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# Infinite root stacks of logarithmic schemes and moduli of parabolic sheaves

Mattia Talpo

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**Relatore:** Prof. Angelo Vistoli

# Contents

# Introduction

This thesis is concerned with some aspects of logarithmic geometry, with a focus on the infinite root stack of a log scheme and the moduli problem for parabolic sheaves.

## State of the art

Logarithmic geometry was firstly inspired by questions of arithmetic geometry and developed by Kazuya Kato ([Kat89]), and later it spread to touch various other areas of algebraic geometry, including moduli theory (for an introduction, see [Ogu] or [ACG$^+$13]). The basic insight was that there are some morphisms of schemes that are not smooth, but for some (for example cohomological) aspects are as good as a smooth morphism; the theory originated as a mean to exploit this "hidden smoothness".

The main objects of the theory are logarithmic schemes: in Kato's formulation, a log scheme is a scheme $X$ together with a sheaf of monoids $M$ on the small étale site $X_{\text{ét}}$ and a map of sheaves of monoids $\alpha\colon M \to \mathcal{O}_X$, where $\mathcal{O}_X$ has its multiplicative structure, such that the restriction $\alpha|_{\alpha^{-1}(\mathcal{O}_X^\times)}\colon \alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism. The idea behind this is that the preimage $\alpha^{-1}(s)$ of a section of $\mathcal{O}_X$ is the set of "logarithms" of such a section (it could be empty), and $\alpha$ is some kind of exponential.

The prototypical example is the following: assume that $X$ is a scheme and $D \subseteq X$ is an effective Cartier divisor. Then we can take $M$ to be the sheaf of functions on $X$ that are invertible outside of $D$, and the map $\alpha\colon M \to \mathcal{O}_X$ to be the inclusion, and we obtain a log scheme. An important instance of this situation is when we have a morphism with semi-stable reduction $X \to \operatorname{Spec}(R)$ with $R$ a discrete valuation ring (i.e. étale locally on $X$ there is a smooth morphism $X \to \operatorname{Spec}(R[x_1, \dots, x_r]/(x_1 \cdots x_r - \pi))$ where $\pi \in R$ is a uniformizer), and we take $D$ to be the special fiber $X_0 \subseteq X$, which is a reduced normal crossing divisor.

One can define morphisms of log schemes, and define various properties of log schemes and morphisms that extend the corresponding non-logarithmic notion. In particular there is a concept of *log smoothness* of a morphism of log schemes; as it happens with ordinary smoothness, one can formulate it via an infinitesimal lifting criterion, or via local freeness of a *sheaf of logarithmic differentials*. For example, all normal toric varieties over a field have a canonical log structure, and are log smooth.

In the case of a variety over a field equipped with the log structure coming from a normal crossing divisor, this sheaf of logarithmic differentials (with respect to the structure morphism to the spectrum of the base field) is exactly the sheaf of 1-forms that have at most poles of order 1 along the divisor. These forms are called *logarithmic forms* because formally one has $d(\log(x)) = dx/x$, and this has a pole of order 1 at zero.

Parabolic sheaves were first introduced by Mehta and Seshadri ([MS80]) in order to generalize to the non-proper case the correspondence between unitary representations of the fundamental group of a smooth complex projective curve $C$ and semi-stable vector bundles of degree 0 on $C$. If $C$ is not proper, one can take a compactification $C \subseteq \overline{C}$ by adding finitely many points $\overline{C} \setminus C = \{p_1, \ldots, p_k\}$. A parabolic bundle as defined by Mehta and Seshadri is a vector bundle $E$ on $\overline{C}$, together with additional data: for every one of the points $p_i$ we have a filtration

$$0 = F_{i,k_i+1} \subset F_{i,k_i} \subset \cdots \subset F_{i,1} = E_{p_i}$$

of the fiber $E_{p_i}$, and a set of real numbers $0 \leq a_{i,1} < \cdots < a_{i,k_i} < 1$ called weights. We remark here that in our work we will use assume that the weights are rational numbers.

Mehta and Seshadri give this definition after showing that a unitary representation of the fundamental group of an open curve leads naturally to such a structure (for example, the weights come from the eigenvalues of the unitary matrix associated by the representation to a small loop around the corresponding point), and after introducing a suitable notion of parabolic degree and (semi-)stability, they prove that there is an equivalence between unitary representations of the fundamental group and semi-stable parabolic bundles of parabolic degree 0.

The definition has been generalized in several steps, by replacing $\overline{C}$ with a projective variety $X$ and the points $p_i$ with an effective Cartier (simple normal crossing) divisor $D \subseteq X$ ([MY92, Bis97, Moc06, IS07, Bor09]). In particular Maruyama and Yokogawa ([MY92]) define a parabolic sheaf on a smooth projective variety $X$ equipped with an effective Cartier divisor as a coherent torsion-free sheaf $E$ on $X$ together with a filtration

$$E(-D) = F_{k+1} \subset F_k \subset \cdots \subset F_1 = E$$

and some weights $0 \leq a_1 < \cdots < a_k < 1$. This is clearly a reformulation of Mehta and Seshadri's definition in the case of curves. This definition corresponds to considering a log structure induced by $D$ that is not the one we described before if $D$ is not smooth, and in some sense is the "wrong" one. Borne ([Bor09]) gives a definition in the case of a simple normal crossing divisor that corresponds to considering the "correct" log structure, by "separating" the components of the divisor $D$. Subsequently, in [BV12], Borne and Vistoli give a definition of a parabolic sheaf on a general coherent log scheme, that gives back the previous definitions in the corresponding particular cases.

This last definition, which is the one we will use throughout this document, requires a slightly different point of view on the concept of a log scheme. For a scheme $X$, call $\mathrm{Div}_X$ the fibered category over the small étale site of $X$, whose objects over $T \to X$ are pairs $(L, s)$ consisting of a line bundle $L$ on $T$ together with a global section $s \in \Gamma(T, L)$, and the arrows are the obvious ones. This fibered category has a tensor product that makes it into a symmetric monoidal fibered category.

A logarithmic scheme can also be defined as a scheme $X$ together with a sheaf of monoids $A$ on the small étale site $X_{\text{ét}}$ and a symmetric monoidal functor $L\colon A \to \mathrm{Div}_X$, where we see $A$ as a discrete monoidal fibered category. One can go from the definition with the map $\alpha\colon M \to \mathcal{O}_X$

to this different one by taking a quotient (in the stacky sense) by $\mathcal{O}_X^\times$, so in this new notation the sheaf $A$ is what usually is denoted $\overline{M}$, i.e. the quotient sheaf $M/\mathcal{O}_X^\times$. This assumes that the action of $\mathcal{O}_X^\times$ on $M$ is faithful, i.e. the log scheme is quasi-integral. We will only be concerned with integral log schemes.

We also need a system of denominators for the sheaf $A$, that plays the role of the weights in Seshadri's definition. This is a second sheaf of monoids $B$ together with an injective morphism of sheaves of monoids $A \to B$ that is moreover of Kummer type. This means that if we take any element $b \in B_x$, where $x$ is a geometric point of $X$, then there exists a positive integer $n$ such that $nb$ is in the image of $A_x \to B_x$. This makes $B$ into a sheaf consisting of "roots" of sections of $A$, in some sense. An important example is given by the maps $A \to \frac{1}{n}A$, where $\frac{1}{n}A$ is just $A$, and the map is multiplication by $n$ (this assumes that $A$ is torsion-free, to ensure injectivity).

Starting from $B$, one defines a fibered category $B^{\mathrm{wt}}$ having as objects sections of $B^{\mathrm{gp}}$ (the associated sheaf of groups), and morphisms $b \to b'$ the sections $b''$ of $B$ such that $b + b'' = b'$. A parabolic sheaf with denominators in $B$ as defined in [BV12] is a cartesian functor $E \colon B^{\mathrm{wt}} \to \mathrm{QCoh}_X$, where $\mathrm{QCoh}_X$ is the fibered category of coherent sheaves restricted to $X_{\text{ét}}$, together with isomorphisms $E_{b+a} \cong E_b \otimes L_a$ for any sections $b$ of $B^{\mathrm{gp}}$ and $a$ of $A$, satisfying some compatibility condition. For example, one of these conditions is that the map $E_b \to E_{b+a}$ coming from the arrow $a \colon b \to b + a$ of $B^{\mathrm{wt}}$ should correspond to multiplication by the distinguished section of $L_a$ (recall that $L$ is a functor $A \to \mathrm{Div}_X$, and in $\mathrm{Div}_X$ each object consists of an invertible sheaf with a specified global section) as a morphism $E_b \to E_b \otimes L_a$.

Note that this gives back the definition of Maruyama and Yokogawa: consider the log structure on $X$ induced by the symmetric monoidal functor $\mathbb{N} \to \mathrm{Div}(X)$ sending 1 to $(\mathcal{O}_X(D), s)$, where $s$ is the canonical section. A parabolic sheaf (say with weights in $\frac{1}{2}\mathbb{N}$) on the resulting log scheme may be visualized as a sequence of coherent sheaves $E_q$ on $X$ parametrized by $q \in \frac{1}{2}\mathbb{Z}$, and with maps

$$\cdots \to E_{-\frac{3}{2}} \to E_{-1} \to E_{-\frac{1}{2}} \to E_0 \to E_{\frac{1}{2}} \to E_1 \to E_{\frac{3}{2}} \to \cdots$$

where we have $E_{q+n} \cong E_q \otimes \mathcal{O}_X(nD)$ for $q \in \frac{1}{2}\mathbb{Z}$ and $n \in \mathbb{Z}$. In particular the piece

$$E_{-1} \to E_{-\frac{1}{2}} \to E_0$$

determines the rest of the sheaf, there is an isomorphism $E_{-1} \cong E_0 \otimes \mathcal{O}_X(-D)$, and the composition $E_{-1} \to E_{-\frac{1}{2}} \to E_0$ coincides with the canonical map $E_0 \otimes \mathcal{O}_X(-D) \to E_0$, so, assuming that all the maps of the parabolic sheaf are injective, from this we get a filtration as in Maruyama and Yokogawa's definition. Injectivity of the maps in this case follows for example from torsion-freenes of the parabolic sheaf, as we define it later in this document.

Parabolic sheaves can be naturally interpreted as quasi-coherent sheaves on a certain algebraic stack over $X$. We denote by $X_{B/A}$ the stack over $X$ that has as objects over $T$ a morphism $T \to X$ together with a symmetric monoidal functor $B_T \to \mathrm{Div}_T$ that lifts the pullback $A_T \to \mathrm{Div}_T$ to $T$ of the log structure of $X$. Objects of this stack are in a sense "roots" of the log structure of $X$ with respect to the system of denominators, and $X_{B/A}$ is called the *root stack* of $X$ with respect to $A \to B$. Although the general definition was first given in [BV12], the idea of the construction is essentially due to Olsson ([Ols07, MO05]).

The main result of [BV12] is that there is an equivalence of tensor categories between quasi-coherent sheaves on $X_{B/A}$ and parabolic sheaves on $X$ with respect to $A \to B$.

The moduli problem for parabolic sheaves has been considered firstly by Mehta and Seshadri for curves ([MS80]), and then by Maruyama and Yokogawa for varieties with an effective divisor ([MY92]). They introduce a notion of parabolic degree (resp. parabolic Hilbert polynomial) and a stability condition, and they construct, using GIT, moduli spaces that parametrize (S-equivalence classes of) (semi-)stable parabolic sheaves.

The original motivation for this work was to generalize these results about moduli of parabolic sheaves to the case of a general log scheme.

# The present work

This thesis is divided into two parts. The first one consists of a treatment of the infinite root stack of a log scheme and the second one is about moduli of parabolic sheaves.

In this document we will always work over a fixed field $k$. The part about the infinite root stack and some parts of the discussion of the moduli theory of parabolic sheaves are also valid without this assumption, but for homogeneity's sake we prefer to make it from the start.

## The infinite root stack

The part about the infinite root stack is joint work with my thesis supervisor Angelo Vistoli.

The infinite root stack of a fine saturated log scheme $X$, denoted by $X_\infty$, can be defined as the inverse limit over all systems of denominators $A \to B$ of the root stacks $X_{B/A}$; in other words, it parametrizes liftings of the symmetric monoidal functor $A \to \mathrm{Div}_X$ along all systems of denominators. Alternatively, it can be described as a root stack relative to the maximal Kummer extension $A \to A_\mathbb{Q} = \varinjlim_{n\in\mathbb{N}} \frac{1}{n}A$ containing all the others. This kind of Kummer extension was not considered in [BV12], since $A_\mathbb{Q}$ is never finitely generated. Consequently the corresponding root stack is more complicated. In fact it is not algebraic (only pro-algebraic) and not of finite type, but despite being quite intimidating at first sight, it is a very natural, functorial object to associate to a log scheme $X$.

We investigate some aspects of its geometry, and the relations with the log geometry of the log scheme $X$. It turns out that there is a very strong relation: we are able to give a criterion for a map $X_\infty \to Y_\infty$ to come from a morphism of log schemes $X \to Y$ (this does not always happen, as simple examples show) and by explicitly describing a method to get back the log structure of $X$ from its infinite root stack $X_\infty$, we show that the root stack determines the log structure uniquely.

**Theorem** (2.3.23)**.** *Let $X$ and $Y$ be fine and saturated log schemes, and assume that we have an isomorphism $f\colon X_\infty \cong Y_\infty$ between the infinite root stacks. Then there exists an isomorphism of log schemes $X \cong Y$ inducing $f$.*

We analyze the local structure of $X_\infty$, showing that locally for the étale topology of $X$ it can be described as a quotient stack by a (non-finite type) diagonalizable group scheme. This shows that $X_\infty$ has an fpqc presentation, that, despite not being as good as a smooth one, allows us to give a natural notion of quasi-coherent sheaf on it. The concept of a coherent sheaf is trickier, since $X_\infty$ is not coherent in general. In fact, we will mostly use finitely presented sheaves instead of coherent ones.

The infinite root stack is the natural environment for parabolic sheaves with arbitrary rational weights. We define those, in the spirit of [BV12], and extend Borne and Vistoli's result on the equivalence with quasi-coherent sheaves on the root stack.

**Theorem** (2.2.48)**.** *Let $X$ be a fine saturated log scheme. Then there is a tensor equivalence between the category of parabolic sheaves with rational weights on $X$ and quasi-coherent sheaves on the infinite root stack $X_\infty$.*

We also investigate an interesting relation with the Kummer-flat topos of Kato ([Kat, Niz08]). The corresponding site is obtained by considering Kummer-flat morphisms $Y \to X$, i.e. morphisms which are (locally) flat morphisms to a base change of $X$ by a Kummer morphism of monoids. By associating with a Kummer-flat map $Y \to X$ the induced morphism between the root stacks $Y_\infty \to X_\infty$, which is representable and fppf, we obtain a functor $\mathrm{kfl}(X) \to \mathrm{fppf}(X_\infty)$ from the Kummer-flat site of $X$ and an opportunely defined fppf site of $X_\infty$. We prove that this functor induces an equivalence between the corresponding topoi.

**Theorem** (2.4.8)**.** *Let $X$ be a fine saturated log scheme. Then there is an equivalence of ringed topoi $(X_\infty)_{\mathrm{fppf}} \cong X_{\mathrm{kfl}}$ between the fppf topos of the infinite root stack $X_\infty$ and the Kummer-flat topos of $X$.*

We also compare quasi-coherent sheaves on the fppf and fpqc topoi of $X_\infty$. Although they are probably not the same thing in general, finitely presented sheaves are in fact the same, and so we obtain an identification between finitely presented sheaves on the Kummer-flat topos and finitely presented sheaves (i.e. finitely presented parabolic sheaves with rational weights) on the infinite root stack. This has some potential application to K-theory of log schemes ([Niz08]) and to a parabolic version of the Riemann-Hilbert correspondence.

## Moduli of parabolic sheaves

The second part of this document is about moduli of parabolic sheaves. We need some additional assumptions for this part: the log scheme $X$ will be a fine and saturated projective log scheme over a field $k$, with a fixed polarization and with a global *simplicial* chart $P \to \mathrm{Div}(X)$ for the log structure. Simpliciality means that the positive rational cone spanned by $P$ in $P^{\mathrm{gp}} \otimes \mathbb{Q}$ is simplicial, i.e. its extremal rays are linearly independent. Furthermore we assume that the log structure is generically trivial, meaning that there is a schematically dense open subscheme $U \subseteq X$ such that the log structure restricted to $U$ is trivial (part of the results actually hold without this last assumption).

We define a notion of (semi-)stability for finitely presented parabolic sheaves with arbitrary rational weights (i.e. finitely presented sheaves on the infinite root stack $X_\infty$), and construct a moduli space. The final result is the following theorem.

**Theorem** (4.3.5)**.** *Let $X$ be a projective fine saturated log scheme over a field $k$ of characteristic $0$ with generically trivial log structure and with a global chart $P \to \mathrm{Div}(X)$ with $P$ simplicial, and $h \in \mathbb{Q}[x]$ a polynomial of degree $\dim(X)$. There is an Artin stack $\mathcal{M}_h^{ss}$ parametrizing semi-stable torsion-free parabolic sheaves with rational weights and reduced Hilbert polynomial $h$ and an open substack $\mathcal{M}_h^{s} \subseteq \mathcal{M}_h^{ss}$ parametrizing stable torsion-free parabolic sheaves with reduced Hilbert polynomial $h$.*

*The stack $\mathcal{M}_h^{ss}$ is locally of finite type and has a good moduli space $M_h^{ss}$ which is a disjoint union of projective schemes, and there is an open subscheme $M_h^s \subseteq M_h^{ss}$ which is a coarse moduli space for $\mathcal{M}_h^s$. Moreover the map $\mathcal{M}_h^s \to M_h^s$ is a $\mathbb{G}_m$-gerbe.*

We will explain later why we need the assumption on the characteristic of $k$. We remark that this result is new also in the case of a projective variety with an effective Cartier divisor: both Mehta and Seshadri ([MS80]) and Maruyama and Yokogawa ([MY92]) fix the weights of the parabolic sheaves when they construct the moduli spaces.

Although it would be nice to have a moduli theory of sheaves directly on $X_\infty$, the fact that it is not of finite type makes this difficult, and we resort to taking a sort of "limit" of moduli theories on the finite root stacks instead. The first step is to give a moduli theory for parabolic sheaves with fixed denominators.

The basic idea is the following: since parabolic sheaves on $X$ with respect to a fixed system of denominators $A \to B$ are equivalent to quasi-coherent sheaves on the root stack $X_{B/A}$, one can do moduli theory of coherent sheaves on $X_{B/A}$. Nironi ([Nir]) developed a moduli theory for coherent sheaves on tame DM stacks over a field by introducing a notion of (modified) Hilbert polynomial and (semi-)stability, by means of a *generating sheaf*, which is a locally free sheaf that contains all representations of the stabilizer group at any point of the stack. We remark that the root stack is DM only if a certain condition on the characteristic of the base field is satisfied. Nironi's machinery should work also for tame Artin stacks, so the results we obtain are probably valid in general. For simplicity in this introduction we assume that the characteristic of our base field is 0.

By comparison with the notion of parabolic Hilbert polynomial defined by Maruyama and Yokogawa in [MY92] we are able to identify a suitable generating sheaf on the root stack and to apply Nironi's machinery, in some cases. More precisely, although root stacks of a projective log scheme are probably always global quotient stacks (and so they will have generating sheaves), to isolate a "canonical" generating sheaf we need additional data, that we identify in what we call a *locally constant sheaf of charts* for the system of denominators $A \to B$. The case in which there is a global chart (i.e. a Kummer morphism of fine saturated monoids $P \to Q$ that induces $A \to B$ via sheafification) is contained in this broader notion. The resulting concept of (semi-)stability of parabolic sheaves does depend on the choice of this additional datum, as we show with an example: a parabolic sheaf can be semi-stable if we use a chart and become unstable if we use another one. This is analogous to what happens when changing the polarization in moduli theory of coherent sheaves.

Here is the result we get by applying Nironi's machinery.

**Theorem** (3.3.37). *Let $X$ be a projective polarized fine saturated log scheme over a field $k$ with a system of denominators $A \to B$ and a locally constant sheaf of charts, and $H \in \mathbb{N}[x]$ a polynomial. Then there is an Artin stack $\mathcal{M}_H^{ss}$ that parametrizes families of semi-stable parabolic sheaves with respect to $A \to B$, with modified Hilbert polynomial $H$. Moreover $\mathcal{M}_H^{ss}$ is of finite type and has a good moduli space $M_H^{ss}$ which is a projective scheme, obtained as a GIT quotient.*

*There is an open substack $\mathcal{M}_H^s \subseteq \mathcal{M}_H^{ss}$ parametrizing stable parabolic sheaves with Hilbert polynomial $H$, and a corresponding open subscheme $M_H^s \subseteq M_H^{ss}$, which is moreover a coarse moduli space. More precisely, the map $\mathcal{M}_H^s \to M_H^s$ is a $\mathbb{G}_m$-gerbe.*

Note that the simpliciality and generic triviality assumptions on the log scheme are absent here. They will be important for the limit process.

We also remark that this is just the final result of [Nir] applied to the situation of parabolic sheaves on a log scheme, and the original contribution here is the determination of the correct generating sheaf. This construction of course gives back Seshadri's and Maruyama and Yokogawa's moduli spaces when applied to a curve with some points or a projective variety with an effective Cartier divisor respectively. This was already briefly noted by Nironi.

The next step is to take a limit of the stacks that we obtain at finite level. Note that it is not clear that we get well-defined maps on the moduli stacks by extending the denominators, and in fact the main question here regards the behavior of (semi-)stability under pullback along maps of finite root stacks.

Now we have to assume that the log structure of $X$ is simplicial, and we consider the minimal Kummer extension of the form $P \subseteq \mathbb{N}^r$ (simpliciality of $P$ ensures that we can find such an extension), and the root stacks $X_n = X_{\frac{1}{n}\mathbb{N}^r/P}$, on which we have "canonical" generating sheaves and the corresponding moduli stacks $\mathcal{M}_n^{ss}$ and $\mathcal{M}_n^{s}$ of (semi-)stable parabolic sheaves. The $X_n$ are a cofinal system among the root stacks, so $X_\infty = \varprojlim_{n\in\mathbb{N}} X_n$, and moreover the transition maps $X_m \to X_n$ when $n \mid m$ are all flat. The flatness, which is one of the reasons for the simpliciality assumption, ensures in particular that pullbacks of pure sheaves remain pure (recall that semi-stable sheaves are always pure).

Let us preliminarily remark that the Hilbert polynomial is *not* preserved by pullback along $X_m \to X_n$, so it is not convenient to fix it in this setting. What is preserved is the *reduced* Hilbert polynomial, i.e. the polynomial $h$ that we obtain by dividing the Hilbert polynomial $H$ by $d!$ times its leading coefficient (where $d$ is the degree of $H$), so that $h$ has leading term $x^d/d!$. We denote by $\mathcal{M}_{h,n}^{ss}$ and $\mathcal{M}_{h,n}^{s}$ the stacks that we obtain by fixing the reduced Hilbert polynomial. They are disjoint unions of (possibly infinitely many of) the previous stacks $\mathcal{M}_{H,n}^{ss}$ and $\mathcal{M}_{H,n}^{s}$ respectively.

We show that in this setting semi-stability is always preserved, so that whenever $n \mid m$ we have maps of moduli stacks $\mathcal{M}_{h,n}^{ss} \to \mathcal{M}_{h,m}^{ss}$ induced by pullback along $X_m \to X_n$, and moreover these are always open immersions. The same is not true for stability, which is not necessarily preserved. When it is preserved, we have corresponding open immersions $\mathcal{M}_{h,n}^{s} \to \mathcal{M}_{h,m}^{s}$, and moreover in this case these maps, together with $\mathcal{M}_{h,n}^{ss} \to \mathcal{M}_{h,m}^{ss}$ and the induced morphisms $M_{h,n}^{ss} \to M_{h,m}^{ss}$ and $M_{h,n}^{s} \to M_{h,m}^{s}$ between the moduli spaces, are all an open and closed immersions. We also show with examples that if stability is not preserved, the open immersion $\mathcal{M}_{h,n}^{ss} \to \mathcal{M}_{h,m}^{ss}$ need not be closed.

If stability is not preserved, it is not clear to us if the maps $M_{h,n}^{ss} \to M_{h,m}^{ss}$ between the good moduli spaces are open and closed immersions. We show that they are always geometrically injective, open and closed. We also do not have any examples where they are not immersions, and it is plausible that this could always be the case.

We give some conditions that ensure that stability is preserved. A notable situation where this holds is when the log structure of $X$ is generically trivial and we are considering torsion free (i.e. pure of maximal dimension) sheaves.

**Theorem** (4.2.33)**.** *Let $X$ be a projective polarized fine saturated log scheme with a simplicial global chart $P \to \mathrm{Div}(X)$ over a field $k$. Assume furthermore that the log structure of $X$ is generically trivial. Then* *the pullback along $X_m \to X_n$ preserves stability of parabolic torsion-free sheaves.*

From now on we restrict to this situation, i.e. we consider only torsion-free sheaves. We gather the results of this discussion in the following theorem.

**Theorem** (4.2.10, 4.2.30)**.** *Let $X$ be a projective polarized fine saturated log scheme over a field $k$ of characteristic 0 with generically trivial log structure and with a global chart $P \to \mathrm{Div}(X)$ with $P$ simplicial, $n, m$ two positive integers with $n \mid m$ and $h \in \mathbb{Q}[x]$ a polynomial of degree $\dim(X)$.*

*Then (semi-)stability of torsion-free sheaves is preserved by pullback along the projection $X_m \to X_n$, and the resulting morphisms $\mathcal{M}^{ss}_{h,n} \to \mathcal{M}^{ss}_{h,m}$ and $\mathcal{M}^{s}_{h,n} \to \mathcal{M}^{s}_{h,m}$ between the moduli stacks, together with the induced maps $M^{ss}_{h,n} \to M^{ss}_{h,m}$ and $M^{s}_{h,n} \to M^{s}_{h,m}$ of their good moduli spaces, are open and closed immersions.*

Finally thanks to these results we can define (semi-)stability for finitely presented sheaves on the infinite root stack, declaring a sheaf to be (semi-)stable if any finitely presented sheaf on a finite root stack that pulls back to it is (semi-)stable. We obtain a stack $\mathcal{M}^{ss}$ (resp. $\mathcal{M}^{s}$) parametrizing families of (semi-)stable parabolic sheaves with rational weights, and we show that it is the direct limit (which is really an increasing union) of the corresponding stacks at finite level. We also construct the good moduli spaces $M^{ss}$ and $M^{s}$ by taking a direct limit.

We remark that the stacks and spaces that we obtain are not of finite type: the space $M^{ss}$ is a union of projective schemes, but it can be an infinite union. We do not know if one can fix more refined invariants than the reduced Hilbert polynomial in order to cut out finite type loci in these stacks and spaces.

We stress once again that, provided that Nironi's machinery also works for tame Artin stacks, the characteristic 0 hypothesis can be omitted, and all the results still hold, up to replacing "good moduli space" with "adequate moduli space" in every instance.

## Future perspectives

Here we discuss some questions left open by the present work, which might be worth pursuing in the future.

One possibly fruitful direction of research is a further study of the infinite root stack of a logarithmic scheme. As we mentioned, the close relation between the geometry of $X_\infty$ and the logarithmic geometry of $X$ itself (for example the relation with the Kummer-flat topos) has potential interesting applications to matters of logarithmic geometry. For instance, the K-theory of logarithmic schemes ([Niz08]), for which the Kummer-flat topos is a fundamental ingredient, can be reinterpreted on the the stack $X_\infty$, and here one can use results on the K-theory of algebraic stacks (with some care, since $X_\infty$ is not algebraic, but only pro-algebraic). Another possible application is to a parabolic version of the Riemann-Hilbert correspondence: in [IKN05], the authors give a version (in characteristic 0) of this correspondence that involves the Kummer-étale site. There should be an analogous result in arbitrary characteristic involving the Kummer-flat site instead, and perhaps by using the equivalence with the fppf topos of $X_\infty$, one can write down a parabolic version.

A natural question left open by Nironi ([Nir]) and by my own work is the following: how do the moduli spaces of sheaves depend on the chosen generating sheaf, and consequently on the chart of the logarithmic structure? What happens to the moduli spaces when one changes them? The corresponding problem for the change of polarization in the moduli theory of coherent sheaves has been studied in some cases, mainly in dimension 2 ([MW97, EG95, Qin93]): there is a chamber decomposition of the ample cone of the variety, and the moduli spaces are constant inside the chambers. Moreover when the polarization crosses a wall, there are interesting flip-like maps connecting the corresponding moduli spaces. It is plausible that something similar happens by varying the generating sheaf: the ample cone should be replaced by a "generating" cone inside the numerical K-theory of the root stack, and one could expect a chamber decomposition and interesting maps when the sheaf crosses a wall. The variation of the moduli spaces of parabolic bundles when one varies the weights was studied in this spirit, on a curve, in [BH95].

Another unresolved question is the formulation of a moduli theory without a global chart (or a locally constant sheaf of charts) for the logarithmic structure. Logarithmic schemes without global charts are common, one example is the projective plane with an irreducible nodal curve, and it would be nice to have a theory that works also in these cases. The main difficulty here is to find a generating sheaf that is "canonically defined" in some sense.

The introduction of a Higgs field to the structure of a parabolic bundle produces what is called a parabolic Higgs bundle. Moduli spaces of these were studied in the case of curves ([Yok93, BY96]), and it is probably worth it to try to apply the same methods we used here for bare parabolic shaves to construct moduli spaces of parabolic Higgs sheaves on more general logarithmic schemes.

Lastly, it would be interesting to have a theory for moduli of parabolic sheaves with real weights. Such a theory seems to be lacking even in the case of a variety equipped with an effective divisor, since in [MY92] at a certain point the authors assume that the weights are rational. Notably, in the case of curves it is proven in [MS80] that a fixed sequence of real weights can be substituted with close enough rational weights without modifying the corresponding notion of semi-stability, so in fact the case with rational weights is sufficient. One can wonder if something like this also happens in general.

## Description of contents

Here we describe the contents of each chapter in some detail.

Chapter 1 contains preliminary notions and results. Most of the results here are well-known or in the literature, and we give references instead of writing down proofs whenever possible. In Section 1.1 we recall basic nomenclature and facts about commutative monoids. Section 1.2 is about logarithmic geometry: for the convenience of the reader we briefly recall the definitions and facts that we will need to use in the rest of the thesis. We also recall the construction of root stacks and their basic properties. Section 1.3 is about parabolic sheaves. We recall the definitions from [BV12], and sketch the proof of the equivalence with quasi-coherent sheaves on the root stack (1.3.8), since the constructions used in it will show up a couple of times in the following chapters.

The remaining chapters can be roughly divided in two parts: the first one introduces the infinite root stack of a log scheme $X$ and studies some aspects of its geometry and its relations to the log geometry of $X$ (Chapter 2), while the second part is focused on the moduli problem for parabolic sheaves on a log scheme (Chapters 3 and 4).

Chapter 2 is about the infinite root stack. After a preliminary section (2.1) about inverse limits of stacks, in Section 2.2 we define the infinite root stack $X_\infty$ and discuss its local geometry. As a consequence we are able to define quasi-coherent sheaves, and we discuss two different sites on $X_\infty$: one is defined by using representable fpqc morphisms (2.2.31) and the other one, that will show up later when we explore the relation with the Kummer-flat topos, by using representable fppf morphisms (2.2.38). We show that finitely presented sheaves are the same on these two sites (2.2.40). In section 2.3 we extend the definitions and results of [BV12] to the infinite root stack and to parabolic sheaves with rational weights. The last two sections of this chapter are about recovering information about the log scheme $X$ from the infinite root stack. We describe a reconstruction method that allows us to recover the log structure from the infinite root stack (2.3.1), showing in particular that log schemes with isomorphic infinite root stacks must be isomorphic themselves, and we show that the fppf topos of $X_\infty$ that we introduced in Section 2.2 is equivalent to the Kummer-flat topos of Kato (2.4). We conclude that finitely presented sheaves on the Kummer-flat topos are exactly finitely presented parabolic sheaves with rational weights (2.4.11).

In Chapter 3 we investigate the moduli theory for parabolic sheaves with respect to a fixed system of denominators. Sections 3.1 and 3.2 contain preliminary discussions of pullbacks of parabolic sheaves along morphisms of log schemes and of various properties, such as coherence, flatness over a base scheme and pureness, that are very important in moduli theory. In section 3.3 we discuss the choice of the generating sheaf on the root stack, that we will use to apply Nironi's machinery. We operate this choice by inspecting Maruyama and Yokogawa's treatment and by finding a generating sheaf that generalizes their definition of parabolic Hilbert polynomial in the case of a variety with an effective Cartier divisor (3.3.1). We also explain how to relax a little the requirement about having a global chart, introducing what we call "locally constant sheaves of charts" (3.3.3). Finally, we apply Nironi's theory and state the results that we get out of it about stacks of parabolic sheaves (3.3.4). In the last section we show with an example that the notion of stability that we get depends on the chart of the log structure that we choose (3.4).

Chapter 4 is about moduli theory for parabolic sheaves with arbitrary rational weights. Our strategy is to take a "limit" of the moduli theories at finite level that we described in the preceding chapter. Section 4.1 is about a simpliciality condition that we have to impose on the log structure of the log scheme $X$ for our methods to work. In particular this ensures that we have a cofinal system of root stacks whose transition maps are all flat (4.1.4). In Section 4.2 we study the behavior of (semi-)stability with respect to pullback along maps between root stacks. We show that semi-stability is always preserved, and stability is preserved in some cases, for example when the log structure of $X$ is generically trivial. We also study the induced maps between the moduli stacks of (semi-)stable sheaves and the corresponding moduli spaces (4.2.4). Finally, the last section is about the resulting moduli theory for finitely presented parabolic sheaves with rational weights (4.3).

## Notations and conventions

We will always work over a fixed field $k$. Most of Chapter 2 and some parts of Chapter 3 make perfect sense for log schemes over $\mathbb{Z}$, but we felt it was more convenient to make this assumption from the start. In Chapter 4 we will also assume that the characteristic of $k$ is zero (see Remark 3.3.24).

Schemes and stacks will always be over $k$. We will denote by $(\mathrm{Sch})$ and $(\mathrm{St})$ the categories of schemes and stacks (for the étale topology) over $k$ respectively. If $S$ is a scheme over $k$, we will denote by $(\mathrm{Sch}/S)$ the category of schemes over $S$, and analogously with other "slice" categories. Furthermore we will denote by $(\mathrm{Aff})$ and $(\mathrm{Algsp})$ the categories of affine schemes and algebraic spaces over $k$ respectively.

The symbol $\times_k$ will denote fibered product over $\mathrm{Spec}(k)$. We will usually omit the subscript in tensor products, unless it is not clear over what the tensor product is taken.

If $\mathcal{C}$ is a category, the symbols $c \in \mathcal{C}$ will mean that $c$ is an object of $\mathcal{C}$. As (almost) everybody does, we will ignore set-theoretic subtleties regarding categories and sites. As usual $\mathcal{C}^{\mathrm{op}}$ will denote the opposite category. For symmetric monoidal categories, we use the same conventions described in [BV12, 2.4], and for fibered categories we refer to [FGI$^+$07, Chapter 1].

The symbol $X$ will most of the times denote a log scheme over $k$, with Deligne–Faltings structure given by $L\colon A \to \mathrm{Div}_X$. In Chapter 3, the symbol $\mathfrak{X}$ will denote the root stack of $X$ with respect to the fixed system of denominators $A \to B$.

A morphism of stacks over $k$ will be *representable* if the base change of an algebraic space is an algebraic space. An *algebraic stack* or *Artin stack* for us will be a stack (in groupoids) with a smooth presentation (i.e. a representable smooth epimorphism from an algebraic space) and representable diagonal. An algebraic stack will be *Deligne–Mumford* (sometimes abbreviated DM) if it has a presentation which is moreover étale. An algebraic stack is *tame* if it has finite inertia and linearly reductive stabilizer groups (see [AOV08]).

We will consider (small) sites and the corresponding topoi of a scheme or stack $\mathcal{X}$; they will be introduced along the way. As for notation, if for example we are considering the étale topology, we will denote by $\text{ét}(\mathcal{X})$ the small étale site and by $\mathcal{X}_{\text{ét}}$ the corresponding topos. If $\mathcal{T}$ is a topos, $\mathrm{QCoh}(\mathcal{T})$ will denote the category of quasi-coherent sheaves on $\mathcal{T}$, and $\mathrm{FP}(\mathcal{T})$ will be the subcategory consisting of finitely presented sheaves.

Whenever we have a groupoid $R \rightrightarrows U$, a superscript $^{eq}$ will denote equivariant objects with respect to the groupoid. For example $\mathrm{FP}^{eq}(U)$ will denote the category of finitely presented equivariant sheaves on $U$.

A *geometric point* of a scheme $X$ will be a morphism $\mathrm{Spec}(K) \to X$ from the spectrum of an algebraically closed field to $X$. It will often be denoted just by $p \to X$. If $x$ is a point of $X$, with $\overline{x}$ we will denote the geometric point lying over $x$ obtained by taking the algebraic closure of the residue field $k(x)$. In particular if $A$ is a sheaf on the small étale site of $X$ and $x$ is a point of $X$, $A_{\overline{x}}$ will denote the stalk of $A$ at the geometric point $\overline{x}$.

A subscript will often be a shorthand for pullback along a morphism of schemes.

The word "morphism" will be interchangeable with both "homomorphism" and "map". The difference between the last two is that "homomorphism" will usually refer to a morphism of algebraic structures (for example groups), whereas "map" will be used mostly for morphisms of geometric objects (for example schemes).

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# Chapter 1

# Preliminaries

In this chapter we collect some basic notions and results that will be used throughout this document. We will mostly give references instead of proofs here.

The first section is about monoids, the second one about logarithmic geometry, and third one treats parabolic sheaves.

## 1.1 Monoids

This section is about the basics of monoids and homomorphisms between them. As general references for monoids we point out the books [Réd65, RS99] and the notes [Ogu].

**Definition 1.1.1.** A *monoid* is a set $P$ together with a binary operation $+\colon P \times P \to P$ that is associative and has a neutral element $0 \in P$.

A *homomorphism* of monoids $f\colon P \to Q$ is a function such that $f(p+p') = f(p) + f(p')$ for all $p, p' \in P$ and $f(0) = 0$.

We will usually write the monoid operation as addition, but sometimes it will be convenient to use a multiplicative notation. For example if $X$ is a scheme, the structure sheaf $\mathcal{O}_X$ is a sheaf of monoids with respect to multiplication.

All our monoids will be commutative, i.e. for any $p, p' \in P$ we have $p + p' = p' + p$. We will denote by (CommMon) the category of commutative monoids.

We will denote by $P^+$ the subset $P \setminus \{0\} \subseteq P$.

**Remark 1.1.2.** If $G$ is a group, then it is also a monoid with respect to its group operation. Whenever we will consider $G$ as a monoid, it will be in this sense.

This gives an inclusion functor (Ab) $\to$ (CommMon) from the category of abelian groups to the category of commutative monoids.

We will denote by $k[P]$ the monoid algebra of $P$. It is defined as the $k$-algebra generated by indeterminates $x^p$ for $p \in P$, and with relations $x^{p+p'} - x^p x^{p'}$ for every pair $p, p' \in P$.

The algebra $k[P]$ is naturally $P$-graded, with the degree defined in the obvious way by $\deg(x^p) = p$.

**Definition 1.1.3.** An ideal $I \subseteq P$ of a monoid $P$ is a subset such that for every $p \in I$ and $q \in P$ we have $p + q \in I$.

There is a bijection between homogeneous ideals of $k[P]$ with respect to the $P$-grading and ideals of the monoid $P$, by taking for an ideal $I \subseteq P$ the ideal of $k[P]$ generated by the elements $x^p$ with $p \in I$.

Given a monoid $P$ we can form the associated group $P^{\mathrm{gp}}$. We start from $P \times P$ and take a quotient by the equivalence relation that identifies pairs $(p_1, p_1')$ and $(p_2, p_2')$ if there exists $q \in P$ such that

$$p_1 + p_2' + q = p_2 + p_1' + q.$$

The idea here is that the pair $(p, p')$ stands for the "difference" $p - p'$.

One checks that $P^{\mathrm{gp}}$, with the induced operation given by

$$[p_1, p_1'] + [p_2, p_2'] = [p_1 + p_2, p_1' + p_2']$$

is an abelian group, and there is a homomorphism $P \to P^{\mathrm{gp}}$ sending $p \in P$ to $[p, 0]$. Moreover this homomorphism is universal with respect to homomorphisms $f \colon P \to G$ with $G$ a group, i.e. every such $f$ factors through $f' \colon P^{\mathrm{gp}} \to G$. The resulting functor $(-)^{\mathrm{gp}} \colon (\mathrm{CommMon}) \to (\mathrm{Ab})$ from commutative monoids to abelian groups is left adjoint to the inclusion functor $(\mathrm{Ab}) \to (\mathrm{CommMon})$.

**Definition 1.1.4.** A monoid $P$ is *integral* if the canonical homomorphism $P \to P^{\mathrm{gp}}$ is injective, or equivalently if $p + q = p + r$ in $P$ implies $q = r$.

We will denote by (IntCommMon) the category of integral commutative monoids.

**Definition 1.1.5.** A *submonoid* $Q \subseteq P$ of a monoid $P$ is a subset that contains the neutral element 0 and such that $q + q' \in Q$ for every $q, q' \in Q$.

If $\{p_i\}_{i \in I}$ is a collection of elements of a monoid $P$, the smallest submonoid of $P$ containing all of the $p_i$'s will be denoted by $\langle p_i \rangle_{i \in I}$, or by $\langle p_1, \dots, p_r \rangle$ in case the collection is finite. This submonoid coincides with the subset of $P$ of elements that can be written as $a_1 p_{i_1} + \cdots + a_k p_{i_k}$ for some $k, a_1, \dots, a_k \in \mathbb{N}$ and $i_1, \dots, i_k \in I$.

**Definition 1.1.6.** A monoid $P$ is *finitely generated* if there exist a finite number of elements $p_1, \dots, p_r \in P$ such that $\langle p_1, \dots, p_r \rangle = P$. A monoid is *fine* if it is both integral and finitely generated.

By a theorem of Rédei [RS99, Theorem 5.12], every finitely generated commutative monoid is also *finitely presented*. This means that the relations between the generators $p_i$ can be described by using finitely many of them. If we want to specify the relation in addition to the generators we will use the following notation: given a finite number of generating elements $p_1, \dots, p_r \in P$, assume that $r_1 = s_1, \dots, r_j = s_j$ is a generating set (as a congruence, i.e. an equivalence relation stable under translations ([Ogu, Section 1.1])) for the relations among the $p_i$'s, where every $r_i$ and $s_i$ is an expression of the form $\sum_{i=1}^r a_i p_i$ for $a_i \in \mathbb{N}$. Then we will write

$$P = \langle p_1, \dots, p_r \mid r_1 = s_1, \dots, r_j = s_j \rangle.$$

This expresses $P$ as a quotient of the free monoid on the generators $p_1, \ldots, p_r$.

A monoid is *free* if it is isomorphic to $\mathbb{N}^r$ for some $r$. Equivalently, it has a presentation with finitely many generators and no relations.

**Example 1.1.7.** The submonoid $P \subseteq \mathbb{N}^2$ generated by $(2,0), (0,2), (1,1)$ can also be described as $\langle p, q, r \mid p + q = 2r \rangle$.

Quotients in the category of monoids are subtler than the ones in more familiar setting (like groups), so we will not go into details (see [Ogu, Section 1.1]). We need only to remark that the category of commutative monoids (CommMon) has all colimits: direct sums are constructed as for abelian groups, and coequalizers by taking quotients. In particular we have amalgamated sums: for a diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g"'] & Q \\ R & \end{tikzcd}$$

of commutative monoids there is commutative monoid $Q \oplus^P R$, unique up to isomorphism, that completes the diagram to a (commutative) square, and such that for any other monoid $P'$ with maps $h\colon Q \to P'$ and $k\colon R \to P'$ such that $h \circ f = k \circ g\colon P \to P'$, there exists a unique $Q \oplus^P R \to P'$ that makes all diagrams commute.

As the notation suggests, $Q \oplus^P R$ is a quotient of the direct sum $Q \oplus R$, but the equivalence relation does not have a nice description, except in particular cases (for example if one of the monoids is a group).

**Definition 1.1.8.** A *unit* in a monoid $P$ is an element $p \in P$ such that there exists $q \in P$ such that $p + q = 0$. A *sharp* monoid is a monoid in which the only unit is 0.

The subset of units $P^\times \subseteq P$ is a subgroup, the *group of units* of $P$. The quotient $P/P^\times$ is usually denoted by $\overline{P}$, and is clearly a sharp monoid.

**Example 1.1.9.** All free monoids are sharp, and so is every submonoid of a free monoid.

A group is sharp as a monoid if and only if it is trivial.

**Definition 1.1.10.** An integral monoid $P$ is *torsion-free* if $P^{\mathrm{gp}}$ is torsion-free as an abelian group, or equivalently if $np = nq$ with $n \in \mathbb{N}$ implies $p = q$.

An integral monoid $P$ is *saturated* if $p \in P^{gp}$ and $np \in P$ for some $n \in \mathbb{N}$ imply $p \in P$.

We denote by (SatCommMon) the category of saturated commutative monoids. If a monoid is both fine and saturated we will usually abbreviate it by saying that it is an *fs monoid*, and we will denote by (FSCommMon) the category of fs monoids.

**Proposition 1.1.11.** *Every fine saturated sharp monoid is torsion-free.*

*Proof.* Assume that $n(p - q) = 0$ in $P^{\mathrm{gp}}$ for $p, q \in P$. Then since $P$ is saturated, we have that $p - q \in P$, and moreover this is a unit, since $p - q + (n-1)(p - q) = 0$, so by sharpness $p - q = 0$, and $p = q$. $\square$

There are operations that make a monoid integral and saturated. If $P$ is any monoid, we set $P^{\mathrm{int}}$ to be the image of $P$ inside $P^{\mathrm{gp}}$ with respect to the natural map $P \to P^{\mathrm{gp}}$. Then $P^{\mathrm{int}}$ is an integral monoid, and for any homomorphism $P \to Q$ with $Q$ integral there is a factorization $P^{\mathrm{int}} \to Q$. In particular a morphism of monoids $P \to Q$ induces a morphism $P^{\mathrm{int}} \to Q^{\mathrm{int}}$, and the resulting functor $(-)^{\mathrm{int}} \colon (\mathrm{CommMon}) \to (\mathrm{IntCommMon})$ is left adjoint to the inclusion functor $(\mathrm{IntCommMon}) \to (\mathrm{CommMon})$.

If $P$ is an integral monoid, we define $P^{\mathrm{sat}}$ to be the submonoid

$$P^{\mathrm{sat}} = \{p \in P^{\mathrm{gp}} \mid np \in P \text{ for some } n \in \mathbb{N}\}$$

of $P^{\mathrm{gp}}$. The monoid $P^{\mathrm{sat}}$ is saturated, and for any homomorphism $P \to Q$ with $Q$ saturated there is a factorization $P^{\mathrm{sat}} \to Q$. This implies that saturation gives a functor $(-)^{\mathrm{sat}} \colon (\mathrm{IntCommMon}) \to (\mathrm{SatCommMon})$ that is left adjoint to the inclusion functor $(\mathrm{SatCommMon}) \to (\mathrm{IntCommMon})$.

If $P$ is finitely generated then $P^{\mathrm{sat}}$ is finitely generated as well, and in this case we will denote it by $P^{\mathrm{fs}}$, to stress the fact that it is going to be fine and saturated.

Fine sharp monoids can be presented in a canonical way using indecomposable elements.

**Definition 1.1.12.** An element $p$ of a monoid $P$ is *indecomposable* if $p = q + r$ in $P$ implies $q = 0$ or $r = 0$.

**Proposition 1.1.13.** *Let $P$ be a fine sharp monoid. Then $P$ has a finite number of indecomposable elements, and they are generators for $P$.*

*Proof.* Let us fix a finite number of generators $p_1, \dots, p_k$ for $P$. Now every $q \in P$ can be written as $q = \sum a_i p_i$, and if $q$ is an indecomposable, then we must have that $a_i = 0$ for all but one $i_0$, and $a_{i_0} = 1$, so $q = p_{i_0}$. In other words every indecomposable must coincide with one of the $p_i's$, and so they are finitely many.

Now let us assume that some $p_k$ is not indecomposable, and let us show that we can omit it from the list. Assume $p_k = p + q$ with $p, q \neq 0$, and let us write $p = \sum a_i p_i$ and $q = \sum b_i p_i$, so $p_k = \sum (a_i + b_i) p_i$. Now if $a_k + b_k \neq 0$, using integrality we could cancel $p_k$ and obtain $0 = \sum_{i \neq k} (a_i + b_i) p_i + (a_k + b_k - 1) p_k$. By sharpness this would imply $a_i = b_i = 0$ for $i \neq k$, and $a_k + b_k = 1$. In other words $p = p_k$ and $q = 0$, or the other way around, contradicting the assumption.

So $a_k = b_k = 0$. This says that $p_k = \sum_{i \neq k} (a_i + b_i) p_i$ lies in the submonoid generated by the remaining $p_i$'s, so we can omit it from the generators.

After finitely many steps, we are left with a finite generating set made up exactly by the indecomposable elements of $P$. □

The following gives embedded models for particularly nice monoids.

**Proposition 1.1.14** ([Ogu, Corollary 2.2.6])**.** *Every fine sharp torsion-free monoid is a submonoid of $\mathbb{N}^r$ for some $r$.*

Let $P$ be a fine saturated torsion-free monoid. Then $P^{\mathrm{gp}}$ is a free abelian group $\mathbb{Z}^r$ for some $r$, that we will call *rank* of $P$. We will denote by $P_{\mathbb{Q}}$ the positive rational cone spanned by $P$ inside $P^{\mathrm{gp}} \otimes \mathbb{Q}$, i.e.

$$P_{\mathbb{Q}} = \{a \in P^{\mathrm{gp}} \otimes \mathbb{Q} \mid na \in P \text{ for some } n \in \mathbb{N}\}$$

where we see $P \subseteq P^{\mathrm{gp}} \otimes \mathbb{Q}$ in the natural way.

If we denote by $\frac{1}{n}P$ the submonoid of $P^{\mathrm{gp}} \otimes \mathbb{Q}$ consisting of the elements $\frac{p}{n}$ for $p \in P$, we have inclusions $\frac{1}{n}P \subseteq \frac{1}{m}P$ whenever $n \mid m$, and $P_{\mathbb{Q}} = \bigcup_n \frac{1}{n}P$. The inclusion $\frac{1}{n}P \subseteq \frac{1}{m}P$ can be seen as multiplication by $k = m/n$ from $P$ to itself.

If $\phi\colon P \to Q$ is a morphism of fs sharp monoids we have an induced morphism $\phi_{\mathbb{Q}}\colon P_{\mathbb{Q}} \to Q_{\mathbb{Q}}$.

This construction makes sense for arbitrary monoids: for every $n$ we take a copy $P_n$ of $P$ and define $P_n \to P_m$ for $n \mid m$ to be multiplication by $m/n$. We can form the direct limit

$$P_{\mathbb{Q}} = \varinjlim_n P_n.$$

If $P$ is not fs and torsion-free the maps of this system might not be injective: for example if $P$ is the monoid with two elements $0, 1$ and $1 + 1 = 1$, then every morphism $P_n \to P_m$ is the identity, and $P \cong P_{\mathbb{Q}}$.

**Remark 1.1.15.** The resulting functor $(-)_{\mathbb{Q}}$ commutes with pushouts of monoids. This follows from the fact that, as we just remarked, $P_{\mathbb{Q}}$ can be written as a direct limit, and direct limits commute with colimits.

**Definition 1.1.16.** Let $P$ and $Q$ be fs monoids. A *Kummer* homomorphism $\phi\colon P \to Q$ is an injective homomorphism such that for any $q \in Q$ there exists a positive $n \in \mathbb{N}$ such that $nq \in \phi(P)$.

Equivalently, if $P$ and $Q$ are torsion-free $\phi$ is Kummer if and only if $\phi_{\mathbb{Q}}\colon P_{\mathbb{Q}} \to Q_{\mathbb{Q}}$ is an isomorphism.

**Example 1.1.17.** A fundamental example of Kummer homomorphism is the one we already described: for $P$ be an fs torsion-free monoid and let $\phi\colon P \to P$ be multiplication by a fixed $n \in \mathbb{N}$.

Here is another (less trivial) example: let $P$ be the submonoid of $\mathbb{N}^2$ generated by $(2,0)$, $(0,2)$ and $(1,1)$, and let $\phi\colon P \to \mathbb{N}^2$ be the inclusion.

Let us now describe a particular kind of quotient maps that are important in logarithmic geometry.

**Definition 1.1.18.** The *kernel* $\ker(\phi)$ of a homomorphism of monoids $\phi\colon P \to Q$ is the submonoid of $P$ consisting of elements $p \in P$ such that $\phi(p) = 0$.

A morphism of monoids $\phi\colon P \to Q$ is a *cokernel* if the induced map $P/\ker(\phi) \to Q$ is an isomorphism. Equivalently, if $\phi(p_1) = \phi(p_2)$ implies that there exist $q_1, q_2 \in \ker(\phi)$ such that $q_1 + p_1 = q_2 + p_2$ in $P$.

Note that, contrarily to what happens with groups, not every surjective map of monoids is a cokernel.

**Example 1.1.19.** The first projection $\mathbb{N}^2 \to \mathbb{N}$ is a cokernel: its kernel is the subset of $\mathbb{N}^2$ of elements of the form $(0,n)$, and if $(a,b)$ and $(c,d)$ have the same image, i.e. $a = c$, we have $(a,b) + (0,d) = (c,d) + (0,b)$.

The morphism $\mathbb{N}^2 \to \mathbb{N}$ that sends $(a,b)$ to $a + b$ is surjective, but is not a cokernel, because its kernel is trivial.

## 1.2 Logarithmic geometry

We will almost always adopt the point of view of [BV12] regarding logarithmic geometry, which differs from the original one of Kato. Other references for the classical point of view on logarithmic geometry are [Kat89, Ogu]. We briefly recall the main definitions and results.

If $X$ is a scheme, we will denote by $\mathrm{Div}(X)$ the symmetric monoidal category of pairs $(L, s)$ with $L$ a line bundle on $X$ and $s$ a global section of $L$. The monoidal structure is given by tensor product in the evident way. Furthermore we will denote by $\mathrm{Div}_X$ the symmetric monoidal fibered category over the small étale site $X_{\text{ét}}$ (i.e. the site that has as objects étale maps $U \to X$ and equipped with the étale topology), whose objects over $U \to X$ are pairs $(L, s)$ consisting of a line bundle on $U$ with a global section, with monoidal operation given again by tensor product. Note that $\mathrm{Div}(X)$ is the category of "global sections" of $\mathrm{Div}_X$.

These should be thought of as a categories of "generalized Cartier divisors". The advantage over ordinary Cartier divisors is that invertible sheaves and sections can be always pulled back, and thus have better functoriality properties. As a fibered category, $\mathrm{Div}_X$ is the restriction of the stacky quotient $[\mathbb{A}^1/\mathbb{G}_m]$ to $X_{\text{ét}}$.

**Remark 1.2.1.** We will be dealing with sheaves of monoids on the small étale site $X_{\text{ét}}$. Whenever we will attach some property to a sheaf of monoids $A$ on $X_{\text{ét}}$, for example integral, saturated, and so on, we always mean that all the geometric stalks (i.e. pullbacks to geometric points $x \to X$ of $X$) of the sheaf $A$ have that property.

**Definition 1.2.2.** A *Deligne–Faltings* (abbreviated DF) *structure* on a scheme $X$ is a symmetric monoidal functor $L\colon A \to \mathrm{Div}_X$ from a sheaf of monoids on $X_{\text{ét}}$, with trivial kernel. A *logarithmic scheme* is a scheme $X$ equipped with a DF structure.

We will usually refer to a DF structure as the functor $L\colon A \to \mathrm{Div}_X$, occasionally as the pair $(A, L)$. Also, we will often abbreviate the word "logarithmic" with just "log".

**Remark 1.2.3.** This definition (as well as everything that follows) makes sense also for $X$ an Artin stack. The only difference is that we have to use the lisse-étale site of $X$ in place of the small étale site. In the case of schemes or DM stacks, using the lisse-étale site or the small étale site produces the same theory if we restrict to fine log structures (Definition 1.2.20 below, and for a proof see [Ols03, Proposition 5.3]).

In the rest of this section and for most of the document we will mainly be concerned with log schemes, but from time to time the wording "log stack" will come up. For precise definitions, see [Ols03].

A *morphism* $(\phi, \Phi)\colon (A, L) \to (B, N)$ of DF structures on a scheme $X$ is a morphism $\phi\colon A \to B$ of sheaves of monoids, together with a natural isomorphism $\Phi\colon L \cong M \circ \phi$. Morphisms can be composed in the obvious way.

The link with the usual definition of a (quasi-integral) log structure as a morphism $\alpha\colon M \to \mathcal{O}_X$ of sheaves of monoids such that $\alpha^{-1}(\mathcal{O}_X^\times) \cong \mathcal{O}_X^\times$ is the following: recall that *quasi-integral* means that the natural action of $\mathcal{O}_X^\times$ on $M$ is faithful. This is implied for example by integrality, as the name suggests.

Starting from $\alpha$ we get $L$ by dividing (in the stacky sense) by $\mathcal{O}_X^\times$, so in particular $A = \overline{M} = M/\mathcal{O}_X^\times$, and $L$ is $A = M/\mathcal{O}_X^\times \to \mathcal{O}_X/\mathcal{O}_X^\times \cong \mathrm{Div}_X$. In other words a section of $A$ is sent by $L$ to the dual $L_a$ of the invertible sheaf associated to the $\mathbb{G}_m$-torsor given by the fiber $M_a$ of $M \to \overline{M} = A$ over $a$, and the restriction of $\alpha$ to $M_a \to \mathcal{O}_X$ gives the section of $L_a$.

In the other direction, given $L$, we get back $\alpha$ by taking the fibered product $M = A \times_{\mathrm{Div}_X} \mathcal{O}_X$, and the induced morphism $M \to \mathcal{O}_X$.

This constructions give an equivalence of between quasi-integral log schemes in the sense of [Kat89] and log schemes in the sense of [BV12]. We will freely pass from one point of view to the other one in the following.

**Remark 1.2.4.** Note that the sheaf $A$ of a log scheme is a sheaf of sharp monoids. This follows from the fact that $L$ has trivial kernel and the units of $\mathrm{Div}_X$ are isomorphic to $(\mathcal{O}_X, 1)$, or alternatively from this description of $A$ as $\overline{M}$.

**Example 1.2.5.** Every scheme $X$ has a trivial log structure, by taking as $A$ the constant sheaf of trivial monoids, or equivalently by taking $M = \mathcal{O}_X^\times$ with the inclusion into $\mathcal{O}_X$.

**Example 1.2.6.** If $k$ is algebraically closed, a log structure on $\mathrm{Spec}(k)$ simply amounts to a monoid $P$, and the morphism $P \to k$ inducing the log structure sends 0 to 1 and everything else to 0. The corresponding "sheaf" of monoids on $\mathrm{Spec}(k)_{\text{ét}}$ is $P \oplus k^\times$.

If $P = \mathbb{N}$, then the resulting log scheme is called the *standard log point*.

**Example 1.2.7.** Let $X$ be a scheme, and $D \subseteq X$ an effective Cartier divisor, seen as a closed subscheme. Then the subsheaf $M \subseteq \mathcal{O}_X$ defined as

$$M(U) = \{f \in \mathcal{O}_X(U) \mid f|_{U \setminus D} \text{ is invertible } \}$$

gives a log structure on $X$. We will call this the *log structure induced by* the divisor $D$.

**Example 1.2.8.** Let $P$ be a monoid. Then the spectrum of the monoid algebra $X = \mathrm{Spec}(k[P])$ has a natural log structure, which is induced by the monoid homomorphism $P \to k[P] = \mathcal{O}_X(X)$.

From now on $\mathrm{Spec}(k[P])$ will always be tacitly equipped with this log structure.

**Notation 1.2.9.** As for notation, if $(X, A, L)$ is a log scheme as above, we will often denote with just $X$ both the log scheme and the underlying scheme. Occasionally it will be important to distinguish between schemes and log schemes: in those occasions, an underlined letter like $\underline{Y}$ will denote a bare scheme, and $Y$ will denote a log scheme.

Regarding the sheaf $A$ and the functor $L$, when we will have several log schemes around we will denote by $A_X$ and $L_X$ the data associated with a log scheme $X$. This subscript notation will also often be a shorthand for pullback, but we are confident that the meaning will always be clear from the context.

If $f\colon X \to Y$ is a morphism of schemes and $L\colon A \to \mathrm{Div}_Y$ is a DF structure on $Y$, then we have a pullback DF structure $f^*L\colon f^*A \to \mathrm{Div}_X$ on $X$.

**Definition 1.2.10.** A morphism of log schemes $X \to Y$ is a morphism $f\colon \underline{X} \to \underline{Y}$ of the underlying schemes, together with a morphism from the pullback DF structure $f^*L_Y\colon f^*A_Y \to \mathrm{Div}_X$ to $L_X\colon A_X \to \mathrm{Div}_X$, i.e. a morphism of sheaves of monoids $f^*A_Y \to A_X$ together with a natural isomorphism of the composite $f^*A_Y \to A_X \to \mathrm{Div}_X$ with $f^*L_Y$.

Log schemes form a category with this notion of morphism, that we will denote by (LogSch).

**Definition 1.2.11.** A morphism of log schemes $X \to Y$ is *strict* if the morphism from $f^*L_X$ to $L_Y$ is an isomorphism.

Strict morphisms are morally morphisms of log schemes where nothing is happening from the "log" point of view.

Now assume that $P$ is a finitely generated monoid, $X$ is a scheme and $P \to \mathrm{Div}(X)$ is a symmetric monoidal functor. Then there is an induced DF structure $A \to \mathrm{Div}_X$, where $A$ is obtained as the quotient of the constant sheaf $P_X$ by the kernel of the induced functor $P_X \to \mathrm{Div}_X$. In particular note that $\ker(P_X \to A) = \ker(P_X \to \mathrm{Div}_X)$.

**Definition 1.2.12.** Let $X$ be a log scheme, and $A$ a sheaf of monoids on $X$. A *global chart* for $A$ is a finitely generated monoid $P$ together with a morphism of monoids $P \to A(X)$ such that the induced morphism of sheaves $P_X \to A$ is a cokernel in the category of sheaves of monoids.

The last sentence means more precisely that if $K$ is the kernel of $P_X \to A$, then there is an induced isomorphism $P_X/K \cong A$, where the left-hand side is the quotient sheaf.

Equivalently we can say that we have a symmetric monoidal functor $P \to \mathrm{Div}(X)$ such that the induced DF structure on $X$ is isomorphic to $L\colon A \to \mathrm{Div}_X$. Moreover one can show ([BV12, Proposition 3.14]) that being a cokernel is something that can be checked on the stalks. In particular if $P \to \mathrm{Div}(X)$ is a chart for $A \to \mathrm{Div}_X$, then for any $x \in X$ the stalk $A_{\overline{x}}$ is a cokernel of $P$.

**Notation 1.2.13.** If $L\colon A \to \mathrm{Div}_X$ is a DF structure and $a \in A(U)$ is a section, we will set $L(a) = (L_a, s_a)$, and we will sometimes call $s_a$ the *distinguished section* of $L_a$. The same notations will be used for a symmetric monoidal functor $P \to \mathrm{Div}(X)$ where $P$ is a monoid.

**Definition 1.2.14.** A sheaf of monoids on a scheme $X$ is *coherent* if étale locally on $X$ it has a chart. A log scheme $X$ is *coherent* if the sheaf $A$ is coherent.

From now on all log schemes will be coherent, unless specified otherwise.

One shows that charts for a coherent log scheme can be obtained from stalks of the sheaf $A$: for every point $x \in X$ there is an étale neighborhood of $x$ where $X$ has a global chart with monoid $A_{\overline{x}}$.

Consequently, a coherent log scheme $X$ has a maximal open subscheme $U \subseteq X$ such that the restriction of the log structure to $U$ is trivial. This open subset coincides with the set of points of $X$ where the stalk of the sheaf $A$ is trivial.

**Definition 1.2.15.** A noetherian log scheme $X$ has *generically trivial* log structure if the open subscheme $U$ where the log structure is trivial is schematically dense (i.e. it contains all associated points of $X$).

**Example 1.2.16.** If $X$ is a noetherian scheme with an effective Cartier divisor $D \subseteq X$, then the log structure induced by $D$ is clearly trivial on $U = X \setminus D$, and since $U$ is schematically dense, the log structure is generically trivial.

If $X$ is any scheme and $P$ is any fine monoid, the log structure induced by the morphism $P \to \mathrm{Div}(X)$ sending 0 to $(\mathcal{O}_X, 1)$ and everything else to $(\mathcal{O}_X, 0)$ is not generically trivial, unless $P$ itself is trivial.

This notion of chart, introduced and studied in [BV12] is slightly different from Kato's one.

**Definition 1.2.17.** A *Kato chart* for a log scheme $(X, M)$ is a finitely generated monoid $P$ together with a homomorphism $P \to M(X)$, such that the induced morphism $P \to \overline{M}(X)$ is a chart for $\overline{M}$.

Clearly a Kato chart induces a chart for $A = \overline{M}$. Moreover it turns out that having Kato charts étale locally is equivalent to having charts étale locally.

**Remark 1.2.18.** The datum of a Kato chart is equivalent to a strict morphism of log schemes $X \to \mathrm{Spec}(k[P])$ (obtained by composing $P \to M(X)$ with $M(X) \to \mathcal{O}_X(X)$), and analogously a chart for $A = \overline{M}$ amounts to a strict morphism $X \to \left[\mathrm{Spec}(k[P])/\widehat{P}\right]$, where $\widehat{P}$ is the diagonalizable group scheme $D[P^{\mathrm{gp}}]$, Cartier dual to the group $P^{\mathrm{gp}}$, and the quotient stack has the log structure induced by descent from the one of $\mathrm{Spec}(k[P])$.

The morphism that sends a Kato chart to the associated chart is given by composition with the strict morphism $\mathrm{Spec}(k[P]) \to \left[\mathrm{Spec}(k[P])/\widehat{P}\right]$.

This explains the fact that charts give us local models for (the log structure of) log schemes and, as we will see, also for natural objects over them, like root stacks.

**Remark 1.2.19.** In [BV12], the authors make a make a point to use charts for $\overline{M}$ instead of Kato charts, and develop the theory by using the quotient stack $\left[\mathrm{Spec}(k[P])/\widehat{P}\right]$ as a local model, instead of the monoid algebra $\mathrm{Spec}(k[P])$.

It turns out that having charts étale locally is the same as having Kato charts étale locally ([BV12, Proposition 3.28]). Because of this there is no loss of generality in using Kato charts when dealing with local problems. Since we feel that some aspects of the treatment are simplified by using Kato charts, in the study of the infinite root stack of a log scheme (Chapter 2) we will usually work with charts coming from Kato charts.

In the chapters about moduli of parabolic sheaves, on the contrary, we will typically use charts that may not come from Kato charts, since we will need to have a global chart on a projective log scheme, and global charts exist more often than global Kato charts (think of the case of a variety with a simple normal crossings divisor, 1.2.21).

The presence of charts will be extremely important for what follows. As is customary, we incorporate it in the notion of fs log scheme.

**Definition 1.2.20.** A log scheme $X$ is *fine* if the sheaf $A$ is coherent and integral.

A log scheme $X$ is *fine and saturated*, abbreviated fs, if the sheaf $A$ is coherent, integral and saturated.

We will mostly be dealing with fine and saturated log schemes. Note that since $A$ is sharp, by proposition 1.1.11 it will also be torsion-free. We will denote the category of fs log schemes by (FSLogSch).

**Example 1.2.21.** Consider again the example of 1.2.7. Let $X$ be a noetherian scheme and $D \subseteq X$ be an effective Cartier divisor. We have a symmetric monoidal functor $\mathbb{N} \to \mathrm{Div}(X)$ sending 1 to $(\mathcal{O}_X(D), s)$, where $s$ is the image of 1 along the natural morphism $\mathcal{O}_X \to \mathcal{O}_X(D)$, and this makes $X$ into a log scheme.

However, the induced DF structure is isomorphic to the one of 1.2.7 only if $D$ is smooth. This is because if $D$ is for example given by $xy = 0$ in $\mathbb{A}^2$, around the origin we should be able to distinguish the two branches of the divisor, and the DF structure of the previous paragraph does not do that.

Instead, assume that $D$ is simple normal crossings and let $D_1, \dots, D_k$ be its irreducible components. Then we have a symmetric monoidal functor $\mathbb{N}^k \to \mathrm{Div}(X)$ sending the $i$-th generator $e_i$ to $(\mathcal{O}_X(D_i), s_i)$, where $s_i$ is again the canonical section, and the induced DF structure on $X$ is isomorphic to the one of example 1.2.7.

