equation implies that

$$0 \leq t^{n-1}W'(w) = q\left(t^{n-1}|w'(t)|^{q-2}w'(t)\right)' = -q(t^{n-1}|w'(t)|^{q-1})'.$$

Both the functions $t^{n-1}|w'(t)|^{q-1}$ and $\frac{1}{t^{n-1}}$ are positive and decreasing. Hence multiplying them we get that $|w'|$ decreases, and since w' is negative we obtain that w is convex. By monotonicity of $|w'|$ and the finiteness of the energy (169),

$$\limsup_{t\to 0} t^n|w'(t)|^q \leq \limsup_{t\to 0} n \int_0^t s^{n-1}|w'(s)|^q ds = 0.$$

Furthermore, since $Z(t) := |w'(t)|^q + W(w(t))$ is decreasing, we have

$$\limsup_{t\to\infty} \frac{1}{n}\left(1 - \frac{1}{2^n}\right) t^n Z(t) \leq \limsup_{t\to\infty} \int_{\frac{t}{2}}^t s^{n-1}Z(s)\, ds \leq \limsup_{t\to\infty} \int_{\frac{t}{2}}^\infty s^{n-1}Z(s)\, ds = 0$$

by the finiteness of the energy (169), which proves (172). Finally in every interval $(a,b) \Subset \{0 < w < 1\}$ we have that $-\infty < w' < w'(b) < 0$, otherwise w would be a positive constant in the half line $(b, +\infty)$. Hence we can extract the $(q-1)$-st root without loosing any smoothness and bootstrap (170):

$$w \in C^{0,\alpha}(a,b) \quad \Rightarrow \quad W'(w) \in C^0(a,b) \quad \Rightarrow \quad w \in C^2(a,b).$$

The same argument shows that $w \in C^1(0,+\infty)$, since $|\cdot|^{\frac{1}{q-1}}$ is continuous. $\square$

In general if $W \in C^k$ and $w \in C^m(a,b)$ then $W'(w) \in C^{m\wedge(k-1)}(a,b)$, hence $w \in C^{(m\wedge(k-1))+2}(a,b)$, hence starting from $m=1$ we obtain $w \in C^{k+1}(\{0<w<1\})$. It is also interesting to analyse whether w touches 0 or not: already in dimension $n=1$ the situation depends on the behaviour of the potential W around 0. Set $\beta > 1$ and $W(f) = |f|^\beta$: the family

$$f_\beta(t) = \begin{cases} \left(1 - \frac{q-\beta}{q(q-1)^{\frac{1}{q}}}t\right)^{\frac{q}{q-\beta}} & \beta < q, \\ e^{-(q-1)^{-\frac{1}{q}}t} & \beta = q, \\ \left(1 + \frac{\beta-q}{q(q-1)^{\frac{1}{q}}}t\right)^{-\frac{q}{\beta-q}} & \beta > q. \end{cases} \tag{173}$$

minimizes $\int_{\mathbb{R}} |f'|^q + |f|^\beta \, dx$ with $f(0)=1$, therefore the threshold exponent for the solution to touch 0 is $\beta = q$. This is also true in higher dimension, although we are not able to produce the explicit expressions of the minimizers:

PROPOSITION 5.3.3. *Let W and w as above, $n \geq 1$ and suppose that $W(f) \sim |f|^\beta$ near $f=0$, for some $\beta > 1$. Then $w(T) = 0$ for some finite $T > 0$ if and only if $\beta < q$.*

PROOF. Multiplying the Euler-Lagrange equation (170) by $t^{n-1}w'$ we obtain after some manipulations

$$0 = -(q-1)(t^{n-1}|w'|^q)' - (n-1)t^{n-2}|w'|^q + t^{n-1}[W(w(t))]'. \tag{174}$$

Set for simplicity $a(t) := (q-1)t^{n-1}|w'|^q$: then

$$0 = -a'(t) - \frac{n-1}{(q-1)t}a(t) + t^{n-1}[W(w(t))]' = -t^{\frac{1-n}{q-1}}[t^{\frac{n-1}{q-1}}a(t)]' + t^{n-1}[W(w(t))]',$$

that is:

$$[t^{\frac{n-1}{q-1}}a(t)]' = t^{q\frac{n-1}{q-1}}[W(w(t))]'.$$

Integrating between t and T we get (recall $w \in C^1$ globally, so $a(T) = W(w(T)) = 0$):

$$-t^{\frac{n-1}{q-1}}a(t) = -t^{q\frac{n-1}{q-1}}W(w(t)) - \int_t^T q\frac{n-1}{q-1}s^{q\frac{n-1}{q-1}-1}W(w(s))ds, \tag{175}$$

or

$$t^{\frac{n-1}{q-1}}a(t) = t^{q\frac{n-1}{q-1}}W(w(t)) + \int_t^T q\frac{n-1}{q-1}s^{q\frac{n-1}{q-1}-1}W(w(s))ds: \tag{176}$$

that is (for $t \to T > 0$)

$$\begin{aligned}(q-1)|w'(t)|^q &= W(w(t)) + t^{q\frac{1-n}{q-1}}\int_t^T q\frac{n-1}{q-1}s^{q\frac{n-1}{q-1}-1}W(w(s))ds \\ &= W(w(t)) + o(W(w(t))).\end{aligned}$$

Therefore
$$\frac{dw}{W^{\frac{1}{q}}(w(t))} \sim dt$$
and $T < \infty$ if and only if $\beta < q$. □

Equation (176) applied to a generic $T > 0$ becomes

$$\begin{aligned} t^{q\frac{n-1}{q-1}}|w'(t)|^q - T^{q\frac{n-1}{q-1}}|w'(T)|^q = t^{q\frac{n-1}{q-1}}W(w(t)) - T^{q\frac{n-1}{q-1}}W(w(T)) \\ + \int_t^T q\frac{n-1}{q-1}s^{q\frac{n-1}{q-1}-1}W(w(s))ds \quad (177) \end{aligned}$$

and yields, for $t \to 0$, that $|w'(t)| = O\left(\frac{1}{t^{\frac{n-1}{q-1}}}\right)$, because W is bounded and $q\frac{n-1}{q-1}-1 > -1$, so the integral is convergent and
$$\lim_{t\to 0} t^{q\frac{n-1}{q-1}}|w'(t)|^q$$
exists finite. As a consequence the bound $|w'(t)| \le Ct^{\frac{1-n}{q-1}}$ can be extended to every finite interval $[0, R]$, for a suitable high constant C: we deduce a local higher integrability property for ∇w_0 in $\mathbb{R}^n$, because
$$\int_{B^n_R} |\nabla w_0|^r = C\int_0^R t^{n-1}|w'(t)|^r dt \le C\int_0^R t^{n-1-r\frac{1-n}{q-1}}dt$$
which implies
$$\nabla w \in L^r(B^n_R, \mathbb{R}^n) \tag{178}$$
for $r < (q-1)\frac{n}{n-1}$. In particular in this range of exponents $t^n|w'(t)|^r \to 0$ for $t \to 0$.

5.4. Γ-lower limit

In this section we aim to prove the first part of Theorem 5.2.8, regarding the Γ-lower limit of the sequence (E_h):

THEOREM 5.4.1. *Let Ω be an open subset of $\mathbb{R}^m$. For every sequence $\big((u_h, v_h)\big) \subset Y(\Omega)$ such that $(u_h, v_h) \to (u, v)$ we have*
$$\liminf_{h\to\infty} E_h(u_h, v_h, \Omega) \ge E(u, v, \Omega);$$
moreover
$$\liminf_{h\to\infty} E_h(u_h, v_h, \Omega) < \infty \quad \Rightarrow \quad u \in GSB_nV(\Omega) \ \textit{and} \ v = 1.$$

The proof will be achieved through a slicing argument, by first proving that in codimension $m - n = 0$ the jacobians Ju_h concentrate around a finite number of points. Our definition of size outlined in the introduction is well-suited to this slicing procedure, and a final localization result yields the proof.

5.4.1. Proof in $\mathbb{R}^n$. Let A be an open subset of $\mathbb{R}^n$: to ease the exposition for any $(u,v) \in Y(A)$ we let

$$G_h(u,v,A) = \int_A (v+k_h)|\det \nabla u|^\gamma dx + \int_A \varepsilon_h^{q-n}|\nabla v|^q + \frac{W(1-v)}{\varepsilon_h^n} dx$$

be the part of energy depending explicitly on v.

