Scuola Normale Superiore di Pisa

Tesi di Perfezionamento in Matematica

Special holonomy and hypersurfaces

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Introduction

The study of special holonomy originated with Berger's classification of the possible holonomy groups of non-locally symmetric Riemannian manifolds. We are interested in the special holonomy groups, i.e. all of the groups in Berger's list except $\mathrm{U}(n)$ and $\mathrm{SO}(n)$. More generally, we relax the holonomy condition and consider, for G a special holonomy group, G-structures that are not necessarily integrable. The resulting theory is often referred to as "special geometry". Integrability is replaced by the vanishing of part of the intrinsic torsion; the part that is required to vanish determines the type of special geometry. Manifolds with weak holonomy and hypersurfaces inside manifolds with special holonomy fall into the category of special geometries.

All special holonomy groups except $\mathrm{Sp}(n)\mathrm{Sp}(1)$ arise as the stabilizer of a point in a spin representation; accordingly, one can view a G-structure as a pair (P, ψ), where P is a spin structure and ψ is a spinor, and a type of special geometry is characterized by conditions on $\nabla\psi$. Alternatively, one can view G as the stabilizer in $\mathrm{GL}(n, \mathbb{R})$ of an exterior form on $\mathbb{R}^n$, and identify a G-structure with a differential form φ; a type of special geometry is then characterized by conditions on $\nabla\varphi$, which is typically determined by $d\varphi$. To be precise, more than one form is needed for some G, but the same principle holds. Notice that reduced holonomy is characterized by the vanishing of $\nabla\psi$ or $\nabla\varphi$.

We are primarily interested in dimensions less than eight. What sets them apart from the general case is the fact that the spin representation is transitive on the sphere; in the language of forms, this correponds to the fact that the defining forms are stable in the sense of Hitchin. This means that a small deformation of the defining form, or spinor, still defines a structure of the same type, although the intrinsic torsion conditions may not be preserved. When the structure is not defined by a single form, but two or more, arbitrary deformations no longer preserve the structure type, leading to technical difficulties in the study of one-parameter families of G-structures. The fact that, as we will see, one can work around these difficulties seems to be related to the fact that these deformations can be viewed as those of the pair (P,ψ).

Having introduced the general language and background, we can give more specific motivation for our work. The first piece of motivation is Hitchin's study of some special geometries defined by stable forms, which we reinterpret using spinors, borrowing terminology from [6]. These are structures defined by a generalized Killing spinor, i.e. a spinor satisfying

$$\nabla_X\psi = \frac{1}{2}A(X)\cdot\psi \ ,$$

where A is a symmetric endomorphism of the tangent bundle and $\cdot$ is Clifford multiplication. The G-structure defined by a generalized Killing spinor is a half-flat $\mathrm{SU}(3)$-structure in six dimensions, and a cocalibrated G_2-structure in seven dimensions. In general, if $\mathbf{M}$ has a parallel spinor ψ, then the restriction of ψ to a hypersurface in $\mathbf{M}$ is a generalized Killing spinor. We say that a Riemannian spin manifold M with a generalized Killing spinor ψ has the *embedding property* if the converse holds, i.e. if M can be isometrically embedded as a hypersurface in a Riemannian spin manifold $\mathbf{M}$ in such a way that ψ extends to a parallel spinor on $\mathbf{M}$. In [25], Hitchin proved that compact half-flat and cocalibrated manifolds have the embedding property, and in these cases the geometry of $\mathbf{M}$ is locally determined by the geometry of M. In [6], Bär, Gauduchon and Moroianu proved an analogous result in arbitrary dimension, but with the hypothesis that ∇A is totally symmetric, generalizing a result of Friedrich regarding surfaces [19].

Another source of motivation is the study of invariant geometric structures on nilmanifolds. Nilmanifolds of dimension up to 7 are completely classified, and there is some interest in classifying those which admit certain types of invariant structures. For example, symplectic or complex structures on 6-dimensional nilmanifolds were listed by Salamon in [32]; on the other hand, half-flat nilmanifolds escape full classification to this day, though classifications have been carried out by Chiossi-Swann [16] and Chiossi-Fino [14] for half-flat structures satisfying additional conditions.

The last piece of motivation is the search for explicit metrics. The above mentioned theory of Hitchin can be effectively used to construct explicit metrics with holonomy G_2 or $\mathrm{Spin}(7)$, starting from a half-flat 6-manifold or a cocalibrated G_2 7-manifold, respectively. However, the resulting manifold is not compact, nor is the metric in general complete. Another technique to construct explicit special metrics was introduced by Salamon in [31]. The idea is to start with a G-structure, and look for invariant special geometries on the total space of an associated vector bundle; this is achieved by writing down a "dictionary" of invariant forms on the vector bundle. This dictionary does not depend on the base manifold, but only on the fibre and tangent space as representations of G; however, the action of d does depend on the geometry of the manifold. This technique, in contrast to the other one we mentioned, can be quite effective when looking for complete metrics: indeed, the first explicit examples of complete G_2-holonomy metrics were obtained in this way.

We now summarize the contents of this thesis. In Chapter 1 we explain in detail the background that we sketched at the beginning of this introduction; no new results appear in this chapter. We explain how in low dimensions, a spinor ψ on a Riemannian manifold M defines a G-structure whose intrinsic torsion can be identified with $\nabla\psi$ (Proposition 1.35). In higher dimensions, the stabilizer of a spinor may vary from point to point: in general, for this result to hold one must require the spinor to lie in the same $\mathrm{Spin}(n)$-orbit at each point (Proposition 1.9). We then illustrate the well-known case of Killing spinors, which can be regarded as generalized Killing spinors with A a constant multiple of the identity. Complete Riemannian manifolds admitting a Killing spinor are classified [5], and they trivially satisfy the Bär-Gauduchon-Morianu theorem: indeed, in this case $\mathbf{M}$ is the cone on M (Theorem 2.22). We also introduce almost-contact metric structures, the odd-dimensional Nijenhuis tensor and Sasaki structures, motivated by the fact that a generalized Killing spinor in dimension 5 defines an almost-contact metric structure.

In Chapter 2 we study the local geometry of hypersurfaces inside spin Riemannian manifolds $\mathbf{M}$ admitting a parallel spinor, as well as abstract G-structures defined by a generalized Killing spinor. We show that the latter are characterized by the condition that their intrinsic torsion is a section of a vector bundle isomorphic to $\mathrm{Sym}(TM)$ (Theorem 2.6). More specifically, for a hypersurface in $\mathbf{M}$ as above, the intrinsic torsion of the G-structure induced by the integrable structure on $\mathbf{M}$ can be identified with the Weingarten tensor (Theorem 2.7). Then, we proceed to study the particular case of "hypo" structures, namely $\mathrm{SU}(2)$-structures defined by a generalized Killing spinor

on a 5-manifold. By construction, hypo structures generalize Einstein-Sasaki structures; more generally, by a theorem of Friedrich and Kim, η-Einstein-Sasaki manifolds carry a generalized Killing spinor (Theorem 1.28), thereby defining a hypo SU(2)-structure, which however is not in general compatible with the Sasaki U(2)-structure. Having identified the Nijenhuis tensor, we prove that compatibility between hypo and Sasaki structures can only occur if the metric is Einstein (see Corollary 2.11, that also characterizes hypo structures which are quasi-Sasakian).
In six dimensions, the existence of a parallel spinor means that the holonomy is contained in SU(3); in other words, parallel spinors correspond to Calabi-Yau geometry. However, the 6-manifolds we consider need not be compact, or simply-connected, and for this reason we shall avoid the term Calabi-Yau. We consider hypo structures on hypersurfaces M inside SU(3)-holonomy manifolds $\mathbf{M}$; we show how the curvature of $\mathbf{M}$ at points of M is determined by the intrinsic torsion of the hypo structure (i.e. the Weingarten tensor) and the curvature of M (Proposition 2.20). Whilst relations of this type exist for general hypersurfaces (namely, the Gauss and Codazzi-Mainardi equations), in this special case the relation is pointwise. This suggests that like in dimensions 6 and 7, one should be able to reconstruct the SU(3)-holonomy manifold from the hypersurface; this idea will be the basis for the discussion of the embedding property in Chapter 5. A review of known results about the embedding property in any dimension follows. In particular, we prove that a generalized Killing spinor in dimension 6 characterizes half-flat SU(3)-structures (Proposition 2.30), so that Hitchin's embedding theorem applies. In fact, we shall define half-flat structures using generalized Killing spinors, and then use Proposition 2.30 to prove consistency with the standard definition.

Chapter 3 is independent of Chapter 2; it is aimed at understanding Salamon's "dictionary" technique from a theoretical point of view. We have explained that special geometries can be defined using differential forms; finding special geometries on a manifold is a problem of solving a system of PDE on the space of differential forms. This problem is made tractable by restricting to a space of invariant forms, as we explained. We did not, however, explain the meaning of "invariant"; in fact, invariance is not explicitly mentioned in [31], and one of the goals of Chapter 3 is to define this notion. We start with a G-structure P on a manifold M, and look for a space of forms on an associated vector bundle. The choice of a connection makes the pullback of P to the bundle into a G-structure; the action on P of its gauge

group yields an action on this G-structure. In Theorem 3.4 we characterize the algebra of forms invariant under this action. It turns out that this algebra is useless from our point of view, because it is not d-closed. We address this issue by considering the subalgebra of invariant parallel forms, where parallel means as sections of a bundle over M; this subalgebra corresponds to Salamon's dictionary. In Theorem 3.8, we prove that it is d-closed provided that the connection on P has parallel torsion. An important example is when M is a homogeneous space: the canonical connection of the second kind has parallel torsion, and invariant parallel forms coincide with left-invariant forms.
In Theorem 3.12 we prove that the algebra of invariant parallel forms is finitely generated over the algebra of invariant parallel functions; this attractive property enables us to convert the system of PDE defining a special geometry into a system of PDE on this algebra of functions. By our characterization, these functions can be identified with G-invariant functions on the fibre. In particular, when the fibre has a one-dimensional section (namely, a submanifold intersecting each orbit), these PDE are really ODE.
In applications, one needs to compute a family of generators for the algebra of invariant parallel forms, and the action of d on them. To achieve this, it is convenient to go back to the notion of dictionary: the philosophy is to define an "alphabet" of invariant, parallel forms, and write each generator as a "word" in that alphabet; this makes things simpler, because in general there are many more independent words than letters. Having produced a list of independent words, one can use Theorem 3.12 to determine whether any generators are missing.
The letters of the alphabet are forms taking values in a vector bundle, for which the usual d is not defined; however, using the pullback connection one can define the exterior covariant derivative D. By construction, in order to compute the action of d, it is sufficient to compute the action of D on the alphabet. In Lemma 3.23 we give an explicit formula for D, in the case of a homogeneous space.

In Chapter 4 we apply the techniques of Chapter 3 to produce invariant special geometries. In all cases considered here, the base manifold is a homogeneous space; as we mentioned, invariance then means invariance under a global action. The algebra of invariant forms can be used to classify special geometries invariant under this action.
We start with the cotangent bundle on the two-sphere, on which Eguchi and Hanson constructed an SO(3)-invariant hyperkähler structure [18]; after giving a characterization of hyperkähler 4-manifolds in terms of differ-

ential forms (Lemma 4.2), we write down this structure in our language. We then apply a technique of Apostolov and Salamon [3] to produce a new G_2-holonomy metric on $T^*S^3 \times \mathbb{R}$, viewed as a three-dimensional bundle over the hyperkähler four-manifold.
The following example is the bundle of anti-selfdual two-forms on the four-sphere, on which Bryant and Salamon constructed an SO(5)-invariant complete G_2-holonomy metric [13], which we write down in our language. The sphere bundle inside this vector bundle is the twistor space $\mathbb{CP}^3$ over S^4 [4]; invariant forms on $\mathbb{CP}^3$ are obtained by restricting invariant forms on the vector bundle (Theorem 4.8). We prove that the standard Kähler structure on $\mathbb{CP}^3$ has no invariant SU(3) reduction; indeed, no invariant SU(3)-structure on $\mathbb{CP}^3$ is a complex structure (Theorem 4.11). On the other hand, invariant SU(3)-structures exist, and consistent with the existence of G_2-holonomy metrics on the 7-dimensional bundle, they are half-flat (Theorem 4.9). One of them is nearly-Kähler; its underlying almost complex structure was studied in [17]. This nearly-Kähler structure gives rise to a conical G_2-holonomy metric on the 7-dimensional bundle which we write down explicitly.

Chapter 5 is independent of Chapter 3, and uses little of Chapter 4; it deals with hypo geometry, i.e. the geometry defined by a generalized Killing spinor in five dimensions. In Proposition 5.7 we characterize hypo SU(2)-structures using differential forms. We go on to discuss the embedding property for compact hypo manifolds. Roughly speaking, one can think of hypo manifolds as Riemannian manifolds (M, g_5) such that $g_6 = g_5 + d\theta^2$ is a half-flat metric on $M \times S^1$. By Hitchin's theorem, there exists a G_2-holonomy metric g_7 on $M \times S^1 \times (a, b)$, and because g_6 is S^1-invariant, so is g_7. One can then take the quotient to obtain a metric on $M \times (a, b)$, which has holonomy SU(3) provided that the S^1 orbits have constant length (see e.g. [3]). This condition turns out to hold for a large class of hypo manifolds, but the proof is more complicated than one might expect. Our strategy is based on Hitchin's: suppose that $M \times (a, b)$ has an integrable SU(3)-structure. Then, each $M \times \{t\}$ has an induced hypo structure; in other words, we obtain a one-parameter family of hypo structures on M. The embedding property translates into the existence of integral lines for a flow on the space of deformations of the starting hypo structure.
This space of deformations is defined in terms of forms, because if we were working with spinors, we would have to allow deformations of the spin structure, which are more complicated to handle; the drawback of this choice is that at least three forms are needed to define an SU(2)-structure. As we

have mentioned, when more than a single form is needed, deformations have to satisfy compatibility conditions; so, it is not automatic that the space of deformations of a hypo structure is smooth. However, if it is, we prove that the embedding property holds (Theorem 5.20).

This result is consistent with Proposition 2.20, because as in the half-flat case, if M is a compact hypersurface in $\mathbf{M}$, then the structure on $M \times (a, b)$ given by the flow coincides with the pulled-back structure relative to the geodesic embedding of $M \times (a, b)$ in a tubular neighbourhood of M, up to restricting (a, b). Thus, the geometry of M determines the geometry of $\mathbf{M}$ in an explicit way.

In the proof of the embedding theorem, $\mathrm{SU}(2)$ is viewed as the intersection of two copies of $\mathrm{SL}(2, \mathbb{C})$; this partly motivates a digression on the intrinsic torsion of $\mathrm{SL}(2, \mathbb{C})$-structures. While this is not a special geometry in our sense, it shares an important feature with special geometries: an $\mathrm{SL}(2, \mathbb{C})$-structure can be defined using two differential forms ω and ψ, and the intrinsic torsion is determined by $d\omega$ and $d\psi$ (Proposition 5.16). In Theorem 5.17 we consider the almost contact structure underlying an $\mathrm{SL}(2, \mathbb{C})$-structure, relating the Nijenhuis tensor of the former to the intrinsic torsion of the latter.

We then give a complete classification of invariant hypo structures on nilmanifolds (Theorem 5.22). This is one on the main results of this thesis; we recall that the analogous problem in six dimensions is still open.

In Proposition 5.29 we argue that in the nilmanifold case, the hypothesis of Theorem 5.20 applies for almost every hypo structure.

Chapter 1

Spinors, G-structures and intrinsic torsion

The purpose of this chapter is to introduce the language and background for the rest of the thesis; it concerns G-structures defined by spinors and their intrinsic torsion.

The first section is purely algebraic; it contains some basic facts in the theory of Clifford algebras and spinor representations, as well as the definition of the groups $\mathrm{Spin}(n)$.

In the second section we apply these notions to define spinors (sections of spinor bundles). We introduce G-structures, and show how spinors, or tensors, may be used to define them. We also introduce holonomy, and state Berger's classification theorem.

In the third section we consider the simplest case of structures defined by a spinor, i.e. the one in which the spinor is parallel with respect to the Levi-Civita connection, so that the structure is integrable. This situation is more special than the one considered in Berger's theorem, and only three possibilities arise: exceptional holonomy, hyperkähler geometry, or Calabi-Yau geometry. We give some basic results on the first two, leaving the third to be discussed in Chapter 2.

The fourth section deals with the next simplest case: structures defined by a Killing spinor, which are also classified. Reflecting the fact that the cone on a manifold with a Killing spinor has a parallel spinor, we also have three possibilities here: nearly-Kähler and nearly-parallel G_2 geometry, corresponding to exceptional holonomy on the cone; 3-Sasaki geometry, corresponding to hyperkähler geometry on the cone; Einstein-Sasaki geometry, corresponding

to Calabi-Yau geometry on the cone. Spheres with the standard metric also carry Killing spinors, but they are somehow a "degenerate" case, because the cone is flat space, which is reducible.
Motivated by the classification of Killing spinors, in the fifth section we define Sasaki structures, and more generally, almost contact structures. In Chapters 2 and 5 we shall study a generalization of η-Einstein-Sasaki structures; in the fifth section we give some definitions motivated by the need to relate this new type of geometry to the known ones.
In the sixth section we compute explicitly the spinor representations in dimension 5, 6, and 7. We exploit the fact that we are only interested in these dimensions to give an ad hoc construction of these representations, based on the action of $\mathrm{SU}(4) = \mathrm{Spin}(6)$ on self-dual 2-forms on $\mathbb{C}^4$.
In the seventh section we define intrinsic torsion, and some related language. In particular we prove that the intrinsic torsion of a G-structure defined by a spinor ψ can be identified with $\nabla\psi$, where ∇ is the Levi-Civita connection. The same holds if ψ is a tensor, or an r-tuple of forms, and G is a subgroup of $\mathrm{SO}(n)$.

1.1 Spinor algebra

Spinors over a manifold M are geometrical entities which arise as sections of a vector bundle over M with structure group $\mathrm{Spin}(n)$. Whilst $\mathrm{Spin}(n)$ is the non-trivial double covering of $\mathrm{SO}(n)$, i.e. its universal covering, the representation we need is not the pullback of a representation of $\mathrm{SO}(n)$. Indeed, it is the restriction of the representation of an algebra, called the Clifford algebra, which also contains $\mathbb{R}^n$, with the consequence that one can "multiply" a spinor by a vector. We shall explain the construction of Clifford algebras, without giving the details, for which we refer to [28] or [7].

Let V be an n-dimensional vector space over $k = \mathbb{R}, \mathbb{C}$ with a fixed symmetric bilinear form $\langle\ ,\rangle$, which we assume to be positive-definite for $k = \mathbb{R}$ and non-degenerate for $k = \mathbb{C}$.