Note that if we want Kato charts for this log scheme, we need to have equations for the irreducible components $D_i$ (because the morphism lands in $\mathcal{O}_X$ rather than in $\mathrm{Div}_X$), so in general charts will exist only locally, whereas we have a global chart for $\overline{M}$.

Charts can be used to describe morphisms $A \to B$ between coherent sheaves of monoids.

**Definition 1.2.22.** Let $A$ and $B$ be sheaves of monoids on $X_{\text{ét}}$, and $j\colon A \to B$ a morphism. A *chart* for $j$ consists of two finitely generated monoids with homomorphisms $P \to A(X)$ and $Q \to B(X)$ giving charts for $A$ and $B$, and a morphism $P \to Q$ that makes the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ A(X) \arrow[r] & B(X) \end{tikzcd}$$

commutative.

One can show ([BV12, Proposition 3.17]) that for any morphism $j\colon A \to B$ between coherent sheaves of monoids on $X$, we can find a chart étale locally on $X$, and moreover we can choose $P$ and $Q$ to be stalks of the sheaves $A$ and $B$ over some point of $X$.

Using a similar definition of chart for morphisms between log schemes, we can describe such morphisms locally as strict morphisms followed by a pullback of a morphisms of monoid algebras.

**Definition 1.2.23.** Let $f\colon X \to Y$ be a morphism of log schemes. A *chart* for $f$ consists of monoids $P, Q$ and morphisms $Q \to A_X(X)$, $P \to A_Y(Y)$ and $P \to Q$ such that the first two morphisms are charts for $X$ and $Y$, and $P \to Q$ induces the given morphism $f^*A_Y \to A_X$.

One can show that morphisms of fs (coherent would be enough) log schemes always admit charts étale locally, and moreover if $X \to Y$ is a morphism of fs log schemes and we have a chart for $Y$ around a point $y \in Y$, we can find a chart for the morphisms that extends the given chart for $Y$. This holds both for charts and Kato charts.

Note that this essentially says that for any morphism $X \to Y$ of fs log schemes, étale locally on $X$ and $Y$ we can find a commutative diagram of log schemes

$$\begin{tikzcd}
X \arrow[dr] \arrow[ddr, bend right] & & \\
& Y \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[Q]) \arrow[r] \arrow[d] & \mathrm{Spec}(k[Q]) \arrow[d] \\
& Y \arrow[r] & \mathrm{Spec}(k[P])
\end{tikzcd}$$

where the two horizontal maps are strict and so is the diagonal one. This presents the morphism $X \to Y$ (étale locally) as a strict morphism followed by a base change of morphisms of monoid algebras.

As we already remarked, charts for coherent log schemes can be produced from stalks of the sheaf $A_X$. This assures that if the stalks of the sheaf $A_X$ have some property (for example are saturated), then we can find charts where the monoid has such property. The converse (if a chart has some property, then all the stalks of $A_X$ have that property) is sometimes subtler, for example for saturation.

Let us now spend some words on fibered products of log schemes. The category (LogSch) has fibered products: given log schemes $X, Y, Z$ and a diagram

$$\begin{tikzcd}
& Y \arrow[d] \\
Z \arrow[r] & X
\end{tikzcd}$$

we can define the fibered product $Y \times_X Z$ as follows: we take the fibered product of the underlying schemes, call it $W$, and pull back to it the three DF structures, obtaining a diagram

$$\begin{tikzcd}
(A_X)_W \arrow[r] \arrow[d] & (A_Z)_W \\
(A_Y)_W &
\end{tikzcd}$$

of sheaves of monoids (here $(-)_W$ stands for pullback to $W$). We then take the pushout of this last diagram, which has a natural symmetric monoidal functor to $\mathrm{Div}_W$, and the induced DF structure. This defines a log scheme with underlying scheme $W$, that is the desired fibered product. Moreover one shows that if $X, Y, Z$ are coherent (i.e. they have charts locally), then also $Y \times_X W$ is, and charts for the three induce a chart for the product.

If one works with fs log schemes, than this construction has to be modified, since if $X, Y, Z$ are fs then the fibered product in (LogSch) is not necessarily fs. The problem is that the amalgamated sum of fs monoids need not be fs itself.

To fix this, one shows that there are left adjoints $(-)^{\mathrm{int}}$ and $(-)^{\mathrm{sat}}$ to the inclusion functors of the category of coherent (resp. fine) log schemes in the category of fine (resp. fs) log schemes. These are constructed locally from the analogous constructions on monoids, and then glued together. Once one has these functors, it is immediate to check that $((Y \times_X Z)^{\mathrm{int}})^{\mathrm{sat}}$ is a fibered product in the category (FSLogSch). Étale locally, the underlying scheme of this fibered product is given by a base change of the ordinary fibered product of the schemes along a morphism of the form $\mathrm{Spec}(k[(P^{\mathrm{int}})^{\mathrm{sat}}]) \to \mathrm{Spec}(k[P])$, which is a finite map.

From now on if $X, Y, Z$ are fs log schemes we will denote their fibered product in (FSLogSch) by $Y \times_X Z$, unless specified otherwise.

### 1.2.1 Root stacks

For proofs and more details about this section we refer to [BV12].

Let $X$ be a log scheme, with DF structure $L\colon A \to \mathrm{Div}_X$. Given a sheaf of monoids $B$ on $X_{\text{ét}}$ containing $A$, we are interested in parametrizing extensions of $L$ to $L'\colon B \to \mathrm{Div}_X$.

**Example 1.2.24.** The basic example of this situation is the following: let $X = \mathbb{A}^1_k$, equipped with the log structure induced by the origin $0 \in X$, seen as an effective Cartier divisor. This has a chart given by $L\colon \mathbb{N} \to \mathrm{Div}(X)$, sending 1 to $(\mathcal{O}_X, x)$ where $\mathbb{A}^1_k = \mathrm{Spec}(k[x])$.

Consider the inclusion $\mathbb{N} \subseteq \frac{1}{n}\mathbb{N}$, and look at extensions $\frac{1}{n}\mathbb{N} \to \mathrm{Div}(X)$ of $L$. These clearly correspond functorially to $n$-th roots of the indeterminate $x$. A more careful analysis shows that the stack parametrizing such extensions is the quotient stack

$$[\mathrm{Spec}(k[x,t]/(t^n - x))/\mu_n]$$

where $\mu_n$ acts by multiplication on $t$.

This is the first example of a root stack, for the log scheme $X$ with respect to the system of denominators (induced by) $\mathbb{N} \subseteq \frac{1}{n}\mathbb{N}$.

**Definition 1.2.25.** Let $X$ be a log scheme. A *system of denominators* on $X$ is a sheaf of monoids $B$ on $X_{\text{ét}}$ with a morphism $j\colon A \to B$, such that $B$ is coherent, and $j$ is Kummer, meaning that for any point $x \in X$ the induced morphism $j_{\overline{x}}\colon A_{\overline{x}} \to B_{\overline{x}}$ is Kummer.

If $j\colon A \to B$ is a system of denominators, by the discussion in the preceding section étale locally we have charts $P \to A(X)$ and $Q \to B(X)$ such that the morphism $P \to Q$ is Kummer. Vice versa, if $P \to A(X)$ gives a chart for $A$ and $P \to Q$ is a Kummer morphism with $P$ and $Q$ fs monoids, then we get a system of denominators $A \to B$, with $Q$ giving a chart for $B$.

**Remark 1.2.26.** Note that the definition does not require $B$ to be saturated. Nevertheless, most of the times we will deal with systems of denominators $A \to B$ on fs log schemes where $B$ is fs as well.

**Definition 1.2.27.** The *root stack* $X_{B/A}$ of $X$ with respect to the system of denominators $j\colon A \to B$ is the following fibered category over $X$: objects over a scheme $\phi\colon T \to X$ are symmetric monoidal functors $\phi^*B \to \mathrm{Div}_T$ together with an isomorphism of the restriction along $\phi^*A \to \phi^*B$ with the pullback DF structure $\phi^*L\colon \phi^*A \to \mathrm{Div}_T$, and arrows are isomorphisms of DF structures, with a compatibility with respect to restriction to $\phi^*A$.

Usually we will refer to an object of $X_{B/A}(T)$ only as the functor $\phi^*B \to \mathrm{Div}_T$, omitting the isomorphism between $\phi^*A \to \phi^*B \to \mathrm{Div}_T$ and $\phi^*A \to \mathrm{Div}_T$. This should cause no confusion.

The root stack has a natural morphism $X_{B/A} \to X$ that we will usually call the *projection* of $X_{B/A}$ to $X$. Over the root stack $X_{B/A}$ we have a tautological DF structure with sheaf of monoids $\pi^*B$, where $\pi\colon X_{B/A} \to X$ is the projection, extending $\pi^*L\colon \pi^*A \to \mathrm{Div}_{X_{B/A}}$. We will usually denote it by $\Lambda\colon \pi^*B \to \mathrm{Div}_{X_{B/A}}$ when there is only one root stack in play.

If the system of denominators $j\colon A \to B$ has a global chart $P \to Q$, its root stack can be described by considering lifts $Q \to \mathrm{Div}(T)$ of the pullback of $P \to \mathrm{Div}(X)$. In this case the root stack will also be denoted by $X_{Q/P}$. Moreover, since étale locally we always have a chart for $j$, locally every root stack $X_{B/A}$ is isomorphic to a root stack of the form $X_{Q/P}$.

**Example 1.2.28.** Let $X = \mathrm{Spec}(k)$ be the standard log point, and take the Kummer extension $\mathbb{N} \subseteq \frac{1}{n}\mathbb{N}$. Then if $X_n$ denotes the corresponding root stack, we have an isomorphism

$$X_n \cong [\mathrm{Spec}(k[t]/(t^n))/\mu_n]\,,$$

where $\mu_n$ acts by multiplication on $t$. This is a particular case of a general description of the root stack as a quotient stack in presence of a global chart (Proposition 1.2.29 below).

This example generalizes to give a quotient description of root stacks of the form $X_{Q/P}$, and thus local models for root stacks in general. Assume that $P \to \mathrm{Div}(X)$ is a global chart for $X$, and fix a Kummer extension $P \to Q$. The chart given by $P$ corresponds to a morphism $X \to \left[\mathrm{Spec}(k[P])/\widehat{P}\right]$, where as usual $\widehat{P} = D[P^{\mathrm{gp}}]$ is the diagonalizable group scheme associated to $P^{\mathrm{gp}}$. The morphism $P \to Q$ induces a morphism of the spectra of the monoid algebras $\mathrm{Spec}(k[Q]) \to \mathrm{Spec}(k[P])$, which (being equivariant with respect to the natural morphism $\widehat{Q} \to \widehat{P}$) in turn gives a map $\left[\mathrm{Spec}(k[Q])/\widehat{Q}\right] \to \left[\mathrm{Spec}(k[P])/\widehat{P}\right]$.

**Proposition 1.2.29** ([BV12, Proposition 4.13])**.** *We have an isomorphism*

$$X_{Q/P} \cong X \times_{\left[\mathrm{Spec}(k[P])/\widehat{P}\right]} \left[\mathrm{Spec}(k[Q])/\widehat{Q}\right].$$

In other words every root stack with respect to $P \to Q$ is a pullback of the quotient stack $\left[\mathrm{Spec}(k[Q])/\widehat{Q}\right]$, which is then some kind of "universal" model.

This gives a quotient stack description of $X_{Q/P}$ itself: call $\eta\colon E \to X$ the $\widehat{P}$-torsor corresponding to the map $X \to \left[\mathrm{Spec}(k[P])/\widehat{P}\right]$, and note that we have a $\widehat{P}$-equivariant map $E \to \mathrm{Spec}(k[P])$. Then we have an isomorphism

$$X_{Q/P} \cong \left[(E \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[Q]))/\widehat{Q}\right]$$

for the natural action.

Moreover $E$ is affine over $X$, and if we set $R = \eta_*\mathcal{O}_E$, then we have $E \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[Q]) \cong \underline{\mathrm{Spec}}_X(R \otimes_{k[P]} k[Q])$. This gives a description of quasi-coherent sheaves on $X_{Q/P}$, that is the key to the relation with parabolic sheaves, and will be important in what follows: quasi-coherent sheaves on $X_{Q/P}$ are $Q^{\mathrm{gp}}$-graded quasi-coherent sheaf on $X$, of modules over the sheaf of rings $R \otimes_{k[P]} k[Q]$. The grading corresponds to $\widehat{Q}$-equivariance.

We have a second description as a quotient stack in presence of a Kato chart: if $P \to \mathrm{Div}(X)$ comes from a Kato chart $P \to \mathcal{O}_X(X)$, then the cartesian diagram expressing $X_{Q/P}$ as a pullback can be broken up

$$\begin{tikzcd}
X_{Q/P} \arrow[r] \arrow[d] & \left[\mathrm{Spec}(k[Q])/\mu_{Q/P}\right] \arrow[r] \arrow[d] & \left[\mathrm{Spec}(k[Q])/\widehat{Q}\right] \arrow[d] \\
X \arrow[r] & \mathrm{Spec}(k[P]) \arrow[r] & \left[\mathrm{Spec}(k[P])/\widehat{P}\right]
\end{tikzcd}$$

in two cartesian squares, where $\mu_{Q/P}$ is the Cartier dual $D[C]$ of the cokernel $C$ of the morphism $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$, a finite abelian group. Consequently, we also have an isomorphism

$$X_{Q/P} \cong \left[(X \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[Q]))/\mu_{Q/P}\right]$$

for the natural action.

**Example 1.2.30.** A particular case that will be important in the following is the one of $n$-th roots.

Given an fs torsion-free monoid $P$, we consider the Kummer extension $P \subseteq \frac{1}{n}P$. In this case we will denote by $P_n$ the monoid $\frac{1}{n}P$ (this is just to remember the denominators, since of course $P_n \cong P$), and the group $\mu_{P_n/P}$ will be denoted just by $\mu_n(P)$. In conclusion the root stack of the monoid algebra $X = \mathrm{Spec}(k[P])$ in this case is

$$X_n = X_{P_n/P} \cong [\mathrm{Spec}(k[P_n])/\mu_n(P)] \,.$$

The following is an immediate consequence of the previous discussion.

**Theorem 1.2.31** ([BV12, Proposition 4.19]). *Let $X$ be a log scheme and $j\colon A \to B$ a system of denominators. The root stack $X_{B/A}$ is a tame Artin stack. It is finite over $X$ (meaning proper and quasi-finite), finitely presented, and if for every geometric point $x \to X$ the order of the group $B_x^{\mathrm{gp}}/A_x^{\mathrm{gp}}$ is prime to the characteristic of $k$ (for example if* $\mathrm{char}(k) = 0$*), then $X_{B/A}$ is Deligne–Mumford.*

Being a tame Artin stack, the root stack $X_{B/A}$ has a coarse moduli space.

**Proposition 1.2.32.** *Assume that $A$ and $B$ are sheaves of fine and saturated monoids. Then the coarse moduli space of $X_{B/A}$ is the morphism $X_{B/A} \to X$.*

*Proof.* This is a local question on $X$, so we can assume to have a chart $P \to \mathrm{Div}(X)$ for $X$ coming from a Kato chart, and a chart $P \to Q$ for the system of denominators. Moreover, since in this case

$$X_{Q/P} \cong X \times_{\mathrm{Spec}(k[P])} \left[\mathrm{Spec}(k[Q])/\mu_{Q/P}\right]$$

with the notation introduced above, by tameness we can reduce to showing that the morphism $\left[\mathrm{Spec}(k[Q])/\mu_{Q/P}\right] \to \mathrm{Spec}(k[P])$ is a coarse moduli space.

This follows from the fact that the invariants of the action of $\mu_{Q/P}$ on $\mathrm{Spec}(k[Q])$ are exactly $\mathrm{Spec}(k[P])$. Recall how the action is constructed: $\mu_{Q/P}$ is the Cartier dual $D[C]$ of the cokernel

$C$ of $P^{\text{gp}} \to Q^{\text{gp}}$, the algebra $k[Q]$ has a natural $Q^{\text{gp}}$-grading that induces a $C$-grading, and this gives the action of $\mu_{Q/P}$.

The invariants are the piece of degree zero with respect to this $C$-grading, and are generated by the $x^q$'s such that $q \in Q$ goes to zero in $C$, i.e. with $q \in P^{\text{gp}} \cap Q = P$, since $P$ and $Q$ are fine and saturated. This concludes the proof. □

Note that in the proof we only used the fact that $P^{\text{gp}} \cap Q = P$, i.e. that the morphism $P \to Q$ is *exact*.

The last proposition implies in particular that $\pi_* \colon \mathrm{QCoh}(X_{B/A}) \to \mathrm{QCoh}(X)$ is exact, since $X_{B/A}$ is tame and $X$ is the coarse moduli space.

This root stack construction has some functoriality properties: if $Y \to X$ is a morphism of log schemes and we have compatible system of denominators on $X$ and $Y$, we get a morphism between the root stack. The following proposition covers the simplified case in which the morphism is strict.

**Proposition 1.2.33.** *Let $X$ be a log scheme with DF structure $L \colon A \to \mathrm{Div}_X$ and $j \colon A \to B$ a system of denominators. If $f \colon Y \to X$ is a strict morphism of log schemes, then we have an isomorphism $Y_{f^*B/f^*A} \cong X_{B/A} \times_X Y$, i.e. the diagram*

$$\begin{tikzcd} Y_{f^*B/f^*A} \arrow[r] \arrow[d] & X_{B/A} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

*is cartesian.*

*Proof.* This is immediate from the functorial description of the root stack: objects of $X_{B/A} \times_X Y$ over a scheme $T$ are given by pairs $(\phi, g, N)$ where $\phi \colon T \to Y$ and $g \colon T \to X$ are morphisms such that $f \circ \phi = g$, and $N \colon g^*B \to \mathrm{Div}_T$ is a lifting of the pullback DF structure $g^*L \colon g^*A \to \mathrm{Div}_T$. Now since $g^* = \phi^* f^*$, these are precisely the objects of the root stack $Y_{f^*B/f^*A}$ over the scheme $T$. □

We also have a functoriality with respect to successive Kummer extensions: if $A \to B$ and $A \to B'$ are systems of denominators with a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B' \\ B \arrow[ur] & \end{tikzcd}$$

then we have a morphism $X_{B'/A} \to X_{B/A}$ defined by restricting the extension of the DF structure $B'_T \to \mathrm{Div}_T$ along $B_T \to B'_T$ for every scheme $T \to X$. We will sometimes call this operation an "extension of denominators".

In fact this morphism is very similar to the projection $X_{B/A} \to X$ from a root stack to the log scheme $X$.

**Proposition 1.2.34.** *Let $j \colon A \to B$ and $j' \colon B \to B'$ be two systems of denominators over the log scheme $X$. Then the root stack $X_{B'/A}$ can be identified with the root stack of the log stack $X_{B/A}$ with respect to the system of denominators $j'$.*

*Proof.* Take a morphism $T \to X_{B/A}$ from a scheme, and equip $T$ with the pullback of the universal DF structure of $X_{B/A}$. Then the following diagram is cartesian

$$\begin{tikzcd} T_{B'/B} \arrow[r] \arrow[d] & X_{B'/A} \arrow[d] \\ T \arrow[r] & X_{B/A} \end{tikzcd}$$

and this clearly implies the conclusion.

The fact that the square is cartesian is an easy verification. □

Because of this, the map $X_{B'/A} \to X_{B/A}$ behaves in some sense as a coarse moduli space. For example, we have a projection formula for quasi-coherent sheaves.

**Proposition 1.2.35** (Projection formula for the root stacks)**.** *With the notation of the preceding proposition, denote by $\pi\colon X_{B'/A} \to X_{B/A}$ the natural map, and assume that $A$, $B$ and $B'$ are fine and saturated. Then:*

- $\mathcal{O}_{X_{B/A}} \cong \pi_* \mathcal{O}_{X_{B'/A}}$,
- *if $F \in \mathrm{QCoh}(X_{B'/A})$ and $G \in \mathrm{QCoh}(X_{B/A})$ we have a functorial isomorphism $\pi_* F \otimes G \cong \pi_*(F \otimes \pi^* G)$,*
- *consequently for $F \in \mathrm{QCoh}(X_{B/A})$ we have an isomorphism $F \cong \pi_* \pi^* F$ on $X_{B/A}$.*

*Proof.* The last bullet is consequence of the first two.

After noting that we have maps $\mathcal{O}_{X_{B/A}} \to \pi_* \mathcal{O}_{X_{B'/A}}$ and $\pi_* F \otimes G \to \pi_*(F \otimes \pi^* G)$, by flat base change along $T \to X_{B/A}$, we reduce to proving the statements for $\pi_T \colon T_{B'/B} \to T$, where the log structure on $T$ is the pullback of the universal DF structure of $X_{B/A}$. Now since $B$ and $B'$ are fine and saturated, the morphism $\pi_T$ is a coarse moduli space of a tame Artin stack, and the claims follow: the first one is a general property of coarse moduli spaces, and the second follows for example from Proposition 4.5 of [Alp12].

□

To conclude, we note that where the log structure is trivial, the root stack is trivial as well.

**Proposition 1.2.36.** *Let $X$ be a log scheme with a DF structure $L\colon A \to \mathrm{Div}_X$, $j\colon A \to B$ be a system of denominators, and let $U \subseteq X$ be the maximal open subset where the log structure is trivial. The the restriction of $\pi\colon X_{B/A} \to X$ to $U$ is an isomorphism $X_{B/A} \times_X U \cong U$.*

*Proof.* This follows from the fact that the inclusion $U \subseteq X$ is strict and the root stack construction is compatible with strict base-change (1.2.33), and from the easy fact that for a trivial log scheme $X$, the projection from the root stack to $X$ is an isomorphism. □

## 1.3 Parabolic sheaves

In this section we will introduce parabolic sheaves on a log scheme, and link them to quasi-coherent sheaves on root stacks. Once again our main reference is [BV12].

Let $X$ be a log scheme with DF structure $L\colon A \to \mathrm{Div}_X$, and $j\colon A \to B$ be a system of denominators. Let us assume first that we have a chart $j_0\colon P \to Q$ for $A \to B$.

Let us introduce a category of weights $Q^{\mathrm{wt}}$ associated to $Q$: objects are elements of $Q^{\mathrm{gp}}$, and an arrow $a \to b$ is an element $q \in Q$ such that $b = a + q$. We will write $a \leq b$ to mean that there is an arrow from $a$ to $b$. Note that if $Q$ is integral (and this will often be the case in our treatment), the element $q$ that gives the arrow is uniquely determined.

The symmetric monoidal functor $L\colon P \to \mathrm{Div}(X)$ giving the DF structure extends to a symmetric monoidal functor $L^{\mathrm{wt}}\colon P^{\mathrm{wt}} \to \mathrm{Pic}(X)$ in the obvious way. If $p \in P^{\mathrm{gp}}$, we denote $L^{\mathrm{wt}}(p)$ simply by $L_p$.

**Definition 1.3.1.** A *parabolic sheaf* on $X$ with denominators in $Q$ is a functor $E\colon Q^{\mathrm{wt}} \to \mathrm{QCoh}(X)$ that we denote by $a \mapsto E_a$, for $a$ an object or an arrow of $Q^{\mathrm{wt}}$, with an additional datum for any $p \in P^{\mathrm{gp}}$ and $a \in Q^{\mathrm{gp}}$ of an isomorphism of $\mathcal{O}_X$-modules

$$\rho^E_{p,a}\colon E_{p+a} \cong L_p \otimes E_a$$

called the pseudo-periods isomorphism.

These isomorphism are required to satisfy some compatibility conditions. Let $p, p' \in P^{\mathrm{gp}}$, $r \in P$, $q \in Q$ and $a \in Q^{\mathrm{gp}}$. Then the following diagrams are commutative

$$\begin{tikzcd}
E_a \arrow[r, "E_r"] \arrow[d] & E_{r+a} \arrow[d, "\rho^E_{r,a}"] \\
\mathcal{O}_X \otimes E_a \arrow[r, "\sigma_r \otimes \mathrm{id}"] & L_r \otimes E_a
\end{tikzcd}$$

$$\begin{tikzcd}
E_{p+a} \arrow[r, "\rho^E_{p,a}"] \arrow[d, "E_q"'] & L_p \otimes E_a \arrow[d, "\mathrm{id} \otimes E_q"] \\
E_{p+q+a} \arrow[r, "\rho^E_{p,q+a}"] & L_p \otimes E_{q+a}
\end{tikzcd}$$

$$\begin{tikzcd}
E_{p+p'+a} \arrow[r, "\rho^E_{p+p',a}"] \arrow[d, "\rho^E_{p,p'+a}"'] & L_{p+p'} \otimes E_a \arrow[d, "\mu_{p,p'} \otimes \mathrm{id}"] \\
L_p \otimes E_{p'+a} \arrow[r, "\mathrm{id} \otimes \rho^E_{p',a}"] & L_p \otimes L_{p'} \otimes E_a,
\end{tikzcd}$$

where $\mu_{p,p'}\colon L_{p+p'} \cong L_p \otimes L_{p'}$ is the natural isomorphism given by $L$, and the composite

$$E_a = E_{0+a} \xrightarrow{\rho^E_{0,a}} L_0 \otimes E_a \cong \mathcal{O}_X \otimes E_a$$

coincides with the natural isomorphism $E_a \cong \mathcal{O}_X \otimes E_a$.

The sheaves $E_a$ will be sometimes called the *pieces* of the parabolic sheaf $E$.

**Remark 1.3.2.** This has an abstract interpretation in terms of module categories. There are natural morphisms $+\colon P^{\mathrm{wt}} \times Q^{\mathrm{wt}} \to Q^{\mathrm{wt}}$ and $\otimes\colon \mathrm{Pic}(X) \times \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$. Then the pseudo-periods isomorphism $\rho^E$ is an isomorphism between the composites $E \circ +$ and $\otimes \circ (L^{\mathrm{wt}} \times E)$ from $P^{\mathrm{wt}} \times Q^{\mathrm{wt}}$ to $\mathrm{QCoh}(X)$, and $E$ is in some sense $P^{\mathrm{wt}}$-equivariant.

There is a notion of morphisms of parabolic sheaves, which is a natural transformation $E \to E'$ between the two functors $Q^{\mathrm{wt}} \to \mathrm{QCoh}(X)$ which is compatible with the pseudo-periods isomorphism in the obvious sense, so we get a category $\mathrm{Par}(X, j_0)$ of parabolic sheaves on $X$ with respect to $j_0\colon P \to Q$. This is in fact an abelian category in the natural way, with a tensor product and internal Homs.

**Example 1.3.3.** Let us examine the case of the standard log point, i.e. $X = \mathrm{Spec}(k)$ with log structure induced by $L\colon \mathbb{N} \to k$ sending 0 to 1 and 1 to 0, and with system of denominators $\mathbb{N} \subseteq \frac{1}{2}\mathbb{N}$.

In this case $Q^{\mathrm{wt}} \cong \frac{1}{2}\mathbb{Z}$ as a partially ordered set in the natural way, and a parabolic sheaf $E\colon \frac{1}{2}\mathbb{Z} \to \mathrm{QCoh}(\mathrm{Spec}(k))$ is determined by its values at 0 and $\frac{1}{2}$, since the pseudo-periods isomorphism gives for any $\frac{1}{2}k$ and $n \in \mathbb{N}$ an isomorphism

$$E_{\frac{1}{2}k+n} \cong E_{\frac{1}{2}k}.$$

In other words we can visualize $E$ as a pair of vector spaces $V_0$ and $V_1$ with maps

$$\begin{array}{ccccc} 0 & & \frac{1}{2} & & 1 \\ \\ V_0 & \xrightarrow{a} & V_1 & \xrightarrow{b} & V_0 \end{array}$$

such that $a \circ b = 0$ and $b \circ a = 0$ (since these compositions have to coincide with multiplication by the image of 1 in $k$, i.e. zero.

Note that this set of data is exactly the same thing as a quasi-coherent sheaf on the root stack $X_{\frac{1}{2}\mathbb{N}/\mathbb{N}}$. In fact we have an isomorphism

$$X_{\frac{1}{2}\mathbb{N}/\mathbb{N}} \cong [\mathrm{Spec}(k[\epsilon])/\mu_2]$$

where $\epsilon^2 = 0$ and $\mu_2$ acts by changing the sign of $\epsilon$. A quasi-coherent sheaf on the root stack is thus a $\mu_2$-equivariant $k[\epsilon]$-module, i.e. a $\mathbb{Z}/2\mathbb{Z}$-graded $k[\epsilon]$-module. The two pieces of the grading correspond to the vector spaces $V_0$ and $V_1$ above, and the two maps correspond to multiplication by $\epsilon$.

This is the simplest example of the correspondence between parabolic sheaves and quasi-coherent sheaves on root stacks of Theorem 1.3.8 below.

**Example 1.3.4.** Let $X$ be a scheme and $D \subseteq X$ an effective Cartier divisor. Consider the log structure on $X$ induced by the symmetric monoidal functor $\mathbb{N} \to \mathrm{Div}_X$ sending 1 to $(\mathcal{O}_X(D), s)$, and the Kummer extension $j\colon \mathbb{N} \subseteq \frac{1}{2}\mathbb{N}$.

Then a parabolic sheaf on $X$ with respect to $j$ consists of quasi-coherent sheaves $E_{\frac{1}{2}k}$ for any $k \in \mathbb{Z}$, and of morphisms $E_{\frac{1}{2}k} \to E_{\frac{1}{2}k+\frac{1}{2}n}$ for any $n \in \mathbb{N}$, with the properties as in the definition. In particular if $m \in \mathbb{Z}$ we have an isomorphism $E_{\frac{1}{2}k+m} \cong E_{\frac{1}{2}k} \otimes \mathcal{O}_X(mD)$, and the map $E_{\frac{1}{2}k} \to E_{\frac{1}{2}k+m}$ for positive $m$ corresponds to multiplication by $s^{\otimes m}$. Note that if the sheaves $E_{\frac{1}{2}k}$ are torsion-free (say $X$ is integral for simplicity) and $s$ is not a zero-divisor, then all these maps will be injective.

Because of the pseudo-periods isomorphism, we can identify a parabolic sheaf with the data consisting of the sheaves $E_0, E_{\frac{1}{2}}$ together with the two maps

$$\begin{tikzcd} 0 & \frac{1}{2} & 1 \\ E_0 \arrow[r] & E_{\frac{1}{2}} \arrow[r] & E_1. \end{tikzcd}$$

The rest of the data is completely determined by this diagram.

Clearly we could have as well chosen the sheaves corresponding to $-1, -\frac{1}{2}, 0$. We will see that for us it will be more convenient to identify a parabolic sheaf with the sheaves and maps in this second range.

**Remark 1.3.5.** The two preceding example suggest a way to visualize parabolic sheaves (at least if $Q^{\mathrm{gp}}$ is free): one should think of the lattice $Q^{\mathrm{gp}}$ inside the vector space $Q^{\mathrm{gp}} \otimes \mathbb{Q}$, and imagine a quasi-coherent sheaf on each point of the lattice. Moreover there is a map (possibly more than one, if the monoids are not integral) from a sheaf in the point $q$ to the one in $q'$ if and only if $q \leq q'$, and if $p \in P$, then the corresponding map from $q$ to $p+q$ coincides with $E_q \to E_q \otimes L_p$ given by multiplication by the section of $L_p$. For example if $Q^{\mathrm{gp}}$ has rank 1, then a parabolic sheaf can be seen as a "sequence" of sheaves arranged on the real line on the integral points, with maps going to the right.

The same definition with minor variations defines a parabolic sheaf in absence of a global chart. Starting from the sheaf $B$ one defines a weight category $B^{\mathrm{wt}}$ in analogy with the preceding case.

**Definition 1.3.6.** A *parabolic sheaf* on $X$ with denominators in the sheaf $B$ is a cartesian functor $E \colon B^{\mathrm{wt}} \to \mathrm{QCoh}_X$, together with the datum for every $U \to X$ étale, any $p \in A^{\mathrm{gp}}(U)$ and $a \in B^{\mathrm{gp}}(U)$ of an isomorphism of $\mathcal{O}_U$-modules

$$\rho^E_{p,a} \colon E_{p+a} \cong L_p \otimes E_a$$

called the pseudo-periods isomorphism.

These morphism satisfy the conditions analogous to those of the preceding definition, and the following one in addition: if $f \colon U \to V$ is a morphism over $X$ and we have $p \in A^{\mathrm{gp}}(V)$ and $a \in B^{\mathrm{gp}}(V)$, then the isomorphism

$$\rho^E_{f^*p, f^*a} \colon E_{f^*(p+a)} \cong L_{f^*p} \otimes E_{f^*a}$$

is the pullback to $U$ of $\rho^E_{p,a} \colon E_{p+a} \cong L_p \otimes E_a$.

As for the preceding case there is a notion of morphism (a natural transformation compatible with the pseudo-periods isomorphisms) that gives a category $\mathrm{Par}(X, j)$ of parabolic sheaves on $X$ with denominators in $B$, and this is an abelian category with a tensor product and internal Homs. This construction has some functoriality property with respect to morphisms of log schemes $X \to Y$. We will discuss this in some detail in Chapter 3.

Furthermore, $\mathrm{Par}(X, j)$ can be extended to a fibered category $\mathfrak{Par}(X, j)$ over the small étale site $X_{\text{ét}}$ by taking over an étale morphism $U \to X$ the category $\mathrm{Par}(U, j|_U)$ where $U$ has the pullback log structure. This fibered category is a stack for the étale topology, by standard arguments of descent theory.

In the case in which we have a global chart, we can use either one of the definitions.

**Proposition 1.3.7** ([BV12, Proposition 5.10])**.** *Let $X$ be a log scheme with a system of denominators $j\colon A \to B$, admitting a global chart $j_0\colon P \to Q$. Then we have an equivalence $\mathrm{Par}(X, j) \cong \mathrm{Par}(X, j_0)$.*

This says that when dealing with local statements about parabolic sheaves, we can assume that they are relative to a chart.

The following is the main result of [BV12], and relates parabolic sheaves on $X$ with respect to $j\colon A \to B$ to quasi-coherent sheaves on the root stack $X_{B/A}$.

**Theorem 1.3.8** ([BV12, Theorem 6.1])**.** *Let $X$ be a log scheme with DF structure $L\colon A \to \mathrm{Div}_X$, and $j\colon A \to B$ a system of denominators. Then there is a tensor equivalence of abelian categories $\Phi\colon \mathrm{QCoh}(X_{B/A}) \to \mathrm{Par}(X, j)$.*

We sketch the proof here, since the definition of the two functors will come up at some point of our treatment.

*Sketch of proof.* Let us denote by $\pi\colon X_{B/A} \to X$ the projection, and by $\Lambda\colon \pi^*B \to \mathrm{Div}_{X_{B/A}}$ the universal DF structure on the root stack $X_{B/A}$.

Let us describe the functor $\Phi$. Given a quasi-coherent sheaf $F$ on $X_{B/A}$, we want to get a parabolic sheaf $\Phi(F)$. We set, for $U \to X$ étale and $b \in B^{\mathrm{gp}}(U)$

$$\Phi(F)_b = \pi_*(F \otimes \Lambda_b).$$

This gives a cartesian functor $B^{\mathrm{wt}} \to \mathrm{QCoh}_X$ by means of the maps $\Lambda_b \to \Lambda_{b+b'}$ for $b \in B^{\mathrm{gp}}$ and $b' \in B$, and there is a pseudo-periods isomorphism, basically coming from the fact that if $a \in A(U)$, then $\Lambda_a \cong \pi^* L_a$, and using the projection formula for $\pi$.

Now since parabolic sheaves on $X$ and quasi-coherent sheaves on $X_{B/A}$ form a stack in the étale topology of $X$, we can construct the quasi-inverse étale locally, and so we can assume that we have a chart $j_0\colon P \to Q$ for the system of denominators. In this case recall that we have an isomorphism

$$X_{Q/P} \cong \left[\underline{\mathrm{Spec}}_X(R \otimes_{k[P]} k[Q])/\widehat{Q}\right]$$

and consequently quasi-coherent sheaves on $X_{Q/P}$ are $Q^{\mathrm{gp}}$-graded quasi-coherent sheaves on $X$, which are modules over the sheaf of rings $R \otimes_{k[P]} k[Q]$.

Starting from a parabolic sheaf $E \in \mathrm{Par}(X, j_0)$, we define $\Psi(E)$ as the direct sum $\bigoplus_{q \in Q^{\mathrm{gp}}} E_q$. This is a $Q^{\mathrm{gp}}$-graded quasi-coherent sheaf on $X$, that has a structure of $R$-module (this uses the pseudo-periods isomorphism). Moreover it is also a sheaf of $k[Q]$-modules in the natural way, and the two actions are compatible over $k[P]$ by the properties of parabolic sheaves. This gives $\Psi(E)$ the structure of a $Q^{\mathrm{gp}}$-graded quasi-coherent sheaf of $R \otimes_{k[P]} k[Q]$-modules, i.e. of a quasi-coherent sheaf on $X_{Q/P}$.

One checks that these two constructions are inverses, and thus give equivalences. □

From the proof of this theorem we see that if the log structure of a noetherian log scheme $X$ is generically trivial, then the maps $E_b \to E_{b'}$ between the pieces of any parabolic sheaf are generically isomorphisms. Moreover in this case if we also assume that the maps $E_b \to E_{b'}$ are injective (this will be automatic for torsion-free parabolic sheaves, see Proposition 3.2.13) the pieces $E_b$ cannot be zero, unless the whole parabolic sheaf is.

This is not true in general, as the following example shows.

**Example 1.3.9.** Let us take a scheme $X$ and the log structure induced by $\mathbb{N} \to \mathrm{Div}(X)$ that sends every non-zero element to $(\mathcal{O}_X, 0)$. Then the following

$$\begin{array}{ccccc} -1 & & -\frac{1}{2} & & 0 \\ \\ E_0 & \longrightarrow & 0 & \longrightarrow & E_0 \end{array}$$

is a perfectly good parabolic sheaf $E$ with weights in $\frac{1}{2}\mathbb{N}$, for $E_0$ a non-zero quasi-coherent sheaf on $X$. In this case the pushforward $\pi_*(E \otimes \Lambda_{-\frac{1}{2}})$ along $\pi\colon X_{\frac{1}{2}\mathbb{N}/\mathbb{N}} \to X$ is the zero sheaf on $X$.

**Notation 1.3.10.** In the following chapters we will always denote by $\Phi$ the mentioned equivalence and by $\Psi$ its quasi-inverse just described, regardless of the log scheme $X$ they refer to. This should cause no confusion.

Moreover we will refer to both these functors as “the BV equivalence”, for Borne-Vistoli.

**Remark 1.3.11.** A variant of this theorem also holds for log stacks: if $X$ is a log stack with a system of denominators $j\colon A \to B$, there is an equivalence between parabolic sheaves on $X$ with respect to $j$ and quasi-coherent sheaves on the root stack $X_{B/A}$. The proof is just a matter of taking an atlas and keeping track of descent data. We will use this without further comment, especially in Chapter 4.

To conclude, let us describe in parabolic terms pushforwards and pullbacks between root stacks: let $j\colon A \to B$ and $j'\colon A \to B'$ be systems of denominators on $X$, with a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B' \\ B \arrow[ur] & \end{tikzcd}$$

and consider the canonical map $\pi\colon X_{B'/A} \to X_{B/A}$.

We have a functor $F\colon \mathrm{Par}(X, j') \to \mathrm{Par}(X, j)$ given by “restriction”: we have an inclusion $B^{\mathrm{gp}} \to B'^{\mathrm{gp}}$, that identifies $B^{\mathrm{wt}}$ with a subcategory of $B'^{\mathrm{wt}}$. Consequently given a parabolic sheaf $E \in \mathrm{Par}(X, j')$ we can restrict the functor $E\colon B'^{\mathrm{wt}} \to \mathrm{QCoh}_X$ to $B^{\mathrm{wt}}$, and one checks that, together with the induced pseudo-periods isomorphism, this gives a parabolic sheaf in $\mathrm{Par}(X, j)$.

**Proposition 1.3.12.** *The functor $F$ described in the preceding discussion corresponds to the pushforward functor $\pi_*\colon \mathrm{QCoh}(X_{B'/A}) \to \mathrm{QCoh}(X_{B/A})$.*

*Proof.* First of all we can assume that we have charts for both $A \to B$ and $B \to B'$, say $P \to Q$ and $Q \to Q'$.

We will use the construction of the equivalence $\Phi$ of 1.3.8. We want to show that for a quasi-coherent sheaf $E$ on $X_{Q'/P}$, we have a natural isomorphism $\Phi_Q(\pi_* E)_q \cong \Phi_{Q'}(E)_q$ compatible with the pseudo-periods isomorphism, where we see $q \in Q \subseteq Q'$ on the right-hand side.

Let us further denote by $p'\colon X_{Q'/P} \to X$ and $p\colon X_{Q/P} \to X$ the two projections, so that $p' = p \circ \pi$, and by $\Lambda'\colon Q' \to \mathrm{Div}(X_{Q'/P})$ and $\Lambda\colon Q \to \mathrm{Div}(X_{Q/P})$ the universal DF structures. Note that if $q \in Q$ we have $\Lambda'_q \cong \pi^*\Lambda_q$.

By definition we have

$$\Phi_{Q'}(E)_q = p'_*(E \otimes \Lambda'_q) \cong p_*(\pi_*(E \otimes \pi^*\Lambda_q)) \cong p_*(\pi_*(E) \otimes \Lambda_q) \cong \Phi_Q(\pi_*(E))_q,$$

where we used the projection formula for $\pi$ (1.2.35).

Compatibility with the pseudo-periods isomorphism is proved with a similar calculation. □

Let us now turn to pullback, whose description is more complicated. We will describe it in the case where we have global charts $j_0\colon P \to Q$ and $j'_0\colon Q \to Q'$ for $A \to B$ and $B \to B'$.

Let us define a functor $G\colon \mathrm{Par}(X, j_0) \to \mathrm{Par}(X, j'_0)$. Start with a parabolic sheaf $E\colon Q^{\mathrm{wt}} \to \mathrm{QCoh}(X)$, and take an element $q' \in Q'^{\mathrm{gp}}$. Denote by $Q_{q'}$ the set

$$Q_{q'} = \{q \in Q^{\mathrm{gp}} \mid q \leq q'\}$$

where $q \leq q'$ means that there exists $a \in Q'$ such that $q + a = q'$. This is naturally a pre-ordered set, and we have a functor $Q_{q'} \to \mathrm{QCoh}(X)$ by restricting $E$.

We define

$$G(E)_{q'} = \varinjlim_{q \in Q_{q'}} E_q$$

which is a quasi-coherent sheaf on $X$, being a colimit of quasi-coherent sheaves.

Note that in particular if $Q_{q'}$ has a maximum $m$ (i.e. if there is an element $m \in Q^{gp}$ such that $q \leq m$ for any $q \in Q_{q'}$), then $G(E)_{q'} = E_m$. Further, if $q \in Q^{\mathrm{gp}}$, we clearly have $G(E)_q = E_q$, where of course we see $q \in Q^{\mathrm{gp}} \subseteq Q'^{\mathrm{gp}}$.

If we have an arrow $q' \to q''$ in $Q'^{\mathrm{wt}}$, i.e. an element $a \in Q'$ such that $q' + a = q''$, then we have a homomorphism $Q_{q'} \to Q_{q''}$ given by inclusion, and this induces a map $G(E)_{q'} \to G(E)_{q''}$. This defines a functor $G(E)\colon Q'^{\mathrm{wt}} \to \mathrm{QCoh}(X)$. Similar reasonings give a pseudo-periods isomorphism, so that $G(E)$ becomes a parabolic sheaf, and one checks that $G$ gives a functor $\mathrm{Par}(X, j_0) \to \mathrm{Par}(X, j'_0)$ as claimed.

It is also immediate to check that $G$ is left adjoint to the $F$ constructed above (in the case where we have global charts), and that the unit of the adjunction $\mathrm{id} \to F \circ G$ is an isomorphism.

**Proposition 1.3.13.** *Assume that we have global charts for $A \to B$ and $B \to B'$ as in the preceding discussion. Then the functor $G$ described in the preceding discussion corresponds to the pullback functor $\pi^*\colon \mathrm{QCoh}(X_{B/A}) \to \mathrm{QCoh}(X_{B'/A})$.*

*Proof.* This follows from uniqueness of adjoint functors and the preceding proposition. $\square$

**Example 1.3.14.** Consider again the situation of Example 1.3.4, and the Kummer extensions $\mathbb{N} \subseteq \frac{1}{2}\mathbb{N} \subseteq \frac{1}{4}\mathbb{N}$. Call $\pi\colon X_4 \to X_2$ the projection, and assume that we have a parabolic sheaf $E$ with respect to $\mathbb{N} \subseteq \frac{1}{2}\mathbb{N}$, given by

$$\begin{tikzcd} 0 & \frac{1}{2} & 1 \\ E_0 \arrow[r] & E_{\frac{1}{2}} \arrow[r] & E_1 \end{tikzcd}$$

as above. Then the pullback $\pi^*E$ on $X_4$ can be described as a parabolic sheaf as

$$\begin{tikzcd} 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ E_0 \arrow[r, equal] & E_0 \arrow[r] & E_{\frac{1}{2}} \arrow[r, equal] & E_{\frac{1}{2}} \arrow[r] & E_1. \end{tikzcd}$$

In fact in this case the set $Q_{q'}$ of the description above has always a maximum, and the direct limit reduces to evaluating the parabolic sheaf $E$ at the maximum. For example if we take $q' = \frac{1}{4}$, then $Q_{q'}$ has 0 as maximum, and consequently $(\pi^*E)_{\frac{1}{4}}$ will be just $E_0$.

**Corollary 1.3.15.** *Let $X$ be a log scheme with DF structure $L\colon A \to \mathrm{Div}_X$, and $A \to B$, $B \to B'$ two systems of denominators. Then pullback along $\pi\colon X_{B'/A} \to X_{B/A}$ is fully faithful.*

*Proof.* Being a local question in the étale topology of $X$, this follows from the previous propositions and from the fact that the unit of the adjunction $G \dashv F$ is an isomorphism. In fact in general if $G$ is left adjoint to $F$ and the unit $\mathrm{id} \to F \circ G$ is an isomorphism, then

$$\mathrm{Hom}(A, B) \cong \mathrm{Hom}(A, F(G(B))) \cong \mathrm{Hom}(G(A), G(B))$$

and the composition coincides with the function induced by $G$.

Alternatively, the conclusion follows directly from the third bullet of Proposition 1.2.35, which says that the unit of the adjunction $\pi^* \dashv \pi_*$ is an isomorphism. $\square$

# Chapter 2

# The infinite root stack of a logarithmic scheme

Let $X$ be an fs log scheme, with log structure $L\colon A \to \mathrm{Div}_X$. For every $n \in \mathbb{N}$ we have a Kummer extension of the sheaf $A$ given by $\{j_n\colon A \to \frac{1}{n}A\}_{n\in\mathbb{N}}$.

The root stacks $\pi_n\colon X_n \to X$ corresponding to these extensions admit natural maps between them. Precisely, whenever $n \mid m$ we have a morphism $\pi_{n,m}\colon X_m \to X_n$, given functorially by taking a lifting $\frac{1}{m}A_T \to \mathrm{Div}_T$ of $L$ over some $T \to X$ to the restriction $\frac{1}{n}A_T \to \mathrm{Div}_X$ to the subsheaf $\frac{1}{n}A_T \subseteq \frac{1}{m}A_T$. These morphisms are compatible in a suitable sense, and make the sequence of root stacks into an inverse system of algebraic stacks, with ordered set the set of non-zero natural numbers and the ordering given by divisibility. As we will see, this is a (locally) cofinal subsystem of a bigger projective system where one considers any Kummer extension of sheaves of monoids.

We want to take the inverse limit of this projective system in order to get a stack $X_\infty$, which we will call the "infinite root stack" of $X$, that parametrizes extensions of the log structure of $X$ with arbitrary denominators. This stack will be non-algebraic and have other nasty properties, but on the bright side it will "embody" parabolic sheaves on $X$ with arbitrary rational weights, and it will have nice local models that resemble the ones of the finite root stacks.

Moreover we will see that the geometry of the infinite root stack is closely related to the logarithmic geometry of the log scheme. Specifically, we will show that there is a reconstruction procedure that gives back the log structure starting from the infinite root stack, and that one can recover the Kummer-flat topos of Kato ([Kat, Niz08]) from an opportunely defined fppf topos of the infinite root stack. We will also see that quasi-coherent sheaves on $X_\infty$ correspond to parabolic sheaves with arbitrary rational weights, so that finitely presented Kummer-flat sheaves on $X$ are the same thing as finitely presented parabolic sheaves with rational weights.

In this chapter (and from here on) we assume that $X$ is fine and saturated. Some parts of the theory make sense without this assumption, but for simplicity we prefer to keep this hypothesis always in the background instead of bringing it out only when it is really needed.

First of all we need some preliminaries on inverse limits of stacks.

## 2.1 Inverse limits of algebraic stacks

There are probably several instances of a definition of an inverse limit of stacks in the literature. We lay them down yet one more time, to establish the notation and for the convenience of the reader.

Assume in this section that $I$ is a partially ordered set, which is moreover filtered, i.e. for every pair $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$. Also, let us fix a category $\mathcal{D}$. Every fibered category in this section will be over $\mathcal{D}$, and will be a category fibered *in groupoids*. This assumption is not crucial and without it one only needs to add the word "cartesian" in the appropriate places, but we will not need this generality.

**Definition 2.1.1.** An *inverse system* of fibered categories indexed by $I$ is the datum of a set $\{\mathcal{C}_i\}_{i\in I}$ of fibered categories indexed by $I$, together with a transition functor $F_{i,j}\colon \mathcal{C}_j \to \mathcal{C}_i$ every time that $j \geq i$. Moreover we have the following data: for every index $i \in I$ an isomorphism $F_{i,i} \cong \mathrm{id}$, and for every triple $i, j, k \in I$ such that $k \geq j \geq i$, we have a natural isomorphism $\alpha_{i,j,k}\colon F_{i,j} \circ F_{j,k} \cong F_{i,k}$ of functors $\mathcal{C}_k \to \mathcal{C}_i$, that satisfies the following compatibility condition: whenever we have $i, j, k, l \in I$ such that $l \geq k \geq j \geq i$ the following diagram of functors $\mathcal{C}_l \to \mathcal{C}_i$ commutes

$$\begin{tikzcd}
F_{i,j} \circ F_{j,k} \circ F_{k,l} \arrow[r, "\alpha_{i,j,k}F_{k,l}"] \arrow[d, "F_{i,j}\alpha_{j,k,l}"'] & F_{i,k} \circ F_{k,l} \arrow[d, "\alpha_{i,k,l}"] \\
F_{i,j} \circ F_{j,l} \arrow[r, "\alpha_{i,j,l}"] & F_{i,l}.
\end{tikzcd}$$

**Remark 2.1.2.** In some situations all the natural isomorphisms $\alpha_{i,j,k}$ are identities. If this happens we will say that the inverse system is *strict*. The inverse system of root stacks of a log scheme will have this property.

We can take the inverse limit of an inverse system of fibered categories. One can give a (2-categorical) universal property that uniquely identifies this limit, we will instead define and use a specific model.

**Definition 2.1.3.** The *canonical inverse limit* $\mathcal{C} = \varprojlim_{i\in I} \mathcal{C}_i$ of an inverse system of fibered categories $\{\mathcal{C}_i, F_{i,j}\}$ is the fibered category defined as follows:

- for an object $d \in \mathcal{D}$, the category $\mathcal{C}(d)$ has as objects collections $\{\xi_i\}_{i\in I}$ of objects $\xi_i \in \mathcal{C}_i(d)$, together with, for every $i, j \in I$ with $j \geq i$, an isomorphism $\phi_{i,j}\colon F_{i,j}(\xi_j) \cong \xi_i$. These isomorphisms satisfy the following compatibility condition: every time that we have $i, j, k \in I$ such that $k \geq j \geq i$, the following diagram in $\mathcal{C}_i$ commutes

$$\begin{tikzcd}
F_{i,j}(F_{j,k}(\xi_k)) \arrow[r, "F_{i,j}(\phi_{j,k})"] \arrow[d, "\alpha_{i,j,k}(\xi_k)"'] & F_{i,j}(\xi_j) \arrow[d, "\phi_{i,j}"] \\
F_{i,k}(\xi_k) \arrow[r, "\phi_{i,k}"] & \xi_i.
\end{tikzcd}$$

- Morphisms in $\mathcal{C}(d)$ from $\{\xi_i, \phi_{i,j}\}$ to $\{\eta_i, \psi_{i,j}\}$ are collections of arrows $f_i\colon \xi_i \to \eta_i$ in $\mathcal{C}_i(d)$ that are compatible with the isomorphisms $\phi_{i,j}$ and $\psi_{i,j}$, in the obvious sense.
- The pullback of $\{\xi_i, \phi_{i,j}\} \in \mathcal{C}(d)$ along $f\colon e \to d$ is defined as $\{f^*\xi_i, f^*\phi_{i,j}\}$, i.e. by pulling back both the objects and the morphisms, in the corresponding category $\mathcal{C}_i$.

Note that for any $i$ there is an obvious projection functor $\pi_i\colon \mathcal{C} \to \mathcal{C}_i$, and for any $i, j \in I$ with $j \geq i$ there is a canonical isomorphism $F_{i,j} \circ \pi_j \cong \pi_i$.

The fibered category $\mathcal{C}$ thus defined has the following universal property.

**Proposition 2.1.4.** *For any fibered category $\mathcal{E}$ with functors $G_i\colon \mathcal{E} \to \mathcal{C}_i$ and for any pair $i, j \in I$ with $j \geq i$, a natural isomorphism $\beta_{i,j}\colon F_{i,j} \circ G_j \cong G_i$, such that for $i, j, k \in I$ with $k \geq j \geq i$ the two morphism of functors $F_{i,k} \circ G_k \to G_i$ given by $\beta_{i,k}$ and by the composition $F_{i,k} \circ G_k \xrightarrow{\alpha^{-1}_{i,j,k}} F_{i,j} \circ F_{j,k} \circ G_k \to F_{i,j} \circ G_j \to G_i$ coincide, there exists a unique functor $G\colon \mathcal{E} \to \mathcal{C}$ such that $\pi_i \circ G = G_i$.*

*Proof.* Since we must have $\pi_i \circ G = G_i$, the action of $G$ on objects is determined by $G(d) = \{G_i(d)\}_{i \in I}$, and the isomorphisms $\phi_{i,j}$ are given by the natural isomorphisms $\beta_{i,j}$. The action on arrows is also given by the $G_i$'s. □

**Definition 2.1.5.** A fibered category $\mathcal{E}$ with the data as in the previous paragraph is an *inverse limit* of an inverse system $\{\mathcal{C}_i, F_{i,j}\}$ if the induced functor $G\colon \mathcal{E} \to \mathcal{C} = \varprojlim_{i \in I} \mathcal{C}_i$ to the canonical inverse limit is an equivalence of fibered categories.

**Definition 2.1.6.** A *cofinal* subset of a filtered partially ordered set $I$ is a subset $J \subseteq I$ such that for any $i \in I$ there exists $j \in J$ with $j \geq i$.

By equipping $J$ with the induced order relation, we can see it as a filtered partially ordered set, and consider $\mathcal{C}_J = \varprojlim_{j \in J} \mathcal{C}_j$. By cofinality, for any $i \in I$ we can choose $j(i) \in J$ such that $j(i) \geq i$, and by composing the projection $\mathcal{C}_J \to \mathcal{C}_{j(i)}$ with $F_{i,j(i)}\colon \mathcal{C}_{j(i)} \to \mathcal{C}_i$, we get a functor $C_J \to C_i$ for each $i$. Note that if $i = j \in J$, we can take the functor $\mathcal{C}_J \to \mathcal{C}_j$ to be just the projection of the inverse limit.

One readily checks that there are natural isomorphisms after composing with the transition maps of the system $\{\mathcal{C}_i\}_{i \in I}$, and thus we get a compatible system of functors to the inverse system, and by Proposition 2.1.4, a functor $F_J\colon \mathcal{C}_J \to \varprojlim_{i \in I} \mathcal{C}_i$.

**Remark 2.1.7.** Note that the functor $F_J$ is compatible with restriction to comma categories on the base category $\mathcal{D}$.

**Proposition 2.1.8.** *For any cofinal subset $J \subseteq I$ in a filtered partially ordered subset, the functor $F_J$ is an equivalence.*

*Sketch of proof.* We can fix an object $d \in \mathcal{D}$ and show that $F_J(d)\colon \mathcal{C}_J(d) \to (\varprojlim_{i \in I} \mathcal{C}_i)(d)$ is an equivalence.

The functor $F_J(d)$ is fully faithful, by looking at the components in $\varprojlim_{i \in I} \mathcal{C}_i$ corresponding to indices in $J$.

It is essentially surjective, since for any family $\{\xi_i\}_{i \in I}$ with isomorphisms $\phi_{i,j}$, corresponding to an object $\xi \in (\varprojlim_{i \in I} \mathcal{C}_i)(d)$, we can just take its "restriction" to the indices in $J$, and this will give an object of $\mathcal{C}_J(d)$, with image isomorphic to the original $\xi$. □

**Proposition 2.1.9.** *Assume that $\mathcal{D}$ is equipped with a Grothendieck topology and all the fibered categories $\mathcal{C}_i$ are stacks over $\mathcal{D}$. Then $\varprojlim_{i \in I} \mathcal{C}_i$ is a stack as well.*

*Proof.* The proof is by standard descent theory arguments. Basically, descent data for $\varprojlim_{i \in I} \mathcal{C}_i$ amount to descent data (for objects and morphisms) for the single $\mathcal{C}_i$'s, so we can glue them at each stage and put everything together.

More succinctly, one could say that limits commute with stackification, so a limit of stacks is a stack. $\square$

## 2.2 The infinite root stack

The root stacks of a log scheme $X$ naturally form an inverse system of stacks over $(\mathrm{Sch}/X)$. Let us consider the set $I = \{\text{Kummer extensions } j\colon A \to B \text{ with } B \text{ coherent}\}$, ordered by $(j', B') \geq (j, B)$ if there is morphism $B \to B'$ with a commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "j"'] \arrow[dr, "j'"] & \\ B \arrow[rr] & & B'. \end{tikzcd}$$

Recall that "coherent" requires $B$ to have charts locally in the étale topology, and this implies in particular that it is finitely generated. For example, $A \to A_{\mathbb{Q}}$ is not an element of $I$.

**Remark 2.2.1.** Note that if $j\colon A \to B$ is Kummer, then $j_{\mathbb{Q}}\colon A_{\mathbb{Q}} \to B_{\mathbb{Q}}$ is an isomorphism, and so $B$ is canonically isomorphic to a subsheaf of $A_{\mathbb{Q}}$. This says that we do not lose anything by restricting to subsheaves of $A_{\mathbb{Q}}$, and from now on in a Kummer extension $A \to B$, the sheaf $B$ will always be a subsheaf of $A_{\mathbb{Q}}$.

Now, for any $j\colon A \to B \in I$, we have the root stack $X_{B/A}$, and when $(j', B') \geq (j, B)$, restriction to $B$ gives a functor $X_{B'/A} \to X_{B/A}$. Moreover these data give an inverse system of stacks over $(\mathrm{Sch}/X)$ indexed by $I$.

The partially ordered set $I$ is filtered: given $j\colon A \to B$ and $j'\colon A \to B'$ in $I$, we take $B'' = B + B' \subseteq A_{\mathbb{Q}}$, and $j''\colon A \to B''$ the induced morphism. Then $(j'', B'')$ is an element of $I$ that dominates both $(j, B)$ and $(j', B')$.

**Definition 2.2.2.** The *infinite root stack* of the logarithmic scheme $X$ is the inverse limit $X_\infty = \varprojlim_{(A \to B \in I)} X_{B/A}$.

By Proposition 2.1.9, $X_\infty$ is a stack over $(\mathrm{Sch}/X)$ (with the fpqc topology or any coarser one). By definition of the inverse limit, the objects of $X_\infty(T)$ for a scheme $T \to X$ are collections $\{L_{A \to B}\colon B_T \to \mathrm{Div}_T\}_{A \to B}$ of liftings of the DF structure of $X$, together with compatibility isomorphisms for any morphism of Kummer extensions, and the arrows are compatible collections of arrows.

We have the following alternative description.

**Proposition 2.2.3.** *There is a natural isomorphism $X_\infty \cong X_{A_{\mathbb{Q}}/A}$ with the root stack with respect to the "maximal" Kummer extension $A \to A_{\mathbb{Q}}$.*

**Remark 2.2.4.** We stress once again that in a system of denominators $A \to B$, the sheaf $B$ is finitely generated, so $A \to A_{\mathbb{Q}}$ is not a system of denominators. Despite this, one can define a root stack $X_{A_{\mathbb{Q}}/A}$ exactly as in the finitely generated case (see Definition 1.2.27).

This means that objects of $X_{A_{\mathbb{Q}}/A}$ over a scheme $T \to X$ are liftings $(A_{\mathbb{Q}})_T \to \mathrm{Div}_T$ of the pullback DF structure $A_T \to \mathrm{Div}_T$, and arrows are morphisms of DF structures.

We also remark here that taking $(-)_{\mathbb{Q}}$ commutes with pullback, i.e. in the situation above we have $(A_{\mathbb{Q}})_T \cong (A_T)_{\mathbb{Q}}$ as sheaves on $T$. This holds basically because colimits commute with left adjoints.

**Lemma 2.2.5.** *Let $T \to X$ be a morphism. Then there is an isomorphism*

$$\varinjlim_{A \to B} B_T \to (A_{\mathbb{Q}})_T$$

*as fibered categories on* $(\mathrm{Sch}\,/T)$.

*Proof.* First of all note that on $(\mathrm{Sch}\,/X)$, the maps $B \to B'$ of the direct system are injective, and since they are compatible with the inclusions $B \subseteq A_{\mathbb{Q}}$ and $B' \subseteq A_{\mathbb{Q}}$, we can identify the direct limit as the ascending union $\bigcup_{A \to B} B \subseteq A_{\mathbb{Q}}$, as a sheaf. By exactness of the pullback, all this remains true if we are on the scheme $T$, so we can assume $T = X$.

Let us prove that the inclusion

$$\bigcup_{A \to B} B \subseteq A_{\mathbb{Q}}$$

is an equality. This means that any section $s \in A_{\mathbb{Q}}(U)$ with $U \to X$ étale, comes étale locally on $U$ from $(\bigcup_{A \to B} B)(U)$, so in particular we can assume that there is a chart $P \to A$ on $X$, where $P$ is a finitely generated monoid.

Since $(P_{\mathbb{Q}})_X \to A_{\mathbb{Q}}$ is surjective, $s$ will locally come from an element $p \in P_{\mathbb{Q}}$, that will lie in some $\frac{1}{n}P \subseteq P_{\mathbb{Q}}$, as $P$ is finitely generated. The image of $p$ in $\frac{1}{n}A$, an element of $\bigcup_{A \to B} B$, will be $s$. $\square$

*Proof of Proposition 2.2.3.* For any system of denominators $A \subseteq B \subseteq A_{\mathbb{Q}}$ we have a restriction morphism $X_{A_{\mathbb{Q}}/A} \to X_{B/A}$, and by varying $B$ we get a map $X_{A_{\mathbb{Q}}/A} \to X_\infty$. Functorially, for a scheme $T \to X$, the morphism above sends a lifting $(A_{\mathbb{Q}})_T \to \mathrm{Div}_T$ of the DF structure on $X$ to the collection of its restrictions to systems of denominators $A \subseteq B \subseteq A_{\mathbb{Q}}$.

Because of the previous lemma, we have $\varinjlim_{A \to B} B_T \cong (A_{\mathbb{Q}})_T$, and consequently

$$\mathrm{Hom}((A_{\mathbb{Q}})_T, \mathrm{Div}_T) = \mathrm{Hom}(\varinjlim_{A \to B} B_T, \mathrm{Div}_T) = \varprojlim_{A \to B} \mathrm{Hom}(B_T, \mathrm{Div}_T).$$

In other words, morphisms of stacks $(A_{\mathbb{Q}})_T \to \mathrm{Div}_T$ correspond to compatible systems of morphisms $B_T \to \mathrm{Div}_T$, and it is clear that symmetric monoidal functors correspond to collections of symmetric monoidal functors. Moreover this equivalence respects the compatibility with the DF structure $A_T \to \mathrm{Div}_T$ coming from $X$. This says that the morphism $X_{A_{\mathbb{Q}}/A} \to X_\infty$ is an isomorphism. $\square$

It is possible to reduce considerably the set of indices over which we take the limit, and still obtain the infinite root stack as a result.

Fix a system of denominators $k\colon A \to A'$ and consider the subset

$$I_k = I_{(k\colon A\to A')} = \left\{ A \to \frac{1}{n}A' \right\}_{n\in\mathbb{N}} \subset I,$$

where the ordering induced by $I$ corresponds to the divisibility ordering on $\mathbb{N}$. The case $k = \mathrm{id}_A$ gives the natural tower of extensions where we just take all sections of $A$ with some fixed denominator $n$.

Note that this subset is not necessarily cofinal, although that this is true if $X$ is quasi-compact.

**Example 2.2.6.** Consider a countable disjoint union of points $X = \bigsqcup_{n\in\mathbb{N}} \operatorname{Spec}(k)$ with $k$ algebraically closed, with the standard rank 1 log structure on each point.

In this case we have $A = \mathbb{N}_X, A_{\mathbb{Q}} = \mathbb{Q}_X$, and, although as sheaves we have $\bigcup \frac{1}{n}A = A_{\mathbb{Q}}$, it is not true that every system of denominators $A \subseteq B \subseteq A_{\mathbb{Q}}$ is contained in some $\frac{1}{n}A$. Indeed, it suffices to take the section $s$ of $A_{\mathbb{Q}}$ that takes the value $\frac{1}{n}$ on the $n$-th copy of $\operatorname{Spec}(k)$, and the submonoid it generates inside $A_{\mathbb{Q}}$.

**Proposition 2.2.7.** *The subset $I_k \subseteq I$ is cofinal if $X$ is quasi-compact.*

*Proof.* Let us fix a system of denominators $A \to B$. Since $X$ is quasi-compact, it has a finite covering by affines where there is a global chart for $A \to B$. On each of these open affines, if $P \to Q$ is the given chart, we can find $n$ such that $Q \subseteq \frac{1}{n}P$, since $Q$ is finitely generated. If we let $N$ be the least common multiple of these finitely many indices, we have $B \subseteq \frac{1}{N}A$ on $X$. □

Despite the fact that the subset $I_k \subseteq I$ is not necessarily cofinal, the induced morphism between the inverse limits is always an isomorphism.

**Proposition 2.2.8.** *The natural functor $A_{I_k}\colon \varprojlim_{n\in\mathbb{N}} X_{\frac{1}{n}A'/A} \to X_\infty$ induced by the inclusion $I_k \subseteq I$ is an isomorphism.*

*Proof.* This follows from the previous lemma and from proposition 2.1.8, using the fact that the two are stacks on $(\mathrm{Sch}\,/X)$. □

From now on we will often see $X_\infty$ as the inverse limit of such a "small" subsystem, typically as $\varprojlim_{n\in\mathbb{N}} X_n$, where $X_n = X_{\frac{1}{n}A/A}$.

The construction of the infinite root stack is functorial, as it is apparent from the interpretation as a root stack for the extension $A \subseteq A_{\mathbb{Q}}$: if $f\colon X \to Y$ is a morphism of log schemes, then there is an induced map $f_\infty\colon X_\infty \to Y_\infty$ that sends a lifting $((A_X)_{\mathbb{Q}})_T \to \mathrm{Div}_T$ (for a scheme $T \to X$) of the DF structure of $X$ to the composition $((f^*A_Y)_{\mathbb{Q}})_T \to ((A_X)_{\mathbb{Q}})_T \to \mathrm{Div}_T$, a lifting of the DF structure of $Y$. This can also be seen as the morphism induced by the morphisms $X_n \to Y_n$ between the intermediate root stacks by taking the inverse limit.

**Remark 2.2.9.** Moreover, as it happens with the finite root stacks, if the morphism $f$ is strict, then the square

$$\begin{tikzcd} X_\infty \arrow[r] \arrow[d] & Y_\infty \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is cartesian. In fact since $f^*A_Y \cong A_X$, we have $(f^*A_Y)_{\mathbb{Q}} \cong (A_X)_{\mathbb{Q}}$ and this implies that an object of $X_\infty(T)$ is an object of $Y_\infty(T)$, plus a map $T \to X$ over $Y$.

The infinite root stack has a natural projection $X_\infty \to X$, which is defined functorially on $X_\infty(T)$ by forgetting everything but the morphism $T \to X$. Clearly this is the same as the composite of the two projections $X_\infty \to X_{B/A} \to X$ for any system of denominator $A \to B$.

**Proposition 2.2.10.** *Assume that $X$ is noetherian and the log structure of $X$ is generically trivial, i.e. the open subscheme $U \subseteq X$ where the log structure is trivial is schematically dense. Then the projection $\pi \colon X_\infty \to X$ is an isomorphism over $U$.*

*Proof.* This follows from the fact that the formation of $X_\infty$ commutes with strict base change, and over $U$ the projection from any root stack is an isomorphism. □

### 2.2.1 Local models

Let us now consider the local version of this construction, where there is a chart. This will lead us to local models for the infinite root stack.