THEOREM 5.4.2. *Let A be an open subset of $\mathbb{R}^n$ and let $((u_h,v_h)) \subset Y(A)$, $(u,v) \in X(\Omega)$ satisfy $(u_h,v_h) \to (u,v)$ and $\|\nabla u_h\|_p \leq C$. Assume also*

$$\liminf_{h\to\infty} G_h(u_h,v_h,A) < \infty. \tag{179}$$

Then $u \in GSB_nV(A,\mathbb{R}^n)$, $v=1$ and

$$\liminf_{h\to\infty} G_h(u_h,v_h,A) \geq \int_A |\det \nabla u|^\gamma dx + \sigma \mathscr{H}^0(A \cap S_u).$$

First of all we extract a subsequence, not relabeled, that achieves the lower limit in (179) and such that $\nabla u_h \rightharpoonup \nabla u$ weakly in L^p. We notice right away that $u \in W^{1,p}$ and $v=1$; also by Proposition 3.1.2 we know that $\mathbf{F}_\Omega^{\mathrm{loc}}(Ju_h \llcorner \Omega - Ju \llcorner \Omega) \to 0$, hence $Ju_h \overset{*}{\rightharpoonup} Ju$ as currents in Ω. We begin with the regular part, disregarding the positive infinitesimal k_h:

LEMMA 5.4.3. *Assume that A is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Then*

$$\liminf_{h\to\infty} \int_A v_h |\det \nabla u_h|^\gamma dx \geq \int_A |\det \nabla u|^\gamma dx. \tag{180}$$

PROOF. Since A is regular and bounded, $q > n$ and the norms $\|\nabla v_h\|_q$ are equibounded by Sobolev embedding Theorem

$$[v_h]_{C^{0,\alpha}(A)} \leq C(A)\varepsilon_h^{-\alpha},$$

where $\alpha = 1 - \frac{n}{q}$ and $C(A)$ depends on the energy and on the regularity of A. We also fix a threshold $t \in (0,1)$: by Hölder continuity there exists $c_0 = c_0(C,t) > 0$ independent of h such that for every $x \in A \cap \{v_h < t\}$

$$A \cap B(x, c_0\varepsilon_h) \subset A \cap \left\{ v_h < \frac{1+t}{2} \right\}. \tag{181}$$

We can then cover $A \cap \{v_h < t\}$ with balls centered at every point having radius $\frac{c_0\varepsilon_h}{5}$: by Vitali's covering Lemma there is a countable disjoint subfamily $\mathcal{F} = \left\{ B(x_i, \frac{c_0\varepsilon_h}{5}) \right\}$ such that

$$\bigcup_i B(x_i, c_0\varepsilon_h) \supset A \cap \{v_h < t\}.$$

Thanks to (181) we can estimate from below MM_{ε_h} of every such small ball:

$$\int_{A\cap B\left(x_i,\frac{c_0\varepsilon_h}{5}\right)} \varepsilon_h^{q-n}|\nabla v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^n} dx \geq W\left(\frac{1-t}{2}\right) \frac{\mathscr{L}^n\left(A \cap B\left(x_i, \frac{c_0\varepsilon_h}{5}\right)\right)}{\varepsilon_h^n}.$$

The latter quantity is bounded below independently of h because the Lipschitz boundary condition on A ensures that $\mathscr{L}^n\left(A \cap B\left(x_i, \frac{c_0\varepsilon_h}{5}\right)\right) \geq c\varepsilon_h^n$. The family $\mathcal{F}$ being disjoint, by the finiteness of the energy we argue that there can be only a finite number N, independent of h, of such balls. Let us then extract a subsequence, not relabeled, along which the balls are in constant number N and the centers $\{x_i^h\}, i = 1, \ldots, N$ converge to points $x_i^\infty \in \overline{A}$. For every open set

$$A' \Subset A \setminus \bigcup_i \{x_i^\infty\}$$

we have that for h sufficiently large:

$$A' \cap \bigcup_i B(x_i^h, c_0 \varepsilon_h) = \emptyset \qquad \text{and} \qquad v_h\big|_{A'} \geq t.$$

The energy bound (179) allows to bound a superlinear power of the jacobians in A'

$$\int_{A'} |\det \nabla u_h|^\gamma \, dx \leq \frac{C}{t+1}$$

hence Theorem 3.4.1 gives

$$\det \nabla u_h \rightharpoonup \det \nabla u \quad \text{weakly in } L^1(A'). \tag{182}$$

By lower semicontinuity

$$\liminf_{h\to\infty} \int_A v_h |\det \nabla u_h|^\gamma dx \geq \liminf_{h\to\infty} \int_{A'} v_h |\det \nabla u_h|^\gamma dx \geq$$
$$\geq \liminf_{h\to\infty} t \int_{A'} |\det \nabla u_h|^\gamma dx \geq t \int_{A'} |\det \nabla u|^\gamma dx. \tag{183}$$

Finally letting $A' \uparrow A \setminus \bigcup_i \{x_i^\infty\}$ and then $t \uparrow 1$ we obtain the result. □

REMARK 5.4.4. The same result of Lemma 5.4.3 holds without the regularity hypothesis on A. In fact it is sufficient to consider a sequence of nested regular open subsets $A_j \subset A$ invading A, apply the Lemma to A_j and then let $A_j \uparrow A$: the left hand side of (180) clearly decreases when restricted to each A_j, and the right hand side by Monotone convergence Theorem increases to $\int_A |\det \nabla u|^\gamma dx$.

Now we analyze the MM_ε term, and prove that around the potentially singular points of the limit function u this energy concentrates. Observe that we still do not know that $u \in GSB_nV$: Ju so far is only a flat current, nevertheless chosen a fixed point x_0 for almost every radius ρ the restriction $Ju \llcorner B_\rho(x_0)$ is meaningful and furthermore $\mathbf{F}(Ju_h \llcorner B_\rho(x_0) - Ju \llcorner B_\rho(x_0)) \to 0$ (see Proposition 2.4.4). With a slight abuse of notation we indicate by $Ju \llcorner B_\rho \ll \mathscr{L}^n$ the fact that $\mathbf{M}(Ju \llcorner B_\rho) < \infty$ and $\|Ju \llcorner B_\rho\| \ll \mathscr{L}^n$: by definition this is satisfied if $u \in R_n$.

LEMMA 5.4.5. *Let $((u_h, v_h))$, u and A as in Theorem 5.4.2, and fix $x_0 \in A$. Suppose $Ju \llcorner B_\rho(x_0) \not\ll \mathscr{L}^n$ for every $\rho > 0$ such that $B_\rho(x_0) \subset A$. Then*

$$\liminf_{h\to\infty} MM_h(v_h, B_\rho(x_0)) \geq \sigma \quad \forall \rho > 0, \tag{184}$$

where σ is defined as in (167).

PROOF. Fix an arbitrary ρ as in the hypotheses and let us suppose for simplicity that $x_0 = 0$: since $Ju \llcorner B_\rho \not\ll \mathscr{L}^n$ we must have

$$\lim_{h\to\infty} \inf_{B_\rho} v_h = 0 \quad \forall \rho > 0.$$

In fact if there were a radius $\bar\rho$ and a subsequence $(v_{\bar h})$ bounded away from zero in $B_{\bar\rho}$, since $Ju_{\bar h} \ll \mathscr{L}^m$ by Theorem 3.4.1 the limit $Ju \llcorner B_{\bar\rho}$ would otherwise be a current in $\mathbf{M}_0(B_{\bar\rho})$ with absolutely continuous mass. The finiteness of the energy (179) guarantees that $v_h \to 1$ in measure in B_ρ. In order to show (184) we modify in B_ρ the asymptotic profiles v_h and we relate them to problem (167). Let us perform the following radial monotone rearrangement of v_h, denoted v_h^*, which preserve the measure of *sublevels*:

$$v_h^*(x) := \inf\big\{t : |\{v_h < t\} \cap B_\rho| > \mathscr{L}^n(B_1)|x|^n\big\} \quad \forall x \in B_\rho.$$

This rearrangement preserves the integral $\int_{B_\rho} W(1-v)$ by the Coarea formula (159), while the L^q norm of the gradient decreases, see [**Tal76, LL01**]. We immediately have that

$$MM_h(v_h, B_\rho) \geq MM_h(v_h^*, B_\rho) \quad \text{and} \quad v_h^* \to 1 \text{ in measure in } B_\rho.$$

In particular $\lambda_h := v_h^*\big|_{\partial B_\rho} \to 1$, and $\mu_h := \inf_{B_\rho} v_h^* = v_h^*(0) \to 0$, hence we can extend v_h^* equal to λ_h for $|x| \geq \rho$. The functions $f_h : \mathbb{R}^n \to \mathbb{R}$,

$$f_h(y) := \frac{1}{\lambda_h - \mu_h}\big(v_h^*(\varepsilon_h y) - \mu_h\big) \tag{185}$$

satisfy

(a) $f_h(0) = 0$,
(b) $\operatorname{spt}(1 - f_h) \subset \overline{B_\rho}$,
(c) $1 - v_h^*(x) \geq \lambda_h - v_h^*(x) = (\lambda_h - \mu_h)\big(1 - f_h(\frac{x}{\varepsilon_h})\big)$.