Definition 1.1. An algebra $\mathrm{Cl}(V)$ is said to satisfy the *Clifford algebra universal property* for V if any linear map $f : V \to \mathcal{A}$ into an associative k-algebra with unit such that

$$f(v) \cdot f(v) = -\langle v, v\rangle 1 \quad \forall\, v \in V$$

extends uniquely to a k-algebra homomorphism $\mathrm{Cl}(V) \to \mathcal{A}$.

It is clear that two algebras satisfying this property are isomorphic. To prove that such algebras exist, consider the tensor algebra $\mathcal{T}(V)$, and let $\mathcal{I}$ be the ideal generated by elements of the form $v \otimes v + \langle v, v \rangle 1$. Then

$$\mathrm{Cl}(V) = \mathcal{T}(V)/\mathcal{I}$$

is easily seen to satisfy Definition 1.1; it is called the *Clifford algebra* of V. One can also check that the inclusion $V \to \mathcal{T}(V)$ induces an inclusion $V \to \mathrm{Cl}(V)$. Hence, every element of $\mathrm{O}(V)$ induces a map $V \to \mathrm{Cl}(V)$ satisfying the universal property, and therefore induces an automorphism of $\mathrm{Cl}(V)$. Moreover, any action of $\mathrm{Cl}(V)$ on a vector space Σ induces by restriction a bilinear map $V \otimes \Sigma \to \Sigma$, known as *Clifford multiplication.*
Now let α be the automorphism of $\mathrm{Cl}(V)$ induced by $-1 : V \to V$; then $\alpha^2 = 1$ and $\mathrm{Cl}(V)$ splits as

$$\mathrm{Cl}(V) = \mathrm{Cl}^0(V) \oplus \mathrm{Cl}^1(V) ,$$

where $\mathrm{Cl}^0(V)$ is the 1 eigenspace and $\mathrm{Cl}^1(V)$ is the -1 eigenspace. This gives $\mathrm{Cl}(V)$ a $\mathbb{Z}_2$ grading, or in other words, a superalgebra structure.

Definition 1.2. The group $\mathrm{Pin}(V)$ is the subgroup of $\mathrm{Cl}(V)$ generated by unit elements of V. The group $\mathrm{Spin}(V)$ is defined by

$$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}^0(V) .$$

We write $\mathrm{Spin}(n)$ for $\mathrm{Spin}(\mathbb{R}^n)$ where $V = \mathbb{R}^n$ with the standard metric; we can now state this fundamental result:

Theorem 1.3. *There is a short exact sequence*

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(n) \to 0 ,$$

where

$$\mathrm{Ad}(g) \cdot v = gvg^{-1} , \quad v \in \mathbb{R}^n .$$

It follows that a representation of $\mathrm{Spin}(n)$ descends to a representation of $\mathrm{SO}(n)$ if and only if -1 acts as the identity.

Remark. The derivative of Ad acts in the following way:

$$\begin{gathered} \mathrm{Ad}_* : \mathfrak{spin}(n) \to \mathfrak{so}(n) \cong \Lambda^2(\mathbb{R}^n) \\ e_i e_j \to 4\, e^i \wedge e^j \end{gathered}$$

where

$$e^i \wedge e^j = \frac{1}{2}\left(e^i \otimes e^j - e^j \otimes e^i\right) \ . \tag{1.1}$$

Other choices of the coefficients also appear in the literature.

Having defined $\mathrm{Spin}(n)$, we need not consider the case $k = \mathbb{R}$ any longer; from now on, let $k = \mathbb{C}$ and $V = \mathbb{C}^n$ with the standard basis $e_1, \dots, e_n$. This case is made simpler by the existence of an element ω of $\mathrm{Cl}(V)$ called the *complex volume element*, defined by

$$\omega = i^{\left[\frac{n+1}{2}\right]} e_1 \cdots e_n \ ,$$

satisfying $\omega^2 = 1$.

There is a classification of the algebras $\mathrm{Cl}(V)$, from which the following follows:

Theorem 1.4. *Every representation of* $\mathrm{Cl}(V)$ *is completely reducible. If n is even, then all irreducible representations of* $\mathrm{Cl}(V)$ *are equivalent, and if n is odd, there are exactly two inequivalent irreducible representations of* $\mathrm{Cl}(V)$*. The complex dimension of an irreducible* $\mathrm{Cl}(V)$*-module is* $2^{[n/2]}$*.*

Remark. In the case of n odd, ω is central and therefore every $\mathrm{Cl}(V)$-module splits into the $+1$ and -1 eigenspaces of ω, so that on every irreducible representation, ω acts as either the identity or minus the identity; Theorem 1.4 asserts that this is all that distinguishes two inequivalent irreducible representations. As $\alpha(\omega) = -\omega$, these representations give one another by composing with α.

Now write $\mathbb{C}^n = \mathbb{C}^{n-1} \oplus \mathbb{C}$, where $\mathbb{C}^{n-1}$ is spanned by $e_1, \dots, e_{n-1}$. The map

$$\mathbb{C}^{n-1} \ni e_i \to e_n e_i \in \mathbb{C}^n$$

is an isometry and extends to an isomorphism

$$\mathrm{Cl}(\mathbb{C}^{n-1}) \cong \mathrm{Cl}^0(\mathbb{C}^n) \ . \tag{1.2}$$

From this fact and Theorem 1.4, it easily follows:

Corollary 1.5. *Let Σ be an irreducible representation of* $\mathrm{Cl}(V)$*; restrict it to a representation* (Σ, ρ) *of* $\mathrm{Cl}^0(V)$*. Then the equivalence class of* (Σ, ρ) *is independent of Σ. If n is odd,* (Σ, ρ) *is also irreducible. If n is even,* (Σ, ρ) *splits up into two inequivalent irreducible representations, which are interchanged by Clifford multiplication by any $v \in V$.*

For n odd, let Σ_n be the irreducible representation of $\mathrm{Cl}(V)$ on which ω acts as the identity. For n even, let Σ_n be the irreducible representation of $\mathrm{Cl}(V)$: under the isomorphism (1.2), using Corollary 1.5 we see that

$$\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$$

where $\Sigma_n^\pm$ is a $\mathrm{Cl}^0(V)$-module on which the complex volume form of $\mathrm{Cl}(\mathbb{C}^{n-1})$ acts as ± 1. By construction, Σ_n^+ pulls back to Σ_{n-1} for n even and Σ_n pulls back to Σ_{n-1} for n odd. Explicitly, $\mathrm{Cl}(\mathbb{C}^{n-1})$ acts on Σ_n by

$$X \odot \phi = e_n \cdot X \cdot \phi \; . \tag{1.3}$$

Consider the sequence of inclusions

$$\mathrm{Spin}(n) \subset \mathrm{Cl}^0(\mathbb{R}^n) \subset \mathrm{Cl}(\mathbb{R}^n) \subset \mathrm{Cl}(\mathbb{C}^n) \; ;$$

the representation Σ_n restricts to

$$\Delta_n : \mathrm{Spin}(n) \to \mathrm{GL}_{\mathbb{C}}(\Sigma_n) \; .$$

By Corollary 1.5, Δ_n does not depend on the choice of Σ; it is called the *complex spinor representation*. Moreover if n is odd, (Σ_n, Δ_n) is irreducible; if n is even, Σ_n splits up into two inequivalent irreducible representations (Σ_n^+, Δ_n^+) and (Σ_n^-, Δ_n^-), where the complex volume form of $\mathrm{Cl}(\mathbb{C}^{n-1})$ acts as ± 1 on $\Sigma_n^\pm$.

Remark. Clifford multiplication can be pulled back to $\mathcal{T}(V)$, and in particular to $\Lambda(V)$; explicitly,

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \cdot \psi = e_{i_1} \cdot \cdots \cdot e_{i_p} \cdot \psi \; .$$

So, there are two actions of $\Lambda^2(V)$ on the space of spinors: Clifford multiplication $\cdot$, and the infinitesimal action Ad_*^{-1} of $\mathfrak{so}(V)$. We have

$$e_{ij} \cdot \psi = 4 \, \mathrm{Ad}_*^{-1}(e_{ij}) \psi \; .$$

1.2 G-structures and holonomy

In this section we explain how spinors can be used to define G-structures; we introduce holonomy and we state Berger's classification theorem.

We assume the reader is familiar with the notion of principal bundle. We recall that if $\rho : H \to G$ is a group homomorphism and P is a principal bundle over M with fibre G, a reduction of P to H via ρ is a principal bundle P_H over M with fibre H with a map $f : P_H \to P$ such that

$$\begin{tikzcd} P_H \arrow[rr, "f"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$

commutes, and such that $R_{\rho(h)} \circ f = f \circ R_h$.
Fix an n-dimensional manifold M; the bundle of frames is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle $P_{\mathrm{GL}(n,\mathbb{R})}$.

Definition 1.6. A spin structure on M is a reduction to $\mathrm{Spin}(n)$ of the bundle of frames $P_{\mathrm{GL}(n,\mathbb{R})}$ via

$$\mathrm{Ad} : \mathrm{Spin}(n) \to \mathrm{SO}(n) < GL(n, \mathbb{R}) ,$$

where Ad is the map defined in Theorem 1.3.

The obstruction to the existence of a spin structure on an orientable M is a topological invariant called the second Stiefel-Whitney class, which is an element w_2 of $H^2(M, \mathbb{Z}_2)$ (see e.g. [24]); an orientable manifold with $w_2 = 0$ is called *spin*.

Remark. Any principal bundle P with fibre $\mathrm{Spin}(n)$ determines a $P_{\mathrm{SO}(n)}$ bundle such that P is a reduction of $P_{\mathrm{SO}(n)}$; if P is a reduction of $P_{\mathrm{GL}(n,\mathbb{R})}$, then so is $P_{\mathrm{SO}(n)}$. Hence, a spin structure determines a metric and orientation.

Now assume M has a spin structure P; for any $\mathrm{Spin}(n)$-module V, we denote by $\underline{V}$ the vector bundle $P \times_{\mathrm{Spin}(n)} V$. Recall the definition of Σ_n from Section 1.1; then $\underline{\Sigma_n}$ is a complex vector bundle called the *complex spinor bundle* and its sections are called *spinors*. One can also define the *complex Clifford bundle*

$$\mathrm{Cl}(M) = P \times_{\mathrm{Spin}(n)} \mathrm{Cl}(\mathbb{C}^n) ,$$

where $\mathrm{Spin}(n)$ acts on $\mathrm{Cl}(\mathbb{C}^n)$ via Ad. Note that Σ_n is not a representation of $\mathrm{SO}(n)$, but $\mathrm{Cl}(\mathbb{C}^n)$ is; therefore one could in theory define $\mathrm{Cl}(M)$ using the $\mathrm{SO}(n)$-bundle of frames, but not $\underline{\Sigma}$.
Since the map $\mathrm{Cl}(\mathbb{C}^n) \otimes \Sigma_n \to \Sigma_n$ is $\mathrm{Spin}(n)$-equivariant, a map

$$\mathrm{Cl}(M) \otimes \underline{\Sigma_n} \to \underline{\Sigma_n}$$

is induced; this gives $\underline{\Sigma_n}$ a structure of $\mathrm{Cl}(M)$-module. Since the map

$$\mathcal{T}(\mathbb{R}^n) \to \mathrm{Cl}(\mathbb{R}^n) \to \mathrm{Cl}(\mathbb{C}^n)$$

is also equivariant, a map $\underline{\mathcal{T}(\mathbb{R}^n)} \otimes \underline{\Sigma_n} \to \underline{\Sigma_n}$ is induced, called *Clifford multiplication* and denoted by $\cdot$; the same holds for $\mathcal{T}((\mathbb{R}^n)^*)$. It will sometimes be useful to view Clifford multiplication as a 1-form $\underline{C}$ in $\Omega^1(M, \operatorname{End}\Sigma_n)$. The notation here is that if $\underline{V}$ is a vector bundle over M,

$$\Lambda^p(M, \underline{V}) \cong \Lambda^p(M) \otimes \underline{V}$$

denotes the bundle of $\underline{V}$-valued p-forms, and $\Omega^p(M, \underline{V})$ the space of its sections.

Our interest in spinors descends from the fact that they can be used to define certain types of G-structures, as we now explain. Recall this standard definition:

Definition 1.7. For G a subgroup of $\mathrm{GL}(n, \mathbb{R})$, a G-structure on M is a reduction to G of the $\mathrm{GL}(n, \mathbb{R})$-bundle of frames.

Now let P_K be a principal bundle over M with fibre K and let V be a K-module (either real or complex). Fix η_0 in V with stabilizer G. Then the following important, though trivial, result holds:

Proposition 1.8. *The sections η of*

$$P_K \times_K (K\eta_0) \subset \underline{V}\ ,$$

where $K\eta_0$ is the orbit of η_0, are in one-to-one correspondence with the reductions P_G of P_K to G. This correspondence, which depends on the choice of η_0, is determined by the condition that for any section s of P_G,

$$\eta = [s, \eta_0]\ .$$

We think of such an η as a section of $\underline{V}$ which satisfies the algebraic condition of being a section of the subbundle $P_K \times_K (K\eta_0)$.

If $\lambda : G \to K$ is a homomorphism, then any principal bundle P_G with fibre G determines a principal bundle P_K with fibre K by

$$P_K = P_G \times_G K\ ,$$

where G acts on K via λ; P_K is called the λ-*extension* of P_G.
Now suppose that G is a subgroup of $\mathrm{Spin}(n)$ which does not contain -1, e.g. $G = \mathrm{G}_2$, $\mathrm{SU}(3)$, $\mathrm{SU}(2)$ in dimension 7, 6, 5 respectively; then Ad is injective on G, which can therefore be viewed as a subgroup of $\mathrm{SO}(n)$. In this case a G-structure P_G determines a spin structure by extension, and P_G can be viewed as a reduction to G of this spin structure. Conversely, a reduction to G of a spin structure on M can be viewed as a G-structure. In conclusion, when considering groups G of this kind we can assume that M has a spin structure P and interpret G-structures as reductions of P. As a consequence:

Proposition 1.9. *If G is the stabilizer of $\psi_0 \in \Sigma_n$, then G-structures on M are in one-to-one correspondence with pairs (P, ψ), where P is a spin structure and ψ a spinor taking values in $P \times_{\mathrm{Spin}(n)} (\mathrm{Spin}(n)\psi_0)$.*

Proof. By construction, -1 does not lie in G and we can apply the above argument. Proposition 1.8 concludes the proof. □

Recall that if P is a principal bundle on M with fibre G, a *connection* on P is an invariant distribution H such that $TP = H \oplus \ker \pi_*$, where $\pi : P \to M$ is the projection; then one can define parallel transport, and the covariant derivative

$$\nabla : \Gamma(TM \otimes \underline{V}) \to \Gamma(\underline{V})$$

for any G-module V, where Γ means "sections of". Any spin structure P carries a canonical connection called the Levi-Civita connection; all the principal bundles we consider in this chapter are reductions $i : P_G \to P$ of a spin structure, and unless otherwise stated, covariant derivative will be taken with respect to the Levi-Civita connection. However, this only makes sense when V is a $\mathrm{Spin}(n)$-module. We say that the Levi-Civita connection H *reduces* to P_G if

$$TP_G = H_G \oplus \ker \pi_* , \qquad \text{where } H_G = i_*^{-1}(H) .$$

In that case, the G-structure P_G is called *integrable*. Since H_G is a connection on P_G, one can then define ∇ for any G-module V.

Definition 1.10. The Levi-Civita connection is said to have *holonomy* G, where G is a subgroup of $\mathrm{SO}(n, \mathbb{R})$, if P has an integrable reduction to G, which admits no integrable reductions in turn.

By definition, the holonomy group G is a subgroup of $\mathrm{SO}(n, \mathbb{R})$ defined only up to conjugation; uniqueness up to conjugation follows from integrable structures being closed under parallel transport, and the $\mathrm{SO}(n)$-invariance of the concept of parallel transport.
We say that a Riemannian manifold is *irreducible* if it is not locally a Riemannian product. We say that it is *locally symmetric* if $\nabla R = 0$, where R is the curvature tensor.

Theorem 1.11 (Berger's Theorem)**.** *Let M be an oriented simply-connected n-dimensional Riemannian manifold which is irreducible and not locally symmetric. Then its holonomy group must equal one of* $\mathrm{SO}(n)$, $\mathrm{U}(\frac{n}{2})$, $\mathrm{SU}(\frac{n}{2})$, $\mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, $\mathrm{Sp}(\frac{n}{4})$, G_2, $\mathrm{Spin}(7)$.

This theorem was first proved in [8]; a different proof can be found in [29]. To be historically correct, Berger's list also contained Spin(9), which was later proved to only appear as the holonomy group of locally symmetric spaces.

1.3 Parallel spinors

In the language of Proposition 1.8, we say that η is *parallel* if $\nabla\eta = 0$. Quite unsurprisingly, a G-structure defined by a parallel η is integrable; we shall prove a stronger result in Section 1.7. Moreover, a tensor η which is parallel with respect to any connection always satisfies the condition of Proposition 1.8, and therefore defines a G-structure.
In particular, G-structures defined by a parallel spinor are classified [33, 24]:

Theorem 1.12. *Let M be an n-dimensional simply connected, irreducible Riemannian spin manifold. Then M carries a parallel spinor if and only if its holonomy group is one of:*

n	*holonomy*
7	G_2
8	$\mathrm{Spin}(7)$
$2m$	$\mathrm{SU}(m)$
$4m$	$\mathrm{Sp}(m)$

Sketch of proof. The proof is based on Berger's theorem, which applies for the following reason: the existence of a parallel spinor implies Ricci-flatness, and a Ricci-flat manifold can only be locally symmetric if it is flat, which is contrary to the hypothesis. Therefore, Berger's theorem provides a list of the possible holonomy groups; the possibilities of $\mathrm{U}(m)$, $\mathrm{SO}(m)$ and $\mathrm{Sp}(m)\mathrm{Sp}(1)$ are ruled out by the lack of fixed points for their action on Σ_n. $\square$

We now list some basic facts about each geometry appearing in Theorem 1.12, except Calabi-Yau geometry, which will be discussed in Chapter 2; we refer to [31] for proofs and details.
We start with G_2.

Definition 1.13. Let $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$ be defined by

$$\varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \tag{1.4}$$

where $e^1, \ldots, e^7$ is the standard basis of $(\mathbb{R}^7)^*$, with the convention that

$$e^{a_1 \ldots a_n} = e^{a_1} \wedge \cdots \wedge e^{a_n} \ .$$

Let $\mathrm{GL}(7, \mathbb{R})$ act on $\Lambda^3(\mathbb{R}^7)^*$ in the usual way. We define $\mathrm{G}_2 = \mathrm{Stab}(\varphi_0)$.

The following is well known (see e.g. [12]):

Proposition 1.14. G_2 *is a connected, simply connected, closed subgroup of* $\mathrm{SO}(7)$.