When $X$ has a global chart $P \to \mathrm{Div}(X)$, following our previous construction, we can consider the set $I = \{Q \subseteq P_{\mathbb{Q}} \mid Q \text{ is finitely generated }\}$, partially ordered by inclusion. As before, $I$ is filtered (by taking $Q + Q' \subseteq P_{\mathbb{Q}}$).

We can consider the inverse limit $(X_\infty)_P = \varprojlim_I X_{Q/P}$, and, as in the discussion of the preceding section, one can show that the objects of $(X_\infty)_P(T)$ for $\phi \colon T \to X$ correspond to liftings $P_{\mathbb{Q}} \to \mathrm{Div}(T)$ of the DF structure of $X$. In other words $(X_\infty)_P$ is isomorphic to the root stack $X_{P_{\mathbb{Q}}/P}$ corresponding to the Kummer extension $P \subseteq P_{\mathbb{Q}}$ (we omit the details).

**Lemma 2.2.11.** *Let $X$ be a log scheme with DF structure $L \colon A \to \mathrm{Div}_X$ and a global chart $P \to \mathrm{Div}(X)$. Then there is an isomorphism $(X_\infty)_P \cong X_\infty$. Moreover, this isomorphism is compatible with the isomorphisms $X_{Q/P} \cong X_{B/A}$, where $A \to B$ is a system of denominators with chart $P \to Q$.*

*Proof.* The proof follows closely the one of Proposition 4.18 of [BV12], with minor modifications. We sketch it briefly for the convenience of the reader.

Let us define a functor $X_\infty \to (X_\infty)_P$ as follows: for a scheme $T \to X$, we send an object $(A_{\mathbb{Q}})_T \to \mathrm{Div}_T$ of $X_\infty(T)$ to the composition $P_{\mathbb{Q}} \to (A_{\mathbb{Q}})_T(T) \to \mathrm{Div}(T)$, which is an object of $(X_\infty)_P$ when equipped with the obvious induced natural isomorphism of the composition $P \to P_{\mathbb{Q}} \to \mathrm{Div}(T)$ with the morphism $P \to \mathrm{Div}(T)$ coming from the chart on $X$. The action on arrows is clear.

The quasi-inverse $(X_\infty)_P \to X_\infty$ associates to an object $P_{\mathbb{Q}} \to \mathrm{Div}(T)$ the induced DF structure $(A_{\mathbb{Q}})_T \to \mathrm{Div}_T$. Here $(A_{\mathbb{Q}})_T$ is the sheaf quotient $((P_{\mathbb{Q}})_T/K)^{sh}$, where $K$ is the kernel of the given functor $P_{\mathbb{Q}} \to \mathrm{Div}(T)$, which coincides with the kernel of $P_{\mathbb{Q}} \to A_{\mathbb{Q}}$, since $\ker(P_T \to A_T) = \ker(P_T \to \mathrm{Div}_T)$.

Compatibility with the isomorphisms $X_{Q/P} \cong X_{B/A}$ is clear from the construction. □

To describe local models for $X_\infty$, let us start from the "universal" case of the spectrum of a monoid algebra.

Assume that $X = \mathrm{Spec}(k[P])$ for a finitely generated monoid $P$, with the natural log structure. Then we have a description of $X_\infty$ as a quotient stack, as it happens with finite root stacks: recall from 1.2.30 that in this case $X_n \cong [\mathrm{Spec}(k[P_n])/\mu_n(P)]$, where we set $P_n = \frac{1}{n}P$ and $\mu_n(P)$ denotes the Cartier dual $D[C_n]$ of the cokernel $C_n$ of the map $P^{\mathrm{gp}} \to P_n^{\mathrm{gp}}$.

Moreover let us denote by $\mu_\infty(P)$ the Cartier dual $D[C_\infty]$ of the cokernel $C_\infty$ of the morphism $P^{\mathrm{gp}} \to P_{\mathbb{Q}}^{\mathrm{gp}}$. There are inclusions $C_n \subseteq C_\infty$ and in fact $C_\infty$ is the ascending union of such subgroups (with respect to divisibility). Correspondingly $\mu_\infty(P) \cong \varprojlim_n \mu_n(P)$.

Note that since $P$ is fine and saturated, by choosing appropriate generators we have $P^{\mathrm{gp}} \cong \mathbb{Z}^r$, consequently $C_n \cong (\mathbb{Z}/n\mathbb{Z})^r$ and $C_\infty \cong (\mathbb{Q}/\mathbb{Z})^r$, and correspondingly $\mu_n(P) \cong (\mu_n)^r$ and $\mu_\infty(P) \cong (\mu_\infty)^r$, where $\mu_\infty = D[\mathbb{Q}/\mathbb{Z}] \cong \varprojlim_n \mu_n$.

**Proposition 2.2.12.** *We have an isomorphism* $X_\infty \cong [\mathrm{Spec}(k[P_{\mathbb{Q}}])/\mu_\infty(P)]$.

*Proof.* The stacks $[\mathrm{Spec}(k[P_n])/\mu_n(P)]$, together with the natural maps

$$[\mathrm{Spec}(k[P_m])/\mu_m(P)] \to [\mathrm{Spec}(k[P_n])/\mu_n(P)]$$

for $n \mid m$ form an inverse system of stacks over $(\mathrm{Sch}\,/\,\mathrm{Spec}(k[P]))$, and for every $n \in \mathbb{N}$ we have from Example 1.2.30 an isomorphism $F_n\colon X_n \to [\mathrm{Spec}(k[P_n])/\mu_n(P)]$. Moreover one checks that these isomorphisms are compatible with the transition maps of the two inverse systems, and thus give a morphism

$$F = \varprojlim_{n\in\mathbb{N}} F_n\colon X_\infty \to \varprojlim_{n\in\mathbb{N}} [\mathrm{Spec}(k[P_n])/\mu_n(P)]\,,$$

which is an isomorphism.

Now it suffices to note that we have an isomorphism

$$\varprojlim_n [\mathrm{Spec}(k[P_n])/\mu_n(P)] \cong [\mathrm{Spec}(k[P_{\mathbb{Q}}])/\mu_\infty(P)]\,.$$

In fact we have a map

$$[\mathrm{Spec}(k[P_{\mathbb{Q}}])/\mu_\infty(P)] \to \varprojlim_{n\in\mathbb{N}} [\mathrm{Spec}(k[P_n])/\mu_n(P)]$$

obtained by change of fiber along $\mu_\infty(P) \to \mu_n(P)$ for every $n$, and this has a quasi-inverse that can be described as follows.

Assume that we have an object of $\varprojlim_{n\in\mathbb{N}} [\mathrm{Spec}(k[P_n])/\mu_n(P)]$ over a scheme $T \to \mathrm{Spec}(k[P])$, i.e. a sequence of $\mu_n(P)$-torsors

$$\begin{tikzcd} Q_n \arrow[r] \arrow[d] & \mathrm{Spec}(k[P_n]) \\ T & \end{tikzcd}$$

with equivariant maps $Q_n \to \mathrm{Spec}(k[P_n])$, and every time that $n \mid m$, an isomorphism $Q_m \times^{\mu_m(P)} \mu_n(P) \cong Q_n$, where as usual $Q_m \times^{\mu_m(P)} \mu_n(P) = (Q_m \times \mu_n(P))/\mu_m(P)$, with the obvious compatibility properties.

Since the maps $Q_m \to Q_n$ are affine, we can take the inverse limit $Q = \varprojlim_n Q_n$ as a scheme over $T$. This has an action of $\mu_\infty(P) = \varprojlim_n \mu_n(P)$, and moreover it is a torsor for $\mu_\infty(P)$, since the morphism $Q \times \mu_\infty(P) \to Q \times_T Q$ is an isomorphism, being the inverse limit of the isomorphisms $Q_n \times \mu_n(P) \cong Q_n \times_T Q_n$.

Finally the $\mu_n(P)$-equivariant maps $Q_n \to \mathrm{Spec}(k[P_n])$ induce a $\mu_\infty(P)$-equivariant morphism $Q \to \varprojlim_n \mathrm{Spec}(k[P_n]) = \mathrm{Spec}(k[P_\mathbb{Q}])$, and this gives an object

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & \mathrm{Spec}(k[P_\mathbb{Q}]) \\ T & \end{tikzcd}$$

of $[\mathrm{Spec}(k[P_\mathbb{Q}])/\mu_\infty(P)]$ over $T$. These two maps are mutually quasi-inverses. $\square$

**Remark 2.2.13.** We have the following description of $k[P_\mathbb{Q}]$: take a finite set of generators $p_1, \dots, p_r$ of $P$, some indeterminates $t_1, \dots, t_r$, and the (finitely many) polynomials $f_i \in k[t_1, \dots, t_r]$ coming from the relations among the generators, so that $k[P] \cong k[t_1, \dots, t_r]/(f_i)$. Then we have

$$k[P_\mathbb{Q}] \cong k[t_1^{\frac{1}{n}}, \dots, t_r^{\frac{1}{n}} \mid n \in \mathbb{N}]/(f_i(t_1^{\frac{1}{n}}, \dots, t_r^{\frac{1}{n}}) \mid n \in \mathbb{N}).$$

For example, if $P = \langle p, q, r \mid p + q = 2r \rangle$, then we have $k[P] \cong k[x, y, z]/(xy - z^2)$, where the equation comes from the relation $p + q = 2r$, and

$$k[P_\mathbb{Q}] \cong k[x^{\frac{1}{n}}, y^{\frac{1}{n}}, z^{\frac{1}{n}} \mid n \in \mathbb{N}]/(x^{\frac{1}{n}} y^{\frac{1}{n}} - z^{\frac{2}{n}} \mid n \in \mathbb{N}).$$

As for the group $\mu_\infty(P)$, we already remarked that $\mu_\infty(P) = D[(\mathbb{Q}/\mathbb{Z})^r] \cong (\mu_\infty)^r$ where $r$ is the rank of $P^{\mathrm{gp}}$.

Now we will see that this description as a quotient stack extends to the case where there is a global chart.

Assume that $X$ is a log scheme with a global chart $P \to \mathrm{Div}(X)$ coming from a Kato chart $P \to \mathcal{O}_X$. Then recall that the root stack $X_n$ fits in a cartesian diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & {[\mathrm{Spec}(k[P_n])/\mu_n(P)]} \arrow[d] \\ X \arrow[r] & \mathrm{Spec}(k[P]). \end{tikzcd}$$

We can obtain a description as a quotient stack stack by pulling the data back to $X$: the stack $X_n$ is isomorphic to the quotient $[U_n/\mu_n(P)]$ where $U_n = X \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[P_n])$ and the action is the natural one on the second factor.

If $n \mid m$ we have natural affine morphisms $f_{n,m} \colon U_m \to U_n$ (induced by $\mathrm{Spec}(k[P_m]) \to \mathrm{Spec}(k[P_n])$) and $\phi_{n,m} \colon \mu_m(P) \to \mu_n(P)$, and moreover $f_{n,m}$ is equivariant with respect to $\phi_{n,m}$, so they fit together in a morphism of groupoids in schemes

$$\begin{tikzcd} U_m \times \mu_m(P) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & U_n \times \mu_n(P) \arrow[d, shift left] \arrow[d, shift right] \\ U_m \arrow[r] & U_n. \end{tikzcd}$$

Moreover if $n \mid m$ and $m \mid k$, the morphism $U_k \to U_n$ coincides with the composition $U_k \to U_m \to U_n$. In other words $\{U_n\}_{n\in\mathbb{N}}$ is an inverse system with index set $\mathbb{N}$ with the divisibility ordering and with affine transition maps, so the inverse limit $U_\infty = \varprojlim_n U_n$ makes sense as a scheme.

We have an action of $\mu_\infty(P)$ on $U_\infty$ obtained as limit of the actions at the finite levels, and we can consider the quotient stack $[U_\infty/\mu_\infty(P)]$.

**Proposition 2.2.14.** *There is an isomorphism $X_\infty \cong [U_\infty/\mu_\infty(P)]$. In particular $X_\infty$ has a representable fpqc morphism from a scheme, $U_\infty \to X_\infty$. Moreover, $X_\infty$ fits in the following cartesian diagram*

$$\begin{tikzcd}
X_\infty \arrow[r] \arrow[d] & {[\mathrm{Spec}(k[P_\mathbb{Q}])/\mu_\infty(P)]} \arrow[d] \\
X \arrow[r] & \mathrm{Spec}(k[P])
\end{tikzcd}$$

*Proof.* For every $n$ we have an isomorphism $X_n \cong [\mathrm{Spec}(k[P_n])/\mu_n(P)] \times_{\mathrm{Spec}(k[P])} X$, and these isomorphisms are compatible with the transition maps $X_m \to X_n$ and

$$[\mathrm{Spec}(k[P_m])/\mu_m(P)] \to [\mathrm{Spec}(k[P_n])/\mu_n(P)]\,.$$

Consequently we have an isomorphism

$$\begin{aligned}
X_\infty \cong \varprojlim_n X_n \cong \varprojlim_n ([\mathrm{Spec}(k[P_n])/\mu_n(P)] \times_{\mathrm{Spec}(k[P])} X) \cong \\
\cong (\varprojlim_n [\mathrm{Spec}(k[P_n])/\mu_n(P)]) \times_{\mathrm{Spec}(k[P])} X \cong [\mathrm{Spec}(k[P_\mathbb{Q}])/\mu_\infty(P)] \times_{\mathrm{Spec}(k[P])} X,
\end{aligned}$$

and the diagram in the statement is cartesian.

Because the diagram is cartesian we have

$$X_\infty \cong \left[X \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[P_\mathbb{Q}])/\mu_\infty(P)\right] = [U_\infty/\mu_\infty(P)]$$

since $U_\infty \cong X \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[P_\mathbb{Q}])$. $\square$

**Example 2.2.15.** Assume $X = \mathrm{Spec}(k)$ is the standard log point. In this case, as is explained in Example 1.2.28, the root stacks of $X$ are given by $X_n \cong [\mathrm{Spec}(k[t]/(t^n))/\mu_n]$, where $\mu_n$ acts by multiplication. Since

$$\varprojlim_n \mathrm{Spec}(k[t]/(t^n)) \cong \mathrm{Spec}(k[t^{\frac{1}{n}} \mid n \in \mathbb{N}]/(t)) = \mathrm{Spec}(k) \times_{\mathrm{Spec}(k[\mathbb{N}])} \mathrm{Spec}(k[\mathbb{Q}_+])$$

the preceding proposition implies that the infinite root stack of $X$ is described as

$$X_\infty \cong \left[\mathrm{Spec}(k[t^{\frac{1}{n}} \mid n \in \mathbb{N}]/(t))/\mu_\infty\right],$$

where $\mu_\infty$ acts via the natural $\mathbb{Q}/\mathbb{Z}$-grading on $k[t^{\frac{1}{n}} \mid n \in \mathbb{N}]/(t)$.

The morphism $X_\infty \to X_n$ to the intermediate $n$-th root stack $X_n \cong [\mathrm{Spec}(k[t]/(t^n))/\mu_n]$ is induced by the homomorphism $k[t]/(t^n) \to k[t^{\frac{1}{n}} \mid n \in \mathbb{N}]/(t)$ sending $t$ to $t^{\frac{1}{n}}$, which is equivariant with respect to the natural morphism $\mu_\infty \to \mu_n$.

**Example 2.2.16.** More generally if $X = \operatorname{Spec}(k)$ with the log structure given by a fine saturated sharp monoid $P$, with $P \to \operatorname{Div}(k)$ that sends 0 to $1 \in k$ and everything else to $0 \in k$, then the infinite root stack is

$$X_\infty \cong \left[\operatorname{Spec}(k[P_{\mathbb{Q}}]/(P^+)/\mu_\infty(P)\right]$$

where recall that $P^+ = P \setminus \{0\}$, and $(P^+) \subseteq k[P_{\mathbb{Q}}]$ is the ideal generated by the variables $x^p$ with $p \in P^+$.

More concretely take a system of generators $p_1, \ldots, p_r$ for $P$, and some indeterminates $t_1, \ldots, t_r$. Call $f_i(t_1, \ldots, t_r)$ the polynomials coming from a finite set of generating relations for the $p_j$'s, so that $k[P] = k[t_1, \ldots t_r]/(f_i)$. Then we have

$$X_\infty \cong \left[\operatorname{Spec}\left(k[t_1^{\frac{1}{n}}, \ldots, t_r^{\frac{1}{n}} \mid n \in \mathbb{N}]/(t_1, \ldots, t_r, f_i(t_1^{\frac{1}{n}}, \ldots, t_r^{\frac{1}{n}}) \mid n \in \mathbb{N})\right)/\mu_\infty^s\right]$$

where $s$ is the rank of $P^{gp}$ and the action is given, as in the previous example, by the natural $(\mathbb{Q}/\mathbb{Z})^s$-grading on the $k$-algebra $k[t_1^{\frac{1}{n}}, \ldots, t_r^{\frac{1}{n}} \mid n \in \mathbb{N}]/(t_1, \ldots, t_r, f_i(t_1^{\frac{1}{n}}, \ldots, t_r^{\frac{1}{n}}) \mid n \in \mathbb{N})$.

**Example 2.2.17.** Let $X$ be a smooth curve and $D \subseteq X$ an effective Cartier divisor, i.e. a finite number of points $\{x_1, \ldots, x_k\}$. Then the projection $X_\infty \to X$ restricted to $U = X \setminus D$ is an isomorphism, and over the points $x_i$ the stack $X_\infty$ has the structure of the infinite root stack of the standard log point, i.e.

$$(X_\infty)_{x_i} \cong \left[\operatorname{Spec}(k(x_i)[t^{\frac{1}{n}} \mid n \in \mathbb{N}]/(t))/\mu_\infty\right].$$

Thus in this case we can see $X_\infty$ as $X$ with added stacky structure on the points $x_i$, with a rather large stabilizer group.

We deduce the following results for general log schemes, without assuming that there is a global chart.

**Corollary 2.2.18.** *The infinite root stack $X_\infty$ of any log scheme $X$ has an étale cover by quotient stacks of the form just described, i.e. it is étale locally a quotient of an affine scheme by a diagonalizable group scheme.*

**Remark 2.2.19.** In this discussion we used the standard root stacks of $X$, but we could have equivalently used the root stacks given by $A \to \frac{1}{n}A'$, where $A \to A'$ is a fixed system of denominators, or even an arbitrary cofinal subset of the partially ordered set of systems of denominators.

This implies that the infinite root stack, even though it is not algebraic in the sense of Artin, still has some kind of (very) weak algebraicity property.

**Definition 2.2.20.** An *fpqc stack* is a stack (in groupoids) $\mathcal{X}$ on (Sch) that has an fpqc presentation (i.e. an fpqc representable morphism $U \to \mathcal{X}$ from a scheme), and such that the diagonal $\mathcal{X} \to \mathcal{X} \times_k \mathcal{X}$ is representable.

**Corollary 2.2.21.** *The infinite root stack $X_\infty$ of an fs log scheme $X$ is an fpqc stack. Moreover it has an fpqc presentation which is an inverse limit of flat (smooth in characteristic 0) presentations for the finite root stacks $X_n$.*

*Proof.* This follows from the previous preposition, by taking a disjoint union of the presentations described above in the local case. Representability of the diagonal follows from the local description as a root stack. □

Apart from this algebraicity property, in what follows we will exploit the fact that $X_\infty$ is an inverse limit of Artin stacks (what we could call a "pro-algebraic stack") and that we can find presentations that are inverse limits of presentations for the finite stacks. When we will refer to presentation of the finite root stacks $X_n$ yielding an fpqc presentation of $X_\infty$, we will write them as $U_n \to X_n$ as before and, in the local case where there is a global chart, $G_n$ will be the group $\mu_n(P)$. The inverse limits will be $U_\infty = \varprojlim_n U_n$ and $G_\infty = \varprojlim_n G_n$.

It is quite clear from its description and simple examples that $X_\infty$ is not going to be "of finite type over $k$", or "noetherian" even if $X$ is. Note that it is not even clear what these adjectives should mean, hence the quotation marks.

In fact, since $X_\infty$ has only an fpqc atlas, we have to be careful when we talk about properties like being noetherian, locally of finite type/presentation and such, since they are not local with respect to the fpqc topology, i.e. if $f\colon X \to Y$ is fpqc and $X$ is say of finite type over $k$, it is not necessarily the case that $Y$ also is, and vice versa. One would like to define such properties on any presentation of the stack, but the fact that one presentation has it will not imply that all presentations do.

About properties of morphisms, whenever we have a representable morphism $\mathcal{X} \to \mathcal{Y}$ between fpqc stacks, we will say that it has some property (for example it is flat, smooth, étale, of finite type, and such) if all base changes by schemes have said property, as usual.

**Example 2.2.22.** Take the standard log point $X = \operatorname{Spec}(k)$. Then the infinite root stack is

$$X_\infty \cong \left[\operatorname{Spec}(k[t^{\frac{1}{n}} \mid n \in \mathbb{N}]/(t))/\mu_\infty\right]$$

and, although the words do not mean anything precise, it is reasonable that it should be considered as non-noetherian and hence not of finite type over $k$, since for example the ideal $(t^{\frac{1}{n}})_{n\in\mathbb{N}}$ is not finitely generated.

The fact that the infinite root stack is not "noetherian" complicates the discussion of coherent sheaves, since "finitely presented" and "coherent" become different concepts. This issue would be absent if $X_\infty$ were at least *coherent*, meaning that $\mathcal{O}_{X_\infty}$ is a coherent sheaf, i.e. every finitely generated ideal $I \subseteq \mathcal{O}_{X_\infty}$ is also finitely presented.

This is true in some cases, but false in general, as the following example shows.

**Example 2.2.23.** Consider the submonoid $P \subseteq \mathbb{Z}^3$ generated by $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ and $e_4 = (1,1,-1)$. The associated rational cone $P_\mathbb{Q} \subseteq \mathbb{Q}^3$ is given by the inequalities

$$P_\mathbb{Q} = \{(a_1,a_2,a_3) \in \mathbb{Q}^3 \mid a_1 \geq 0,\ a_2 \geq 0,\ a_1 + a_3 \geq 0,\ a_2 + a_3 \geq 0\}.$$

Let us consider the spectrum $X = \operatorname{Spec}(k[P])$ of the monoid algebra of $P$.

From what we discussed, the infinite root stack $X_\infty$ has a flat presentation $U_\infty \to X_\infty$ where $U_\infty = \operatorname{Spec}(k[P_\mathbb{Q}])$, and the natural way to prove that $X_\infty$ is coherent would be to prove that $U_\infty$ is, but this is not the case.

Let us set $R = k[P_{\mathbb{Q}}]$ and let $x_i = x^{e_i} \in k[P_{\mathbb{Q}}]$ be the element corresponding to $e_i$, and consider the ideal $I \subseteq R$ generated by $x_1$ and $x_3$. We will show that $I$ is not finitely presented, by showing that the kernel

$$K = \{(f_1, f_3) \in R^2 \mid x_1 f_1 + x_3 f_3 = 0\}$$

of the presentation of $I$ is not finitely generated.

To check this, we will show that its image $J \subseteq R$ along the first projection $R^2 \to R$ is not finitely generated. Since $J$ is a homogeneous ideal, it corresponds to an ideal $A \subseteq P_{\mathbb{Q}}$ (Definition 1.1.3), the set of degrees of non-zero elements in $J$.

Let us check that we can describe $A$ as

$$A = \{(a_1, a_2, a_3) \in \mathbb{Q}^3 \mid a_1 \geq 0,\ a_2 \geq 0,\ a_1 + a_3 \geq 0,\ a_2 + a_3 \geq 1\}.$$

In fact, if $a \in A$ then there exist $f_1, f_3 \in R$ such that $x_1 f_1 + x_3 f_3 = 0$, with $f_1$ of degree $a$. Note that necessarily $f_3 \neq 0$, and call $b$ the degree of $f_3$. Then we conclude that $a + e_1 = b + e_3$, and consequently $a - e_3 + e_1$ is in $P_{\mathbb{Q}}$.

Conversely if $a - e_3 + e_1 \in P_{\mathbb{Q}}$ and $a \in P_{\mathbb{Q}}$, we have that $x_1 x^a - x_3 x^{a - e_3 + e_1} = 0$ (where as usual $x^p$ denotes the element of $k[P_{\mathbb{Q}}]$ corresponding to $p \in P_{\mathbb{Q}}$), so $a \in A$. Finally, one checks easily that $a - e_3 + e_1 \in P_{\mathbb{Q}}$ and $a \in P_{\mathbb{Q}}$ are equivalent to the inequalities above.

Now consider

$$\begin{aligned} A_0 &= \{a = (a_1, a_2, a_3) \in A \mid a_1 = 0,\ a_2 + a_3 = 1\} \\ &= \{(0, a_2, a_3) \in \mathbb{Q}^3 \mid a_2 \geq 0,\ a_3 \geq 0, a_2 + a_3 = 1\}. \end{aligned}$$

It is easy to check that $a + b \in A_0$ implies $a = 0$ for $a \in P_{\mathbb{Q}}$ and $b \in A$, and this says that any set of generators of $A$ as an ideal of $P_{\mathbb{Q}}$ must contain a set of generators of $A_0$, and thus must be infinite. In conclusion the ideal $J$ is not finitely generated.

The ideal $I \subseteq k[P_{\mathbb{Q}}]$ descends to give an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{X_\infty}$ on $X_\infty$, and since finite presentation is fpqc local, $\mathcal{I}$ is not finitely presented, and $X_\infty$ is not coherent.

**Remark 2.2.24.** Note that the monoid in the last example is not simplicial (meaning that the rational cone it generates in $P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is not simplicial). This is not a coincidence: we will see that if the log structure has a simplicial global chart in this sense, then there is a cofinal system of root stacks with flat transition maps and $\mathcal{O}_{X_\infty}$ is coherent. We will return on this once we have discussed quasi-coherent sheaves on $X_\infty$ with some detail (see Proposition 2.2.46).

Note that the infinite root stack $X_\infty$ has some natural DF structures. We need to be careful when we talk about sheaves of monoids, since $X_\infty$ is not algebraic. A sheaf of monoids on $X_\infty$ for us will be a sheaf of monoids on a small étale site of $X_\infty$, with objects isomorphism classes of étale representable morphisms $\mathcal{A} \to X_\infty$ and arrows classes of morphisms over $X_\infty$, with coverings given by families of classes of jointly surjective étale morphisms.

With this definition, $X_\infty$ has a DF structure $\Lambda_n \colon \pi^* \frac{1}{n} A \to \mathrm{Div}_{X_\infty}$ for any $n$ (where $\pi \colon X_\infty \to X$ is the projection), that consists of the pullback of the universal DF structure of the finite root stack $X_n$, and moreover we have a universal DF structure $\Lambda_\infty \colon \pi^* A_{\mathbb{Q}} \to \mathrm{Div}_{X_\infty}$ that extends all the $\Lambda_n$ simultaneously.

Here, in analogy with the case of schemes and algebraic stacks, $\mathrm{Div}_{X_\infty}$ is the fibered category, on the small étale site described above, consisting of invertible sheaves with sections. We will define precisely quasi-coherent sheaves on $X_\infty$ shortly.

To conclude this section, we show that, as it happens with the intermediate root stacks, the infinite root stack of $X$ can be seen as the infinite root stack of any intermediate root stack $X_{B/A}$.

**Proposition 2.2.25.** *Let $X$ be a log scheme, and fix a system of denominators $A \to B$. Then $X_{B/A}$ is a log stack, with the tautological log structure given by the universal lifting $\Lambda\colon B_{X_{B/A}} \to \mathrm{Div}_{X_{B/A}}$, and we can consider its infinite root stack $(X_{B/A})_\infty$. Then the natural map $(X_{B/A})_\infty \to X_\infty$ induced by the projection $X_{B/A} \to X$ is an isomorphism.*

*Proof.* This follows immediately from the fact that the morphism $A \to B$ induces an isomorphism $A_\mathbb{Q} \cong B_\mathbb{Q}$ (and likewise on any base change along $T \to X$), and the log structure on $X_{B/A}$ is given by the tautological DF structure $\Lambda\colon B_{X_{B/A}} \to \mathrm{Div}_{X_{B/A}}$.

With some more detail, we can define a morphism $X_\infty \to (X_{B/A})_\infty$ by sending an object $N\colon (A_\mathbb{Q})_T \to \mathrm{Div}_T$ of $X_\infty(T)$ to the induced $(B_\mathbb{Q})_T \to \mathrm{Div}(T)$, together with the morphism $T \to X_{B/A}$ determined by the restriction of $N$ to $B_T \subseteq (A_\mathbb{Q})_T$.

One checks that this is a quasi inverse to the map $(X_{B/A})_\infty \to X_\infty$. $\square$

### 2.2.2 Sheaves on the infinite root stack

Let us give a definition of quasi-coherent sheaf on an arbitrary fibered category over (Sch). Let us denote by QCoh the fibered category of quasi-coherent sheaves on (Sch). In other words for a scheme $T$, the category $\mathrm{QCoh}(T)$ is the category of quasi-coherent sheaves on $T$, and for a morphism $S \to T$ we have the usual pullback functor $\mathrm{QCoh}(T) \to \mathrm{QCoh}(S)$.

In what follows we will repeatedly use the fact that QCoh is a stack for the fpqc topology of (Sch). For a proof of this see the first chapter of [FGI+07].

**Definition 2.2.26.** Let $\mathcal{X} \to (\mathrm{Sch})$ be a fibered category. A *quasi-coherent sheaf* on $\mathcal{X}$ is a cartesian functor $\mathcal{X} \to \mathrm{QCoh}$ of fibered categories over (Sch).

Equivalently, a quasi-coherent sheaf on $\mathcal{X}$ assigns to every morphism $\phi\colon T \to \mathcal{X}$ of fibered categories from a scheme $T$ a quasi-coherent sheaf $E_\phi \in \mathrm{QCoh}(T)$ on $T$, and for any factorization

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "\psi"'] & T \arrow[d, "\phi"] \\ & \mathcal{X} \end{tikzcd}$$

we have an isomorphism $\alpha_f\colon f^*E_\phi \cong E_\psi$ of quasi-coherent sheaves on $S$. These are required to satisfy some compatibility properties that we leave to the reader to spell out.

This definition applies in particular to the infinite root stack $X_\infty$ of a log scheme, and gives us a notion of quasi-coherent sheaf. Let us introduce two equivalent notions, that use the fact that $X_\infty$ is an fpqc stack.

Assume more generally that $\mathcal{X}$ is any fpqc stack over (Sch), and fix an fpqc presentation $U \to \mathcal{X}$, with $R = U \times_\mathcal{X} U$ where $U$ is a scheme. Using the presentation we can give the following definition for quasi-coherent sheaves on $\mathcal{X}$.

**Definition 2.2.27.** A quasi-coherent sheaf $(E, \alpha)$ on $\mathcal{X}$ is a quasi-coherent sheaf $E$ on $U$, together with descent data with respect to the groupoid $R \rightrightarrows U \to \mathcal{X}$, i.e. an isomorphism $\alpha\colon \pi_1^* E \cong \pi_2^* E$, where $\pi_1, \pi_2\colon R \to U$ are the two projections, satisfying the cocycle condition on $R \times_{\mathcal{X}} R \times_{\mathcal{X}} R$. A morphism of quasi-coherent sheaves $f\colon (E, \alpha) \to (F, \beta)$ is a morphism $f\colon E \to F$ of quasi-coherent sheaves on $U$ which is compatible with the descent data.

We will write $\mathrm{QCoh}(\mathcal{X})$ for the category of quasi-coherent sheaves on $\mathcal{X}$. It is, as usual, an abelian category, and it is independent of the chosen fpqc presentation for $\mathcal{X}$.

**Proposition 2.2.28.** *Let $\mathcal{X} \to (\mathrm{Sch})$ be an fpqc stack and let us fix an fpqc presentation $U \to \mathcal{X}$. Then the two notions we gave above for quasi-coherent sheaves on $\mathcal{X}$ agree.*

*Proof.* This follows directly from the fact that quasi-coherent sheaves on schemes satisfy fpqc descent. □

**Definition 2.2.29.** A quasi-coherent sheaf $(E, \alpha)$ on $\mathcal{X}$ is *finitely presented* if the sheaf $E$ is finitely presented on $U$.

We will denote by $\mathrm{FP}(\mathcal{X})$ the full subcategory consisting of finitely presented $(E, \alpha)$. As the notation suggests, this full subcategory is also independent of the choice of the fpqc presentation. This relies on the standard fact that if $f\colon X \to Y$ is fpqc and $F \in \mathrm{QCoh}(Y)$, then $F$ is finitely presented if and only if $f^* F$ is.

**Remark 2.2.30.** As we already mentioned, the fact that $X_\infty$ is not coherent makes finitely presented sheaves, and not coherent ones, the right object for our purposes.

A third way of defining quasi-coherent sheaves is by defining a "small fpqc site" of $\mathcal{X}$, in analogy with the lisse-étale site of an Artin stack, and take quasi-coherent sheaves on this site.

**Definition 2.2.31.** The *small fpqc site* $\mathrm{fpqc}(\mathcal{X})$ of $\mathcal{X}$ has as objects isomorphism classes of representable morphisms of stacks $\mathcal{A} \to \mathcal{X}$, morphisms are commutative diagrams and coverings are families $\{\mathcal{A}_i \to \mathcal{A}\}_{i \in I}$ of classes of jointly surjective fpqc morphisms. We will denote by $\mathcal{X}_{\mathrm{fpqc}}$ the corresponding topos of sheaves.

**Remark 2.2.32.** The point of taking isomorphism classes of maps is that we want to get a 1-category. Representable maps into $\mathcal{X}$ form a 2-category, but this is equivalent to the 1-category that we get by taking isomorphism classes. From now on for simplicity and to avoid making notations more complicated we will pretend that the objects of the site are actual morphisms to $X_\infty$ (and this will also happen with the fppf and étale variations).

Let us show that this last concept of quasi-coherent sheaf is the same as the one that uses an fpqc presentation.

**Proposition 2.2.33.** *There is an equivalence of categories between the category $\mathrm{QCoh}(\mathcal{X}_{\mathrm{fpqc}})$ of quasi-coherent sheaves on the small fpqc topos of $\mathcal{X}$ and the category $\mathrm{QCoh}(\mathcal{X})$ of quasi-coherent sheaves defined using an fpqc presentation.*

*Moreover, this equivalence restricts to an equivalence $\mathrm{FP}(\mathcal{X}_{\mathrm{fpqc}}) \cong \mathrm{FP}(\mathcal{X})$ between the subcategories of finitely presented sheaves.*

*Proof.* Let us introduce a smaller fpqc topos for $\mathcal{X}$: we will denote by fpqcsch$(\mathcal{X})$ the subcategory of fpqc$(\mathcal{X})$ of objects $V \to \mathcal{X}$ of fpqc$(\mathcal{X})$ where $V$ is a scheme. It is a site with the induced topology. The inclusion fpqcsch$(\mathcal{X}) \to$ fpqc$(\mathcal{X})$ is a morphism of sites, and by descent for quasi-coherent sheaves it induces an equivalence of categories $\mathrm{QCoh}(\mathcal{X}_{\mathrm{fpqcsch}}) \to \mathrm{QCoh}(\mathcal{X}_{\mathrm{fpqc}})$.

Now let us show that there is an equivalence $\mathrm{QCoh}(\mathcal{X}_{\mathrm{fpqcsch}}) \cong \mathrm{QCoh}(\mathcal{X})$: we have a functor $\mathrm{QCoh}(\mathcal{X}_{\mathrm{fpqcsch}}) \to \mathrm{QCoh}(\mathcal{X})$ that sends a quasi-coherent sheaf on the topos $\mathcal{X}_{\mathrm{fpqcsch}}$ to its restriction to an fpqc presentation $U \to \mathcal{X}$, together with the associated descent data. By descent of quasi-coherent sheaves along fpqc morphisms, this functor is an equivalence.

Finally, finitely presented sheaves are clearly preserved in each of the two steps. □

Note that the small fpqc topos has some functoriality properties. Namely, if $f\colon \mathcal{X} \to \mathcal{Y}$ is a representable morphism of fpqc stacks, there are the usual pushforward and pullback functors $f_*\colon \mathcal{X}_{\mathrm{fpqc}} \to \mathcal{Y}_{\mathrm{fpqc}}$ and $f^*\colon \mathcal{Y}_{\mathrm{fpqc}} \to \mathcal{X}_{\mathrm{fpqc}}$, together with an adjunction $f^* \dashv f_*$. Moreover $f^*$ is right exact, so it preserves quasi-coherence, finite generation and presentation. We will still denote its restriction by $f^*\colon \mathrm{QCoh}(\mathcal{Y}_{\mathrm{fpqc}}) \to \mathrm{QCoh}(\mathcal{X}_{\mathrm{fpqc}})$.

**Remark 2.2.34.** If $\mathcal{X}$ is a scheme, then quasi-coherent sheaves in the small fpqc topos are the same as Zariski quasi-coherent sheaves (and, in fact, the same is true for all "intermediate" topologies, for example the étale topology). For this we refer to [Sta13, Tag 03DR].

Now we specialize the situation back to the infinite root stack of a log scheme $X$. The fact that $X_\infty$ is an inverse limit implies that every finitely presented sheaf on it comes from some finite level.

**Proposition 2.2.35.** *Let $X$ be a fs log scheme. The pullback morphisms $\mathrm{FP}(X_n) \to \mathrm{FP}(X_m)$ for $n, m$ with $n \mid m$ fit into a direct system of categories. Moreover the pullbacks $\mathrm{FP}(X_n) \to \mathrm{FP}(X_\infty)$ along the projection $X_\infty \to X_n$ are compatible with the structure maps of the system, and if in addition $X$ is quasi-compact the induced functor $\varinjlim \mathrm{FP}(X_n) \to \mathrm{FP}(X_\infty)$ is an equivalence.*

This follows directly from the following lemma.

**Lemma 2.2.36.** *Consider the presentations $U_n \to X_n$ and $U_\infty \to X_\infty$ discussed in Section 2.2.1. Then we have an equivalence*

$$\mathrm{FP}^{eq}(U_\infty) = \varinjlim_n \mathrm{FP}^{eq}(U_n),$$

*where $(-)^{eq}$ denotes the category of equivariant sheaves with respect to the corresponding groupoid.*

*Proof.* We will use the approximation properties of finitely presented sheaves on an inverse limit, as discussed in EGA IV-3 [Gro67], Section 8. Namely we will use that if $\{T_i\}_{i\in I}$ is an inverse system of schemes with affine transition maps and quasi-compact and quasi-separated base scheme $T_0$, we have

$$\mathrm{FP}(\varprojlim_i T_i) = \varinjlim_i \mathrm{FP}(T_i),$$

i.e. finitely presented sheaves on the limit come from a scheme $T_i$, "uniquely", meaning that two such sheaves on $T_i$ and $T_j$ become isomorphic on some $T_k$, and likewise for morphisms.

First of all from $U_\infty = \varprojlim_n U_n$ we have that $\mathrm{FP}(U_\infty) = \varinjlim_n \mathrm{FP}(U_n)$, and secondly, since

$$U_\infty \times_{X_\infty} U_\infty \cong \varprojlim_n (U_n \times_{X_n} U_n)$$

and

$$U_\infty \times_{X_\infty} U_\infty \times_{X_\infty} U_\infty \cong \varprojlim_n (U_n \times_{X_n} U_n \times_{X_n} U_n),$$

we also have identifications

$$\mathrm{FP}(U_\infty \times_{X_\infty} U_\infty) = \varinjlim_n \mathrm{FP}(U_n \times_{X_n} U_n)$$

and

$$\mathrm{FP}(U_\infty \times_{X_\infty} U_\infty \times_{X_\infty} U_\infty) = \varinjlim_n \mathrm{FP}(U_n \times_{X_n} U_n \times_{X_n} U_n).$$

From this it is easy to see that descent data at the infinite level must come, uniquely, from finite level. □

**Remark 2.2.37.** This also holds if we use the subsystem $\left\{A \to \frac{1}{n}A'\right\}_{n\in\mathbb{N}}$ for some fixed system of denominators $A \to A'$, or any cofinal subset of the set of systems of denominators on $X$.

There is another natural site over $X_\infty$, obtained by using fppf morphisms instead of fpqc ones, that will be related to the Kummer-flat site of the log scheme $X$ later in this chapter (Section 2.4). We will prove here that finitely presented sheaves on this new site are the same as finitely presented fpqc sheaves.

Again, we give the definition for a general fpqc stack $\mathcal{X}$.

**Definition 2.2.38.** The *small fppf site* $\mathrm{fppf}(\mathcal{X})$ of $\mathcal{X}$ is the site defined as follows: objects are isomorphism classes of representable fppf morphisms of stacks $\mathcal{A} \to \mathcal{X}$, the morphisms are classes of morphisms of stacks $\mathcal{A} \to \mathcal{B}$ over $\mathcal{X}$, and the covers are collections of classes of jointly surjective representable fppf morphisms. The associated *fppf topos* will be denoted by $\mathcal{X}_{\mathrm{fppf}}$.

Note that this time $\mathcal{X}$ may have no fppf morphism from a scheme, so this topos is subtler than the small fpqc topos.

Since any representable fppf morphism $\mathcal{A} \to \mathcal{X}$ is also fpqc, we have an inclusion functor $i\colon \mathrm{fppf}(\mathcal{X}) \to \mathrm{fpqc}(\mathcal{X})$, which is continuous and induces a morphism of topoi $(i_*, i^*)\colon \mathcal{X}_{\mathrm{fpqc}} \to \mathcal{X}_{\mathrm{fppf}}$. Moreover, if $f\colon \mathcal{X} \to \mathcal{Y}$ is a representable morphism of fpqc stacks, then we have pushforward and pullback functors $f_*\colon \mathcal{X}_{\mathrm{fppf}} \to \mathcal{Y}_{\mathrm{fppf}}$ and $f^*\colon \mathcal{Y}_{\mathrm{fppf}} \to \mathcal{X}_{\mathrm{fppf}}$, together with an adjunction $f^* \dashv f_*$.

**Remark 2.2.39.** Additionally, pullback on quasi-coherent sheaves is compatible with the morphism $(i_*, i^*)$, i.e. the following diagram is 2-commutative

$$\begin{tikzcd}
\mathrm{QCoh}(\mathcal{Y}_{\mathrm{fppf}}) \arrow[r, "i^*"] \arrow[d, "f^*"'] & \mathrm{QCoh}(\mathcal{Y}_{\mathrm{fpqc}}) \arrow[d, "f^*"] \\
\mathrm{QCoh}(\mathcal{X}_{\mathrm{fppf}}) \arrow[r, "i^*"] & \mathrm{QCoh}(\mathcal{X}_{\mathrm{fpqc}})
\end{tikzcd}$$

where we used the same letter $i^*$ for $\mathcal{X}$ and $\mathcal{Y}$.

In general there is no reason for $(i_*, i^*)$ this to be an isomorphism, even if we restrict to quasi-coherent sheaves. Nonetheless, we can say something if we restrict to finitely presented sheaves on an infinite root stack $X_\infty$.

**Proposition 2.2.40.** *The morphism of topoi $(i_*, i^*)$ induces an equivalence*

$$\mathrm{FP}((X_\infty)_{\mathrm{fppf}}) \cong \mathrm{FP}(X_\infty).$$

**Remark 2.2.41.** As we will see in the proof, here it is crucial to use the inverse system defining $X_\infty$. Also, we believe that this equivalence can not be extended to quasi-coherent sheaves without any finiteness hypothesis, and in fact we believe that quasi-coherent sheaves in the two topoi should be different.

The philosophical reason is that when one proves that Zariski quasi-coherent sheaves are a stack for the fpqc topology (and from this follows that quasi-coherent sheaves are the same in all topologies), one defines quasi-coherence on the same topology for which the objects are locally rings (i.e. the Zariski topology), when in our case we have fppf sheaves on an object that is only fpqc-locally a ring.

**Remark 2.2.42.** While the fact that $i^*$ preserves finitely presented sheaves is standard, the corresponding fact for $i_*$ is not obvious, and will follow from the proof.

*Proof.* We start by giving an alternative description of the two functors $i_*$ and $i^*$: let us define $\alpha\colon \mathrm{FP}((X_\infty)_{\mathrm{fppf}}) \to \mathrm{FP}(X_\infty)$, and $\beta\colon \mathrm{FP}(X_\infty) \to \mathrm{FP}((X_\infty)_{\mathrm{fppf}})$. Let us fix a presentation $\phi\colon U_\infty = \varprojlim_n U_n \to X_\infty$ coming from flat presentations of the finite root stacks (as in the discussion preceding Proposition 2.2.14) for $X_\infty$, and recall that $\mathrm{FP}(X_\infty)$ is by definition the category $\mathrm{FP}^{eq}(U_\infty)$ of finitely presented sheaves over $U_\infty$, equivariant with respect to the groupoid $R = U_\infty \times_{X_\infty} U_\infty \rightrightarrows U_\infty$. For the rest of the proof we will denote $U_\infty$ just by $U$, to ease the notation.

Given a finitely presented sheaf $F \in \mathrm{FP}((X_\infty)_{\mathrm{fppf}})$, we consider the pullback $\phi^* F \in \mathrm{FP}(U_{\mathrm{fppf}}) = \mathrm{FP}(U)$. This sheaf comes naturally equipped with descent data with respect to the groupoid $R \rightrightarrows U$, and this gives an object $\alpha(F) \in \mathrm{FP}^{eq}(U)$.

Conversely, let us assume that $G \in \mathrm{FP}^{eq}(U)$, and that $f\colon \mathcal{A} \to X_\infty$ is an object of $(X_\infty)_{\mathrm{fppf}}$. We consider the pullback $R_{\mathcal{A}} \rightrightarrows U_{\mathcal{A}} \to \mathcal{A}$ of the groupoid $R \rightrightarrows U$, and the pullback $G_{\mathcal{A}}$ of $G$ to $U_{\mathcal{A}}$, together with the pullback of the descent data. This gives an object $G_{\mathcal{A}}$ of $\mathrm{FP}^{eq}(U_{\mathcal{A}})$, and we define

$$\beta(G)(\mathcal{A}) = \mathrm{Hom}^{eq}(\mathcal{O}_{U_{\mathcal{A}}}, G_{\mathcal{A}}).$$

Let us check that $\beta(G)$ is a sheaf on $(X_\infty)_{\mathrm{fppf}}$. This follows from descent of quasi-coherent sheaves on schemes: given a fppf morphism $\mathcal{A} \to \mathcal{B}$ in $(X_\infty)_{\mathrm{fppf}}$, we have to check that $\beta(G)(\mathcal{B})$ is the equalizer of the two pullback maps $\beta(G)(\mathcal{A}) \rightrightarrows \beta(G)(\mathcal{A} \times_{\mathcal{B}} \mathcal{A})$ (it is clear that $\beta(G)$ carries disjoint unions into products).

By following the construction of $\beta$, we pull back the presentation $U$ from $X_\infty$, obtaining the following diagram

$$\begin{tikzcd} U_{\mathcal{A}\times_{\mathcal{B}}\mathcal{A}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[rrr, bend left, "k_U"] & U_{\mathcal{A}} \arrow[r] \arrow[d] \arrow[rr, bend left, "h_U"] & U_{\mathcal{B}} \arrow[r, "g_U"] \arrow[d] & U \arrow[d] \\ \mathcal{A}\times_{\mathcal{B}}\mathcal{A} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{A} \arrow[r] & \mathcal{B} \arrow[r] & X_\infty \end{tikzcd}$$

and what we have to prove is that $\mathrm{Hom}^{eq}(\mathcal{O}_{U_{\mathcal{B}}}, g_U^*G)$ is the equalizer of the two maps

$$\mathrm{Hom}^{eq}(\mathcal{O}_{U_{\mathcal{A}}}, h_U^*G) \rightrightarrows \mathrm{Hom}^{eq}(\mathcal{O}_{U_{\mathcal{A}\times_{\mathcal{B}}\mathcal{A}}}, k_U^*G),$$

This is true by descent properties of quasi-coherent sheaves on schemes, and the fact that $U_{\mathcal{A}} \to U_{\mathcal{B}}$ is fppf.

Now let us check that $\beta(G)$ is finitely presented, and that $\alpha\circ\beta\cong \mathrm{id}$. We will use the fact that $\mathrm{FP}^{eq}(U) = \varinjlim_n \mathrm{FP}^{eq}(U_n)$ (see Proposition 2.2.36). This gives us a finitely presented equivariant sheaf on some $U_n$, whose pullback on $U$ is $G$. We also consider the fpqc stack $Y_n$ defined by the cartesian square in the diagram

$$\begin{tikzcd} U \arrow[dr, "\pi"] \arrow[drr, bend left] \arrow[ddr, bend right, "\phi"'] & & \\ & Y_n \arrow[r] \arrow[d, "h"] & U_n \arrow[d] \\ & X_\infty \arrow[r] & X_n. \end{tikzcd}$$

The fpqc stack $Y_n$ together with the morphism $h$ is an object of $\mathrm{fppf}(X_\infty)$, and on it we have a finitely presented (fpqc) sheaf $G_n$ (pulled back from $U_n$) with an isomorphism $\pi^*G_n \cong G$.

(Moreover, $U = \varprojlim_n Y_n$. This is easy to see functorially, since (locally on $X$) $Y_n$ parametrizes extensions of the DF structure together with a Kato chart, and $U$ parametrizes extensions to $P_{\mathbb{Q}}$ together with a Kato chart, and a collection of compatible Kato charts at finite levels is the same as a Kato chart for $P_{\mathbb{Q}}$.)

To see that $\alpha\circ\beta\cong\mathrm{id}$, note that $\alpha(\beta(G)) = \phi^*(\beta(G)) = \pi^*h^*(\beta(G))$ for $G\in\mathrm{FP}^{eq}(U)$, so it is enough to show that $h^*(\beta(G))\cong G_n$ as a sheaf on the fppf site of $Y_n$ (note that since $h$ is fppf, $h^*$ is just a restriction). This will also show that $\beta(G)$ is finitely presented in $(X_\infty)_{\mathrm{fppf}}$, since $Y_n\to X_\infty$ is an fppf morphism and $G_n$ is finitely presented on $Y_n$.

To check this, note that there is a natural map $a\colon G_n \to h^*(\beta(G))$, defined as follows: for $j\colon \mathcal{V}\to Y_n$ in $\mathrm{fppf}(Y_n)$, and the usual diagram

$$\begin{tikzcd} U_{\mathcal{V}} \arrow[r] \arrow[d] & U_{Y_n} \arrow[d] \\ \mathcal{V} \arrow[r] & Y_n \end{tikzcd}$$

we set

$$a(\mathcal{V})\colon G_n(\mathcal{V}) \to h^*(\beta(G))(\mathcal{V}) = \mathrm{Hom}^{eq}(\mathcal{O}_{U_{\mathcal{V}}}, p^*G_n)$$

where $p\colon U_{\mathcal{V}} \to Y_n$ is the composition $U_{\mathcal{V}} \to U_{Y_n} \to Y_n$, as the natural map sending a section of $G_n$ over $\mathcal{V}$, seen as a morphism $\mathcal{O}_{\mathcal{V}} \to G_n|_{\mathcal{V}}$, to its pullback along $U_{\mathcal{V}} \to \mathcal{V}$. The fact that this map is a bijection follows from the fact that $G_n$ is an fpqc sheaf on $Y_n$, and $U_{\mathcal{V}} \to \mathcal{V}$ is fpqc.

Finally, the fact that $\beta \circ \alpha \cong \mathrm{id}$ follows from the following lemma.

**Lemma 2.2.43.** *There are natural isomorphisms of functors $\alpha \cong i^*\colon \mathrm{FP}((X_\infty)_{\mathrm{fppf}}) \to \mathrm{FP}(X_\infty)$ and $\beta \cong i_*\colon \mathrm{FP}(X_\infty) \to \mathrm{FP}((X_\infty)_{\mathrm{fppf}})$.*

*Proof of lemma.* Note that we implicitly use the equivalence $\mathrm{FP}(X_\infty) \cong \mathrm{FP}((X_\infty)_{\mathrm{fpqc}})$ of proposition 2.2.33 restricted to finitely presented sheaves. Let us first consider $\beta$: if $G \in \mathrm{FP}^{eq}(U)$ corresponds to $\widetilde{G} \in \mathrm{FP}(X_\infty)$ and $f\colon \mathcal{A} \to X_\infty$ is fppf, then by descent for fpqc sheaves with respect to the fpqc groupoid $R_{\mathcal{A}} \rightrightarrows U_{\mathcal{A}} \to \mathcal{A}$, we have $\beta(G)(\mathcal{A}) = \widetilde{G}(\mathcal{A}) = i_*\left(\widetilde{G}\right)(\mathcal{A})$, and this gives $\beta \cong i_*$.

As for $\alpha$, let us show first that we have a morphism $a\colon i^* \to \alpha$: given $F \in \mathrm{FP}((X_\infty)_{\mathrm{fppf}})$, we have $i^*F(\mathcal{A}) = \varinjlim F(\mathcal{B})$ where the limit runs through the diagrams

$$\begin{tikzcd} \mathcal{A} \arrow[rr] \arrow[dr] & & \mathcal{B} \arrow[dl] \\ & X_\infty & \end{tikzcd}$$

and $\mathcal{B} \to X_\infty$ is fppf. Now for a fixed $\mathcal{B}$, we have a map $F(\mathcal{B}) \to \alpha(F)(\mathcal{A})$ given by pullback $\mathrm{Hom}(\mathcal{O}_{\mathcal{B}}, F|_{\mathcal{B}}) \to \mathrm{Hom}^{eq}(\mathcal{O}_{U_{\mathcal{B}}}, (\phi^*F)|_{U_{\mathcal{B}}}) \to \mathrm{Hom}^{eq}(\mathcal{O}_{U_{\mathcal{A}}}, (\phi^*F)|_{U_{\mathcal{A}}}) = \alpha(F)(\mathcal{A})$, where the last equality follows from fpqc descent. These maps are compatible with restrictions, and thus induce a map $i^*F(\mathcal{A}) \to \alpha(F)(\mathcal{A})$, and this gives the natural transformation $a$.

Now consider the pullback $\phi^*(i^*F) \to \phi^*(\alpha(F))$ to $U$ of the morphism $a(F)$ just defined. By compatibility of the pullback with the morphisms of topoi, $\phi^*(i^*F)$ is just $\phi^*F$, and on the other hand $\phi^*(\alpha(F))$ is by definition $\phi^*F$. After these identifications the morphism $\phi^*(a(F))$ is the identity, and since $i^*F$ and $\alpha(F)$ are fpqc sheaves and $\phi\colon U \to X_\infty$ is fpqc, this implies that $a(F)$ is an isomorphism. □

Now $i_* \circ i^* \cong \mathrm{id}$ is an easy check since $i_*$ is a restriction, and this concludes the proof. □

The following gives a projection formula (as in Proposition 1.2.35) for the infinite root stack.

**Proposition 2.2.44** (Projection formula for the infinite root stack)**.** *Let $X$ be a fine saturated log scheme, $A \subseteq B$ a system of denominators with $B$ saturated, and denote by $\pi\colon X_\infty \to X_{B/A}$ be the canonical projection. Then:*

- $\pi_*\colon \mathrm{QCoh}(X_\infty) \to \mathrm{QCoh}(X_{B/A})$ *is exact,*
- $\mathcal{O}_{X_{B/A}} \cong \pi_*\mathcal{O}_{X_\infty}$,
- *if $F \in \mathrm{QCoh}(X_{B/A})$ and $G \in \mathrm{QCoh}(X_\infty)$ we have a functorial isomorphism $F \otimes \pi_*G \cong \pi_*(\pi^*F \otimes G)$,*
- *consequently for $F \in \mathrm{QCoh}(X_{B/A})$ we have an isomorphism $F \cong \pi_*\pi^*F$ on $X_{B/A}$.*

*Proof.* The last bullet is consequence of the second and third.

Recall (Proposition 2.2.25) that the projection $X_{B/A} \to X$ induces an isomorphism $(X_{B/A})_\infty \to X_\infty$. Consequently, if $T \to X_{B/A}$ is a morphism from a scheme, then the following diagram is cartesian

$$\begin{tikzcd} T_\infty \arrow[r] \arrow[d] & X_\infty \arrow[d] \\ T \arrow[r] & X_{B/A} \end{tikzcd}$$

where $T$ has the pullback log structure, from $B_{X_{B/A}} \to \mathrm{Div}_{X_{B/A}}$.

After noting that we have maps $\mathcal{O}_{X_{B/A}} \to \pi_*\mathcal{O}_{X_\infty}$ and $F \otimes \pi_* G \to \pi_*(\pi^* F \otimes G)$, by flat base change we can reduce to proving the same statements for $\pi_T \colon T_\infty \to T$. After further étale shrinking on $T$ we can assume that $T$ is affine and that we have a chart $P \to \mathrm{Div}(T)$ for the log structure induced by a Kato chart.

By Proposition 2.2.14 we have an isomorphism $T_\infty = [U_\infty/G_\infty]$, where as usual $U_\infty = T \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[P_\mathbb{Q}])$ and $G_\infty = D[C_\infty]$ is the diagonalizable group Cartier dual of the cokernel $C_\infty$ of $P^{\mathrm{gp}} \to P^{\mathrm{gp}}_{\mathbb{Q}}$.

Let us prove first that $\mathcal{O}_T \to (\pi_T)_*\mathcal{O}_{T_\infty}$ is an isomorphism. Clearly it is sufficient to prove that $\mathcal{O}_T(T) \to \mathcal{O}_{T_\infty}(T_\infty)$ is a bijection, for then the same reasoning will apply to an étale morphism $S \to T$. This map coincides with the natural map $\mathrm{Hom}(T, \mathbb{A}^1) \to \mathrm{Hom}(T_\infty, \mathbb{A}^1)$ given by composition. Now note that, since $T_\infty = \varprojlim_n T_n$, we have a natural function

$$\varinjlim_n \mathrm{Hom}(T_n, \mathbb{A}^1) \to \mathrm{Hom}(T_\infty, \mathbb{A}^1)$$

which is moreover a bijection. This follows from the fact that morphisms $T_n \to \mathbb{A}^1$ are precisely morphisms $U_n \to \mathbb{A}_1$ that are $G_n$-invariant (where we are using the notation of Section 2.2.1), and the same holds for $T_\infty$. Furthermore $U_\infty = \varprojlim_n U_n$, $G_\infty = \varprojlim_n G_n$ and $\mathbb{A}^1$ is finitely presented over $k$, so $\varinjlim_n \mathrm{Hom}^{eq}(U_n, \mathbb{A}^1) = \mathrm{Hom}^{eq}(U_\infty, \mathbb{A}^1)$. Finally we have $\mathrm{Hom}(T_n, \mathbb{A}^1) = \mathrm{Hom}(T, \mathbb{A}^1)$ since $T$ is the coarse moduli space of $T_n$.

For the first and third bullets it suffices to note that $T_\infty$ is a quotient stack of an affine scheme by a diagonalizable group and in this situation pushforward corresponds to taking invariants. By the discussion in SGA3 [ABD+66], in particular Expose I Théorème 5.3.3, taking invariants is exact, and proposition 4.5 of [Alp12] implies our thesis. $\square$

As a consequence, we see that $X$ is a coarse moduli space of $X_\infty$, at least with respect to maps to schemes.

**Corollary 2.2.45.** *Let $X$ be a fs log scheme. The morphism $X_\infty \to X$ has the following property: for any map $X_\infty \to T$ to a scheme $T$, there exists a unique factorization $X_\infty \to X \to T$.*

*Proof.* Observe first of all that the morphism $\pi \colon X_\infty \to X$ is a homeomorphism. This follows from the fact that all projections from finite root stacks $X_n \to X$ are homeomorphisms, since they are coarse moduli spaces by 1.2.32.

Now if $T$ is affine, the conclusion is immediate from the fact that $\pi_*\mathcal{O}_{X_\infty} \cong \mathcal{O}_X$. The general case follows from this by covering $T$ with affines $T_i$ and considering the inverse images $\mathcal{X}_i \subseteq X_\infty$, which will come from unique open subschemes $X_i \subseteq X$. $\square$

To conclude this section, let us prove what we anticipated in Remark 2.2.24 about coherence of the infinite root stack. This is the analogue of Proposition 3.3 of [Niz08].

**Proposition 2.2.46.** *Let $X$ be a fine and saturated log scheme with a global chart $P \to \mathrm{Div}(X)$. Assume that there is a cofinal system $P \subseteq Q_i \subseteq P_{\mathbb{Q}}$ of Kummer extensions such that the transition maps $X_{Q_j/P} \to X_{Q_i/P}$ are flat for every $j \geq i$. Then the infinite root stack $X_\infty$ is coherent.*

*Proof.* We have an inverse system of flat presentations $\{U_i\}_{i\in I}$ of $X_i = X_{Q_i/P}$, such that the transition maps are affine and flat, and the inverse limit $U_\infty = \varprojlim_i U_i$ is an fpqc presentation of $X_\infty$. This implies that for any $i$ the projection $\pi_i \colon X_\infty \to X_i$ is flat. Moreover by cofinality of the subsystem $\{Q_i\}_{i\in I}$ we have $\mathrm{FP}(X_\infty) \cong \varinjlim_i \mathrm{FP}(X_i)$.

Take a finitely generated sheaf of ideals $I \subseteq \mathcal{O}_{X_\infty}$, and call $Q$ the cokernel, so we have an exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_{X_\infty} \xrightarrow{p} Q \longrightarrow 0.$$

The sheaf $Q$ on $X_\infty$ is finitely presented, so by the analogue of Lemma 2.2.36 it comes from some $Q_i \in \mathrm{FP}(X_i)$, and moreover we have a morphism $p_i \colon \mathcal{O}_{X_i} \to Q_i$ that pulls back to $p$. If we denote by $I_i$ the kernel of $p_i$, a finitely presented sheaf on $X_i$, by flatness of $\pi_i$ we have that $I \cong \pi_i^* I_i$, and from this follows that $I$ is finitely presented.

$\square$

### 2.2.3 Parabolic sheaves with rational weights

In this section we extend the BV equivalence to parabolic sheaves with arbitrary rational weights. The definitions and results are immediate generalizations of the ones of [BV12] that we recalled in Chapter 1 for finite root stacks (corresponding to "coherent" systems of denominators). We spell them out anyway for the convenience of the reader.

Let us fix a log scheme $X$ with DF structure $L \colon A \to \mathrm{Div}_X$. Assume first that there is a global chart $P \to \mathrm{Div}_X$.

As we already noted in Chapter 1, recall that $L \colon P \to \mathrm{Div}(X)$ extends to $L^{\mathrm{wt}} \colon P^{\mathrm{wt}} \to \mathrm{Pic}(X)$. For $p \in P^{\mathrm{gp}}$, we will denote $L^{\mathrm{wt}}(p)$ just by $L_p$.

**Definition 2.2.47.** A *parabolic sheaf* $(E, \rho^E)$ with rational weights on the log scheme $X$ is a functor $E \colon P^{\mathrm{wt}}_{\mathbb{Q}} \to \mathrm{QCoh}(X)$ that we denote by $a \mapsto E_a$, for $a$ an object or an arrow of $P^{\mathrm{wt}}_{\mathbb{Q}}$, with an additional datum for any $p \in P^{\mathrm{gp}}$ and $a \in P^{\mathrm{gp}}_{\mathbb{Q}}$ of an isomorphism of $\mathcal{O}_X$-modules

$$\rho^E_{p,a} \colon E_{p+a} \cong L_p \otimes E_a$$

called the pseudo-periods isomorphism.

These isomorphism are required to satisfy some compatibility conditions. Let $p, p' \in P^{\mathrm{gp}}$, $r \in P$, $q \in P_{\mathbb{Q}}$ and $a \in P^{\mathrm{gp}}_{\mathbb{Q}}$. Then the following diagrams are commutative

$$\begin{tikzcd} E_a \arrow[r, "E_r"] \arrow[d] & E_{r+a} \arrow[d, "\rho^E_{r,a}"] \\ \mathcal{O}_X \otimes E_a \arrow[r, "\sigma_r \otimes \mathrm{id}"] & L_r \otimes E_a \end{tikzcd}$$

$$\begin{tikzcd} E_{p+a} \arrow[r, "\rho^E_{p,a}"] \arrow[d, "E_q"'] & L_p \otimes E_a \arrow[d, "\mathrm{id} \otimes E_q"] \\ E_{p+q+a} \arrow[r, "\rho^E_{p,q+a}"] & L_p \otimes E_{q+a} \end{tikzcd}$$

$$\begin{tikzcd} E_{p+p'+a} \arrow[r, "\rho^E_{p+p',a}"] \arrow[d, "\rho^E_{p,p'+a}"'] & L_{p+p'} \otimes E_a \arrow[d, "\mu_{p,p'} \otimes \mathrm{id}"] \\ L_p \otimes E_{p'+a} \arrow[r, "\mathrm{id} \otimes \rho^E_{p',a}"] & L_p \otimes L_{p'} \otimes E_a, \end{tikzcd}$$

where $\mu_{p,p'} \colon L_{p+p'} \cong L_p \otimes L_{p'}$ is the natural isomorphism given by $L$, and the composite

$$E_a = E_{0+a} \xrightarrow{\rho^E_{0,a}} L_0 \otimes E_a \cong \mathcal{O}_X \otimes E_a$$

coincides with the natural isomorphism $E_a \cong \mathcal{O}_X \otimes E_a$.

A morphism of parabolic sheaves with rational weights is a natural transformation compatible with the pseudo-periods isomorphism.

As in the case of parabolic sheaves with fixed weights, the definition extends to the general case (without a global chart), where one requires the commutativity of the diagrams and compatibility of $\rho^E$ with pullback. One shows that in the presence of a global chart, the corresponding categories are equivalent (the analogous of Proposition 5.10 of [BV12].

This gives an abelian category $\mathrm{Par}(X)$ of parabolic sheaves with rational weights on $X$, with a tensor product and internal Homs.

Recall that on the infinite root stack $X_\infty$ we have a universal DF structure $L_\infty \colon \pi^* A_{\mathbb{Q}} \to \mathrm{Div}_{X_\infty}$, and by restriction to $\pi^* \frac{1}{n} A \subseteq \pi^* A_{\mathbb{Q}}$, for every $n$ we get a DF structure $L_n \colon \pi^* \frac{1}{n} A \to \mathrm{Div}_{X_\infty}$,

The following is an analogue of Theorem 6.1 in [BV12], and has the same exact proof, by using the natural DF structure of $X_\infty$.

**Proposition 2.2.48.** *There is a tensor equivalence of abelian categories* $\mathrm{Par}(X) \cong \mathrm{QCoh}(X_\infty)$.

*Proof.* See the proof of Theorem 6.1 in [BV12], or the sketch of proof in 1.3.8. □

**Proposition 2.2.49.** *Let $X$ be a log scheme with DF structure $L \colon A \to \mathrm{Div}_X$, and $j \colon A \to B$ be a system of denominators. Then pullback along $\pi \colon X_\infty \to X_{B/A}$ is fully faithful.*

*Proof.* This is proved as the corresponding statement in the case of finite root stacks. We refer to the discussion in Section 1.3.

As for the case of finite root stacks, alternatively this follows from Proposition 2.2.44, which proves that the unit of the adjunction $\pi^* \dashv \pi_*$ is an isomorphism, so $\pi^*$ is fully faithful. □

This says that parabolic sheaves with respect to some system of denominators can be seen inside the category of parabolic sheaves with arbitrary rational weights.

**Example 2.2.50.** Let us see how this happens in a simple case: assume that $X$ is a scheme and $D \subseteq X$ is an effective Cartier divisor, and consider the log structure given by $\mathbb{N} \to \mathrm{Div}(X)$ sending 1 to $(\mathcal{O}_X(D), s)$. Then a parabolic sheaf $E$ on $X_2 = X_{\frac{1}{2}\mathbb{N}/\mathbb{N}}$ is determined by three sheaves and two maps

$$
\begin{array}{ccccc}
-1 & & -\frac{1}{2} & & 0 \\
 & & & & \\
E \otimes \mathcal{O}_X(-D) & \longrightarrow & E_1 & \longrightarrow & E.
\end{array}
$$

A parabolic sheaf on $X_\infty$ is determined by a sheaf $E$ corresponding to 0 (and in $-1$ there will still be $E \otimes \mathcal{O}_X(-D)$), but this time we have a sheaf $E_q$ for any rational number $q \in (-1, 0)$, and an arrow $E_q \to E_{q'}$ every time that $q \leq q'$, and those will be compatible with compositions.

If we follow the recipe for the pullback along $\pi \colon X_\infty \to X_2$, it is easy to see that the pullback of the parabolic sheaf $E$ will have

$$
\begin{aligned}
&(\pi^* E)_0 = E \\
&(\pi^* E)_q = E \otimes \mathcal{O}_X(-D) \text{ for } -1 \leq q < -\frac{1}{2} \\
&(\pi^* E)_q = E_1 \text{ for } -\frac{1}{2} \leq q < 1
\end{aligned}
$$

and the morphisms that are not the identity are given by the maps of $E$.