Let us now evaluate the MM_h energy (recall W is monotone increasing):

$$\begin{aligned} MM_h(v_h, B_\rho) \geq MM_h(v_h^*, B_\rho) &= \int_{B_\rho} \varepsilon_h^{q-n}|\nabla v_h^*(x)|^q + \frac{W(1 - v_h^*(x))}{\varepsilon_h^n}\, dx \\ &\overset{(c)}{\geq} \int_{B_\rho} \varepsilon_h^{q-n}|\nabla v_h^*(x)|^q + \frac{W(\lambda_h - v_h^*(x))}{\varepsilon_h^n}\, dx \\ &\geq \int_{\mathbb{R}^n} (\lambda_h - \mu_h)^q|\nabla f_h|^q + W((\lambda_h - \mu_h)(1 - f_h))dx. \end{aligned} \tag{186}$$

By properties (a) and (b) the functions $1 - f_h$ are competitors for problem (167) and $\lambda_h - \mu_h \to 1$, hence the last integral is asymptotically greater or equal than the infimum σ. □

Proof of Theorem 5.4.2. Let $\Sigma = \{x \in A : Ju \llcorner B_\rho(x) \not\ll \mathscr{L}^n \text{ for all } B_\rho(x) \subset A\}$. Then the superadditivity of the lim inf together with (179) and Lemma 5.4.5 gives

$$\mathscr{H}^0(\Sigma) \leq \frac{1}{\sigma} \liminf_h G_h(u_h, v_h, A).$$

Moreover Lemma 5.4.3 showed the existence of another finite set Υ such that $Ju \llcorner (A \setminus \Upsilon) \ll \mathscr{L}^n$. Hence necessarily $\Sigma \subset \Upsilon$ and the flat defect current

$$T := (Ju - \det \nabla u\, \mathbf{E}^m) \llcorner A$$

is supported in Υ. By the general theory of flat currents presented in chapter 2 (see Theorem 2.6.3) $\mathbf{M}(T) < \infty$ and $T = \sum_{x_i \in \Sigma} a_i [\![x_i]\!]$. In particular $u \in GSB_nV(A)$ and

$S_u \subset \Sigma$, so

$$\mathscr{H}^0(S_u \cap A) \le \mathscr{H}^0(\Sigma).$$

Taking $B \Subset A \setminus \Sigma$ open, and applying the superadditivity of the lower limit on open disjoint sets, as well as (180) to B we obtain for some ρ sufficiently small (so that $B \cap \bigcup_{x\in\Sigma} B_\rho(x) = \emptyset$)

$$\begin{aligned}\liminf_h G_h(u_h, v_h, A) &\ge \liminf_h \int_B v_h |\det \nabla u_h|^\gamma dx \\ &\quad + \sum_{x\in S_u\cap A} \liminf_h \int_{B_\rho(x)\cap A} \varepsilon_h^{q-2} |\nabla v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^2} dx \\ &\ge \int_B |\det \nabla u|^\gamma dx + \sigma \mathscr{H}^0(S_u \cap A).\end{aligned}$$

Letting $B \uparrow A$ concludes the proof. □

5.4.2. Reduction argument and proof of Theorem 5.4.1 for general m, n. In this paragraph we prove Theorem 5.4.1 from the results obtained in the previous paragraph in dimension n. We will first use the slicing properties of the jacobians to reduce to the n-dimensional case discussed above, and then we will optimize the choices of the slicing directions to conclude.

Proof. As a preliminary step let us extract a subsequence out of $((u_h, v_h))$ such that the lower limit $\liminf_h E_h(u_h, v_h, \Omega)$ is attained and such that $(u_h, v_h) \to (u, 1)$ rapidly in $X(\Omega)$:

$$\sum_h \|u_h - u\|_{L^s} + d(v_h, 1) < \infty.$$

This implies that given an orthogonal projection $\pi \in \mathbf{O}_{m-n}$, for $\mathscr{L}^{m-n}$-almost every $x \in \pi(\Omega)$

$$u_h(x, \cdot) \to u(x, \cdot) \text{ in } L^s(\Omega^x, \mathbb{R}^n) \qquad \text{and} \qquad v_h(x, \cdot) \to 1 \text{ in measure in } \Omega^x,$$

where we put $\Omega^x := \Omega \cap \pi^{-1}(x)$. Let us consider an arbitrary open subset $A \subset \Omega$ and let us fix a projection π as above. Observe that the energy E_h is bounded along (u_h, v_h): using Fatou's Lemma we obtain

$$\begin{aligned}&\lim_{h\to\infty} E_h(u_h, v_h, \Omega) \ge \liminf_{h\to\infty} E_h(u_h, v_h, A) \ge \\ &\ge \int_{\pi(A)} \liminf_{h\to\infty} \left\{ \int_{A^x} |\nabla_y u_h|^p + v_h |\det \nabla_y u_h|^\gamma + \varepsilon_h^{q-n} |\nabla_y v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^n} dy \right\} dx.\end{aligned}$$

In particular for $\mathscr{L}^{m-n}$-almost every $x \in \pi(A)$

$$\liminf_h \int_{A^x} |\nabla_y u_h(x, y)|^p \, dy + G_h(u_h, v_h, A^x) \le C(x) < \infty.$$

For these x we can extract a subsequence $(u_{h(k)})$, a priori depending on the point x, along which both the L^p norm of $\nabla u_{h(k)}$ and the n-dimensional energy G_h are bounded:

$$\sup_k \int_{A^x} |\nabla_y u_{h(k)}|^p \, dy + G_h(u_{h(k)}, v_{h(k)}, A^x) < \infty. \tag{187}$$

This implies that

$$\nabla_y u_{h(k)}(x,\cdot) \rightharpoonup \nabla_y u(x,\cdot) \text{ in } L^p(A^x). \tag{188}$$

Finally observe that the slicing Theorem 3.2.9 immediately gives that $(u_h(x,\cdot), v_h(x,\cdot)) \in Y(A^x)$ almost everywhere. Theorem 5.4.2 implies that $u(x,\cdot) \in GSB_nV(A^x)$ and that

$$\liminf_h G_h(u_h, v_h, A^x) \geq \int_{A^x} |\det \nabla_y u(x,\cdot)|^\gamma dy + \sigma \mathscr{H}^0(A^x \cap S_{u(x,\cdot)});$$

integrating and applying Fatou's Lemma on the left hand side we have

$$\begin{aligned}\liminf_h E_h(u_h, v_h, A) \geq \int_A |\nabla u|^p + \int_{\pi(A)} \liminf_h G_h(u_h(x,\cdot), v_h(x,\cdot), A^x)\, dx \geq \\ \geq \int_A |\nabla u|^p + \int_A |\det \nabla_y u(x,\cdot)|^\gamma dy dx + \sigma \int_{\pi(A)} \mathscr{H}^0(A^x \cap S_{u(x,\cdot)}) dx.\end{aligned}$$

Let us call

$$\tau_\pi(A) := \int_A |\det \nabla_y u(x,y)|^\gamma dy dx + \sigma \int_{\pi(A)} \mathscr{H}^0(A^x \cap S_{u(x,\cdot)}) dx.$$

the right hand side and $\underline{E}(A) = \liminf_h E_h(u_h, v_h, A)$. $\underline{E}(\cdot)$ is a superadditive set function on open sets such that $\underline{E}(A) \leq \underline{E}(\Omega) < \infty$ and each single τ_π is a finite Borel measure; therefore taking disjoint open sets $A_1, \ldots, A_k$ and orthogonal projections $\pi_1, \ldots, \pi_k$ we have that

$$\sum_i \tau_{\pi_i}(A_i) \leq \sum_i \underline{E}(A_i) \leq \underline{E}(\Omega). \tag{189}$$

By inner and outer regularity of τ_{π_i} inequality (189) holds for generic disjoint Borel sets B_i instead of A_i, hence the supremum

$$\tau := \bigvee_\pi \tau_\pi \tag{190}$$

is a finite Borel measure. In particular $M_n \nabla u \in L^\gamma$ and since for every projection π slice and jacobian commute according to Theorem 3.2.9, we have that the current $T = (Ju - M_n \nabla u\, \mathbf{E}^m) \llcorner \Omega$ satisfies $\operatorname{spt}(\langle T, \pi, x\rangle) \subset S_{u(x,\cdot)}$ almost everywhere, so its size is finite. Hence $u \in GSB_nV(\Omega)$. Finally since by Definition 3.3.1 the measures μ_T and $|M_n \nabla u|^\gamma \mathscr{L}^m$ are mutually singular and $|M_n L| = \sup_\pi |M_n L \llcorner d\pi|$, it is not difficult to prove that the supremum τ equals

$$\tau = |M_n \nabla u|^\gamma \mathscr{L}^m + \sigma \mathscr{H}^{m-n} \llcorner S_u,$$

which concludes the proof. □

5.5. Γ-upper limit

This section is devoted to the proof of the upper limit inequality: our construction of the recovery sequence will mimic the truncation argument presented in [**AT90**] and [**AT92**]. Note that we only assume a mild geometric property on the singular set S_u expressed in terms of its Minkowski content. We provide an interior statement as well as boundary statement, where differently from [**AT90**] we need to take care of any possible accumulation of the singular set at the boundary. The limit energy must account for the possible loss of mass in the Modica-Mortola term, due to the transition of v happening partially outside the domain. We finally generalize the form of the functional in which the size term is weighted by a continuous density.