Clearly G_2, as a closed subgroup of $\mathrm{SO}(7)$, is compact; indeed, this argument works for all holonomy groups.
Proposition 1.8 gives a one-to-one correspondence between G_2-structures on a manifold and three-forms which pointwise "look like φ_0". It should be noted that the representation of G_2 on $\mathbb{R}^7$ we are considering is the same that appears in Berger's theorem. So in dimension 7, in the hypotheses of Theorem 1.12, a parallel spinor determines a parallel 3-form of correct algebraic type and vice versa.

Theorem 1.15 (Fernández-Grey)**.** *Let φ define a G_2-structure. Then $\nabla\varphi$ is determined by $d\varphi$ and $d * \varphi$; in particular,*

$$\nabla\varphi = 0 \iff d\varphi = 0,\ d * \varphi = 0 \ .$$

Notice that the conditions $\nabla\varphi = 0$ and $d * \varphi = 0$ are not linear in φ, because ∇ as well as the Hodge operator depend on the metric, which in turn depends on φ; for this reason, finding explicit G_2 holonomy metrics is not an easy task.

We now turn our attention to Spin(7). With notation from Section 1.6, define a real 4-form on V by

$$\Omega = \operatorname{Re} u_{0123} - \frac{1}{2}(w^{12} + w^{34} + w^{56} + w^{78})^2 \ ;$$

then Spin(7) fixes Ω. More precisely, the spinor representation determines an inclusion $\mathrm{Spin}(7) < \mathrm{SO}(8)$ such that $\mathrm{Spin}(7) = \operatorname{Stab}\Omega$. Moreover, Spin(7) acts transitively on the sphere, with stabilizer G_2; this can be viewed as a consequence of the fact that relative to the decomposition $\mathbb{R}^8 = \mathbb{R}^7 \oplus \mathbb{R}$,

$$\Omega = *\varphi_0 + e^8 \wedge \varphi_0$$

up to a change of basis. We have the following (see [31]):

Theorem 1.16. *Let Ω define a* Spin(7)*-structure. Then $\nabla\Omega = 0$ if and only if $d\Omega = 0$.*

We recall that $\mathrm{Sp}(n)$ is the group of invertible n by n quaternionic matrices; holonomy $\mathrm{Sp}(n)$ corresponds to hyperkähler geometry:

Definition 1.17. A *hyperkähler* structure on a Riemannian manifold (M, g) is a triple of complex structures J_1, J_2, J_3 such that $J_3 = J_1J_2$ and (M, g, J_i) is Kähler for all i.

It follows from the definition that $J_3 = -J_2J_1$; cyclic permutations of this equation also hold. Let ω_i be the Kähler form relative to J_i; then $\omega_1, \omega_2, \omega_3$ are clearly closed. The converse also holds: if one has a metric g and almost complex structures J_i with $J_3 = J_1J_2$ such that (g, J_i) is an almost-Kähler structure, then M is hyperkähler [31].

Proposition 1.18. *A Riemannian manifold (M, g) has holonomy contained in* $\mathrm{Sp}(n)$ *if and only if it admits a hyperkähler structure.*

Proof. A triple J_1, J_2, J_3 as in Definition 1.17 corresponds to a quaternionic structure on the tangent bundle, which in turn corresponds to an $\mathrm{Sp}(n)$-structure. The condition of (M, g, J_i) being Kähler can be stated as $\nabla J_i = 0$, which is equivalent to the integrability of the $\mathrm{Sp}(n)$-structure. □

1.4 Killing spinors

Fix a spin structure P on M such that the induced metric on M is *complete*.

Definition 1.19. A *Killing spinor* is a spinor Ψ such that

$$\nabla_X \Psi = \lambda\, X \cdot \Psi$$

for some constant λ, called the Killing constant of Ψ.

Killing spinors with Killing constant λ are parallel with respect to the modified connection $\tilde{\nabla} = \nabla - \lambda \underline{C}$; Therefore, Killing spinors have constant norm and they define G-structures. Moreover, Killing spinors with a given Killing constant form a finite-dimensional vector space. The following is well known [7]:

Theorem 1.20. *Let M have a non-trivial Killing spinor. Then M is Einstein with scalar curvature $4n(n-1)\lambda^2$.*

It follows that if there exists a non-trivial Killing spinor on M with Killing constant λ, then every Killing spinor has Killing constant $\pm\lambda$. There are three cases:

- $\lambda = 0$: ψ is a parallel spinor.
- $\lambda \in \mathbb{R} \setminus \{0\}$: ψ is called a *real Killing spinor*.
- $\lambda \in i\mathbb{R} \setminus \{0\}$: ψ is called an *imaginary Killing spinor*.

So, parallel spinors only exist on Ricci-flat manifolds, as previously remarked. By Theorem 1.20 and Myers' theorem, non-trivial real Killing spinors only exist on compact, positive-scalar-curvature M. On the other hand, non-trivial imaginary Killing spinors only exist on non-compact, negative-scalar-curvature M.

Remark. In even dimension, let $\psi = \psi^+ + \psi^-$ be a Killing spinor; then

$$\nabla_X \psi^+ = \lambda X \cdot \psi^- \,, \qquad \nabla_X \psi^- = \lambda X \cdot \psi^+ \,,$$

because Clifford multiplication by a vector swaps Σ_n^+ and Σ_n^-. Therefore, $\psi^+ - \psi^-$ is a distinct Killing spinor with Killing constant $-\lambda$. In general, for any nonzero real constant k one can replace $P_{\mathrm{SO}(n)}$ with the bundle $kP_{\mathrm{SO}(n)}$, defined as

$$(kP_{\mathrm{SO}(n)})_x = \{ku : \mathbb{R}^n \to T_x M \mid u \in (P_{\mathrm{SO}(n)})_x\} \ ;$$

this amounts to rescaling the metric, as well as changing the orientation if $k < 0$ and n is odd. Now, if one identifies the spinor $[s, \psi_0]$ with $[ks, \psi_0]$, $\nabla_X \Psi$ remains unchanged, but $X \cdot \Psi$ is divided by k; hence, the Killing constant of $[ks, \psi_0]$ is $k\lambda$. In other words, one can always adjust things to normalize the Killing constant.

There is a classification of manifolds carrying real Killing spinors, which was completed in [5]. In order to explain the classification, we need to recall two definitions:

Definition 1.21. A *nearly Kähler* structure on a Riemannian manifold is a compatible almost complex structure J such that $(\nabla_X J)X = 0$, but $\nabla J \neq 0$.

With slight abuse of notation, we shall say that a Riemannian manifold M is nearly Kähler if it admits a compatible nearly Kähler structure. With a similar abuse of notation, we shall provisionally define Sasaki and 3-Sasaki manifolds using cones (for a more precise definition, see Section 1.5). If (M, g) is a Riemannian manifold and I is an interval, we define the *warped product*

$$M \times_f I$$

as the Riemannian manifold

$$(M, f(r)^2 g + dr^2) \ ,$$

where r is the coordinate on I. We write $\mathbb{R}^+$ for the interval $(0, +\infty)$.

Definition 1.22. A Riemannian manifold M is *Sasaki* if $M \times_r \mathbb{R}^+$ is Kähler; if M is also Einstein, it is called *Einstein-Sasaki.*

Definition 1.23. A Riemannian manifold M is *3-Sasaki* if $M \times_r \mathbb{R}^+$ is hyperkähler.

Definition 1.24. A G_2-structure is *nearly-parallel* if $d\varphi = *\varphi$.

Now suppose M is simply connected; let the space of spinors with Killing constant $1/2$ have dimension p and the space of spinors with Killing constant $-1/2$ have dimension q. The only possibilities are listed in the following table (where $m \geq 1$):

n	p, q	geometry
n	$2^{[n/2]}, 2^{[n/2]}$	S^n
6	$1, 1$	nearly-Kähler
7	$1, 0$	nearly-parallel G_2
$4m+1$	$1, 1$	Einstein-Sasaki
$4m+3$	$2, 0$	Einstein-Sasaki
$4m+3$	$m, 0$	3-Sasaki

The next-to-last line of the table, for instance, should be read as follows: if M has dimension $4m+3$, then M is Einstein-Sasaki if and only if $p \geq 2$. This is consistent with the fact that 3-Sasaki manifolds and odd-dimensional spheres are Einstein-Sasaki.

It may be worth mentioning that Einstein-Sasaki manifolds, as well as 6-dimensional nearly-Kähler manifolds, are known to have scalar curvature equal to $n(n-1)$; this can be seen as a consequence of the classification and Theorem 1.20, because we are assuming the Killing constant to be $\pm 1/2$. On the other hand if one has a non-trivial space of real Killing spinors with $\lambda \neq \pm 1/2$, one may rescale the metric and then apply the classification.

1.5 Sasaki structures

The definition of Sasaki geometry we have given is not very satisfactory for two reasons: first, it is rather unnatural to define a property of a Riemannian manifold M in terms of a property of a different Riemannian manifold, in this case $M \times_r \mathbb{R}^+$; second, we have defined as a property of the metric what is really the property of a G-structure. Sasaki structures are the odd-dimensional equivalent of Kähler structures, in the same sense that contact structures are the odd-dimensional equivalent of symplectic structures. In this section we give the precise definition of Sasaki structures; we also introduce some related facts and notation which will be used in the sequel.

An *almost contact metric structure* on a manifold M of dimension $2m+1$ is a $\mathrm{U}(m)$-structure. As

$$\mathrm{U}(m) = \mathrm{Sp}(m, \mathbb{R}) \cap \mathrm{SO}(2m+1) ,$$

we shall think of an almost contact metric structure as a triple (g, α, ω), where g is a Riemannian metric, α is a unit 1-form and ω is a 2-form with norm $\sqrt{m/2}$, such that ω^m is nowhere-vanishing and $\alpha \lrcorner \omega = 0$, where $\lrcorner$ denotes interior product. Locally, this implies the existence of a basis $e^1, \ldots, e^n$ (where $n = 2m+1$) such that

$$\alpha = e^n , \qquad\qquad \omega = e^{12} + \cdots + e^{n-2,n-1} .$$

An almost contact metric structure is *contact* if $d\alpha = -2\omega$. A contact metric structure is *K-contact* if the dual vector to α, which we denote by ξ, is a Killing vector field. The following is well known

Proposition 1.25. *An almost contact metric structure is K-contact if and only if $\nabla\alpha = -2\omega$.*

The condition in Proposition 1.25 means $(\nabla_X \alpha)Y = -2\omega(X, Y)$.
To an almost contact metric structure, much like to a almost-hermitian structure, one can associate the *Nijenhuis tensor*, which is a tensor of type $(2, 1)$. Since $\mathrm{U}(2)$ is a subgroup of $\mathrm{U}(3)$, an almost contact metric structure on M defines an almost-hermitian structure on the product $M \times \mathbb{R}$; then the Nijenhuis tensors of M and $M \times \mathbb{R}$ can be identified. If we define a TM-valued 1-form J by

$$g(\underline{J}(X), \cdot) = X \lrcorner \omega ,$$

N is characterized by

$$N(X, Y) = (\nabla_{\underline{J}X} \underline{J})Y - (\nabla_{\underline{J}Y} \underline{J})X + (\nabla_X \underline{J})(\underline{J}Y) - (\nabla_Y \underline{J})(\underline{J}X) + \\ - \alpha(Y)\nabla_X \xi + \alpha(X)\nabla_Y \xi \quad (1.5)$$

An almost contact metric structure is *normal* if N vanishes, i.e. the induced almost complex structure on $M \times \mathbb{R}$ is integrable.

An almost contact metric structure is *quasi-Sasaki* if N vanishes and ω is closed. A quasi-Sasaki structure is *Sasaki* if it is contact. Sasaki manifolds are K-contact, and can be characterized in the following way:

Theorem 1.26. *An almost contact metric structure is Sasaki if and only if*

$$\nabla_X \omega = g(X, \alpha \wedge \underline{\theta}) , \qquad (1.6)$$

where θ is the solder form.

An almost contact metric structure is *η-Einstein* if

$$\mathrm{Ric} = a\,\alpha \otimes \alpha + b\,g\ ,$$

where a and b are functions. There are two well-known constraints on a and b: their sum must equal $n-1$, and we have the following:

Theorem 1.27. *If M is a K-contact η-Einstein manifold of dimension greater than 3, then a and b are constants.*

The following characterization was proved in [20];

Theorem 1.28 (Friedrich-Kim)**.** *Let M be a simply connected Sasaki manifold. There exists a spinor ψ satisfying*

$$\nabla_X \psi = \lambda\, X \cdot \psi + \mu\, \alpha(X)\alpha \cdot \psi$$

for some constants λ, μ if and only if M is η-Einstein.

1.6 Explicit spinor representations

In this section we give explicit formulae for the spinor representations. We shall start with the 6-dimensional case, which is quite central in the thesis. For this reason, we shall be detailed.

In dimension 6, $\mathrm{Spin}(6) = \mathrm{SU}(4)$; setting $V = \mathbb{C}^4$, the spin representation is $V \oplus \overline{V}$. We recall the construction of $\mathrm{Ad} : \mathrm{SU}(4) \to \mathrm{SO}(6)$ given in [1]. Let V be a 8-dimensional real vector space with a fixed SU(4)-module structure; in other words, we fix a metric, an almost complex structure and a complex form on V of type $(4,0)$. The space of complex 1-forms $\Sigma = \Lambda^1_{\mathbb{C}} V^*$, splits up as $V^{1,0} \oplus V^{0,1}$. Fix a basis u_0, u_1, u_2, u_3 of $V^{1,0}$ and take the conjugate basis u_4, u_5, u_6, u_7 of $V^{0,1}$, so that the complex volume form is u_{0123}, and $\overline{u_k} = u_{k+4}$. The hermitian product and the complex volume form on V define the Hodge duality $*$, which is an antilinear automorphism of $\Lambda^{2,0}V$. The space of self-dual forms W is a real 6-dimensional $SU(4)$-module spanned by

$$\begin{array}{lll} u_{01} + u_{23}\ , & u_{02} + u_{31}\ , & u_{03} + u_{12}\ , \\ i(u_{01} - u_{23})\ , & i(u_{02} - u_{31})\ , & i(u_{03} - u_{12})\ . \end{array}$$

Let T be $(W + \overline{W}) \cap \Lambda^2_{\mathbb{R}} V^*$. The action of SU(4) on T preserves the metric and the orientation of T, therefore giving a map $\mathrm{SU}(4) \to \mathrm{SO}(6)$, which being injective at the Lie algebra level is a covering map. By the simply connectedness of SU(4), we conclude that Spin(6) = SU(4) and this map coincides with Ad. Explicitly, write

$$u_0 = w^1 + iw^2 \,, \quad u_1 = w^3 + iw^4 \,, \quad u_2 = w^5 + iw^6 \,, \quad u_3 = w^7 + iw^8 \,;$$

then T is spanned by

$$\begin{aligned} e_1 &= w^{13} + w^{42} + w^{57} + w^{86} \,, & e_2 &= w^{14} + w^{23} - w^{58} - w^{67} \,, \\ e_3 &= w^{15} + w^{62} + w^{73} + w^{48} \,, & e_4 &= w^{16} + w^{25} - w^{74} - w^{83} \,, \\ e_5 &= w^{17} + w^{82} + w^{35} + w^{64} \,, & e_6 &= w^{18} + w^{27} - w^{36} - w^{45} \,. \end{aligned}$$

We define

$$\lrcorner : V_{\mathbb{C}} \otimes \Lambda^{p+1}_{\mathbb{C}} V^* \to \Lambda^p_{\mathbb{C}} V^*$$

by extending the usual interior product

$$\lrcorner : V \otimes \Lambda^{p+1} V^* \to \Lambda^p V^*$$

linearly. Using the (antilinear) isomorphism $\Lambda^1_{\mathbb{C}} V^* \cong V_{\mathbb{C}}$ given by the hermitian metric, we obtain a map

$$\cdot : T \otimes \Sigma \to \Sigma \,, \qquad \text{where } \omega \cdot \phi = \overline{\phi \lrcorner \omega} \,,$$

which is $\mathbb{R}$-linear in the first variable and $\mathbb{C}$-linear in the second variable. Note that

$$\omega \cdot \overline{\phi} = \overline{\overline{\phi} \lrcorner \omega} = \phi \lrcorner \overline{\omega} = \phi \lrcorner \omega = \overline{\omega \cdot \phi} \,;$$

therefore $\cdot$ preserves the real structure of Σ (i.e. conjugation). Moreover, since elements of T have type $(2,0) + (0,2)$, $\cdot$ interchanges $V^{0,1}$ and $V^{1,0}$. If viewed as a one-form taking values in $\mathrm{End}_{\mathbb{C}}(\Sigma)$, with respect to the basis $u_0, \dots, u_7$ the Clifford multiplication form C reads $\left(\begin{smallmatrix} 0 & \overline{A} \\ A & 0 \end{smallmatrix}\right)$, where

$$A = \begin{pmatrix} 0 & -e^1 - ie^2 & -e^3 - ie^4 & -e^5 - ie^6 \\ e^1 + ie^2 & 0 & -e^5 + ie^6 & e^3 - ie^4 \\ e^3 + ie^4 & e^5 - ie^6 & 0 & -e^1 + ie^2 \\ e^5 + ie^6 & -e^3 + ie^4 & e^1 - ie^2 & 0 \end{pmatrix} \,;$$

in particular A is skew-symmetric and satisfies

$$A(v)\overline{A}(w) + A(w)\overline{A}(v) = -2\langle v, w \rangle I$$

for all $v, w \in T$. Hence $\cdot$ extends to a representation of $\mathrm{Cl}(T)$; by dimension count and by Theorem 1.4, it can be identified with the spinor representation. The fact that Clifford multiplication commutes with conjugation corresponds to the fact that in dimension 6 the spinor representation is real. It is easy to see that the complex volume form acts as $+1$ on $V^{1,0}$ and as -1 on $V^{0,1}$, implying

$$V^{1,0} = \Sigma_6^+ \,, \qquad\qquad V^{0,1} = \Sigma_6^- \,.$$

From (1.3), we deduce that $\Sigma_5 = V^{1,0}$, with Clifford action given by

$$\begin{pmatrix} ie^5 & -e^4 - ie^3 & e^2 + ie^1 & 0 \\ e^4 - ie^3 & -ie^5 & 0 & e^2 + ie^1 \\ -e^2 + ie^1 & 0 & -ie^5 & e^4 + ie^3 \\ 0 & -e^2 + ie^1 & -e^4 + ie^3 & ie^5 \end{pmatrix} .$$

We also have $\Sigma_7 = \Sigma$. Indeed, let J be the complex structure on V, and let e_7 act on Σ as $-J$. Using (1.3), we find the following Clifford form with respect to the basis $u_0, \ldots, u_7$:

$$\begin{pmatrix} -ie^7\,\mathrm{Id} & \overline{iA} \\ iA & ie^7\,\mathrm{Id} \end{pmatrix}$$

where A is the same as before. Again, Clifford multiplication commutes with conjugation, so the spinor representation is real (in fact, this holds in all dimensions n such that $n = 6, 7, 8 \mod 8$).