This situation is particularly simple because for any given rational $q \in (-1, 0)$, the set $\{q' \in \frac{1}{2}\mathbb{Z} \mid q' \leq q\}$ has a maximum, and the direct limit in the construction of the pullback is trivial. In the case of more complicated log structures one would need to take more complicated colimits.

Note that it is clear from this description that $\mathrm{Hom}(\pi^* E, \pi^* E) = \mathrm{Hom}(E, E)$, and also that we have $\pi_* \pi^* E \cong E$, from the description of the pushforward.

In conclusion the infinite root stack allows us to interpret parabolic sheaves with arbitrary rational weights as quasi-coherent sheaves.

**Remark 2.2.51.** There are definition of parabolic sheaves with arbitrary rational weights in the literature (for example in [Bor09]), but as far as we know they all assume that the parabolic sheaf is completely determined by a finite set of rational numbers (as in the example above), and in our definition this might not happen.

In fact in some situations (for example on a variety with a simple normal crossings divisor) the sheaves that are determined by a finite number of (finitely presented) pieces and with a "semicontinuity from the left" are exactly the ones that are finitely presented, and they come from a finitely presented sheaf on some finite root stack $X_n$.

## 2.3 The infinite root stack determines the log scheme

In the last two sections of this chapter we will consider the following question:

**Question 2.3.1.** what can we deduce about the fs log scheme $X$ from its infinite root stack $X_\infty$?

In the present section we will describe a reconstruction procedure that gives back the log structure of $X$ in terms of the infinite root stack $X_\infty$, and we will apply this to show that two log schemes $X$ and $Y$ having isomorphic infinite root stacks must be isomorphic themselves. As an application we will also give conditions for a morphism between two infinite root stacks $X_\infty \to Y_\infty$ to come from a morphism of log schemes $X \to Y$. In the next section (2.4) we will show that the Kummer-flat topos of a log scheme can be identified with the fppf topos of its infinite root stack.

To investigate these questions it is natural to consider the functor $(-)_\infty \colon (\mathrm{FSLogSch}) \to (\mathrm{St})$ that associates to an fs log scheme $X$ its infinite root stack $X_\infty$. A natural way to show that log schemes with isomorphic infinite root stacks must be isomorphic would be to show that this functor is fully faithful. Although we will show that it is faithful, it is not true that it is full.

**Example 2.3.2.** Consider the standard log point $X = \mathrm{Spec}(k)$ with $k$ algebraically closed and DF structure $L \colon \mathbb{N} \to \mathrm{Div}(k)$ sending 1 to 0. Then it is easy to see that the monoid of endomorphisms $X \to X$ of log schemes over $k$ is a semidirect product of $\mathbb{N}$ and $k^\times$, where $n \in \mathbb{N}$ acts by raising to the $n$-th power. The component in $\mathbb{N}$ of an endomorphism gives the homomorphism $\mathbb{N} \to \mathbb{N}$, the component in $k^\times$ corresponds to the natural isomorphism between the two functors $\mathbb{N} \to \mathrm{Div}(k)$.

Moreover, we saw in Example 2.2.15 that the infinite root stack can be described as the quotient

$$X_\infty \cong \left[\mathrm{Spec}(k[t^{\frac{1}{n}} \mid n \in \mathbb{N}]/(t))/\mu_\infty\right].$$

This is of course non-reduced, and its reduction $(X_\infty)_{\mathrm{red}} \subseteq X_\infty$ is the quotient $[\mathrm{Spec}(k)/\mu_\infty] = B\mu_\infty$. One checks that the morphism $X \to X$ corresponding to $(n, a)$, where $n \in \mathbb{N}$ and $a \in k^\times$, induces the morphism $X_\infty \to X_\infty$ determined by $t^{\frac{1}{m}} \mapsto a^{\frac{1}{m}} t^{\frac{n}{m}}$, where we chose a compatible system of $n$-th roots of $a \in k^\times$ (a different choice yields the same map, up to isomorphism). Note that this kills all the $t^{\frac{1}{k}}$ with $k \leq n$ (since $t = 0$), but the ones with $k > n$ are not killed.

Now note that the inclusion $k \subseteq k[t^{\frac{1}{n}} \mid n \in \mathbb{N}]$ induces a morphism $X_\infty \to (X_\infty)_{\mathrm{red}}$ and the composition $X_\infty \to (X_\infty)_{\mathrm{red}} \subseteq X_\infty$ does not come from a morphism $X \to X$, since all $t^{\frac{1}{n}}$ are killed.

We give a name to morphisms of root stacks coming from a morphism between the log schemes.

**Definition 2.3.3.** Let $X$ and $Y$ be fs log schemes. We say that a morphism $\phi \colon X_\infty \to Y_\infty$ is *logarithmic* if there exists a morphism of log schemes $f \colon X \to Y$ such that $\phi \cong f_\infty \colon X_\infty \to Y_\infty$.

The morphism constructed in the example above is not logarithmic. We will give a characterization of logarithmic morphisms (Proposition 2.3.20), which will generalize in some sense Example 2.3.2. This characterization will imply in particular that isomorphisms $X_\infty \cong Y_\infty$ do come from morphisms $X \to Y$ (that have to be isomorphisms themselves).

### 2.3.1 Recovering the log structure

We will now describe the reconstruction process that will let us recover the log structure from the infinite root stack. From now on for a while we will focus on DF structures on a single scheme

$X$. Because of this, in the discussion that follows we will use the notation $\sqrt[\infty]{(A, L)}$ for the infinite root stack of the log scheme $X$ with DF structure $L\colon A \to \mathrm{Div}_X$.

Let us first give an abstract definition of an "infinite root stack" over $X$.

**Definition 2.3.4.** An *infinite root stack* over a scheme $X$ is a stack $\mathcal{X}$ on $X_{\text{ét}}$ with a morphism $\mathcal{X} \to X$, that étale locally on $X$ is the infinite root stack of some fs DF structure.

More precisely, there is an étale covering $\{U_i \to X\}_{i \in I}$ and fs monoids $\{P_i\}_{i \in I}$ with morphisms $U_i \to \mathrm{Spec}(k[P_i])$ and an isomorphism

$$\mathcal{X} \times_X U_i \cong (\mathrm{Spec}(k[P]))_\infty \times_X U_i$$

over $U_i$.

An infinite root stack over $X$ is an fpqc stack, in the sense of Definition 2.2.20. Of course, if $(A, L)$ is a DF structure on $X$, then $\sqrt[\infty]{(A, L)}$ is an infinite root stack on $X$. Moreover, we will show that every infinite root stack is of this form (Theorem 2.3.11).

Let us explain how to construct a DF structure on $X$, starting from an infinite root stack $\mathcal{X} \to X$.

If $\mathcal{X}$ is an infinite root stack over $X$, we will use the notation $\mathrm{Div}_{\mathcal{X}_{\text{ét}}}$ for the symmetric monoidal fibered category over $X_{\text{ét}}$ whose objects over $U \to X$ are the objects of $\mathrm{Div}(\mathcal{X}_U)$, where $\mathcal{X}_U$ denotes the fibered product $\mathcal{X} \times_X U$.

**Definition 2.3.5.** Let $\pi\colon \mathcal{X} \to X$ be an infinite root stack. Consider the symmetric monoidal fibered category $\mathcal{A}_{\mathcal{X}} \to X_{\text{ét}}$ defined as follows. For each étale map $U \to X$, the objects of $\mathcal{A}_{\mathcal{X}}(U)$ are of the form $(\Lambda, \Lambda_n, \phi, \alpha_{m,n})$, where:

(a) $\Lambda$ is an object of $\mathrm{Div}(U)$.

(b) For each positive integer $n$, $\Lambda_n$ is an object of $\mathrm{Div}(\mathcal{X}_U)$.

(c) $\phi\colon \Lambda_1 \cong \pi^*\Lambda$ is an isomorphism in $\mathrm{Div}(\mathcal{X}_U)$.

(d) For each $m \mid n$, $\alpha_{m,n}\colon \Lambda_n^{\otimes(n/m)} \cong \Lambda_m$ is an isomorphism in $\mathrm{Div}(\mathcal{X}_U)$.

(e) Suppose that $p$ is a point of $X$; denote by $\mathcal{X}_p$ the fiber of $\mathcal{X}$ over $p$. If $n$ is sufficiently divisible and $\Lambda_n = (L_n, s_n)$, then the restriction of $s_n$ to $\mathcal{X}_p$ is nonzero.

We require the isomorphisms $\alpha_{m,n}$ to be subject to the following compatibility conditions.

(i) $\alpha_{n,n} = \mathrm{id}_{\Lambda_n}$ for any $n$.

(ii) if $m \mid n$ and $n \mid p$, then

$$\alpha_{m,p} = \alpha_{m,n}^{\otimes(p/n)} \circ \alpha_{n,p}\colon \Lambda_p^{\otimes(p/n)} \cong \Lambda_m.$$

The arrows $(\Lambda, \Lambda_n, \phi, \alpha_{m,n}) \to (\Lambda', \Lambda'_n, \phi', \alpha'_{m,n})$ are given by isomorphisms $\Lambda \cong \Lambda'$ and $\Lambda_n \cong \Lambda'_n$ compatible with the $\phi$'s and the $\alpha_{m,n}$'s. The fibered structure is obtained from the evident pseudo-functor structure.

We call the objects of $\mathcal{A}_{\mathcal{X}}(U)$ *infinite roots*.

We will see that this fibered category gives a DF structure on $X$, and that if we started from an infinite root stack of a DF structure, we get back the original DF structure. In the rest of this section we sketch how the proof works. Some of the following statement will be proved later, after we discussed the notion of an infinite root in a monoid.

**Lemma 2.3.6.** *$(\Lambda, \Lambda_n, \phi, \alpha_{m,n})$ and $(\Lambda', \Lambda'_n, \phi', \alpha'_{m,n})$ be infinite roots in an infinite root stack $\mathcal{X}$. Then the tensor product*

$$(\Lambda \otimes \Lambda', \Lambda_n \otimes \Lambda'_n, \phi \otimes \phi', \alpha_{m,n} \otimes \alpha'_{m,n})$$

*is also an infinite root.*

*Proof.* Here this essential point is to show that if we set $\Lambda_n = (L_n, s_n)$ and $\Lambda'_n = (L'_n, s'_n)$, then the restriction of $s_n \otimes s'_n$ to any geometric fiber is nonzero for sufficiently divisible $n$. This follows from the second statement in Lemma 2.3.18 below. □

This gives $\mathcal{A}_{\mathcal{X}}$ a symmetric monoidal structure by tensor product.

**Proposition 2.3.7.** *Let $\mathcal{X} \to X$ be an infinite root stack. Then the symmetric monoidal category $\mathcal{A}_{\mathcal{X}}$ is fibered in equivalence relations.*

Hence by dividing by isomorphism we obtain a sheaf of monoids on $X_{\text{ét}}$, call it $A_{\mathcal{X}} \colon X_{\text{ét}} \to (\text{CommMon})$, and the projection $\mathcal{A}_{\mathcal{X}} \to A_{\mathcal{X}}$ is an equivalence. By choosing a symmetric monoidal quasi-inverse $A_{\mathcal{X}} \to \mathcal{A}_{\mathcal{X}}$ and composing with the obvious symmetric monoidal functor $\mathcal{A}_{\mathcal{X}} \to \text{Div}_X$ that sends $(\Lambda, \Lambda_n, \phi, \alpha_{m,n})$ to $\Lambda$, we obtain a symmetric monoidal functor $L_{\mathcal{X}} \colon A_{\mathcal{X}} \to \text{Div}_X$, unique up to a unique isomorphism.

**Proposition 2.3.8.** *The sheaf $A_{\mathcal{X}}$ is fine saturated.*

*Proof.* Since being fine saturated is a local condition in the étale topology, and étale-locally $\mathcal{X}$ comes from a fine saturated DF structure, this follows from Proposition 2.3.9 below. □

Hence, from an infinite root stack $\mathcal{X} \to X$ we obtain a fine saturated DF structure $(A_{\mathcal{X}}, L_{\mathcal{X}})$.

Now suppose that $(A, L)$ is a DF structure on $X$, and $\mathcal{X} = \sqrt[\infty]{(A, L)}$. Recall moreover that there exist a symmetric monoidal functor $\widetilde{L} \colon A_{\mathbb{Q}} \to \text{Div}_{\mathcal{X}_{\text{ét}}}$, and an isomorphism of symmetric monoidal functors between the restriction of $\widetilde{L}$ to $A$, and the composite of $L$ with the pullback $\text{Div}_X \to \text{Div}_{\mathcal{X}_{\text{ét}}}$. We will describe how to get a morphism $A \to \mathcal{A}_{\mathcal{X}}$ of symmetric monoidal categories on $X_{\text{ét}}$.

Let $U \to X$ be an étale map, and $a \in A(U)$. Then we obtain an object $L(a)$ of $\text{Div}(U)$. Furthermore, for each positive integer $n$ we also obtain an object $\widetilde{L}(a/n) \in \text{Div}(\mathcal{X}_U)$. The fact that the functor is symmetric monoidal gives, for each $m \mid n$, isomorphisms $\alpha_{m,n} \colon \widetilde{L}(a/n)^{\otimes(n/m)} \cong \widetilde{L}(a/m)$ in $\text{Div}(\mathcal{X}_U)$. Furthermore, the isomorphism between the restriction of $L$ to $A$, and the composite of $L$ with the pullback $\text{Div}(U) \to \text{Div}(\mathcal{X}_U)$ yields an isomorphism $\phi \colon \Lambda_1 = \widetilde{L}(a) \cong L(a)$. This gives a symmetric monoidal functor $A \to \mathcal{A}_{\mathcal{X}}$; by definition, the composite of $A \to \mathcal{A}_{\mathcal{X}}$ with $\mathcal{A}_{\mathcal{X}} \to \text{Div}_X$ is precisely $L$.

**Proposition 2.3.9.** *Suppose that $(A, L)$ is a fine saturated DF structure on $X$, and set $\mathcal{X} = \sqrt[\infty]{(A, L)}$. Then the composite $A \to \mathcal{A}_{\mathcal{X}} \to A_{\mathcal{X}}$ is an isomorphism.*

**Corollary 2.3.10.** *The DF structures $(A, L)$ and $(A_{\mathcal{X}}, L_{\mathcal{X}})$ are isomorphic.*

Now conversely, let us show how to compare the infinite root stack $\sqrt[\infty]{(\mathcal{A}_\mathcal{X}, L_\mathcal{X})}$ of the DF structure $(\mathcal{A}_\mathcal{X}, L_\mathcal{X})$ with $\mathcal{X}$ itself.

Given an infinite root stack $\mathcal{X}$, let us produce a functor $\mathcal{X} \to \sqrt[\infty]{(A_\mathcal{X}, L_\mathcal{X})}$. Let $f\colon T \to \mathcal{X}$ be a morphism; we need to construct a morphism $T \to \sqrt[\infty]{(A, L)}$, that is, an extension $(f^*A)_\mathbb{Q} \to \mathrm{Div}(T)$ of the DF structure $f^*L\colon f^*A \to \mathrm{Div}_T$. Call $f^{-1}A$ the pullback presheaf on $T_{\text{ét}}$; its sections on an étale map $V \to T$ are colimits $\varinjlim A(U)$, where the colimit is taken over all factorizations $V \to U \to X$, with $U \to X$ étale, of the composite $V \to T \to \mathcal{X} \to X$. The sheafification of the presheaf $B_T$ on $T_{\text{ét}}$ sending $V$ to $(f^{-1}A)(V) \otimes \mathbb{Q}$ is the sheaf $(f^*A)_\mathbb{Q}$; by [BV12, Proposition 3.3], every symmetric monoidal functor $B_T$ extends uniquely to a symmetric monoidal functor $(f^*A)_\mathbb{Q} \to \mathrm{Div}(T)$.

Consider the filtered category $I_V$ defines as follows. The objects are pairs $(m, V \to U \to X)$, where $m$ is a positive integer and $V \to U \to X$ is a factorization of the composite $V \to T \to \mathcal{X} \to X$, with $U \to X$ étale. An arrow $\phi\colon (m, V \to U \to X) \to (n, V \to U' \to X)$ exists only when $m \mid n$, in which case it consists of a morphism $\phi\colon U \to U'$ such that the diagram

$$\begin{tikzcd} & U \arrow[dd, "\phi"] \arrow[dr] & \\ V \arrow[ur] \arrow[dr] & & X \\ & U' \arrow[ur] & \end{tikzcd}$$

commutes. Composition is the obvious one.

There is a lax 2-functor from $I_V^{\mathrm{op}}$ into the 2-category of symmetric monoidal categories, sending each $(m, V \to U \to X)$ into $\mathcal{A}_\mathcal{X}(U)$, and each morphism $\phi\colon (m, V \to U \to X) \to (n, V \to U' \to X)$ into the composite of the pullback $\phi^*\colon \mathcal{A}_\mathcal{X}(U') \to \mathcal{A}_\mathcal{X}(U)$ with the functor $\mathcal{A}_\mathcal{X}(U) \to \mathcal{A}_\mathcal{X}(U)$ given by raising to the $(n/m)^{\text{th}}$ power. We have a canonical equivalence of symmetric monoidal categories between $\varinjlim_{I_V} \mathcal{A}_\mathcal{X}(U)$ and the monoid $B_T(V)$.

There is also a symmetric monoidal functor $\varinjlim_{I_V} \mathcal{A}_\mathcal{X}(U) \to \mathrm{Div}(V)$ that sends an object $(\Lambda, \Lambda_n, \phi, \alpha_{m,n})$ over $(m, V \to U \to X)$ to $h^*\Lambda_m$, where $h\colon V \to \mathcal{X}_U$ is the morphism induced by $V \to U$ and the composite $V \to T \to \mathcal{X}$. By composing this with a quasi-inverse of the equivalence $\varinjlim_{I_V} \mathcal{A}_\mathcal{X}(U) \to B_T(V)$ we obtain a symmetric monoidal functor $B_T(V) \to \mathrm{Div}(V)$. This induces the desired symmetric monoidal functor $B_T \to \mathrm{Div}(T)$.

**Theorem 2.3.11.** *The resulting functor $\mathcal{X} \to \sqrt[\infty]{(A_\mathcal{X}, L_\mathcal{X})}$ is an equivalence.*

*Proof.* The statement is local in the étale topology on $X$, so we may assume that $\mathcal{X} = \sqrt[\infty]{(A, L)}$ for a DF structure $L\colon A \to \mathrm{Div}_X$. In this case the result follows immediately from Proposition 2.3.9. □

To prove these facts we need to study the corresponding notion of an infinite root for a monoid $P$.

### 2.3.2 Infinite quotients in sharp fine saturated monoids

Let $P$ be a sharp fine saturated monoid, and assume that $P^{\mathrm{gp}}$ has rank $r$. In other words we have $P^{\mathrm{gp}} \cong \mathbb{Z}^r$, and consequently $P^{\mathrm{gp}}_\mathbb{Q} \cong \mathbb{Q}^r$. Moreover $P^{\mathrm{gp}}_\mathbb{Q}/P^{\mathrm{gp}} = P^{\mathrm{gp}} \otimes (\mathbb{Q}/\mathbb{Z})$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^r$. We consider $P^{\mathrm{gp}}_\mathbb{Q}$ as a topological space via the usual metric topology on $\mathbb{Q}^r$.

Set

$$\breve{P} = \varprojlim_n (P^{\mathrm{gp}}_{\mathbb{Q}}/P^{\mathrm{gp}})[n] = P^{\mathrm{gp}} \otimes \widehat{\mathbb{Z}} \cong \widehat{\mathbb{Z}}^r,$$

where the square brackets denote the $n$-torsion, the map $(P^{\mathrm{gp}}_{\mathbb{Q}}/P^{\mathrm{gp}})[n] \to (P^{\mathrm{gp}}_{\mathbb{Q}}/P^{\mathrm{gp}})[m]$ for $m \mid n$ is given by multiplication by $n/m$, and by $\widehat{\mathbb{Z}}$ we denote the profinite completion of $\mathbb{Z}$. An element of $\breve{P}$ consists of a collection $\{\lambda_n\}$ of elements of $P^{\mathrm{gp}}_{\mathbb{Q}}/P^{\mathrm{gp}}$ such that $\lambda_1 = 0$, and $(n/m)\lambda_n = \lambda_m$ whenever $m \mid n$.

Set

$$\Delta_P = P_{\mathbb{Q}} \setminus (P^+ + P_{\mathbb{Q}}),$$

where recall that $P^+ = P \setminus \{0\}$. Since $\Delta_P$ is the complement of an ideal in $P_{\mathbb{Q}}$, it has the property that if $\gamma, \delta \in P_{\mathbb{Q}}$ and $\gamma + \delta \in \Delta_P$, then $\gamma$ and $\delta$ are in $\Delta_P$. Hence, if $n$ is a positive integer, $\gamma \in P^{\mathrm{gp}}_{\mathbb{Q}}$, and $n\gamma \in \Delta_P$, then $\gamma \in \Delta_P$.

The set $\Delta_P$ is clearly bounded in $P^{\mathrm{gp}}_{\mathbb{Q}}$. Also, if $v_1, \dots\, v_m$ are the indecomposable elements of $P$ we have

$$\Delta_P = P_{\mathbb{Q}} \setminus \bigcup_{i=1}^{m} (v_i + P_{\mathbb{Q}});$$

since $P_{\mathbb{Q}}$ is closed in $P^{\mathrm{gp}}_{\mathbb{Q}}$, we have that $\Delta_P$ is open in $P_{\mathbb{Q}}$.

We set

$$\Delta^0_P = \{\gamma \in \Delta_P \mid (\gamma + P^{\mathrm{gp}}) \cap \Delta_P = \{\gamma\}\}.$$

By definition, the restriction of the projection $P^{\mathrm{gp}}_{\mathbb{Q}} \to P^{\mathrm{gp}}_{\mathbb{Q}}/P^{\mathrm{gp}}$ to $\Delta^0_P$ is injective.

**Lemma 2.3.12.** *The set $\Delta^0_P$ is a neighborhood of $0$ in $P_{\mathbb{Q}}$.*

*Proof.* It is easy to see that $0 \in \Delta^0_P$. Also, we have

$$\Delta^0_P = \bigcap_{\gamma \in P^{\mathrm{gp}} \setminus \{0\}} \big(\Delta_P \setminus (\gamma + \Delta_P)\big);$$

but $\Delta_P$ is bounded, so there exists a finite number of $\gamma \in P^{\mathrm{gp}} \setminus \{0\}$ such that $\Delta_P \cap (\gamma + \Delta_P) \neq \varnothing$. So it is enough to prove that $\Delta_P \setminus (\gamma + \Delta_P)$ is neighborhood of 0 in $P_{\mathbb{Q}}$ for all $\gamma \in P^{\mathrm{gp}} \setminus \{0\}$.

If $\gamma \in -P_{\mathbb{Q}}$ we have

$$\Delta_P \setminus (\gamma + \Delta_P) = \gamma + \big(\Delta_P \setminus (-\gamma + \Delta_P)\big) = \varnothing.$$

Otherwise, we have $0 \notin \gamma + P_{\mathbb{Q}}$, so $\Delta_P \setminus (\gamma + P_{\mathbb{Q}})$ is neighborhood of 0 in $P_{\mathbb{Q}}$, and $\Delta_P \setminus (\gamma + P_{\mathbb{Q}}) \subset \Delta_P \setminus (\gamma + \Delta_P)$. This finishes the proof. ☐

There is a group homomorphism $P^{\mathrm{gp}} \to \breve{P}$ sending each $p \in P^{\mathrm{gp}}$ into the element $p/\infty = \{[p/n]\} \in \breve{P}$. This is easily seen to be injective. Consider the restriction $P \to \breve{P}$.

We need to recognize elements in $\breve{P}$ that come from $P$. To do so, we introduce the following definition.

**Definition 2.3.13.** Let $\{\lambda_n\}$ be an element of $\breve{P}$. A *determination function for* $\{\lambda_n\}$ is a function $\Phi\colon \mathbb{N}^+ \to \{0, 1\}$ with the following properties

(a) If $m \mid n$, then $\Phi(m) \leq \Phi(n)$.

(b) $\Phi(m) = 1$ for some $m \in \mathbb{N}^+$.

(c) For every positive integer $m$ the following holds: assume there exists a positive integer $k$ and a sequence $\gamma_1, \ldots, \gamma_k$ of elements of $\lambda_{km} \cap \Delta_P$ such that we have $\gamma_1 + \cdots + \gamma_k \notin \Delta_P$. Then $\Phi(m) = 0$.

An element $\{\lambda_n\}$ of $\breve{P}$ is *an infinite quotient* if it admits a determination function. We denote the set of infinite quotients in $P$ by $P/\infty$.

To motivate this definition let us note that the image $p/\infty = \{[p/n]\}$ of an element $p \in P$ has a determination function. Let us define $\Phi$ by

$$n \longmapsto \begin{cases} 1 & \text{if } p/n \in \Delta_P^0 \\ 0 & \text{if } p/n \notin \Delta_P^0 . \end{cases}$$

Let us check that this is a determination function. The first two conditions are immediate, and for the third one, note that if $\Phi(m) = 1$, then $p/km \in \Delta_P^0$ for any $k$. Consequently if we take a sequence $\gamma_1, \ldots, \gamma_k$ of elements of $[p/km] \cap \Delta_P$, we will necessarily have $\gamma_i = p/km$ for all $i$ (basically by definition of $\Delta_P^0$), and the sum $\gamma_1 + \cdots + \gamma_k$ will be $p/m \in \Delta_P^0 \subseteq \Delta_P$. This shows that also the third condition is satisfied.

The following proposition says in particular that the converse holds, i.e. infinite quotients in $\breve{P}$ correspond exactly to elements of $P$.

**Proposition 2.3.14.**

(a) *Let $\{\lambda_n\}$ be an infinite quotient in $P$. For every sufficiently divisible $n$ we have $\lambda_n = [\gamma_n]$ for some $\gamma_n \in \Delta_P^0$.*

(b) *Let $\{\lambda_n\}$ and $\{\lambda'_n\}$ be infinite quotients in $P$. For every sufficiently divisible $n$ we have $\lambda_n = [\gamma_n]$ and $\lambda'_n = [\gamma'_n]$ with $\gamma_n + \gamma'_n \in \Delta_P^0$.*

(c) *The image of $P$ in $\breve{P}$ is precisely $P/\infty$.*

Thus $P/\infty$ is a submonoid of $\breve{P}$, which is isomorphic to $P$.

*Proof.* Let us show that there is a norm $|-|$ on $P_{\mathbb{Q}}^{\mathrm{gp}}$ with the property that $|\gamma + \delta| = |\gamma| + |\delta|$ for any $\gamma$ and $\delta$ in $P_{\mathbb{Q}}$. For this, notice that there is basis $v_1, \ldots\ v_r$ of $P_{\mathbb{Q}}^{\mathrm{gp}} \cong \mathbb{Q}^r$ with the property that every vector in $P_{\mathbb{Q}}$ has non-negative coordinates (in fact, since $P$ is sharp the cone in the dual space $(P_{\mathbb{Q}}^{\mathrm{gp}})^\vee$ that is dual to $P_{\mathbb{Q}}$ has nonempty interior, so it contains a basis of $P_{\mathbb{Q}}^\vee$, and the dual basis in $P_{\mathbb{Q}}^{\mathrm{gp}}$ has this property). Then the norm $|x_1 v_1 + \cdots + x_r v_r| = |x_1| + \cdots + |x_n|$ has this property.

Now choose a positive integer $m$ such that $\Phi(m) = 1$ and a positive real number $\epsilon$ such that every $\gamma \in P_{\mathbb{Q}}$ with $|\gamma| \leq \epsilon$ is in $\Delta_P^0$. Since $\Delta_P$ is bounded in $P_{\mathbb{Q}}^{\mathrm{gp}}$, choose $N > 0$ with the property that $N\epsilon$ is larger than the diameter of $\Delta_P$. If $n$ is divisible by $m$ and $n/m > N$, then we claim that $|\gamma_n| \leq \epsilon$ (where $\gamma_n \in \Delta_P$ is such that $[\gamma_n] = \lambda_n$), so that $\gamma_n \in \Delta_P^0$, which will conclude the proof of part (a).

In fact if $|\gamma_n| > \epsilon$, then we can take $k = n/m$ and the sequence $\gamma_n, \dots, \gamma_n$ of $k$ copies of $\gamma_n \in \lambda_{km} \cap \Delta_P$. For the sum of these elements we have $|k\gamma_n| > k\epsilon > N\epsilon$, and consequently $k\gamma_n \notin \Delta_P$. This contradicts the third condition in the definition of a determination function and the fact that we chose $m$ to satisfy $\Phi(m) = 1$.

By a similar reasoning and by choosing $\epsilon$ such that every $\gamma \in P_{\mathbb{Q}}$ with $|\gamma| \leq 2\epsilon$ is in $\Delta_P^0$, we see that (b) holds.

For (c), for any $p \in P$ we already gave a determination function for $p/\infty$, and so $p/\infty \in P/\infty$.

Conversely, suppose that $\{\lambda_n\} \in P/\infty$, and fix a determination function $\Phi\colon \mathbb{N}^+ \to \{0,1\}$. Choose $m$ such that $\Phi(m) = 1$ and $\lambda_{km} = [\gamma_{km}]$ with $\gamma_{km} \in \Delta_P^0$ for all $k$ (this is possible by the first part of the proof). For every positive integer $k$ we have $k\gamma_{km} = \gamma_m + p_k$ for some $p_k \in P^{\mathrm{gp}}$. If $p_k \neq 0$ we would have $\gamma_m + p_k \notin \Delta_P$, because $\gamma_m \in \Delta_P^0$; but this implies $\Phi(m) = 0$, by the third condition in the definition of a determination function. Hence $\gamma_{km} = \gamma_m/k$ for all $k$. Since $q = m\gamma_m \in P$ we have $\gamma_n = q/n$ for all $n$ divisible by $m$, which implies that this is true for all $n$, so that $\{\lambda_n\} = q/\infty$. □

### 2.3.3 Picard groups of infinite root stacks over geometric points

Next we need some results on the Picard group of an infinite root stack over a geometric point. We will show that it can be identified with the quotient $P_{\mathbb{Q}}^{\mathrm{gp}}/P^{\mathrm{gp}}$, where $P$ is the stalk of the sheaf $A$ at the point.

If $k$ is a field, we will denote by $\sqrt[\infty]{P/k}$ the infinite root stack of the DF structure $(A, L)$ on $\operatorname{Spec}(k)$, where $A$ is the constant sheaf of monoids on $(\operatorname{Spec}(k))_{\text{ét}}$ corresponding to $P$, and $L\colon A \to \operatorname{Div}_{\operatorname{Spec}(k)}$ corresponds to the homomorphism $\Lambda\colon P \to k$ that sends 0 into 1 and everything else into 0.

In other words, $\sqrt[\infty]{P/k}$ is the fiber product $\operatorname{Spec}(k) \times_{\operatorname{Spec}(k[P])} \bigl[\operatorname{Spec}(k[P_{\mathbb{Q}}])/\mu_\infty(P)\bigr]$, where $\operatorname{Spec}(k) \to \operatorname{Spec}(k[P])$ corresponds to the ring homomorphism $k[P] \to k$ determined by $\Lambda$. Or, again, we have

$$\sqrt[\infty]{P/k} = \bigl[\operatorname{Spec}(k[P_{\mathbb{Q}}]/(P^+))/\mu_\infty(P)\bigr]$$

where the action of $\mu_\infty(P)$ on $\operatorname{Spec}(k[P_{\mathbb{Q}}]/(P^+))$ is determined by the natural $P_{\mathbb{Q}}^{\mathrm{gp}}/P^{\mathrm{gp}}$-grading on $k[P_{\mathbb{Q}}]/(P^+)$. Moreover the reduced substack $(\sqrt[\infty]{P/k})_{\mathrm{red}}$ is the classifying stack $\mathcal{B}_k\mu_\infty(P) = \bigl[\operatorname{Spec}(k)/\mu_\infty(P)\bigr]$.

Set $R = k[P_{\mathbb{Q}}]/(P^+)$. This is a $P_{\mathbb{Q}}^{\mathrm{gp}}$-graded algebra. If $\gamma \in P_{\mathbb{Q}}^{\mathrm{gp}}$, we have $\dim_k R_\gamma = 0$ if $\gamma \notin \Delta_P$, and $\dim_k R_\gamma = 1$ if $\gamma \in \Delta_P$. We will use the induced $P_{\mathbb{Q}}^{\mathrm{gp}}/P^{\mathrm{gp}}$-grading, so that for any $\lambda \in P_{\mathbb{Q}}^{\mathrm{gp}}/P^{\mathrm{gp}}$ we have $R_\lambda = \oplus_{\gamma \in \lambda \cap \Delta_P} R_\gamma$.

Invertible sheaves on $\sqrt[\infty]{P/k}$ correspond to $P_{\mathbb{Q}}^{\mathrm{gp}}/P^{\mathrm{gp}}$-graded invertible modules on $R$; this gives a very concrete description of $\operatorname{Pic}(\sqrt[\infty]{P/k})$. There is a natural homomorphism

$$P_{\mathbb{Q}}^{\mathrm{gp}}/P^{\mathrm{gp}} = \operatorname{Hom}\bigl(\mu_\infty(P), \mathbb{G}_{\mathrm{m}}\bigr) \longrightarrow \operatorname{Pic}(\sqrt[\infty]{P/k})$$

that sends $\gamma \in P_{\mathbb{Q}}^{\mathrm{gp}}/P^{\mathrm{gp}}$ into the graded $R$-module $R(\lambda)$, where $R(\lambda) = R$ as an $R$-module, but the $P_{\mathbb{Q}}^{\mathrm{gp}}/P^{\mathrm{gp}}$-grading is defined by $R(\lambda)_\mu = R_{\lambda+\mu}$.

Since $R$ is the inductive limit of the local artinian rings $k[\frac{1}{n}P]/(P^+)$, every invertible module on $R$ is trivial; hence, every $P_{\mathbb{Q}}^{\mathrm{gp}}/P^{\mathrm{gp}}$-graded invertible module on $R$ is of the form $R(\lambda)$ for

$\lambda \in P^{\mathrm{gp}}_{\mathbb{Q}}/P^{\mathrm{gp}}$. So the homomorphism above is surjective. Since $R(\lambda) \otimes_R k = k(\gamma)$ is a $P^{\mathrm{gp}}_{\mathbb{Q}}/P^{\mathrm{gp}}$-graded vector space, we see that $R(\lambda) \cong R(\mu)$ if and only if $\lambda = \mu$, and the homomorphism is also injective.

Let us record this in a lemma.

**Lemma 2.3.15.** *The natural homomorphism*

$$P^{\mathrm{gp}}_{\mathbb{Q}}/P^{\mathrm{gp}} = \mathrm{Hom}\big(\mu_\infty(P), \mathbb{G}_{\mathrm{m}}\big) \longrightarrow \mathrm{Pic}(\sqrt[\infty]{P/k})$$

*is an isomorphism.*

Furthermore, if $\lambda = [L] \in \mathrm{Pic}(\sqrt[\infty]{P/k}) = P^{\mathrm{gp}}_{\mathbb{Q}}/P^{\mathrm{gp}}$, we have $\mathrm{H}^0(\sqrt[\infty]{P/k}, L) = R(\lambda)_0 = R_\lambda$. So $\dim_k \mathrm{H}^0(\sqrt[\infty]{P/k}, L) = \sharp(\lambda \cap \Delta_P)$.

Let $(\Lambda, \Lambda_n, \phi, \alpha_{m,n})$ and $(\Lambda', \Lambda'_n, \phi', \alpha'_{m,n})$ be infinite roots on $\sqrt[\infty]{P/k}$, and set $\Lambda_n = (L_n, s_n)$ and $\Lambda'_n = (L'_n, s'_n)$. The following will be used later.

**Lemma 2.3.16.** *For sufficiently divisible $n$, we have* $\dim_k \mathrm{H}^0(\sqrt[\infty]{P/k}, L_n) = 1$, *and the multiplication map*

$$\mathrm{H}^0(\sqrt[\infty]{P/k}, L_n) \otimes_k \mathrm{H}^0(\sqrt[\infty]{P/k}, L'_n) \longrightarrow \mathrm{H}^0(\sqrt[\infty]{P/k}, L_n \otimes L'_n)$$

*is an isomorphism.*

*Proof.* This follows from 2.3.14(a) and (b). □

### 2.3.4 Proofs

**Lemma 2.3.17.** *Let $\pi\colon \mathcal{X} \to X$ be an infinite root stack. Assume that $X$ is locally noetherian, and let $F$ be a finitely presented sheaf on $\mathcal{X}$. Then $\pi_* F$ is coherent.*

*Proof.* The statement is local in the étale topology on $X$, so we may assume that $X = \mathrm{Spec}(A)$, and that $\mathcal{X}$ is an infinite root stack coming from a DF structure endowed with a Kato chart $L\colon P \to A$. For each $n > 0$ we have a factorization

$$\sqrt[\infty]{(P, L)} \xrightarrow{\ \rho\ } \sqrt[n]{(P, L)} \xrightarrow{\ \phi\ } X\,,$$

where $\sqrt[n]{(P, L)}$ denotes the $n$-th root stack.

The sheaf $F$ is finitely presented, so for some $n$ there exists a finitely presented sheaf $G$ on $\sqrt[n]{(P, L)}$ and an isomorphism $F \cong \rho^* G$. Since $\sqrt[\infty]{(P, L)}$ is fppf locally an infinite root stack over $\sqrt[n]{(P, L)}$, we have that $G = \rho_* \rho^* G$ by 2.2.44, so $\pi_* F = \phi_* G$, and the statement is clear. □

**Lemma 2.3.18.** *Let $\pi\colon \mathcal{X} \to X$ be an infinite root stack, and let $(\Lambda, \Lambda_n, \phi, \alpha_{m,n})$ be in infinite root on $\mathcal{X}$. Set $\Lambda_n = (L_n, s_n)$.*

*If $X$ is quasi-compact, then for sufficiently divisible $n$ the sheaf $\pi_* L_n$ is an invertible sheaf on $X$, and the section $s_n \in \mathrm{H}^0(X, \pi_* L_n)$ does not vanish anywhere.*

*Furthermore, let $(\Lambda', \Lambda'_n, \phi', \alpha'_{m,n})$ be another infinite root on $\mathcal{X}$, and set $\Lambda'_n = (L'_n, s'_n)$. Then for sufficiently divisible $n$ the multiplication map*

$$\pi_* L_n \otimes \pi_* L'_n \longrightarrow \pi_*(L_n \otimes L'_n)$$

*is an isomorphism.*

*Proof.* Since formation of $\mathcal{A}_{\mathcal{X}}$ commutes with base change on $X$, the pushforward $\pi_*$ also commutes with base change, the statement is local in the étale topology, and every DF structure is obtained étale-locally by base change from a scheme of finite type over $\mathbb{Z}$, we may assume that $X$ is noetherian. Each $L_n$ is invertible on $\mathcal{X}$ and $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$, so we see that the annihilator of $\pi_* L_n$ is trivial. Since each $\pi_* L_n$ is coherent, by Lemma 2.3.17, to prove the statement it is enough to check that $s_n$ generates all the fibers of $\pi_* L_n$. Again because $\pi_*$ commutes with base change, and by Nakayama's lemma, we can reduce to the case that $X = \operatorname{Spec}(k)$, where $k$ is a field.

We can also assume that $k$ is algebraically closed. Then $\mathcal{X} = \sqrt[\infty]{P/k}$ for a certain sharp fine saturated monoid $P$. Then it is enough to show that $\dim_k \mathrm{H}^0(\sqrt[\infty]{P/k}, L_n) = 1$ for sufficiently divisible $n$; this is the content of Lemma 2.3.16. $\square$

*Proof of Proposition 2.3.7.* Since the category $\mathcal{A}_{\mathcal{X}}$ is fibered in groupoids, it is enough to show that an object of some $\mathcal{A}_{\mathcal{X}}(U)$ ha no non-trivial automorphisms. We may assume that $X = U$, and $X$ is quasi-compact. Choose an object $(\Lambda, \Lambda_n, \phi, \alpha_{m,n})$ of $\mathcal{A}_{\mathcal{X}}(X)$, and set $\Lambda = (L, s)$ and $\Lambda_n = (L_n, s_n)$. An automorphism of $(\Lambda, \Lambda_n, \phi, \alpha_{m,n})$ is given by a sequence of elements $\xi_n \in \mathcal{O}_{\mathcal{X}}^{\times}(\mathcal{X}) = \mathcal{O}_X^{\times}(X)$ with $\xi_n s_n = s_n$ for all $n$, and such that $\xi_n^{(m/n)} = \xi_m$ whenever $m \mid n$. From Lemma 2.3.18 we see that $\xi_n = 1$ when $n$ is sufficiently divisible, and this implies that $\xi_n = 1$ for all $n$. $\square$

*Proof of Proposition 2.3.9.* The statement can be checked on the geometric stalks; since formation of $A_{\mathcal{X}}$ commutes with base change, we may assume that $X = \operatorname{Spec}(k)$ is the spectrum of an algebraically closed field $k$, so that the logarithmic structure is given by a sharp fine saturated monoid $P$ and the monoidal functor $L\colon P \to \operatorname{Div}(k)$ sending 0 to $(\mathcal{O}_{\operatorname{Spec}(k)}, 1)$ and anything else to $(\mathcal{O}_{\operatorname{Spec}(k)}, 0)$. Then the root stack $\mathcal{X} = \sqrt[\infty]{(P, L)}$ equals $\mathcal{X} = \left[\operatorname{Spec}(k[P_{\mathbb{Q}}]/(P^+))/\mu_\infty(P)\right]$ for a sharp fine saturated monoid $P$.

Let us identify $\operatorname{Pic}(\mathcal{X})$ with $P_{\mathbb{Q}}^{\mathrm{gp}}/P^{\mathrm{gp}}$. We have a homomorphism of monoids $A_{\mathcal{X}} \to \breve{P}$ sending an infinite root $(\Lambda, \Lambda_n, \phi, \alpha_{m,n})$ to $\{[L_n]\}$, where $\Lambda_n = (L_n, s_n)$. The element $\{[L_n]\}$ has a determination function $\mathbb{N}^+ \to \{0, 1\}$, sending $n$ to 0 if $s_n = 0$ and to 1 otherwise; hence this gives a homomorphism $A_{\mathcal{X}} \to P/\infty$.

Let $(\Lambda, \Lambda_n, \phi, \alpha_{m,n})$ and $(\Lambda', \Lambda'_n, \phi', \alpha'_{m,n})$ be two infinite roots on $\mathcal{X}_P$. Assume that $[L_n] = [L'_n]$ for all $n$; then from Lemma 2.3.16 we see that for sufficiently divisible $n$ there is a unique isomorphism $L_n \cong L'_n$ carrying $s_n$ to $s'_n$. These give an isomorphism of the two infinite roots. This implies that the homomorphism $A_{\mathcal{X}} \to P/\infty$ is injective.

Now consider the composite $P = A \to A_{\mathcal{X}} \to P/\infty$, which is easily seen to send $p \in P$ into $p/\infty \in P/\infty$. Since $A_{\mathcal{X}} \to P/\infty$ is injective and $P \to P/\infty$ is an isomorphism, by Proposition 2.3.14(c), the result follows. $\square$

### 2.3.5 Morphisms of infinite root stacks

In this section we characterize morphisms of infinite root stacks that come from morphisms of DF structures, by means of infinite roots.

Let $(\phi, \Phi)\colon (A, L) \to (B, M)$ be a morphism of DF structures on $X$. Recall that this means that $\phi\colon A \to B$ is a homomorphism of sheaves of monoids on $X_{\text{ét}}$, while $\Phi\colon L \cong M \circ \phi$ is a base-preserving isomorphism of symmetric monoidal functors $A \to \operatorname{Div}_X$. A morphism of fine saturated DF structures as above induces a morphism of fibered categories $\sqrt[\infty]{\phi}\colon \sqrt[\infty]{(B,M)} \to \sqrt[\infty]{(A,L)}$ by composition.

It is not true however that any morphism $\sqrt[\infty]{(B,M)} \to \sqrt[\infty]{(A,L)}$ of stacks over $X$ comes from a morphism of DF structures (for an example, see 2.3.2).

**Definition 2.3.19.** As in Definition 2.3.3, we call *logarithmic* the morphisms $\sqrt[\infty]{(B,M)} \to \sqrt[\infty]{(A,L)}$ between two infinite root stacks over $X$ that come from morphisms of the corresponding DF structures.

We have the following characterization of logarithmic morphisms.

**Proposition 2.3.20.** *A morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of infinite root stack over $X$ is logarithmic if and only if for any geometric point $p \to X$ and any infinite root $\lambda$ on the geometric fiber $\mathcal{X}_p$, the pullback $f_p^*\lambda$ is again an infinite root on $\mathcal{Y}_p$.*

Note that this excludes precisely what happens in example 2.3.2, where the map $X_\infty \to X_\infty$ kills all the elements $t^{\frac{1}{n}}$.

*Proof.* The "only if" part follows from Proposition 2.3.9, and the "if" part is also immediate from the previous discussion: a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ satisfying the condition on infinite roots induces a morphism of monoids $A_{\mathcal{X}} \to A_{\mathcal{Y}}$ by pullback, and in turn this fits into a morphism $(\phi_f, \Phi_f)\colon (A_{\mathcal{X}}, L_{\mathcal{X}}) \to (A_{\mathcal{Y}}, L_{\mathcal{Y}})$ of DF structures in the evident way.

Finally, it is easy to see that the morphism $\sqrt[\infty]{\phi_f}$ coincides with $f$. □

The composite of two logarithmic morphisms of infinite root stack is logarithmic. Thus, infinite root stacks over $X$ with logarithmic morphisms form a 2-category; we will call it the *2-category of infinite root stacks*. Taking the infinite root stack defines a functor $F$ from the opposite of the category of DF structures to this 2-category of infinite root stacks over $X$.

**Proposition 2.3.21.** *The functor $F$ described above is faithful.*

*Proof.* Assume that we have two morphisms of DF structures $(\phi, \Phi), (\psi, \Psi)\colon (A,L) \to (B,M)$ that induce isomorphic maps between the root stacks $f \cong g\colon \mathcal{Y} \to \mathcal{X}$, where $\mathcal{X} = \sqrt[\infty]{(A,L)}$, $\mathcal{Y} = \sqrt[\infty]{(B,M)}$ and $f = \sqrt[\infty]{\phi}$, $g = \sqrt[\infty]{\psi}$. Now $f$ and $g$ will induce the same morphism between the DF structures $f^* = g^*\colon (A_{\mathcal{X}}, L_{\mathcal{X}}) \to (A_{\mathcal{Y}}, L_{\mathcal{Y}})$, and since the diagram

$$\begin{tikzcd} (A,L) \arrow[r, "(\phi,\Phi)"] \arrow[d] & (B,M) \arrow[d] \\ (A_{\mathcal{X}}, L_{\mathcal{X}}) \arrow[r, "f^*"] & (A_{\mathcal{Y}}, L_{\mathcal{Y}}) \end{tikzcd}$$

commutes, along with the analogous one with $(\psi, \Psi)$, and the vertical maps are isomorphisms by 2.3.10, the conclusion follows. □

As a corollary, we see that the functor that sends a fs log scheme to its infinite root stack is faithful, and that the infinite root stack determines the log scheme.

**Corollary 2.3.22.** *The functor $X \mapsto X_\infty$ from* (FSLogSch) *to the category* (St) *of stacks over $k$ is faithful.*

*Proof.* Assume that we have two morphisms of log schemes $f, g\colon X \to Y$ that induce the same map $f_\infty \cong g_\infty\colon X_\infty \to Y_\infty$. First of all from the fact that $X$ and $Y$ are “coarse moduli spaces” of $X_\infty$ and $Y_\infty$ respectively (see Corollary 2.2.45) we conclude that the two morphisms of schemes $\underline{f}, \underline{g}\colon \underline{X} \to \underline{Y}$ are equal.

Once we have this, the result follows from 2.3.21 after pulling back to $\underline{X}$. $\square$

**Corollary 2.3.23.** *Let $X$ and $Y$ be fs log schemes. Assume that we have an isomorphism $X_\infty \cong Y_\infty$ of stacks over $k$. Then this is induced by an isomorphism of log schemes $X \cong Y$.*

*Proof.* From Corollary 2.2.45 we see that the isomorphism $X_\infty \cong Y_\infty$ induces an isomorphism of the schemes $\underline{X} \cong \underline{Y}$. After pulling everything back to $\underline{X}$, we just have to note that isomorphisms of infinite root stacks are logarithmic by Proposition 2.3.20 (since the condition about infinite roots is trivially satisfied). Consequently the given isomorphism comes from a morphism of DF structures and this has to be an isomorphism, once again by Proposition 2.3.21. $\square$

To conclude, we prove a lemma which will be useful in the next section.

**Lemma 2.3.24.** *Let $(\phi, \Phi)\colon (A, L) \to (B, M)$ and $(\psi, \Psi)\colon (A, L) \to (C, N)$ be morphisms of DF structures on a scheme $X$, such that $\psi$ is Kummer. Suppose that $f\colon \sqrt[\infty]{(C, N)} \to \sqrt[\infty]{(B, M)}$ is a morphism of stacks over $X$ making the diagram*

$$\begin{tikzcd} \sqrt[\infty]{(C,N)} \arrow[rr, "f"] \arrow[dr, "\sqrt[\infty]{\phi}"'] & & \sqrt[\infty]{(B,M)} \arrow[dl, "\sqrt[\infty]{\psi}"] \\ & \sqrt[\infty]{(A,M)} & \end{tikzcd}$$

*commute. Then $f$ is logarithmic.*

*Proof.* We need to check that $f$ sends infinite roots in geometric fibers to infinite roots; by base change, we may assume that $X = \operatorname{Spec}(k)$, where $k$ is an algebraically closed field. For consistency with the previous notation, set $P = A$, $Q = B$ and $R = C$; we need to check that the homomorphism $f^*\colon R^{\mathrm{gp}}_{\mathbb{Q}}/R^{\mathrm{gp}} \to Q^{\mathrm{gp}}_{\mathbb{Q}}/Q^{\mathrm{gp}}$ induced by $f$ sends $R/\infty$ into $Q/\infty$ (here we are using the identification $Q^{\mathrm{gp}}_{\mathbb{Q}}/Q\,\mathrm{gp} \cong \operatorname{Pic}(\sqrt[\infty]{Q/k})$). Taking projective limits and using the identifications $\check{P} \cong P^{\mathrm{gp}} \otimes \widehat{\mathbb{Z}}$ we obtain a commutative diagram

$$\begin{tikzcd} & P^{\mathrm{gp}} \otimes \widehat{\mathbb{Z}} \arrow[dl] \arrow[dr] & \\ R^{\mathrm{gp}} \otimes \widehat{\mathbb{Z}} \arrow[rr, "f^*"] & & Q^{\mathrm{gp}} \otimes \widehat{\mathbb{Z}} \end{tikzcd}$$

in which the two diagonal arrows take $P$ into $R$ and $Q$ respectively. We need to show that $f^*$ takes $R$ into $Q$. Since the homomorphism $P \to R$ is Kummer, given $r \in R$ we can find a positive integer $n$ such that $nr$ comes from $P$; this implies that $nf^*(r) = f^*(nr)$ is in $Q$. Since $\widehat{\mathbb{Z}}/\mathbb{Z}$ is torsion free, we see that $f^*(r) \in Q^{\mathrm{gp}}$; since $Q$ is saturated this implies $f^*(r) \in Q$, and this concludes the proof. $\square$

**Corollary 2.3.25.** *Let $X, Y$ and $Z$ be fs log schemes with two maps $Z \to X$ and $Y \to X$, the first one being Kummer, and let $F\colon Z_\infty \to Y_\infty$ be a morphism of infinite root stacks over $X_\infty$. Then $F$ is logarithmic.*

*Proof.* From Corollary 2.2.45 we see that $Z_\infty \to Y_\infty$ induces a morphism of schemes $\underline{Z} \to \underline{Y}$ that makes the diagram

$$\begin{tikzcd} \underline{Z} \arrow[rr] \arrow[dr] & & \underline{Y} \arrow[dl] \\ & \underline{X} & \end{tikzcd}$$

commute.

Now we can pullback the log structures of $Y$ and $X$ (together with their infinite root stacks) to $\underline{Z}$, and the result follows from the preceding proposition. □

## 2.4 The infinite root stack recovers the Kummer-flat topos

In this section we will show that the Kummer-flat topos of a log scheme ([Kat, Niz08, INT13]) can be recovered as the fppf topos of the corresponding infinite root stack.

We briefly recall the construction of the Kummer-flat topos of a log scheme.

**Definition 2.4.1.** A morphism of fine saturated log schemes $f\colon Y \to X$ is *Kummer-flat* if it is log-flat and Kummer, and the underlying map of schemes is locally of finite presentation.

Recall that a morphism $f\colon Y \to X$ is log-flat if the following holds: fppf locally on $X$ and $Y$ we can find Kato charts $P \to M_X$ and $Q \to M_Y$ and a morphism $P \to Q$ such that the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \mathrm{Spec}(k[Q]) \arrow[d] \\ X \arrow[r] & \mathrm{Spec}(k[P]) \end{tikzcd}$$

commutes, and the induced map $Y \to X \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[Q])$ is flat. A morphism $f\colon Y \to X$ is Kummer if the corresponding $f^*A_X \to A_Y$ is Kummer, meaning that the homomorphism of monoids $(f^*A_X)_y \to (A_Y)_y$ is Kummer for any geometric point $y \to Y$.

Since charts can be made up from stalks, if $f\colon Y \to X$ is Kummer-flat, then locally we can find charts as above such that in addition $P \to Q$ is Kummer, and it is proven in [INT13] that we can also make $Y \to X \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[Q])$ locally of finite presentation.

For a log scheme $X$, there is a site, called the *Kummer-flat site* and denoted by $\mathrm{kfl}(X)$, whose objects are morphisms of log schemes $U \to X$ that are Kummer-flat, with morphisms of log schemes over $X$ as arrows, and with jointly surjective families $\{U_i \to U\}_{i \in I}$ of Kummer-flat morphisms as coverings. The corresponding topos $X_{\mathrm{kfl}}$ is called the *Kummer-flat topos* of $X$.

**Remark 2.4.2.** The site $X_{\mathrm{kfl}}$ has a final object and fibered products. Given a diagram

$$\begin{tikzcd} & V \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

in $X_{\text{kfl}}$, the fibered product is given by the fibered product $V \times_Y Z$ in the category of fine saturated log schemes over $k$, together with the induced Kummer-flat map $V \times_Y Z \to X$.

**Remark 2.4.3.** If we have two objects $Y \to X$ and $Z \to X$ of $X_{\text{kfl}}$, then any morphism $Z \to Y$ in $X_{\text{kfl}}$ is also Kummer. This follows from the fact that if two morphisms of fs torsion-free monoids $P \to Q$ and $P \to R$ are Kummer and we have a commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & R \\ Q \arrow[ur] & \end{tikzcd}$$

then also $Q \to R$ is Kummer.

Indeed, any $r \in R$ has some multiple $nr$ coming from $p$, which means that it also comes from $Q$. Moreover the map is injective: if $q$ and $q'$ go to the same element $r$, take $n \in \mathbb{N}$ such that $nr, nq$ and $nq'$ all come from $P$. Then if say $p$ goes to $nq$ and $p'$ goes to $nq'$, since $P \to R$ is injective and $p, p'$ both go to $nr$, we must have $p = p'$, which means $nq = nq'$, and so $q = q'$ by torsion-freeness.

One can also restrict to considering Kummer-étale morphisms, where the definitions are the analogous ones, with "flat" replaced by "étale" in all the instances. The results are the *Kummer-étale site* $\text{két}(X)$ and the corresponding *Kummer-étale topos* $X_{\text{két}}$. In the characteristic zero case these étale variants are usually enough for applications.

**Proposition 2.4.4.** *Let $f\colon Y \to X$ be a Kummer-flat (resp. Kummer-étale) morphism of log schemes. Then the induced morphism $f_\infty\colon Y_\infty \to X_\infty$ between the infinite root stacks is representable and fppf (resp. representable and étale).*

*Proof.* Since the question is local for the fppf topology of $X$ and $Y$, we can assume that we have a diagram

$$\begin{tikzcd} Y \arrow[dr] \arrow[ddr, bend right] & & \\ & X \times_{\operatorname{Spec}(k[P])} \operatorname{Spec}(k[Q]) \arrow[r] \arrow[d] & \operatorname{Spec}(k[Q]) \arrow[d] \\ & X \arrow[r] & \operatorname{Spec}(k[P]) \end{tikzcd}$$

where $Y \to X \times_{\operatorname{Spec}(k[P])} \operatorname{Spec}(k[Q])$ is fppf (resp. étale) and strict, and $P \to Q$ is Kummer.

Now in turn this means that we have a commutative diagram

$$\begin{tikzcd} & X_{Q/P} \arrow[d] \\ Y \arrow[ur] \arrow[r] & X \end{tikzcd}$$

where the map $Y \to X_{Q/P}$ is strict and flat (resp. étale). Consequently by taking infinite root stacks we have a cartesian diagram

$$\begin{tikzcd} Y_\infty \arrow[r] \arrow[d] & (X_{Q/P})_\infty \cong X_\infty \arrow[d] \\ Y \arrow[r] & X_{Q/P} \end{tikzcd}$$

where we used Remark 2.2.9 and Proposition 2.2.25, and from this we see that $Y_\infty \to X_\infty$ is flat (resp. étale).

Representability follows from the local description of the map $Y_\infty \to X_\infty$ as a map between quotient stacks, and the following lemma. ☐

**Lemma 2.4.5.** *Let $G$ be a diagonalizable group (not necessarily of finite type) over a ring $R$, and $H \subseteq G$ be a diagonalizable subgroup. Assume that $H$ acts on an affine scheme $T$ and $G$ acts on an affine scheme $S$ over $R$, and we have a morphism $T \to S$ that is equivariant with respect to the immersion $H \subseteq G$. Then the induced morphism*

$$[T/H] \to [S/G]$$

*between the quotient stacks is affine.*

*Proof.* By fpqc descent, to check that the map is affine (as well as any map between fpqc stacks) we can reduce to checking it for a particular fpqc presentation of the target stack.

Now note that we have a diagram

$$\begin{tikzcd} {[T/H]} \arrow[r] & {[S/H]} \arrow[r] \arrow[d] & {[S/G]} \arrow[d] \\ & B_R H \arrow[r] & B_R G \end{tikzcd}$$

where the square is cartesian, and the conclusion follows from the two cartesian diagrams

$$\begin{tikzcd} {[G/H]} \arrow[r] \arrow[d] & \mathrm{Spec}(R) \arrow[d] \\ B_R H \arrow[r] & B_R G \end{tikzcd}$$

and

$$\begin{tikzcd} T \arrow[r] \arrow[d] & S \arrow[d] \\ {[T/H]} \arrow[r] & {[S/H]} \end{tikzcd}$$

and the fact that the quotient $[G/H]$ is represented by a diagonalizable group, and in particular is affine. ☐

Because of proposition 2.4.4 there is a natural functor $F\colon \mathrm{kfl}(X) \to \mathrm{fppf}(X_\infty)$ from the Kummer-flat site of $X$ to the small fppf site of $X_\infty$, acting on objects by taking $f\colon Y \to X$ to $f_\infty\colon Y_\infty \to X_\infty$, and on arrows by taking $g\colon Z \to Y$ over $X$ to $g_\infty\colon Z_\infty \to Y_\infty$ over $X_\infty$.

**Lemma 2.4.6.** *The functor $F$ preserves fibered products.*

*Proof.* The statement means that if

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

is a cartesian diagram in $X_{\mathrm{kfl}}$, then the diagram

$$\begin{tikzcd} W_\infty \arrow[r] \arrow[d] & V_\infty \arrow[d] \\ Z_\infty \arrow[r] & Y_\infty \end{tikzcd}$$

is 2-cartesian, i.e. the induced morphism $W_\infty \to Z_\infty \times_{Y_\infty} V_\infty$ is an equivalence.

Recall first of all that the morphisms $Z \to Y$ and $V \to Y$ are Kummer, and denote by $A, B, C, D$ the sheaves of monoids giving the log structures of $Y, V, Z, W$ respectively. Recall that $W$ is obtained in the following way: we first form the fibered product of the underlying schemes $\underline{V} \times_{\underline{Y}} \underline{Z}$ and, locally where we have charts $P \to A, Q \to B, R \to C$, equip it with the log structure coming from the pushout $Q \oplus^P R$ of the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ R \arrow[r] & Q \oplus^P R \end{tikzcd}$$

and then base change along $\mathrm{Spec}(k[(Q \oplus^P R)^{\mathrm{fs}}]) \to \mathrm{Spec}(k[Q \oplus^P R])$. Now note that since the functor $P \mapsto P_{\mathbb{Q}}$ preserves pushouts (Remark 1.1.15), the diagram

$$\begin{tikzcd} P_{\mathbb{Q}} \arrow[r] \arrow[d] & Q_{\mathbb{Q}} \arrow[d] \\ R_{\mathbb{Q}} \arrow[r] & (Q \oplus^P R)_{\mathbb{Q}} \end{tikzcd}$$

is also a pushout, but in this case the maps $P_{\mathbb{Q}} \to Q_{\mathbb{Q}}$ and $P_{\mathbb{Q}} \to R_{\mathbb{Q}}$ are isomorphisms, since $P \to Q$ and $P \to R$ are Kummer. Consequently the remaining two maps in the diagram are also isomorphisms, and we have $(Q \oplus^P R)_{\mathbb{Q}} \cong P_{\mathbb{Q}}$.

Now we construct a quasi-inverse to the natural functor $W_\infty \to Z_\infty \times_{Y_\infty} V_\infty$. Take an object of $(Z_\infty \times_{Y_\infty} V_\infty)(T)$, i.e. a triple $(\xi, \eta, f)$ where $\xi\colon (B_T)_{\mathbb{Q}} \to \mathrm{Div}_T$ and $\eta\colon (C_T)_{\mathbb{Q}} \to \mathrm{Div}_T$ are liftings of the DF structures coming from $V$ and $Z$ respectively, and $f$ is an isomorphism between their restrictions to $(A_T)_{\mathbb{Q}}$. Call $E$ the pushout of the diagram

$$\begin{tikzcd} A_T \arrow[r] \arrow[d] & B_T \\ C_T & \end{tikzcd}$$

of sheaves of monoid over $T$. The preceding remarks imply that $(A_T)_{\mathbb{Q}}, (B_T)_{\mathbb{Q}}, (C_T)_{\mathbb{Q}}$ are all isomorphic, and they are also isomorphic to $E_{\mathbb{Q}}$, so we have an induced DF structure $E_{\mathbb{Q}} \to \mathrm{Div}_T$. Moreover since $E_{\mathbb{Q}} = (A_T)_{\mathbb{Q}}$ is integral and saturated, the map $E \to E_{\mathbb{Q}}$ factors through $E \to E^{\mathrm{fs}}$, the fine saturation of the sheaf $E$. By restriction along $E^{\mathrm{fs}} \to E_{\mathbb{Q}}$, this gives a log structure on $T$ that makes the diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & V \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

a commutative diagram of fs log schemes. Consequently there is an induced (strict) morphism $T \to W$, and together with the lifting $(D_T)_{\mathbb{Q}} \cong E_{\mathbb{Q}} \to \mathrm{Div}_T$ of the DF structure coming from $W$ this gives our object of $W_\infty(T)$. We leave the remaining verifications to the reader. □

**Remark 2.4.7.** The stronger statement that the functor $X \mapsto X_\infty$ from fine saturated log schemes over $k$ to stacks over $k$ preserves fibered products is probably false. For the preceding proof it is essential that the morphisms involved are Kummer.

**Proposition 2.4.8.** *The functor $F$ gives a morphism of topoi $X_{\mathrm{kfl}} \to (X_\infty)_{\mathrm{fppf}}$, which is an equivalence.*

*Proof.* We will apply the following lemma from the Stacks Project (Lemma 7.27.1 in [Sta13, Tag 039Z]).

**Lemma 2.4.9.** *Let $C, D$ be sites and $u\colon C \to D$ a functor. If*

1. *$u$ is continuous and cocontinuous*
2. *given $a, b\colon U' \to U$ in $C$ such that $u(a) = u(b)$, then there exists a covering $\{f_i\colon U'_i \to U'\}$ in $C$ such that $a \circ f_i = b \circ f_i$ for every $i$,*
3. *given $U, U' \in C$ and a morphism $c\colon u(U') \to u(U)$ in $D$, then there exists a covering $\{f_i\colon U'_i \to U'\}$ in $C$ and morphisms $c_i\colon U'_i \to U$ such that $u(c_i) = c \circ u(f_i)$ for every $i$,*
4. *given $V \in D$, then there exists a covering of $V$ in $D$ of the form $\{u(U_i) \to V\}$,*

*then there is an equivalence $\mathrm{Sh}(C) \cong \mathrm{Sh}(D)$.*

Note that, as remarked in the proof in the Stacks Project, the functor $\alpha\colon \mathrm{Sh}(D) \to \mathrm{Sh}(C)$ that plays the role of the "pullback functor" of the mentioned equivalence is defined in the natural way by $\alpha(G)(c) = G(u(c))$ for an object $c \in C$.

Back to the proof, the fact that $F$ is continuous follows from Proposition 2.4.4 and Lemma 2.4.6. Showing that it is cocontinuous amounts to proving that for any object $Z_\infty \to X_\infty$ of

$(X_\infty)_{\mathrm{fppf}}$, where $Z \to X$ is Kummer-flat, any covering $\{\mathcal{A}_i \to Z_\infty\}$ in $(X_\infty)_{\mathrm{fppf}}$ can be refined by the family of maps $\{(Z_i)_\infty \to Z_\infty\}$, for some Kummer-flat covering $\{Z_i \to Z\}$. Clearly this will follow from item number 4 applied to $Z$ in place of $X$.

Note that item number 2 (local faithfulness) follows directly from Corollary 2.3.22, and item number 3 (local fullness) from Corollary 2.3.25, that implies that every morphism $Z_\infty \to Y_\infty$ in $(X_\infty)_{\mathrm{fppf}}$ is logarithmic, i.e. comes from a morphism $Z \to Y$ of log schemes. All that is left is to prove item number 4.

Let us fix an object $\mathcal{A} \to X_\infty$ of $(X_\infty)_{\mathrm{fppf}}$. After étale-shrinking $X$, we can assume that we have a Kato chart $P \to \mathcal{O}_X$ for the log structure of $X$. We will find a Kummer-flat morphism $Y \to X$ with a factorization $Y_\infty \to \mathcal{A} \to X_\infty$, such that $Y_\infty \to \mathcal{A}$ is fppf and surjective.

Take a presentation $U_\infty \to X_\infty$ coming from a compatible system of presentations $U_n \to X_n$, as in the discussion preceding 2.2.14, and the groups $G_n$ and $G_\infty$. Consider the pullback of $U_\infty$ to $\mathcal{A}$, as in the cartesian square

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U_\infty \arrow[d] \\ \mathcal{A} \arrow[r] & X_\infty. \end{tikzcd}$$

where $V$ is an algebraic space and $V \to U_\infty$ is fppf, since $\mathcal{A} \to X_\infty$ is representable and fppf.

Note that the description of $X_\infty$ as a quotient $[U_\infty/G_\infty]$ gives a presentation of $\mathcal{A}$ as $[V/G_\infty]$ for the induced action, and we have a groupoid presentation of the form

$$V \times G_\infty \rightrightarrows V \to \mathcal{A}.$$

Since the morphism $V \to U_\infty$ is fppf and $U_\infty = \varprojlim_n U_n$, we have an fppf morphism $V_n \to U_n$ such that the diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V_n \arrow[d] \\ U_\infty \arrow[r] & U_n \end{tikzcd}$$

is cartesian.

Now we claim that in fact the whole groupoid presentation of $\mathcal{A}$ comes from some finite level: in fact, also the action of $G_\infty$ on $V$, which can be seen as a morphism $V \times G_\infty \to V \times G_\infty$ (over $U \times G_\infty$), must come from some morphism $V_n \times G_\infty \to V_n \times G_\infty$ (over $U_n \times G_\infty$) where $V_n$ is as above, since $V \times G_\infty \to U \times G_\infty$ is fppf and $U \times G_\infty = \varprojlim_n (U_n \times G_\infty)$. Moreover by increasing $n$ we may assume that this last morphisms also gives an action of $G_\infty$ on $V_n$.

The action of $G_\infty = \varprojlim_n G_n$ factors through some finite stage $G_m$, because $V_n$ is of finite type. This simply follows from looking at the coaction of the Hopf algebra of $G_\infty$. This gives us an action $G_m \times V_n \to V_n$.

Now denote by $k$ the least common multiple of $n$ and $m$. We first pull the action back along $V_k \to V_n$ (where $V_k = V_n \times_{U_n} U_k$), obtaining an action $G_m \times V_k \to V_k$, and finally by means of the map $G_k \to G_m$, we get an action $G_k \times V_k \to V_k$, and we take the quotient stack $\mathcal{A}_k = [V_k/G_k]$.