THEOREM 5.5.1. *Suppose $\Omega \subset \mathbb{R}^m$ is a bounded set of class C^1 and $u \in GSB_nV(\Omega)$ with constraints*

$$s \geq \frac{np}{n-p}, \qquad 1 < \gamma \leq \frac{1}{\frac{n-1}{p} + \frac{1}{s}}.$$

Let also (k_h) be a positive sequence such that $k_h = o(\varepsilon_h^\gamma)$. If

$$E(u, 1, \Omega) < \infty, \qquad \mathcal{M}_\Omega^{*m-n}(S_u) = \mathscr{H}^{m-n}(S_u) \tag{191}$$

then there exists a sequence $\big((u_h, v_h)\big) \subset Y(\Omega)$ such that

$$(u_h, v_h) \to (u, 1) \quad \text{and} \quad \limsup_{h\to\infty} E_h(u_h, v_h, \Omega) \leq E(u, 1, \Omega).$$

5.5.1. Proof of Theorem 5.5.1. We start by setting the approximating sequence (v_h) for a generic $\mathscr{L}^m$-null and closed set $S \subset \Omega$ satisfying

$$\mathcal{M}_\Omega^{*m-n}(S) = \mathscr{H}^{m-n}(S) < \infty. \tag{192}$$

Let $w(t)$ be the optimal profile of problem 5.3.1 and choose $\delta_h \downarrow 0$ such that $k_h(\varepsilon_h\delta_h)^{-\gamma} \to 0$. Let

$$w_h(t) := \min\left\{\frac{w(t)}{w(\delta_h)}, 1\right\}$$

so that $w_h(|x|) = 1$ in $B_{\delta_h}(0)$. Clearly $w_h'(\delta_h)$ is finite and $I(w_h) \to I(w)$ as $h \to \infty$. Moreover by the proof of Lemma 5.3.2

$$|w'(t)|^q + W(w(t))$$

is C^1 and decreases to 0 for $t \to \infty$: these properties hold true in $(\delta_h + \infty)$ for w_h. Set

$$v_h(x) = 1 - w_h\left(\frac{d(x,S)}{\varepsilon_h}\right), \tag{193}$$

where $d(x, S) = \operatorname{dist}(x, S)$: note that $v_h \to 1$ in measure and that by equation (158)

$$|\nabla v_h(x)| = \frac{1}{\varepsilon_h}\left|w_h'\left(\frac{d(x,S)}{\varepsilon_h}\right)\right|$$

at almost every point x. Recall the notations $S_r = \{x \in \Omega : 0 < \operatorname{dist}(x, S) \leq r\}$ and $V(r) = \mathscr{L}^m(S_r)$.

PROPOSITION 5.5.2. *The functions (v_h) satisfy*

$$v_h = 0 \quad on \quad S_{\varepsilon_h\delta_h}$$

and

$$\limsup_{h\to\infty} MM_h(v_h, \Omega) \leq \sigma \mathcal{M}_\Omega^{*m-n}(S).$$

PROOF. The first statement is true by the definition (193). Looking at the energy

$$MM_h(v_h, \Omega) = \int_\Omega \varepsilon_h^{q-n}|\nabla v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^n} dx$$

we observe right away that the integration on the set $S_{\varepsilon_h\delta_h}$ in infinitesimal, since there v_h is identically 0 and so

$$\int_{S_{\varepsilon_h\delta_h}} \varepsilon_h^{q-n}|\nabla v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^n}dx = W(1)\frac{V(\varepsilon_h\delta_h)}{\varepsilon_h^n} \to 0.$$

Applying the Coarea formula (159) on the level sets of the distance function $d(\cdot, S)$ we can write

$$\begin{aligned} MM_h(v_h,\Omega) &= o(1) + \int_{\Omega\setminus S_{\varepsilon_h\delta_h}} \varepsilon_h^{q-n}|\nabla v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^n}dx \\ &= o(1) + \int_{\varepsilon_h\delta_h}^{+\infty} \left[\left|w_h'(\tfrac{t}{\varepsilon_h})\right|^q + W\left(w_h(\tfrac{t}{\varepsilon_h})\right)\right]\frac{V'(t)}{\varepsilon_h^n}dt \\ &= o(1) + \int_{\delta_h}^{+\infty} \left[|w_h'(s)|^q + W(w_h(s))\right]\frac{[V(\varepsilon_h s)]'}{\varepsilon_h^n}ds. \end{aligned}$$

Since $Z_h(s) := |w_h'(s)|^q + W(w_h(s))$ is C^1 we can integrate by parts

$$\int_{\delta_h}^{+\infty} Z_h(s)\frac{[V(\varepsilon_h s)]'}{\varepsilon_h^n}ds = -\int_{\delta_h}^{+\infty} Z_h'(s)\frac{V(\varepsilon_h s)}{\varepsilon_h^n}ds + \frac{Z_h(+\infty)V(+\infty) - Z_h(\delta_h)V(\varepsilon_h\delta_h)}{\varepsilon_h^n}.$$

As previously outlined $Z_h(+\infty) = 0$ and $V(+\infty) = \mathscr{L}^m(\Omega)$, hence the second addendum is null; moreover $\frac{V(\varepsilon_h\delta_h)}{\varepsilon_h^n} \le (\mathcal{M}_\Omega^{*m-n}(S)+1)\mathscr{L}^n(B_1)\delta_h^n$ and $Z_h(\delta_h) = |w_h'(\delta_h)|^q + W(w_h(\delta_h))$, so also the third term goes to 0 by Lemma 5.3.2. The basic assumption (192) on the Minkowski content of S implies that there exist infinitesimal numbers ξ_h such that

$$V(s) \le \mathscr{L}^n(B_1)\mathcal{M}_\Omega^{*m-n}(S)s^n + \xi_h s^n \quad \forall s \in [0, \varepsilon_h \mathrm{diam}(\Omega)]. \tag{194}$$

Recall that $Z_h'(s) \le 0$ in $[\delta_h, \infty)$ and $I(w_h) \to I(w) = \sigma$:

$$\begin{aligned} MM_h(v_h,\Omega) &= o(1) - \int_{\delta_h}^{+\infty} Z_h'(s)\frac{V(\varepsilon_h s)}{\varepsilon_h^n}ds \\ &\le o(1) - \int_{\delta_h}^{+\infty} Z_h'(s)(\mathscr{L}^n(B_1)\mathcal{M}_\Omega^{*m-n}(S)+\xi_h)s^n ds \qquad (195) \\ &\overset{(171)(172)}{=} o(1) + n(\mathscr{L}^n(B_1)\mathcal{M}_\Omega^{*m-n}(S)+\xi_h)\int_{\delta_h}^{+\infty} s^{n-1}Z_h(s)ds \\ &= o(1) + (\mathscr{H}^{n-1}(S^{n-1})\mathcal{M}_\Omega^{*m-n}(S) + n\xi_h)\int_{\delta_h}^{+\infty} s^{n-1}Z_h(s)ds \\ &= o(1) + \mathcal{M}_\Omega^{*m-n}(S)\cdot I(w_h) = o(1) + \sigma\mathcal{M}_\Omega^{*m-n}(S). \end{aligned}$$

□

Remark 5.5.3. Observe that the same Proposition proves something more general, that will be useful in the sequel: if $\bar{w}$ is a radial profile such that $\bar{Z}(t) := |\bar{w}'(t)|^q + W(\bar{w}(t))$ is decreasing, then the sequence (v_h) constructed from $\bar{w}$ as in (193) satisfies:

$$\limsup_h MM_h(v_h,\Omega) \le I(\bar{w}(|x|))\mathcal{M}_\Omega^{*m-n}(S).$$

We now show how to construct the sequence (u_h). Outside $S_{\varepsilon_h\delta_h}$ the jacobian Ju is absolutely continuous, hence there is no need to modify u there. We will only change u inside $S_{\varepsilon_h\delta_h}$ with the scope of keeping

$$\int_\Omega |\nabla(u-u_h)|^p\,dx$$

infinitesimal, and letting

$$\int_{S_{\varepsilon_h\delta_h}} |M_n\nabla u_h|^\gamma\,dx$$

diverge at a controlled rate, independently of the function u. Note that this is equivalent to show

$$E_h(u_h,v_h,S_{\varepsilon_h\delta_h}) = \int_{S_{\varepsilon_h\delta_h}} |\nabla u_h|^p + k_h|M_n\nabla u_h|^\gamma\,dx + W(1)\frac{\mathscr{L}^m(S_{\varepsilon_h\delta_h})}{\varepsilon_h^n} \to 0$$

for suitable k_h, because the last term is infinitesimal by (191). Suppose ϕ^1 is a smooth function. If we multiply only the first coordinate by ϕ^1 and compute the jacobian determinant we obtain

$$\nabla(\phi^1u^1,u^2,\dots,u^n) = (\phi^1\nabla u^1,\nabla u^2,\dots,\nabla u^n) + (u^1\nabla\phi^1,\nabla u^2,\dots,\nabla u^n),$$

hence

$$J(\phi^1u^1,u^2,\dots,u^n) = \phi^1 Ju + u^1 J(\phi^1,u^2,\dots,u^n) \tag{196}$$

in the sense of currents; also the following pointwise estimate holds for $1\le k\le n$:

$$\begin{aligned}|M_k&\nabla(\phi^1u^1,u^2,\dots,u^n)| \\ &\le \left|\binom{n-1}{k}|M_k\nabla u|^2 + \binom{n-1}{k-1}(\|\phi^1\|_{L^\infty}|M_k\nabla u| + \|\nabla\phi^1\|_{L^\infty}|u^1||M_{k-1}\nabla u|)^2\right|^{\frac12} \\ &\qquad\le c_{n,k}\big((1+\|\phi^1\|_{L^\infty})|M_k\nabla u| + \|\nabla\phi^1\|_{L^\infty}|u^1||M_{k-1}\nabla u|\big).\end{aligned}$$

Therefore if we truncate u by multiplying each component u^i by smooth functions ϕ^i which satisfy $\operatorname{spt}(\nabla\phi^i)\cap\operatorname{spt}(\nabla\phi^j)=\emptyset$ for $i\ne j$, we obtain that

$$\begin{aligned}&\phi\bowtie u := (\phi^1u^1,\phi^2u^2,\dots,\phi^nu^n) = 0 \qquad \text{in } \{\phi=0\} = \bigcap_i\{\phi^i=0\},\\ &|M_k\nabla(\phi\bowtie u)| \le c_{n,k}\big((1+\|\phi\|_{L^\infty})|M_k\nabla u| + \|\nabla\phi\|_{L^\infty}|u||M_{k-1}\nabla u|\big)\end{aligned} \tag{197}$$

because at each point for only one index j the gradient row $\nabla(\phi^ju^j)$ will present the non zero extra term $u^j\nabla\phi^j$. Observe also that (196) implies that

$$\text{if}\quad S_u \Subset \{\phi=0\} \qquad \text{then}\quad J(\phi^1u^1,\dots,\phi^nu^n) \ll \mathscr{L}^m.$$

Finally note that if the supports of the gradients $\operatorname{spt}(\nabla\phi^j)$ overlap then the jacobian of $u\bowtie\phi$ will in general be bounded by the full vector of minors $M\nabla u$; however the particular choice where all ϕ^i's are equal restores the dependence of the bound only on the precedent order minor, since the choice of $\nabla\phi$ in two rows annihilates the minor.