For completeness' sake, we give general formulae for the spinor representation in arbitrary dimension. Let $v_0 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $v_1 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$; a basis $u_0, \ldots, u_{2^m-1}$ of $\Sigma_m = (\mathbb{C}^2)^{\otimes m}$ is given by

$$u_k = v_{a_{m-1}} \otimes \cdots \otimes v_{a_0} \,, \qquad\qquad \text{where} \quad k = \sum_{0 \le r < k} a_r 2^r \,.$$

We think of the a_r as elements of $\mathbb{Z}/2\mathbb{Z}$. The Clifford algebra $\mathrm{Cl}(2m)$ acts irreducibly on Σ_m by

$$\begin{aligned} e_{2j} \cdot u_k &= -(-1)^{a_{j-1} + \cdots + a_0} v_{a_{m-1}} \otimes \cdots \otimes v_{a_j} \otimes v_{1-a_{j-1}} \otimes v_{a_{j-2}} \otimes \cdots \otimes v_{a_0} \\ e_{2j-1} \cdot u_k &= i(-1)^{a_{j-2} + \cdots + a_0} v_{a_{m-1}} \otimes \cdots \otimes v_{a_j} \otimes v_{1-a_{j-1}} \otimes v_{a_{j-2}} \otimes \cdots \otimes v_{a_0} \end{aligned}$$

It is easy to see that $\omega \cdot u_k = (-1)^{a_{m-1}+\cdots+a_0} u_k$; the complex volume form of $\mathbb{C}^{2m-1}$ acts as ω, showing that

$$\begin{aligned}\Sigma_{2m}^{+} &= \operatorname{span}\{u_k \mid a_{m-1} + \cdots + a_0 = 0 \mod 2\} \ ,\\ \Sigma_{2m}^{-} &= \operatorname{span}\{u_k \mid a_{m-1} + \cdots + a_1 = 1 \mod 2\} \ ,\end{aligned}$$

and it is obvious that Clifford multiplication by e_j maps $\Sigma_{2m}^{\pm}$ to $\Sigma_{2m}^{\mp}$.

Now consider $\mathrm{Cl}(2m+1)$; let $\Sigma_{2m+1} = \Sigma_{2m}$, and denote the Clifford multiplication of $\mathrm{Cl}(2m)$ by $\odot$. Then the action of $\mathrm{Cl}(2m+1)$ on Σ_{2m+1} is given by

$$\begin{aligned} e_{2m+1} \cdot u_k &= (-1)^{a_{m-1}+\cdots+a_0} (-i)\, u_k \\ e_j \cdot u_k &= -e_{2m+1} \cdot e_j \odot u_k \ , \qquad\qquad j = 0, \ldots, 2m \end{aligned}$$

This representation is consistent with (1.3); moreover, ω acts as the identity, as required in the definition of Σ_{2m+1}.

1.7 Intrinsic torsion

In this section we introduce the intrinsic torsion of a G-structure; for G-structures defined by a spinor ψ, we prove that the intrinsic torsion can be identified with $\nabla\psi$. We recall the following [26]:

Definition 1.29. If P is a principal bundle on M with fibre G, and

$$\rho : G \to \mathrm{GL}(V)$$

is a representation of G, a pseudotensorial p-form of type (ρ, V) is a p-form $\alpha \in \Omega(P, V)$ such that for all $g \in G$ one has

$$R_g^* \alpha = \rho(g^{-1})\alpha \ .$$

A tensorial form is a pseudotensorial form α such that $X \lrcorner\, \alpha$ vanishes for all vertical X.

In this language, a *connection form* on P is a $\mathfrak{g}$-valued pseudotensorial 1-form which extends the canonical isomorphism $\ker \pi_* \cong (P \times \mathfrak{g})$. A connection form ω defines a connection $H = \ker \omega$, and vice versa.

Proposition 1.30. *The space of tensorial p-forms of type (ρ, V) can be identified with $\Omega^p(M, \underline{V})$.*

Proof. Let α be a tensorial form; let $u \in P$, $X_1, \dots, X_p \in T_u P$. An element $\underline{\alpha}$ in $\Omega^p(M, P \times_G V)$ is defined by

$$\underline{\alpha}_{\pi(u)}(\pi_* X_1, \dots, \pi_* X_p) = [u, \alpha_u(X_1, \dots, X_p)]$$

For details, we refer to [26]. $\square$

In the sequel we shall always write α for the tensorial form and $\underline{\alpha}$ for the corresponding element of $\Omega^p(M, \underline{V})$. When P is a reduction of the frame bundle, so that e.g. $\underline{T} = TM$, to a tensorial p-form α one can also associate a tensorial (i.e. equivariant) map

$$\overline{\alpha} : P \to \Lambda^p T^* \otimes V \ .$$

We shall not adhere to this convention when dealing with tensors, spinors or $\mathbb{R}$-valued forms (e.g. we write ψ for spinors instead of $\underline{\psi}$).

Now let (V, ρ) be a G-module.

Definition 1.31. Fix a connection on P. For α a pseudotensorial form, the *exterior covariant derivative* of α is $D\alpha$, given by:

$$D\alpha(X_1, \dots, X_p) = d\alpha(\pi_H X_1, \dots, \pi_H X_p) \ ,$$

where $\pi_H : TP \to H$ is the projection.

Using the identification between tensorial p-forms and $\underline{V}$-valued p-forms, we can interpret D as an operator

$$D : \Omega^p(M, \underline{V}) \to \Omega^{p+1}(M, \underline{V}) \ .$$

For 0-forms, D coincides with ∇ as defined by parallel transport. Explicitly, if h is a V-valued function and s a local section of P, for the section $[s, h]$ of $\underline{V}$ we have

$$\nabla[s, h] = [s, dh + s^* \omega h] \ ,$$

where ω acts on h by the infinitesimal action $\mathfrak{g} \to \mathrm{End}(V)$. For instance, if $G = \mathrm{Spin}(n)$, $V = \Sigma_n$ we get

$$\nabla_X \psi = X(\psi) + \frac{1}{4} \omega(X) \cdot \psi \ .$$

A more general formula holds, which will be needed in the sequel:

Proposition 1.32. *If ω is a connection form on P and α is a tensorial p-form, then with respect to the connection determined by ω,*

$$D\alpha = d\alpha + \omega \wedge \alpha \ ,$$

where the wedge implies contracting $\mathfrak{g} \oplus V \to V$ by the infinitesimal action.

Proof. Let ρ be the representation $G \to \mathrm{GL}(V)$. One has to prove that for all $X_0, \ldots, X_p$ in T_xP,

$$D\alpha(X_0, \ldots, X_p) = d\alpha(X_0, \ldots, X_p) + (\omega \wedge \alpha)(X_0, \ldots, X_p) \ .$$

This equation being linear, one has to consider two cases: if all the X_i's are horizontal, $(\omega \wedge \alpha)(X_0, \ldots, X_p)$ vanishes and the equation is satisfied; if X_0 is vertical, we can extend the X_i so that $X_0 = A^*$ is the fundamental vector field corresponding to $A \in \mathfrak{g}$ and $X_1, \ldots, X_p$ are invariant. Then the left-hand side vanishes, and using the fact that α horizontal and ω vertical, the right-hand side becomes

$$\begin{aligned}
&\frac{1}{p}\left(A^*\alpha(X_1, \ldots, X_p) + \sum_{i=1}^{p}(-1)^i \alpha\big([A^*, X_i], \ldots, X_p\big)\right) + \\
&\qquad + \frac{1}{(p+1)!}\rho_*(\omega(A^*))\, p!\, \alpha(X_1, \ldots, X_p) = \\
&\qquad\qquad = \frac{1}{p}\big(A^*\alpha(X_1, \ldots, X_p) + \rho_*(A)\alpha(X_1, \ldots, X_p)\big)
\end{aligned}$$

because the X_i's are invariant and therefore $[A^*, X_i] = 0$. On the other hand, if a_t is the one-parameter subgroup of G generated by A, using the invariance of the X_i one finds

$$\begin{aligned}
&A^*_x(\alpha(X_1, \ldots, X_p)) = \\
&\qquad \lim_{t \to 0} \frac{1}{t}\left(\alpha_{xa_t}(X_1, \ldots, X_p) - \alpha_x(X_1, \ldots, X_p)\right) = \\
&\qquad \lim_{t \to 0} \frac{1}{t}\big((R^*_{a_t}\alpha)_x(X_1, \ldots, X_p) - \alpha_x(X_1, \ldots, X_p)\big) = \\
&\qquad \lim_{t \to 0} \frac{1}{t}\big(\rho(a_t^{-1})(\alpha_x(X_1, \ldots, X_p)) - \alpha_x(X_1, \ldots, X_p)\big) = \\
&\qquad\qquad - \rho_*(A)\alpha_x(X_1, \ldots, X_p) \quad \square
\end{aligned}$$

Now let P be a reduction to G of the bundle of frames; in our language, this includes the spin bundle. Let $T = \mathbb{R}^n$ with the G-action induced by the standard action of $\mathrm{GL}(n, \mathbb{R})$; then $\underline{T} = TM$. The tensorial map

$$\overline{\theta} : P \to (T)^* \otimes T \cong \mathrm{GL}(T)$$

given by the identity corresponds to a T-valued tensorial 1-form θ on P, called the *canonical form*, or *solder form*.

Definition 1.33. The *torsion* of a connection is $\Theta = D\theta$.

So Θ is a T-valued tensorial 2-form; it is well known that

$$2\,\underline{\Theta}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]\ .$$

Fix a connection form ω_0. Any connection form ω is a pseudotensorial 1-form taking values in $\mathfrak{g}$; as such, using ω_0 it can be identified with a tensorial map

$$\overline{\omega} : P \to (\mathfrak{g} \oplus T)^* \otimes \mathfrak{g} = \operatorname{End}\mathfrak{g} \oplus T^* \otimes \mathfrak{g}\ ;$$

accordingly, $\overline{\omega}$ decomposes as $1 + \overline{\xi}$, where 1 is the identity of $\operatorname{End}\mathfrak{g}$, and $\xi = \omega - \omega_0$. Now consider the skew-symmetrization map ∂:

$$T^* \otimes \mathfrak{g} \to T^* \otimes T^* \otimes T \to \Lambda^2 T^* \otimes T\ .$$

An exact sequence is induced:

$$0 \to \mathfrak{g}^{(1)} \to T^* \otimes \mathfrak{g} \xrightarrow{\partial} \Lambda^2 T^* \otimes T \to \operatorname{Coker}\partial \to 0\ ,$$

where the *first prolongation* $\mathfrak{g}^{(1)}$ is defined as the kernel of ∂. Then

$$\partial \circ \overline{\xi} = \overline{\Theta - \Theta_0}\ . \tag{1.7}$$

Definition 1.34. The *intrinsic torsion* of P is the image $[\overline{\Theta}]$ of $\overline{\Theta}$ in $\operatorname{Coker}\partial$, where Θ is the torsion of any connection on P.

Some remarks:

- Well-definedness follows from (1.7).
- Suppose $[\overline{\Theta}] = 0$; then $\overline{\Theta} = \partial\overline{\xi}$, and $\omega - \xi$ is torsion-free. This means that the intrinsic torsion of P is zero if and only if P admits a torsion-free connection.

- Two connection forms ω and ω_0 have the same torsion if and only if ξ lies in $\mathfrak{g}^{(1)}$.

- If $\mathfrak{g} = \mathfrak{so}(n)$, ∂ is the inverse of

$$\begin{aligned} \lambda : \Lambda^2 T^* \otimes T &\to T^* \otimes \Lambda^2 T^* \\ e^{ij} \otimes e_k &\to e^i \otimes e^{jk} - e^j \otimes e^{ik} - e^k \otimes e^{ij} \end{aligned}$$

  In particular, ∂ is an isomorphism, and by the above remarks, this gives the existence and uniqueness of the Levi-Civita connection.

- If $\mathfrak{g} \subseteq \mathfrak{so}(n)$, then λ induces an isomorphism

$$\hat{\lambda} : \operatorname{Coker} \partial \to T^* \otimes \mathfrak{g}^\perp$$

  where $\mathfrak{g}^\perp$ is the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{so}(n)$. For an element v of $T^* \otimes \mathfrak{so}(n)$, let $v^\perp$ be the projection on $T^* \otimes \mathfrak{g}^\perp$; then

$$\hat{\lambda}[\overline{\Theta}] = (\lambda \overline{\Theta})^\perp \,. \tag{1.8}$$

Now let P be the spin bundle of M, and let η_0 be as in Proposition 1.8 (e.g. a spinor), with an action $\rho : \mathrm{Spin}(n) \to \mathrm{GL}_C(V)$. It is well known that V has an inner product preserved by $\mathrm{Spin}(n)$. The infinitesimal action of $\mathrm{Spin}(n)$ on η_0 gives a map

$$\rho_* : \mathfrak{spin}(n) \to T_{\eta_0} V = V$$

with kernel $\mathfrak{h}$. Using the fact that ρ preserves an inner product, we see that on restricting to $\mathfrak{h}^\perp$ we get an inclusion

$$\underline{\rho_*} : \underline{\mathfrak{h}}^\perp \to \underline{\eta_0^\perp} \subset \underline{V} \,.$$

In a later section we shall look at some significant cases where $\underline{\rho_*}$ is an isomorphism, but we do not assume this to hold for the moment.
Let ω be a connection form on P_H, which we extend to P; let ω_0 be the Levi-Civita connection form on P.

Proposition 1.35. *The map $\rho_* \circ \hat{\lambda}$ is injective, and the intrinsic torsion satisfies*

$$\underline{\rho_*} \, \underline{\hat{\lambda}} \, [\underline{\Theta}] = -\nabla_0 \underline{\eta} \,.$$

Proof. Let s be a section of P_H; then

$$\nabla_0\eta = \nabla_0\eta - \nabla\eta = [s, s^*(\omega_0 - \omega)\cdot\eta_0] = [s, -s^*\xi\cdot\eta_0]$$

From $\Theta_0 = 0$ and (1.7), it follows that

$$\overline{\xi} = \lambda \circ \overline{\Theta}\,.$$

Hence, by (1.8), $\lambda[\overline{\Theta}] = \overline{\xi}^{\perp}$. But

$$\underline{\rho_*}\,\underline{\xi}^{\perp} = \underline{\rho_*}\,\underline{\xi} = [s, s^*\xi\cdot\eta_0] = -\nabla_0\eta \qquad \square$$

Proposition 1.35 asserts that the intrinsic torsion of the G-structure defined by a tensor or spinor η can be identified with the covariant derivative of η. If one is only interested in tensors, one can replace the spin structure with the bundle of orthonormal frames.

Chapter 2

Some hypersurface geometry

In this chapter we study the local geometry of hypersurfaces inside Riemannian spin manifolds with a parallel spinor.

The first section begins with the equations of Gauss and Codazzi-Mainardi; we formulate them in an abstract form, which will be useful later to compute the curvature of the connection defined in the second section. We introduce generalized Killing spinors, and prove that the restriction of a parallel spinor to a hypersurface is a generalized Killing spinor. We characterize G-structures defined by a generalized Killing spinor in terms of intrinsic torsion. Putting these two facts together, we prove that the intrinsic torsion of the G-structure induced on the hypersurface of a manifold with a parallel spinor can be identified with the Weingarten tensor. As an example, we illustrate the well-known case of cocalibrated G_2 hypersurfaces inside Spin(7)-holonomy manifolds.
In the second section we define hypo structures on 5-manifolds, and introduce a canonical connection on these structures, called the hypo connection. We show that hypo is a generalization of Einstein-Sasaki, and all hypo Sasaki structures are Einstein.
The third section consists in an example of a hypo 5-manifold, whose curvature we compute. We write down explicitly an integrable SU(3)-structure on a 6-manifold which contains this 5-manifold as a hypersurface.
In the fourth section we study the intrinsic torsion of hypo and half-flat manifolds.
In the fifth section we study the Levi-Civita curvature of hypo and half-flat manifolds. We exploit the fact that $\mathfrak{su}(3)$ contains no simple forms to prove that the curvature of an SU(3)-holonomy manifold is pointwise determined

by the curvature and intrinsic torsion of a hypo hypersurface. In the opposite direction, we show how the curvature of the 6-manifold and the Levi-Civita curvature of the hypersurface determine the curvature of the hypo connection.
In the sixth section, we discuss the embedding property in general.
The last section provides a bridge with Chapters 3 and 4: we show how SU(3)-structures on a 6-manifold can be defined using forms, and characterize generalized Killing spinors in terms of the forms they define through this correspondence. From the point of view of Chapter 2, this completes the proof that 6-dimensional manifolds with a generalized Killing spinor have the embedding property, reducing this fact to a theorem of Hitchin. From the point of view of Chapter 3, this result motivates the study of structures defined by differential forms.

2.1 Generalized Killing spinors and hypersurfaces

In this section we introduce generalized Killing spinors, which appear naturally on hypersurfaces of Riemannian manifolds with a parallel spinor. We study the G-structures defined by a generalized Killing spinor, showing that they are characterized by the condition that their intrinsic torsion lies in $\mathrm{Sym}(TM)$.
We shall start by explaining how one can modify a given connection (e.g. the Levi-Civita one) to obtain a connection of prescribed holonomy; this construction will be applied to introduce a canonical connection on the structure defined by a spinor, and also to characterize the Levi-Civita connection on a hypersurface in $\mathbf{M}$ in terms of the Levi-Civita connection on $\mathbf{M}$. In particular, we give formulae for the curvature which, applied in the latter case, give the Gauss and Codazzi-Mainardi equations.

Let P_G be a G-structure, $G < \mathrm{SO}(n)$; let ω be a connection form on the bundle of orthonormal frames. Under the adjoint action of G,

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^{\perp} \; ;$$

for any $\mathfrak{so}(n)$-valued form α, we write $\alpha = [\alpha]_{\mathfrak{g}} + [\alpha]_{\mathfrak{g}^{\perp}}$ accordingly, with the exception of ω, for which we write $\omega = \omega_{\mathfrak{g}} + \omega^{\perp}$.

Proposition 2.1. *The restriction to P_G of $\omega_{\mathfrak{g}}$ is a connection form. The curvatures of ω and $\omega_{\mathfrak{g}}$ are related by*

$$\Omega = \Omega_{\mathfrak{g}} + D_{\mathfrak{g}}\omega^{\perp} + \frac{1}{2}[\omega^{\perp}, \omega^{\perp}]$$

where $D_{\mathfrak{g}}$ is exterior covariant differentiation with respect to $\omega_{\mathfrak{g}}$.