The morphism $V_k \to U_k$ induces a representable and fppf map $\mathcal{A}_k = [V_k/G_k] \to [U_k/G_k] =$

$X_k$, and the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{A}_k \arrow[d] \\ X_\infty \arrow[r] & X_k. \end{tikzcd}$$

is cartesian.

In fact, the morphism $X_\infty \to X_k$ factors as

$$X_\infty = [U_\infty / G_\infty] \to [U_k / G_\infty] \to [U_k / G_k] = X_k$$

and we can calculate the fibered product in two steps. For the rightmost map, we have $\mathcal{A} \times_{X_k} [U_k / G_\infty] \cong [V_k / G_\infty]$, since in the diagram

$$\begin{tikzcd} {[V_k / G_\infty]} \arrow[r] \arrow[d] & {[V_k / G_k]} \arrow[d] \\ {[U_k / G_\infty]} \arrow[r] \arrow[d] & {[U_k / G_k]} \arrow[d] \\ BG_\infty \arrow[r] & BG_k \end{tikzcd}$$

the two squares are cartesian.

For the second base change, note that the following square is cartesian

$$\begin{tikzcd} {[V / G_\infty]} \arrow[r] \arrow[d] & {[V_k / G_\infty]} \arrow[d] \\ {[U / G_\infty]} \arrow[r] & {[U_k / G_\infty]} \end{tikzcd}$$

since $V \cong V_k \times_{U_k} U$ by construction. In conclusion $\mathcal{A} = [V / G_\infty] \cong \mathcal{A}_k \times_{X_k} X_\infty$.

Moreover if we equip $\mathcal{A}_k$ with the pullback of the log structure of $X_k$ along the map $\mathcal{A}_k \to X_k$ we have an isomorphism $(\mathcal{A}_k)_\infty \cong \mathcal{A}$. This is just because by construction $\mathcal{A}_k \to X_k$ is strict, and so we have an isomorphism $(\mathcal{A}_k)_\infty \cong \mathcal{A}_k \times_{X_k} (X_k)_\infty \cong \mathcal{A}$, since $(X_k)_\infty \cong X_\infty$.

Now if we also endow $V_k$ with the pullback log structure from $X_k$ the composition $V_k \to X$ will be Kummer-flat, and we have a factorization $(V_k)_\infty \to (\mathcal{A}_k)_\infty \cong \mathcal{A} \to X_\infty$. All that is left is to note that the map $(V_k)_\infty \to \mathcal{A}$ is surjective, fppf and representable (i.e. a cover in the fppf site). This follows from the fact that $V_k \to \mathcal{A}_k$ has those properties and is strict, so the map between the infinite root stacks is a base change, by Remark 2.2.9. This concludes the proof. □

The same line of reasoning proves the analogue of these results for the Kummer-étale topos, which is particularly relevant in characteristic zero. In this case the étale variant of the topos is typically sufficient for applications, whereas in the positive characteristic case one often has to look at the Kummer-flat one.

We state the result in this case.

Let $X$ be a log scheme over a field of characteristic 0, and let $\text{ét}(X_\infty)$ denote the small étale site of $X_\infty$, consisting of the category of isomorphism classes of representable étale morphisms $\mathcal{A} \to X_\infty$ endowed with the étale topology. Denote by $(X_\infty)_{\text{ét}}$ the corresponding topos. By Proposition 2.4.4 the association $(Y \to X) \mapsto (Y_\infty \to X_\infty)$ gives a functor $\text{két}(X) \to \text{ét}(X_\infty)$.

**Theorem 2.4.10.** *This functor induces an equivalence of ringed topoi* $X_{\text{két}} \cong (X_\infty)_{\text{ét}}$.

Back to the general situation, by combining Proposition 2.4.8 with 2.2.40 we see that finitely presented (and in particular, for example, locally free of finite rank) Kummer-flat sheaves on a log scheme are precisely finitely presented sheaves on its infinite root stack, i.e. finitely presented parabolic sheaves.

**Corollary 2.4.11.** *There is an equivalence of categories* $\mathrm{FP}(X_{\text{kfl}}) \cong \mathrm{FP}(X_\infty)$.

*Proof.* This follows formally from the fact that the equivalence of Proposition 2.4.8 is an isomorphism of *ringed* topoi, after we equip them with the structure sheaf $\mathcal{O}$ on both sides. This is immediate from the description of the functor $\alpha\colon (X_\infty)_{\text{fppf}} \to X_{\text{kfl}}$ as $\alpha(G)(Y \to X) = G(Y_\infty \to X_\infty)$ (see the proof of 2.4.8), and the fact that for $\pi\colon Y_\infty \to Y$ we have $\pi_*\mathcal{O}_{Y_\infty} \cong \mathcal{O}_Y$ (Proposition 2.2.44). $\square$

# Chapter 3

# Moduli of parabolic sheaves with fixed weights

The subject of this chapter is the moduli theory for parabolic sheaves on a log scheme $X$, with a fixed system of denominators.

The strategy will be the following: by the BV equivalence, parabolic sheaves on $X$ with denominators in $B/A$ correspond to quasi-coherent sheaves on the root stack $X_{B/A}$. We can therefore study the moduli theory of coherent sheaves on said root stack.

The moduli theory for coherent sheaves on a tame DM stack has been developed in [Nir]. Apart from assuming that $X$ is projective over a field and choosing a polarization, one also has to choose a generating sheaf on the root stack. In general there are many choices for such a sheaf, and it is not clear which of them is best suited to generalize the notion of stability given by Seshadri and Maruyama and Yokogawa in the case of curves and varieties with a divisor.

One case where it is possible to find a distinguished generating sheaves that generalizes the situations in the literature is the case in which we have a global chart $P \to \mathrm{Div}(X)$ for the log structure of $X$, or more generally what we call a locally constant sheaf of charts on $X$. In this case we isolate a generating sheaf that gives back the stability notions already present in the literature in the particular cases, and we get moduli spaces for pure parabolic sheaves with weights in $B/A$.

We will also see that the choice of the sheaf of charts changes the notion of stability. This phenomenon is analogous to the change of the stability when one changes the polarization in the context of moduli of coherent sheaves on a projective scheme.

For the whole chapter, $X$ will be a fine and saturated log scheme with a DF structure $L\colon A \to \mathrm{Div}_X$, and $j\colon A \to B$ will be our fixed system of denominators. Moreover $X$ will be projective over $k$, and from Section 3.2 on $\mathfrak{X}$ will denote the root stack $X_{B/A}$. To apply Nironi's machinery we will also have to assume the root stack $X_{B/A}$ is Deligne–Mumford. This is automatic if for example $\mathrm{char}(k)$ does not divide the order of the group $B_x^{\mathrm{gp}}/A_x^{\mathrm{gp}}$ for any geometric point of $X$ (see 1.2.31).

## 3.1 Pullbacks of parabolic sheaves

Let $X$ be a log scheme with log structure $L\colon A \to \mathrm{Div}_X$, and $f\colon \underline{Y} \to \underline{X}$ a morphism of schemes. Equip $\underline{Y}$ with the pullback log structure, and assume furthermore that $j\colon A \to B$ is a system of denominators. We want to define a pullback functor $f^*$ from parabolic sheaves on $X$ with respect to $j$ to parabolic sheaves on $Y$ with respect to the pullback system of denominators $f^*j\colon f^*A \to f^*B$. The BV equivalence suggests a natural way to do it: recall that we denote the functors giving the equivalence by $\Phi\colon \mathrm{QCoh}(X_{B/A}) \to \mathrm{Par}(X, j)$ and $\Psi\colon \mathrm{Par}(X, j) \to \mathrm{QCoh}(X_{B/A})$ (see Section 1.2.1 for details).

Assume more generally that $Y$ is a log scheme with log structure $N\colon C \to \mathrm{Div}_Y$, $h\colon C \to D$ is a system of denominators, $f\colon Y \to X$ is a morphism of log schemes and the morphism $f^*A \to C$ fits in a commutative diagram

$$\begin{tikzcd} f^*A \arrow[r] \arrow[d] & C \arrow[d] \\ f^*B \arrow[r] & D. \end{tikzcd}$$

Then we have a natural morphism of root stacks $\Pi\colon Y_{D/C} \to X_{B/A}$ and a commutative diagram

$$\begin{tikzcd} Y_{D/C} \arrow[r] \arrow[d] & X_{B/A} \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

sending an object $\xi\colon \phi^*D \to \mathrm{Div}_T$ of $Y_{D/C}(T)$, where $\phi\colon \underline{T} \to \underline{Y}$, to the composition $\Pi(\xi)\colon (f \circ \phi)^*B \to \phi^*D \to \mathrm{Div}_T$, which is an object of $X_{B/A}(T)$.

In case $N = f^*L$ and $h = f^*j$, the diagram is also cartesian, so that $Y_{f^*B/f^*A} \cong X_{B/A} \times_X Y$.

**Definition 3.1.1.** Given a parabolic sheaf $E \in \mathrm{Par}(X, j)$, the *pullback* $f^*E$ of $E$ along $f$ is the parabolic sheaf $\Phi(\Pi^*(\Psi(E))) \in \mathrm{Par}(Y, h)$ corresponding via the BV equivalence to the pullback of the quasi-coherent sheaf $\Psi(E) \in \mathrm{QCoh}(X_{B/A})$ along $\Pi$.

The pullback $f^*E$ is unique up to isomorphism, and functorial, in the sense that $f^*\colon \mathrm{Par}(X, j) \to \mathrm{Par}(Y, h)$ is a functor. Moreover by definition the diagram

$$\begin{tikzcd} \mathrm{Par}(X, j) \arrow[r, "f^*"] \arrow[d, "\Psi"'] & \mathrm{Par}(Y, h) \arrow[d, "\Psi"] \\ \mathrm{QCoh}(X_{B/A}) \arrow[r, "\Pi^*"] & \mathrm{QCoh}(Y_{D/C}) \end{tikzcd}$$

is 2-commutative.

We can now define a fibered category $\underline{\mathrm{Par}}(X, j) \to (\mathrm{Sch}/X)$ of parabolic sheaves on $X$ by taking as $\underline{\mathrm{Par}}(X, j)(\underline{T})$, where $\phi\colon \underline{T} \to \underline{X}$, the category $\mathrm{Par}(T, \phi^*j)$ of parabolic sheaves over $\underline{T}$, equipped with the pullback log structure, and with respect to $\phi^*j$. The arrows of $\underline{\mathrm{Par}}(X, j)$ are defined using the notion of pullback just described.

On the other hand we also have the fibered category $\underline{\mathrm{QCoh}}(X_{B/A}) \to (\mathrm{Sch}\,/X)$, whose fiber category over $\phi\colon \underline{T} \to \underline{X}$ is $\mathrm{QCoh}(T_{\phi^*B/\phi^*A})$ and the arrows are again defined by pullback.

Basically by definition, we have the following extension of Theorem 1.3.8.

**Proposition 3.1.2.** *There are equivalences of fibered categories $\Phi\colon \underline{\mathrm{QCoh}}(X_{B/A}) \to \underline{\mathrm{Par}}(X, j)$ and $\Psi\colon \underline{\mathrm{Par}}(X, j) \to \underline{\mathrm{QCoh}}(X_{B/A})$ that coincide with the BV equivalences on every fiber category.*

*Proof.* The functors $\Phi(\underline{T})$ and $\Psi(\underline{T})$ for $\underline{T} \to \underline{X}$ are defined by the functors of the BV equivalence on $T$ (equipped with the pullback log structure), and the resulting $\Phi$ and $\Psi$ are cartesian by construction. Finally, they are equivalences since they are so fiberwise. □

This implies in particular that $\underline{\mathrm{Par}}(X, j)$ is a stack for the fpqc (or any coarser) topology of $(\mathrm{Sch}\,/X)$, as one can verify directly by standard arguments of descent theory.

In the case where $X$ and the system of denominators have a global chart, the pullback of a parabolic sheaf along a strict morphism $f\colon Y \to X$ has a simple "purely parabolic" description (which seems to be lacking for example for a non-strict morphism): assume that we have charts $L_0\colon P \to \mathrm{Div}(X)$ for $L$ and $j_0\colon P \to Q$ for $j$. Then $P$ and $j_0$ also give charts on $Y$ for $f^*A$ and $f^*j$, and given a parabolic sheaf $E\colon Q^{\mathrm{wt}} \to \mathrm{QCoh}(X)$, we can define $f^*E$ as the composition $Q^{\mathrm{wt}} \to \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$, where the last functor is pullback of quasi-coherent sheaves.

**Proposition 3.1.3.** *The functor $f^*E$ is a parabolic sheaf on $Y$, and the corresponding quasi-coherent sheaf $\Psi(f^*E)$ on the root stack $Y_{f^*B/f^*A}$ is naturally isomorphic to the pullback along $\Pi\colon Y_{f^*B/f^*A} \to X_{B/A}$ of the quasi-coherent sheaf $\Psi(E)$ on $X_{B/A}$ corresponding to $E$.*

*Proof.* It is clear that $f^*E$ is a parabolic sheaf, by applying $f^*$ to the pseudo-periods isomorphism $\rho^E$ and all the relevant diagrams.

Let us now show that the parabolic sheaf $\Phi(\Pi^*\Psi(E)) \in \mathrm{Par}(Y, f^*j)$ is isomorphic to $f^*E$ as defined above.

For $q \in Q^{\mathrm{gp}}$, let us calculate

$$(f^*E)_q = f^*E_q = f^*\pi_*(\Psi(E) \otimes \Lambda_q)$$

and

$$\Phi(\Pi^*\Psi(E))_q = (\pi_Y)_*(\Pi^*\Psi(E) \otimes (\Lambda_Y)_q).$$

Note now that $(\Lambda_Y)_q = \Pi^*\Lambda_q$, so

$$(\pi_Y)_*(\Pi^*\Psi(E) \otimes (\Lambda_Y)_q) \cong (\pi_Y)_*\Pi^*(\Psi(E) \otimes \Lambda_q).$$

Now we apply Proposition 1.5 of [Nir] to the cartesian diagram

$$\begin{tikzcd} Y_{f^*B/f^*A} \arrow[r, "\Pi"] \arrow[d, "\pi_Y"'] & X_{B/A} \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

to get a functorial base change isomorphism $f^*\pi_* \to (\pi_Y)_*\Pi^*$ of functors $\mathrm{QCoh}(X_{B/A}) \to \mathrm{QCoh}(Y)$.

This gives an isomorphism $(f^*E)_q \to \Phi(\Pi^*\Psi(E))_q$ for any $q \in Q$. By functoriality, by putting all these isomorphisms together we get a natural isomorphism of functors $Q^{\mathrm{wt}} \to \mathrm{QCoh}(X)$, which moreover respects the pseudo-periods isomorphisms, and so is an isomorphism of parabolic sheaves. □

This also gives a local description of the pullback as defined in general, using local charts for the Kummer morphism $j$.

When there is no global chart, it is still possible to give a parabolic description of the pullback along a strict morphism, even though it is more complicated. Since we are not going to use this description, we only sketch it briefly for completeness.

With notations as above, assume that $E\colon B^{\mathrm{wt}} \to \mathrm{QCoh}_X$ is a parabolic sheaf on $X$. Both $B^{\mathrm{wt}}$ and $\mathrm{QCoh}_X$ are stacks on the small étale site $X_{\text{ét}}$, and we can pull them back together with the morphism $E$ using $f$, thus obtaining a morphism $f^*(B^{\mathrm{wt}}) \to f^*\mathrm{QCoh}_X$ of stacks over $Y_{\text{ét}}$. The composition with the natural morphism $f^*\mathrm{QCoh}_X \to \mathrm{QCoh}_Y$ will be our desired $f^*E$.

**Proposition 3.1.4.** *We have a natural equivalence of symmetric monoidal stacks $f^*(B^{\mathrm{wt}}) \cong f^*(B)^{\mathrm{wt}}$ on $Y_{\text{ét}}$, and the resulting functor $f^*E\colon f^*(B)^{\mathrm{wt}} \to \mathrm{QCoh}_Y$ is a parabolic sheaf on $Y$.*

*Proof.* To see that $f^*(B^{\mathrm{wt}}) \cong f^*(B)^{\mathrm{wt}}$ note that $B^{\mathrm{wt}}$ has a presentation (as a stack over $X_{\text{ét}}$) as $B^{\mathrm{gp}} \times B \rightrightarrows B^{\mathrm{gp}}$, and by pulling back everything we get a presentation for $f^*(B^{\mathrm{wt}})$. On the other hand $f^*(B^{\mathrm{gp}}) \cong f^*B^{\mathrm{gp}}$, and the presentation $f^*B^{\mathrm{gp}} \times f^*B \rightrightarrows f^*B$ gives $f^*B^{\mathrm{wt}}$.

Let us show that $f^*E$ is a parabolic sheaf. Recall that the pseudo-periods isomorphism $\rho^E$ can be seen an an isomorphism of functors $E \circ + \cong \otimes \circ (L^{\mathrm{wt}} \times E)\colon A^{\mathrm{wt}} \times B^{\mathrm{wt}} \to \mathrm{QCoh}_X$, and the only other condition for $E$ to be a parabolic sheaf is that it should be $A^{\mathrm{wt}}$-equivariant. Pulling back $\rho^E$ along $f$, and using the fact that the diagram

$$\begin{tikzcd} f^*\mathrm{Pic}_X \times f^*\mathrm{QCoh}_X \arrow[r, "f^*\otimes_X"] \arrow[d] & f^*\mathrm{QCoh}_X \arrow[d] \\ \mathrm{Pic}_Y \times \mathrm{QCoh}_Y \arrow[r, "\otimes_Y"] & \mathrm{QCoh}_Y \end{tikzcd}$$

is 2-commutative, we get a pseudo-periods isomorphism for $f^*E$. Finally, $f^*E$ is clearly $f^*A^{\mathrm{wt}}$-equivariant, so it is a parabolic sheaf. □

One can show that the pullback of parabolic sheaves thus defined is compatible with the BV equivalence and pullback of quasi-coherent sheaves on the root stacks, so it coincides (as always up to isomorphism) with the one we defined before.

## 3.2 Families of parabolic sheaves

Using the notion of pullback, we can now define families of parabolic sheaves on a fixed log scheme $X$, and with respect to a Kummer morphism $j\colon A \to B$. We define a fibered category $\underline{\mathrm{Par}}_X \to (\mathrm{Sch})$ by setting, for a scheme $T$, $\underline{\mathrm{Par}}_X(T) = \mathrm{Par}(T \times_k X, \pi_T^* j)$ (with $\pi_T\colon T \times_k X \to X$ the second projection), and by declaring that pullback along a morphism $f\colon S \to T$ over $k$ is given by the pullback of parabolic sheaves along the induced morphism $S \times_k X \to T \times_k X$ of log schemes.

Of course here $T \times_k X$ plays the role of a trivial family with fiber $X$, and a parabolic sheaf over $T \times_k X$ is seen as a "naive" (i.e. without any flatness hypothesis) family of parabolic sheaves over $X$.

On the other hand we have a second fibered category $\underline{\mathrm{QCoh}}_{X_{B/A}} \to (\mathrm{Sch})$ of quasi-coherent sheaves over the root stack $X_{B/A}$, where for a scheme $T$, we have

$$\underline{\mathrm{QCoh}}_{X_{B/A}}(T) = \mathrm{QCoh}(T \times_k X_{B/A}) = \mathrm{QCoh}((T \times_k X)_{\pi_T^* B/\pi_T^* A})$$

with pullback along $S \to T$ defined by the induced morphism $S \times_k X_{B/A} \to T \times_k X_{B/A}$.

To ease the notation, for the rest of this chapter $\mathfrak{X}$ will denote the root stack $X_{B/A}$, and for a scheme $T$, a subscript $(-)_T$ will denote a base change to $T$, or to the fibered product $X_T = T \times_k X$ along the projection $\pi_T \colon X_T \to X$ (this ambiguity should cause no real confusion). For example $L_T \colon A_T \to \mathrm{Div}_{X_T}$ will denote the pullback log structure.

Also, in what follows we will repeatedly consider properties of parabolic sheaves on $X_T$ relative to the base $T$. It will be useful to keep in mind the following diagram, where all the squares are cartesian.

$$\begin{tikzcd}
\mathfrak{X}_T \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\
X_T \arrow[r] \arrow[d] & X \arrow[d] \\
T \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

Note that $\mathfrak{X}_T \cong (X_T)_{B_T/A_T}$, from Proposition 1.2.33. Moreover the projection $\mathfrak{X}_T \to X_T$ is a coarse moduli space for any $T$, since the log structure of $T$ is fine and saturated (see 1.2.32).

The proof of Proposition 3.1.2 yields, with minor modifications, the following result.

**Proposition 3.2.1.** *There are equivalences* $\Phi \colon \underline{\mathrm{QCoh}}_{\mathfrak{X}} \to \underline{\mathrm{Par}}_X$ *and* $\Phi \colon \underline{\mathrm{Par}}_X \to \underline{\mathrm{QCoh}}_{\mathfrak{X}}$*, restricting to the BV equivalences on the fiber categories.*

This allows us to systematically transport various (absolute and relative) notions from ordinary quasi-coherent sheaves to parabolic sheaves. Some of the notions that we will examine will also have a parabolic interpretation.

**Definition 3.2.2** (Meta-definition)**.** A parabolic sheaf $E \in \mathrm{Par}(X_T, j_T)$ has some property, absolute or over the base $T$ (for example is *coherent, finitely generated, finitely presented, locally free, flat over* $T$), if the corresponding $\Psi(E) \in \mathrm{QCoh}(\mathfrak{X}_T)$ has said property.

These definitions also make sense for an arbitrary log scheme, not of the form $X_T$. We restrict to this case because we are interested in families of parabolic sheaves over a fixed log scheme $X$.

Let us now focus on moduli theory for sheaves on $\mathfrak{X}$. This requires some hypotheses.

**Assumptions 3.2.3.** From now on $X$ will be projective over $k$, with a fixed very ample line bundle $\mathcal{O}_X(1)$. This polarization will be fixed throughout all that follows, so we will omit it from the notations.

As sketched in the introduction, the strategy is to reduce to moduli of coherent sheaves on the root stack $\mathfrak{X}$. For this we need to introduce some properties of parabolic sheaves, that will single out the correct concept of "family of coherent parabolic (pure) sheaves".

Let us start with coherence: the definition is contained in 3.2.2, but let us spell it out again.

**Definition 3.2.4.** A parabolic sheaf $E \in \mathrm{Par}(X_T, j_T)$ is *coherent* if the corresponding $\Psi(E) \in \mathrm{QCoh}(\mathfrak{X}_T)$ is coherent.

Here is a parabolic interpretation of coherence.

**Proposition 3.2.5.** *A parabolic sheaf $E \in \mathrm{Par}(X_T, j_T)$ is coherent if and only if for every étale morphism $U \to X_T$ and every section $b \in B_T^{\mathrm{wt}}(U)$, the quasi-coherent sheaf $E_b \in \mathrm{QCoh}(U)$ is coherent.*

*Proof.* Assume first that $E$ is coherent. Then, since $\pi_T \colon \mathfrak{X}_T \to T$ is proper, the pushforward $E_b = (\pi_T)_*(E \otimes \Lambda_b)$ is still coherent.

In the other direction, this is a local problem so we can assume that there is a global chart. In this case, recall that the quasi-coherent sheaf on $\mathfrak{X}$ corresponding to $E$ is obtained by taking $\bigoplus_{v \in Q^{\mathrm{gp}}} E_v$ as a sheaf on $X$, with an action of the sheaf of algebras $A = \bigoplus_{u \in P^{\mathrm{gp}}} L_u$, and then by descending it on the quotient stack $\mathfrak{X}$. Now it suffices to notice that a finite number of the $E_v$ generate the direct sum as a sheaf of $A$-modules (thanks to the pseudo-periods isomorphism), and since the $E_v$'s are coherent we are done. $\square$

Now let us turn to flatness.

**Definition 3.2.6.** A parabolic sheaf $E \in \mathrm{Par}(X_T, j_T)$ is *flat* over $T$ if the corresponding $\Psi(E) \in \mathrm{QCoh}(\mathfrak{X}_T)$ is flat over $T$.

The following proposition gives a parabolic interpretation of flatness.

**Proposition 3.2.7.** *A parabolic sheaf $E \in \mathrm{Par}(X_T, j_T)$ is flat over $T$ if and only if for every étale morphism $U \to X_T$ and every section $b \in B_T^{\mathrm{wt}}(U)$ the quasi-coherent sheaf $E_b \in \mathrm{QCoh}(U)$ is flat over $T$.*

*Proof.* Assume first that the sheaf $\Psi(E) \in \mathrm{QCoh}(\mathfrak{X}_T)$ is flat over $T$. Given $f \colon U \to X_T$ étale, call $\mathfrak{U}$ the base change of $\mathfrak{X}_T$ to $U$, $\pi_U \colon \mathfrak{U} \to U$ the projection to the coarse moduli space, and $f_{\mathfrak{U}} \colon \mathfrak{U} \to \mathfrak{X}_T$ the base change of $f$.

$$\begin{tikzcd} \mathfrak{U} \arrow[r, "f_{\mathfrak{U}}"] \arrow[d, "\pi_U"'] & \mathfrak{X}_T \arrow[d, "\pi_T"] \\ U \arrow[r, "f"] & X_T \end{tikzcd}$$

Now for $b \in B_T^{\mathrm{wt}}(U)$ we have

$$E_b = (\pi_U)_*(f_{\mathfrak{U}}^* \Psi(E) \otimes (\Lambda_T)_b) \in \mathrm{QCoh}(U).$$

Since $(\Lambda_T)_b$ is invertible (so flat over $T$, since $\mathfrak{X}_T$, and hence $\mathfrak{U}$, are) and $\Psi(E)$ is flat over $T$ by assumption (so that also its pullback to $\mathfrak{U}$ is), the sheaf $f_{\mathfrak{U}}^* \Psi(E) \otimes (\Lambda_T)_b$ is flat over $T$. Furthermore since $\mathfrak{X}$ is tame, so that $\mathfrak{U}$ also is, the pushforward along the projection to the coarse moduli space

$\pi_U : \mathfrak{U} \to U$ preserves flatness over the base $T$ (Corollary 1.3 of [Nir]). In conclusion $E_b$ is flat over $T$.

Conversely, since the question is local we can assume that $X$ has a global chart.

Now recall once again that the sheaf $\Psi(E)$ is defined by forming $\bigoplus_{q \in Q^{\mathrm{gp}}} E_q$, a quasi-coherent sheaf on $X$. Then this is regarded as a quasi-coherent sheaf on the relative spectrum of a sheaf of algebras on $X$, and by descent this gives a quasi-coherent sheaf on the root stack $\mathfrak{X}$ (which is a quotient stack of this relative spectrum). Now since by assumption all the $E_q$ are flat over $T$, their direct sum is flat over $T$, and the quasi-coherent sheaf induced on the relative spectrum mentioned above will be as well. Finally by descent $\Psi(E)$ itself will be flat over $T$.

□

In moduli theory of coherent sheaves (and in all of moduli theory, in fact), flatness is a crucial condition to impose on a family.

**Definition 3.2.8.** A *family of parabolic sheaves* on $X$ with denominators in $B/A$ over a base scheme $T$ is a coherent parabolic sheaf $E \in \mathrm{Par}(X_T, j_T)$ that is flat over $T$.

From now on the wording "family of parabolic sheaves" will always include this flatness condition.

The last important concept in moduli of coherent sheaves is pureness.

For the definition of a pure sheaf on an algebraic stack we refer to [Nir] and [Lie07]. It is the natural generalization of the concept for schemes; one possible definition is that a coherent sheaf on a (noetherian) Artin stack is pure if and only if its pullback to a smooth presentation is. Moreover one can define the support of a coherent sheaf $F$ on an algebraic stack $\mathcal{X}$ as the closed substack defined by the kernel of the morphism $\mathcal{O}_{\mathcal{X}} \to \mathrm{End}(F)$. The *dimension* of $F$ will be the dimension of the support, and a sheaf is pure of dimension $d$ if and only if all of its subsheaves have dimension $d$.

We declare the zero sheaf to be pure of arbitrary dimension. This allows us to simplify some statements and does no harm, since the property of being zero for a flat sheaf over a projective family if open and closed.

We will say that a coherent sheaf $F$ is *torsion-free* on a noetherian algebraic stack $\mathcal{X}$ if it is pure of maximal dimension (the dimension of $\mathcal{X}$). Note that this does not imply that $F$ is supported everywhere unless $\mathcal{X}$ is integral.

**Definition 3.2.9.** A parabolic sheaf $E \in \mathrm{Par}(X_T, j_T)$ is *pure of dimension $d$* if the corresponding $\Psi(E) \in \mathrm{QCoh}(\mathfrak{X}_T)$ is pure of dimension $d$.

As for the preceding properties, there is a parabolic interpretation of pureness.

**Proposition 3.2.10.** *A parabolic sheaf $E \in \mathrm{Par}(X_T, j_T)$ is pure of dimension $d$ if and only if for every étale morphism $U \to X_T$ and every section $b \in B_T^{\mathrm{wt}}(U)$ the quasi-coherent sheaf $E_b \in \mathrm{QCoh}(U)$ is pure of dimension $d$.*

Here we use the convention that the zero sheaf is pure of arbitrary dimension. Let us first prove the following lemma.

**Lemma 3.2.11.** *Let $E$ be a coherent sheaf on a noetherian scheme $X$, and set $d = \dim(F)$. Then $E$ is pure of dimension $d$ if and only if for every open subset $U \subseteq X$ such that $\dim(X \setminus U) < d$, the adjunction map $\sigma\colon E \to i_* i^* E$ is injective, where $i\colon U \to X$ is the inclusion.*

*Proof.* First note that the injectivity of $E \to i_* i^* E$ is equivalent to the fact that if $V \subseteq X$ is open, and $f \in E(V)$ is such that $f|_{U \cap V} = 0$, then $f = 0$.

Assume first that the second condition holds, and by contradiction that $E$ is not pure, so that there is a non-zero subsheaf $G \subseteq E$ with $\dim(G) < d$. Take $U = X \setminus \mathrm{Supp}(G)$, which is an open subscheme of $X$ with $\dim(X \setminus U) < d$. Now by assumption if $V \subseteq X$ is any open subset and $f \in E(V)$, if the restriction of $f$ to $U \cap V$ is zero, then $f$ itself is. In particular every section of $G(V) \subseteq E(V)$ will be zero, since it restricts to zero on $U \cap V$ by construction of $U$. So $G(V) = 0$ for any $V$, against the fact that $G$ was non-zero.

Vice versa, assume that there is an open subset $U \subseteq X$ with $\dim(X \setminus U) < d$ and such that $\sigma\colon E \to i_* i^* E$ is not injective. Set $G = \ker(\sigma)$, and notice that this is a subsheaf of $E$ of dimension strictly less than $d$, so that $E$ is not pure. In fact we have $\mathrm{Supp}(G) \subseteq X \setminus U$, since if $x \in U$, then the map $\sigma_x\colon E_x \to (i_* i^* E)_x \cong E_x$ is the identity, so $G_x = 0$. $\square$

**Corollary 3.2.12.** *If $\mathcal{X}$ is a noetherian DM stack and $E \in \mathrm{Coh}(\mathcal{X})$ is pure of dimension $d$ then for every open substack $\mathcal{U} \subseteq \mathcal{X}$ such that $\dim(\mathcal{X} \setminus \mathcal{U}) < d$, the adjunction map $\sigma\colon E \to i_* i^* E$ is injective, where $i\colon \mathcal{U} \to \mathcal{X}$ is the inclusion.*

*Proof.* One can take a groupoid presentation $R \rightrightarrows U \to \mathcal{X}$ and repeat the proof of the above lemma on $U$ with "equivariant" (with respect to the groupoid) sheaves and open subsets. $\square$

*Proof of Proposition 3.2.10.* Assume first that $G = \Psi(E)$ is pure of dimension $d$, and fix $f\colon U \to X$ étale, and $b \in B^{\mathrm{wt}}(U)$. Moreover call $\mathfrak{U}$ the base change of $\mathfrak{X}$ to $U$, $\pi_U\colon \mathfrak{U} \to U$ the projection, and $f_{\mathfrak{U}}$ the base change of $f$.

$$\begin{tikzcd} \mathfrak{U} \arrow[r, "f_{\mathfrak{U}}"] \arrow[d, "\pi_U"'] & \mathfrak{X} \arrow[d, "\pi"] \\ U \arrow[r, "f"] & X \end{tikzcd}$$

We will show that the coherent sheaf $E_b = (\pi_U)_*(f_{\mathfrak{U}}^*(G) \otimes \Lambda_b) \in \mathrm{Coh}(U)$ is pure of dimension $d$. Set $G' = f_{\mathfrak{U}}^*(G)$, and notice that this is still pure of dimension $d$ on $\mathfrak{U}$ (it may be zero).

Notice that since $\pi_U$ is proper and quasi-finite, if $E_b$ is not zero (in which case there is nothing to prove) than it has dimension $d$. Take an open subset $V \subseteq U$ with $\dim(U \setminus V) < d$, and call $i\colon V \to U$ the inclusion. We will show that the adjunction map $\sigma\colon E_b \to i_* i^* E_b$ is injective, and by Lemma 3.2.11 this will prove that $E_b$ is pure of dimension $d$.

Call $\mathfrak{V}$ the fiber product $V \times_U \mathfrak{U}$, denote by $j\colon \mathfrak{V} \to \mathfrak{U}$ the base change of the inclusion $V \subseteq U$ and $\pi_V\colon \mathfrak{V} \to V$ the base change of the projection. The map $j$ is an open immersion, so $\mathfrak{V}$ is an open substack of $\mathfrak{U}$, with complement of dimension less than $d$. Since $G'$ is pure, by Corollary 3.2.12 the map $\sigma'\colon G' \to j_* j^* G'$ is injective.

Now the pushforward $(\pi_U)_*$ and tensor product with $\Lambda_b$ are both exact functors, so the induced map

$$E_b = (\pi_U)_*(G' \otimes \Lambda_b) \to (\pi_U)_*(j_* j^*(G') \otimes \Lambda_b)$$

is still injective. Now note that by the projection formula we have

$$j_*j^*(G') \otimes \Lambda_b \cong j_*(j^*(G') \otimes j^*(\Lambda_b)) \cong j_*j^*(G' \otimes \Lambda_b).$$

From the cartesian diagram

$$\begin{tikzcd} \mathfrak{V} \arrow[r, "j"] \arrow[d, "\pi_V"'] & \mathfrak{U} \arrow[d, "\pi_U"] \\ V \arrow[r, "i"] & U \end{tikzcd}$$

we first see that $(\pi_U)_* j_* = i_*(\pi_V)_*$, and since $i$ is flat, by base change (Proposition 1.5 of [Nir]) we also have a canonical isomorphism $(\pi_V)_* j^* \cong i^*(\pi_U)_*$.

By putting everything together we have that the composite

$$E_b = (\pi_U)_*(G' \otimes \Lambda_b) \to (\pi_U)_* j_* j^*(G' \otimes \Lambda_b) \cong i_* i^*(\pi_U)_*(G' \otimes \Lambda_b) \cong i_* i^* E_b,$$

which coincides with the adjunction map of $E_b$, is injective, and this is what we had to show.

Note that some of these $E_b$ can be zero (see Example 1.3.9), which is consistent with our convention about the zero sheaf, but if $E$ is not zero as a parabolic sheaf, than necessarily we will have $E_b \neq 0$ for some $U \to X$ étale and $b \in B^{\mathrm{wt}}(U)$.

Now for the converse, assume that all the $E_b$'s are pure of dimension $d$, and that $E$ is not zero (otherwise there is nothing to prove). If by contradiction $\Psi(E)$ is not pure, then there is a non zero pure subsheaf $\Psi(G) \subseteq \Psi(E)$, of dimension strictly less than $d$, say $d' \geq 0$. Now pick $U \to X$ étale and $b \in B^{\mathrm{wt}}(U)$ such that $G_b \neq 0$.

By the first part of the proof $0 \neq G_b \subseteq E_b$ is pure of dimension $d'$, so $E_b$ is non zero and thus of dimension $d > d'$. In particular $E_b$ is not pure, and this contradicts the assumption. □

In case the log structure of $X$ is generically trivial, the maps between the pieces of a torsion-free parabolic sheaf are injective.

**Proposition 3.2.13.** *Let $X$ be a noetherian log scheme with generically trivial log structure and with a chart $P \to \mathrm{Div}(X)$, and let $j\colon P \to Q$ be a Kummer extension of fine saturated monoids. Take a torsion-free parabolic sheaf $E \in \mathrm{Par}(X, j)$. Then for any pair $q, q' \in Q^{\mathrm{wt}}$ such that $q \leq q'$, the morphism $E_q \to E_{q'}$ is injective.*

*Proof.* If $E_q$ is zero there is nothing to prove.

Otherwise, by assumption we have a schematically dense open subscheme $U \subseteq X$ on which the log structure is trivial. Consequently the restriction of the projection $X_{Q/P} \to X$ to $U$ is an isomorphism $U_{Q/P} \cong U$, and the morphism $E_q \to E_{q'}$ is an isomorphism on $U$. If $K$ is the kernel of this morphism, it follows that $K$ has dimension strictly less then the dimension of $X$, but then since $E_q$ is pure of maximal dimension by 3.2.10, we must have $K = 0$, i.e. the map is injective. □

Finally, we give the definition of a family of pure parabolic sheaves.

**Definition 3.2.14.** A *family of pure $d$-dimensional parabolic sheaves* on $X$ with denominators in $B/A$ over a base scheme $T$ is a coherent sheaf $E \in \mathrm{Par}(X_T, j_T)$ that is flat over $T$, and such that for any geometric point $t \to T$, the pullback $E_t$ on $X_t$ is pure of dimension $d$.

Now that we have these basic properties laid out we will discuss (semi-)stability, in order to get a well-behaved moduli stack.

## 3.3 Generating sheaves and stability conditions

As in the case of sheaves on schemes, if we want to construct moduli spaces we need to come up with some good notion of stability. The equivalence with coherent sheaves on an algebraic stack together with the theory for moduli of coherent sheaves on algebraic stacks of [Nir] suggest that a way to do this is to find a (canonical, ideally) generating sheaf on the root stack $X_{B/A}$. For generalities about generating sheaves see [Nir] and the references therein.

Although root stacks are probably always global quotient stacks, so that they will have generating sheaves, in general there does not seem to be a canonical choice of such a sheaf. It is possible to single out distinguished generating sheaves in presence of a global chart for the logarithmic structure, or in slight greater generality, of a locally constant sheaf of charts.

Another important aspect of this is the behavior of the stability with respect to the maps between the root stacks, when we have a morphism of Kummer extensions. This will be the main topic of the next chapter, where we will discuss a moduli theory with varying denominators.

We will recall Nironi's method along the way. We first give the definition of a generating sheaf, and recall how it is used to give a notion of Hilbert polynomial and slope. The definition makes sense for $\mathfrak{X}$ any tame Artin stack.

**Definition 3.3.1.** A locally free sheaf $\mathcal{E}$ of finite rank on $\mathfrak{X}$ is a *generating sheaf* if for any geometric point $x \to \mathfrak{X}$ the fiber $\mathcal{E}_x$ contains every irreducible representation of $\mathrm{Stab}(x)$.

Once we have a generating sheaf, we can define what Nironi calls the *modified Hilbert polynomial*. We will drop the adjective "modified" for brevity.

**Definition 3.3.2.** The *Hilbert polynomial* (with respect to $\mathcal{E}$) of a coherent sheaf $F \in \mathrm{Coh}(\mathfrak{X})$ is the Hilbert polynomial

$$P_{\mathcal{E}}(F) = P(\pi_*(F \otimes \mathcal{E}^\vee)) \in \mathbb{Q}[m]$$

of the coherent sheaf $\pi_*(F \otimes \mathcal{E}^\vee)$ on $X$, with respect to $\mathcal{O}_X(1)$.

Note that, since $\pi_*(- \otimes \mathcal{E}^\vee)$ preserves the dimension (see [Nir], Proposition 3.6), $P_{\mathcal{E}}(F)$ will be a polynomial of degree $d$ where $d = \dim(F)$, and as usual we can write it as

$$P_{\mathcal{E}}(F)(m) = \sum_{i=0}^{d} \frac{\alpha^i(F)}{i!} m^i$$

where $\alpha^i(F)$ are rational numbers, that depend also on $\mathcal{E}$ of course. Sometimes, when the sheaf $F$ is clear from the context, we will denote these coefficients just by $\alpha^i$

The number $\alpha^d(F)$, which is always positive, will be called the *multiplicity* of the sheaf $F$. If $X$ is integral and $F$ has maximal dimension, it is strictly related to the rank of the sheaf $\pi_*(F \otimes \mathcal{E}^\vee)$ on $X$.

**Definition 3.3.3.** The *reduced Hilbert polynomial* or *(generalized) slope* of a coherent sheaf $F \in \mathrm{Coh}(\mathfrak{X})$ is the polynomial

$$p_{\mathcal{E}}(F) = \frac{P_{\mathcal{E}}(F)}{\alpha^d(F)} = \frac{1}{d!} m^d + \ldots + \frac{\alpha^0(F)}{\alpha^d(F)} \in \mathbb{Q}[x].$$

**Remark 3.3.4.** We are aware that the word "slope" is usually reserved to the quotient

$$\mu(F) = \frac{\alpha^{d-1}(F)}{\alpha^d(F)},$$

that in the case of curves is closely related to the ratio $\deg(F)/\operatorname{rk}(F)$, but nonetheless in this document we will use it to mean the reduced Hilbert polynomial, since we will never have to mention the "real" slope.

This slope will give a notion of (semi-)stable parabolic sheaves, and we will restrict to that class in order to get well-behaved moduli stacks and moduli spaces. Before describing how this happens (which we will do in Section 3.3.4), let us focus on the choice of the generating sheaf.

### 3.3.1 The case of a variety with a divisor

To get some clues for the choice of the generating sheaf, let us look at the case of a projective variety $X$ over $k$, equipped with the log structure induced by an effective Cartier divisor. In this case, moduli spaces of parabolic sheaves with rational weights have been constructed in [MY92], by generalizing the classical GIT construction of moduli spaces of (semi-)stable coherent sheaves on a projective scheme. Their result in turn generalizes the first results of Seshadri [Ses82] on curves.

Let us recall their definition of a parabolic sheaf. Let $X$ be a projective smooth (connected) scheme over $k$, and $D \subseteq X$ an effective Cartier divisor.

**Definition 3.3.5.** A *MY-parabolic sheaf* $E_*$ on $X$ is given by the following data:

- a coherent torsion-free sheaf $E \in \operatorname{Coh}(X)$,
- a sequence of real numbers $a_1, \ldots, a_k$ called *weights*, such that $0 \leq a_1 < a_2 < \cdots < a_k < 1$, and
- a filtration $E(-D) = F_{k+1}(E) \subset F_k(E) \subset \cdots \subset F_1(E) = E$ of $E$, where $E(-D)$ is the image of the natural map $\mathcal{O}_X(-D) \otimes E \to E$.

The *rank* of $E_*$ will be the rank of the torsion-free sheaf $E$.

From now on we will assume that the weights $a_1, \ldots, a_k$ are rational numbers. This is crucial for the correspondence with quasi-coherent sheaves on a root stack to work, and also for the moduli theory developed in [MY92] (the rationality assumption is on page 94). Moreover, in light of what follows it is more convenient to think about the opposites $-1 < -a_k < \cdots < -a_1 \leq 0$.

Let us now describe explicitly how this definition is connected with our definition of a parabolic sheaf.

First observe that the divisor $D$ induces a log structure on $X$, given by the morphism $L\colon \mathbb{N} \to \operatorname{Div}(X)$ that sends $1 \in \mathbb{N}$ to $(\mathcal{O}_X(D), s)$, where $s$ is the section of $\mathcal{O}_X(D)$ corresponding to the natural map $\mathcal{O}_X \to \mathcal{O}_X(D)$. This coincides with what we called the "induced log structure" (which is "finer") in Example 1.2.7 only if $D$ is irreducible.

Given a MY-parabolic sheaf $E$, let us take $n$ to be the least common multiple of the denominators of the weights $a_i$, and consider the system of denominators $j\colon \mathbb{N} \to \frac{1}{n}\mathbb{N}$.

Now we define a parabolic sheaf, that we still denote by $E \in \mathrm{Par}(X, j)$, as follows. Set $a_i = \frac{b_i}{n}$ with $b_i \in \mathbb{N}$, and for $q = \frac{a}{n}$ with $-n < a \leq 0$ define $E_{\frac{a}{n}} = F_i(E)$, where $-b_i \leq a < -b_{i-1}$, with the convention that $b_0 = 1$ and $b_{k+1} = -1$. For an arbitrary $\frac{a}{n} \in \mathbb{Q}$, we set $E_{\frac{a}{n}} = E_{\frac{a'}{n}} \otimes \mathcal{O}_X(bD)$, where $-n < a' \leq 0$, $b \in \mathbb{N}$ and $\frac{a}{n} = \frac{a'}{n} + b$, and for $\frac{a}{n} \leq \frac{a'}{n}$ there is an obvious morphism $E_{\frac{a}{n}} \to E_{\frac{a'}{n}}$, which is either the identity or an inclusion. Moreover by construction there is a pseudo-periods isomorphism, and this gives a parabolic sheaf in our sense.

Conversely, given a parabolic sheaf $E \in \mathrm{Par}(X, j)$ such that the maps $E_{\frac{a}{n}} \to E_{\frac{a'}{n}}$ are all injective, we obtain a MY-parabolic sheaf by taking as weights the opposites of the numbers $\frac{i}{n} \in \mathbb{Q} \cap (-1, 0]$ such that $E_{\frac{i-1}{n}} \to E_{\frac{i}{n}}$ is not an isomorphism, and the filtration consisting of the sheaves $E_{\frac{a}{n}}$ with $-n \leq a \leq 0$, but without repetitions.

This gives an equivalence between MY-parabolic sheaves and parabolic sheaves with injective maps. The injectivity condition is implied by torsion-freeness of $E$, see Proposition 3.2.13.

From this description we see that the weights of the MY definition are nothing else than (the opposites of) what we could call "jumping numbers" for a parabolic sheaf with injective maps, i.e. the numbers $\frac{i}{n} \in (-1, 0]$ where the subsheaf $E_{\frac{i}{n}} \subseteq E$ "jumps" with respect to the preceding one.

**Remark 3.3.6.** If $E_*$ is a MY-parabolic sheaf and $L \in \mathrm{Pic}(X)$ is an invertible sheaf, then there is a natural MY-parabolic sheaf $E_* \otimes L$ obtained by tensoring everything with $L$. In particular this gives for any $m \in \mathbb{Z}$ the MY-parabolic sheaf $E_*(m) = E_* \otimes \mathcal{O}_X(m)$.

In [MY92], in order to construct moduli spaces, they define a parabolic Hilbert polynomial. Let us briefly recall their definitions and results.

**Definition 3.3.7.** The *MY-parabolic Euler characteristic* of a MY-parabolic sheaf $E_*$ is the rational number

$$\chi_{MY}(E_*) = \chi(E(-D)) + \sum_{i=1}^{k} a_i \chi(G_i),$$

where $G_i$ is the quotient $F_i(E)/F_{i+1}(E)$.

The *MY-parabolic Hilbert polynomial* of $E_*$ is the polynomial with rational coefficients given by

$$P_{MY}(E_*)(m) = \chi_{MY}(E_*(m)),$$

where $E_*(m) = E_* \otimes \mathcal{O}_X(m)$.

The *MY-reduced parabolic Hilbert polynomial* of $E_*$ is the polynomial

$$p_{MY}(E_*) = \frac{P_{MY}(E_*)}{\mathrm{rk}(E_*)}.$$

**Remark 3.3.8.** The rank of $E_*$ is the rank of the torsion-free sheaf $E$. Note that, since $G_i = F_i(E)/F_{i+1}(E)$ is generically zero, if instead of using $\mathrm{rk}(E_*)$ we use the leading coefficient of $P_{MY}(E_*)$ (as one does when dealing with pure sheaves, not necessarily torsion-free), we would get a scalar multiple of $p_{MY}(E_*)$, which of course would then give the same stability condition.

**Definition 3.3.9.** A *parabolic subsheaf* $F_* \subseteq E_*$ of a MY-parabolic sheaf $E_*$ is a MY-parabolic sheaf $F_*$ such that

- $F \subseteq E$ is a subsheaf with $E/F$ torsion-free,
- $F_{b_i} \subseteq E_{a_j}$ for every $i$, where $b_i$ are the weights of $F_*$, and $a_j$ is the smallest weight of $E_*$ such that $a_j \geq b_i$.

**Remark 3.3.10.** This is slightly different from the stronger definition given in [MY92], and on the other hand agrees with the one given later in [Yok93].

**Definition 3.3.11.** A MY-parabolic sheaf $E_*$ is (semi-)stable if for any parabolic subsheaf $F_* \subseteq E_*$ we have

$$p_{MY}(F_*) \ (\leq) \ p_{MY}(E_*).$$

This notion of (semi-)stability has many properties resembling the ones of classical (semi-)stability for coherent sheaves, for example the existence of Harder-Narasimhan and Jordan-Hölder filtrations.

Now let us define the moduli functor for parabolic sheaves. Fix finitely many rational numbers $a_1, \dots, a_k$ with $0 \leq a_1 < \dots < a_k < 1$, polynomials $H, H_1, \dots, H_k$ and define for a scheme $T$

$$\overline{MY}(T) \quad = \quad \left\{ \begin{array}{c} \text{flat families of MY-parabolic sheaves } E_* \text{ on } X_T \\ \text{with weights } a_1, \dots, a_k \text{ and property (*)} \end{array} \right\} \Big/ \sim .$$

where property (*) is: for any geometric point $t \to T$, the restriction $(E_*)_t$ is semi-stable, $P((E)_t) = H$ and $P((E)_t/F_{i+1}(E)_t) = H_i$, where $P$ denotes the Hilbert polynomial on $X$. We omit the dependence of the functor on the polynomials and the weights for brevity.

The equivalence relation $\sim$ is defined as follows: $E_* \sim F_*$ if there are global filtrations of the two sheaves such that:

- they restrict to Jordan-Hölder filtrations on every geometric point,
- the associated graded parabolic sheaves are flat over the base scheme, and differ by an invertible sheaf coming from the base scheme.

Note that the polynomials $H_i$ and $H$ determine the parabolic Hilbert polynomial of $(E_*)_t$. Vice versa, if we fix the parabolic Hilbert polynomial we have finitely many choices for $H, H_1, \dots, H_k$, since fixing these is equivalent to fixing the Hilbert polynomials of the pieces of the sheaf in $[-1, 0)$. We also have a subfunctor $MY \subseteq \overline{MY}$, corresponding to families of stable MY-parabolic sheaves.

These functors satisfy some boundedness and openness properties. The following is the main result of [MY92] and [Yok93] (in the latter the result is stated more generally for parabolic Higgs bundles).

**Theorem 3.3.12** ([MY92, Yok93]). *The functor $\overline{MY}$ has a coarse moduli space $\overline{M}$ which is locally of finite type and separated. If the family of parabolic sheaves with fixed data $H, H_1, \dots, H_k$ and $a_1, \dots, a_k$ is bounded (for example if* $\mathrm{char}(k) = 0$*), then $\overline{M}$ is projective over $k$.*

*The subfunctor $MY$ has a coarse moduli space $M$, which is an open subscheme of $\overline{M}$.*

To extend these result to general log schemes we aim to find in this particular case a generating sheaf $\mathcal{E}$ on $\mathfrak{X} = X_{\frac{1}{n}\mathbb{N}/\mathbb{N}}$ (for some fixed $n$) that gives the parabolic Hilbert polynomial of Maruyama and Yokogawa, where the parabolic sheaves have weights in $\frac{1}{n}\mathbb{N}$.

A little thought produces the locally free sheaf

$$\mathcal{E} = \mathcal{O}_{\mathfrak{X}}(\mathcal{D}) \oplus \mathcal{O}_{\mathfrak{X}}(2\mathcal{D}) \cdots \oplus \mathcal{O}_{\mathfrak{X}}(n\mathcal{D}) = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathfrak{X}}(i\mathcal{D}),$$

where $\mathcal{D}$ is the universal root on $\mathfrak{X}$ of the pullback of the divisor $D$: in fact assume we have a torsion-free parabolic sheaf $E \in \mathrm{Par}(X, j)$, with weights $a_1, \ldots, a_k$, and thus jumping numbers $-a_k, \ldots, -a_1 \in (-1, 0] \cap \frac{1}{n}\mathbb{N}$. Let us write $a_j = \frac{b_j}{n}$ for $b_j \in \mathbb{N}$.

The MY-parabolic Hilbert polynomial of the MY-parabolic sheaf corresponding to $E$ is

$$\begin{aligned} P_{MY}(E_*)(m) &= \chi(E(-D)(m)) + \sum_{i=1}^{k} \frac{b_i}{n} \chi(G_i(m)) \\ &= \frac{1}{n}\left( n\chi(E(-D)(m)) + \sum_{i=1}^{k} b_i \chi(F_i(E)(m)/F_{i+1}(E)(m)) \right) \\ &= \frac{1}{n}\left( n\chi(E(-D)(m)) + \sum_{i=1}^{k} b_i (\chi(F_i(E)(m)) - \chi(F_{i+1}(E)(m))) \right) \\ &= \frac{1}{n}\left( (n - b_k)\chi(E(-D)(m)) + \sum_{i=0}^{k-1} (b_{i+1} - b_i) \chi(F_{i+1}(E)(m))) \right) \end{aligned}$$

where in the last line $b_0 = 0$.

On the other hand the Hilbert polynomial we get by using the sheaf $\mathcal{E}$ above is

$$\begin{aligned} P_{\mathcal{E}}(E)(m) = P(\pi_*(E \otimes \mathcal{E}^\vee))(m) &= P\left( \bigoplus_{i=1}^{n} \pi_*(E \otimes \mathcal{O}_{\mathfrak{X}}(-i\mathcal{D})) \right)(m) \\ &= \sum_{i=1}^{n} P\left(E_{-\frac{i}{n}}\right)(m) = \sum_{i=1}^{n} \chi\left(E_{-\frac{i}{n}}(m)\right) \end{aligned}$$

(recall that $E_{-\frac{i}{n}} = \pi_*(E \otimes \mathcal{O}_{\mathfrak{X}}(-i\mathcal{D})))$ and this last expression coincides with $P_{MY}(E_*)(m)$ after dividing by $n$, since among the sheaves $E_{-\frac{i}{n}}$, with $1 \leq i \leq n$ there are exactly $n - b_k$ copies of $E(-D) = F_{k+1}(E)$ (just to the right of $-1$), exactly $b_k - b_{k-1}$ copies of $F_k(E)$, and so on. Of course the constant factor $\frac{1}{n}$ does not affect the notion of (semi-)stability that we get.

**Remark 3.3.13.** This says that the notion of (semi-)stability for MY-parabolic sheaves is equivalent to the notion of stability when we use the generating sheaf $\mathcal{E}$ introduced above, so the two moduli theories that we get should be the same.

There is a minor detail, though, related to the fact that by working on the root stack $\mathfrak{X} = X_{\frac{1}{n}\mathbb{N}/\mathbb{N}}$ we only bound the denominators of the weights (in the divisibility sense), when in

[MY92] and [Yok93], the authors fix the jumps of the parabolic sheaves, and the parabolic Hilbert polynomials of the quotients $F_i(E)/F_{i+1}(E)$.

We will return later on this point, and describe a comparison between our moduli stacks and spaces and Maruyama and Yokogawa's (Section 3.3.5).

### 3.3.2 The general case

Now we turn to the general case of a log scheme $X$ with a global chart $L\colon P \to \mathrm{Div}(X)$, and a Kummer extension $j\colon P \to Q$, that gives a chart for the system of denominators $A \to B$. The previous example suggests the following construction: since $Q$ is sharp and fine, its finite number of indecomposable elements are a minimal set of generators (see Proposition 1.1.13), call them $q_1, \dots, q_r$. Moreover, call $d_i$ the order of the image of $q_i \in Q^{\mathrm{gp}}$ in the quotient $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$.

If $\Lambda\colon Q \to \mathrm{Div}(\mathfrak{X})$ is the universal lifting of the log structure of $X$, for any $q_i$ we have an associated invertible sheaf $\Lambda_i = \Lambda(q_i)$ on $\mathfrak{X}$, and we consider the locally free sheaf

$$\mathcal{E} = \mathcal{E}_{Q/P} = \bigoplus_{1 \leq a_i \leq d_i} \Lambda\left(\sum_i a_i q_i\right) = \bigotimes_{i=1,\dots,r}\left(\bigoplus_{j=1,\dots,d_i} \Lambda_i^{\otimes j}\right).$$

Note that if $X$ is the log scheme given by a variety with an effective Cartier divisor, then this sheaf corresponds to the one described in the last section: in fact for $\mathbb{N} \subseteq \frac{1}{n}\mathbb{N}$, we have the only indecomposable element $\frac{1}{n}$, and the order $d$ is exactly $n$, so that

$$\mathcal{E} = \bigoplus_{1 \leq i \leq n} \Lambda\left(\frac{i}{n}\right) = \bigoplus_{1 \leq i \leq n} \mathcal{O}_{\mathfrak{X}}(i\mathcal{D}),$$

since in this case the universal DF structure of $\mathfrak{X}$ is the functor $\Lambda\colon \frac{1}{n}\mathbb{N} \to \mathrm{Div}(\mathfrak{X})$ that sends $\frac{1}{n}$ to the universal root $(\mathcal{O}_{\mathfrak{X}}(\mathcal{D}), s)$ of the pullback of the divisor $D$.

We will denote this sheaf by $\mathcal{E}$ when the Kummer extension is clear, and by $\mathcal{E}_{Q/P}$ when it needs to be specified. In particular we will write $\mathcal{E}_n$ for $\mathcal{E}_{\frac{1}{n}P/P}$, or more generally $\mathcal{E}_{\frac{1}{n}Q/P}$ for a fixed Kummer extension $P \subseteq Q$, which will be clear from the context.

**Remark 3.3.14.** One could argue that the sheaf

$$\mathcal{E}' = \bigoplus_{0 \leq a_i < d_i} \Lambda\left(\sum_i a_i q_i\right) = \bigotimes_{i=1,\dots,r}\left(\bigoplus_{j=0,\dots,d_i-1} \Lambda^{\otimes j}\right).$$

in which we take $\mathcal{O}_{\mathfrak{X}}$ instead of $\Lambda_i^{\otimes d_i}$ (which is the pullback of something from $X$, so corresponds to the trivial representation of the stabilizer of any point of $\mathfrak{X}$) in each summand would be somewhat more natural. In fact one could twist different pieces of the direct sum with an invertible sheaf coming from $X$ and still have a perfectly good generating sheaf.

The choice of the one we singled out is guided by the fact that in the case of a variety with a divisor it gives back the (semi-)stability of Maruyama and Yokogawa, and, as we will see in the next chapter, it will allow semi-stability to be preserved after changing denominators, something that does not happen for example with the generating sheaf written down in the last formula. Actually we will also need to use the alternative sheaf $\mathcal{E}'$ in that instance, but only as an accessory.

**Proposition 3.3.15.** *The locally free sheaf $\mathcal{E}$ is a generating sheaf on $\mathfrak{X}$.*

*Proof.* Recall from that the global chart for the system of denominators gives the following description for the stack of roots: the map $X \to [\mathrm{Spec}(k[P])/\widehat{P}]$ corresponding to the chart $P \to \mathrm{Div}(X)$ for the logarithmic structure of $X$ gives a $\widehat{P}$-torsor $\eta\colon E \to X$, and $\mathfrak{X}$ is isomorphic to the quotient stack $[E \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[Q])/\widehat{Q}]$, where the action of $\widehat{Q}$ on the first factor is induced by the action of $\widehat{P}$ on $E$ and the natural homomorphism $\widehat{Q} \to \widehat{P}$. In particular a quasi-coherent sheaf on $\mathfrak{X}$ corresponds to a $\widehat{Q}$-equivariant quasi-coherent sheaf on $E \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[Q])$, or equivalently to a $Q^{\mathrm{gp}}$-graded quasi-coherent sheaf of modules over the sheaf of rings $B = A \otimes_{k[P]} k[Q]$, where $A = \eta_* \mathcal{O}_E$.

Now fix a geometric point $p \to \mathfrak{X}$. We will show that the fiber $\mathcal{E}_p = p^*\mathcal{E}$ at $p$ of $\mathcal{E}$ contains every irreducible representation of the stabilizer group $\mathrm{Stab}(p) \subseteq \widehat{G} \subseteq \widehat{Q}$. Notice that being a closed subgroup of a diagonalizable group we will have $\mathrm{Stab}(p) = \mathrm{D}[M]$ for a quotient $M$ of the group $G$, and the action of $\mathrm{Stab}(p)$ on $\mathcal{E}_p$ will correspond to an $M$-grading. Moreover since $\mathrm{Stab}(p)$ is diagonalizable, irreducible representations correspond to characters, so what we need to verify is that in the $M$-grading every piece is non zero.

This grading is obtained as follows: the $Q^{\mathrm{gp}}$-grading on the sheaf corresponding to $\mathcal{E}$ on $E \times_{\mathrm{Spec}(k[P])} \mathrm{Spec}(k[Q])$ is inherited by the various summands, and by construction the sheaf corresponding to $\Lambda_i = \Lambda(q_i)$ is in degree $q_i$. This gives a $Q^{\mathrm{gp}}$ grading on $\mathcal{E}_p$ by pulling back, and we finally get the $M$-grading by means of the homomorphism $Q^{\mathrm{gp}} \to G \to M$.

More explicitly, following through the above we find

$$\mathcal{E}_p \cong \bigoplus_{m \in M} k(p)^{\oplus a(m)}$$

where $a(m)$ is the number of $r$-tuples $(e_1, \dots, e_r)$ of integers such that $0 < e_i \leq d_i$ and $e_1 m_1 + \dots + e_r m_r = m$, where $m_i$ is the image of $q_i$ in $M$. Since the $q_i$'s generate $G$, the $m_i$'s will generate $M$ (and still have order at most $d_i$), so $a(m) \geq 1$ for any $m$. This means that every character of $\mathrm{Stab}(p)$ appears in $\mathcal{E}_p$, so $\mathcal{E}$ is a generating sheaf on $\mathfrak{X}$. □

This settles the choice of a generating sheaf in the case where there is a global chart. One would hope that this construction could be generalized to an arbitrary log scheme, by patching the local generating sheaves on open subsets where there is a chart. Unfortunately, it is not so clear how to do this.

**Example 3.3.16.** Let us look at the simplest example of a log scheme coming from a normal crossing divisor, but not simple normal crossing.

Take $X$ to be a projective smooth surface over $k$, with an irreducible curve $D$ with one ordinary node $p \in D \subseteq X$ as effective divisor, inducing a log structure $L\colon A \to \mathrm{Div}_X$. In this example $X$ does not have a global chart: it has a chart with monoid $\mathbb{N}$ on the complement of the node $U = X \setminus \{p\}$, and one with $\mathbb{N}^2$ in some étale neighborhood $V \to X$ of the node $p$, where the two branches are separated, call them $D_1, D_2$.

Let us say we are considering square roots, so let $\mathfrak{X}, \mathfrak{U}, \mathfrak{V}$ denote the root stacks of $X, U, V$ (the last two with the pullback log structure) with respect to the Kummer extensions $A \to \frac{1}{2}A$, $\mathbb{N} \to \frac{1}{2}\mathbb{N}$ and $\mathbb{N}^2 \to \frac{1}{2}\mathbb{N}^2$ respectively. Our construction gives us generating sheaves $\mathcal{E}_{\mathfrak{U}}$ and $\mathcal{E}_{\mathfrak{V}}$

on the root stacks of $U$ and $V$, and the idea would be to glue them along the intersection $\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{V}$, which is none other than the root stack of $V \setminus \{q\}$, where $q$ is the preimage of the node.

It is clear though that this can not work for a pretty stupid reason: the sheaf $\mathcal{E}_{\mathfrak{U}} = \mathcal{O}_{\mathfrak{U}}(\mathcal{D}) \oplus \mathcal{O}_{\mathfrak{U}}(2\mathcal{D})$ has rank 2 and $\mathcal{E}_{\mathcal{V}} = \mathcal{O}_{\mathfrak{V}}(\mathcal{D}_1 + \mathcal{D}_2) \oplus \mathcal{O}_{\mathfrak{V}}(2\mathcal{D}_1 + \mathcal{D}_2) \oplus \mathcal{O}_{\mathfrak{V}}(\mathcal{D}_1 + 2\mathcal{D}_2) \oplus \mathcal{O}_{\mathfrak{V}}(2\mathcal{D}_1 + 2\mathcal{D}_2)$ has rank 4.

Then one could think of constructing a generating sheaf on $\mathfrak{X}$ by taking $\mathcal{O}_{\mathfrak{X}}(\mathcal{D}) \oplus M \oplus N$, where $M$ and $N$ have ranks 2 and 1 respectively, and are obtained by descent from $\mathcal{O}_{\mathfrak{V}}(2\mathcal{D}_1 + \mathcal{D}_2) \oplus \mathcal{O}_{\mathfrak{V}}(\mathcal{D}_1 + 2\mathcal{D}_2)$ and $\mathcal{O}_{\mathfrak{U}}(2\mathcal{D}) \oplus \mathcal{O}_{\mathfrak{U}}(\mathcal{D})$, and $\mathcal{O}_{\mathfrak{V}}(2\mathcal{D}_1 + 2\mathcal{D}_2)$ and $\mathcal{O}_{\mathfrak{U}}(2\mathcal{D})$ respectively (we will see that something like this works in the equivariant case described below).

This attempt also fails: the sheaves $\mathcal{O}_{\mathfrak{V}}(2\mathcal{D}_1 + 2\mathcal{D}_2)$ and $\mathcal{O}_{\mathfrak{U}}(2\mathcal{D})$ are naturally identified after restricting to $\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{V}$, so they give the desired sheaf $N$ by descent, but there is no natural way to identify the restrictions of $\mathcal{O}_{\mathfrak{V}}(2\mathcal{D}_1 + \mathcal{D}_2) \oplus \mathcal{O}_{\mathfrak{V}}(\mathcal{D}_1 + 2\mathcal{D}_2)$ and $\mathcal{O}_{\mathfrak{U}}(2\mathcal{D}) \oplus \mathcal{O}_{\mathfrak{U}}(\mathcal{D})$. The "moral" reason for this is that in $V$ one can tell apart the branches of the curve around the node, and in $U$ one can not do it.

This shows that the obvious strategy will not work in general. The next example will demonstrate that if we add some structure to the situation, then we can obtain a generating sheaf.

**Example 3.3.17.** A case where we get a generating sheaf is the following: assume that the $X$ described in the last example admits a $\mu_2$-cover $Y \to X$ (assume $\mathrm{char}(k) \neq 2$, for simplicity) from another surface $Y$, that has two irreducible smooth curves $D_1, D_2 \subseteq Y$ that are exchanged by the action, and that map to $D \subseteq X$. In other words there is an involution $\mu \colon Y \to Y$ that exchanges $D_1$ and $D_2$, and we have $X = Y/\mu$ and $D = (D_1 \cup D_2)/\mu$.

For example, we could take a curve $C$ of genus 2 and fix two Weierstrass points $p, q \in C$. Then we can embed $C$ in its Jacobian $J$, a surface, by $c \mapsto \mathcal{O}_C(c - p)$, and consider the translation $\widetilde{C}$ of $C$ by the point of order 2 given by $\mathcal{O}_C(q - p)$. Since the self intersection $C^2$ is 2 and $C \cap \widetilde{C}$ contains $\mathcal{O}_C$ and $\mathcal{O}_C(q - p)$, these are the only points in the intersection and the intersection is transverse. The quotient of $J$ by the translation by $\mathcal{O}_C(q - p)$ gives our $X$, and the image of the union $C \cup \widetilde{C}$ is the nodal curve.

Back to the general situation, note first of all that the morphism of log schemes $f \colon Y \to X$ (where the log structures are given by the divisors) is strict, so the following diagram is cartesian

$$\begin{tikzcd}
\mathfrak{Y} \arrow[r, "\overline{f}"] \arrow[d, "\pi_Y"'] & \mathfrak{X} \arrow[d, "\pi_X"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

where $\mathfrak{Y}$ and $\mathfrak{X}$ are the root stacks parametrizing square roots of the log structures.