Choose functions $\phi_h = (\phi_h^1,\dots,\phi_h^n)$ such that

- $0\le\phi_h^i\le 1$;
- $\phi_h^i = 1$ outside $S_{(2^{-1}+2^{-i})\varepsilon_h\delta_h}$;

- $\phi_h^i = 0$ inside $S_{(2^{-1}+2^{-i-1})\varepsilon_h\delta_h}$;
- $|\nabla\phi_h^i| \leq 2^{i+2}(\varepsilon_h\delta_h)^{-1}$

and set $u_h := \phi_h \bowtie u$: Then clearly $(u_h, v_h) \in Y(\Omega)$; note also that $u_h \to u$ in L^s by dominated convergence. Moreover by the conditions on (ϕ_h^i), estimate (197) applied to $k = 1$ (with the convention $M_0\nabla u = 1$) reduces to $|\nabla u_h| \leq c_n(|\nabla u| + (\varepsilon_h\delta_h)^{-1}|u|)$ and yields

$$\begin{aligned}\int_\Omega |\nabla(u_h - u)|^p dx &\leq c_{n,p}\int_{S_{\varepsilon_h\delta_h}} |\nabla u|^p dx + c_{n,p}(\varepsilon_h\delta_h)^{-p}\int_{S_{\varepsilon_h\delta_h}} |u|^p dx \leq \\ &\leq c_{n,p}\int_{S_{\varepsilon_h\delta_h}} |\nabla u|^p dx + \frac{c_{n,p}}{(\varepsilon_h\delta_h)^p}\left(\int_{S_{\varepsilon_h\delta_h}} |u|^{\frac{np}{n-p}}\,dx\right)^{\frac{n-p}{n}} \mathscr{L}^m(S_{\varepsilon_h\delta_h})^{\frac{p}{n}} \\ &\leq c_{n,p}\int_{S_{\varepsilon_h\delta_h}} |\nabla u|^p dx + c_{n,p}\|u\|^p_{L^{\frac{np}{n-p}}(S_{\varepsilon_h\delta_h})}(1 + \mathcal{M}^{*m-n}(S_u))^{\frac{p}{n}}.\end{aligned} \tag{198}$$

Therefore u_h is close to u in $W^{1,p}$. Regarding the jacobian term:

PROPOSITION 5.5.4. *If $k_h(\varepsilon_h\delta_h)^{-\gamma} \to 0$ then*

$$\limsup_{h\to\infty}\int_\Omega (v_h + k_h)|M_n\nabla u_h|^\gamma\,dx = \int_\Omega |M_n\nabla u|^\gamma\,dx. \tag{199}$$

PROOF. By construction $u_h = u$ outside $S_{\varepsilon_h\delta_h}$ and by Lebesgue dominated convergence Theorem

$$\int_{\Omega\setminus S_{\varepsilon_h\delta_h}} v_h|M_n\nabla u_h|^\gamma\,dx \to \int_\Omega |M_n\nabla u|^\gamma\,dx.$$

On the other hand inside $S_{\varepsilon_h\delta_h}$ v_h is identically zero, hence we are left with the estimate of $k_h\int_{S_{\varepsilon_h\delta_h}} |M_n\nabla u_h|^\gamma dx$. Thanks to (197) we know that

$$\begin{aligned}\int_{S_{\varepsilon_h\delta_h}} |M_n\nabla u_h|^\gamma dx \leq c_{m,n,\gamma}(1 + \|\phi_h\|_{L^\infty})^\gamma &\int_{S_{\varepsilon_h\delta_h}} |M_n\nabla u|^\gamma\,dx \\ + c_{m,n,\gamma}\|\nabla\phi_h\|^\gamma_{L^\infty} &\int_{S_{\varepsilon_h\delta_h}} |u|^\gamma|M_{n-1}\nabla u|^\gamma\,dx.\end{aligned}$$

The first term is infinitesimal by the absolute continuity of the integral. The second one can be estimated applying Hölder's inequality with exponents $\frac{s}{\gamma}$ and $\frac{p}{\gamma(n-1)}$: this can be done because

$$\frac{\gamma(n-1)}{p} + \frac{\gamma}{s} \leq 1.$$

Recalling Hadamard's inequality $|M_k\nabla u| \leq c_k|\nabla u|^k$ we get

$$\int_{S_{\varepsilon_h\delta_h}} |u|^\gamma|M_{n-1}\nabla u|^\gamma\,dx \leq c_k\|u\|^\gamma_{L^s(S_{\varepsilon_h\delta_h})}\|\nabla u\|^{\gamma(n-1)}_{L^p(S_{\varepsilon_h\delta_h})}.$$

Since $\|\nabla\phi_h\|^\gamma_{L^\infty} \leq c(\varepsilon_h\delta_h)^{-\gamma}$ our assumption on k_h allows to conclude. □

Putting Propositions 5.5.2, 5.5.4 and (198) together we conclude the proof of Theorem 5.5.1.

Remark 5.5.5. From the proof of Theorem 5.2.8 we deduce that

$$\liminf_h F_h(u_h, v_h, A) \geq F(u, A),$$

for every open set $A \subset \Omega$, and

$$\limsup_h F_h(u_h, v_h, \Omega) \leq F(u, \Omega).$$

This entails that $F_h(u_h, v_h, A) \to F(u, A)$ whenever $F(u, \partial A) = 0$ and $(u_h, v_h) \to (u, 1)$ with equibounded energies. Note that $A \mapsto F(u, A)$ is the restriction to open sets of an absolutely continuous measure, hence it does not charge the boundary of any regular open set.

5.5.2. Further observations. It is interesting to notice that the exponent γ is bounded above by $\frac{p}{n-1}$, in order for Theorem 5.5.1 to hold. There is however a trick allowing to overcome this bound, if we assume the Lagrangian to contain a nonlinear power of the full vector of minors $M\nabla u$. Retaining the structure of the size and phase transition terms as in Definition 5.2.7, the bulk energy

$$\tilde{F}(u, \Omega) = \int_\Omega |\nabla u|^p + |M\nabla u|^\gamma \, dx \tag{200}$$

can be approximated by

$$\tilde{F}_\varepsilon(u, v, \Omega) = \int_\Omega |\nabla u|^p + \Big|\sum_{k=1}^{n-1} |M_k \nabla u|^2 + (v + k_\varepsilon)|M_n \nabla u|^2\Big|^{\frac{\gamma}{2}} \, dx. \tag{201}$$

Although $p > n - 1$ guarantees that the same approximation holds, we can observe the following: applying Minkowski's inequality to (201) we have

$$\begin{aligned} k_h^{\frac{\gamma}{2}} \int_{S_{\varepsilon_h \delta_h}} |M_n \nabla u_h|^\gamma &\leq C_\gamma k_h^{\frac{\gamma}{2}} \int_{S_{\varepsilon_h \delta_h}} |M_n \nabla u|^\gamma + |\nabla \phi_h|^\gamma |u|^\gamma |M_{n-1} \nabla u|^\gamma \, dx \\ &\leq C_\gamma (1 + k_h^{\frac{\gamma}{2}} \|\nabla \phi_h\|_\infty^\gamma \|u\|_\infty^\gamma) \tilde{F}(u, S_{\varepsilon_h \delta_h}). \end{aligned}$$

Again if k_h goes to 0 sufficiently fast then $k_h^{\frac{\gamma}{2}} \|\nabla \phi_h\|_\infty^\gamma \to 0$ and we get the Γ-upper limit statement, at least when $u \in L^\infty$. More generally the Lagrangian can feature different summability exponents on every order of the minors considered. In the model case

$$\tilde{\tilde{F}}(u, \Omega) := \int_\Omega |\nabla u|^p + \sum_{k=2}^n |M_k \nabla u|^{p_k} \, dx$$

Theorem 5.2.8 can be proved if we assume $p_n > 1$ and (here $p_1 = p$)

$$\frac{1}{s} + \frac{n-1}{p} \leq 1, \qquad \frac{1}{s} + \frac{1}{p_{k-1}} \leq \frac{1}{p_k}.$$

In particular if we impose $p < n$ to retain the possibility of Ju having a singular part, for the price of a very large s we can take the p_k's arbitrarily close to the threshold n.