Proof. Follows from the structure equation

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

and Proposition 1.32. □

Remark. By (1.7), $\omega^{\perp}$ (or more precisely, its image under the isomorphism ∂) is the difference of the torsions. In particular if $\omega^{\perp}$ is zero on P_G, it means that ω reduces to P_G, i.e. its holonomy is contained in G. If ω is the Levi-Civita connection, $\partial\omega^{\perp}$ is the torsion of $\omega_{\mathfrak{g}}$.

Remark. Since $\mathfrak{g}^{\perp}$ is a G-module, $D_{\mathfrak{g}}\omega^{\perp}$ lies in $\omega^{\perp}$, so the formula for the curvature in Proposition 2.1 can be rewritten:

$$[\Omega]_{\mathfrak{g}} = \Omega_{\mathfrak{g}} + \frac{1}{2}[\omega^{\perp}, \omega^{\perp}]_{\mathfrak{g}} \tag{2.1}$$

$$[\Omega]_{\mathfrak{g}^{\perp}} = D_{\mathfrak{g}}\omega^{\perp} + \frac{1}{2}[\omega^{\perp}, \omega^{\perp}]_{\mathfrak{g}^{\perp}} \tag{2.2}$$

Now let $(\mathbf{M}, \mathbf{g})$ be an $(n+1)$-dimensional Riemannian manifold and let $i : M \to \mathbf{M}$ be an oriented hypersurface; assume $\mathbf{M}$ is oriented and spin. The unit normal vector ν, as a section of $i^*T(\mathbf{M})$, induces a map i_ν from the frame bundle F of M to the pullback $i^*\mathbf{F}$ of the frame bundle of $\mathbf{M}$:

$$i_\nu(e_1, \ldots, e_n) \to (i_*e_1, \ldots, i_*e_n, \nu) \ .$$

The bundle map i_ν gives a reduction of $i^*\mathbf{F}$ to F relative to the inclusion

$$\mathrm{GL}(n, \mathbb{R}) \ni A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+1, \mathbb{R}) \ ; \tag{2.3}$$

we may also think of this reduction as determined by ν in the sense of Proposition 1.8.

The same construction can be carried out using the bundles of oriented frames (provided ν is chosen with the correct sign).

Proposition 2.2. *Let* $\mathbf{P}$ *be a reduction of* $\mathbf{F}$ *to* G*. Then one has the following diagram:*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathbf{P} \arrow[d, "\pi_{\mathbf{P}}"] \\ F \arrow[r, "i_\nu"] & \mathbf{F} \end{tikzcd}$$

where $P = \pi_{\mathbf{P}}^{-1}(i_\nu(F))$*, and* P *is a reduction of* F *to* $G \cap \mathrm{GL}(n, \mathbb{R})$*.*

Proof. Obvious. □

From now on, $\mathbf{P}$ will be a spin structure on $\mathbf{M}$ and P the induced spin structure on M. Denote by $\underline{A}$ the TM-valued form on M defined by

$$\boldsymbol{\nabla}_X Y = \nabla_X Y + g(\underline{A}(X), Y)\nu \ ;$$

then $\underline{A} = -\boldsymbol{\nabla}\nu$; moreover, $\underline{A}$ is symmetric, i.e. it is a section of $\mathrm{Sym}(TM)$. We shall refer to A as the *Weingarten form.* In the language of Proposition 2.1, A is the component in $\mathfrak{so}(n)^\perp$ of the connection form, and one has a decomposition

$$i_\nu^* \boldsymbol{\omega} = \begin{pmatrix} \omega & -A \\ A & 0 \end{pmatrix} \ .$$

Setting $A^i = e^i(A)$, equation (2.1) gives the classical Gauss' equation:

Proposition 2.3. *If* A *is the Weingarten form,*

$$[i_\nu^* \Omega]_{\mathfrak{so}(n)} = \Omega_{\mathfrak{so}(n)} + \sum_{i,j} A^i \wedge A^j \otimes e^{ij} \tag{2.4}$$

Now write Σ for Σ_n and $\boldsymbol{\Sigma}$ for Σ_{n+1}; by the remarks in Section 1.1, we can identify $\underline{\Sigma}$ with $i^*\underline{\boldsymbol{\Sigma}}$ if n is odd and with $i^*\underline{\boldsymbol{\Sigma}^+}$ if n is even. We write $\cdot$ for Clifford multiplication on Σ_{n+1} and $\odot$ for Clifford multiplication on Σ_n. On the spinor bundle the following holds:

$$\boldsymbol{\nabla}_X \psi = \nabla_X \psi - \frac{1}{2} \nu \cdot \underline{A}(X) \cdot \psi \ .$$

This motivates the following generalization of Definition 1.19, due to Bär, Gauduchon and Moroianu [6]:

Definition 2.4. A spinor ψ is a *generalized Killing spinor* if there exists a section $\underline{A}$ of $\mathrm{Sym}(TM)$ such that

$$\nabla_X \psi = \frac{1}{2}\underline{A}(X) \cdot \psi \ .$$

Generalized Killing spinors with the additional property that the trace of A is constant arise naturally in the study of the Dirac operator [20]; however in the study of hypersurfaces this additional condition plays no evident role, and for this reason we opt for the terminology in [6].
The following is now a straightforward consequence of (1.3):

Proposition 2.5. *If ψ is a parallel spinor on $\mathbf{M}$, then its restriction to M is a generalized Killing spinor, and A is the Weingarten form.*

Remark. The coefficient 1/2 in Definition 2.4 is not only motivated by the fact that it allows to identify A with W, but it actually simplifies some formulae.

A generalized Killing spinor is parallel with respect to the covariant derivative $\nabla_X - \frac{1}{2}\underline{C}(\underline{A}(X))$; therefore, Proposition 1.9 is satisfied, and the spinor defines a G-structure with $G < \mathrm{SO}(n)$. From Proposition 1.35, one immediately proves:

Theorem 2.6. *The G-structure on M defined by a spinor ψ satisfies*

$$\underline{\rho_*}\,\underline{\hat{\lambda}}\,[\underline{\Theta}\ ] = -\frac{1}{2}\underline{A} \odot \psi$$

for some section $\underline{A}$ of $\mathrm{Sym}(TM)$ *if and only if ψ is a generalized Killing spinor.*

This theorem characterizes geometries defined by a generalized Killing spinor by their intrinsic torsion, which takes values in $\mathrm{Sym}(TM)$.

Now consider a parallel spinor ψ on $\mathbf{M}$; we know from section 1.3 that ψ defines an integrable structure on $\mathbf{M}$. The restriction of ψ to M is a generalized Killing spinor, and therefore defines a G-structure P_G. On the other hand, by Proposition 2.2, the integrable structure on $\mathbf{M}$ also induces a G-structure on M; it can be easily verified that these structures are the same. We can then restate Theorem 2.6 as follows:

Theorem 2.7. *If* $\mathbf{M}$ *has a parallel spinor* ψ*, then the intrinsic torsion of* P_G *satisfies*

$$\underline{\rho_*}\,\underline{\hat{\lambda}}\,[\underline{\Theta}] = -\frac{1}{2}\nu \cdot A \cdot \psi \ .$$

We next give an example of Theorem 2.7 for 7-dimensional M; the cases of dimension 5 and 6 will be treated in Section 2.4. Let $\mathbf{M}$ be 8-dimensional, with an integrable Spin(7)-structure, and let $i : M \to \mathbf{M}$ be an oriented hypersurface. By Proposition 2.2, a G_2-structure is induced on $\mathbf{M}$; Theorem 2.7 asserts that its intrinsic torsion lies in $\mathrm{Sym}(T)$. In order to reinterpret this fact, we recall that a Spin(7)-structure is defined by a 4-form Ω, and a G_2-structure is defined by a 3-form φ. In our setup, they are linked by

$$*\varphi = i^*\Omega \ ;$$

then $*\varphi$ is closed because, by Theorem 1.16, $d\Omega = 0$.
The intrinsic torsion of a G_2-structure takes values in

$$\mathfrak{g}_2^\perp \otimes T^* = T^* \otimes T^* = S^2T^* \oplus \Lambda^2 T^* = \mathbb{R} \oplus S^2_0 \oplus \mathfrak{g}_2 \oplus T^*$$

Theorem 1.15 can be reformulated by saying that the component $\mathbb{R}\oplus S^2_0 \oplus T^*$ is determined by $d\varphi$, whereas the component $\mathfrak{g}_2 \oplus T^*$ is determined by $d*\varphi$; these components have non-zero intersection because $d\varphi$ and $d*\varphi$ are not independent. In conclusion, $d*\varphi = 0$ forces the intrinsic torsion to take values in $\mathbb{R} \oplus S^2_0$.

2.2 Five dimensions

We study 5-dimensional manifolds M with a spin structure P and a generalized Killing spinor ψ. The stabilizer of a non-zero element of Σ_5 is SU(2), so ψ defines an SU(2)-structure $P_{\mathrm{SU}(2)}$. We shall call $P_{\mathrm{SU}(2)}$ a *hypo* structure on M, and *hypo connection* the connection on $P_{\mathrm{SU}(2)}$ defined in Proposition 2.1.

Let $e_1, \dots, e_5$ be a basis of $T = \mathbb{R}^5$, and $e^1, \dots, e^5$ the dual basis of T^*. Under the action of SO(4), and consequently of SU(2):

$$T = \langle e_1, \dots, e_4\rangle \oplus \langle e_5 \rangle$$

which we write

$$T = \Lambda^1 \oplus \mathbb{R} \ .$$

Consistently, we write Λ^k for $\Lambda^k(\Lambda^1)$. A basis of $\Lambda^2 T^*$ is given by

$$\begin{aligned}
\omega_1 &= e^{12} + e^{34} & \omega_2 &= e^{13} + e^{42} & \omega_3 &= e^{14} + e^{23}\\
\sigma_1 &= e^{12} - e^{34} & \sigma_2 &= e^{13} - e^{42} & \sigma_3 &= e^{14} - e^{23}\\
v_i &= e^{i5}\ ,\ \text{for } i = 1 \dots 4
\end{aligned}$$

We identify $\Lambda^2 T^*$ with $\mathfrak{so}(5)$ so that e.g.

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}$$

As an SO(4)-module, $\mathfrak{so}(5)$ splits as

$$\mathfrak{so}(5) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_- \oplus \Lambda^1\ ,$$

and the basis we have given respects this splitting, in the sense that

$$\begin{aligned}
\mathfrak{su}(2)_+ &\cong \langle \omega_1, \omega_2, \omega_3\rangle = \Lambda^2_+\\
\mathfrak{su}(2)_- &\cong \langle \sigma_1, \sigma_2, \sigma_3\rangle = \Lambda^2_-\\
\Lambda^1 &\cong \langle v_1, \dots, v_4\rangle
\end{aligned}$$

Let σ^i, ω^i, v^i be the dual basis of $\mathfrak{so}(5)^*$.

On any SO(5)-module, we define an endomorphism J by the action of $2\omega_1$. In particular, J is an almost complex structure on Λ^1 with $Jv_1 = v_2$, $Jv_3 = v_4$, and acts in a similar fashion on T, T^*, consistently with the notation of Section 1.5.
Now consider the action of $\mathfrak{so}(5)$ on Σ_5 induced by the action of $\mathfrak{spin}(5)$; if viewed as an element of $\mathfrak{so}(5)^* \otimes \mathrm{End}(\Sigma_5)$, in terms of the basis $u_0, \dots, u_3$ of Σ_5 from Section 1.6, this action is given by:

$$\mathrm{Ad}_*^{-1} = \frac{1}{4}\begin{pmatrix} -2i\,\omega^1 & -v^3 + i\,v^4 & v^1 - i\,v^2 & -2\omega^2 + 2i\,\omega^3\\ v^3 + i\,v^4 & -2i\,\sigma^1 & -2\sigma^2 - 2i\,\sigma^3 & -v^1 + i\,v^2\\ -v^1 - i\,v^2 & 2\sigma^2 - 2i\,\sigma^3 & 2i\,\sigma^1 & -v^3 + i\,v^4\\ 2\omega^2 + 2i\,\omega^3 & v^1 + i\,v^2 & v^3 + i\,v^4 & 2i\,\omega^1 \end{pmatrix} \tag{2.5}$$

Let $\psi_0 = u_0$. If one writes

$$\mathrm{SU}(2)_+\mathrm{SU}(2)_- = \mathrm{SO}(4) < \mathrm{SO}(5)\ ,$$

the stabilizer of ψ_0 is $\mathrm{SU}(2)_-$; we shall write simply $\mathrm{SU}(2)$ when there is no ambiguity.
By looking at the first column of (2.5), and using $\rho_*\omega^\perp = \frac{1}{2}A\cdot\psi_0$, we deduce

Proposition 2.8. *If ω is the Levi-Civita connection form and $\omega_{\mathfrak{su}(2)}$ is the hypo connection form, then $\omega - \omega_{\mathfrak{su}(2)}$ is given by*

$$\omega^\perp = -A^5\otimes\omega_1 - 2\sum_{i=1}^{4} A^i\otimes Jv_i \tag{2.6}$$

where $A^k(X) = e^k(A(X))$.

Remark. This formula is consistent with Theorem 2.6. Indeed, from (2.5), we see that

$$(\mathrm{Ad}_*^{-1}\,\omega^\perp)\psi_0 = \frac{1}{2}\left(iA^5u_0 + (-A^2+iA^1)u_2 + (A^4-iA^3)u_1\right)\ .$$

On the other hand, from the formulae in Section 1.6,

$$A\cdot\psi_0 = iA^5u_0 + (-A^2+iA^1)u_2 + (A^4-iA^3)u_1\ .$$

Since $-\partial\omega^\perp$ is the torsion of the hypo connection, we obtain the expected result.

Hypo structures are $\mathrm{SU}(2)$-structures on a 5-manifold; as such, it is quite natural to ask what sort of $\mathrm{U}(2)$-structures are induced by extension, i.e. what the underlying almost contact metric structures are like. For the next results, we refer the reader back to the definitions in Section 1.5.

Theorem 2.9. *The Nijenhuis tensor of a hypo manifold is*

$$N(X,Y) = \alpha(X)\left(A(JY) - (JA)Y\right)\ .$$

Proof. Omitting summation over $i=1,\dots,4$, we have

$$g((\nabla\underline{J})Y,Z) = 2v_i(Y,Z)A^i\ ,\qquad\qquad \nabla\xi = A^i\otimes Je_i\ ;$$

therefore (1.5) yields

$$\begin{aligned}g(N(X,Y),Z) = 2[&A^i(JX)v_i(Y,Z) - A^i(JY)v_i(X,Z) + A^i(X)v_i(JY,Z)+\\ &- A^i(Y)v_i(JX,Z)] + \alpha(Y)A(X,JZ) - \alpha(X)A(Y,JZ)\end{aligned}$$

For $X, Y, Z \in \ker \alpha$:

$$\begin{aligned} g(N(\xi, Y), Z) &= A(JY, Z) - A(Y, JZ) \\ g(N(\xi, Y), \xi) &= A(JY, \xi) \\ g(N(X, Y), \xi) &= A(JX, Y) - A(JY, X) + A(X, JY) - A(Y, JX) \end{aligned}$$

proving the theorem by the symmetry of A. □

Lemma 2.10. *A hypo structure on M is contact if and only if*

$$A = \begin{pmatrix} -1 + a_1 & a_2 & 0 & 0 & 0 \\ a_2 & -1 - a_1 & 0 & 0 & 0 \\ 0 & 0 & -1 + a_3 & a_4 & 0 \\ 0 & 0 & a_4 & -1 - a_3 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{pmatrix}$$

where the a_i are functions on M. It is K-contact if and only if in addition $a_2 = a_4 = 0$.

Proof. The proof follows immediately from $\nabla\alpha = A^i \otimes Je^i$ and the remarks in Section 1.5. □

Remark. We can restate the first part of Lemma 2.10 as follows: a hypo structure is contact if and only if

$$\omega_1(X, AY) + \omega_1(AX, Y) = -2\omega_1(X, Y) \quad \forall X, Y \ .$$

For M a hypersurface in a holonomy SU(3) 6-manifold, we recover the condition on the Weingarten tensor characterizing contact hypersurfaces of a Kähler manifold found by Okumura [10].

We can now characterize hypo SU(2)-structures which are reductions of Sasaki or quasi-Sasakian U(2)-structures:

Corollary 2.11. *A hypo structure is quasi-Sasakian if and only if A commutes with J. A hypo structure is Sasaki if and only if $A = -\mathrm{id}$; a Sasaki* SU(2)*-structure is hypo if and only if it is Einstein.*

Proof. The first statement is an immediate consequence of Theorem 2.9. From Lemma 2.10, it follows that a hypo structure is Sasaki if and only if

$$A = -\mathrm{id} + a\alpha \otimes \xi$$

for some function a; we must prove that this condition implies $a=-1$. We have

$$R(X,\xi)\psi = \nabla_X\nabla_\xi\psi - \nabla_\xi\nabla_X\psi - \nabla_{[X,\xi]}\psi = \\ = \frac{1}{2}\left((Xa)\xi + (a+1)\nabla_X\xi - a\,\xi\cdot X\right)\cdot\psi\ , \quad (2.7)$$

and on the other hand (see e.g. [7]):

$$\frac{1}{2}\operatorname{Ric}(X)\cdot\psi = \sum_{i=1}^{5} e_i\cdot R(e_i,X)\psi\ ,$$

where e_i is a local orthonormal basis. Using (2.7), we find

$$\operatorname{Ric}(\xi)\cdot\psi = \sum_{i=1}^{5}\left((\partial_i a)e_i\cdot\xi + (a+1)e_i\cdot\nabla_{e_i}\xi - a\,\xi\right)\cdot\psi\ .$$

By Proposition 1.25, the middle summand acts on ψ like a multiple of ω_1, and therefore trivially. On the other hand, every Sasaki 5-manifold satisfies $\operatorname{Ric}(\xi)=4\xi$ [10], so the first summand has to vanish and $a=-1$. $\square$

Remark. Recall Friedrich and Kim's characterization of η-Einstein-Sasaki structures (Theorem 1.28). Our result is more special because we only consider spinors which are preserved by the U(2)-structure; in the proof of Corollary 2.11, this hypothesis appears in the form $\omega_1\cdot\psi=0$. We conclude that while a quasi-Sasakian Killing spinor defines both a hypo and an η-Einstein-Sasaki structure, these two structures are not compatible, unless the metric is Einstein. They are nonetheless related, suggesting that hypo geometry may be used to study η-Einstein-Sasaki geometry.

We conclude this section by giving an explicit formula which will be useful later.