In particular $\overline{f} \colon \mathfrak{Y} \to \mathfrak{X}$ is a $\mu_2$-torsor (call $\overline{\mu} \colon \mathfrak{Y} \to \mathfrak{Y}$ the corresponding involution), and we can use descent for quasi-coherent sheaves with respect to $\overline{f}$: note that we have natural isomorphisms $\overline{\mu}^* \mathcal{O}_{\mathfrak{Y}}(\mathcal{D}_1) \cong \mathcal{O}_{\mathfrak{Y}}(\mathcal{D}_2)$ and $\overline{\mu}^* \mathcal{O}_{\mathfrak{Y}}(\mathcal{D}_2) \cong \mathcal{O}_{\mathfrak{Y}}(\mathcal{D}_1)$, so both $\mathcal{O}_{\mathfrak{Y}}(2\mathcal{D}_1 + \mathcal{D}_2) \oplus \mathcal{O}_{\mathfrak{Y}}(\mathcal{D}_1 + 2\mathcal{D}_2)$ and $\mathcal{O}_{\mathfrak{Y}}(2\mathcal{D}_1 + 2\mathcal{D}_2)$ will be $\mu_2$-equivariant. By descent we get two locally free sheaves $M$ and $N$ on $\mathfrak{X}$ of rank 2 and 1 respectively, and by construction the locally free sheaf

$$\mathcal{E} = \mathcal{O}_{\mathfrak{X}}(\mathcal{D}) \oplus M \oplus N$$

is a generating sheaf on $\mathfrak{X}$.

This example can be generalized to a situation in which we have what we call a *locally constant sheaf of charts* for the log structure of $X$, which is something that binds together charts on different open subsets of $X$, and consequently will bind together the corresponding generating sheaves on such opens.

In the example above, this sheaf is obtained by descent from the constant sheaf of monoids $\mathbb{N}^2_Y$ on $Y$, glued to itself along $f$, using the morphism $\mathbb{N}^2 \to \mathbb{N}^2$ that switches the two coordinates.

### 3.3.3 Locally constant sheaves of charts

We introduce the additional structure that allows us to define a generating sheaf.

**Definition 3.3.18.** Let $X$ be a log scheme with DF structure $L\colon A \to \mathrm{Div}_X$ and fix a system of denominators $j\colon A \to B$. A *locally constant sheaf of charts* for this data is a sheaf of monoids $Q$ on $X$, together with the following:

- an étale covering $\{U_i \to X\}_{i \in I}$ and isomorphisms $\phi_i\colon Q|_{U_i} \cong (Q_0)_{U_i}$, where $Q_0$ is a fixed fine sharp monoid (hence the "locally constant"),
- a morphism of sheaves of monoids $\alpha\colon Q \to B$, which is a cokernel (hence the "sheaf of charts"),
- for every $i \in I$, a fine monoid $P_i$, a Kummer morphism $\beta_i\colon P_i \to Q_0$ and a morphism of monoids $P_i \to A(U_i)$, that together with $\alpha$ and the $\phi_i$'s gives a chart for $j$ on $U_i$.

We will refer to a locally constant sheaf of charts as above by writing $(Q, Q_0, \{U_i \to X\}_{i \in I}, P_i \to Q_0)$.

**Remark 3.3.19.** Clearly a global chart for $A \to B$ gives a locally constant sheaf of charts.

Another particular case of interest is the following: the sheaf $A$ has a locally constant sheaf of charts, i.e. a sheaf of monoids $P$ which is locally $(P_0)_{U_i}$ for some fixed fine and sharp monoid $P_0$ and with a cokernel $P \to A$, as in the definition, and $B = \frac{1}{n}A$ for some $n \in \mathbb{N}$.

The datum of a locally constant sheaf of charts is essentially equivalent to that of a torsor $\varphi\colon Y \to X$ for a finite group $G$, such that $\varphi^*B$ has a global chart, which is in some sense equivariant with respect to $G$, as in example 3.3.17. This is the content of the construction that follows.

Suppose we have a locally constant sheaf of charts, with the same notation as above. Consider the sheaf $F = \underline{\mathrm{Isom}}(Q, Q_0)$ on the big étale site of $X$, which associates to a map $f\colon T \to X$ the set $\underline{\mathrm{Isom}}(Q, Q_0)(T) = \mathrm{Isom}(f^*Q, (Q_0)_T)$ of isomorphisms of sheaves of monoids, and acts in the obvious way on the morphisms. Notice that this is a locally constant sheaf for the étale topology on $X$, and specifically we have isomorphisms $F|_{U_i} \cong \mathrm{Isom}((Q_0)_{U_i}, (Q_0)_{U_i}) = \mathrm{Aut}(Q_0)_{U_i}$.

Set $G = \mathrm{Aut}(Q_0)$, the group of monoid automorphisms of $Q_0$, and notice that this is a finite group: in fact since $Q_0$ is fine and sharp, it has a finite number of indecomposable elements (see Proposition 1.1.13), and those must be permuted by any automorphism, which in turn is completely determined by the induced permutation.

Being a finite locally constant sheaf, $F$ is represented by a scheme $Y$ over $X$, call $\varphi\colon Y \to X$ the structure morphism.

**Remark 3.3.20.** In other words, by the Yoneda Lemma, we have a functorial bijection

$$\mathrm{Hom}_X(T,Y) = \{(f,\eta) \text{ where } f\colon T \to X \text{ and } \eta\colon f^*Q \cong (Q_0)_T\}$$

and the identity $\mathrm{id}_Y \in \mathrm{Hom}_X(Y,Y)$ corresponds to a universal object $(\varphi,\xi)$, with $\xi\colon \varphi^*Q \cong (Q_0)_Y$, such that for any morphism $f\colon T \to X$ and $\eta\colon f^*Q \cong (Q_0)_T$ there is a unique morphism $\overline{f}\colon T \to Y$ such that $f = \varphi \circ \overline{f}$ and the following diagram commutes

$$\begin{tikzcd} \overline{f}^*\varphi^*Q \arrow[r, "\overline{f}^*\xi"] \arrow[d] & (Q_0)_T \\ f^*Q, \arrow[ur, "\eta"'] & \end{tikzcd}$$

where the vertical arrow is the canonical isomorphism.

In particular note that on $Y$ we have $\xi\colon \varphi^*Q \cong (Q_0)_Y$ and $\varphi^*\alpha\colon \varphi^*Q \to \varphi^*B$ is still a cokernel, since $\alpha$ is. Composing the two we get a cokernel $(Q_0)_Y \to \varphi^*B$, corresponding to a chart $Q_0 \to (\varphi^*B)(Y)$ for the pullback $\varphi^*B$.

Moreover since $F$ is a $G$-torsor by means of the obvious left action obtained by composition, $\varphi$ is also a $G$-torsor. Denote by $\mathfrak{Y}$ the stack of roots of the DF structure on $Y$ with respect to the kummer morphism $\varphi^*(A) \to \varphi^*(B)$, which is just the fibered product $\mathfrak{X} \times_X Y$. Notice that the induced map $\widetilde{\varphi}\colon \mathfrak{Y} \to \mathfrak{X}$ is a representable $G$-torsor, for the $G$-action induced on $\mathfrak{Y}$ by that on $Y$.

Now our strategy is to take a "naturally defined" generating sheaf on $\mathfrak{Y}$, which will be $G$-equivariant, by generalizing slightly the construction of 3.3.2, and then get by descent a sheaf on $\mathfrak{X}$, which will be our generating sheaf.

We start by defining the generating sheaf $\mathcal{E}$ on $\mathfrak{Y}$: call $\{q_1,\dots,q_r\}$ the indecomposable elements of $Q_0$. Notice that this time the monoid giving the chart for the DF structure $\varphi^*A$ of $Y$ is not the same over all of $Y$: we have an étale covering $\{Y_i \to Y\}_{i\in I}$ induced by the covering $\{U_i \to X\}_{i\in I}$ in the definition of the sheaf of charts, and on each $Y_i$ we have a chart $\beta_i\colon P_i \to Q_0$ for the kummer morphism $\varphi^*A \to \varphi^*B$. Let us set $G_i = {Q_0}^{\mathrm{gp}}/P_i^{\mathrm{gp}}$ with projection $\pi_i\colon Q_0^{\mathrm{gp}} \to G_i$, and this time put

$$d_i = \gcd\{\mathrm{ord}(\pi_i(f(q_i))) \text{ for } f \in \mathrm{Aut}(Q_0)\},$$

where ord is the order of an element in the finite group $G_i$.

Finally denote by $\Lambda\colon Q_0 \to \mathrm{Div}(\mathfrak{Y})$ the pullback of the universal DF structure on $\mathfrak{X}$ along the projection $\widetilde{\varphi}\colon \mathfrak{Y} \to \mathfrak{X}$, and set $\Lambda_i = \Lambda(q_i)$, and

$$\overline{\mathcal{E}} = \bigoplus_{1\le a_i\le d_i} \Lambda\left(\sum_i a_i q_i\right) = \bigotimes_{i=1,\dots,r}\left(\bigoplus_{j=1,\dots,d_i} \Lambda_i^{\otimes j}\right).$$

as in 3.3.2.

Since $\widetilde{\varphi}\colon \mathfrak{Y} \to \mathfrak{X}$ is a representable $G$-torsor, to give $\overline{\mathcal{E}}$ the structure of a $G$-equivariant sheaf we have to give an isomorphism $\lambda\colon \alpha^*\overline{\mathcal{E}} \cong \pi_2^*\overline{\mathcal{E}}$, where $\alpha, \pi_2\colon \mathfrak{Y} \times_{\mathfrak{X}} G \to \mathfrak{Y}$ are the action and the second projection, that satisfies a compatibility condition on the triple product $\mathfrak{Y} \times_{\mathfrak{X}} G \times_{\mathfrak{X}} G$. Here we are considering the finite group $G$ as a relative group scheme over $\mathfrak{X}$ in the usual way, as $G = \coprod_{g\in G} \mathfrak{X}$, so in particular we have $\mathfrak{Y} \times_{\mathfrak{X}} G \cong \coprod_{g\in G} \mathfrak{Y}$, and the action corresponds to morphisms $\psi(g)\colon \mathfrak{Y} \to \mathfrak{Y}$ for $g \in G$. The resulting map $\psi\colon G \to \mathrm{Aut}_{\mathfrak{X}}(\mathfrak{Y})$ is an injective group homomorphism, so in particular all the $\psi(g)$ are automorphisms. All of this holds for $Y$ too, and from now on we will make the following abuse of notation: we will write simply $g$ in place of $\psi(g)$, and also to denote the corresponding automorphism $Y \to Y$ over $X$.

The above discussion shows that in order to give the isomorphism $\lambda$ as above, we can equivalently give isomorphisms $\lambda_g\colon g^*\overline{\mathcal{E}} \cong \overline{\mathcal{E}}$ for $g \in G$, satisfying the natural compatibility property with respect to composition.

**Proposition 3.3.21.** *There are canonical isomorphisms $\lambda_g\colon g^*\overline{\mathcal{E}} \cong \overline{\mathcal{E}}$ for $g \in G$, such that for any $g, h \in G$ we have $\lambda_g \circ g^*\lambda_h = \lambda_{hg}\colon (hg)^*\overline{\mathcal{E}} \cong \overline{\mathcal{E}}$.*

*Proof.* We will show that there are canonical isomorphisms $g^*\Lambda(q_i) \cong \Lambda(g(q_i))$ for each $i$, compatible with composition in $G$. Putting all of those and their various tensor powers together, we will get isomorphisms $\lambda_g\colon g^*\overline{\mathcal{E}} \cong \overline{\mathcal{E}}$ with the desired properties.

First of all let us fix $g \in G$, and describe the pullback $g^*F$ for a quasi-coherent sheaf on $\mathfrak{Y}$: given an étale map $f\colon U \to \mathfrak{Y}$ from a scheme, we have $(g^*F)(U) = F(U \xrightarrow{g\circ f} \mathfrak{Y})$ as an $O_U(U)$-module.

Secondly, by unraveling the definitions one checks that the map $g(U)\colon \mathfrak{Y}(U) \to \mathfrak{Y}(U)$ takes an object of $\mathfrak{Y}$, which will be a morphism $a\colon U \to Y$ together with a lifting $a^*\varphi^*B \to \mathrm{Div}_U$ of the pullback of the DF structure $\varphi^*A \to \mathrm{Div}_Y$ to $U$, to the composition $g \circ a$, together with the induced lifting $a^*g^*\varphi^*B \to \mathrm{Div}_U$, obtained using the canonical isomorphism $a^*g^*\varphi^*B \cong a^*\varphi^*B$ (recall that $\varphi \circ g = \varphi$).

Putting these facts together, the conclusion follows from the claim that we have a commutative diagram

$$\begin{tikzcd} (Q_0)_Y \arrow[r] \arrow[d, "g_Y"'] & \varphi^*Q \arrow[d, "\mathrm{can}"] \\ (Q_0)_Y \arrow[r] & g^*\varphi^*Q \end{tikzcd}$$

of sheaves of monoids on $Y$, where all the maps are isomorphisms, can stands for the canonical isomorphism (coming from $\varphi \circ g = \varphi$), and the horizontal arrows are the maps corresponding to $\mathrm{id}_Y$ (top one) and $g\colon Y \to Y$ (bottom one) in the Yoneda correspondence described in Remark 3.3.20. In other words the top arrow is $\xi^{-1}$, and the bottom one is $g^*\xi^{-1}$, and the equality to prove is

$$\mathrm{can} \circ \xi^{-1} = g^*\xi^{-1} \circ g_Y. \tag{3.3.22}$$

Let us show that the conclusion follows from this: in fact, $g^*\Lambda(q_i)(U)$ for an étale $f\colon U \to \mathfrak{Y}$ will be $\Lambda(q_i)$ applied to the composition $U \to \mathfrak{Y} \xrightarrow{g} \mathfrak{Y}$, so we have to ask ourselves what is the image of $q_i \in Q_0$ in $\mathrm{Div}(U)$, with respect to the morphism $Q_0 \to \mathrm{Div}(U)$ coming from the composition

$$(Q_0)_U \cong f^*\phi^*Q \cong (g \circ f)^*\varphi^*Q \to \mathrm{Div}_U$$

where the first two maps are the top row and the right one of the preceding diagram, pulled back to $U$. The above claim shows that this image is precisely $\Lambda(g(q_i))$.

To prove the claim we need to give names to various morphisms of sheaves of monoids: recall from above that a morphism $f\colon Y \to Y$ over $X$ corresponds to an isomorphism $f^\sharp\colon \varphi^*Q \cong (Q_0)_Y$. Let us put $\alpha(f) = f^\sharp \circ \xi^{-1}$, where $\xi\colon \varphi^*Q \cong (Q_0)_Y$ is the universal isomorphism of the Yoneda correspondence. Then one can check that $\alpha\colon \mathrm{Aut}_X(Y) \to \mathrm{Aut}((Q_0)_Y)$ is an isomorphism of groups, and in particular for $g \in G$, we have $\alpha(\psi(g)) = g_Y\colon (Q_0)_Y \to (Q_0)_Y$.

Writing down how $f^\sharp$ is obtained as a pullback of the universal object $\xi$, we get that, if we denote by $\beta(f)\colon \varphi^*Q \cong f^*\varphi^*Q$ the canonical isomorphism, then $\beta(f) = f^*(\xi^{-1}) \circ f^\sharp$.

Now using these equalities we get

$$\beta(f) \circ \xi^{-1} = f^*\xi^{-1} \circ f^\sharp \circ \xi^{-1} = f^*\xi^{-1} \circ \alpha(f),$$

and applying this to $g \in G$ seen as the corresponding $g\colon Y \to Y$, we get exactly the equality 3.3.22.

The statement about the composition can again be checked on the single $L_{Q_0}(q_i)$, and boils down to a similar calculation, using the commutative diagram

$$\begin{tikzcd}
(Q_0)_Y \arrow[r] \arrow[d, "g_Y"'] & \varphi^*Q \arrow[d] \\
(Q_0)_Y \arrow[r] \arrow[d, "h_Y"'] & g^*\varphi^*Q \arrow[d] \\
(Q_0)_Y \arrow[r] & g^*h^*\varphi^*Q.
\end{tikzcd}$$

□

By descent along torsors, the data given in the previous proposition give a locally free sheaf $\mathcal{E} \in \mathrm{Coh}(\mathfrak{X})$.

**Proposition 3.3.23.** *The sheaf $\mathcal{E}$ is a generating sheaf on $\mathfrak{X}$.*

*Proof.* Take a geometric point $p \to \mathfrak{X}$. Since the map $\mathfrak{Y} \to \mathfrak{X}$ is étale and surjective there exists a lifting $q \to \mathfrak{Y}$ of $p$, so that $\mathcal{E}_p = \overline{\mathcal{E}_q}$, and moreover there is an index $i \in I$ such that the image of $p$ in $X$ is in the image of $U_i \to X$, where $\{U_i \to X\}_{i\in I}$ is an étale covering that satisfies the requirement in the definition of a locally constant sheaf of charts.

By construction of $\mathfrak{Y}$ we have $\mathrm{Stab}(q) = \mathrm{Stab}(p) \subseteq \widehat{G_i} \subseteq \widehat{Q_0}$, so it suffices to verify that every character of $\mathrm{Stab}(q) = \mathrm{D}[M]$ appears in the decomposition of the $k(q)$-vector space $\mathcal{E}_q$ for the action of $\mathrm{Stab}(q)$, and this is true by the same argument used in the proof of proposition 3.3.15, since the images of the $q_i$'s will generate both $G_i$ and its quotient $M$. □

### 3.3.4 Results

From now on we will assume that the log scheme $X$ has a locally constant sheaf of charts $(Q, Q_0, \{U_i \to X\}_{i\in I}, P_i \to Q_0)$, that may in particular be a global chart $P \to Q$ for the Kummer extension $j\colon A \to B$. In fact this last situation will come up more often than the more general one, since we will be able to say more with a global chart.

In this situation we can produce a generating sheaf $\mathcal{E}$ on the root stack $\mathfrak{X} = X_{B/A}$, and we can apply Nironi's theory [Nir] for moduli of coherent sheaves on an algebraic stack. In this section we will summarize the notions and results that we get from it.

The proof of the results that are simply stated in this section can all be found in [Nir].

**Remark 3.3.24.** In order to apply Nironi's theory we have to assume that the root stack $X_{B/A}$ is Deligne–Mumford. For example, this is assured by the condition the $\mathrm{char}(k)$ does not divide the order of the quotient $B_x^{\mathrm{gp}}/A_x^{\mathrm{gp}}$ for any geometric point $x \to X$ (as in 1.2.31). We will include this assumption in our treatment from now on.

This will force us to assume that $\mathrm{char}(k) = 0$ in the next chapter, since we will have to consider a cofinal system of root stacks, and, for example if we consider the system of root stacks $X_n$, it would do no good to exclude indices divisible by some fixed prime $p$.

We remark that it seems likely that Nironi's theory also applies to tame Artin stacks, without the Deligne–Mumford assumption. If this were true our results would hold in arbitrary characteristic.

**Notation 3.3.25.** From now on we will use the same letter to denote a coherent sheaf $E \in \mathrm{Coh}(\mathfrak{X})$ on the root stack and the corresponding parabolic sheaf $\Phi(E) \in \mathrm{Par}(X, j)$. In particular we will denote by $E_b$ the piece $\pi_*(\Phi(E) \otimes \Lambda_b) \in \mathrm{Coh}(X)$ of the parabolic sheaf corresponding to the element $b \in B^{\mathrm{wt}}(U)$.

From now on we will be drawing parabolic sheaves more often. We recall how to visualize them, in the case where there is a global chart $P \to Q$ for the system of denominators: one has to picture the lattice $Q^{\mathrm{gp}}$, and imagine a quasi-coherent sheaf $E_q$ on $X$ on each point $q$ of the lattice. Moreover there are maps $E_q \to E_{q'}$ exactly when $q \leq q'$, in the sense that there exists $q'' \in Q$ such that $q' = q + q''$, and if $p \in P$, then the sheaf $E_{q+p}$ is isomorphic to $E_q \otimes L_p$, and the map $E_q \to E_{q+p}$ corresponds to multiplication by the distinguished section of $L_p$. In practice it will be enough to draw a small portion of the sheaf, which will determine it uniquely.

The starting point is the definition of the generalized slope $p_{\mathcal{E}}(E) \in \mathbb{Q}[m]$ for a parabolic sheaf $E \in \mathrm{Par}(X, j)$ (Definition 3.3.3). We recall that it is defined as

$$p_{\mathcal{E}}(E)(m) = \frac{P_{\mathcal{E}}(E)(m)}{\alpha^d(E)} = \frac{\chi(\pi_*(E \otimes \mathcal{E}^\vee)(m))}{\alpha^d(E)}$$

where $d$ is the degree of the Hilbert polynomial $P_{\mathcal{E}}(E) = P(\pi_*(E \otimes \mathcal{E}^\vee))$, and $\alpha^d(E)$ is $d!$ times its leading coefficient, a positive rational number.

From now on this will be simply called the *slope* of $E$, and if there is no risk of confusion we will omit to mention $\mathcal{E}$ in Euler characteristics, Hilbert polynomials and slopes.

If the Kummer extension $j\colon A \to B$ has a global chart $P \to Q$, then $\mathcal{E}$ is a sum of line bundles and $\pi_*(E \otimes \mathcal{E}^\vee)$ also splits as a direct sum

$$\begin{aligned}\pi_*(E \otimes \mathcal{E}^\vee) &= \pi_*(E \otimes (\bigoplus_{1 \leq a_i \leq d_i} \Lambda_1^{\otimes a_1} \otimes \cdots \otimes \Lambda_r^{\otimes a_r})^\vee) \\ &= \bigoplus_{1 \leq a_i \leq d_i} \pi_*(E \otimes (\Lambda_1^{\otimes a_1} \otimes \cdots \otimes \Lambda_r^{\otimes a_r})^\vee) = \bigoplus_{1 \leq a_i \leq d_i} E_{-\sum a_i q_i}\end{aligned}$$

of pieces of $E$ in some kind of (negative) "fundamental region" for the Kummer extension $P \to Q$.

We will give a name to the pieces of $E$ that show up in this decomposition.

**Definition 3.3.26.** The *fundamental pieces* of $E$ are the pieces $E_q$ with $q = -\sum a_i q_i$ and $1 \le a_i \le d_i$, where $q_i$ are the indecomposable elements of $Q$ and $d_i$ is the order of the image of $q_i$ in the quotient $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$.

**Example 3.3.27.** For example, if $P = \mathbb{N}^2$ and we are considering the extension $\mathbb{N}^2 \to \frac{1}{2}\mathbb{N}^2$, then for a parabolic sheaf $E \in \mathrm{Par}(X, j)$ the fundamental pieces are the four sheaves in the "negative unit square"

$$\begin{tikzcd}
-1 & -\frac{1}{2} & \\
E_{-1,-\frac{1}{2}} \arrow[r] & E_{-\frac{1}{2},-\frac{1}{2}} & -\frac{1}{2} \\
E_{-1,-1} \arrow[r] \arrow[u] & E_{-\frac{1}{2},-1} \arrow[u] & -1
\end{tikzcd}$$

A similar description holds if $P$ is free and we are considering the extension $P \to \frac{1}{n}P$.

From the fundamental pieces of a parabolic sheaf we can reconstruct all of its pieces, since for any $q \in Q^{\mathrm{gp}}$ there is a $p \in P^{\mathrm{gp}}$ such that $q + p = -\sum a_i q_i$ for $1 \le a_i \le d_i$, and consequently $E_q \cong E_{-\sum a_i q_i} \otimes L_p^{\vee}$. We can even reconstruct the morphisms between the pieces of $E$ (and then the whole sheaf), from the morphisms between the fundamental pieces and, for example, all morphisms

$$E(-\sum_{i \neq i_0} a_i q_i - q_{i_0}) \to E(-\sum_{i \neq i_0} a_i q_i)$$

for varying $i_0$ and $a_i$.

In example 3.3.27, these additional morphisms would be the ones going up and right of the negative unit square, to the pieces of $E$ "lying on the coordinate axes", i.e. the thicker arrows in the following picture

$$\begin{tikzcd}
-1 & -\frac{1}{2} & 0 & \\
E_{-1,-1} \otimes L_{0,1} \arrow[r] & E_{-\frac{1}{2},-1} \otimes L_{0,1} \arrow[r] & E_{-1,-1} \otimes L_{1,1} & 0 \\
E_{-1,-\frac{1}{2}} \arrow[r] \arrow[u, Rightarrow] & E_{-\frac{1}{2},-\frac{1}{2}} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & E_{-1,-\frac{1}{2}} \otimes L_{1,0} \arrow[u] & -\frac{1}{2} \\
E_{-1,-1} \arrow[r] \arrow[u] & E_{-\frac{1}{2},-1} \arrow[r, Rightarrow] \arrow[u] & E_{-1,-1} \otimes L_{1,0} \arrow[u] & -1.
\end{tikzcd}$$

**Remark 3.3.28.** Note that if $P$ is not free, then it is not necessarily the case that every fundamental piece shows up exactly once in $\pi_*(E \otimes \mathcal{E}^{\vee})$. Take for example

$$P = \langle p, q, r \mid p + q = 2r \rangle,$$

the Kummer extension $P \subseteq \frac{1}{4}P$, and denote by $\Lambda_1 = \Lambda(\frac{1}{2}p)$, $\Lambda_2 = \Lambda(\frac{1}{2}q)$ and $\Lambda_3 = \Lambda(\frac{1}{2}r)$, where $\Lambda\colon \frac{1}{2}P \to \mathrm{Div}_{\mathfrak{X}}$ is the universal DF structure on the root stack $\mathfrak{X}$. Note that since $p+q=2r$, we have $\Lambda_1 \otimes \Lambda_2 \cong \Lambda_3^{\otimes 2}$. Then the generating sheaf is

$$\mathcal{E} = \left(\bigoplus_{i=1}^{4} \Lambda_1^{\otimes i}\right) \otimes \left(\bigoplus_{i=1}^{4} \Lambda_2^{\otimes i}\right) \otimes \left(\bigoplus_{i=1}^{4} \Lambda_3^{\otimes i}\right)$$

and for example the piece $\Lambda_1^{\otimes 2} \otimes \Lambda_2^{\otimes 2} \otimes \Lambda_3$ shows up also as $\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3^{\otimes 3}$, since in $\frac{1}{4}P$ we have

$$2\left(\frac{1}{4}p\right) + 2\left(\frac{1}{4}q\right) + \frac{1}{4}r = \frac{1}{4}p + \frac{1}{4}q + 3\left(\frac{1}{4}r\right).$$

So accordingly, the fundamental piece $E(-2\left(\frac{1}{4}p\right) - 2\left(\frac{1}{4}q\right) - \frac{1}{4}r)$ will appear twice in $\pi_*(E \otimes \mathcal{E}^\vee)$.

From the splitting of $\pi_*(E \otimes \mathcal{E}^\vee)$ described above we also see that

$$P_{\mathcal{E}}(E) = P(\pi_*(E \otimes \mathcal{E}^\vee)) = P(\bigoplus_{1 \le a_i \le d_i} E_{-\sum a_i q_i}) = \sum_{1 \le a_i \le d_i} P(E_{-\sum a_i q_i})$$

is the sum of the Hilbert polynomials of the fundamental pieces of $F$. Consequently, assuming that the fundamental pieces of $F$ all have dimension $d$ (and recall that by our conventions the zero sheaf is pure of any dimension), for the slope of $F$ we have

$$p_{\mathcal{E}}(E) = \frac{\sum_{1 \le a_i \le d_i} P(E_{-\sum a_i q_i})}{\sum_{1 \le a_i \le d_i} \alpha^d(E_{-\sum a_i q_i})} = \sum_{1 \le a_i \le d_i} \gamma_{(a_i)} p(E_{-\sum a_i q_i}) \tag{3.3.29}$$

where $d$ is the dimension of $E$, and

$$\gamma_{(a_i)} = \frac{\alpha^d(E_{-\sum a_i q_i})}{\sum_{1 \le a_i \le d_i} \alpha^d(E_{-\sum a_i q_i})}$$

are rational numbers such that $0 \le \gamma_{(a_i)} \le 1$ and $\sum_{1 \le a_1 \le d_i} \gamma_{(a_i)} = 1$.

In other words the slope of the parabolic sheaf $E$ (provided that all its non-zero pieces are of the same dimension) is a weighted mean of the slopes of its non-zero fundamental pieces. The condition about the pieces is satisfied in particular if $E$ is pure, as we saw in Proposition 3.2.10.

**Definition 3.3.30.** A parabolic sheaf is *(semi-)stable* if it is pure, and for any subsheaf $G \subseteq E$ we have

$$p_{\mathcal{E}}(G)\ (\le)\ p_{\mathcal{E}}(E).$$

As is usually done in moduli theory of sheaves, we write $(\le)$ to indicate that one should read $\le$ when he considers semi-stability, and $<$ when he considers stability.

This notion of stability has many properties of the classical notion of Gieseker stability on a projective scheme.

**Remark 3.3.31.** For example, as in the classical case, (semi-)stability can be checked on *saturated* subsheaves $G \subseteq E$, i.e. subsheaves such that the quotient $E/G$ is pure of the same dimension as $E$. This implies that line bundles are all stable.

Moreover, a direct sum $E_1 \oplus E_2$ is never stable, and is semi-stable if and only if $E_1$ and $E_2$ are semi-stable of the same slope.

**Example 3.3.32.** It is clear that if the fundamental pieces of a parabolic sheaf $E$ are all (Gieseker) semi-stable (as coherent sheaves on $X$, with respect to the same polarization that we fixed at the beginning), then $E$ will be semi-stable. In fact, every fundamental piece $F_q$ of a subsheaf $F \subseteq E$ is a subsheaf $F_q \subseteq E_q$ of the corresponding fundamental piece of $E$, and by (semi-)stability of $E_q$ we have

$$P(F_q)\alpha^d(E_q) \ (\leq) \ P(E_q)\alpha^d(F_q)$$

where $P$ is the ordinary Hilbert polynomial on $X$, and $\alpha^d$ is its leading coefficient.

By summing on the fundamental pieces and diving by the sum of the $\alpha^d$'s, we get exactly

$$p_{\mathcal{E}}(F) \ (\leq) \ p_{\mathcal{E}}(E)$$

so $E$ is (semi-)stable. In particular if all the fundamental pieces of a parabolic sheaf are line bundles, then it is semi-stable.

The following two results are also identical to the corresponding ones for classical moduli theory of sheaves. They provide filtrations that "break up" a parabolic sheaf in semi-stable pieces, and a semi-stable parabolic sheaf in stable pieces.

**Proposition 3.3.33** (Harder-Narasimhan filtration)**.** *For any parabolic sheaf $E \in \mathrm{Par}(X, j)$ there is a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k \subset E_{k+1} = E$$

*such that*

- *the quotients $E_i/E_{i-1}$ are semi-stable for $i = 1, \ldots, k+1$; call $p_i$ the slope $p_{\mathcal{E}}(E_i/E_{i-1})$,*
- *the slopes are such that $p_1 > \cdots > p_{k+1}$.*

*Moreover this filtration is unique, and it is called the* Harder-Narasimhan filtration *of the parabolic sheaf $E$.*

**Proposition 3.3.34** (Jordan-Hölder filtration)**.** *For any semi-stable parabolic sheaf $E \in \mathrm{Par}(X, j)$ there is a filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_h \subset F_{h+1} = E$$

*such that the quotients $F_i/F_{i-1}$ are stable with slope $p_{\mathcal{E}}(E)$ for $i = 1, \ldots, h+1$.*

*This filtration is not unique, but the set $\{F_i/F_{i-1}\}_{i=1,\ldots,h+1}$ of partial quotients of the filtration is unique, as is their direct sum*

$$\mathrm{gr}(E) = \bigoplus_{i=1}^{h+1} F_i/F_{i-1},$$

*sometimes called the* associated graded sheaf *of $E$.*

*Any such filtration is called a* Jordan-Hölder filtration *of the parabolic sheaf $E$.*

Note that since the quotients $F_i/F_{i-1}$ of the above filtration are stable with the same slope, the parabolic sheaf $\mathrm{gr}(E)$ is semi-stable with the same slope as $E$.

**Definition 3.3.35** (S-equivalence)**.** Two parabolic sheaves $E, E' \in \mathrm{Par}(X, j)$ are said to be *S-equivalent* if their associated graded sheaves $\mathrm{gr}(E)$ and $\mathrm{gr}(E')$ are isomorphic.

Equivalently one can say that the sets $\{F_i/F_{i-1}\}$ and $\{F_i'/F_{i-1}'\}$ of quotients of a Jordan-Hölder filtrations of the two sheaves are the same, i.e. such quotients are pairwise isomorphic.

Recall that a semi-stable sheaf is called *polystable* if it is a direct sum of stable sheaves, which then must all have the same slope. Every parabolic semi-stable sheaf $E \in Par(X, j)$ is S-equivalent to exactly one polystable sheaf, the sheaf $\mathrm{gr}(E)$.

The notion of (semi-)stability satisfies some openness and boundedness conditions, as shown in [Nir]. We summarize the final product of the theory.

Fix a polynomial $H \in \mathbb{Z}[x]$, and define the stack $\mathcal{M}_H^{ss}$ over (Sch) having as objects of $\mathcal{M}_H^{ss}(T)$ for a scheme $T$ families of parabolic sheaves $E \in \mathrm{Par}(X_T, j_T)$ such that for every geometric point $t \to T$, the restriction $E_t \in \mathrm{Par}(X_t, j_t)$ is pure and semi-stable with Hilbert polynomial $H$, and as arrows isomorphisms of parabolic sheaves. The pullback $\mathcal{M}_H^{ss}(T) \to \mathcal{M}_H^{ss}(S)$ for $S \to T$ is the pullback of parabolic sheaves we discussed earlier.

Note that of course this stack also depends on the system of denominators $A \to B$, but we omitted it in the notation to keep it lighter.

Denote by $\mathcal{M}_H^{s} \subseteq \mathcal{M}_H^{ss}$ the subcategory parametrizing families of parabolic sheaves that are stable on the fibers, instead of merely semi-stable. This is an open substack.

**Remark 3.3.36.** To define (semi-)stability on the base change $X_t = X \times_k \mathrm{Spec}(k(t))$ we use the pullback of the generating sheaf $\mathcal{E}$ that we have on $\mathfrak{X}$ along the natural map $(X_t)_{B/A} \to X_{B/A} = \mathfrak{X}$.

Here is the result that we obtain from [Nir, Section 6].

**Theorem 3.3.37.** *Let $X$ be a projective log scheme with a DF structure $L\colon A \to \mathrm{Div}_X$ and $j\colon A \to B$ a system of denominators with a locally constant sheaf of charts. Moreover assume that the root stack $X_{B/A}$ is Deligne–Mumford.*

*Then the stack $\mathcal{M}_H^{ss}$ of semi-stable parabolic sheaves is an Artin stack of finite type over k, and it has a presentation as a global quotient stack $[\mathcal{Q}/GL_{N,k}]$, where $\mathcal{Q}$ is an open subscheme of a certain quot scheme. Moreover it has a good (resp. adequate, in positive characteristic) moduli space in the sense of Alper [Alp12, Alp], that we denote by $M_H^{ss}$. This moduli space is a projective scheme, constructed with GIT techniques.*

*The open substack $\mathcal{M}_H^{s} \subseteq \mathcal{M}_H^{ss}$ of stable sheaves also has a good moduli space $M_H^{s}$, which is an open subscheme of $M_H^{ss}$, and the map $\mathcal{M}_H^{s} \to M_H^{s}$ is a $\mathbb{G}_m$-gerbe.*

Some comments about this theorem.

**Remark 3.3.38.** We should have included also the locally constants sheaf of charts in the notation for the stack $\mathcal{M}_H^{ss}$, since stability is not independent of the choice, as we will see shortly.

**Remark 3.3.39.** We chose to fix the Hilbert polynomial in this formulation, but one can also fix other invariants of coherent sheaves, for example Chern classes, or the reduced Hilbert polynomial $h$. The corresponding moduli stacks are defined analogously, and the results one obtains translate verbatim.

In particular, in the next chapter we will fix the reduced Hilbert polynomial $h \in \mathbb{Q}[x]$, that is obtained from $H$ by dividing it by $d!$ times its leading coefficient, $d$ being the degree, and consider the corresponding stacks $\mathcal{M}_h^s \subseteq \mathcal{M}_h^{ss}$. Note that for example $\mathcal{M}_h^{ss}$ will be a disjoint union

$$\mathcal{M}_h^{ss} = \bigsqcup_{\overline{H}=h} \mathcal{M}_H^{ss}$$

where $\overline{H}$ is $H$ divided by $d!$ times its leading coefficient. This has a good moduli space, the disjoint union of the corresponding moduli spaces, and the same is true for the substack of stable sheaves.

In the same spirit one can form the disjoint union $\mathcal{M}^{ss} = \bigsqcup_{H \in \mathbb{Z}[x]} \mathcal{M}_H^{ss}$ and the analogous one for stable sheaves. This stack, that parametrizes parabolic sheaves on $X$ with respect to the system of denominators $A \to B$ without fixing invariants, will still have a good moduli space, the disjoint union of the $M_H^{ss}$, which of course will not be projective anymore.

**Remark 3.3.40.** The points of the good moduli space $M_H^{ss}$ do not correspond to isomorphism classes of semi-stable sheaves, but rather to S-equivalence classes, or, in other words, to isomorphism classes of polystable sheaves. This follows from the GIT construction.

Moreover, a point of the stack $\mathcal{M}_H^{ss}$ is closed if and only if the corresponding sheaf is polystable. This follows from the description as a quotient and from the fact that an orbit of a point is closed if and only if it is polystable (see Theorem 6.20 of [Nir]).

### 3.3.5 Comparison with Maruyama and Yokogawa's theory

In this short section we remark that this construction recovers the moduli spaces of Maruyama and Yokogawa. Recall from 3.3.1 that they considered the case of a projective variety $X$ with an effective Cartier divisor $D \subseteq X$. This induces a log structure on $X$, given by the global chart $\mathbb{N} \to \mathrm{Div}(X)$ sending 1 to $(\mathcal{O}_X(D), s)$. Recall also their definition of a parabolic sheaf on $X$ as a torsion-free sheaf $E$ with a filtration $E(-D) = F_{k+1}(E) \subset F_k(E) \subset \cdots \subset F_1(E) = E$ and rational weights $0 \leq a_1 < \cdots < a_k < 1$.

Let us fix a common denominator $n \in \mathbb{N}$ for the weights $a_i$, and consider the root stack $X_n = X_{\frac{1}{n}\mathbb{N}/\mathbb{N}}$. Torsion-free quasi-coherent sheaves on $X_n$ correspond to parabolic sheaves of the form

$$\begin{array}{ccccccccc} -1 & & -\frac{n-1}{n} & & \dots & & -\frac{1}{n} & & 0 \\ & & & & & & & & \\ E \otimes \mathcal{O}_X(-D) & \xrightarrow{f_{n-1}} & E_{n-1} & \xrightarrow{f_{n-2}} & \dots & \xrightarrow{f_1} & E_1 & \xrightarrow{f_0} & E. \end{array}$$

where every sheaf is torsion-free and the maps are injective by 3.2.10 and 3.2.13.

This resembles closely the definition of a MY-parabolic sheaf. The difference is that Maruyama and Yokogawa fix the weights, i.e. the sequence of numbers $-\frac{a}{n}$ corresponding to maps $E_{a+1} \to E_a$ that are not isomorphisms. We will check now that fixing the weights gives a component of our moduli stack of parabolic sheaves, and that this component gives back the moduli spaces of Maruyama and Yokogawa.

Let us fix a sequence of rational weights $0 \leq a_1 < \cdots < a_k < 1$ and a polynomial $H \in \mathbb{Q}[x]$. Write $a_i = \frac{b_i}{n}$, and let us denote by $\mathcal{M}^{ss}(a_1, \dots, a_k)$ the stack of families of parabolic torsion free sheaves on $X$, such that the morphism $E_{j+1} \to E_j$ is an isomorphism if and only if $j$ is not in $\{b_1, \dots, b_k\}$ (in other words, $-a_i$ are exactly the indices where the sheaf jumps).

This clearly gives a subcategory $\mathcal{M}^{ss}(a_1, \dots, a_k) \subseteq \mathcal{M}^{ss}$ of the moduli stack we defined above, and moreover this map of stacks is an open and closed immersion.

To see this we need the following lemma, which says moreover that the flatness of the cokernels that is required in [MY92] in the definition of the moduli functor is actually automatic.

**Lemma 3.3.41.** *Let $X$ be a noetherian log scheme with generically trivial log structure, $j\colon P \to Q$ a chart for a system of denominators on $X$ and $E \in \mathrm{Par}(X_T, j_T)$ be a family of torsion-free parabolic sheaves on a scheme $T$. Then for every pair $q \leq q'$ in $Q^{\mathrm{gp}}$ the cokernel of the map $E_q \to E_{q'}$ is flat over $T$.*

*Proof.* Note first of all that the map $E_q \to E_{q'}$ is injective. This follows from the fact that it is injective on the geometric fibers, by 3.2.13, from projectivity of $X_T \to T$ and flatness of the sheaves over $T$.

The conclusion now follows from the local criterion of flatness: if we pull back the exact sequence

$$0 \longrightarrow E_q \longrightarrow E_{q'} \longrightarrow Q \longrightarrow 0$$

along a point $t$ of $T$, then the map $E_q \otimes k(t) \to E_{q'} \otimes k(t)$ is injective by 3.2.13, since it is a map between pieces of a parabolic sheaf over a log scheme with generically trivial log structure (the fiber $X_t$). Consequently, since by flatness of $E_{q'}$ we have $\mathrm{Tor}_1(E_{q'}, k(t)) = 0$, we also have $\mathrm{Tor}^1(Q, k(t)) = 0$, and by the local criterion of flatness this shows that $Q$ is flat over $T$. □

What we stated above follows from the fact that if $E$ is a family of torsion-free parabolic sheaves on $X_n$ over a scheme $T$, then the cokernels of the injective maps $E_{a+1} \to E_a$ are flat over $T$, and consequently the locus where they are trivial (which is the locus where these maps are isomorphisms) is open and closed in $T$ (since it coincides with the locus where the Hilbert polynomial of the cokernels is zero, for example).

On the other hand we have an obvious "projection" map $\mathcal{M}^{ss}(a_1, \dots, a_k) \to \overline{MY}$ to the moduli functor of Maruyama and Yokogawa (we recalled their definition in Section 3.3.1; note that here we did not fix Hilbert polynomials). Moreover one can check that, if we denote by $M^{ss}(a_1, \dots, a_k)$ the good moduli space of $\mathcal{M}^{ss}(a_1, \dots, a_k)$, there is a factorization

$$\begin{tikzcd} \mathcal{M}^{ss}(a_1, \dots, a_k) \arrow[r] \arrow[d] & \overline{MY} \arrow[dl] \\ M^{ss}(a_1, \dots, a_k) & \end{tikzcd}$$

of the map $\mathcal{M}^{ss}(a_1, \dots, a_k) \to M^{ss}(a_1, \dots, a_k)$, and this implies that $M^{ss}(a_1, \dots, a_k)$ is the moduli space constructed by Maruyama and Yokogawa.

In conclusion the moduli spaces of Maruyama and Yokogawa are open and closed subschemes of the moduli spaces that we produce.

**Remark 3.3.42.** In this discussion we did not fix Hilbert polynomials for our parabolic sheaves nor for MY-parabolic sheaves. Nevertheless the arguments carry through if one fixes them (the ones that Maruyama and Yokogawa fix determine our parabolic Hilbert polynomial), and the corresponding spaces will be open and closed in the ones we considered here, as usual.

We also considered only semi-stable sheaves, but the same conclusions hold for stable sheaves.

## 3.4 Dependance of stability on the chart

Since there are many choices for a chart or a locally constant sheaf of charts of a logarithmic structure with a kummer morphism (when they exist, of course), the problem of the dependance of the (semi-)stability of a parabolic sheaf on the chart or the sheaf of charts is a very natural one. It turns out that the (semi-)stability is not independent of the chart (or the locally constant sheaf of charts), as we will show with the following example.

Take $X = \mathbb{P}^1 \times \mathbb{P}^1$, with effective divisor $D = D_1 + D_2$ where $D_1$ and $D_2$ are two distinct closed fibers of the first projection $X \to \mathbb{P}^1$, so that $\mathcal{O}(D_1) \cong \mathcal{O}(D_2) \cong \mathcal{O}(1,0)$ and $\mathcal{O}(D) \cong \mathcal{O}(2,0)$. The DF structure induced by $D$, call it $L\colon A \to \mathrm{Div}_X$, has two natural charts $l\colon \mathbb{N} \to \mathrm{Div}(X)$, sending 1 to $(\mathcal{O}(D), s_D)$, with $s_D$ the canonical section of $\mathcal{O}(D)$ as usual, and $l'\colon \mathbb{N}^2 \to \mathrm{Div}(X)$, sending $(1,0)$ to $(\mathcal{O}(D_1), s_{D_1})$ and $(0,1)$ to $(\mathcal{O}(D_2), s_{D_2})$.

Notice that any cokernel of monoids $P \to \mathbb{N}^2$ would give us a new chart $P \to \mathrm{Div}(X)$ for the DF structure $L$, since the composite of two cokernels is still a cokernel (this is an easy verification). The simplest case of this is a projection $\mathbb{N}^r \to \mathbb{N}^2$ with $r \geq 3$, but in this case a calculation shows that the (semi-)stability does not change.

Take instead $P = \mathbb{N}^4/(e_1 + e_2 = e_3 + e_4)$, where the $e_i$'s are the canonical basis, call $p_i$ the image of $e_i$ in $P$, and consider the morphism $\phi\colon P \to \mathbb{N}^2$ determined by

$$\begin{aligned}
\phi(p_1) &= (1,0)\\
\phi(p_2) &= (0,1)\\
\phi(p_3) &= (1,1)\\
\phi(p_4) &= (0,0).
\end{aligned}$$

We claim that $\phi$ is a cokernel, and so the composition $P \to \mathbb{N}^2 \to \mathrm{Div}_X$ gives a chart for the DF structure $L$.

**Lemma 3.4.1.** *The map $\phi$ is a cokernel.*

*Proof.* We denote an element $p$ of $P$ by a quadruple $(a,b,c,d)$, where $a,b,c,d > 0$ and $p = ap_1 + bp_2 + cp_3 + dp_4$. Clearly such a quadruple is not unique, as we have $(a,b,c,d) = (a+1, b+1, c-1, d-1)$ if $c,d > 1$, and the analogue for $a,b > 1$. In this representation, the map $\phi$ sends $(a,b,c,d)$ to $(a+c, b+c)$.

With a simple computation one sees that $\phi^{-1}(0) = \langle p_4 \rangle \subseteq P$, so to show that $\phi$ is a cokernel we have to verify that if $\phi(p) = \phi(p')$, then there exist $e, e' \in \mathbb{N}$ such that $p + ep_4 = p' + e'p_4$.

Now $\phi(p) = \phi(p')$ means $(a+c, b+c) = (a'+c', b'+c')$. Assume without loss of generality that $a \geq b$; then $(a,b,c,d) = (a-b, 0, c+b, d+b)$, and since

$$a' - b' = (a'+c') - (b'+c') = (a+c) - (b+c) = a - b,$$

we also have $a' \geq b'$ and $(a', b', c', d') = (a' - b', 0, c' + b', d' + b')$. Since $a - b = a' - b'$ and $c + b = c' + b'$, we only have to worry about the last term, and this is easy: if $d + b \leq d' + b'$, just take $e = (d' + b') - (d + b)$ and $e' = 0$, otherwise take $e = 0$ and $e' = (d + b) - (d' + b')$. □

**Remark 3.4.2.** The map $\phi$ above is an example of a cokernel $P \to \mathbb{N}^2$ that "does not split", i.e. such that $\mathbb{N}^2$ is not a direct summand of $P$: there is a natural section $\mathbb{N}^2 \to P$ of $\phi$, but the resulting map $\mathbb{N}^2 \oplus \langle p_3, p_4 \rangle \to P$ is not an isomorphism, so that $\phi$ does not "split".

Now take the kummer morphism $j\colon A \to \frac{1}{2}A$, and as usual call $\mathfrak{X}$ the stack of roots of $X$ with respect to $j$, denote by $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$ the universal square roots of $D_1, D_2$ and $D$ respectively, and call $\mathcal{E}$ and $\mathcal{E}'$ the two generating sheaves associated to the charts $l'\colon \mathbb{N}^2 \to \mathrm{Div}(X)$ and $l' \circ \phi\colon P \to \mathrm{Div}(X)$. By following the construction of the generating sheaf we get

$$\mathcal{E} = \mathcal{O}(\mathcal{D}) \oplus \mathcal{O}(2\mathcal{D}_1 + \mathcal{D}_2) \oplus \mathcal{O}(\mathcal{D}_1 + 2\mathcal{D}_2) \oplus \mathcal{O}(2\mathcal{D})$$

and, noting that the indecomposable elements of $P$ are precisely the $p_i$'s, we get $\mathcal{E}' = \mathcal{E}'' \oplus \mathcal{E}''$ (since $\Lambda_{p_4} \cong \Lambda_{2p_4} \cong \mathcal{O}$), where

$$\begin{aligned}\mathcal{E}'' = \mathcal{O}(2\mathcal{D}) \otimes (\mathcal{O} \oplus \mathcal{O}(\mathcal{D}_1) \oplus \mathcal{O}(\mathcal{D}_2) \oplus \mathcal{O}(\mathcal{D}) \oplus \\ \oplus \mathcal{O}(2\mathcal{D}_1 + \mathcal{D}_2) \oplus \mathcal{O}(\mathcal{D}_1 + 2\mathcal{D}_2) \oplus \mathcal{O}(\mathcal{D}) \oplus \mathcal{O}(2\mathcal{D}));\end{aligned}$$

in particular $p_{\mathcal{E}'}(F) = p_{\mathcal{E}''}(F)$ for any parabolic sheaf $F \in \mathrm{Par}(X, j)$.

Our objective is to find a parabolic sheaf $F \in \mathrm{Par}(X, j)$ that is $\mathcal{E}$-(semi-)stable but not $\mathcal{E}'$-(semi-)stable. Our example will be an extension of two line bundles $L$ and $L'$ on the root stack $\mathfrak{X}$: the point will be that such an extension is semi-stable (and in fact stable, if is not trivial) if and only if $L$ and $L'$ have the same slope. To find an $F$ with the property we want, it will suffice then to find $L$ and $L'$ such that

$$p_{\mathcal{E}}(L) = p_{\mathcal{E}}(L')$$

but

$$p_{\mathcal{E}'}(L) \neq p_{\mathcal{E}'}(L').$$

Now recall that to give a torsion-free parabolic sheaf $F \in \mathrm{Par}(X, j)$, it suffices to give a torsion-free coherent sheaf $F_0 \in \mathrm{Coh}(X)$, together with a subsheaf $F_1 \subseteq F_0$ such that $F_0(-D) \subseteq F_1$. In particular we can take $F_1 = F_0$, and we get a (somewhat trivial) parabolic sheaf, we will denote it by $\widetilde{F_0} \in \mathrm{Par}(X, j)$. For such a parabolic sheaf, using

$$\begin{aligned}\pi_*\left(\widetilde{F_0} \otimes \mathcal{O}(-\mathcal{D}_i)\right) &= F_0 \\ \pi_*\left(\widetilde{F_0} \otimes \mathcal{O}(-\mathcal{D})\right) &= F_0 \\ \pi_*\left(\widetilde{F_0} \otimes \mathcal{O}(-2\mathcal{D})\right) &= F_0(-D) \\ \pi_*\left(\widetilde{F_0} \otimes \mathcal{O}(-2\mathcal{D}_1 - \mathcal{D}_2)\right) &= F_0(-D_1) \\ \pi_*\left(\widetilde{F_0} \otimes \mathcal{O}(-\mathcal{D}_1 - 2\mathcal{D}_2)\right) &= F_0(-D_2)\end{aligned}$$

we find

$$p_{\mathcal{E}}\left(\widetilde{F_0}\right) = \frac{1}{4}(p(F_0) + p(F_0(-D_1)) + p(F_0(-D_2)) + p(F_0(-D)))$$

$$= \frac{1}{4}(p(F_0) + 2p(F_0(-D_1)) + p(F_0(-D)))$$

where $p$ denotes the usual Gieseker (generalized) slope of the coherent sheaf $F_0$ on $X$ with respect to the fixed polarization, that in our case will be $H = \mathcal{O}(1,1)$, and $D_1 \sim D_2$ in our case.

The calculation for $\mathcal{E}''$ gives

$$\begin{aligned} p_{\mathcal{E}''}\left(\widetilde{F_0}\right) &= \frac{5}{8}p(F_0(-D)) + \frac{1}{8}\left(p(F_0(-2D)) + p(F_0(-D-D_1)) + p(F_0(-D-D_2))\right) \\ &= \frac{5}{8}p(F_0(-D)) + \frac{1}{8}\left(p(F_0(-2D)) + 2p(F_0(-D-D_1))\right). \end{aligned}$$

Now take $L_0 = \mathcal{O}(2,0)$ and $L_0' = \mathcal{O}(1,1)$. A straightforward calculation using

$$\chi(\mathcal{O}(a,b)(m)) = (a+1+m)(b+1+m)$$

and recalling that $\mathcal{O}(-D) \cong \mathcal{O}(-2,0)$ and $\mathcal{O}(-D_1) \cong \mathcal{O}(-1,0)$ , yields

$$p_{\mathcal{E}}\left(\widetilde{L_0}\right) = m^2 + 3m + 2 = p_{\mathcal{E}}\left(\widetilde{L_0'}\right)$$

but

$$p_{\mathcal{E}''}\left(\widetilde{L_0}\right) = m^2 + \frac{3}{2}m + \frac{1}{2}$$

and

$$p_{\mathcal{E}''}\left(\widetilde{L_0'}\right) = m^2 + \frac{3}{2}m - 1$$

as we wanted.

**Remark 3.4.3.** This shows that it is not true that (semi-)stability is independent of the chart, but it could well be that nonetheless the moduli spaces of (semi-)stable sheaves are isomorphic. For example there could be some autoequivalence $f_{\mathcal{E},\mathcal{E}'} \colon \mathrm{Coh}(\mathfrak{X}) \to \mathrm{Coh}(\mathfrak{X})$ such that $F \in \mathrm{Coh}(\mathfrak{X})$ is $\mathcal{E}$-(semi-)stable if and only if $f_{\mathcal{E},\mathcal{E}'}(F)$ is $\mathcal{E}'$-(semi-)stable.

For example, if $X$ is a log scheme coming from a divisor $D$ and $\mathcal{D}$ denotes the universal square root on $\mathfrak{X} = X_2$, then instead of the sheaf $\mathcal{O}(\mathcal{D}) \oplus \mathcal{O}(2\mathcal{D})$ we could take $\mathcal{O} \oplus \mathcal{O}(\mathcal{D})$ as generating sheaf, as we remarked in 3.3.14. In this case though there is a very simple autoequivalence as above, namely tensoring by $\mathcal{O}(-\mathcal{D})$, that does the trick.

In the example we just gave, using the fact that every invertible sheaf on $\mathfrak{X}$ is of the form $L = \pi^* M \otimes \mathcal{O}(a\mathcal{D}_1 + b\mathcal{D}_2)$ for $M \in \mathrm{Pic}(X)$ and $a, b$ are either 1 or 0, a computation shows that tensoring by any $L$ does not make the slopes corresponding to $\mathcal{E}$ and $\mathcal{E}'$ equal. We do not know if in this case there is some other autoequivalence of $\mathrm{Coh}(\mathfrak{X})$ that identifies the moduli spaces.

# Chapter 4

# Moduli of parabolic sheaves with varying weights

In this chapter we consider the moduli problem of parabolic sheaves with rational weights on a log scheme $X$, without bounding the denominators or fixing a finitely generated Kummer extension. For this, the infinite root stack $X_\infty$ is a natural object to consider, in view of the correspondence between quasi-coherent sheaves on it and parabolic sheaves with rational weights.

The natural approach to this problem is to take a limit of the moduli theory at finite levels, and this is what we will do in this chapter. In particular this will require $X$ to have a global chart $P \to \mathrm{Div}(X)$, giving us, as we saw in the last chapter, the generating sheaves $\mathcal{E}_n$ on the root stacks $X_n = X_{\frac{1}{n}P/P}$, and the moduli spaces and stacks $\mathcal{M}_n^s \subseteq \mathcal{M}_n^{ss}$ and $M_n^s \subseteq M_n^{ss}$ of (semi-)stable parabolic sheaves on $X$ (here the subscript keeps track of the denominators, and the Hilbert polynomial is not fixed for now).

In order to have this for every $n$, in this chapter we will assume that the characteristic of $k$ is zero. As remarked in the last chapter (see 3.3.24), this would be unnecessary if we knew that Nironi's machinery works on projective tame Artin stacks.

The ideal situation to take a limit would be the following.

**Ideal Theorem 4.0.1** (?)**.** *Let $X$ be a projective log scheme over $k$ with a global chart $P \to \mathrm{Div}(X)$. Then for every pair $n, m \in \mathbb{N}$ with $n \mid m$ there is a morphism $\iota_{n,m} \colon \mathcal{M}_n^{ss} \to \mathcal{M}_m^{ss}$, that induces $i_{n,m} \colon M_n^{ss} \to M_m^{ss}$ between the good moduli spaces, given by the pullback along $X_m \to X_n$. Moreover these morphisms are open and closed immersions.*

This would allow us to make sense of the direct limit $\varinjlim_n M_n^{ss}$ as a scheme, which would be a moduli space for parabolic sheaves with arbitrary rational weights on $X$.

Even without the last statement about the morphisms, the direct limit would still make sense as an ind-scheme, and we will see that it would be a good candidate for a "moduli space" of parabolic sheaves with arbitrary rational weights.

Unfortunately even the first part of Ideal Theorem 4.0.1 does not always hold. We will see that semi-stability is preserved by pullback along $X_m \to X_n$ if $P$ is free, and the following example shows that in general this need not necessarily be.

**Example 4.0.2.** Let us take $P = \langle p, q, r \mid p + q = 2r \rangle$ and a projective curve $X$ over $k$ with log structure induced by the morphism $P \to \mathrm{Div}(X)$ that sends $p, q$ and $r$ to invertible sheaves $L_p, L_q, L_r$, together with the zero section.

Their degrees need to satisfy $\deg(L_p) + \deg(L_q) = 2\deg(L_r)$ since $L_r^{\otimes 2} \cong L_p \otimes L_q$; let us assume that $\deg(L_p) = 0, \deg(L_q) = 2$ and $\deg(L_r) = 1$. Let us consider a fourth line bundle $L$ on $X$, of degree $d$. The sheaf $L$ on $X$ can be seen as a parabolic sheaf on $X_1 = X$, and as such, being a line bundle, it is stable. We will show that its pullback along $\pi \colon X_2 \to X$ is not semi-stable.

Note that parabolic sheaves on $X_2$ can be visualized as in the following diagram

$$\begin{tikzcd}
-2 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & \\
F_{(0,0)} \otimes L_p^\vee \arrow[rr] & & F_{(-1,0)} \arrow[rr] & & F_{(0,0)} & 0 \\
& F_{(-\frac{3}{2},-\frac{1}{2})} \arrow[ur] \arrow[rr] & & F_{(-\frac{1}{2},-\frac{1}{2})} \arrow[ur] & & -\frac{1}{2} \\
& & F_{(0,0)} \otimes L_r^\vee \arrow[uu] \arrow[ur] \arrow[rr] & & F_{(0,-1)} \arrow[uu] & -1 \\
& \cdots & & F_{(-\frac{1}{2},-\frac{3}{2})} \arrow[uu] \arrow[ur] & & -\frac{3}{2} \\
& & & & F_{(0,0)} \otimes L_q^\vee \arrow[uu] & -2
\end{tikzcd}$$

where we have a sheaf for every point $(\frac{a}{2}, \frac{b}{2})$ with $a$ and $b$ integers which are congruent modulo 2, and all the sheaves are uniquely determined by the ones in the diagram. In particular since we chose the zero sections when we defined the log structure, note that all the maps $F_b \to F_b \otimes L_p$ are zero, and the same holds with $L_q$ and $L_r$.

One calculates the pullback $\pi^* L$ (as explained in Section 1.3), and checks that in the diagram above it has

$$F_{(0,0)} = L$$
$$F_{(-\frac{1}{2},-\frac{1}{2})} = L \otimes L_r^\vee$$

$$
\begin{aligned}
F_{(-1,0)} &= L \otimes L_p^\vee \oplus L \otimes L_r^\vee \\
F_{(0,-1)} &= L \otimes L_q^\vee \oplus L \otimes L_r^\vee \\
F_{(-\frac{3}{2},-\frac{1}{2})} &= L \otimes L_p^\vee \otimes L_r^\vee \oplus L \otimes (L_r^\vee)^{\otimes 2} \\
F_{(-\frac{1}{2},-\frac{3}{2})} &= L \otimes L_q^\vee \otimes L_r^\vee \oplus L \otimes (L_r^\vee)^{\otimes 2}
\end{aligned}
$$

(the direct sums come from taking a direct limit, of course). One also checks that the pieces that contribute to the slope (i.e. the fundamental pieces), using the generating sheaf

$$\mathcal{E} = \left(\Lambda\left(\frac{p}{2}\right) \oplus \Lambda(p)\right) \otimes \left(\Lambda\left(\frac{q}{2}\right) \oplus \Lambda(q)\right) \otimes \left(\Lambda\left(\frac{r}{2}\right) \oplus \Lambda(r)\right),$$

are two copies of each of the following

$$
\begin{aligned}
& L \otimes (L_r^\vee)^{\otimes 2} \\
& L \otimes (L_r^\vee)^{\otimes 3} \\
& L \otimes (L_r^\vee)^{\otimes 3} \oplus L \otimes L_q^\vee \otimes (L_r^\vee)^{\otimes 2} \\
& L \otimes (L_r^\vee)^{\otimes 3} \oplus L \otimes L_p^\vee \otimes (L_r^\vee)^{\otimes 2}
\end{aligned}
$$

and so the parabolic degree of $\pi^* L$ is

$$2(d-2+d-3+d-3+d-4+d-3+d-2) = 12d - 34$$

and its parabolic rank is 12, so that the parabolic slope (the ratio of degree divided by the rank) is $\mu_{\mathcal{E}}(\pi^* L) = d - \frac{17}{6}$. In the present case (semi-)stability can be described by using this slope, since we are on a curve.

Finally one sees that $\pi^* L$ has a parabolic subsheaf $G$ were the only relevant pieces for the slope are two copies of $L \otimes L_p^\vee \otimes (L_r^\vee)^{\otimes 2}$, and the remaining ones are all zero, so that its parabolic degree is $2(d-2) = 2d-4$. Since its parabolic rank is 2, its parabolic slope will be $\mu_{\mathcal{E}}(G) = d-2$, which is greater than $\mu_{\mathcal{E}}(\pi^* L)$.

In conclusion $\pi^* L$ is not semi-stable.

This example leaves us with two choices: either we put additional hypotheses on the monoid, or we choose a different cofinal system of submonoids of $P_{\mathbb{Q}}$ with better properties.

Our solution is a mix of these two strategies: we will assume that $P$ is what we call a *simplicial* monoid, and we will take a slightly different cofinal system, made up of the monoids $\frac{1}{n}\mathbb{N}^r$ for a Kummer extension $P \subseteq \mathbb{N}^r$.

## 4.1 Simplicial logarithmic structures

In this section we briefly describe simplicial monoids and logarithmic structures.

**Definition 4.1.1.** A monoid $P$ is simplicial if it is fine, saturated, sharp and the positive rational cone $P_{\mathbb{Q}}$ it generates inside $P_{\mathbb{Q}}^{\mathrm{gp}}$ is simplicial, meaning that its extremal rays are linearly independent.

**Definition 4.1.2.** An indecomposable element $p \in P$ that lies on an extremal ray of the rational cone $P_{\mathbb{Q}}$ will be called *extremal*. Non-extremal indecomposables will be called *internal*.

In other words, an indecomposable $p \in P$ is extremal if $q + r \in \langle p \rangle$ implies $q, r \in \langle p \rangle$.

Assume $P$ is a simplicial monoid, and call $p_1, \ldots, p_r$ its extremal indecomposable elements, and $q_1, \ldots, q_s$ its internal ones. For any $q \in P$, we can write $q = \sum_i a_i p_i$ in $P_{\mathbb{Q}}$, where $a_i \in \mathbb{Q}$, and by simpliciality of $P$ the $a_i$ are uniquely determined.

In particular for every $q_j$ we have get a relation $c_j q_j = \sum_i a_{ij} p_i$ in $P$ where $(c_j, \{a_{ij}\}) = 1$. These relations will be called the *standard relations* of $P$.

**Proposition 4.1.3.** *Every simplicial monoid has a Kummer morphism to some free monoid $\mathbb{N}^r$. Viceversa, if a fine saturated monoid $P$ has a Kummer morphism $P \subseteq \mathbb{N}^r$, then $P$ is simplicial.*

In fact we will see that there is a minimal such Kummer extension, that we well call the *free envelope* of $P$.

The preceding proposition is the reason for introducing this simpliciality hypothesis. The Kummer extension $P \subseteq \mathbb{N}^r$ gives us a sequence $P \subseteq \frac{1}{n}\mathbb{N}^r = P_n$ of finitely generated Kummer extensions such that $\bigcup_n P_n = P_{\mathbb{Q}}$, and since $\mathbb{N}^r$ is free the transition maps $X_m \to X_n$ between the corresponding root stacks are flat, as the following lemma shows.

**Lemma 4.1.4.** *Let $X$ be a log stack with a global chart $\mathbb{N}^r \to \mathrm{Div}(X)$. Then for any $n$, the projection $\pi \colon X_n \to X$ is flat.*

This implies that all projections $X_m \to X_n$ between root stacks are flat as well, and actually for this one needs to assume that the log structure of $X$ is locally free, in the sense that the stalks of $A$ are all free monoids. For example if $D \subseteq X$ is a normal crossings divisor, then the induced log structure on $X$ is locally free, but does not necessarily have a global chart.

*Proof.* We can assume that $X$ is a log scheme. Then this follows from the fact that the projection $X_n \to X$ is a base change of the morphism $[\mathbb{A}^r / \mu_n^r] \to \mathbb{A}^r$ induced by the map $\mathbb{A}^r \to \mathbb{A}^r$ given by raising the variables to the $n$-th power. This last morphism is flat, and the conclusion follows. □

This assures that purity of coherent sheaves is preserved by pullback (recall that a semi-stable sheaf is pure). This cofinal system of root stacks will also be crucial for the arguments that we will use in the rest of this chapter.

Let us remark that this simpliciality assumption is forced if we want to find a cofinal system of root stacks with flat transition maps on the universal model.

Recall the following criterion from [Kat89].

**Proposition 4.1.5.** *If $P$ and $Q$ are integral monoids and $h \colon P \to Q$ is an injective morphism, then the induced map $\mathbb{Z}[P] \to \mathbb{Z}[Q]$ is flat if and only if the following condition is satisfied: for any $x_1, x_2 \in P$, $y_1, y_2 \in Q$ such that $h(x_1)y_1 = h(x_2)y_2$, there exist $x_3, x_4 \in P$ and $y \in Q$ such that $y_1 = h(x_3)y$, $y_2 = h(x_4)y$, (and then automatically $x_1x_3 = x_2x_4$).*

**Proposition 4.1.6.** *Let $P$ be a fine saturated torsion-free sharp monoid. If the natural morphism $\mathbb{Z}[P] \to \mathbb{Z}[P_{\mathbb{Q}}]$ is flat, then $P$ is a free monoid.*

*Proof.* Let $p_1, \ldots, p_k$ denote the indecomposable elements of $P$, so that $P$ has a presentation with generators the $p_i$'s and some relations. We need to show that there are no (nontrivial) relations.

First notice that we can assume that every $p_i$ is in some nontrivial relation, otherwise we can write $P = P' \oplus \mathbb{N}^h$ where $P'$ satisfies this condition, and focus on $P'$.

Let us now embed $P$ in some $\mathbb{N}^r$ (using 1.1.14), so that every $p \in P$ can be identified with a vector with $r$ coordinates. Thus we can write $p_i = (p_{i1}, \ldots, p_{ir})$ for $p_{ij} \in \mathbb{N}$, and for every $p_i$ we can consider the sum $s(p_i) = \sum_j p_{ij} \in \mathbb{N}$ of its components. Among the $p_i$'s there will be one, assume it is $p_1$, such that $s(p_i) \geq s(p_1)$ for every $i$. Notice that this implies that for every $i \neq 1$ there exists an $m \in \{1, \ldots, r\}$ such that $p_{im} > p_{1m}$. This embedding also allows us to define a monoid homomorphism $\lambda \colon P \to \mathbb{N}$, by composing the embedding in $\mathbb{N}^r$ with the map that takes the sum of the coordinates, landing in $\mathbb{N}$. This map has the property that $\lambda(p) = 0$ if and only if $p = 0$.

By assumption $p_1$ will show up in some relation $\sum_i a_i p_i = \sum_i b_i p_i$ with $a_1 \neq b_1$. In particular we can assume (by integrality of $P$) that (exactly) one among $a_1$ and $b_1$ is zero, say $b_1 = 0$, and more generally for any $i$, at least one of $a_i$ and $b_i$ is zero. Among all such relations, we can consider one in which $\lambda(\sum_i a_i p_i) = \lambda(\sum_i b_i p_i) \in \mathbb{N}$ is minimal.