5.6. Boundary constraints

In this section we analyse the behaviour of the previous Γ-convergence Theorems first when we compute the energy on subsets of the domain and then when we impose a boundary condition for u at $\partial\Omega$ to be preserved by the approximating sequence. We start by applying the "free" version of the Theorem and combine it with Remark 5.2.2. If we want to prescribe a fixed trace at $\partial\Omega$ as observed in chapter 4 the Sobolev trace constraint is not sufficient to properly set our problem, due to possible dependence of Ju on the exterior extension. We therefore set $U \Supset \Omega$ open and fix $\phi \in W^{1,n}(U, \mathbb{R}^n)$ such that $\phi|_{\partial\Omega} \in W^{1,p}(\partial\Omega, \mathbb{R}^n)$: our approximating sequences (u_h, v_h) will belong to

$$Y^\phi = \{(u, v) \in Y(U) : u = \phi \text{ in } U \setminus \Omega\}.$$

Recall the previous result establishes the variational approximation of the energy on open sets: potential losses of mass due to presence of singular set at the boundary are disregarded in the lower limit, and a priori excluded in the upper limit by the hypothesis $\mathcal{M}^{*m-n}_{\Omega}(S_u) = \mathscr{H}^{m-n}(S_u)$.

The following proposition is an easy consequence of Theorem 5.2.8.

Proposition 5.6.1. *Suppose $\big((u_h, v_h)\big) \subset Y^\phi$ such that $(u_h, v_h) \to (u, v)$ and $\liminf_{h\to\infty} E_h(u_h, v_h, U) < \infty$. Then $v = 1$ and*

$$\liminf_{h\to\infty} F_h(u_h, v_h, \Omega) + MM_h(v_h, U) \geq E(u, 1, \overline{\Omega}) = F(u, 1, \Omega) + \sigma\mathscr{H}^{m-n}(S_u \cap \overline{\Omega}). \tag{202}$$

Proof. The statement follows straightforward from Theorem 5.2.8 applied to the domain U, since $S_u \subset \overline{\Omega} \Subset U$, hence $\mathscr{H}^{m-n}(S_u \cap U) = \mathscr{H}^{m-n}(S_u \cap \overline{\Omega})$. Moreover Remark 5.5.5 entails that

$$\liminf_h F_h(u_h, v_h, \Omega) \geq F(u, 1, \Omega),$$

thus the proof is complete. □

Similarly we can prove the upper limit analog:

Proposition 5.6.2. *Suppose that Ω is of class C^2, $E(u, 1, U) < \infty$ and $\mathcal{M}^{*m-n}(S_u) = \mathscr{H}^{m-n}(S_u)$. Then there exists $\big((u_h, v_h)\big) \subset Y^\phi$ such that*

$$\limsup_h E_h(u_h, v_h, \Omega) \leq E(u, 1, \overline{\Omega}).$$

Proof. Denote $\Omega_s = \{x \in U : \text{sgndist}(x, \partial\Omega) \leq s\}$, where sgndist is the signed distance from $\partial\Omega$, positive outside Ω and negative inside. The C^2 regularity of Ω ensures the existence of a tubular neighborhood of $\partial\Omega$, namely there exists s_0 (depending on the C^2 norm of $\partial\Omega$) and a C^1 diffeomorphism

$$\partial\Omega \times (-s_0, s_0) \ni (y, t) \mapsto x = y + t\nu(y) \in (\partial\Omega)_{s_0}$$

build up via the normal map ν to $\partial\Omega$. With the help of this map one can construct, for any given $s \in (-s_0, s_0)$, Lipschitz diffeomorphisms $T_s : U \to U$ deforming Ω_s to Ω and satisfying $T_0 = id$ and

$$\|T_s - T_{s'}\|_{W^{1,\infty}(U,U)} + \|T_s^{-1} - T_{s'}^{-1}\|_{W^{1,\infty}(U,U)} \leq C|s - s'| \tag{203}$$

for every s, s'. We also point out that the existence of the tubular neighborhood gives a reflection map

$$\Pi_{s_0} : (\partial\Omega)_{s_0} \ni (y,t) \mapsto (y,-t) \in (\partial\Omega)_{s_0}$$

of class C^1 such that $\lim_{s_0\to 0} \|\Pi_{s_0} - id\|_{C^1} = 0$. Since the energy is finite $u \in GSB_nV(U)$: given $\eta > 0$ we let

$$u_\eta = \begin{cases} u \circ T_{-\eta} & \text{in } \Omega_{-\eta}, \\ u \circ T_s & \text{on } \partial\Omega_s,\ -\eta < s < 0, \\ \phi & \text{in } U \setminus \Omega. \end{cases} \tag{204}$$

Notice that $u_\eta = \phi$ outside Ω and $u_\eta \in W^{1,n}(U \setminus \Omega_{-\eta}, \mathbb{R}^n)$, hence $u_\eta \in X^\phi$ and $S_{u_\eta} \subset \overline{\Omega_{-\eta}} \Subset \Omega$. Moreover it is not difficult to use (203) to show $E(u_\eta, 1, U) \to E(u, 1, U)$ for $\eta \downarrow 0$: in fact the energy in $U \setminus \Omega$ is fixed, the one in $\overline{\Omega_{-\eta}}$ after a change of variables equals to

$$\int_\Omega \Big\{ |\nabla u \cdot (DT_{-\eta} \circ T_{-\eta}^{-1})|^p \\ + \Big| \sum_{|I|=|J|=n} \Big| \sum_{|K|=n} \det(\nabla u)_K^I \det(DT_{-\eta} \circ T_{-\eta}^{-1})_J^K \Big|^2 \Big|^{\frac{\gamma}{2}} \Big\} |\det DT_{-\eta}^{-1}|\, dx \\ + \int_{\overline{\Omega} \cap S_u} |\langle \Lambda_{m-n} DT_{-\eta}^{-1}, \tau_{S_u} \rangle| d\mathscr{H}^{m-n}, \tag{205}$$

which is asymptotically equal to $E(u, \overline{\Omega})$ thanks to (203); finally in the annulus $\Omega \setminus \overline{\Omega_{-\eta}}$, u_η being a constant extension along the trajectories $s \mapsto T_s(x)$, enjoys

$$\int_{\Omega \setminus \overline{\Omega_{-\eta}}} |\nabla u_\eta|^p\, dx \le C(\partial\Omega)\eta \int_{\partial\Omega} |\nabla_\tau \phi|^p\, dx$$

and $M_n \nabla u_\eta = 0$, hence $E(u, 1, \Omega \setminus \overline{\Omega_{-\eta}}) \to 0$. Thanks to Remark 5.2.2 and Proposition 5.6.1 it is sufficient to prove the $\Gamma - \limsup$ for u_η. Theorem 5.5.1 ensures the existence of $(u_h, v_h) \in Y^\phi$ satisfying

$$\limsup_h E_h(u_h, v_h, U) \le E(u, 1, U) :$$

subtracting the constant term $F(\phi, 1, U \setminus \Omega)$ we have the thesis. □

Propositions 5.6.1 and 5.6.2 are only in part satisfactory, since in (202) we took into account some energy outside Ω. We want to refine these results assessing the quantitative loss of energy due to exterior phase transition in MM_h.

PROPOSITION 5.6.3. *With the same hypotheses of Proposition 5.6.1 it holds:*

$$\liminf_h MM_h(v_h, \Omega) \ge \sigma \mathscr{H}^{m-n}(S_u \cap \Omega) + \frac{1}{2} \sigma \mathscr{H}^{m-n}(S_u \cap \partial\Omega).$$

PROOF. Let us start from the codimension zero case $m = n$. The proof stems from Lemma 5.4.5, applied to the larger domain U, whose argument we here retrace. Since we are evaluating the energy $MM_h(v_h, \Omega \cap B_{2\rho}(x_0))$ we can suppose $x_0 = 0 \in S_u \cap \partial\Omega$, as the interior case is already contained in Lemma 5.4.5. Recall the proof showed that in every ball $B_\rho(0) \subset U$ the sequence satisfies $\lim_h \inf_{B_\rho} v_h = 0$. We actually know that

$$\lim_h \inf_{B_\rho \cap \overline{\Omega}} v_h = 0,$$

because every u_h equals ϕ in $U \setminus \Omega$ and $J\phi \ll \mathscr{L}^n$. Let (x_h) be one of the minimum points of v_h in B_ρ: we have two cases.

Case 1: $(\liminf_h \frac{|x_h|}{\varepsilon_h}) < \infty$. In this case scaling back v_h by a factor ε_h we obtain

$$MM_h(v_h, \Omega \cap B_{2\rho}) = MM_1(v_h(\varepsilon_h x), \tfrac{\Omega \cap B_{2\rho}}{\varepsilon_h}).$$

Using a diagonal argument and reasoning as in Proposition 5.3.1 we produce a limit f_∞ such that

$$v_h(\varepsilon_h x) \to 1 - f_\infty(x) \qquad \text{locally uniformly in } \mathbb{R}^n$$

and

$$\min_{\mathbb{R}^n}\{1 - f_\infty\} = 0.$$

Fix a compact $K \Subset H := \{\langle x, \nu(0)\rangle < 0\}$: by C^1 regularity $\frac{\Omega}{\varepsilon_h} \to H$ locally in the Hausdorff metric and $K \subset \frac{1}{\varepsilon_h}(\Omega \cap B_{2\rho})$ for h large enough. By lower semicontinuity

$$\liminf_h MM_1(v_h(\varepsilon_h x), \tfrac{\Omega \cap B_{2\rho}}{\varepsilon_h}) \geq \liminf_h MM_1(v_h(\varepsilon_h x), K) \geq MM_1(1 - f_\infty, K) = I(f_\infty, K)$$

and letting $K \uparrow H$ we entail

$$\liminf_h MM_h(v_h, \Omega \cap B_{2\rho}) \geq I(f_\infty, H).$$

Therefore if we redefine f_∞ in $\mathbb{R}^n \setminus H$ by reflection with respect to ∂H we obtain $I(f_\infty, H) = \frac{1}{2} I(f_\infty, \mathbb{R}^n)$. A radial rearrangement f_∞^* of f_∞ decreases the energy and gives $f_\infty^*(0) = 1$, hence by Proposition 5.3.1 $I(f_\infty, H) \geq \frac{1}{2}\sigma$.