Lemma 2.12. *The curvature Ω of the Levi-Civita connection is related to the curvature $\Omega_{\mathfrak{su}(2)}$ of the hypo connection by*

$$[\Omega]_{\mathfrak{su}(2)} = \Omega_{\mathfrak{su}(2)} + (A^1\wedge A^2 - A^3\wedge A^4)\otimes\sigma_1 + \\ + (A^1\wedge A^3 - A^4\wedge A^2)\otimes\sigma_2 + (A^1\wedge A^4 - A^2\wedge A^3)\otimes\sigma_3 \quad (2.8)$$

Proof. Using (2.6) and $[v_i, v_j] = \frac{1}{2}e^{ij}$, a straightforward calculation yields

$$\frac{1}{2}[\omega^\perp, \omega^\perp] = -\sum_i A^5 \wedge A^i \otimes v_i + \sum_{i,j} A^i \wedge A^j \otimes (Je^i \wedge Je^j) ,$$

In particular

$$\frac{1}{2}[\omega^\perp, \omega^\perp]_{\mathfrak{su}(2)_-} = (A^1 \wedge A^2 - A^3 \wedge A^4)\sigma_1 + (A^1 \wedge A^3 - A^4 \wedge A^2)\sigma_2 + \\ + (A^1 \wedge A^4 - A^2 \wedge A^3)\sigma_3 ,$$

and (2.1) proves the Lemma. □

2.3 An example

Consider the 5-manifold $M = T^2 \times N$, where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the torus and $N = H/\Gamma$ is a discrete compact quotient of the 3-dimensional Heisenberg group; M has a global basis of 1-forms $e^1, \dots, e^5$, with $e^1, \dots, e^4$ closed and $de^5 = e^{12}$. We view it as an $\{e\}$-structure, which we extend to a hypo SU(2)-structure. A standard formula shows that the Levi-Civita connection form is

$$\omega_{\mathfrak{so}(5)} = e^5 \otimes e^{12} + e^2 \otimes e^{15} - e^1 \otimes e^{25} ,$$

so clearly

$$\omega_{\mathfrak{su}(2)} = \frac{1}{2}e^5 \otimes \sigma_1 .$$

By Equation 2.5, M is hypo and

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

The hypo curvature is clearly

$$\Omega_{\mathfrak{su}(2)} = \frac{1}{2}e^{12} \otimes \sigma_1 .$$

The Levi-Civita curvature is

$$\Omega_{\mathfrak{so}(5)} = \frac{3}{2}e^{12} \otimes e^{12} - \frac{1}{2}e^{15} \otimes e^{15} - \frac{1}{2}e^{25} \otimes e^{25} .$$

In particular

$$[\Omega]_{\mathfrak{su}(2)} = \frac{3}{4}e^{12}\otimes\sigma_1 ,$$

consistently with Lemma 2.12.

Remark. In this example, the curvature tensor is not symmetric. In general, one has the first Bianchi identity

$$D_{\mathfrak{su}(2)}\Theta = \Omega_{\mathfrak{su}(2)}\wedge\theta ;$$

when the left-hand side is zero, e.g. when the hypo structure is integrable, it follows that the curvature tensor is symmetric.

Now let $\mathbf{M} = M\times(-\infty, 2/3)$, with t a coordinate on $(-\infty, 2/3)$, and define an $\{e\}$-structure by

$$\left(-\frac{3}{2}t+1\right)^{1/3}e_1, \left(-\frac{3}{2}t+1\right)^{1/3}e_2, e_3, e_4, \left(-\frac{3}{2}t+1\right)^{-1/3}e_5, dt ;$$

we claim that the SU(3)-structure obtained by extension is integrable. In fact, in Chapter 5 we shall introduce a general technique to produce integrable SU(3)-structures of this kind; for the moment, we simply write down the curvature form:

$$\Omega_{\mathfrak{su}(3)} = 2(e^{12}-e^{56})\otimes(e^{12}-e^{56})-(e^{15}-e^{62})\otimes(e^{15}-e^{62})-(e^{16}-e^{25})\otimes(e^{16}-e^{25})$$

This equation tells us that the holonomy is $\mathrm{SU}(2) < \mathrm{SU}(3)$. Moreover, it is consistent with the Gauss equation, giving

$$[i_\nu^*\Omega_{\mathfrak{su}(3)}]_{\mathfrak{so}(5)} = 2e^{12}\otimes e^{12} - e^{15}\otimes e^{15} - e^{25}\otimes e^{25} .$$

2.4 Intrinsic torsion for $G = \mathrm{SU}(2), \mathrm{SU}(3)$

We study the intrinsic torsion of SU(2)-structures on 5-manifolds and SU(3)-structures on 6-manifolds. The intrinsic torsion of an SU(2)-structure takes values in the SU(2)-module $T^*\otimes\mathfrak{su}(2)^\perp$; we want to decompose this space into irreducible components. First, we list some well-known facts:

- Over $\mathbb{C}$, for each $k\geq 0$, there is a unique $(k+1)$-dimensional irreducible SU(2)-module V^k.

- For k even, V^k is the complexification of an irreducible real representation of SU(2).
- For k odd, $V^k \oplus V^k$ is the complexification of an irreducible real representation of SU(2).
- For $h \leq k$, $V^k \otimes V^h = V^{k-h} \oplus \cdots \oplus V^{k+h}$.
- $S^2(V^k) = V^{2k} \oplus V^{2k-4} \oplus \ldots$.
- $\Lambda^2(V^k) = V^{2k-2} \oplus V^{2k-6} \oplus \ldots$.

So the 3-dimensional SU(2)-module $\mathfrak{su}(2)$ is isomorphic to V^2, and Λ^1 is isomorphic to V^3. Recall that we identify $\mathfrak{su}(2)$ with $\mathfrak{su}(2)_-$; therefore, $\mathfrak{su}(2)_+$ is a trivial representation.

Proposition 2.13. *The intrinsic torsion of an* SU(2)*-structure takes values in the* SU(2)*-module*

$$7\mathbb{R} \oplus 4\Lambda^1 \oplus 4\Lambda^2_- \ .$$

By Theorem 2.6, hypo SU(2)-structures are characterized by the condition that their intrinsic torsion lie in $\mathrm{Sym}(T)$. By the above remarks, we conclude:

Proposition 2.14. *The intrinsic torsion of a hypo structure takes values in the* SU(2)*-module*

$$\mathbb{R} \oplus \Lambda^1 \oplus 3\Lambda^2_- \ .$$

We now turn to SU(3)-structures. Let (T, J, ω) be a 6-dimensional vector space with a U(n)-structure; on $T_{\mathbb{C}} = T_{1,0} \oplus T_{0,1}$, J acts as $i \oplus -i$. We say that an element α of $\Lambda^*_{\mathbb{C}}T$ has type (p, q) if

$$\begin{cases} X_1 \lrcorner \ldots \lrcorner X_{p+1} \lrcorner \alpha = 0 & \forall X_1, \ldots, X_{p+1} \in T_{1,0} \\ X_1 \lrcorner \ldots \lrcorner X_{q+1} \lrcorner \alpha = 0 & \forall X_i, \ldots, X_{q+1} \in T_{0,1} \end{cases}$$

We denote $\Lambda^{p,q}$ the space of forms of type (p, q); each of these spaces is a U(n)-module. There is a natural real structure on $\Lambda^*_{\mathbb{C}}T$, induced by the conjugation on $T_{\mathbb{C}}$, which swaps $\Lambda^{p,q}$ and $\Lambda^{q,p}$. We use this real structure to define real spaces $[\![\Lambda^{p,q}]\!]$, $[\Lambda^{p,p}]$ such that

$$\Lambda^{p,q} \oplus \Lambda^{q,p} \cong [\![\Lambda^{p,q}]\!] \otimes \mathbb{C} \ , \quad p \neq q$$
$$\Lambda^{p,p} \cong [\Lambda^{p,p}] \otimes \mathbb{C} \ .$$

Consider the natural map $\Lambda^{p-1,q-1} \to \Lambda^{p,q}$ given by wedging with ω; we denote its cokernel by $\Lambda_0^{p,q}$. Then $\Lambda_0^{p,q}$ is irreducible, not only as a $\mathrm{U}(n)$-module, but also as a $\mathrm{SU}(n)$-module. The Lie algebra $\mathfrak{su}(3)$ is isomorphic to $[\Lambda_0^{1,1}]$.

Proposition 2.15. *The intrinsic torsion of an* $\mathrm{SU}(3)$*-structure takes values in the* $\mathrm{SU}(3)$*-module*

$$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$$

where

$$\mathcal{W}_1 \cong 2\mathbb{R}\,, \qquad \mathcal{W}_2 \cong 2[\Lambda_0^{1,1}]\,, \qquad \mathcal{W}_3 \cong [\![\Lambda_0^{2,1}]\!]\,, \qquad \mathcal{W}_4 \cong 2\Lambda^1\,.$$

We define an $\mathrm{SU}(3)$-structure to be *half-flat* if it is defined by a generalized Killing spinor. This is not the classical definition, but we shall prove equivalence in Section 2.7. For the moment, we recall that as an $\mathrm{SU}(3)$-module, $\Lambda_0^{2,1} \cong S^{2,0}$; the same argument as before gives

Proposition 2.16. *The intrinsic torsion of a half-flat structure takes values in the* $\mathrm{SU}(3)$*-module*

$$\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3\,,$$

where $\mathcal{W}_1^-$ *is a submodule of* $\mathcal{W}_1 \cong 2\mathbb{R}$ *isomorphic to* $\mathbb{R}$*, and* $\mathcal{W}_2^-$ *is a submodule of* $\mathcal{W}_2 \cong 2[\Lambda_0^{1,1}]$ *isomorphic to* $[\Lambda_0^{1,1}]$*.*

For more details on the intrinsic torsion of $\mathrm{SU}(3)$-structures, see [22, 15].

2.5 $\mathrm{SU}(3)$-holonomy manifolds, hypersurfaces, curvature

We compute the curvature of a generic hypo manifold M. If M arises as a hypersurface in a $\mathrm{SU}(3)$-holonomy manifold, we relate the curvature of M to the curvature of the $\mathrm{SU}(3)$-holonomy manifold which contains M as a hypersurface. We know from Equations 2.1 and the identification of the intrinsic torsion with the Weingarten form that the curvature and intrinsic torsion of M determine the curvature of $\mathbf{M}$, as sections of a bundle over M. In this special case, we prove that this dependence is pointwise; in other words, we are dealing with a bundle map. For details on the general theory of $\mathrm{SU}(n)$-holonomy manifolds, we refer to [31].

Let M be a 5-manifold with a SO(4)-structure $P_{\mathrm{SO}(4)}$. Eventually, we will assume $P_{\mathrm{SO}(4)}$ to be obtained from a hypo structure by extension, but we do not need this hypothesis for the moment. Extend $P_{\mathrm{SO}(4)}$ to a bundle $P_{\mathrm{SO}(5)}$ with fibre SO(5), and consider the Levi-Civita connection form ω on $P_{\mathrm{SO}(5)}$; the curvature form is $D\omega$, corresponding to a map

$$\overline{R} : P \to \Lambda^2 T^* \otimes \mathfrak{so}(5) .$$

Following the notation of [26], we set $R = 2D\omega$, so that

$$\underline{R}(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} .$$

By the first Bianchi identity, $\overline{R}$ really takes values in the kernel $\mathcal{R}$ of the natural map

$$a : S^2(\Lambda^2 T^*) \to \Lambda^4 T^* ,$$

having identified $\mathfrak{so}(5)$ with $\Lambda^2 T^*$. Recall that the Ricci tensor, taking values in $\mathrm{Ric} = S^2 T^*$, is defined by the contraction

$$\overline{\mathrm{Ric}} = \sum_i \overline{R}(e_i, \cdot, \cdot, e_i) ;$$

it is customary to write $\mathrm{Ric} = \mathrm{Ric}_0 + \mathbb{R}$, separating the traceless part. Accordingly, we have an exact sequence of SO(5)-modules

$$0 \to \mathrm{Weyl} \to \mathcal{R} \to \mathrm{Ric}_0 \oplus \mathbb{R} \to 0 .$$

Now restrict the curvature to a tensorial form on $P_{\mathrm{SO}(4)}$; then $\mathcal{R}$ is to be viewed as an SO(4)-module.

Write $T = \mathbb{R} \oplus \Lambda^1$, where Λ^1 represents the standard 4-dimensional representation of SO(4); we shall write Λ^k for $\Lambda^k(\Lambda^1)$ and S^2 for $S^2(\Lambda^1)$. Recall that irreducible 8-dimensional modules $K_\pm$ are defined by the exact sequence of SO(4)-modules:

$$0 \to K_\pm \to \Lambda^1 \otimes \Lambda^2_\pm \to \Lambda^1 \to 0 .$$

Let $W_\pm$ be the irreducible 5-dimensional module given by $S^2_0(\Lambda^2_\pm)$.

Proposition 2.17. *The components of the Levi-Civita curvature of a five-manifold with an* SO(4)*-structure take values in*

$$\mathrm{Ric}_0 = \mathbb{R} \oplus \Lambda^1 \oplus S^2_0 , \qquad \mathrm{Weyl} = S^2_0 \oplus W_+ \oplus W_- \oplus K_+ \oplus K_- .$$

Proof. Clearly, $\mathrm{Ric} = 2\mathbb{R} \oplus \Lambda^1 \oplus S^2_0$, proving the first equation. From

$$\Lambda^2 T^* \cong \Lambda^1 \oplus \Lambda^2_+ \oplus \Lambda^2_- ,$$

it follows:

$$\begin{aligned} S^2(\Lambda^2 T^*) &= 3\mathbb{R} \oplus S^2_0 \oplus S^2_0(\Lambda^2_+) \oplus S^2_0(\Lambda^2_-) \oplus \Lambda^1 \otimes \Lambda^2_+ \oplus \Lambda^1 \otimes \Lambda^2_- \oplus \Lambda^2_+ \otimes \Lambda^2_- = \\ &= 3\mathbb{R} \oplus 2\Lambda^1 \oplus 2S^2_0 \oplus W_+ \oplus W_- \oplus K_+ \oplus K_- \end{aligned}$$

On the other hand $\Lambda^4 T^* \cong T^* = \mathbb{R} \oplus \Lambda^1$, so by Schur's lemma

$$\mathcal{R} = 2\mathbb{R} \oplus \Lambda^1 \oplus 2S^2_0 \oplus W_+ \oplus W_- \oplus K_+ \oplus K_- . \qquad \square$$

From the remarks of Section 2.4, it is easy to deduce that as SU(2)-modules, Λ^1, Λ^2_-, W_- and K_- are irreducible, and W_+, K_+, Λ^2_+ are trivial. We immediately conclude:

Corollary 2.18. *As* SU(2)*-modules,*

$$\mathrm{Ric}_0 = \mathbb{R} \oplus \Lambda^1 \oplus 3\Lambda^2_- , \qquad \mathrm{Weyl} = 5\mathbb{R} \oplus 3\Lambda^2_- \oplus W_- \oplus 2\Lambda^1 \oplus K_- .$$

Now suppose M is a hypersurface in a holonomy SU(3) manifold $\mathbf{M}$; we want to relate the curvature of M to the curvature of $\mathbf{M}$. For simplicity's sake, we do not refer explicitly to the principal bundles, but we have the diagram of Proposition 2.2 in mind.
Let $\mathbf{T} = \mathbb{R}^6$; we write $\mathbf{\Lambda}^p$ for $\Lambda^p \mathbf{T}$. Define the exterior powers $\mathbf{\Lambda}^{p,q}$ as usual and the symmetric powers $\mathbf{S}^{p,q}$ in the same way. Write

$$[\mathbf{S}^{2,2}] = \mathbf{B} \oplus [\mathbf{\Lambda}^{1,1}_0] \oplus \mathbb{R} ;$$

then $\mathcal{R}_{CY} = \mathbf{B}$. We shall compute $\mathcal{R}_{CY}$ as an SU(2)-module, and more importantly, prove that the normal part of the curvature is pointwise determined by the tangential part. Both tasks require the following:

Lemma 2.19. *The projection $\Lambda^2(\mathbf{T}) \to \Lambda^2(T)$ is injective on $\mathfrak{su}(3)$; its image, as an* SU(2)*-module, is*

$$\mathfrak{su}(3) = \mathbb{R} \oplus \Lambda^1 \oplus \Lambda^2_- .$$

Proof. As SU(2)-modules, we can write

$$\Lambda^2(\mathbf{T}) = \Lambda^2(T) \oplus T \otimes \mathbb{R} \ ;$$

the kernel of the projection is therefore $T \otimes \mathbb{R}$. So, its restriction is injective on $\mathfrak{su}(3) \subset \Lambda^2(\mathbf{T})$ if and only if

$$\mathfrak{su}(3) \cap (T \otimes \mathbb{R}) = 0 \ .$$

Since $\mathfrak{su}(3) = [\Lambda_0^{1,1}]$ contains no simple forms, injectivity holds.
The second part of the Lemma is straightforward. □

Remark. An analogous result holds for $\mathfrak{g}_2$ and $\mathfrak{spin}(7)$, which also contain no simple forms. A similar property was considered by Reyes-Carrión in [30] in the context of complexes of differential operators.

We can now prove:

Proposition 2.20. *The curvature of a 6-manifold* $\mathbf{M}$ *with holonomy* SU(3) *takes values in a space which as an* SU(2)*-module is*

$$\mathcal{R}_{CY} = W_- \oplus K_- \oplus 3\Lambda^2_- \oplus \Lambda^1 \oplus \mathbb{R} \ .$$

If M *is a hypersurface in* $\mathbf{M}$*, the curvature of* $\mathbf{M}$ *at a point of* M *is determined by the tangential curvature at that point.*

Recall that the tangential curvature at x is $([i_\nu^*\Omega]_{\mathfrak{so}(5)})_x$; in terms of Riemann tensors, this means that all components with one or more normal indices are discarded. Once again, we stress the fact that this statement is pointwise, i.e. no derivatives are involved, unlike in (2.2).

Proof. Using Lemma 2.19 and $S_0^2(\Lambda^1) = 3\Lambda^2_-$,

$$S^2(\mathfrak{su}(3)) = 3\mathbb{R} \oplus W_- \oplus 2\Lambda^1 \oplus 4\Lambda^2_- \oplus K_- \ ;$$

The image of $S^2(\mathfrak{su}(3))$ through a is $\mathbb{R} \oplus \mathfrak{su}(3)$ (see [31]); Lemma 2.19 gives the first statement.
The second statement is also a straightforward consequence of Lemma 2.19. □

The relevance of this result comes from the fact that by the Gauss equation (2.4), the tangential part of the curvature of $\mathbf{M}$ is algebraically determined by the Weingarten form and the curvature of M. So, the relation between the curvature of $\mathbf{M}$, the curvature of M and the Weingarten form is purely algebraic. That is to say, given a hypo manifold M which can be embedded in a SU(3)-holonomy 6-manifold $\mathbf{M}$, we can compute the curvature of $\mathbf{M}$ along M without having to compute derivatives.