So we have a relation in $P$ of the form

$$\sum_{i \in I} a_i p_i = \sum_{j \in J} b_j p_j \tag{4.1.7}$$

where $I, J \subseteq \{1, \ldots, r\}$ are non-empty, $I \cap J = \emptyset$, $1 \in I$ and $a_i, b_j > 0$ for any $i \in I$, $j \in J$. Now we pick and element of $J$, say it is 2, and call $d = \sum_{i \neq 1} a_i p_i$, $d' = \sum_{j \neq 2} b_j p_j$, so that our relation becomes $a_1 p_1 + d = b_2 p_2 + d'$.

Relation 4.1.7 gives, for any positive integer $n$, the relation

$$\sum_{i \in I} a_i \frac{p_i}{n} = \sum_{j \in J} b_j \frac{p_j}{n} \tag{4.1.8}$$

in $P_{\mathbb{Q}}$. This can also be written as

$$a_1 \frac{p_1}{n} + \frac{d}{n} = b_2 \frac{p_2}{n} + \frac{d'}{n}.$$

Using the last line, we get the following equality in $P_{\mathbb{Q}}$:

$$a_1 p_1 + \frac{d}{n} + (n-1) b_2 \frac{p_2}{n} = b_2 p_2 + \frac{d'}{n} + (n-1) a_1 \frac{p_1}{n}.$$

This is a relation in $P_{\mathbb{Q}}$ of the form $x_1 + y_1 = x_2 + y_2$, with $x_i \in P$ and $y_i \in P_{\mathbb{Q}}$, where $x_1 = a_1 p_1 + \lfloor \frac{(n-1)b_2}{n} \rfloor p_2$, $x_2 = b_2 p_2 + \lfloor \frac{(n-1)a_1}{n} \rfloor p_1$, $y_1 = \frac{d}{n} + \left\{ \frac{(n-1)b_2}{n} \right\} p_2$ and $y_2 = \frac{d'}{n} + \left\{ \frac{(n-1)a_1}{n} \right\} p_1$, where as usual $\lfloor \cdot \rfloor$ and $\{\cdot\}$ are the floor and fractional part of a rational number. From the flatness hypothesis and proposition 4.1.5, we know that there exist $x_3, x_4 \in P$ and $y \in P_{\mathbb{Q}}$ such that

$$\frac{d}{n} + \left\{ \frac{(n-1)b_2}{n} \right\} p_2 \quad = \quad x_3 + y$$

$$\frac{d'}{n} + \left\{ \frac{(n-1)a_1}{n} \right\} p_1 \quad = \quad x_4 + y.$$

Now the claim is that for $n$ big enough, we necessarily have $x_4 = 0$ or $x_4 = p_1$. This would conclude the proof: notice that for $n$ big enough, $\left\{ \frac{(n-1)b_2}{n} \right\} = 1 - \frac{b_2}{n}$, and $\left\{ \frac{(n-1)a_1}{n} \right\} = 1 - \frac{a_1}{n}$. So if $x_4 = 0$, the equalities above would give $d - d' = nx_3 + (n - a_1)p_1 - (n - b_2)p_2$ in $P^{\mathrm{gp}}$. But we also know that $d - d' = b_2 p_2 - a_1 p_1$ in $P^{\mathrm{gp}}$, and this gives $n(x_3 + p_1) = np_2$ in $P$, and by torsion-freeness we finally get $p_2 = x_3 + p_1$, a contradiction since $p_1$ and $p_2$ are distinct indecomposable elements of $P$.

In case $x_4 = p_1$, we get $d' = a_1 p_1 + ny$ (notice that $ny \in P$ by saturation of $P$, since $ny$ is both in $P^{\mathrm{gp}}$ and in $P_{\mathbb{Q}}$, and so it has a multiple in $P$). Since this is a relation in $P$ involving $p_1$ (recall $d' = \sum_{j \neq 2} b_j p_j$, and $1 \notin J \setminus \{2\}$) and clearly $\lambda(d') < \lambda(\sum_i b_i p_i)$, this contradicts the minimality of $\lambda(\sum_i b_i p_i)$ among such relations. Notice that $\lambda(d') = 0$ also gives a contradiction with the sharpness of $P$, since then $a_1 p_1 + ny = 0$.

To prove that $x_4 = 0$ or $p_1$ for $n$ big, let us write $x_4 = \sum_i c_i p_i$. Notice that for any $m \in \{1, \ldots, r\}$, the $m$-th coordinate of $d'/n$ converges to zero as $n$ grows. Since there are a finite number of coordinates, we can take $n$ large enough so that $\left(\frac{d'}{n}\right)_m < 0,000001$ for any $m$. Let us show that $c_i$ has to be zero for $i \neq 1$: if $c_i \geq 1$, pick $m$ such that $p_{im} > p_{1m}$, and consider the $m$-th coordinate in the equality defining $x_4$.

We get $\left(\frac{d'}{n}\right)_m + p_{1m} > \left(\frac{d'}{n}\right)_m + \left(1 - \frac{a_1}{n}\right) p_{1m} = \sum_i c_i p_{im} + y_m \geq p_{im}$, and since $\left(\frac{d'}{n}\right)_m$ is small and $p_{1m}, p_{im}$ are integers, we can conclude $p_{1m} \geq p_{im}$, a contradiction. For the same reason $c_1 \leq 1$, and so $x_4 = 0$ or $x_4 = p_1$, concluding the proof. □

**Proposition 4.1.9.** *If there is a sequence of monoids $Q_n \subseteq P_{\mathbb{Q}}$ containing $P$, and such that $Q_n \subseteq Q_m$ every time that $n|m$, $\bigcup_n Q_n = P_{\mathbb{Q}}$ and $\mathbb{Z}[P] \to \mathbb{Z}[Q_n]$ is flat for every $n$ (or even for $n$ very divisible), then $\mathbb{Z}[P] \to \mathbb{Z}[P_{\mathbb{Q}}]$ is flat as well.*

*Proof.* This follows immediately from the flatness criterion recalled above. □

**Corollary 4.1.10.** *If $P$ is such that there exists a sequence of monoids $Q_n \subseteq P_{\mathbb{Q}}$ containing $P$, and such that $Q_n \subseteq Q_m$ every time that $n|m$, $\bigcup_n Q_n = P_{\mathbb{Q}}$ and $\mathbb{Z}[Q_n] \to \mathbb{Z}[Q_m]$ is flat every time that $n|m$, then $P$ is simplicial.*

This shows that simpliciality of $P$ is forced if we want to have flat transition maps in the universal model.

Let us now construct for a simplicial monoid $P$ the minimal Kummer extension to a free monoid.

*Proof of Proposition 4.1.3.* Assume that $c_j q_j = \sum_i a_{ij} p_i$ be the standard relations of $P$, and let $b_{ij} = c_i / \gcd(c_i, a_{ij})$, a positive integer. The standard relations can be rewritten as follows

$$q_j = \sum_i \frac{a_{ij}}{\gcd(c_i, a_{ij})} \cdot \frac{p_i}{b_{ij}}.$$

Finally, let $d_i = \operatorname{lcm}(b_{ij} \mid j = 1, \ldots, r)$, and let $F(P)$ be the (free) submonoid of $P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the elements $\frac{p_1}{d_1}, \ldots, \frac{p_r}{d_r}$. By construction we have $P \subseteq F(P)$, and $P_{\mathbb{Q}} = F(P)_{\mathbb{Q}}$, so the morphism is Kummer.

The converse is clear, since if we have a Kummer morphism $P \subseteq \mathbb{N}^r$, then $P_{\mathbb{Q}} \cong \mathbb{Q}^r_+$, which is a simplicial cone. $\square$

**Definition 4.1.11.** We will call the monoid $F(P)$ constructed in the proof the *free envelope* of $P$. The *rank* of $P$ will be the rank of the free monoid $F(P)$, or equivalently of the free abelian group $P^{\mathrm{gp}}$.

**Example 4.1.12.** Let $P = \langle p, q, r \mid p + q = 2r \rangle$. Then $p$ and $q$ are the extremal indecomposables, and the only standard relation

$$r = \frac{p}{2} + \frac{q}{2}$$

gives the two generators $\frac{p}{2}$ and $\frac{q}{2}$ for the free envelope $F(P)$.

If we identify $P$ with the submonoid of $\mathbb{N}^2$ generated by $(2,0), (1,1), (0,2)$, then $F(P)$ coincides with $\mathbb{N}^2$, as $\frac{p}{2} = (1,0)$ and $\frac{q}{2} = (0,1)$.

The free envelope has the following universal property.

**Proposition 4.1.13.** *For any Kummer homomorphism $\phi\colon P \to \mathbb{N}^r$ there exists a unique (injective) homomorphism $\bar{\phi}\colon F(P) \to \mathbb{N}^r$ extending $\phi$.*

The proof is easy and left to the reader.

One can give the following definition of a simplicial log scheme.

**Definition 4.1.14.** A fs log scheme $X$ is simplicial if for any geometric point $x \to X$ the stalk $(A_X)_x$ is a simplicial monoid.

Since charts can be made up from stalks, a simplicial log scheme has local charts $P \to \mathrm{Div}(X)$ with $P$ a simplicial monoid.

The converse (if there are simplicial charts, then the stalks are simplicial) is also true, and follows from the fact that the kernel of a morphism $P \to Q$ from a simplicial monoid to a sharp fs monoid is generated by extremal indecomposables. From this one sees that the map $P_{\mathbb{Q}} \to Q_{\mathbb{Q}}$ corresponds to a quotient by the span of a subset of a basis of $P^{\mathrm{gp}}_{\mathbb{Q}}$, and consequently $Q_{\mathbb{Q}}$ is still a simplicial cone inside $Q^{\mathrm{gp}}_{\mathbb{Q}}$.

**Remark 4.1.15.** Despite this general definition, for the rest of this chapter we will assume that $X$ has a global chart $P \to \mathrm{Div}(X)$, in which $P$ is moreover simplicial.

## 4.2 (semi-)stability and extension of denominators

For the rest of this chapter, $X$ will be a projective simplicial log scheme with a global chart $P \to \mathrm{Div}(X)$, where $P$ is a simplicial monoid of rank $r$.

The first thing we want to do is to replace $X$ by the root stack $X_1 = X_{F(P)/P}$, where $F(P) \cong \mathbb{N}^r$ is the free envelope of $P$ introduced in the last section. After we have done this, when considering parabolic sheaves on $X_n = X_{\frac{1}{n}F(P)/P} = (X_1)_{\frac{1}{n}F(P)/F(P)}$ we can see them as parabolic sheaves on the log stack $X_1$, where the log structure has a free chart, and the transition maps $X_m \to X_n$ will be flat (see 4.1.4). This way we can effectively argue as if the log structure on $X$ itself had a free chart to start with.

Denote by $\mathcal{E}_n$ the generating sheaf on $X_n$ coming from the root stack structure over $X$, and $\widetilde{\mathcal{E}_n}$ the generating sheaf that comes from seeing it as a root stack over $X_1$. We would like to say that these two generating sheaves give the same stability. This is true, provided that we equip $X_1$ with the right generating sheaf relative to $X$.

The following lemma relates the generating sheaves of two root stacks of $X$, where one of them is obtained by taking $n$-th roots over the other one.

**Lemma 4.2.1.** *Let $X$ be a log scheme with a global chart $P \to \mathrm{Div}(X)$, and let $P \subseteq Q$ be a Kummer extension. Consider the commutative diagram*

$$\begin{tikzcd} X_{\frac{1}{n}Q/P} \arrow[rr, "\pi"] \arrow[dr, "p'"'] & & X_{Q/P} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

*and the generating sheaves $\mathcal{E}_n$ on $X_{\frac{1}{n}Q/P}$ and $\mathcal{E}$ on $X_{Q/P}$ relative to $X$, and $\mathcal{E}_{rel}$ on $X_{\frac{1}{n}Q/P}$ obtained by seeing it as a root stack over $X_{Q/P}$ with respect to the Kummer extension $Q \subseteq \frac{1}{n}Q$. Denote by $L\colon Q \to \mathrm{Div}(X_{Q/P})$ the universal DF structure on $X_{Q/P}$ and by $p_i$ the indecomposable elements of $Q$.*

*Then we have an isomorphism*

$$\mathcal{E}_n \cong \pi^*\mathcal{E} \otimes \mathcal{E}_{rel} \otimes \pi^* M$$

*where $M = (\bigotimes_{i=1}^r L(p_i))^\vee = L(\sum_i -p_i)$.*

*Proof.* Denote by $L_n\colon \frac{1}{n}Q \to \mathrm{Div}(X_{\frac{1}{n}Q/P})$ the universal DF structure on $X_{\frac{1}{n}Q/P}$, and by $d_i$ the order of the image of $p_i$ in $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$.

Let us write down the generating sheaves. We have

$$\mathcal{E}_n = \bigoplus_{0<a_i\leq nd_i} L_n\left(\sum_i a_i \frac{p_i}{n}\right)$$

$$\mathcal{E} = \bigoplus_{0<b_i\leq d_i} L\left(\sum_i b_i p_i\right)$$

$$\mathcal{E}_{rel} = \bigoplus_{0<c_i\leq n} L_n\left(\sum_i c_i \frac{p_i}{n}\right)$$

so that, since $\pi^* L\left(p_i\right) \cong L_n\left(n\frac{p_i}{n}\right)$,

$$\pi^*\mathcal{E} = \bigoplus_{0<b_i\leq d_i} L_n\left(\sum_i n b_i \frac{p_i}{n}\right).$$

In conclusion

$$\pi^*\mathcal{E} \otimes \mathcal{E}_{rel} \otimes \pi^* M \cong \bigoplus_{\substack{0<b_i\leq d_i\\ 0<c_i\leq n}} L_n\left(\sum_i (nb_i + c_i - n)\frac{p_i}{n}\right)$$

and this is $\mathcal{E}_n$, since every $0 < a_i \leq nd_i$ arises exactly once as $nb_i + c_i - n$ for $0 < b_i \leq d_i$ and $0 < c_i \leq n$.

□

**Remark 4.2.2.** The locally free sheaf $\mathcal{E} \otimes M$ on $X_{Q/P}$ of the previous lemma is still a generating sheaf, and it is precisely the generating sheaf $\mathcal{E}'$ of Remark 3.3.14.

If we equip $X_1$ with the generating sheaf $\mathcal{E} \otimes M$, then the stability notions on $X_n$ corresponding to $\mathcal{E}_n$ relative to $X$ and $\widetilde{\mathcal{E}_n}$ relative to $X_1$ are the same. Indeed if $F \in \mathrm{Coh}(X_n)$, then (keeping the notation of the lemma, with $Q = F(P)$) from the previous lemma and the projection formula for $\pi$ we see that

$$\begin{aligned} p_{\mathcal{E}_n}(F) &= p(p'_*(F \otimes \mathcal{E}_n^\vee)) \\ &= p(p_*\pi_*(F \otimes \pi^*(\mathcal{E} \otimes M)^\vee \otimes \widetilde{\mathcal{E}_n}^\vee)) \\ &= p(p_*(\pi_*(F \otimes \widetilde{\mathcal{E}_n}^\vee) \otimes (\mathcal{E} \otimes M)^\vee)) \\ &= p_{\mathcal{E}\otimes\mathcal{M}}(\pi_*(F \otimes \widetilde{\mathcal{E}_n}^\vee)) \end{aligned}$$

where $p$ denotes the reduced Hilbert polynomial on $X$. Note also that if $P$ is already free, then $\mathcal{E} \otimes M$ is indeed trivial. In conclusion we can replace $X$ by $X_1$ in what follows, even though we will keep this notation for clarity.

**Notation 4.2.3.** From now on we will fix an isomorphism $F(P) \cong \mathbb{N}^r$, and denote the canonical log structure on $X_n$ by $L_n \colon \frac{1}{n}\mathbb{N}^r \to \mathrm{Div}(X_n)$. Moreover $p_i$ for $i = 1, \dots, r$ will denote the indecomposable elements of $F(P) \cong \mathbb{N}^r$, and for any $r$-tuple of integers $(a_1, \dots, a_r)$, we will denote by $L_n^{(a_i)}$ the invertible sheaf $L_n\left(\sum_i a_i \frac{p_i}{n}\right)$ on the root stack $X_n$. In particular $L_n^i$ will be the invertible sheaf $L_n^{(0,\dots,1,\dots,0)} = L_n\left(\frac{p_i}{n}\right)$ on $X_n$.

In the same spirit, if $E$ is a parabolic sheaf on $X_n$ and $(e_1, \dots, e_r)$ is an element of $\mathbb{Z}^r$, we denote by $E_{(e_i)}$ the piece of the parabolic sheaf $E$ corresponding to the element $(\frac{e_1}{n}, \dots, \frac{e_r}{n})$ of $\frac{1}{n}\mathbb{Z}^r$.

We will consider the generating sheaves $\mathcal{E}_n$ on $X_n$ that we introduced in the last chapter, the notion of (semi-)stability defined by them, the corresponding moduli stacks $\mathcal{M}_n^{ss}$ and $\mathcal{M}_n^s$ of (semi-)stable sheaves, with good moduli spaces $M_n^{ss}$ and $M_n^s$. If $F$ is a coherent sheaf on $X_n$, with $p_n(F)$ we will denote the reduced Hilbert polynomial $p_{\mathcal{E}_n}(F)$ obtained by using the generating sheaf $\mathcal{E}_n$. We will also denote just by $p$ the reduced Hilbert polynomial on $X_1$, with respect to the generating sheaf $\mathcal{E} \otimes M$ discussed above.

We summarize here the results of this section.

**Theorem 4.2.4.** *Let $X$ be a projective simplicial log scheme over $k$ with a global chart, and $n, m$ two natural numbers with $n \mid m$. Then:*

- *the pullback along $\pi \colon X_m \to X_n$ of a semi-stable sheaf is semi-stable (with the same reduced Hilbert polynomial), so we get a morphism $\iota_{n,m} \colon \mathcal{M}_n^{ss} \to \mathcal{M}_m^{ss}$. This morphism in turn induces $i_{n,m} \colon M_n^{ss} \to M_m^{ss}$ between the good moduli spaces.*

- *$\iota_{n,m}$ is always an open immersion, and $i_{n,m}$ is proper, open and injective on geometric points (in particular it is also finite).*

- *if pullback along $\pi$ preserves stability (for example this happens if the log structure of $X$ is generically trivial), then $\iota_{n,m}$ restricts to a morphism $\iota^{o}_{n,m}\colon \mathcal{M}^{s}_{n} \to \mathcal{M}^{s}_{m}$ between the stacks of stable sheaves, and correspondingly $i_{n,m}$ restricts to $i^{o}_{n,m}\colon M^{s}_{n} \to M^{s}_{m}$ between the good moduli spaces. Moreover in this case all the maps are open and closed immersions.*

**Remark 4.2.5.** The fact that $i_{n,m}$ is an open and closed immersion (i.e. an immersion of a union of connected components) will allow us to make sense of the direct limit of the moduli spaces as a scheme.

We will see that the pullback along $\pi$ does not preserve stability in general, and if this happens we will not be able to take the direct limit of the stack/space of stable sheaves, nor to conclude that $i_{n,m}$ is an immersion of a union of connected components. Being open, closed and injective, its image will still be a union of connected components, but it may not be an isomorphism onto the image. We do not have any example where it does not happen, but we also have no reason for why it could not happen.

Let us see with a simple example why semi-stability is preserved

**Example 4.2.6.** Assume that $X$ is an integral projective log scheme with a rank 1 free chart $\mathbb{N} \to \mathrm{Div}(X)$, and denote by $(L, s) \in \mathrm{Div}(X)$ the image of $1 \in \mathbb{N}$, and say we are considering pure sheaves of maximal dimension. Let us consider the extension of denominators $\pi\colon X_4 \to X_2$, and a semi-stable coherent sheaf $F \in \mathrm{Coh}(X_2)$. Let us give names to the fundamental pieces of $F$, say

$$\begin{tikzcd} & -1 & -\frac{1}{2} & 0 \\ F = & F_0 \arrow[r] & F_1 \arrow[r] & F_0 \otimes L \end{tikzcd}$$

so that

$$p_2(F) = \frac{p(F_0) + p(F_1)}{\mathrm{rk}(F_0) + \mathrm{rk}(F_1)}.$$

The pullback $\pi^* F \in \mathrm{Coh}(X_4)$ corresponds to the parabolic sheaf

$$\begin{tikzcd} & -1 & -\frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 \\ \pi^* F = & F_0 \arrow[r, equal] & F_0 \arrow[r] & F_1 \arrow[r, equal] & F_1 \arrow[r] & F_0 \otimes L \end{tikzcd}$$

and notice that

$$p_4(\pi^* F) = \frac{2(p(F_0) + p(F_1))}{2(\mathrm{rk}(F_0) + \mathrm{rk}(F_1))} = p_2(F).$$

Now take a subsheaf $G \subseteq \pi^* F$, corresponding to the following diagram

$$\begin{tikzcd}
 & -1 & -\frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 \\
\pi^* F = & F_0 \arrow[r, equal] & F_0 \arrow[r] & F_1 \arrow[r, equal] & F_1 \arrow[r] & F_0 \otimes L \\
G = & G_0 \arrow[u, hook] \arrow[r, "f_0"] & G_1 \arrow[u, hook] \arrow[r, "f_1"] & G_2 \arrow[u, hook] \arrow[r, "f_2"] & G_3 \arrow[u, hook] \arrow[r, "f_3"] & G_0 \otimes L \arrow[u, hook]
\end{tikzcd}$$

and with

$$p_4(G) = \frac{p(G_0) + p(G_1) + p(G_2) + p(G_3)}{\mathrm{rk}(G_0) + \mathrm{rk}(G_1) + \mathrm{rk}(G_2) + \mathrm{rk}(G_3)}.$$

Now note that from $G$ we can get the following two subsheaves $G', G''$ of our original $F$

$$\begin{tikzcd}
 & -1 & -\frac{1}{2} & 0 \\
F = & F_0 \arrow[r] & F_1 \arrow[r] & F_0 \otimes L \\
G' = & G_0 \arrow[u, hook] \arrow[r, "f_1 \circ f_0"] & G_2 \arrow[u, hook] \arrow[r, "f_3 \circ f_2"] & G_0 \otimes L \arrow[u, hook]
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & -1 & -\frac{1}{2} & 0 \\
F = & F_0 \arrow[r] & F_1 \arrow[r] & F_0 \otimes L \\
G'' = & G_1 \arrow[u, hook] \arrow[r, "f_2 \circ f_1"] & G_3 \arrow[u, hook] \arrow[r, "f_0 \circ f_3"] & G_1 \otimes L \arrow[u, hook]
\end{tikzcd}$$

where $f_0 \circ f_3 \colon G_3 \to G_1 \otimes L$ denotes the composition of $f_3$ with $f_0 \otimes \mathrm{id} \colon G_0 \otimes L \to G_1 \otimes L$.

We have

$$p_2(G') = \frac{p(G_0) + p(G_2)}{\mathrm{rk}(G_0) + \mathrm{rk}(G_2)} \leq p_2(F)$$
$$p_2(G'') = \frac{p(G_1) + p(G_3)}{\mathrm{rk}(G_1) + \mathrm{rk}(G_3)} \leq p_2(F)$$

since $F$ is semi-stable, and it is easy to see that $p_4(G) = \alpha_1 p_2(G') + \alpha_2 p_2(G'')$, where $0 \leq \alpha_i \leq 1$ and $\alpha_1 + \alpha_2 = 1$. In conclusion $p_4(G) \leq \alpha_1 p_2(F) + \alpha_2 p_2(F) = p_2(F) = p_4(\pi^* F)$, so $\pi^* F$ is semi-stable on $X_4$.

Note that $G'$ and $G''$ can be zero, but the argument still works. If they are both zero, then $G$ itself is zero and there is nothing to prove. Otherwise assume that $G'$ is zero and $G''$ is not. In this case $G'$ doesn't contribute to the reduced Hilbert polynomial of $G$ at all, and in fact $p_4(G) = p_2(G'')$, and the rest of the argument applies.

The following lemma relates the generating sheaves of $X_n$ and $X_m$, and is the starting point of the proof.

**Lemma 4.2.7.** *Set $m = nk$, and consider the commutative diagram*

$$\begin{tikzcd} X_m \arrow[rr, "\pi"] \arrow[dr, "p'"'] & & X_n \arrow[dl, "p"] \\ & X_1. & \end{tikzcd}$$

*We have an isomorphism*

$$\mathcal{E}_m \cong \pi^* \mathcal{E}_n \otimes \mathcal{E}_{n,m} \otimes M$$

*where $\mathcal{E}_{n,m}$ is the generating sheaf of $X_m$ as a root stack of $X_n$ and $M = \left(\bigotimes_{i=1}^r L_m^i\right)^{\otimes(-k)} = L_m(\sum_i -k\frac{p_i}{m})$.*

*Proof.* This is a particular case of Lemma 4.2.1. □

**Lemma 4.2.8.** *With the notation of the previous lemma, let $G \in \mathrm{Coh}(X_m)$ be a coherent sheaf on $X_m$. Then $p_m(G)$ is a weighted mean of the reduced Hilbert polynomials of the non-zero sheaves among $\pi_*\left(G \otimes L_m^{(d_i)}\right)$ on $X_n$, with $0 \leq d_i < k$.*

*Proof.* Let us compute $p_m(G)$, using the previous lemma and the projection formula for the morphism $\pi$ (Proposition 1.2.35):

$$\begin{aligned} p_m(G) &= p(p'_*(G \otimes \mathcal{E}_m^\vee)) \\ &= p(p_*\pi_*(G \otimes \pi^*\mathcal{E}_n^\vee \otimes \mathcal{E}_{n,m}^\vee \otimes M^\vee)) \\ &= p(p_*(\pi_*(G \otimes \mathcal{E}_{n,m}^\vee \otimes M^\vee) \otimes \mathcal{E}_n^\vee)) \\ &= p_n(\pi_*(G \otimes \mathcal{E}_{n,m}^\vee \otimes M^\vee)) \end{aligned}$$

(where $p$ denotes the reduced Hilbert polynomial on $X_1$) and since we have $\mathcal{E}_{n,m}^\vee \otimes M^\vee = \bigoplus_{0 \leq d_i < k} L_m\left(\sum_i d_i \frac{p_i}{m}\right) = \bigoplus_{0 \leq d_i < k} L_m^{(d_i)}$, the last expression is equal to

$$p_n\Big(\bigoplus_{0 \leq d_i < k} \pi_*(G \otimes L_m^{(d_i)})\Big)$$

and this is a weighted mean of the polynomials $p_n(\pi_*(G \otimes L_m^{(d_i)}))$, as claimed.

Note that if for some $(d_i)$ the sheaf $\pi_*(G \otimes L_m^{(d_i)})$ is zero, then the corresponding Hilbert polynomial will not contribute to the reduced Hilbert polynomial of $G$ (this accounts for the "non-zero" part of the statement). □

**Remark 4.2.9.** Let us describe the sheaf $G^{(d_i)} = \pi_*(G \otimes L_m^{(d_i)})$ in a more concrete way as a parabolic sheaf on $X_1$. This will be important for the proof of the next results.

Let us take $(e_i) \in \mathbb{Z}^r$ with $0 \leq e_i < n$, and let us calculate the component $(G^{(d_i)})_{(e_i)} \in \mathrm{Coh}(X_1)$.

We have

$$
\begin{aligned}
(G^{(d_i)})_{(e_i)} &= p_*(\pi_*(G \otimes L_m^{(d_i)}) \otimes L_n^{(e_i)}) \\
&= p_*(\pi_*(G \otimes L_m^{(d_i)} \otimes \pi^* L_n^{(e_i)})) \\
&= p_*(\pi_*(G \otimes L_m^{(d_i)} \otimes L_m^{(ke_i)})) \\
&= p_*\pi_*(G \otimes L_m(\textstyle\sum_i (d_i + ke_i)\frac{p_i}{m})) \\
&= p'_*(G \otimes L_m(\textstyle\sum_i (d_i + ke_i)\frac{p_i}{m})) \\
&= G_{(d_i + ke_i)}.
\end{aligned}
$$

This calculation has the following "pictorial" interpretation: the parabolic sheaf $G^{(d_i)}$ is obtained by dividing the unit hypercube in $\frac{1}{m}\mathbb{N}^r$ in $n^r$ smaller hypercubes (of "size" $k^r$), by subdividing each segment in $n$ pieces, and then by picking the pieces of $G$ in position $(d_i)$ in each of these hypercubes, together with the induced maps.

Let us clarify this with a simple example in rank 2: let us assume that $X$ has a free log structure $L\colon \mathbb{N}^2 \to \mathrm{Div}(X)$, and take $m = 4$, $n = 2$ and a parabolic sheaf $F \in \mathrm{Coh}(X_4)$. If we take $(d_i) = (1,1)$, then by the calculation above the parabolic sheaf $F^{(1,1)} = \pi_*(F \otimes L_4^{(1,1)}) \in \mathrm{Coh}(X_2)$ (where $\pi\colon X_4 \to X_2$) takes the following form

$$
\begin{tikzcd}
-1 & -\frac{1}{2} & 0 & \\
F_{-3,-3} \otimes L_{0,1} \arrow[r] & F_{-1,-3} \otimes L_{0,1} \arrow[r] & F_{-3,-3} \otimes L_{1,1} & 0 \\
F_{-3,-1} \arrow[r] \arrow[u] & F_{-1,-1} \arrow[r] \arrow[u] & F_{-3,-1} \otimes L_{1,0} \arrow[u] & -\frac{1}{2} \\
F_{-3,-3} \arrow[r] \arrow[u] & F_{-1,-3} \arrow[r] \arrow[u] & F_{-3,-3} \otimes L_{1,0} \arrow[u] & -1
\end{tikzcd}
$$

and we see that it is obtained by subdividing the (negative) unit square of $\frac{1}{4}\mathbb{N}^2$ in four smaller squares and looking at the top right sheaf, corresponding to $(1,1)$, in each of these squares, together with the maps between them:

$$\begin{tikzcd}
-1 & -\frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 & \\
F_{-4,-4}\otimes L_{0,1} \arrow[r] & F_{-3,-4}\otimes L_{0,1} \arrow[r] & F_{-2,-4}\otimes L_{0,1} \arrow[r] & F_{-1,-4}\otimes L_{0,1} \arrow[r] & F_{-4,-4}\otimes L_{1,1} & 0 \\
F_{-4,-1} \arrow[r] \arrow[u] & F_{-3,-1} \arrow[r] \arrow[u] & F_{-2,-1} \arrow[r] \arrow[u] & F_{-1,-1} \arrow[r] \arrow[u] & F_{-4,-1}\otimes L_{1,0} \arrow[u] & -\frac{1}{4} \\
F_{-4,-2} \arrow[r] \arrow[u] & F_{-3,-2} \arrow[r] \arrow[u] & F_{-2,-2} \arrow[r] \arrow[u] & F_{-1,-2} \arrow[r] \arrow[u] & F_{-4,-2}\otimes L_{1,0} \arrow[u] & -\frac{1}{2} \\
F_{-4,-3} \arrow[r] \arrow[u] & F_{-3,-3} \arrow[r] \arrow[u] & F_{-2,-3} \arrow[r] \arrow[u] & F_{-1,-3} \arrow[r] \arrow[u] & F_{-4,-3}\otimes L_{1,0} \arrow[u] & -\frac{3}{4} \\
F_{-4,-4} \arrow[r] \arrow[u] & F_{-3,-4} \arrow[r] \arrow[u] & F_{-2,-4} \arrow[r] \arrow[u] & F_{-1,-4} \arrow[r] \arrow[u] & F_{-4,-4}\otimes L_{1,0} \arrow[u] & -1.
\end{tikzcd}$$

From this example we see that $F$ is literally "made up" from the sheaves $\pi_*(F \otimes L_m^{(d_i)})$. This will be useful in some of the following arguments.

**Proposition 4.2.10.** *With the notation of the previous lemmas, let $F \in \mathrm{Coh}(X_n)$ be a coherent sheaf on $X_n$. Then $p_m(\pi^* F) = p_n(F)$, and if $F$ is semi-stable, then $\pi^* F$ is semi-stable as well.*

For the proof, we will need the following lemma.

**Lemma 4.2.11.** *Let $Y$ be a log stack with a free global chart $L\colon \mathbb{N}^r \to \mathrm{Div}(Y)$, and consider the root stack $\pi\colon Y_n \to Y$, with $L_n\colon \frac{1}{n}\mathbb{N}^r \to \mathrm{Div}(Y_n)$ the canonical lifting of the log structure of $Y$. Then for any $0 \le d_i < n$, we have*

$$\pi_* L_n^{(d_i)} \cong \mathcal{O}_Y.$$

*Proof.* This is a calculation on the universal model for the root stack.

First of all by taking a presentation we can assume that $Y$ is a log scheme with a free global chart. The chart gives a cartesian diagram

$$\begin{tikzcd}
Y_n \arrow[r] \arrow[d, "\pi"'] & {[\mathbb{A}^r/\mu_n^r]} \arrow[d, "\pi'"] \\
Y \arrow[r] & \mathbb{A}^r
\end{tikzcd}$$

where the vertical map $\pi'$ is induced by raising the variables to the $n$-th power. Now $\pi'$ is a coarse moduli space of a tame DM stack and the diagram is cartesian, so we have a base change formula (Proposition 1.5 of [Nir]), and $L_n^{(d_i)}$ is a pullback of the corresponding sheaf on $[\mathbb{A}^r/\mu_n^r]$, so we can reduce to proving the statement in the universal case.

In this case, the invertible sheaf $L_n^{(d_i)}$ over $[\mathbb{A}^r/\mu_n^r]$ corresponds to the module of rank one over $A = k[x_1, \dots, x_r]$ generated by $x_1^{d_1} \cdots x_r^{d_r}$. Pushing forward amounts to taking invariants for $\mu_n^r$, and if $0 \leq d_i < n$ it is clear that the invariants are $k[x_1^n, \dots, x_r^n]$. This shows that $\pi'_* L_n^{(d_i)} \cong \mathcal{O}_{\mathbb{A}^r}$ in this case, and concludes the proof. □

*Proof of Proposition 4.2.10.* We will apply the last lemma to the morphism $\pi\colon X_m \to X_n$, which is a relative root stack morphism.

First let us prove that $p_m(\pi^* F) = p_n(F)$: by Lemma 4.2.8, $p_m(\pi^* F)$ is a weighted mean of the polynomials $p_n(\pi_*(\pi^* F \otimes L_m^{(d_i)}))$. But in this case by the projection formula for $\pi$ (Proposition 1.2.35) and the previous lemma we have

$$p_n(\pi_*(\pi^* F \otimes L_m^{(d_i)})) = p_n(F \otimes \pi_* L_m^{(d_i)}) = p_n(F)$$

so that $p_m(\pi^* F) = p_n(F)$.

Now let us show that if $F$ is semi-stable on $X_n$, then $\pi^* F$ is semi-stable on $X_m$. For any subsheaf $G \subseteq \pi^* F$, we know that $p_m(G)$ is a weighted mean of the non-zero ones among the reduced Hilbert polynomials $p_n(\pi_*(G \otimes L_m^{(d_i)}))$ for $0 \leq d_i < k$. Now note that by exactness of $\pi_*$ the inclusion $G \otimes L_m^{(d_i)} \subseteq \pi^* F \otimes L_m^{(d_i)}$ will induce

$$\pi_*(G \otimes L_m^{(d_i)}) \subseteq \pi_*(\pi^* F \otimes L_m^{(d_i)}) \cong F \otimes \pi_* L_m^{(d_i)} \cong F$$

and by semi-stability of $F$ we see that if $\pi_*(G \otimes L_m^{(d_i)})$ is non-zero, then

$$p_n(\pi_*(G \otimes L_m^{(d_i)})) \leq p_n(F).$$

This in turn implies that $p_m(G) \leq p_n(F) = p_m(\pi^* F)$, so we conclude that $\pi^* F$ is semi-stable on $X_m$. □

**Corollary 4.2.12.** *The pullback functor along $\pi\colon X_m \to X_n$ induces a morphism $\iota_{n,m}\colon \mathcal{M}_n^{ss} \to \mathcal{M}_m^{ss}$ of stacks over $(\mathrm{Aff})^{op}$, and a corresponding morphism $i_{n,m}\colon M_n^{ss} \to M_m^{ss}$ between the good moduli spaces.*

*Proof.* The functor $\iota_{n,m}(S)\colon \mathcal{M}_n^{ss}(S) \to \mathcal{M}_m^{ss}(S)$ is defined as pullback along the morphism $X_m \times_k S \to X_n \times_k S$. It is well-defined because of the preceding proposition and of the fact that $X_m \to X_n$ is flat, so in particular it preserves purity. The morphism $i_{n,m}$ is defined by the universal property of $M_n^{ss}$ as a good moduli space. □

**Remark 4.2.13.** Proposition 4.2.10 shows that the reduced Hilbert polynomial (unlike the non-reduced one) is preserved by pullback, so that the morphism $\iota_{n,m}$ restricts to $\mathcal{M}_{h,n}^{ss} \to \mathcal{M}_{h,m}^{ss}$ for any fixed $h \in \mathbb{Q}[x]$.

**Proposition 4.2.14.** *The morphism $\iota_{n,m}$ is an open immersion.*

*Proof.* Let us consider a morphism $f\colon S \to \mathcal{M}_m^{ss}$ from a scheme, and the cartesian diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[d] & S \arrow[d, "f"] \\ \mathcal{M}_n^{ss} \arrow[r, "\iota_{n,m}"] & \mathcal{M}_m^{ss}. \end{tikzcd}$$

The morphism $f$ corresponds to a family $F \in \mathrm{Coh}(X_m \times_k S)$ of semi-stable sheaves on $X_m$, and by construction $\mathfrak{X}(T)$ is the category of triples $(\phi\colon T \to S, G, \beta)$ with $G \in \mathrm{Coh}(X_n \times_k T)$ a family of semi-stable sheaves and $\beta\colon \iota_{n,m}(G) \cong \phi^* F$ as coherent sheaves on $X_m \times_k T$. Note that by adjunction we have a map $\alpha\colon \pi_S^* \pi_{S*} F \to F$ of sheaves on $X_m \times_k S$.

Consider the locus $S^0 \subseteq S$ of points where $\alpha$ is an isomorphism. We will prove that this is an open subscheme of $S$, and that it represents the fibered product $\mathfrak{X}$.

First of all observe that if $f\colon X \to Y$ is a proper morphism and $F$ is a quasi-coherent sheaf of finite type on $X$, then the locus of points of $Y$ such that $F_y = 0$ is an open subset of $Y$. This is because the support of $F$ is a closed subset of $X$, and its image in $Y$, which is closed by properness, is the complement of the locus where $F_y = 0$. In fact: it is clear that if $y$ is not in $f(\mathrm{Supp}(F))$, then $F_y = 0$. On the other hand, if $y \in f(\mathrm{Supp}(F))$, let us take a point in the preimage and localize $X$ and $Y$. We end up with a local morphism of local rings $A \to B$, a $B$-module $M$ such that $M/\mathfrak{m}_B M \neq 0$ (by Nakayama), and we need to show that $M/\mathfrak{m}_A M \neq 0$. This is clear from $\mathfrak{m}_A M \subseteq \mathfrak{m}_B M \subseteq M$.

Let us look at the kernel and cokernel of $\alpha$,

$$0 \longrightarrow K \longrightarrow \pi_S^* \pi_{S*} F \longrightarrow F \longrightarrow Q \longrightarrow 0.$$

Since tensor product is right exact, the locus on $S$ where $\alpha$ is surjective is exactly the locus where $Q_s = 0$. Let us call this locus $S' \subseteq S$, an open subscheme. Once we restrict to $S'$ the map $\alpha$ is surjective, so $K$ satisfies base change over points of $S$, since $F$ is flat over $S$. Now the locus in $S'$ where $\alpha$ is an isomorphism, our $S^0$, is exactly the locus where $K_s = 0$, which is therefore open, both in $S'$ and in $S$.

From the preceding discussion, on $S_0$ we have an isomorphism $\alpha^0\colon \pi_{S^0}^*(\pi_{S^0})_* F^0 \cong F^0$ where $F^0$ is the restriction of $F$ to $X_m \times_k S^0$, so the object $(S^0 \to S, (\pi_{S^0})_* F^0, \alpha^0)$ is an object of $\mathfrak{X}(S_0)$ (recall that $\iota_{n,m}$ is pullback along $\pi$). This object corresponds to a map $g\colon S^0 \to \mathfrak{X}$ (which coincides with the one induced by the two maps $S_0 \to S$, $S_0 \to \mathcal{M}_n^{ss}$). We claim that $g$ is an equivalence.

Indeed, it is essentially surjective because if $(\phi\colon T \to S, G, \beta)$ is an object of $\mathfrak{X}(T)$, then the map $\phi$ will factor through $S^0$, since a parabolic sheaf $F \in \mathrm{Coh}(X_m)$ comes from $X_n$ if and only if $\pi^*\pi_* F \to F$ is an isomorphism (by the projection formula), and the sheaf $\pi_S^* \pi_{S*} F$ satisfies base change. This gives us an object of $S^0(T)$, whose image is readily checked to be isomorphic to $(\phi\colon T \to S, G, \beta)$. On the other hand one checks that for a fixed scheme $T$ over $k$, objects $(\phi\colon T \to S, G, \beta)$ and $(\psi\colon T \to S, H, \gamma)$ are isomorphic if and only if $\phi = \psi$ and there are no non-trivial automorphisms, so $g$ is also fully faithful. □

**Remark 4.2.15.** The locus $S^0$ can also be described as the locus where all the maps that are identities in the pullback of a parabolic sheaf from $X_n$ are isomorphisms.

Now we turn our attention to the behavior of stable sheaves.

**Proposition 4.2.16.** *Assume that the pullback along $\pi$ of any stable sheaf is still stable. Then $\iota_{n,m}$ restricts to an open immersion $\iota^o_{n,m}\colon \mathcal{M}_n^s \to \mathcal{M}_m^s$, inducing $i^o_{n,m}\colon M_n^s \to M_m^s$ (which coincides with the restriction of $i_{n,m}$).*

We will need a couple of lemmas.

**Lemma 4.2.17.** *If $G \in \mathrm{Coh}(X_n)$ is a sheaf such that $\pi^* G \in \mathrm{Coh}(X_m)$ is stable, then $G$ is stable on $X_n$.*

*Proof.* Let $F \subseteq G$ be a non-zero proper subsheaf. Then since $\pi$ is flat and $\pi^*$ is fully faithful, $\pi^* F \subseteq \pi^* G$ is a non-zero proper subsheaf, and thus $p_n(F) = p_m(\pi^* F) < p_m(\pi^* G) = p_n(G)$, since $\pi^* G$ is stable. $\square$

**Lemma 4.2.18.** *The square*

$$\begin{tikzcd} \mathcal{M}_n^s \arrow[r, "\iota^o_{n,m}"] \arrow[d] & \mathcal{M}_m^s \arrow[d] \\ \mathcal{M}_n^{ss} \arrow[r, "\iota_{n,m}"] & \mathcal{M}_m^{ss} \end{tikzcd}$$

*is cartesian.*

*Proof.* Denote by $\mathfrak{X}$ the fibered product $\mathcal{M}_n^{ss} \times_{\mathcal{M}_m^{ss}} \mathcal{M}_m^s$. The objects of $\mathfrak{X}(T)$ for $T$ a scheme over $k$ are triples $(G, H, \alpha)$, where $G \in \mathrm{Coh}(X_n \times_k T)$ is a family of semi-stable sheaves, $F \in \mathrm{Coh}(X_m \times_k T)$ is a family of stable sheaves, and $\alpha \colon \pi_T^* G \cong H$ is an isomorphism in $\mathrm{Coh}(X_m \times_k T)$. We have a map $g \colon \mathcal{M}_n^s \to \mathfrak{X}$ sending a family of stable sheaves $F \in \mathrm{Coh}(X_n \times_k T)$ over $T$ to the object $(F, \pi_T^* F, \mathrm{id})$ of $\mathfrak{X}(T)$, and we will prove that this is an equivalence.

Now take an object $(G, H, \alpha)$ of $\mathfrak{X}(T)$. The previous lemma implies that the fibers of $G$ are stable, since their pullback to $X_m$ is stable. This says that $G$ is an object of $\mathcal{M}_{h,n}^s$, and one checks that $g(G) = (G, \pi_T^* G, \mathrm{id})$ is isomorphic to the original $(G, H, \alpha)$, so $g$ is essentially surjective. For fully faithfullness, it is sufficient to notice that given a morphism $(\phi, \psi) \colon (F, \pi_T^* F, \mathrm{id}) \to (G, \pi_T^* G, \mathrm{id})$ the component $\psi \colon \pi_T^* F \to \pi_T^* G$ has to coincide with $\pi_T^* \phi$. $\square$

**Remark 4.2.19.** From the proof it is clear that this lemma will also hold if we are considering variants with fixed reduced Hilbert polynomial $h \in \mathbb{Q}[x]$ (or with some other fixed datum, compatible with pullback), i.e. the square

$$\begin{tikzcd} \mathcal{M}_{h,n}^s \arrow[r, "\iota^o_{n,m}"] \arrow[d] & \mathcal{M}_{h,m}^s \arrow[d] \\ \mathcal{M}_{h,n}^{ss} \arrow[r, "\iota_{n,m}"] & \mathcal{M}_{h,m}^{ss} \end{tikzcd}$$

is cartesian as well.

*Proof of Proposition 4.2.16.* The fact that pullback preserves stability implies that $\iota_{n,m}$ maps $\mathcal{M}_n^s$ to $\mathcal{M}_m^s$, so the map $\iota^o_{n,m}$ is well-defined. Lemma 4.2.18 implies that $\iota^o_{n,m}$ is an open immersion, since we know that $\iota_{n,m}$ is an open immersion, and the statement for $i^o_{n,m}$ follows from the properties of good moduli spaces. $\square$

The morphism $\iota_{n,m} \colon \mathcal{M}_n^{ss} \to \mathcal{M}_m^{ss}$ is not always closed.

**Example 4.2.20.** Consider the case of the standard log point, i.e. $X = \mathrm{Spec}(k)$ with the log structure $L \colon \mathbb{N} \to k$, sending $0$ to $1 \in k$ and everything else to zero. Consider the projection

$\pi\colon X_2 \to X$, and the family of parabolic sheaves $\{E_t\}_{t\in k}$ with weights in $\frac{1}{2}\mathbb{N}$, over $\mathbb{A}^1_k$, given by $(E_t)_a = k$ for any $a \in \frac{1}{2}\mathbb{Z}$, and maps

$$(E_t)_{-1} \to (E_t)_{-1/2} \to (E_t)_0 = k \xrightarrow{\cdot t} k \xrightarrow{0} k$$

This is a flat family of semi-stable sheaves over $\mathbb{A}^1_k$, i.e. an object of $\mathcal{M}_2^{ss}(\mathbb{A}^1_k)$. Notice also that the diagram

$$\begin{tikzcd} k \arrow[r, "\cdot t"] \arrow[d, equal] & k \arrow[r, "0"] & k \arrow[d, equal] \\ k \arrow[r, equal] & k \arrow[u, "\cdot t"] \arrow[r, "0"] & k \end{tikzcd}$$

shows that for $t \neq 0$, $E_t$ is isomorphic to the pullback of the unique invertible sheaf on $\mathrm{Spec}(k)$, but when $t = 0$ this is clearly not true.

This essentially shows that the following diagram is cartesian

$$\begin{tikzcd} \mathbb{A}^1_k \setminus \{0\} \arrow[r] \arrow[d] & \mathbb{A}^1_k \arrow[d, "E_t"] \\ \mathcal{M}_1^{ss} \arrow[r, "\iota_{1,2}"] & \mathcal{M}_2^{ss} \end{tikzcd}$$

and this implies that $\iota_{1,2}$ is not closed in this case.

In this example the pullback of a stable sheaf need not be stable in general. Let us examine directly a larger class of examples where this happens.

**Example 4.2.21.** Assume that $X$ is a log scheme with a chart $L\colon \mathbb{N} \to \mathrm{Div}(X)$ such that $L(1) = (L_1, 0) \in \mathrm{Div}(X)$, and let $F$ be a stable sheaf on $X_1$. Then for every $0 \le i < n$ we have the stable parabolic sheaf

$$\begin{array}{cccccccccc} & -1 & & \cdots & & -\frac{i}{n} & & \cdots & & 0 \\ F_i = & 0 \longrightarrow 0 & \longrightarrow & \cdots & \longrightarrow 0 \longrightarrow & F & \longrightarrow 0 \longrightarrow & \cdots & \longrightarrow 0 \longrightarrow & 0 \end{array}$$

on $X_n$, with one copy of $F$ in place $-\frac{i}{n}$, and 0 everywhere else (of course when $i = 0$ we will have $F \otimes L_1^{-1}$ in place $-1$).

Given $m = nk$, the pullback along $\pi\colon X_m \to X_n$ of $F_i$ is given by

$$\begin{array}{cccccccc} & -1 & \cdots & -\frac{ki}{m} & \cdots & -\frac{ki-k+1}{m} & \cdots & 0 \\ \pi^* F_i = & 0 \longrightarrow & \cdots \longrightarrow & F = F = & \cdots & = F = F & \longrightarrow \cdots \longrightarrow & 0, \end{array}$$

or, in other words

$$(\pi^* F_i)_a = \begin{cases} F & \text{for } -\frac{ki}{m} \leq a \leq -\frac{ki-k+1}{m} \\ 0 & \text{otherwise} \end{cases}$$

with the obvious maps (so there are $k$ copies of $F$). The sheaves $\pi^* F_i$ are semi-stable, but not stable anymore: for example we have a subsheaf $G \subseteq \pi^* F$ given by

$$\begin{tikzcd}
 & -1 & \cdots & -\frac{ki}{m} & \cdots & -\frac{ki-k+2}{m} & -\frac{ki-k+1}{m} & -\frac{k(i-1)}{m} & \cdots & 0 \\
\pi^* F_i = & 0 \arrow[r] & \cdots \arrow[r] & F \arrow[r, equal] & \cdots \arrow[r, equal] & F \arrow[r, equal] & F \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \\
G = & 0 \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & 0 \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & 0 \arrow[r] \arrow[u, hook] & F \arrow[r] \arrow[u, hook] & 0 \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & 0, \arrow[u, hook]
\end{tikzcd}$$

that is

$$G_a = \begin{cases} F & \text{for } a = -\frac{ki-k+1}{m} \\ 0 & \text{otherwise,} \end{cases}$$

and clearly $p_m(G) = p_m(\pi^* F_i) = p_1(F)$.

Moreover it is easy to describe the stable factors of $\pi^* F_i$: they are precisely the sheaves $G_j$ with one copy of $F$ in place $-\frac{ki-j}{m}$ for $0 \leq j \leq k-1$, and all zeros otherwise. Note that all of the sheaves on $X_n$ described in Remark 4.2.9 (obtained starting from one of the stable factors $G_j$) coincide with the original $F_i$ or are zero, in this case.

Note also that $\pi^* F_i$ is not even polystable: of the sheaves $G_j$ just described, only $G = G_{k-1}$ is a subsheaf of $\pi^* F_i$. The polystable sheaf $\bigoplus G_j$ which is S-equivalent to $\pi^* F_i$ is

$$\begin{tikzcd}
 & -1 & \cdots & -\frac{ki}{m} & & \cdots & & -\frac{ki-k+1}{m} & \cdots & 0 \\
\bigoplus G_j = & 0 \arrow[r] & \cdots \arrow[r] & F \arrow[r, "0"] & F \arrow[r, "0"] & \cdots \arrow[r, "0"] & F \arrow[r, "0"] & F \arrow[r] & \cdots \arrow[r] & 0
\end{tikzcd}$$

where all the maps are zero.

This example can be generalized in arbitrary rank. For example if $X$ has a chart $L\colon \mathbb{N}^2 \to \mathrm{Div}(X)$ with both $(1,0)$ and $(0,1)$ going to $(L_{1,0}, 0)$ and $(L_{0,1}, 0)$, this example carries through verbatim (so that again there will be stable sheaves that become strictly semi-stable after pullback), but we can also do something different.

Let us introduce some notation first.

**Notation 4.2.22.** We need to draw parabolic sheaves on $X_n$, where $X$ is a log stack with a free log structure $\mathbb{N}^r \to \mathrm{Div}(X)$.

When $r = 1$, we can draw parabolic sheaves easily as a the segment in $[-1, 0]$

$$\begin{tikzcd}
\cdots & -1 & -\frac{n-1}{n} & \cdots & -\frac{1}{n} & 0 & \cdots \\
\cdots \arrow[r] & F_0 \arrow[r] & F_1 \arrow[r] & \cdots \arrow[r] & F_{n-1} \arrow[r] & F_0 \otimes L \arrow[r] & \cdots
\end{tikzcd}$$

of a "sequence" of sheaves arranged on the real line.

If $r = 2$ we can draw the sheaf as the square $[-1, 0]^2$ (for example if we are taking square roots)

$$\begin{tikzcd}
-1 & -\frac{1}{2} & 0 & \\
F_{-1,-1} \otimes L_{0,1} \arrow[r] & F_{-\frac{1}{2},-1} \otimes L_{0,1} \arrow[r] & F_{-1,-1} \otimes L_{1,1} & 0 \\
F_{-1,-\frac{1}{2}} \arrow[r] \arrow[u] & F_{-\frac{1}{2},-\frac{1}{2}} \arrow[r] \arrow[u] & F_{-1,-\frac{1}{2}} \otimes L_{1,0} \arrow[u] & -\frac{1}{2} \\
F_{-1,-1} \arrow[r] \arrow[u] & F_{-\frac{1}{2},-1} \arrow[r] \arrow[u] & F_{-1,-1} \otimes L_{1,0} \arrow[u] & -1
\end{tikzcd}$$

inside a "diagram" on the plane with a sheaf on every point with integral coordinates and maps going up and to the right.

If the rank is bigger this becomes less feasible, but we have an "inductive" way of drawing parabolic sheaves in higher rank. For example, the sheaf with $r = 2$ above can be drawn in the following way: say that the DF structure is given by $L\colon \mathbb{N}^2 \to \mathrm{Div}(X)$, and consider the new DF structure given by the composition $\mathbb{N} \subseteq \mathbb{N}^2 \to \mathrm{Div}(X)$, where $\mathbb{N} \subseteq \mathbb{N}^2$ is the inclusion of the first or second component. Call the resulting log schemes $\widetilde{X}_1$ and $\widetilde{X}_2$ respectively.

Then a parabolic sheaf on the root stack $X_2$ can be drawn as a diagram

$$\begin{tikzcd}
\cdots & -1 & -\frac{1}{2} & 0 & \cdots \\
\cdots \arrow[r] & F_0 \arrow[r] & F_1 \arrow[r] & F_0 \otimes L_{(1,0)} \arrow[r] & \cdots
\end{tikzcd}$$

with the formal properties of a parabolic sheaf on $\widetilde{X}_1$, but where the sheaves $F_0$ and $F_1$ are parabolic sheaves on $\widetilde{X}_2$. In other words we are "collapsing" the vertical direction, and the price is to use parabolic sheaves in place of quasi-coherent sheaves.

In general if $X$ is a log scheme with a chart $L\colon \mathbb{N}^r \to \mathrm{Div}_X$, let us consider the DF structure given by the inclusion $\mathbb{N}^{r-1} \subseteq \mathbb{N}^r \to \mathrm{Div}_X$ that omits the $i$-th standard generator $e_i$. Call $\widetilde{X}$ the resulting log scheme. Then a parabolic sheaf on $X_n$ can be seen as a diagram

$$\begin{tikzcd}
-1 & \cdots & & -\frac{j}{n} & & \cdots & & 0 & \\
& & & & & & & & \circ \arrow[llllllll, "i\text{-th direction}"'] \arrow[d, "\text{other directions}"'] \\
F_0 \arrow[r] & F_1 \arrow[r] & \cdots \arrow[r] & F_{n-j-1} \arrow[r] & F_{n-j} \arrow[r] & F_{n-j+1} \arrow[r] & \cdots \arrow[r] & F_{n-1} \arrow[r] & F_0 \otimes L_{e_i}
\end{tikzcd}$$

where each of the $F_i$'s is a parabolic sheaf on the root stack $\widetilde{X}_n$.

We will use this notation several times in the following arguments.

**Example 4.2.23.** Take a log scheme $X$ with a global chart $L\colon \mathbb{N}^2 \to \mathrm{Div}(X)$, and now assume only that $L((0,1)) = (L_{0,1}, 0) \in \mathrm{Div}(X)$, and the section of $L_{1,0}$ can be non-zero, and let us fix $n = 2$ for simplicity. Consider as in the preceding discussion the log scheme $\widetilde{X}$, which has the same underlying scheme as $X$, but the log structure is given by $\mathbb{N} \subseteq \mathbb{N}^2 \to \mathrm{Div}(X)$, where the map is the immersion as the first component, so the image of 1 is $(L_{1,0}, s_{1,0})$.

Take a stable sheaf $F \in \mathrm{Coh}(\widetilde{X}_2)$, say

$$\begin{tikzcd}
& -1 & -1/2 & 0 \\
F = & F_0 \arrow[r] & F_1 \arrow[r] & F_0 \otimes L_{1,0}
\end{tikzcd}$$

and form the following parabolic sheaf, call it $F'$, on $X_2$

$$\begin{tikzcd}
& -1 & -\frac{1}{2} & 0 & \\
& 0 \arrow[r] & 0 \arrow[r] & 0 & 0 \\
F' = & F_0 \arrow[r] \arrow[u] & F_1 \arrow[r] \arrow[u] & F_0 \otimes L_{1,0} \arrow[u] & -\frac{1}{2} \\
& 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] & -1
\end{tikzcd}$$

or, with the notation of 4.2.22,

$$
\begin{tikzcd}
 & -1 & -\frac{1}{2} & 0 & \\
 & & & & \circ \arrow[llll, "\text{vertical direction}"'] \arrow[d, "\text{horizontal direction}"'] \\
F' = & 0 \arrow[r] & F \arrow[r] & 0. &
\end{tikzcd}
$$

Note that this is well defined because the given section of $L_{0,1}$ is zero. It is clear that $F'$ is also stable on $X_2$, since its subsheaves correspond exactly to subsheaves of $F$ on $\widetilde{X}_2$, and the slopes are the same. Assume also that $\widetilde{\pi}^* F$ is stable on $\widetilde{X}_4$, where $\widetilde{\pi}\colon \widetilde{X}_4 \to \widetilde{X}_2$ is the projection.

Now consider the pullback of $F'$ along $\pi\colon X_4 \to X_2$

$$
\begin{tikzcd}
 & -1 & -\frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 & \\
 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 & 0 \\
 & F_0 \arrow[u] \arrow[r, equal] & F_0 \arrow[u] \arrow[r] & F_1 \arrow[u] \arrow[r, equal] & F_1 \arrow[u] \arrow[r] & F_0 \otimes L_{1,0} \arrow[u] & -\frac{1}{4} \\
\pi^* F' = & F_0 \arrow[u, equal] \arrow[r, equal] & F_0 \arrow[u, equal] \arrow[r] & F_1 \arrow[u, equal] \arrow[r, equal] & F_1 \arrow[u, equal] \arrow[r] & F_0 \otimes L_{1,0} \arrow[u, equal] & -\frac{1}{2} \\
 & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] & -\frac{3}{4} \\
 & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] & -1
\end{tikzcd}
$$

and notice that this is not stable, because the following sheaf $G \in \mathrm{Coh}(X_4)$

$$G = \begin{tikzcd}
-1 & -\frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 & \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 & 0 \\
F_0 \arrow[u] \arrow[r, equal] & F_0 \arrow[u] \arrow[r] & F_1 \arrow[u] \arrow[r, equal] & F_1 \arrow[u] \arrow[r] & F_0 \otimes L_{1,0} \arrow[u] & -\frac{1}{4} \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] & -\frac{1}{2} \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] & -\frac{3}{4} \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] & -1
\end{tikzcd}$$

is a subsheaf of $\pi^* F'$, and clearly has $p_4(G) = p_4(\pi^* F') = p_2(F')$.

Once again we can easily describe the stable factors: they are the sheaf $G$, and the analogous one with the rows corresponding to $-\frac{1}{4}$ and $-\frac{1}{2}$ switched (note that this is not a subsheaf of $\pi^* F'$), so that the polystable sheaf S-equivalent to $\pi^* F'$ is

$$\begin{tikzcd}
-1 & -\frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 & \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 & 0 \\
F_0 \arrow[u] \arrow[r, equal] & F_0 \arrow[u] \arrow[r] & F_1 \arrow[u] \arrow[r, equal] & F_1 \arrow[u] \arrow[r] & F_0 \otimes L_{1,0} \arrow[u] & -\frac{1}{4} \\
F_0 \arrow[u, "0"] \arrow[r, equal] & F_0 \arrow[u, "0"] \arrow[r] & F_1 \arrow[u, "0"] \arrow[r, equal] & F_1 \arrow[u, "0"] \arrow[r] & F_0 \otimes L_{1,0} \arrow[u, "0"] & -\frac{1}{2} \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] & -\frac{3}{4} \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] & -1.
\end{tikzcd}$$

As in the previous example, we can completely reconstruct $F'$ from any of its stable factors, for example from $G$, as $\pi_*(G \otimes L_4^{(d_i)})$ for some $(d_i)$ (for example $(1,1)$ works). Finally note that $\pi_*(G \otimes L_4^{(d_i)})$ is either isomorphic to $F'$, or is zero.

We will see now that the behavior in the previous example is in fact typical for stable sheaves with non-stable pullback.

**Notation 4.2.24.** We denote by $X^i$ (for $i = 1, \dots, r$) the log stack given by $X_1$, together with the log structure induced by the composition $\mathbb{N}^{r-1} \subseteq \mathbb{N}^r \to \mathrm{Div}(X_1)$, where $\mathbb{N}^{r-1} \subseteq \mathbb{N}^r$ is the inclusion that omits the $i$-th basis element.

Let $\mathrm{Coh}((X^i)_n)_{s_i}$ denote the subcategory of $\mathrm{Coh}((X^i)_n)$ of sheaves annihilated by the section $s_i$ of $L_i$ coming from the log structure (meaning that every component of the parabolic sheaf is annihilated by $s_i$). We define fully faithful functors $I^i_{n,j} \colon \mathrm{Coh}((X^i)_n)_{s_i} \to \mathrm{Coh}(X_n)$ for $i = 1, \dots, r$, and $j = 1, \dots, n$ as follows: for $F \in \mathrm{Coh}((X^i)_n)_{s_i}$, we set

$$I^i_{n,j}(F)_{a_1,\dots,a_r} = \begin{cases} F_{a_1,\dots,\widehat{a_i},\dots,a_r} & \text{for } a_i = -\frac{j}{n} \\ 0 & \text{otherwise.} \end{cases}$$

with maps

$$I^i_{n,j}(F)_{a_1,\dots,a_r} \to I^i_{n,j}(F)_{a_1,\dots,a_k+\frac{1}{n},\dots,a_r}$$

defined to be zero, except if $a_i = -\frac{j}{n}$ and $k \neq i$, in which case it is defined as the corresponding map

$$F_{a_1,\dots,\widehat{a_i},\dots,a_r} \to F_{a_1,\dots,a_k+\frac{1}{n},\dots,\widehat{a_i},\dots,a_r}$$

of the sheaf $F$.

In other words, $I^i_{n,j}(F)$ is obtained by placing $F$ in the "slice" $a_i = -\frac{j}{n}$, and filling the rest with zeros. Note that this is well defined only if the components of $F$ are annihilated by $s_i$.

If we use the notation of 4.2.22, we can draw $I^i_{n,j}(F)$ as

$$\begin{array}{cccccccccc} & -1 & & \cdots & & -\frac{j}{n} & & \cdots & & 0 \end{array}$$

$$\xleftarrow{\quad\quad\quad\quad\quad i\text{-th direction}\quad\quad\quad\quad\quad} \circ \qquad \text{other directions} \downarrow$$

$$I^i_{n,j}(F) = \qquad 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow F \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0$$

and from this description, it is clear that:

- we have $p_n(I^i_{n,j}(F)) = p_n(F)$,
- subsheaves of $I^i_{n,j}(F)$ correspond bijectively to subsheaves of $F$ via $I^i_{n,j}$,
- so $I^i_{n,j}(F)$ is (semi-)stable on $X_n$ if and only if $F$ is (semi-)stable on $(X^i)_n$.

Now let us set $m = nk$ and assume that $F$ is stable, so that $I^i_{n,j}(F)$ is also stable. Consider the pullback of $I^i_{n,j}(F)$ along $\pi\colon X_m \to X_n$, and consider also the projection $\pi_i\colon (X^i)_m \to (X^i)_n$. We can write the pullback as

$$\begin{array}{ccccccc} -1 & \cdots & -\frac{kj}{m} & \cdots & -\frac{kj-k+1}{m} & \cdots & 0 \end{array}$$

$$\xleftarrow{\quad i\text{-th direction}\quad} \circ \qquad \text{other directions} \downarrow$$

$$\pi^* I^i_{n,j}(F) = \qquad 0 \longrightarrow \cdots \longrightarrow \pi_i^* F = \pi_i^* F = \cdots = \pi_i^* F = \pi_i^* F \longrightarrow \cdots \longrightarrow 0,$$

and, as in example 4.2.21, we see that $\pi^* I^i_{n,j}(F)$ is not stable: the sheaf $I^i_{m,kj-k+1}(\pi_i^* F) \in \mathrm{Coh}(X_m)$ having one copy of $\pi_i^* F$ in the "slice" $a_i = -\frac{kj-k+1}{m}$ is a proper subsheaf of the pullback $\pi^* I^i_{n,j}(F)$

$$\begin{array}{ccccccc} -1 & \cdots & -\frac{kj}{m} & \cdots & -\frac{kj-k+1}{m} & \cdots & 0 \end{array}$$

$$\xleftarrow{\quad i\text{-th direction}\quad} \circ \qquad \text{other directions} \downarrow$$

$$\begin{tikzcd}[column sep=small]
\pi^* I^i_{n,j}(F) = & 0 \arrow[r] & \cdots \arrow[r] & \pi_i^* F \arrow[r, equal] & \cdots \arrow[r, equal] & \pi_i^* F \arrow[r, equal] & \pi_i^* F \arrow[r] & \cdots \arrow[r] & 0 \\
I^i_{m,kj-k+1}(\pi_i^* F) = & 0 \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & 0 \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & 0 \arrow[r] \arrow[u, hook] & \pi_i^* F \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & 0 \arrow[u, hook]
\end{tikzcd}$$

and has $p_m(I^i_{m,kj-k+1}(\pi_i^* F)) = p_m(\pi^* I^i_{n,j}(F))$, as they are both equal to $p_n(F)$ on $(X^i)_n$.

We can describe the stable factors of $\pi^* I^i_{n,j}(F)$ if $\pi_i^* F$ if stable on $(X^i)_m$ (which is not always the case): the quotient $\pi^* I^i_{n,j}(F) / I^i_{m,kj-k+1}(\pi_i^* F)$ is the sheaf

$$\begin{array}{ccccccc} -1 & \cdots & -\frac{kj}{m} & \cdots & -\frac{kj-k+1}{m} & \cdots & 0 \end{array}$$

$$\xleftarrow{\quad i\text{-th direction}\quad} \circ \qquad \text{other directions} \downarrow$$

$$0 \longrightarrow \cdots \longrightarrow \pi_i^* F = \pi_i^* F = \cdots = \pi_i^* F \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

with one less copy of $\pi_i^* F$ at the end, and it has $I^i_{m,kj-k+2}(\pi_i^* F)$ as a subsheaf with the same slope. Inductively, we see that the stable factors of $\pi^* I^i_{n,j}(F)$ are the sheaves $I^i_{m,kj-h}(\pi_i^* F)$ for

$h = 0, \dots, k-1$, and the semi-stable sheaf S-equivalent to $\pi^* I^i_{n,j}(F)$ is

$$
\begin{array}{ccccccccc}
-1 & \cdots & -\frac{kj}{m} & & \cdots & & -\frac{kj-k+1}{m} & \cdots & 0
\end{array}
$$

$$
\begin{tikzcd}[column sep=large]
\bullet & & & & & & \circ \arrow[llllll, "i\text{-th direction}"'] \arrow[d, "\text{other directions}"'] \\
& & & & & & {}
\end{tikzcd}
$$

$$
0 \longrightarrow \cdots \longrightarrow \pi_i^* F \xrightarrow{0} \pi_i^* F \xrightarrow{0} \cdots \xrightarrow{0} \pi_i^* F \xrightarrow{0} \pi_i^* F \longrightarrow \cdots \longrightarrow 0
$$

with zeros instead of identity maps.

The next proposition says that every stable sheaf $F \in \operatorname{Coh}(X_n)$ such that $\pi^* F \in \operatorname{Coh}(X_m)$ is not stable is of this form.

**Proposition 4.2.25.** *Let $F \in \operatorname{Coh}(X_n)$ be a stable sheaf. Then $\pi^* F \in \operatorname{Coh}(X_m)$ is not stable if and only if $F$ is in the image of one of the functors $I^i_{n,j}$, for some $i, j$.*

*Proof.* The "if" part is contained in the previous discussion.

For the other direction, let us consider a subsheaf $G \subseteq \pi^* F$, along with the subsheaves $\pi_*(G \otimes L_m^{(d_i)}) \subseteq F$ for $0 \leq d_i < k$. Recall that by proposition 4.2.8, the slope $p_m(G)$ is a weighted mean of the polynomials $p_n(\pi_*(G \otimes L_m^{(d_i)}))$, with $\pi_*(G \otimes L_m^{(d_i)})$ non-zero.