Case 2: $\lim_h \frac{|x_h|}{\varepsilon_h} = \infty$. In this situation we blow-up around x_h and obtain that

$$\frac{\Omega \cap B_{2\rho}(0) - x_h}{\varepsilon_h} \supset B_{\frac{\rho}{\varepsilon_h}}(0) \to \mathbb{R}^n$$

in the same sense as before. The limit f_∞ of the translated sequence $(v_h(x_h + \varepsilon_h y))$ will now satisfy $f_\infty(0) = 0$, hence by lower semicontinuity

$$\liminf_h MM_1(v_h(x_h + \varepsilon_h y), \tfrac{\Omega \cap B_{2\rho}(0) - x_h}{\varepsilon_h}) \geq I(f_\infty, \mathbb{R}^n) \geq \sigma.$$

The case $m > n$ can be treated as in (190), where now the projection measures τ_π contain the extra term $\frac{1}{2}\sigma \mathscr{H}^{m-n} \llcorner (S_u \cap \partial\Omega)$. □

Similarly we have a statement for the upper limit:

Proposition 5.6.4. *For every $u \in X^\phi$ such that $E(u, 1, \Omega) < \infty$, $\mathcal{M}^*(S_u) = \mathscr{H}^{m-n}(S_u)$ and $\mathscr{H}^{m-n}(\overline{S_u \cap \Omega} \cap \partial\Omega) = 0$ there exists a sequence $\big((u_h, v_h)\big) \subset Y^\phi$ such that $(u_h, v_h) \to (u, 1)$ and*

$$\limsup_{h\to\infty} E_h(u_h, v_h, \Omega) \leq E(u, 1, \Omega) + \frac{1}{2}\sigma \mathscr{H}^{m-n}(S_u \cap \partial\Omega).$$

In order to prove this result we begin with a Lemma:

LEMMA 5.6.5. *Let $\tau > 0$ be a given positive number: there exists a profile $\bar{w} : [0,\infty) \to [0,1]$ such that*

(1) $|I(\bar{w}(|x|)) - \sigma| < \tau$;
(2) $\bar{Z}(t) := |\bar{w}'(t)|^q + W(\bar{w}(t))$ *is decreasing;*
(3) $\bar{w} \in \mathrm{Lip}([0,\infty))$ *and* $\bar{w} = 0$ *in* $[R,\infty)$ *for some* R;

PROOF. Using the optimal profile w given by Proposition 5.3.1, it is sufficient to take into account the continuity of I along the family of profiles

$$\frac{w(t+\lambda)}{w(\lambda)}, \qquad \lambda \geq 0 \tag{206}$$

and choose a $\lambda > 0$ satisfying $|I(\frac{w(|x|+\lambda)}{w(\lambda)}) - \sigma| < \tau$. We name $\bar{w}$ the profile (206) relative to such choice: $\bar{w}$ is clearly Lipschitz by Lemma 5.3.2. The second property follows from the fact that both $w(t)$ and $|w'(t)|$ are decreasing. The third one can be obtained again by dilating the new profile around 1 and truncate it to 0 changing the energy I only by a small amount. $\square$

We will also use the following fact, whose proof we leave to the reader:

LEMMA 5.6.6. *If $S \subset \Omega$ is countably $\mathscr{H}^k$-rectifiable and satisfies $\mathcal{M}^{*k}_{\Omega}(S) = \mathscr{H}^k(S)$ then the same is true for every $S' \subset S$ such that $\mathscr{H}^k(S \cap (\overline{S'} \setminus S')) = 0$.*

We can now prove Proposition 5.6.4.

PROOF. By the finiteness of the energy $u \in GSB_nV(U)$ and $S_u \subset \overline{\Omega}$. Let $\eta_h \downarrow 0$ to be chosen later. We can consider the tilted sequence u_{η_h} described in (204): we have

$$\lim_h F(u_{\eta_h}, 1, \Omega) = F(u, 1, \Omega).$$

For an arbitrary τ let $\bar{w}$ be a function as in Lemma 5.6.5: by Proposition 5.5.2 and Remark 5.5.3 we can construct a sequence (v_h) of approximating functions such that

$$\limsup_h MM_h(v_h, \Omega) = I(\bar{w}(|x|))\mathcal{M}^{*m-n}_{\Omega}(S_u) < (\sigma + \tau)\mathcal{M}^{*m-n}_{\Omega}(S_u).$$

Denote $v_{\eta_h,h} = v_h \circ T_{-\eta_h}$: recall that $v_h = 0$ in $S_{\varepsilon_h \delta_h}$ and because of (203) we have that

$$||T^{-1}_{-\eta_h}(x) - T^{-1}_{-\eta_h}(y)| - |x-y|| \leq \mathrm{Lip}(T^{-1}_{-\eta_h} - id)|x-y| \leq C\eta_h|x-y|,$$

therefore $v_{\eta_h,h} = 0$ on $(T^{-1}_{-\eta_h}(S_u))_{\varepsilon_h\delta_h(1-C\eta_h)}$, thus eventually in $(T^{-1}_{-\eta_h}(S_u))_{\varepsilon_h\delta_h/2}$.

Let us analyse the bulk part first. Since the null set of $v_{\eta_h,h}$ has width at least $\varepsilon_h\delta_h/2$ we can apply Theorem 5.5.1 relative to the limit u_{η_h} in the domain U and define $u_{\eta_h,h}$ such that

- $(u_{\eta_h,h}, v_{\eta_h,h}) \in Y(U)$,
- $|u_{\eta_h,h}| \leq |u_{\eta_h}|$ pointwise almost everywhere,
- $u_{\eta_h,h} = u_{\eta_h}$ outside $(T^{-1}_{-\eta_h}(S_u))_{\varepsilon_h\delta_h/2} \subset \{v_{\eta_h,h} = 0\}$.

In particular $u_{\eta_h,h} \to u$ in L^s. Moreover the construction guarantees that

$$\left| F_h(u_{\eta_h,h}, v_h, U) - \int_U |\nabla u_{\eta_h}|^p + v_h |M_n \nabla u_{\eta_h}|^\gamma \, dx \right| \\ \leq \int_U k_h |M_n \nabla u_{\eta_h,h}|^\gamma \, dx + \int_{(T^{-1}_{-\eta_h}(S_u))_{\varepsilon_h \delta_h/2}} |\nabla u_{\eta_h,h}|^p + |\nabla u_{\eta_h}|^p \, dx.$$

The same estimates yielding (198) and (199) show that the right hand side is infinitesimal. Furthermore the constraint $(u_{\eta_h,h}, v_{\eta_h,h}) \in Y^\phi$ is satisfied once we choose $\eta_h = \varepsilon_h \delta_h$. Observing that

$$\int_U |\nabla u_{\eta_h}|^p + v_h |M_n \nabla u_{\eta_h}|^\gamma \, dx \leq F(u_{\eta_h}, 1, U) \to F(u, 1, U)$$

the previous two equations entail

$$\limsup_h F_h(u_{\eta_h,h}, v_h, U) \leq F(u, 1, U).$$

Subtracting the constant term $F(u, 1, U \setminus \Omega)$ we remain with

$$\begin{aligned} F(u,1,\Omega) &\geq \limsup_h F_h(u_{\eta_h,h}, v_h, U) - F(u,1,U \setminus \Omega) \\ &= \limsup_h F_h(u_{\eta_h,h}, v_h, \Omega) + F_h(u_{\eta_h,h}, v_h, U \setminus \Omega) - F(u,1,U \setminus \Omega) \\ &= \limsup_h F_h(u_{\eta_h,h}, v_h, \Omega) + F_h(u, v_h, U \setminus \Omega) - F(u,1,U \setminus \Omega) \\ &= \limsup_h F_h(u_{\eta_h,h}, v_h, \Omega) + \int_{U \setminus \Omega} (v_h + k_h - 1) |M_n \nabla u|^\gamma \, dx \\ &= \limsup_h F_h(u_{\eta_h,h}, v_h, \Omega). \end{aligned}$$

It remains to evaluate the asymptotic of $MM_h(v_{\eta_h,h}, \Omega)$. First of all changing back variables we have that

$$MM_h(v_{\eta_h,h}, \Omega) = \int_{T_{\eta_h}(\Omega)} \left\{ \varepsilon_h^{q-n} |(DT_{-\eta_h} \circ T^{-1}_{-\eta_h}) \nabla v_h|^q + \frac{W(1-v_h)}{\varepsilon_h^n} \, dx \right\} |\det DT^{-1}_{-\eta_h}| \, dx$$

and by (203) and Lemma 5.6.5 this is asymptotic to $MM_h(v_h, T_{\eta_h}(\Omega))$: we now show that if $\frac{\eta_h}{\varepsilon_h} = \delta_h \to 0$ sufficiently fast then the last energy is asymptotically equal to $MM_h(v_h, \Omega)$, namely $MM_h(v_h, T_{\eta_h}(\Omega) \setminus \Omega) \to 0$.