Remark. We do not make a distinction between Ric and Weyl for the Levi-Civita connection on $\mathbf{M}$ because we are assuming SU(3) holonomy, and so the curvature only consists of the Weyl tensor, or, more precisely, the Bochner tensor [31].

So far in this section, we have only used the Levi-Civita connection on M. For an embedded M, we can also use the Gauss equation to compute the hypo curvature (namely, the curvature of the hypo connection). Whilst the hypo connection depends on the SU(2)-structure, the hypo curvature only depends on the Riemannian metrics:

Proposition 2.21. *The hypo curvature of M is given by*

$$\Omega_{\mathfrak{su}(2)} = 2[\Omega_{\mathfrak{so}(5)}]_{\mathfrak{su}(2)} - [i_\nu^*\Omega]_{\mathfrak{su}(2)} \ .$$

Proof. Let $\omega_{\mathfrak{so}(5)}$ be the Riemannian connection induced by $\boldsymbol{\omega}$. The tangential curvature of $\boldsymbol{\omega}$ is related to the curvature of $\omega_{\mathfrak{so}(5)}$ by Gauss' equation (2.4), which upon restriction to $\mathfrak{su}(2)_-$ gives

$$\begin{aligned}[i_\nu^*\Omega]_{\mathfrak{su}(2)} = [\Omega_{\mathfrak{so}(5)}]_{\mathfrak{su}(2)} &+ (A^1 \wedge A^2 - A^3 \wedge A^4)\sigma_1 + \\ &+ (A^1 \wedge A^3 - A^4 \wedge A^2)\sigma_2 + (A^1 \wedge A^4 - A^2 \wedge A^3)\sigma_3\end{aligned}$$

Comparing this equation with Lemma 2.12 completes the proof. □

Let us check this result in the example of Section 2.3. Indeed, we have

$$\Omega_{\mathfrak{su}(2)} = \frac{1}{2}e^{12} \otimes \sigma_1 \ , \quad [\Omega_{\mathfrak{so}(5)}]_{\mathfrak{su}(2)} = \frac{3}{4}e^{12} \otimes \sigma_1 \ , \quad [i_\nu^*\Omega_{\mathfrak{su}(3)}]_{\mathfrak{su}(2)} = e^{12} \otimes \sigma_1 \ ,$$

so Proposition 2.21 clearly holds.

2.6 Embedding as hypersurfaces

So far we have seen that if M is an immersed hypersurface in $\mathbf{M}$, where $\mathbf{M}$ has a spin structure $\mathbf{P}$ and a parallel spinor $\boldsymbol{\psi}$, then a spin structure P and a spinor ψ are induced on M; since ψ is a generalized Killing spinor, we have formulae for the curvature of the induced metric, and the intrinsic torsion of the structure defined by ψ. We now consider the inverse problem: fix (M, P, ψ, A), where M is a compact manifold, P is a spin structure, ψ is a spinor and $\underline{A}$ is a symmetric tensor as in Definition 2.4. Is it possible to embed M as a hypersurface in a spin manifold $\mathbf{M}$ with a spin structure $\mathbf{P}$ and a parallel spinor $\boldsymbol{\psi}$, such that ψ is the restriction of $\boldsymbol{\psi}$ and P is induced by $\mathbf{P}$? We shall say that (M, P, ψ, A) has the *embedding property* if the answer is positive.

Remark. The reason for looking at embeddings rather than immersions is that with this condition, for compact M, one can replace $\mathbf{M}$ with a tubular neighbourhood of M in $\mathbf{M}$, which topologically is just $M \times (a, b)$. Indeed, what we aim at is a recipe to construct a metric on $M \times (a, b)$ which gives the required embedding. To this end, it seems essential that M be compact: otherwise, the interval (a, b) where the metric on $\mathbf{M}$ is defined might shrink to zero when one goes to infinity along M.

The case of Killing spinors is settled by the following result [5]:

Theorem 2.22 (Bryant, Bär)**.** *If ψ is a Killing spinor, then (M, P, ψ, A) has the embedding property. In particular if $A = -1/2\mathrm{Id}$, $\mathbf{M} = M \times_r \mathbb{R}^+$.*

There is a converse to the second statement: if $\mathbf{M}$ is the cone over M, then a parallel spinor on $\mathbf{M}$ restricts to a Killing spinor with Killing constant $-1/2$ on M. However, this is obvious from the identification of A with the Weingarten form W.

Theorem 2.23 (Bär-Gauduchon-Moroianu)**.** *If $\nabla\underline{A}$ is totally symmetric, then (M, P, ψ, A) has the embedding property.*

In dimension 2, $\nabla\underline{A}$ is always symmetric, and one recovers the following theorem [19]:

Theorem 2.24 (Friedrich)**.** *Every two-dimensional (M, P, ψ, A) has the embedding property.*

However, in this case the geometry of $\mathbf{M}$ is trivial, because by Theorem 1.12 a parallel spinor gives flatness in dimension 3. A more recent result is the following [25]:

Theorem 2.25 (Hitchin)**.** *Every six- or seven-dimensional* (M, P, ψ, A) *has the embedding property.*

In the case of dimension 6, the holonomy of $\mathbf{M}$ is contained in G_2. Indeed, we know that the spinor representation in dimension 7 is real, so that a parallel spinor may be assumed to be real; since the stabilizer of a real spinor is G_2, a parallel spinor defines an integrable G_2 structure. One might also view this fact as a consequence of Theorem 1.12: the holonomy of $\mathbf{M}$ must be one of G_2, $\mathrm{SU}(3)$, $\mathrm{Sp}(2)$, $\mathrm{Sp}(1)$, $\{e\}$, and all of these are subgroups of G_2. Similarly, if M has dimension 7 then $\mathbf{M}$ has holonomy $\mathrm{Spin}(7)$.

Theorem 2.22 can be generalized in another direction, not related to the embedding property; this result was proved by Bilal and Metzger in the 6-dimensional nearly-Kähler case [9, 2]:

Theorem 2.26. *Let* (M, P, ψ) *be a manifold with a Killing spinor, and Killing constant equal to* $-1/2$*. Then* $\mathbf{M} = M \times_{\sin(t)} (0, \pi)$ *also has a Killing spinor with Killing constant equal to* $-1/2$*.*

Proof. Take the cone $M \times_y \mathbb{R}^+_y$, and then the product with $\mathbb{R}$, to define a metric h on $M \times \mathbb{R}_x \times \mathbb{R}^+_y$, which then admits a parallel spinor by Theorem 2.22. Consider the inclusion

$$\mathbb{R}_x \times \mathbb{R}^+_y \subset \mathbb{R} \times \mathbb{R} = \mathbb{C} \ ;$$

then $\theta \to e^{i\theta}$ defines an embedding

$$j : M \times (0, \pi) \to M \times \mathbb{R} \times \mathbb{R}^+ \ ,$$

and h pulls back to a metric j^*h on $\mathbf{M}$. Consequently, the parallel spinor restricts to a generalized Killing spinor ψ on $\mathbf{M}$. Writing $x + iy = \rho e^{i\theta}$,

$$h = y^2 g + dx^2 + dy^2 = (\sin^2\theta)\rho^2 g + d\rho^2 \ .$$

This clearly shows that j^*h is indeed the warped product metric, and that h is the conical metric over j^*h, i.e. the Weingarten tensor of $\mathbf{M}$ is minus the identity, where the minus sign is a consequence of $\nu = -\partial/\partial_\rho$. Therefore ψ is a Killing spinor with Killing constant equal to $-1/2$. $\square$

2.7 Half-flat geometry

Let us return to SU(3)-structures in dimension 6. In this section we show that these structures can be defined using differential forms; in particular, we study structures defined by a generalized Killing spinor, that we call half-flat. We shall prove that this definition matches with the standard definition.
Let $T = \mathbb{R}^6$ with the action of $\mathrm{Spin}(6) = \mathrm{SU}(4)$ given in Section 1.6. We write $\Lambda^{p,q}$ for $\Lambda^{p,q}(V^*)$, where V is the 8-dimensional space with an SU(4) action we used to define the spinor representation. We have seen that

$$\Sigma_6^+ = \Lambda^{1,0} \cong \Lambda^{0,3} , \qquad\qquad \Sigma_6^- = \Lambda^{0,1} \cong \Lambda^{3,0} .$$

Therefore

$$\begin{aligned} \Sigma_6^+ \otimes \Sigma_6^+ &= \Lambda^{2,0} \oplus S^{2,0} \\ \Sigma_6^+ \otimes \Sigma_6^- &= \Lambda_0^{1,1} \oplus \mathbb{R} \end{aligned}$$

Now, we have

$$\Lambda^p T^* = \begin{cases} \mathbb{R} , & p = 0, 6 \\ [\![\Lambda^{2,0}]\!] , & p = 1, 5 \\ [\Lambda_0^{1,1}] , & p = 2, 4 \\ 2[\![S^{2,0}]\!] , & p = 3 \end{cases}$$

Therefore, as Spin(6)-modules,

$$\Sigma_6 \otimes \Sigma_6 = \Lambda^* T .$$

To compute explicitly this isomorphism, it helps to consider eigenvalues and eigenspaces for the Clifford multiplication.

Lemma 2.27. *For a real spinor ψ, the space of two-forms whose Clifford action has eigenvalues $-3i$ and i, with corresponding eigenspaces $\langle\psi\rangle$ and $\psi^\perp$, is one-dimensional.*

By a real spinor we mean a spinor of the form $\psi + \overline{\psi}$, with ψ in Σ^+; Lemma 2.27 also holds for chiral spinors (i.e. elements of $\Sigma^+ \cup \Sigma^-$). On the other hand, a two-form does not determine a real spinor uniquely, but only up to the action of the complex structure on V, corresponding to the fact that a two-form by itself is not sufficient to reduce from SO(5) to SU(3).

Lemma 2.28. *For a real spinor ψ, the space of three-forms whose Clifford action has eigenvalues -4 and 0, with corresponding eigenspaces $\langle\psi\rangle$ and $\psi^\perp$, is one-dimensional.*

Proposition 2.29. *A spinor ψ defines differential forms (ω, ψ^+) characterized up to multiple by*

$$\begin{aligned} &\omega \cdot \psi = -3i\psi \,, && \psi^+ \cdot \psi = -4\psi \,, \\ &\omega \cdot \phi = i\phi; , && \psi^+ \cdot \phi = 0 \quad \forall \phi \in \psi^\perp \,. \end{aligned}$$

Let u be a frame such that $u^*\psi = u_0$; then

$$u^*\omega = e^{12} + e^{34} + e^{56} \,, \qquad u^*\psi^+ = e^{135} - e^{146} - e^{245} - e^{236} \,. \tag{2.9}$$

Now fix a generalized Killing spinor ψ. In analogy with Section 2.2, we define the *half-flat connection*, which is a connection on the SU(3)-structure. Explicitly, the Levi-Civita connection differs from the hypo connection by

$$\begin{aligned} \omega^\perp = \frac{1}{4}\big(& A^1 \otimes (-e^{35} + e^{46}) + A^2 \otimes (-e^{36} - e^{45}) + A^3 \otimes (e^{15} - e^{26}) + \\ &+ A^4 \otimes (e^{16} + e^{25}) + A^5 \otimes (-e^{13} + e^{24}) + A^6 \otimes (-e^{14} - e^{23})\big) \end{aligned}$$

Recall that we defined a half-flat SU(3)-structure as the structure defined on a 6-manifold by a generalized Killing spinor. However, we can think of an SU(3)-structure as defined by forms (ω, ψ^+) which in a frame look like (2.9); half-flatness can then be characterized in these terms:

Proposition 2.30. *An* SU(3)*-structure is half-flat if and only if ω^2 and ψ^+ are closed.*

Proof. By Proposition 2.16 and [15], each condition means that the intrinsic torsion is forced to lie in an SU(3)-module isomorphic to

$$\mathbb{R} \oplus [\Lambda_0^{1,1}] \oplus [\![\Lambda_0^{2,1}]\!] \,.$$

A priori these isomorphic modules need not coincide; however, they have to coincide if one of them is contained in the other one. It is therefore sufficient to prove the "only if" part of the statement.
Now consider a half-flat structure; using the explicit formula for $\omega^\perp$, it is easy to check that ψ^+ and ω^2 have to be closed. $\square$

Chapter 3

Invariant differential forms

In this chapter we study invariant differential forms on an associated vector bundle $P \times_G V$, where P is a G-structure and V a representation of G.

In the first section, we introduce the gauge group of P and we compute its action on the manifold $P \times_G V$. We conclude that the space of forms which are invariant under this action is not interesting to us, because it does not reflect the geometry of the base manifold.
In the second section, we define another action of the gauge group; compared with the first one, this action is less natural, because it depends on the choice of a connection on P. However, the resulting space of invariant forms is geometrically more interesting.
Using the algebra of invariant forms, we can define invariant G-structures; to impose intrinsic torsion conditions we must be able to compute the action of the exterior derivative d on these forms. Unfortunately, the space of invariant forms is not closed under d. In the third section we solve this problem by imposing parallelism conditions: we assume that the connection has parallel torsion, and we consider the algebra of invariant parallel forms.
In the fourth section we study this algebra as a module over the algebra of invariant parallel functions. If G is compact and connected, this module is finitely generated; we provide an effective characterization of systems of generators.
In the fifth section, we specialize these results to the case $V = \mathfrak{so}(3)$, obtaining more explicit formulae.
Our characterization of systems of generators is thus far not constructive, and it does not help to compute the action of d; in the sixth section we discuss a technique to produce generators, and compute the action of d on

them.
The seventh section is devoted to the particular case of homogeneous spaces.

3.1 The gauge group

Let P be a G-structure on M. Consider the bundle $\underline{G} = P \times_G G$, where G acts on itself by conjugation; the space $\mathcal{G}$ of its sections is called the *gauge group* of P. Indeed, the bundle map

$$\begin{gathered} \underline{G} \times \underline{G} \to \underline{G} \\ [u, g] \times [u, h] \to [u, gh] \end{gathered}$$

induces a group structure on $\mathcal{G}$. Now let (V, ρ_V) be a G-module, and let X be the total space of the vector bundle

$$\underline{V} = P \times_G V \ ;$$

then we have a bundle map

$$\underline{G} \times \underline{V} \to \underline{V} \tag{3.1}$$

$$[u, g] \times [u, v] \to [u, gv] \tag{3.2}$$

which is easily seen to be well-defined; in this way, $\mathcal{G}$ acts on X as a group of diffeomorphisms. Whilst we are ultimately interested in the geometry of X, we shall reserve the notation with the underscore for bundles on M constructed from P, because eventually everything will be expressed in terms of bundles on M.

Remark. The construction of $\underline{V}$, and the action of $\mathcal{G}$ on it, generalizes to arbitrary manifolds with a G action, finite-dimensional or not.

We must define the action of $\mathcal{G}$ on $\Omega(X)$; there are at least two possible choices:

1. The action induced by the action on X in the following way:

$$\underline{g}\alpha = (\underline{g}^{-1})^* \alpha \ , \quad \underline{g} \in \mathcal{G} \ , \ \alpha \in \Omega(X) \ . \tag{3.3}$$

2. The action induced by the action on $\Gamma(P \times V)$:

$$[u, g](u, v) = (ug, v) \ . \tag{3.4}$$

To see how this induces an action on $\Omega(X)$, let $\pi : X \to M$ be the projection; then the pullback to X of P is

$$\pi^* P = P \times V \ ,$$

with principal H action given by $(R_h, \rho_V(h^{-1}))$. As we shall see, by choosing a connection on P we can make $\pi^* P$ into a G-structure on X. Then, an action on $\Gamma(\pi^* P)$ gives an action on $\Omega(X)$.

Whilst (3.3) seems the more natural action to consider, it is also the least interesting from our point of view. We shall spend the rest of this section explaining why. Consider the map

$$\begin{array}{ccc} TP \times TV \times TG & \longrightarrow & TX \\ \downarrow & & \downarrow \\ P \times V \times G & \longrightarrow & X \end{array}$$

given by

$$\begin{array}{ccc} \big((u;u'),(v;v'),(g;g')\big) & \longrightarrow & \big([u,gv];[u',gv'+g'v]\big) \\ \downarrow & & \downarrow \\ (u,v,g) & \longrightarrow & [u,gv] \end{array}$$

Our notation is that, say, an element u' of $T_u P$ is denoted $(u;u')$; for the moment, we think of TX as determined by regarding X as a quotient of $P \times V$. Clearly, the map on the top row is the derivative of the map on the bottom row. To compute the action of $\mathcal{G}$ on TX, we proceed as follows. Take a curve in X and lift it to a curve (u_t, v_t) in $P \times V$. Consider a generic element $\underline{g}$ of $\mathcal{G}$; we view $\underline{g}$ as a map $M \to \underline{G}$ which defines a curve (u_t, g_t) in $P \times G$ by

$$[u_t, g_t] = \underline{g}\left(\pi\big([u_t, v_t]\big)\right) \ ,$$

that one might also view as a section of the pullback of $\underline{G}$ to the interval on which our curves are defined. The above diagram shows that $\underline{g}$ acts by

$$\big([u_0, v_0];[u_0', v_0']\big) \to \big([u_0, g_0 v_0];[u_0', g_0 v_0' + g_0' v_0]\big) \tag{3.5}$$

Now take a section s of TX and write

$$s(x) = \big([u_0, v_0];[u_0', v_0']\big) \quad \text{for some } x = [u_0, v_0], v_0 \neq 0 \ .$$

Choose $\underline{g}$ such that $g_0 = e$, and suppose that s is invariant under $\underline{g}$; then $g_0' v_0 = v_0$. Hence, a section of TX invariant under all of $\mathcal{G}$ can only exist if the (infinitesimal) action of G on V is trivial. In general, in order to have invariant sections, one would have to restrict $\mathcal{G}$. A valid option would be to consider parallel elements of $\mathcal{G}$, i.e. sections of $\underline{G}$ which locally have the form $[u, g]$ where g is a constant and u is horizontal with respect to some connection. However, it is not hard to check that with this choice, the pullback of a form on M is invariant; moreover, with a little bit of work one can prove that the space of invariant forms is simply

$$\Omega(M) \otimes \Omega(V)^G .$$

This space is not interesting because it does not quite reflect the geometry of the situation: indeed, this space can be defined for an arbitrary principal bundle P and not only for G-structures. From our point of view, this means losing significant information, i.e. the fact that the gauge group acts infinitesimally on the manifold, beside acting on the fibre.