The only possibility for $G$ to be destabilizing is that $p_n(\pi_*(G \otimes L_m^{(d_i)})) = p_n(F)$ for all non-zero $\pi_*(G \otimes L_m^{(d_i)})$, and by stability of $F$ this implies $\pi_*(G \otimes L_m^{(d_i)}) = F$ for those values of $(d_i)$.

Nota also that if $\pi_*(G \otimes L_m^{(d_i)}) = F$ for all values of $(d_i)$, then we must have $G = \pi^* F$: this can be seen directly from the description of the sheaves $\pi_*(G \otimes L_m^{(d_i)})$ given in Remark 4.2.9, or from the fact that the direct sum $\mathcal{E} = \bigoplus_{0 \leq d_i < 0} L_m^{(d_i)}$ is a generating sheaf for the relative root stack $\pi \colon X_m \to X_n$, and the cokernel of $G \subseteq \pi^* F$ would be sent to zero by $\pi_*(- \otimes \mathcal{E})$ ([Nir, Lemma 3.4]).

This implies that if $\pi^* F$ is not stable, then there is a subsheaf $G$ with

- $\pi_*(G \otimes L_m^{(d_i)}) = 0$ or $\pi_*(G \otimes L_m^{(d_i)}) = F$ for all $0 \leq d_i < k$, and
- each of the two cases occur for at least one $(d_i)$.

Now we will see that this implies that $F$ is in the image of one of the functors $I^i_{n,j}$. From now on for brevity we will write $G^{(d_i)} = \pi_*(G \otimes L_m^{(d_i)}) \in \operatorname{Coh}(X_n)$.

Observe first that if $G^{(d_i)} = F$ for some $(d_i)$, then $G^{(e_i)} = F$ also for any $(e_i) \geq (d_i)$ in the componentwise order. This is because, since $G$ is a subsheaf of $\pi^* F$, the diagram

$$
\begin{tikzcd}
F = (\pi^* F)^{(d_i)} \arrow[r, equal] & (\pi^* F)^{(e_i)} = F \\
F = G^{(d_i)} \arrow[u, hook] \arrow[r] & G^{(e_i)} \arrow[u, hook]
\end{tikzcd}
$$

commutes, and this forces $G^{(e_i)} = F$. This implies that if $G^{(d_i)} = 0$, then $G^{(e_i)} = 0$ for any $(e_i) \leq (d_i)$. In particular we necessarily have $G^{(0,\dots,0)} = 0$ and $G^{(k-1,\dots,k-1)} = F$. Now we justify the fact that there is a direction $i_0 \in \{1, \dots, r\}$ such that $G^{(d_i)} = 0$ and $F$ both occur for $d_i$ with $i \neq i_0$ fixed, and $d_{i_0}$ ranging from 0 to $k-1$.

Look first at the sheaves $G^{(a,0,\dots,0)}$ for $0 \leq a < k$: if $G^{(k-1,0\dots,0)} = F$, we are done. Otherwise, all the sheaves of this form are 0, and we look at $G^{(k-1,a,0,\dots,0)}$ for $0 \leq a < k$, and so on. If we are unlucky, at the $(r-1)$-th step we will find $G^{(k-1,\dots,k-1,0)} = 0$, and so the sheaves $G^{(k-1,\dots,k-1,a)}$ satisfy the requirement, since for $a = 0$ we have 0 and for $a = k-1$ we have $F$.

Now we claim that all the maps of the parabolic sheaf $F$ in the direction $i_0$ are necessarily zero: in fact take $(a_1, \dots, a_r) \in ([-1, 0) \cap \frac{1}{n}\mathbb{Z})^r$, and consider the map

$$f\colon F_{(a_1,\dots,a_r)} \to F_{(a_1,\dots,a_{i_0}+\frac{1}{n},\dots,a_r)}.$$

By looking at the hypercubes corresponding to these two sheaves in the pullback $\pi^*F$, along with the subsheaf $G$ and its property that in the direction $i_0$ it has both $F$ and 0, we see that the following diagram commutes (the top row is in $\pi^*F$, the bottom in the subsheaf $G$)

$$\begin{tikzcd}
F_{(a_1,\dots,a_r)} \arrow[r, "f"] \arrow[d, equal] & F_{(a_1,\dots,a_{i_0}+\frac{1}{n},\dots,a_r)} \arrow[r, equal] & F_{(a_1,\dots,a_{i_0}+\frac{1}{n},\dots,a_r)} \arrow[d, equal] \\
F_{(a_1,\dots,a_r)} \arrow[r] & 0 \arrow[r] \arrow[u, hook] & F_{(a_1,\dots,a_{i_0}+\frac{1}{n},\dots,a_r)}
\end{tikzcd}$$

from which we deduce that $f$ is zero.

**Example 4.2.26.** Let us examine a simple example more closely. Assume that $r = 2$, set $L((1,0)) = (L_{1,0}, s_{1,0})$ and $L((0,1)) = (L_{0,1}, s_{0,1})$ and take a parabolic sheaf

$$\begin{tikzcd}
 & -1 & -\frac{1}{2} & 0 & \\
 & F_{-1,-1} \otimes L_{0,1} \arrow[r] & F_{-\frac{1}{2},-1} \otimes L_{0,1} \arrow[r] & F_{-1,-1} \otimes L_{1,1} & 0 \\
F = & F_{-1,-\frac{1}{2}} \arrow[r] \arrow[u] & F_{-\frac{1}{2},-\frac{1}{2}} \arrow[r] \arrow[u] & F_{-1,-\frac{1}{2}} \otimes L_{1,0} \arrow[u] & -\frac{1}{2} \\
 & F_{-1,-1} \arrow[r] \arrow[u] & F_{-\frac{1}{2},-1} \arrow[r] \arrow[u] & F_{-1,-1} \otimes L_{1,0} \arrow[u] & -1
\end{tikzcd}$$

on $X_2$, and its pullback $\pi^*F$ along $\pi\colon X_4 \to X_2$,

$$
\begin{tikzcd}
-1 & -\frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 & \\
F_{-1,-1}\otimes L_{0,1} \arrow[r, equal] & F_{-1,-1}\otimes L_{0,1} \arrow[r] & F_{-\frac{1}{2},-1}\otimes L_{0,1} \arrow[r, equal] & F_{-\frac{1}{2},-1}\otimes L_{0,1} \arrow[r] & F_{-1,-1}\otimes L_{1,1} & 0 \\
F_{-1,-\frac{1}{2}} \arrow[u] \arrow[r, equal] \arrow[d, equal] & F_{-1,-\frac{1}{2}} \arrow[u] \arrow[r] \arrow[d, equal] & F_{-\frac{1}{2},-\frac{1}{2}} \arrow[u] \arrow[r, equal] \arrow[d, equal] & F_{-\frac{1}{2},-\frac{1}{2}} \arrow[u] \arrow[r] \arrow[d, equal] & F_{-1,-\frac{1}{2}}\otimes L_{1,0} \arrow[u] \arrow[d, equal] & -\frac{1}{4} \\
F_{-1,-\frac{1}{2}} \arrow[r, equal] & F_{-1,-\frac{1}{2}} \arrow[r, mapsto] & F_{-\frac{1}{2},-\frac{1}{2}} \arrow[r, equal] & F_{-\frac{1}{2},-\frac{1}{2}} \arrow[r, mapsto] & F_{-1,-\frac{1}{2}}\otimes L_{1,0} & -\frac{1}{2} \\
F_{-1,-1} \arrow[u] \arrow[r, equal] \arrow[d, equal] & F_{-1,-1} \arrow[u] \arrow[r] \arrow[d, equal] & F_{-\frac{1}{2},-1} \arrow[u] \arrow[r, equal] \arrow[d, equal] & F_{-\frac{1}{2},-1} \arrow[u] \arrow[r] \arrow[d, equal] & F_{-1,-1}\otimes L_{1,0} \arrow[u] \arrow[d, equal] & -\frac{3}{4} \\
F_{-1,-1} \arrow[r, equal] & F_{-1,-1} \arrow[r, mapsto] & F_{-\frac{1}{2},-1} \arrow[r, equal] & F_{-\frac{1}{2},-1} \arrow[r, mapsto] & F_{-1,-1}\otimes L_{1,0} & -1.
\end{tikzcd}
$$

Now assume that $\pi^*F$ is not stable, so that we have a subsheaf $G \subseteq \pi^*F$ with $p_4(G) = p_4(\pi^*F) = p_2(F)$. As we discussed, this means that the four sheaves

$$
\pi_*(G \otimes L_4^{(0,0)}), \pi_*(G \otimes L_4^{(1,0)}), \pi_*(G \otimes L_4^{(0,1)}), \pi_*(G \otimes L_4^{(1,1)}) \subseteq F
$$

on $X_2$ are either $F$ or $0$, and both cases occur.

Let us look at the square $[-1, -\frac{3}{4}]^2$ of $G$. We have the following cases:

$$
\begin{tikzcd}
-1 & -\frac{3}{4} & \\
F_{-1,-1} \arrow[r, equal] & F_{-1,-1} \arrow[d, equal] & -\frac{3}{4} \\
0 \arrow[u] \arrow[r] & F_{-1,-1} & -1
\end{tikzcd}
$$

or

$$\begin{tikzcd}
-1 & -\frac{3}{4} & \\
0 \arrow[r] & F_{-1,-1} & -\frac{3}{4} \\
0 \arrow[r] \arrow[u] & F_{-1,-1} \arrow[u, equal] & -1
\end{tikzcd}$$

or

$$\begin{tikzcd}
-1 & -\frac{3}{4} & \\
F_{-1,-1} \arrow[r, equal] & F_{-1,-1} & -\frac{3}{4} \\
0 \arrow[r] \arrow[u] & 0 \arrow[u] & -1,
\end{tikzcd}$$

or

$$\begin{tikzcd}
-1 & -\frac{3}{4} & \\
0 \arrow[r] & F_{-1,-1} & -\frac{3}{4} \\
0 \arrow[r] \arrow[u] & 0 \arrow[u] & -1,
\end{tikzcd}$$

and, in each of these cases, the pattern will be the same in each of the other three squares of $[-1,0]^2$.

Now assume we are in the first case, and look at the segment $[-1,\frac{1}{4}] \times \{-1\}$ of $G$, along with the inclusion in the same line of $F$

$$\begin{tikzcd}
 & -1 & -\frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 & \frac{1}{4} & \\
F = & F_{-1,-1} \arrow[r, equal] & F_{-1,-1} \arrow[r] & F_{-\frac{1}{2},-1} \arrow[r, equal] & F_{-\frac{1}{2},-1} \arrow[r] & F_{-1,-1} \otimes L_{1,0} \arrow[r, equal] & F_{-1,-1} \otimes L_{1,0} & -1 \\
G = \arrow[u, hook] & 0 \arrow[r] \arrow[u, hook] & F_{-1,-1} \arrow[r] \arrow[u, equal] & 0 \arrow[r] \arrow[u, hook] & F_{-\frac{1}{2},-1} \arrow[r] \arrow[u, equal] & 0 \arrow[r] \arrow[u, hook] & F_{-1,-1} \otimes L_{1,0} \arrow[u, equal] & -1.
\end{tikzcd}$$

From this diagram we can conclude that the maps $F_{-1,-1} \to F_{-\frac{1}{2},-1}$ and $F_{-\frac{1}{2},-1} \to F_{-1,-1} \otimes L_{1,0}$ are zero. By looking at the line $[-1,0] \times \{-\frac{1}{2}\}$ we see analogously that also the maps $F_{-1,-\frac{1}{2}} \to$

$F_{-\frac{1}{2},-\frac{1}{2}}$ and $F_{-\frac{1}{2},-\frac{1}{2}} \to F_{-1,-\frac{1}{2}} \otimes L_{1,0}$ are zero and so the parabolic sheaf $F$

$$
\begin{tikzcd}
 & -1 & -\frac{1}{2} & 0 & \\
 & F_{-1,-1} \otimes L_{0,1} \arrow[r, "0"] & F_{-\frac{1}{2},-1} \otimes L_{0,1} \arrow[r, "0"] & F_{-1,-1} \otimes L_{1,1} & 0 \\
F = & F_{-1,-\frac{1}{2}} \arrow[r, "0"] \arrow[u] & F_{-\frac{1}{2},-\frac{1}{2}} \arrow[r, "0"] \arrow[u] & F_{-1,-\frac{1}{2}} \otimes L_{1,0} \arrow[u] & -\frac{1}{2} \\
 & F_{-1,-1} \arrow[r, "0"] \arrow[u] & F_{-\frac{1}{2},-1} \arrow[r, "0"] \arrow[u] & F_{-1,-1} \otimes L_{1,0} \arrow[u] & -1
\end{tikzcd}
$$

has zero maps in the horizontal direction.

The other cases are similar.

Note that if the square $\left[-1,-\frac{3}{4}\right]^2$ was the last one in our list (the one with three zeros), then the conclusion would be that the maps of the sheaf in both the horizontal and vertical direction are zero. This is a case in which $F$ is of the form $I^i_{n,j}(F')$ for a sheaf $F' \in (X^i)_n$, but $F'$ will again have non-stable pullback to $(X^i)_m$, and thus we can apply the whole procedure again to $F'$.

Returning to the proof, the above discussion implies that, using the notation of 4.2.22, the sheaf $F$ can be written as

$$
\begin{tikzcd}
 & -1 & -\frac{n-1}{n} & \cdots & -\frac{1}{n} & 0 & \\
 & & & & & & \circ \arrow[llllll, "i_0\text{-th direction}"'] \arrow[d, "\text{other directions}"'] \\
F = & F_0 \arrow[r, "0"] & F_1 \arrow[r, "0"] & \cdots \arrow[r, "0"] & F_{n-1} \arrow[r, "0"] & F_0 \otimes L_{i_0} &
\end{tikzcd}
$$

where all the parabolic sheaves $F_h \in \mathrm{Coh}((X^{i_0})_n)$ are annihilated by the section $s_{i_0}$, so in fact $F_h \in \mathrm{Coh}((X^{i_0})_n)_{s_{i_0}}$. Now note that unless only one of the $F_h$'s is non-zero, a sheaf of this form cannot be stable, since it is the direct sum of the parabolic sheaves having $F_h$ in place $-\frac{n-h}{n}$ (in the direction $i_0$) and zero everywhere else, and a semi-stable direct sum is stable if and only if there is only one factor, which moreover is stable.

In conclusion $F$ is of the form

$$\begin{tikzcd} & -1 & -\frac{n-1}{n} & \cdots & -\frac{n-h+1}{n} & -\frac{n-h}{n} & \frac{n-h-1}{n} & \cdots & 0 & \\ & & & & & & & & & \circ \arrow[llllllll, "i_0\text{-th direction}"'] \arrow[d, "\text{other directions}"'] \\ F = & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & F_h \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 & {} \end{tikzcd}$$

or, in other words, $F = I^{i_0}_{n,n-h}(F_h)$ with $F_h \in \mathrm{Coh}((X^{i_0})_n)_{s_{i_0}}$ a stable sheaf, and this concludes the proof. □

**Lemma 4.2.27.** *Let $F \in \mathrm{Coh}(X_n)$ be a stable sheaf and let $F' \in \mathrm{Coh}(X_m)$ be one of the stable factors of $\pi^*F$. Then for any $0 \leq d_i < k$ the sheaf $\pi_*(F' \otimes L_m^{(d_i)}) \in \mathrm{Coh}(X_n)$ is isomorphic to $F$ or zero (and both cases occur if $\pi^*F$ is not stable).*

*Proof.* If $\pi^*F$ is still stable, this is clear from the description of the pullback and by remark 4.2.9. In the other case we know that $F$ must be of the form $I^i_{n,j}(G)$ with $G$ stable on $(X^i)_n$ from the preceding proposition, and we know the stable factors $F'$ of $\pi^*F$ from the discussion preceding the proof of 4.2.25, if the pullback of $G$ along $(X^i)_m \to (X^i)_m$ is stable. If this pullback is not stable we can apply Proposition 4.2.25 again to $G$, and after a finite number of steps we will get down to a stable sheaf.

From the explicit form of the stable factors and the description of the sheaves $\pi_*(F' \otimes L_m^{(d_i)}) \in \mathrm{Coh}(X_n)$ of Remark 4.2.9, the conclusion follows. □

**Lemma 4.2.28.** *Let $F, G \in \mathrm{Coh}(X_n)$ be stable sheaves such that $\pi^*F$ and $\pi^*G$ are S-equivalent on $X_m$. Then $F \cong G$ on $X_n$.*

*Proof.* If one of $\pi^*F$ or $\pi^*G$ is stable, then $\pi^*F \cong \pi^*G$ (since S-equivalent implies isomorphic, if one of the sheaves is stable) and since $\pi^*$ is fully faithful we conclude that $F \cong G$.

If both $\pi^*F$ and $\pi^*G$ are not stable, denote by $F_i$ and $G_j$ their stable factors. Since $\pi^*F$ and $\pi^*G$ are S-equivalent, they have the same stable factors, so for some $i$ and $j$ we have $F_i \cong G_j$. Now by Lemma 4.2.27, the shaves $\pi_*(F_i \otimes L_m^{(d_i)})$ and $\pi_*(G_j \otimes L_m^{(d_i)})$ for $0 \leq d_i < k$ are isomorphic to $F$ or $G$ respectively, or zero. Since $F$ and $G$ are not zero, and the isomorphism $F_i \cong G_j$ will induce isomorphisms $\pi_*(F_i \otimes L_m^{(d_i)}) \cong \pi_*(G_j \otimes L_m^{(d_i)})$ for any $(d_i)$, we get an isomorphism $F \cong G$. □

**Proposition 4.2.29.** *The morphism $i_{n,m} \colon M_n^{ss} \to M_m^{ss}$ between the good moduli spaces is geometrically injective. In particular, being proper, it is also finite.*

*Proof.* Fix an algebraically closed extension $k \subseteq K$, and let us show that $M_n^{ss}(K) \to M_m^{ss}(K)$ is injective. This means that if $F, G \in \mathrm{Coh}((X_K)_n)$ are semi-stable sheaves such that $\pi^*F, \pi^*G \in \mathrm{Coh}((X_K)_n)$ are S-equivalent, then $F$ and $G$ are S-equivalent themselves. We can assume that $F$ and $G$ are polystable, and write $F = \bigoplus_i F_i$ and $G = \bigoplus_j G_j$ as sums of stable sheaves on $(X_K)_n$. We will proceed by induction on $N = \max\{\#\text{stable factors of } F, \#\text{stable factors of } G\}$.

For $N = 1$, this is the previous lemma, applied to $X_K$.

If $N > 1$, write $F_{i,h}$ and $G_{j,k}$ for the stable factors of $\pi^* F_i$ and $\pi^* G_j$ respectively. Since

$$\pi^* F = \bigoplus_i \pi^* F_i$$

and

$$\pi^* G = \bigoplus_j \pi^* G_j$$

are S-equivalent, they will have the same stable factors, so for some $i, j, k, h$ we have $F_{i,h} \cong G_{j,k}$.

Now let us look at the sheaves $\pi_*(F_{i,h} \otimes L_m^{(d_i)})$ and $\pi_*(G_{j,k} \otimes L_m^{(d_i)})$ on $X_n$ for all $0 \leq d_i < k$: by Lemma 4.2.27, they are isomorphic to $F_i$, and $G_j$ respectively, or zero. But since neither of $F_i$ or $G_j$ is zero, as in the proof of the previous lemma, we can conclude that the isomorphism $F_{i,h} \cong G_{j,k}$ induces $F_i \cong G_j$.

After erasing these two factors from $F$ and $G$, we end up with two polystable sheaves $F'$ and $G'$ with $\max\{\#\text{stable factors of } F', \#\text{stable factors of } G'\} = N - 1$ and such that $\pi^* F'$ and $\pi^* G'$ are S-equivalent. By the induction hypothesis $F'$ and $G'$ are isomorphic, and this concludes the proof.

The part about finiteness follows from Chevalley's theorem. $\square$

**Proposition 4.2.30.** *If the pullback of a stable sheaf is stable, then all the maps $\iota_{n,m}, \iota^o_{n,m}, i_{n,m}, i^o_{n,m}$ are open and closed immersions.*

*Proof.* We already know that $\iota_{n,m}$ is an open immersion. The fact that stable sheaves go to stable sheaves implies that polystables go to polystables, and this says that $\iota_{n,m}$ sends closed points to closed points (recall that the closed points of $\mathcal{M}_n^{ss}$ correspond to polystable sheaves). Since we know that the induced map $i_{n,m}$ on the good moduli spaces is finite, by Proposition 6.4 of [Alp12] we can conclude that $\iota_{n,m}$ is also finite, and in particular closed. This shows that $\iota_{n,m}$ is an open and closed immersion, and this implies the conclusion also for $i_{n,m}$.

Finally, the same conclusion for $\iota^o_{n,m}$ and $i^o_{n,m}$ holds because of Lemma 4.2.18. $\square$

It is not clear to us that this should hold in general. Example 4.2.20 showed that $\iota_{n,m}$ need not be closed in general.

**Example 4.2.31.** As in 4.2.20 consider the standard log point, i.e. $X = \mathrm{Spec}(k)$ with the log structure $L \colon \mathbb{N} \to k$, sending 0 to $1 \in k$ and everything else to zero, and the map $\pi \colon X_2 \to X$. We showed that $\iota_{1,2} \colon \mathcal{M}_1^{ss} \to \mathcal{M}_2^{ss}$ is not closed.

Note first of all that in case everything is semi-stable, since the reduced Hilbert polynomial (of a non-zero sheaf) is always 1, and the only stable sheaves are those without proper non-zero subsheaves. Furthermore $\mathcal{M}_1^{ss}$ and $M_1^{ss}$ are disjoint union of substacks/schemes parametrized by a natural number $r \in \mathbb{N}$ (the rank). Let us restrict to the component $(\mathcal{M}_1^{ss})_1$ parametrizing semi-stable sheaves of rank 1. This lands in the component $(\mathcal{M}_2^{ss})_2$ parametrizing semi-stable sheaves on $X_2$ of parabolic rank 2.

Let us show that $i_{1,2} \colon (M_1^{ss})_1 \to (M_2^{ss})_2$ is an open and closed immersion. It is clear that $(\mathcal{M}_1^{ss})_1 \cong B\mathbb{G}_m$, since we are just parametrizing invertible sheaves, and so $(M_1^{ss})_1 = \mathrm{Spec}(k)$. On the other hand $(\mathcal{M}_2^{ss})_2$ has three connected components, parametrizing sheaves of the following three kinds:

$$\begin{array}{ccccc} -1 & & -\frac{1}{2} & & 0 \\ \\ k \oplus k & \longrightarrow & 0 & \longrightarrow & k \oplus k \\ \\ 0 & \longrightarrow & k \oplus k & \longrightarrow & 0 \\ \\ k & \longrightarrow & k & \longrightarrow & k. \end{array}$$

The first two kinds correspond to components of the form $B\,\mathrm{GL}_2$, that have $\mathrm{Spec}(k)$ has moduli space, and the third one parametrizes pairs $(L, M)$ of invertible sheaves with two maps $a\colon L \to M$ and $b\colon M \to L$ such that $a \circ b = 0$ and $b \circ a = 0$.

From this description we see that this component can be identified with the quotient stack $[\mathrm{Spec}(k[x,y]/(xy))/\mathbb{G}_m \times \mathbb{G}_m]$ for the action defined by $(\alpha, \beta) \cdot (x, y) = (\alpha\beta^{-1}x, \alpha^{-1}\beta y)$. From this we see that the moduli space of this component is also $\mathrm{Spec}(k)$. In conclusion the morphism $(M_1^{ss})_1 \to (M_2^{ss})_2$ is the inclusion of one component $\mathrm{Spec}(k) \to \mathrm{Spec}(k) \coprod \mathrm{Spec}(k) \coprod \mathrm{Spec}(k)$.

We have no examples in which $i_{n,m}$ is not an open and closed immersion, so one could conjecture that this is always the case. At the very least, the fact that on each connected component the inductive limit stabilizes (in the sense that $i_{n,m}$ is an isomorphism for $n, m$ big enough) seems very reasonable.

The following example shows that the fact that the map between spaces is an immersion will not follow from general facts on good moduli spaces.

**Example 4.2.32.** We will construct an example of an open immersion between Artin stacks with good moduli spaces, that does not induce an immersion on the moduli spaces.

Let us consider the action of $\mathbb{G}_m$ on the second component of $X = \mathbb{A}^1 \times \mathbb{G}_m = \mathrm{Spec}(k[x, t^{\pm 1}])$, and the natural action on $Y = \mathrm{Spec}(A)$ where $A \subseteq k[x,t]$ is the subring generated by monomials of total degree at least 2, i.e. $x^2, xt, t^2, x^3, \ldots$. We have a $\mathbb{G}_m$ equivariant map $X \to Y$ which is an open immersion (since outside the origin $A$ is just $\mathbb{A}^2 \setminus 0$), and therefore induces an open immersion between the quotient stacks $[X/\mathbb{G}_m] \to [Y/\mathbb{G}_m]$.

The induced morphism between the good moduli spaces is the normalization map $\mathbb{A}^1 \to C$ where $C$ is the standard cuspidal curve, and therefore is not an immersion.

The following proposition gives sufficient conditions that ensure that stability is preserved under pullback.

**Proposition 4.2.33.** *The pullback of a stable sheaf is stable in each of the following cases:*

- *we are considering torsion-free sheaves and the log structure on $X$ is generically trivial (as in 1.2.15);*
- *we look at components corresponding to a reduced Hilbert polynomial $h \in \mathbb{Q}[x]$, which is not the reduced Hilbert polynomial of a stable parabolic sheaf on one of the log stacks $X^i$.*

*Proof.* This is immediate from the previous discussion: a stable sheaf with non-stable pullback will have a lot of zeros, but the maps of a torsion-free parabolic sheaf on a log scheme with generically trivial log structure are injective (see 3.2.13), and this is it for the first part.

As for the second part, a stable sheaf with a non-stable pullback is of the form $I^i_{n,j}(F)$ for some $F \in \mathrm{Coh}((X^i)_n)_{s_i}$, and recall that $p_n(I^i_{n,j}(F)) = p_n(F)$. □

**Remark 4.2.34.** Let us briefly discuss the significance of the second condition. Clearly, it will only be meaningful if the set of reduced Hilbert polynomials of stable parabolic sheaves on $X$ is not entirely contained in the set of reduced Hilbert polynomials of stable sheaves on one of the $X^i$.

We feel that this should be the case in general: the reduced Hilbert polynomial of a parabolic sheaf is in particular the reduced Hilbert polynomial of a sheaf on $X$ (the sum of its fundamental pieces), but this sheaf on $X$ is typically not even semi-stable. Moreover, adding generators to the log structure should give more freedom for stable sheaves, and thus for the set of their Hilbert polynomials. For example if the log structure has rank 1, the pieces of a parabolic stable sheaf need not be stable on $X$.

Anyways, to completely understand this problem first of all one should understand the problem of which polynomials in $\mathbb{Q}[x]$ can be realized as reduced Hilbert polynomials of some stable sheaf, which is non-trivial even in the classical (non-parabolic) setting.

With that said, let us look at the example of curves, where one can say something.

If $X = \mathbb{P}^1$, say with the log structure corresponding to the divisor given by 0, then the reduced Hilbert polynomial of any coherent sheaf is of the form $h(x) = x + q$ with $q \in \mathbb{Q}[x]$, and since the only stable sheaves are the line bundles, their reduced Hilbert polynomials are exactly those for which $q$ is an integer.

In this case for any fixed $q \in \mathbb{Q}$ there are stable parabolic sheaves on $X$ that have $x + q$ as their reduced Hilbert polynomial (it suffices to consider parabolic sheaves whose pieces are all line bundles), so that the second condition in the last proposition is meaningful in this case.

On the other hand if the genus of $X$ is at least 1, reduced Hilbert polynomials are still of the form $x + q$ with $q \in \mathbb{Q}$, but now any one of these polynomials is a reduced Hilbert polynomial of a stable sheaf on $X$. In fact it known that for any fixed degree and rank (that we may assume coprime) an a curve of genus at least 1 there is a stable sheaf of the fixed degree and rank. In this case the second condition of the last proposition cannot be applied in a meaningful way.

## 4.3 Limit moduli theory on $X_\infty$

In this section we will use the notations of the last one, and moreover we will denote by $\pi_{n,m}\colon X_m \to X_n$ the natural projection for $n \mid m$, and by $\pi_n\colon X_\infty \to X_n$ the projection from the infinite root stack. Note that, being an inverse limit of flat morphisms, $\pi_n$ is flat.

The subject of this section is the moduli theory for finitely presented sheaves on $X_\infty$ that we get by taking a limit of the theories at finite levels. In our setting $X_\infty$ is coherent by 2.2.46, so that finitely presented sheaves are the same as coherent sheaves, but since this does not hold in general we will formulate everything using finitely presented sheaves.

Recall from 2.2.35 that $\mathrm{FP}(X_\infty) = \varinjlim_n \mathrm{FP}(X_n)$, and this means that

- every finitely presented sheaf $F \in \mathrm{FP}(X_\infty)$ is of the form $\pi_n^* F_n$ for some $n$ and $F_n \in \mathrm{FP}(X_n)$,

- for any $n, m$ and $F_n \in \mathrm{FP}(X_n)$, $F_m \in \mathrm{FP}(X_m)$ such that $F \cong \pi_n^* F_n \cong \pi_m^* F_m$, there exists $k \geq n, m$ such that $\pi_{n,k}^* F_n \cong \pi_{m,k}^* F_m$ on $X_k$.

**Definition 4.3.1.** The *reduced Hilbert polynomial* $p(F)$ of $F \in \mathrm{FP}(X_\infty)$ is the reduced Hilbert polynomial $p_n(F_n)$ of any finitely presented sheaf $F_n \in \mathrm{FP}(X_n)$ such that $\pi_n^* F_n \cong F$.

Since $\pi_n^*$ is fully faithful and $p_m(\pi_{n,m}^*(F_n)) = p_n(F_n)$ by Proposition 4.2.10, the reduced Hilbert polynomial of $F$ is well-defined.

**Definition 4.3.2.** A finitely presented sheaf $F \in \mathrm{FP}(X_\infty)$ is *pure* if it comes from a pure sheaf on one of the $X_n$.

A finitely presented pure sheaf $F \in \mathrm{FP}(X_\infty)$ is *(semi-)stable* if for any finitely presented subsheaf $G \subseteq F$ we have

$$p(G) \ (\leq) \ p(F).$$

**Proposition 4.3.3.** *Let $F \in \mathrm{FP}(X_\infty)$, and assume $F_n \in \mathrm{FP}(X_n)$ is such that $\pi_n^* F_n \cong F$. Then $F$ is semi-stable if and only if $F_n$ is semi-stable on $X_n$. The "only if" part is true with "semi-stable" replaced by "stable".*

*Proof.* If $\pi_n^* F_n$ is (semi-)stable, then since $\pi_n^*$ is fully faithful and $\pi_n$ is flat, if $G \subseteq F_n$ is a non-zero proper subsheaf, then $\pi_n^* G \subseteq \pi_n^* F_n$ is also a non-zero proper subsheaf, and

$$p_n(G) = p(\pi_n^* G) \ (\leq) \ p(\pi_n^* F_n) = p_n(F_n).$$

On the other hand, if $F_n$ is semi-stable, consider a finitely presented subsheaf $G \subseteq \pi^* F_n$. Since it is finitely presented, $G$ will come from some $G_m \in \mathrm{FP}(X_m)$. By pushing forward the inclusion

$$\pi_m^* G_m \subseteq \pi_n^* F_n$$

to $X_k$ where $k = \mathrm{lcm}(m, n)$ and using the projection formula for $\pi_k$, we see that $G$ comes from $\pi_{m,k}^* G_m \subseteq \pi_{n,k}^* F_n$. Since by Proposition 4.2.10 $\pi_{n,k}^* F_n$ is semi-stable on $X_k$, we see that

$$p(G) = p_k(\pi_{m,k}^* G_k) \leq p_k(\pi_{n,k}^* F_n) = p(\pi_n^* F_n)$$

so $\pi_n^* F_n$ is semi-stable. □

**Example 4.3.4.** The previous statement is false for stable sheaves in general, and there are stable sheaves $F_n \in \mathrm{FP}(X_n)$ such that $\pi_n^* F_n$ is not stable. Indeed, this will happen if $\pi_{n,m}^* F_n$ is not stable for some $m = kn$, and we saw examples where this happens in the last section.

We consider the stack $\mathcal{FP}_{X_\infty}$ over $(\mathrm{Aff})^{op}$ of finitely presented sheaves on $X_\infty$, defined as follows: an object over $A \in (\mathrm{Aff})^{op}$ is a finitely presented sheaf on $(X_\infty)_A = X_\infty \times_k \mathrm{Spec}(A)$, flat over $A$, and arrows are defined using pullback along $(X_\infty)_B \to (X_\infty)_A$ for a homomorphism $A \to B$.

Inside $\mathcal{FP}_{X_\infty}$ there is a subcategory parametrizing families of semi-stable sheaves: define $\mathcal{M}^{ss}$ (resp. $\mathcal{M}^s$) as the stack over $(\mathrm{Aff})^{op}$ with objects over $A \in (\mathrm{Aff})^{op}$ finitely presented sheaves $F$ on $(X_\infty)_A$, flat over $\mathrm{Spec}(A)$, and such that for every geometric point $p \to \mathrm{Spec}(A)$, the pullback of $F$ to $(X_\infty)_p$ is semi-stable (resp. stable).

In the rest of this chapter we will prove the following theorem.

**Theorem 4.3.5.** *Let $X$ be a projective simplicial log scheme over $k$ with a global simplicial chart $P \to \mathrm{Div}(X)$. The stack $\mathcal{M}^{ss}$ is an Artin stack, locally of finite presentation, and it has an open substack $\mathcal{M}^s \subseteq \mathcal{M}^{ss}$ parametrizing stable sheaves.*

*If in addition stability is preserved by pullback along the projections $X_m \to X_n$ between the root stacks of $X$ (for example if the log structure of $X$ is generically trivial, and we are considering pure sheaves of maximal dimension), then $\mathcal{M}^{ss}$ has a good moduli space $M^{ss}$, which is a disjoint union of projective schemes. Moreover there is an open subscheme $M^s \subseteq M^{ss}$ that is a coarse moduli space for the substack $\mathcal{M}^s$, and $\mathcal{M}^s \to M^s$ is a $\mathbb{G}_m$-gerbe.*

Let us start by relating the stack of parabolic sheaves on $X_\infty$ with the ones at finite level.

**Proposition 4.3.6.**

- *We have a natural isomorphism of stacks over $(\mathrm{Aff})^{op}$*
$$\varinjlim_n \mathcal{M}_n^{ss} \to \mathcal{M}^{ss}.$$
- *If pullbacks preserve stability, then we also have an isomorphism*
$$\varinjlim_n \mathcal{M}_n^{s} \to \mathcal{M}^{s}$$
*which is compatible with the previous one. Moreover, in this last case the transition maps are open and closed immersions, so $\mathcal{M}^{ss}$ and $\mathcal{M}^s$ are in fact a union of connected components of the stacks at finite level.*

*Proof.* Let us recall first of all how to define the direct limit $\varinjlim_n \mathcal{M}_n^{ss}$.

Given in general a filtered directed system $\{\mathcal{C}_i\}_{i\in I}$ of fibered categories over some category $\mathcal{D}$, we can define the direct limit $\mathcal{C} = \varinjlim_{i\in I} \mathcal{C}_i$ as a fibered category over $\mathcal{D}$ as follows: objects are pairs $(d, c)$, where $d \in \mathcal{D}$ and $c \in \mathcal{C}_i(d)$ for some $i \in I$, and a morphism $(d, c) \to (d', c')$ is a pair $(f, g)$ where $f\colon d \to d'$ is a morphism in $\mathcal{D}$, and $g$ is an element of the direct limit $\varinjlim_{i\geq i_0} \mathrm{Hom}(\phi_{i_0,i}(f^*c'), \phi_{i_0,i}(c))$, where $i_0$ is an index where both $c$ and $c'$ are defined. In other words we are taking the disjoint union of the objects and the direct limit for morphisms, fiberwise. If $\mathcal{D}$ is a site and $\mathcal{C}_i$ are stacks, we can stackify $\mathcal{C}$ to get the direct limit as a stack.

In our particular case note that the direct limit is already a stack: this is because, since we're working on $(\mathrm{Aff})^{op}$, every covering has a finite refinement, so we can reduce effectivity of descent data and the fact that Hom is a sheaf to some finite level. Moreover, since all the maps $\iota_{n,m}$ are fully faithful, in the direct limit we have $\mathrm{Hom}(F_n, F_m) = \mathrm{Hom}_{\mathcal{M}_h^{ss}}(\pi_{n,h}^* F_n, \pi_{m,h}^* F_m)$, where $h = \mathrm{lcm}(n, m)$.

Now for every $n \in \mathbb{N}$ the pullback along $\pi_n\colon X_\infty \to X_n$ induces $\iota_n\colon \mathcal{M}_n^{ss} \to \mathcal{M}^{ss}$, and moreover these maps are compatible with the transition maps of the system $\iota_{n,m}$. Thus we have a morphism
$$\iota\colon \varinjlim_n \mathcal{M}_n^{ss} \to \mathcal{M}^{ss}.$$
We will check that this is fully faithful and essentially surjective.

Take a $k$-algebra $A$, and consider the map $(\varinjlim_n \mathcal{M}_n^{ss})(A) \to \mathcal{M}^{ss}(A)$. If $F$ is an object of $\mathcal{M}^{ss}(A)$, i.e. a finitely presented sheaf on $(X_\infty)_A = X_\infty \times_k \operatorname{Spec}(A)$, then since $(X_\infty)_A = \varprojlim_n (X_n)_A$, we have $\mathrm{FP}((X_\infty)_A) = \varinjlim \mathrm{FP}((X_n)_A)$, and $F$ comes from some $F_n \in \mathrm{FP}((X_n)_A)$. Moreover by possibly increasing $n$ we can assume that $F_n$ is flat over $A$, and its fibers over $(X_n)_p$ for geometric points $p \to \operatorname{Spec}(A)$ will be semi-stable by Proposition 4.3.3, since their pullback to $(X_\infty)_p$ is. In other words $F_n$ is an object of $\mathcal{M}_n^{ss}(A)$, and its image in $(\varinjlim_n \mathcal{M}_n^{ss})(A)$ via $\iota$ will be isomorphic to $F$.

For full faithfulness, if $F_n$ and $F_m$ are two objects of $(\varinjlim_n \mathcal{M}_n^{ss})(A)$, as noted above we have $\operatorname{Hom}(F_n, F_m) = \operatorname{Hom}_{\mathcal{M}_h^{ss}}(\pi_{n,h}^* F_n, \pi_{m,h}^* F_m)$ with $h = \operatorname{lcm}(n, m)$, and since pullback along $(X_\infty)_A \to (X_h)_A$ is fully faithful, the conclusion follows.

The same line of reasoning works for the statement about stable sheaves, and compatibility of the maps is immediate from the compatibility at finite level. □

**Remark 4.3.7.** What perhaps is not clear enough, is that $j\colon \mathcal{M}^s \subseteq \mathcal{M}^{ss}$ is an open substack. This holds even if stability is not preserved by pullback.

In fact, take a morphism $f\colon T \to \mathcal{M}^{ss}$, and note that we can assume that $T$ is affine, say $T = \operatorname{Spec}(A)$. The map $f$ corresponds to a sheaf $F \in \mathcal{M}^{ss}(A)$, and by the preceding proposition $F$ will come from some $F_n \in \mathcal{M}_n^{ss}(A)$.

From this, and the observation that $j^{-1}\mathcal{M}_n^{ss} \cong \mathcal{M}_n^s$, we see that the fibered product $\mathcal{M}^s \times_{\mathcal{M}^{ss}} T$ coincides with $\mathcal{M}^s \times_{\mathcal{M}_n^{ss}} T = j^{-1}\mathcal{M}_n^{ss} \times_{\mathcal{M}_n^{ss}} T = \mathcal{M}_n^s \times_{\mathcal{M}_n^{ss}} T$, and this is open in $T$ because $\mathcal{M}_n^s \to \mathcal{M}_n^{ss}$ is an open immersion.

**Proposition 4.3.8.** *The stack $\mathcal{M}^{ss}$ is an Artin stack, locally of finite presentation over $k$. Being an open substack, $\mathcal{M}^s$ has the same properties.*

**Lemma 4.3.9.** *For any $n \in \mathbb{N}$ the morphism $\iota_n\colon \mathcal{M}_n^{ss} \to \mathcal{M}^{ss}$ induced by pullback is an open immersion.*

*Proof.* This goes exactly as the proof of Lemma 4.2.14. The main point is that, by the projection formula for $\pi_n\colon X_\infty \to X_n$, a finitely presented sheaf $F \in \mathrm{FP}(X_\infty)$ comes from $X_n$ if and only if the adjunction morphism $\pi_n^*\pi_{n*}F \to F$ is an isomorphism. □

*Proof of Proposition 4.3.8.* Let us fix a smooth presentation $A_n \to \mathcal{M}_n^{ss}$ for every $n \in \mathbb{N}$. We have a natural induced map $A = \bigsqcup_n A_n \to \varinjlim_n \mathcal{M}_n^{ss} = \mathcal{M}^{ss}$, and this is a smooth presentation for $\mathcal{M}^{ss}$.

Indeed, the map is an epimorphism since $\mathcal{M}^{ss}$ is a union of the open substacks $\mathcal{M}_n^{ss}$, and $A_n \to \mathcal{M}_n^{ss}$ is an epimorphism, and for a morphism $f\colon T \to \mathcal{M}^{ss}$ from a scheme $T$ we have $A \times_{\mathcal{M}^{ss}} T = \bigsqcup_n (A_n \times_{\mathcal{M}^{ss}} T)$, so we can consider a single piece $A_n \times_{\mathcal{M}^{ss}} T$. Now it suffices to note that the map $A_n \to \mathcal{M}_n^{ss} \subseteq \mathcal{M}^{ss}$ is a composition of two smooth representable morphisms.

Let us now show that the diagonal $\Delta\colon \mathcal{M}^{ss} \to \mathcal{M}^{ss} \times_k \mathcal{M}^{ss}$ is representable. Let us take a morphism $f\colon T \to \mathcal{M}^{ss} \times_k \mathcal{M}^{ss}$ from a scheme, and consider the fibered product over $\Delta$. Since $\mathcal{M}^{ss} \times_k \mathcal{M}^{ss}$ is the union of its open substacks $\{\mathcal{M}_n^{ss} \times_k \mathcal{M}_n^{ss}\}_{n\in\mathbb{N}}$, we have a Zariski cover $\{T_n = f^{-1}(\mathcal{M}_n^{ss} \times_k \mathcal{M}_n^{ss})\}_{n\in\mathbb{N}}$ of $T$, and the question is Zariski-local, so we can replace $T$ with one $T_n$. Consequently, $f$ factors as $T \to \mathcal{M}_n^{ss} \times_k \mathcal{M}_n^{ss} \subseteq \mathcal{M}^{ss} \times_k \mathcal{M}^{ss}$, and since the diagram

$$\begin{tikzcd}
\mathcal{M}_n^{ss} \arrow[r] \arrow[d] & \mathcal{M}_n^{ss} \times_k \mathcal{M}_n^{ss} \arrow[d] \\
\mathcal{M}^{ss} \arrow[r] & \mathcal{M}^{ss} \times_k \mathcal{M}^{ss}
\end{tikzcd}$$

is cartesian, the fibered product $\mathcal{M}^{ss} \times_{\mathcal{M}^{ss} \times_k \mathcal{M}^{ss}} T = \mathcal{M}^{ss}_n \times_{\mathcal{M}^{ss}_n \times_k \mathcal{M}^{ss}_n} T$ is representable by an algebraic space. □

### 4.3.1 What invariants can we fix?

Before we go further, let us briefly consider the following problem: can we fix some invariants for finitely presented sheaves on $X_\infty$, in order to cut out a finite-type moduli stack inside $\mathcal{M}^{ss}$? Ideally, since we are taking a limit of the theories at finite level, we would like to fix some invariant of coherent sheaves on the $X_n$'s, that is preserved by pullback along the maps $\pi_{n,m} \colon X_m \to X_n$.

The stacks $\mathcal{M}^{ss}_n$ we considered up to this point are not of finite type themselves, and the standard solution when one studies moduli of coherent sheaves is to fix the Hilbert polynomial $H \in \mathbb{Q}[x]$. This gives finite-type components, both of the corresponding moduli stack and of its good moduli space (the components of the moduli space are actually even projective). There are other things one can fix, for example Chern classes, but here we will focus mainly on Hilbert polynomials.

It is clear that we cannot fix the Hilbert polynomial at the limit, since it is not preserved by pullback. Rather, we have $P_m(\pi^*_{n,m}(F)) = k \cdot P_n(F)$, where $k$ is such that $m = nk$. On the other hand, we saw in Proposition 4.2.10 that the *reduced* Hilbert polynomial $h$ is preserved by pullback, i.e. $p_m(\pi^*_{n,m}F) = p_n(F)$ for any $F \in \mathrm{FP}(X_n)$. This also follows immediately from the formula for the Hilbert polynomial, which implies that we have $\alpha^m(\pi^*_{n,m}(F)) = k \cdot \alpha^n(F)$, where $\alpha^n(F)$ denotes the multiplicity of the sheaf $F$ on $X_n$, and so

$$p_m(\pi^*_{n,m}F) = \frac{P_m(\pi^*_{n,m}F)}{\alpha^m(\pi^*_{n,m}F)} = \frac{k \cdot P_n(F)}{k \cdot \alpha^n(F)} = p_{n(F)}.$$

**Notation 4.3.10.** We denote by $\mathcal{M}^{ss}_{h,n}$ and $\mathcal{M}^{s}_{h,n}$ the stacks that parametrize families of (semi-)stable parabolic sheaves on $X_n$ with reduced Hilbert polynomial $h \in \mathbb{Q}[x]$. They will have good moduli spaces $M^{ss}_{h,n}$ and $M^{s}_{h,n}$, and since the reduced Hilbert polynomial is preserved by pullback, the morphisms $\iota_{n,m} \colon \mathcal{M}^{ss}_n \to \mathcal{M}^{ss}_m$ will restrict to morphisms $\mathcal{M}^{ss}_{h,n} \to \mathcal{M}^{ss}_{h,m}$, which we will still denote by $\iota_{n,m}$. The same goes for the morphism $i_{n,m}$, and also for $\iota^o_{n,m}$, $i^o_{n,m}$ when they are defined.

Exactly as in Proposition 4.3.6, we have an isomorphism

$$\varinjlim_n \mathcal{M}^{ss}_{h,n} \cong \mathcal{M}^{ss}_h$$

and the analogous one for stable sheaves if stability is preserved by pullback.

This all works well with the direct limit, but there is an issue: $\mathcal{M}^{ss}_{h,m}$ is not necessarily of finite type. In fact, fixing the reduced Hilbert polynomial $h$ does not fix the rank (say we are considering torsion-free sheaves), like it happens with the ordinary Hilbert polynomial, and the rank can become arbitrarily large, without changing $h$. In other words, we have $\mathcal{M}^{ss}_{h,n} = \bigsqcup_H \mathcal{M}^{ss}_{H,n}$ where the union ranges over $H \in \mathbb{Q}[x]$ of degree $d$ such that $H/\alpha = h$, where $\alpha$ is $d!$ times the leading coefficient of $H$.

**Example 4.3.11.** In the case of $X = \mathrm{Spec}(k)$, the standard log point, the Hilbert polynomial coincides with the rank, the reduced Hilbert polynomial is always 1 (and everything is semi-stable), so $\mathcal{M}^{ss}_{1,n}$ is the only non-empty stack at level $n$, and it decomposes as a disjoint union $\bigsqcup_H \mathcal{M}^{ss}_{H,n}$, where $H$ is an integer.

Moreover all of the pieces $\mathcal{M}^{ss}_{H,n}$ are non-empty, as it is easy to write a parabolic sheaf on $X_n$ of arbitrary rank, and even in the limit $\mathcal{M}^{ss}_1 = \varinjlim_n \mathcal{M}^{ss}_{1,n}$ there are infinitely many connected components: for any $n$, the parabolic sheaf

$$\begin{array}{ccccccccc} -1 & & -\frac{n-1}{n} & & -\frac{n-2}{n} & & \cdots & & 0 \\ & & & & & & & & \\ 0 & \longrightarrow & k & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \end{array}$$

in $\mathcal{M}^{ss}_{1,n}(k)$, having one copy of $k$ in position $-\frac{n-1}{n}$ and all zeros elsewhere, is not in the closure of any point of the stacks $\mathcal{M}^{ss}_{1,h}$ with $h \leq n$, so it will belong to a connected component which is outside of the image of those stacks.

In the case where the logarithmic structure of $X$ is generically trivial, $X$ is integral and we are considering torsion-free sheaves, there is another thing that we can fix and that gives intermediate components of finite type, namely the rank of the pushforward of the sheaf to $X$. In fact, since $\pi \colon X_n \to X$ is generically an isomorphism, if $F \in \mathrm{FP}(X_n)$ has rank $r$, the pushforward $\pi_* F$ will still have rank $r$. Moreover the "parabolic" rank of $\pi^*_{n,m}(F)$ is easily seen to be $m \cdot r$, so fixing $h$ and $r$ is equivalent to fixing $H$, and thus will give a finite-type union of components $\mathcal{M}^{ss}_{h,r,n}$ of $\mathcal{M}^{ss}_n$.

With these assumptions, the moduli stacks $\mathcal{M}^{ss}_{h,r,n}$ and $\mathcal{M}^{s}_{h,r,n}$ are of finite type, and the good moduli space $M^{ss}_{h,r,n}$ (resp. $M^{s}_{h,r,n}$) is projective (resp. quasi-projective).

We remark that even in this case, the "limit" moduli stack $\mathcal{M}^{ss}_{h,r}$ is not necessarily of finite type, and it can have infinitely many connected components.

**Example 4.3.12.** Take $X = \mathbb{P}^1$, with the log structure induced by the divisor $0 + \infty$, and let us fix the reduced Hilbert polynomial $h(x) = x + n$ for $n \in \mathbb{Z}$, and rank $r = 1$.

For any $m \in \mathbb{N}$, the parabolic sheaf

$$\begin{array}{cccccccccccccc} & -1 & & -\frac{m-1}{m} & & -\frac{m-2}{m} & & \cdots & & -\frac{2}{m} & & -\frac{1}{m} & & 0 \\ & & & & & & & & & & & & & \\ F = & \mathcal{O}(n-2) & \longrightarrow & \mathcal{O}(n-1) & = & \mathcal{O}(n-1) & = & \cdots & = & \mathcal{O}(n-1) & \longrightarrow & \mathcal{O}(n) & = & \mathcal{O}(n) \end{array}$$

on $X_m$ has reduced Hilbert polynomial

$$p_m(F_m)(x) = \frac{x + n - 1 + (m-2)(x+n) + x + n + 1}{m} = x + n = h(x)$$

and rank 1, so it gives is a point of $\mathcal{M}^{ss}_{h,1}$, and it sits in the substack $\mathcal{M}^{ss}_{h,1,m}$. Moreover, it is not in any of the $\mathcal{M}^{ss}_{h,1,j}$ with $j \mid m$ (otherwise the only two non-identity maps would need to be the identity), and so it not in a connected component coming from lower levels, since in this case the immersions are open and closed. This shows that there are infinitely many components in $\mathcal{M}^{ss}_{h,1}$, and so it is not of finite type.

### 4.3.2 Taking the limit

Motivated by the previous discussion on how to see the stack $\mathcal{M}^{ss}$ of semi-stable sheaves on $X_\infty$ as a direct limit of the stacks $\mathcal{M}_n^{ss}$ of semi-stable sheaves on the finite root stacks, we want to justify the fact that the direct limit of the moduli spaces at finite levels is a good candidate for a moduli space for the stack $\mathcal{M}^{ss}$. There is no reason for this direct limit to be a scheme in general, but rather only an ind-scheme. For this reason we have to consider ind-moduli spaces as well.

Ind-schemes in the literature are usually required to have closed embeddings as transition maps of their defining filtered system, and this is not necessarily the case in our situation, as far as we know. We will use the following definition.

**Definition 4.3.13.** An *ind-algebraic space* over $k$ is a presheaf on $(\mathrm{Aff})^{op}$ that can be written as a filtered direct limit $\varinjlim_{i\in I} X_i$ of a directed filtered system of sheaves, which are moreover algebraic spaces.

A morphism of ind-algebraic spaces is a morphisms of sheaves over $(\mathrm{Aff})^{op}$, and there is a category of ind-algebraic spaces over $k$, which we denote $(\text{Ind-algsp})$. There is a fully faithful functor $(\mathrm{Algsp}) \to (\text{Ind-algsp})$ that takes an algebraic space to the functor it represents on $(\mathrm{Aff})^{op}$. In particular, an ind-algebraic space $X$ gives a presheaf on $(\mathrm{Algsp})^{op}$ defined as $X(T) = \mathrm{Hom}(T, X)$, where the Hom is taken as presheaves on $(\mathrm{Aff})^{op}$.

The point that we want to make is that if $\mathcal{M}^{ss}$ admits a good moduli space (which is usually the case for stacks parametrizing (semi-)stable sheaves, so it seems the right object to look for), than it has to be isomorphic to the direct limit $\varinjlim_n M_n^{ss}$.

First of all let us show that if $\varinjlim_n M_n^{ss}$ is an algebraic space, then it has the factorization property that good moduli spaces possess.

**Proposition 4.3.14.** *Let $\{\mathcal{M}_i\}_{i\in I}$ be a directed system of locally noetherian Artin stacks with good moduli spaces $\{M_i\}_{i\in I}$, and assume that $\varinjlim_i M_i$ is an algebraic space. Then it has the following universal property: for every morphism $\varinjlim_i \mathcal{M}_i \to N$ to an algebraic space there exist a unique morphism $\varinjlim_i M_i \to N$ that completes the diagram*

$$\begin{tikzcd} \varinjlim_i \mathcal{M}_i \arrow[d] \arrow[dr] & \\ \varinjlim_i M_i \arrow[r] & N. \end{tikzcd}$$

*In particular if $\varinjlim_i \mathcal{M}_i$ is locally noetherian and has a good moduli space, then this is canonically isomorphic to $\varinjlim_i M_i$.*

*Proof.* This follows directly from

$$\mathrm{Hom}(\varinjlim_i \mathcal{M}_i, N) = \varprojlim_i \mathrm{Hom}(\mathcal{M}_i, N) = \varprojlim_i \mathrm{Hom}(M_i, N) = \mathrm{Hom}(\varinjlim_i M_i, N)$$

where the second equality is by the factorization property of the good moduli spaces. $\square$

This implies that if $\varinjlim_n M_n^{ss}$ is an algebraic space and $\mathcal{M}^{ss}$ has a good moduli space, then this has to be isomorphic to $\varinjlim_n M_n^{ss}$.

Let us show now that if stability is preserved, then $\varinjlim_n M_n^{ss}$ is indeed a good moduli space for $\mathcal{M}^{ss}$. In fact, after a lemma about direct limits, we will complete the proof of 4.3.5.

**Lemma 4.3.15.** *Let $\{M_i\}_{i\in I}$ be a filtered directed system of schemes, where every transition map is an open and closed immersion. Then the ind-scheme $\varinjlim_{i\in I} M_i$ is isomorphic as an ind-scheme to a disjoint union of components of the $M_i$'s, and in particular it is a scheme.*

*Proof.* Let us write $A_i$ for the set of connected components of the scheme $M_i$. The open and closed immersion $M_i \to M_j$ for $i \leq j$ induces a function of sets $A_i \to A_j$, and these functions for varying $i$ and $j$ form a filtered directed system of sets. Let $A$ be the direct limit of this directed system, and for every $a \in A$ fix a component $X_a$ of some $M_i$ that goes to $a$ in the limit. We claim that the scheme

$$M = \coprod_{a\in A} X_a$$

is the direct limit of the system.

In fact we have natural maps $M_i \to M$ that induce a map of presheaves $\varinjlim_i M_i \to M$ on $(\mathrm{Aff})^{op}$. We have to verify that this is a natural isomorphism: injectivity is clear, since every $M_i$ is an open and closed subscheme of $M$, and surjectivity follows from the fact that the image of a morphism $\mathrm{Spec}(A) \to M$ is contained in finitely many $M_a$'s, by quasi-compactness of $\mathrm{Spec}(A)$. $\square$

*Proof of 4.3.5.* The fact that $\mathcal{M}^{ss}$ is an Artin stack locally of finite presentation is in Proposition 4.3.8, and the fact that stable sheaves form an open substack is explained in Remark 4.3.7.

As for the second part, assume that stability is preserved by pullback, so that all the maps between the stacks and moduli spaces are open and closed immersions. In particular $\mathcal{M}^{ss}$ and $\mathcal{M}^s$ will be ascending unions of (open and closed) substacks, where at each step we might add some new connected components. Let $M^{ss}$ be the direct limit of the system $\{M^{ss}_{\overline{n}}\}_{n\in\mathbb{N}}$ of good moduli spaces at finite level. By the preceding lemma, this will be a disjoint union $M^{ss} = \coprod_i M_i$, where each $M_i$ is a connected component of some $M^{ss}_{\overline{n}}$.

We have a natural map $\mathcal{M}^{ss} \to M^{ss}$, and since being a good moduli space is a local property, we can restrict to a single component $M_i$, say it comes from $M^{ss}_{\overline{n}}$. Now the fibered product $\mathcal{M}^{ss} \times_{M^{ss}} M_i$ will clearly be the connected component of $\mathcal{M}^{ss}$ (coming from $\mathcal{M}^{ss}_{\overline{n}}$ and) corresponding to $M_i$, and so the projection $\mathcal{M}^{ss} \times_{M^{ss}} M_i \to M_i$ is a good moduli space, because it is a good moduli space at level $\overline{n}$.

The remaining statements about the substack of stable sheaves follow in the same way from the corresponding statement for the stacks $\mathcal{M}^s_{\overline{n}}$ at finite level. $\square$

In the remaining few pages we will show that, even without the assumption that $\varinjlim_n M_n$ is an algebraic space, if $\mathcal{M}^{ss}$ has a good moduli space $M$ then there is an isomorphism $M \cong \varinjlim_n M_n$. This hints at the fact that the direct limit of the moduli spaces at finite level gives the correct moduli space at the limit, even though it might not be an algebraic space.

**Definition 4.3.16.** Let $\mathcal{M}$ be an Artin stack over $(\mathrm{Aff})^{op}$. A *naive ind-moduli space* for $\mathcal{M}$ is an ind-algebraic space $M$ with a morphism $\mathcal{M} \to M$ such that for any other morphism $\mathcal{M} \to N$ to an ind-algebraic space, there is a unique factorization $\mathcal{M} \to M \to N$.

**Remark 4.3.17.** The previous definition will only play a role in the heuristics that justify the fact that we want to look at the direct limit of the moduli spaces at finite level, and is not meant to be particularly meaningful otherwise.

**Remark 4.3.18.** As usual with objects defined by a universal property, if an Artin stack $\mathcal{M}$ admits a naive ind-moduli space, then this is unique up to isomorphism.

Now we need a couple of facts about ind-algebraic spaces. Note that by definition if $Y = \operatorname{Spec}(A)$ is affine, then we have

$$\operatorname{Hom}(Y, \varinjlim_{i \in I} X_i) = \varinjlim_{i \in I} \operatorname{Hom}(Y, X_i).$$

**Lemma 4.3.19.** *If $Y$ is a qcqs (quasi-compact and quasi-separated) scheme over $k$ and $\varinjlim_{i \in I} X_i$ is an ind-algebraic space, we have*

$$\operatorname{Hom}(Y, \varinjlim_{i \in I} X_i) = \varinjlim_{i \in I} \operatorname{Hom}(Y, X_i).$$

*Proof.* The proof is by gluing along affines. Write $Y$ as a union of finitely many affine open subsets $Y_1, \dots, Y_n$.

We have a natural function

$$\varinjlim_{i \in I} \operatorname{Hom}(Y, X_i) \to \operatorname{Hom}(Y, \varinjlim_{i \in I} X_i)$$

defined by $[Y \to X_i] \mapsto (Y \to X_i \to \varinjlim_i X_i)$. Let us show that it is bijective.

If $[Y \to X_i]$ and $[Y \to X_j]$ have the same image $Y \to \varinjlim_i X_i$, then for any $h \in \{1, \dots, n\}$ the restrictions of the two maps to the open subset $Y_h \subseteq Y$ will be equal after composition with the map to $\varinjlim_i X_i$, and so they will be also equal after composition with the map to some finite $X_k$, because in this case we know that $\operatorname{Hom}(Y_h, \varinjlim_i X_i) = \varinjlim_i \operatorname{Hom}(Y_h, X_i)$. Since the $Y_h$'s are finitely many, we can find an index that works for all of them, and this shows that $[Y \to X_i] = [Y \to X_j]$.

As for surjectivity, take a morphism $f\colon Y \to \varinjlim_i X_i$, and restrict it to $f_h\colon Y_h \to \varinjlim_i X_i$. Every $f_h$ will come from some finite level, and since they are finitely many there will be a $k$ such that $f_h$ comes from $g_h\colon Y_h \to X_k$. Now we examine the intersections $Y_{hk} = Y_h \cap Y_k$: since $Y$ is quasi-separated, we can cover each $Y_{hk}$ with finitely many affines, and since the restrictions of $g_h$ and $g_k$ to any of these affines will give the same map to $\varinjlim_i X_i$, we can find a bigger index $k$ that renders them equal as maps to $X_k$. In finitely many steps, the maps $g_h$ will agree on the double intersections $Y_{hk}$, and will yield a map $g\colon Y \to X_k$ that is in the preimage of $f$. $\square$

**Lemma 4.3.20.** *Let $\mathcal{M}$ be a qcqs Artin stack over $k$ and $\varinjlim_i X_i$ be an ind-algebraic space. Then we have a bijection* $\operatorname{Hom}(\mathcal{M}, \varinjlim_i X_i) = \varinjlim_i \operatorname{Hom}(\mathcal{M}, X_i)$.

*Proof.* The proof mimics the one of the preceding lemma, with respect to a smooth presentation $U \to \mathcal{M}$ which is a disjoint union of finitely many affines. $\square$

**Lemma 4.3.21.** *Any ind-algebraic space $X = \varinjlim_i X_i$ is a sheaf in the étale topology of* $(\mathrm{Aff})^{op}$.

*Proof.* Let $A$ be a $k$-algebra, and $A \to A_j$ morphisms that give a covering of $\operatorname{Spec}(A)$ in the étale topology. Note that since $\operatorname{Spec}(A)$ is quasi-compact we can extract a finite subcovering $\{A_1, \dots, A_n\}$, and it is sufficient to show that the sheaf condition holds for it. Now it suffices to note that the diagrams

$$X_i(A) \to \prod_{j=1}^{n} X_i(A_j) \rightrightarrows \prod_{j,k=1,\dots,n} X_i(A_j \otimes_A A_k)$$

are equalizers for all $i$, since the $X_i$'s are sheaves for the étale topology, and filtered directed limits are exact and commute with finite products, so that also the diagram

$$X(A) \to \prod_{j=1}^{n} X(A_j) \rightrightarrows \prod_{j,k=1,\dots,n} X(A_j \otimes_A A_k)$$

is an equalizer. □

**Lemma 4.3.22.** *Let $X$ be an ind-algebraic space. Then the functor* $\mathrm{Hom}(-, X)\colon (\mathrm{Algsp})^{op} \to (\mathrm{Set})$ *is a sheaf for the étale topology.*

*Proof.* Let $T$ be an algebraic space and $\{T_i \to T\}_{i\in I}$ an étale covering. We have to show that a compatible collection of morphisms $\phi\colon T_i \to X$ of presheaves on $(\mathrm{Aff})^{op}$ yields a unique morphism $T \to X$.

Let us consider a $k$-algebra $A$ together with a morphism $\mathrm{Spec}(A) \to T$. By base change, from the covering $\{T_i \to T\}_{i\in I}$ we obtain an étale covering $\{Y_i \to \mathrm{Spec}(A)\}_{i\in I}$, that we can refine to a covering $\{\mathrm{Spec}(B_j) \to \mathrm{Spec}(A)\}_{j\in J}$ by affines. Now the morphisms $T_i \to X$ restrict to the $B_j$ to give elements $\xi_j \in X(B_j)$, and the compatibility on the fibered products $T_i \times_T T_{i'}$ gives equality of the restrictions of $\xi_j$ and $\xi_{j'}$ to $X(B_j \otimes_A B_{j'})$. From Lemma 4.3.21 we obtain a unique element of $X(A)$, and this construction gives a morphism of presheaves $T \to X$. It is immediate to check that this morphism restricts to the given ones over the $T_i$'s, and is unique. □

**Proposition 4.3.23.** *A good moduli space $M$ for a locally noetherian Artin stack $\mathcal{M}$ is also a naive ind-moduli space.*

*Proof.* Let us fix an affine étale covering $\{U_i \to M\}_{i\in I}$ of the algebraic space $M$. By the properties of good moduli spaces, the restriction $\mathcal{M} \times_M U_i \to U_i$ is still a good moduli space, so it enjoys the universal property for maps to algebraic spaces, since $\mathcal{M}$ is locally noetherian.

Now fix an ind-algebraic space $N = \varinjlim_j N_j$. We have to show that the map $\mathrm{Hom}(M, N) \to \mathrm{Hom}(\mathcal{M}, N)$ is a bijection. Since $\mathrm{Hom}(-, N)$ is a sheaf on $(\mathrm{Sch})^{op}$ with the étale topology by Lemma 4.3.22, to verify this we can pass to the étale cover $U_i$. Now since $U_i$ and $\mathcal{M} \times_M U_i$ are qcqs, by 4.3.20 and the universal property of good moduli spaces we have

$$\begin{aligned}\mathrm{Hom}(U_i, N) = \mathrm{Hom}(U_i, \varinjlim N_j) = \varinjlim \mathrm{Hom}(U_i, N_i) = \\ = \varinjlim \mathrm{Hom}(\mathcal{M} \times_M U_i, N_i) = \mathrm{Hom}(\mathcal{M} \times_M U_i, N).\end{aligned}$$

and this concludes the proof. □

**Proposition 4.3.24.** *Let $\{\mathcal{M}_i\}_{i\in I}$ be a filtered directed system of locally noetherian Artin stacks with good moduli spaces $\mathcal{M}_i \to M_i$. Assume moreover that each $\mathcal{M}_i$ and $M_i$ is a disjoint union of qcqs stacks (resp. spaces), in a compatible way. Then $\varinjlim_i \mathcal{M}_i \to \varinjlim_i M_i$ is a naive ind-moduli space.*

*Proof.* Since the stacks $\mathcal{M}_i$ are locally noetherian, their good moduli spaces will be universal with respect to maps to algebraic spaces. Let us write $\mathcal{M}_{i,k}$ for the (qcqs) components of $\mathcal{M}_i$ and $M_{i,k}$ for the ones of $M_i$, so that the maps $\mathcal{M}_{i,k} \to M_{i,k}$ are good moduli spaces.

Let us consider an ind-algebraic space $N = \varinjlim_j N_j$, and let us calculate

$$\begin{aligned}\mathrm{Hom}(\varinjlim_i \mathcal{M}_i, N) = \varprojlim_i \mathrm{Hom}(\mathcal{M}_i, \varinjlim_j N_j) = \\ = \varprojlim_i \bigsqcup \mathrm{Hom}(\mathcal{M}_{i,k}, \varinjlim_j N_j) = \varprojlim_i \bigsqcup \varinjlim_j \mathrm{Hom}(\mathcal{M}_{i,k}, N_j)\end{aligned}$$

where we used in the last equality that that $\mathcal{M}_{i,k}$ are qcqs. Now by the universality property of the good moduli spaces we have $\mathrm{Hom}(\mathcal{M}_{i,k}, N_j) = \mathrm{Hom}(M_{i,k}, N_j)$, and then we can repack everything together

$$\begin{aligned}\varprojlim_i \bigsqcup \varinjlim_j \mathrm{Hom}(M_{i,k}, N_j) = \varprojlim_i \bigsqcup \mathrm{Hom}(M_{i,k}, \varinjlim_j N_j) = \\ = \varprojlim_i \mathrm{Hom}(M_i, \varinjlim_j N_j) = \mathrm{Hom}(\varinjlim_i M_i, N)\end{aligned}$$

where we used the fact that $M_{i,k}$ is also qcqs, and the preceding lemmas. □

This discussion applies to the moduli stack of parabolic sheaves with rational weights, since the stacks $\mathcal{M}_n^{ss}$, together with their good moduli spaces, are disjoint unions of qcqs stacks (resp. spaces) and locally noetherian.

**Corollary 4.3.25.** *If the moduli stack of parabolic sheaves with rational weights $\mathcal{M}^{ss}$ admits a good moduli space $M$, then there is an isomorphism $M \cong \varinjlim_n M_n^{ss}$.*

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