Fix a radius R such that $\mathrm{spt}(\bar{w}) \subset B^n_R$ and $\mathscr{L}^m(B_{2\varepsilon_h R} \cap (T_{\eta_h}(\Omega) \setminus \Omega)) \leq C(\varepsilon_h R)^{m-1} \eta_h$. We can cover $S_{\varepsilon_h R}$ with (closed) balls of radius $\varepsilon_h R$ centered at $x_0 \in S_u$:

$$S_{\varepsilon_h R} \subset \bigcup_{x_0 \in S_u} \overline{B_{\varepsilon_h R}(x_0)}.$$

By Besicovitch's covering Lemma there are N disjoint subfamilies $\mathcal{F}_i$ that still cover the set of old centers, namely S_u: by triangle inequality

$$S_{\varepsilon_h R} \subset \bigcup_{i=1}^N \bigcup_{\mathcal{F}_i} B_{2\varepsilon_h R},$$

and the assumption of $\mathcal{M}^{*m-n}(S_u)$ implies that $\#\mathcal{F}_i \leq C(\varepsilon_h R)^{n-m}$; as a consequence the family of double balls $\{B_{2\varepsilon_h R}\}$ has bounded overlap. Without loss of generality we can also assume that $\mathcal{M}^{*m-n}(S_u \cap B_{2\varepsilon_h R}(x_0)) = \mathscr{H}^{m-n}(S_u \cap B_{2\varepsilon_h R}(x_0))$, recalling that this is true at almost every radius. For any of such double ball

$$\frac{MM_h(v_h, B_{2\varepsilon_h R}(x_0) \cap (T_{\eta_h}(\Omega) \setminus \Omega))}{\varepsilon_h^{m-n}} = \int_{B_{2R}(0) \cap \frac{(T_{\eta_h}(\Omega)\setminus\Omega)-x_0}{\varepsilon_h}} |\nabla \psi_h|^q + W(\psi_h)\, dy \quad (207)$$

with

$$\psi_h(y) = \bar{w}_h\left(d\big(y, \frac{S_u - x_0}{\varepsilon_h}\big)\right).$$

The integral can be simply bounded by

$$(\mathrm{Lip}(\bar{w}_h)^q + \|W\|_\infty)\mathscr{L}^m\left(B_{2R}(0) \cap \frac{(T_{\eta_h}(\Omega) \setminus \Omega) - x_0}{\varepsilon_h}\right) \leq C(\mathrm{Lip}(\bar{w}_h)^q + \|W\|_\infty)R^{m-1}\frac{\eta_h}{\varepsilon_h}.$$

Recall the construction of $\bar{w}_h$ from $\bar{w}$ in (193) gives that $\mathrm{Lip}(\bar{w}_h) \leq C\mathrm{Lip}(\bar{w})$. Summing on the number of balls we have

$$\begin{aligned} MM_h(v_h, \Omega_{\eta_h} \setminus \Omega) &\leq C\varepsilon_h^{m-n}(\varepsilon_h R)^{n-m}(\mathrm{Lip}(\bar{w}_h)^q + \|W\|_\infty)R^{m-1}\frac{\eta_h}{\varepsilon_h} \\ &= C(\mathrm{Lip}(\bar{w})^q + \|W\|_\infty)R^{n-1}\frac{\eta_h}{\varepsilon_h} \to 0. \end{aligned}$$

Suppose now that $\mathscr{H}^{m-n}(\overline{S_u \cap \Omega} \cap \partial\Omega) = 0$: then $\mathcal{M}_U^{*m-n}(S_u \cap \Omega) = \mathscr{H}^{m-n}(S_u \cap \Omega)$ by Lemma 5.6.6 (applied to $S' = S_u \cap \Omega$). Moreover:

$$\mathcal{M}_\Omega^{*m-n}(S_u) \leq \mathcal{M}_U^{*m-n}(S_u \cap \Omega) + \mathcal{M}_\Omega^{*m-n}(S_u \cap \partial\Omega) = \mathscr{H}^{m-n}(S_u \cap \Omega) + \mathcal{M}_\Omega^{*m-n}(S_u \cap \partial\Omega).$$

Regarding the last term, by (203) the reflection map Π_{s_0} that swaps Ω and $U \setminus \Omega$ has a jacobian uniformly close to 1 as we move close to $\partial\Omega$ and therefore

$$\mathcal{M}_\Omega^{*m-n}(S_u \cap \partial\Omega) = \frac{1}{2}\mathcal{M}_U^{*m-n}(S_u \cap \partial\Omega) = \frac{1}{2}\mathscr{H}^{m-n}(S_u \cap \partial\Omega).$$

In conclusion

$$\limsup_h MM_h(v_h, \Omega) \leq (\sigma + \tau)(\mathscr{H}^{m-n}(S_u \cap \Omega) + \frac{1}{2}\mathscr{H}^{m-n}(S_u \cap \partial\Omega))$$

and the assertion follows by letting $\tau \to 0$. □

5.7. General Lagrangians

The Γ-convergence Theorem 5.2.8 proved in the previous sections can be extended, always in the setting of higher codimension singular sets, to polyconvex Lagrangians of more general form than Definition (5.1.1).

Indeed the key ingredients for the $\Gamma - \liminf$ are again the compactness Theorem 3.4.1, which is at the heart of Theorem 5.1.2, as well as the lower semicontinuity of the energy for the convergence provided by it. Regarding the $\Gamma - \limsup$ in order to approximate the size term we rely on the same Modica-Mortola approximation of before. The recovery sequence is obtained via an approximation in measure of the limit function u, with regular functions $u_\varepsilon \in R_n$ coinciding with u outside the narrow sets S_ε. The proof of Proposition 5.5.4 amounts to show that the contribution to the bulk energy in S_ε is infinitesimal.

Both these arguments can be adapted to a broader class of Lagrangians, related to the one described in Theorem 4.1.1. Let s, p satisfy (87), and assume that the following hypotheses on the functions $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^\kappa \to [0, +\infty)$ and $g : \Omega \to [0, \infty)$ are satisfied:

(a) f is $\mathscr{L}_m \times \mathscr{B}(\mathbb{R}^{n+\kappa})$-measurable;
(b) for $\mathscr{L}^m$-a.e. $x \in \Omega$, $(u, w) \mapsto f(x, u, w)$ is lower semicontinuous;
(c) for $\mathscr{L}^m$-a.e. $x \in \Omega$ and for every $u \in \mathbb{R}^n$ the map $w \mapsto f(x, u, w)$ is convex in $\mathbb{R}^\kappa$;
(d) $c\big(|w_1|^p + \Psi(|w_n|)\big) \le f(x, u, w) \le C\big(1 + |u|^s + |w_1|^p + |w_n|^\gamma\big)$ for Ψ convex and superlinear at infinity and for some constants $\gamma > 1$, $c, C > 0$;

and $g \in C^0(\overline{\Omega})$, $g \ge c > 0$. (Compare the hypotheses of the compactness Theorem: (a), (b), (c) are identical, (d) instead is more stringent since we now require a growth bound from above of the integrand). Then thanks to the Theorem 3.4.1 the energy

$$\mathcal{E}(u, \Omega) = \int_\Omega f(x, u, M\nabla u)\, dx + \sigma \int_{\Omega \cap S_u} g\, d\mathscr{H}^{m-n}. \tag{208}$$

is lower semicontinuous along sequences converging strongly in L^s and with equibounded energies. The upper bound on f on the other side allows to prove the upper limit statement. The approximating energies will be

$$\mathcal{E}_\varepsilon(u, v, \Omega) := \int_\Omega f(x, u, \nabla u, \ldots, M_{n-1}\nabla u, (v + k_\varepsilon)M_n \nabla u)\, dx + \int_\Omega g(x) \left(\varepsilon^{q-n} |\nabla v|^q + \frac{W(1-v)}{\varepsilon^n} \right) dx.$$

We therefore have:

Theorem 5.7.1. *Let Ω be a bounded open subset of class C^1 of $\mathbb{R}^m$ and suppose*

$$s \ge \frac{np}{n-p}, \qquad 1 < \gamma \le \frac{1}{\frac{n-1}{p} + \frac{1}{s}}, \qquad q > n, \qquad k_\varepsilon = o(\varepsilon).$$

Suppose the integrands f, g satisfy the assumptions above. Then:

(a) *For every sequence $\big((u_h, v_h)\big) \subset Y(\Omega)$ such that $\liminf_{h\to\infty} \mathcal{E}_h(u_h, v_h, \Omega) < \infty$ and $(u_h, v_h) \to (u, v)$ in $X(\Omega)$ we have*
$$u \in GSB_nV(\Omega),\ v = 1 \text{ and } \liminf_{h\to\infty} \mathcal{E}_h(u_h, v_h, \Omega) \ge \mathcal{E}(u, \Omega).$$
(b) *For every $u \in GSB_nV(\Omega)$ such that $\mathcal{E}(u, 1, \Omega) < \infty$ and $\mathcal{M}^{*m-n}_\Omega(S_u) = \mathscr{H}^{m-n}(S_u)$ there exists a sequence $\big((u_h, v_h)\big) \subset Y(\Omega)$ such that $(u_h, v_h) \to (u, 1)$ in $X(\Omega)$ and*
$$\limsup_{h\to\infty} \mathcal{E}_h(u_h, v_h, \Omega) \le \mathcal{E}(u, \Omega).$$

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