3.2 Gauge-invariant forms

The conclusion of Section 3.1 was that we are not interested in action (3.3); for the rest of this chapter, the action of the gauge group will be (3.4).
In order to define this action, we have to make $\pi^* P$ into a G-structure on X, i.e. determine an isomorphism

$$\lambda : TX \cong \pi^* P \times_G (T \oplus V) ,$$

where $T = \mathbb{R}^n$, so that $\underline{T} = TM$. Such an identification will give an inclusion

$$\pi^* TM = \pi^* P \times_G T \subset \pi^* P \times_G (T \oplus V) = TX ;$$

so having a G-structure on $\pi^* P$ is essentially the same as having a notion of horizontal lift for sections of $\underline{V}$. Now fix a connection on P, with connection form ω; this choice determines an isomorphism λ as above which we now describe explicitly. Let θ be the solder form on P, taking values in T. Define

$$\begin{aligned} \lambda : T(P \times V) &\to \pi^* P \times (T \oplus V) \\ (u, v; u', v') &\to \big(([u, v], u), (\theta_u(u'), \omega_u(u')v + v')\big) \end{aligned}$$

Since the solder form and the connection form are pseudotensorial, λ is G-equivariant:

$$\lambda\left(ug, g^{-1}v; (R_g)_{*u}u', g^{-1}v'\right) = \\ \left(\left([ug, g^{-1}v], ug\right), \left((R_g^*\theta)_u(u'), (R_g^*\omega)_u(u')g^{-1}v + g^{-1}v'\right)\right) = \\ = g\lambda(u, v; u', v') \ .$$

Therefore λ induces a map $TX \to \pi^*P \times (T \oplus V)$. Moreover, a section $[u, v]$ of $\pi^*P \times (T \oplus V)$ is a section of $\pi^*P \times T$, if and only if, restricting to every curve $[u(t), v(t)]$, one has $\omega_u(u')v + v' = 0$. This is exactly the same thing as saying that $\nabla[u, v] = 0$; in other words, λ preserves the notion of horizontality. In conclusion, λ is the required isomorphism; in particular, it induces an isomorphism

$$\lambda : \Lambda(X) \cong \pi^*P \times_G \Lambda^*(T \oplus V) \ , \tag{3.6}$$

the notation being that for a vector space W, Λ^*W is the exterior algebra on W^*, which we also denote ΛW^*.

Remark. The gauge group $\mathcal{G}$ acts on P on the right by

$$u[u, g] = ug$$

and this is consistent with its action on $\underline{V}$, i.e.

$$[u, g][u, v] = [u[u, g], v] \ .$$

The action of $\mathcal{G}$ on a form on M can be written

$$[u, g][u, \alpha] = [ug, \alpha] \ , \quad \alpha \in \Lambda^*T \ .$$

More generally, if we consider a W-valued pseudotensorial form α on M, we can use the connection to write it as an equivariant map

$$\overline{\alpha} : P \to \Lambda^*T \otimes \Lambda^*\mathfrak{g} \otimes W \ ;$$

the gauge group acts on α in the usual way.

By way of example, we can now prove the following:

Proposition 3.1. *The connection form ω is $\mathcal{G}$-invariant.*

Proof. As a map on P, $\overline{\omega}$ is constant; this implies $\mathcal{G}$-invariance. $\square$

We now express $\Omega(X)$ as the space of sections of a bundle over M. Let $R = C^\infty(V, G)$ be the space of G-valued functions on V; G acts on the left on R by

$$(gr)v = g(r(g^{-1}v)) \ .$$

Consider the map $\mu : \underline{R}_x \to \Gamma(X_x, \pi^* P)$ given by

$$\mu\big([u,r]\big)_{[u,v]} = \big([u,v], ur(v)\big) \ ;$$

to check well-definedness, one needs $\mu\big([u,r]\big) = \mu\big([ug, g^{-1}r]\big)$, which follows immediately from the definition of the action on R. For every open $U \subset M$, μ induces a map

$$\mu_U : \Gamma(U, \underline{R}) \to \Gamma(\pi^{-1}(U), \pi^* P) \ .$$

By (3.1), the gauge group acts on sections of $\underline{R}$.

Lemma 3.2. *The maps μ_U are $\mathcal{G}$-equivariant isomorphisms.*

Proof. The only non-obvious part of the statement is equivariance. We can work on a fibre X_x; the action of $[u,g]$ on $\underline{R}$ satisfies

$$\mu\big([u,g][u,r]\big)_{[u,gv]} = \big([u,gv], ugr(v)\big) \ .$$

On the other hand, we have

$$\mu\big([u,r]\big)_{[u,v]} = \big([ur(v), r^{-1}(v)v], ur(v)\big) \ ,$$

so the action of $[u,g]$ on sections of $\pi^* P$ is given by

$$[ur(v), r^{-1}(v)gr(v)]\big([ur(v), r^{-1}(v)v], ur(v)\big) = \big([ugr(v), r^{-1}(v)v], ugr(v)\big) \ ,$$

clearly showing that μ is equivariant. $\square$

We establish the following notation: if f is a map defined on V, then

$$(g^* f)v = f(gv) \ ;$$

we define a left G action on $C^\infty(V)$ by

$$gf = (g^{-1})^* f \ .$$

Using

$$\Omega(V) = C^\infty(V) \otimes \Lambda^* V \ , \tag{3.7}$$

we can extend this action to $\Omega(V)$. Equivalently, we can use the pullback and set

$$g\alpha = (g^{-1})^* \alpha \ .$$

Here, we are regarding V as a manifold: the space $\Omega(V)$ represents the space of sections of a (trivial) bundle on V. However, to avoid ambiguity we shall *not* call this bundle $\Lambda(V)$.

Proposition 3.3. *The maps μ_U induce isomorphisms*

$$\Gamma(TX) \cong \Gamma(\underline{C^\infty(V) \otimes (T \oplus V)}) \tag{3.8}$$

$$\Omega(X) \cong \Gamma(\underline{\Lambda^* T \otimes \Omega(V)}) \tag{3.9}$$

Proof. Represent a section of $\pi^* P$ over X_x by $[u, r]$. Smooth functions on X can be identified with sections of $\underline{C^\infty(V)}$. We have a pairing

$$\begin{aligned} \Gamma(X_x, \pi^* P \times_G (T \oplus V)) \times (\underline{C^\infty(V) \otimes (T^* \oplus V^*)})_x &\to (\underline{C^\infty(V)})_x \\ [[u, r], (w \oplus v')] \times [u, f \otimes (\eta \oplus \xi)] &\to [u, f\eta(rw) + f\xi(rv')] \end{aligned}$$

This pairing is the bundle version of the pairing

$$(T \oplus V) \times (T^* \oplus V^*) \to \mathbb{R} \ .$$

Using (3.7), (3.9) follows. Equation 3.8 is proved in a similar manner. □

Now identify $\Lambda^* T \otimes \Omega(V)$ with the space $\Omega(V, \Lambda^* T)$ of $\Lambda^* T$-valued forms on V.

Theorem 3.4. *Under the identification (3.9), the action of $\mathcal{G}$ on $\Omega(X)$ is given by*

$$[u, g][u, \alpha \otimes \beta] = [u, g\alpha \otimes g\beta] \ .$$

In particular, the space of invariant forms is

$$\Omega(X)^{\mathcal{G}} = C^\infty(M) \otimes_{\mathbb{R}} \Omega(V, \Lambda^* T)^G \ .$$

Proof. By construction,

$$[u,g][u,\alpha\otimes\beta]=[ug,\alpha\otimes\beta]=[u,g\alpha\otimes g\beta]\ ,$$

proving the first part of the theorem. As a consequence, the space of invariant forms is

$$\Gamma(P\times_G(\Omega(V,\Lambda^*T)^G))\ .$$

On the other hand, $\underline{\Omega(V,\Lambda^*T)^G}$ is a trivial bundle, because by construction the structure group G acts trivially on its fibre. Therefore

$$\Gamma(\underline{\Omega(V,\Lambda^*T)^G})=C^\infty(M)\otimes_{\mathbb{R}}\Omega(V,\Lambda^*T)^G\ ,$$

concluding the proof. □

The space of invariant forms we have found does not have the desirable property of being closed under d. Indeed, it contains all functions on M, but it does not contain all exact forms on M. In the next section we shall see that imposing appropriate conditions on ω, the space of parallel invariant forms is d-closed.

3.3 Closure under d

We have seen that the algebra of invariant forms is not closed under d. We can work around this problem by discarding the $C^\infty(M)$ factor, and consider

$$\Omega(V,\Lambda^*T)^G\cong 1\otimes\Omega(V,\Lambda^*T)^G\subseteq\Omega(X)^{\mathcal{G}}\ .$$

In this section we prove that this subalgebra of $\Omega(X)$ is indeed closed under d, provided that the torsion of ω is parallel.

First, we characterize this subalgebra as follows:

Lemma 3.5. *A $\mathcal{G}$-invariant form on X lies in $\Omega(V,\Lambda^*T)^G$ if and only if it is parallel as a section of $\underline{\Omega(V,\Lambda^*T)}$.*

Proof. A section of $\underline{\Omega(V,\Lambda^*T)}$ can locally be written $[s,\alpha]$, where s is a section of P and α takes values in $\Omega(V,\Lambda^*T)$. Then,

$$\nabla[s,\alpha]=[s,d\alpha+s^*\omega\,\alpha]=[s,d\alpha]\ ,$$

because α is G-invariant. Therefore $[s,\alpha]$ is parallel if and only if α is constant. □

Remark. The curvature form Ω cannot be expected to lie in $\Omega(V, \Lambda^* T)^G$. For instance, if P is the bundle of orthonormal frames and ω is the Levi-Civita connection, then the curvature can lie in $\Omega(V, \Lambda^* T)^G$ only if M is locally symmetric.

Proposition 3.6. *Let W be a G-module, and let the torsion of ω satisfy*

$$\nabla \Theta = 0 \ .$$

Immerse $(\Lambda^* T \otimes W)^G$ into $\Omega(M, \underline{W})$. Then $(\Lambda^* T \otimes W)^G$ is D-closed.

Proof. Let $\overline{\alpha}$ be an element of $(\Lambda^* T \otimes W)^G$. By Lemma 3.5, $\nabla \overline{\alpha} = 0$. At each point, $D\alpha$ is determined by the torsion and $\nabla \alpha$ in a G-equivariant way; the statement follows. $\square$

We can identify $\Omega(X)$ with $\Omega(M, \underline{\Omega(V)})$. So, we identify $C^\infty(M)$ with sections of $\underline{C^\infty(V)}$, and $\Omega^1(M)$ with

$$\Gamma(\underline{\Omega^1(V)}) \oplus \Omega^1(M, \underline{C^\infty(V)}) \ .$$

We define the covariant derivative

$$\Gamma(\underline{C^\infty(V)}) \to \Omega^1(M, \underline{C^\infty(V)})$$

in the usual way. The standard operator

$$d : C^\infty(V) \to \Omega^1(V)$$

is G-equivariant, and it therefore induces a bundle map $\underline{d}$.

Lemma 3.7. *The map $d : C^\infty(X) \to \Omega^1(X)$ satisfies*

$$dh = \underline{d}h + \nabla h \tag{3.10}$$

where h is a section of $\underline{C^\infty(V)}$.

Proof. It is sufficient to prove the statement at a point $[u, v]$ of X; let s be a local section of P with $u = s(x)$ for some x in M, and write $h = [s, f]$. We must evaluate both sides of (3.10) on a vector

$$[u, w] \in T_{[u,v]} X \ , \quad w \in T \oplus V \ .$$

Suppose first that w lies in V. Let σ be a curve on V with

$$\sigma(0) = v \ , \quad \sigma'(0) = w \ ;$$

then $\tilde{\sigma} = [u, \sigma]$ is a curve on X satisfying

$$\partial_t \tilde{\sigma}(0) = [u, w] \ .$$

Clearly, we have

$$\partial_t (\tilde{\sigma}^* h)(0) = df_v(w) \ ,$$

so (3.10) holds in this case.

Now let w lie in T. Take a curve $\sigma(t)$ on M satisfying

$$\sigma(0) = x \ , \quad \sigma'(0) = [u, w] \in T_x M \ ;$$

then $\tilde{\sigma} = [\sigma^* s, v]$ is a curve on X. We have

$$\lambda(u, v; s', 0) = \big(([u, v], u), (w, \omega_u(s')v)\big) \ , \quad s' = \partial_t(\sigma^* s)(0) \ ;$$

this implies

$$\partial_t \tilde{\sigma}(0) = [u, w] + [u, \omega_u(s')v] \in \pi^* TM \oplus \ker \pi_* \ ,$$

which means that

$$\partial_t (\tilde{\sigma}^* h)(0) = dh_{[u,v]}\big([u, w]\big) + dh_{[u,v]}\big([u, \omega_u(s')v]\big) \ . \tag{3.11}$$

From the general formula for the covariant derivative:

$$\nabla_{[u,w]}[s, f] = \big[u, \partial_{[u,w]} f + (s^*\omega)\big([u, w]\big) f\big] = \big[u, \partial_{[u,w]} f\big] + \big[u, \omega_u(s')f\big] \ , \tag{3.12}$$

where $\partial_{[u,w]} f$ is the derivative along $[u, w]$ of $f : M \to C^\infty(V)$. By construction,

$$\partial_{[u,w]} f(v) = \partial_t f(v)(0) = \partial_t (\tilde{\sigma}^* h)(0) \ ;$$

substituting (3.11) into (3.12) and using the fact that the infinitesimal action of $\mathfrak{g}$ on $C^\infty(V)$ is minus the derivation, we conclude that (3.10) also holds at horizontal vectors $[u, w]$. $\square$

We can now prove the following:

Theorem 3.8. *If ω has parallel torsion, the space $\Omega(V, \Lambda^*T)^G$ is a differential graded subalgebra of $\Omega(X)$.*

Proof. We extend (3.10) to differential forms. Like for functions, we have bundle maps

$$\underline{d} : \underline{\Omega^q(V)} \to \underline{\Omega^{q+1}(V)} ,$$

and differential operators

$$D : \Omega^p(M, \underline{\Omega^q(V)}) \to \Omega^{p+1}(M, \underline{\Omega^q(V)}) ;$$

we combine them into a differential operator

$$\hat{d} : \Gamma(\Omega(M, \underline{\Omega(V)})) \to \Gamma(\Omega(M, \underline{\Omega(V)}))$$

given by

$$\hat{d}\alpha = \underline{d}\alpha + (-1)^p D\alpha ,$$

where p is the degree on α as a form on M. This operator satisfies the Leibnitz rule, and by Lemma 3.7 it agrees with d on functions; therefore, $\hat{d} = d$.
Since the differential operator d on $\Omega(V)$ is G-equivariant, $\underline{d}$ maps invariant forms to invariant forms. By Proposition 3.6, applied to $W = \Omega(V)$, the same holds for D. □

3.4 Invariant forms on a vector space

By Theorem 3.4, the problem of determining $\Omega(X)^{\mathcal{G}}$ is reduced to determining the space of G-invariant Λ^*T-valued differential forms on V. In this section we discuss this problem in a quite general form: we only assume that G is compact and connected.

We rewrite $\Omega(V, \Lambda^*T)$ as

$$\mathcal{F} = \{\alpha : V \to S\} , \quad S = \Lambda^*T \otimes \Lambda^*V .$$

Let $\mathcal{F}^G$ be the space of G-equivariant elements of $\mathcal{F}$; we view $\mathcal{F}^G$ as a $C^\infty(V)^G$-module. For every subspace W of $\mathcal{F}$ and v in V, we set

$$W_v = \{\alpha(v) \mid \alpha \in W\} .$$

Lemma 3.9. *Let W be a finitely generated submodule of $\mathcal{F}^G$, such that*

$$W_v = \mathcal{F}^G_v \ , \quad \forall v \in V \ .$$

Then $W = \mathcal{F}^G$.

Proof. We construct elements $w_1, \ldots w_k$ of W such that for all v in V, the non-zero $w_i(v)$ constitute a basis of W_v; we shall then prove that $w_1, \ldots, w_k$ generate $\mathcal{F}^G$. Let $w_1, \ldots, w_k$ be any set of generators of W. Consider the set

$$U = \{v \in V \mid w_k(v) \text{ is linearly dependent on } w_1(v), \ldots, w_{k-1}(v)\} \ ;$$

let f be a smooth function on V with $f^{-1}(0) = U$. For instance, we can take f to be the distance between $w_k(v)$ and the vector space spanned by $w_1(v), \ldots, w_{k-1}(v)$. Now define

$$g(x) = \int_G f(gx) \mu_G \ ,$$

where μ_G is the Haar measure on G. Clearly, g is a G-invariant function with $g^{-1}(0) = U$. Thus, we have obtained elements

$$w_1, \ldots, w_{k-1}, g w_k \in W$$

such that at each v, the last element is either zero or linearly independent of $w_1, \ldots, w_{k-1}$. Iterating this procedure for $w_{k-1}, \ldots, w_2$, we find elements with the required property.
At each v, since $w_1(v), \ldots, w_k(v)$ is a basis of $\mathcal{F}^G_v$, we can choose a dual basis $w_1^*(v), \ldots, w_k^*(v)$. The w_i^* can be extended to smooth, equivariant maps $V \to S^*$. Then, for every α in $\mathcal{F}^G$, we have

$$\alpha(v) = \sum_{i=1}^k w_i^*(v)(\alpha(v)) w_i(v) \ ,$$

i.e. $\alpha = \sum w_i^*(\alpha) w_i$. By construction, the $w_i^* \alpha$ are smooth invariant functions; so, α is linearly dependent on the w_i. □

We write $H \leq G$ if H is a closed subgroup of G. We say that V has a finite number of orbit types if

$$\{\mathrm{Stab}(v) \mid v \in V\}$$

contains a finite number of conjugacy classes. The following is well known (see [11]):

Lemma 3.10. *An orthogonal representation of a compact group has a finite number of orbit types.*

We shall also need the following:

Theorem 3.11. *Let M and N be manifolds on which a compact Lie group G acts, and let A be a closed invariant subspace of M. Then every smooth equivariant map $\phi : A \to N$ can be extended to a smooth equivariant map $\overline{\phi} : M \to N$.*

Proof. By a theorem of Tietze-Gleason, we can extend ϕ to a continuous equivariant $\tilde{\phi} : M \to N$. By Bredon's Smooth Approximation Theorem, we can approximate $\tilde{\phi}$ with a smooth $\overline{\phi}$ which on A coincides with $\tilde{\phi}$, and therefore ϕ. For details, we refer to [11]. □

We can now prove the main theorem of this section:

Theorem 3.12. *$\mathcal{F}^G$ is finitely generated. Let W be a finitely generated submodule of $\mathcal{F}^G$; then W coincides with all of $\mathcal{F}^G$ if and only if*

$$W_v = S^{\mathrm{Stab}\, v}\ \forall v \in V\ . \tag{3.13}$$

Proof. Let W be a finitely generated submodule of $\mathcal{F}^G$ satisfying (3.13). Let $\alpha : V \to S$ be invariant; clearly, $\alpha(v)$ is fixed by $\mathrm{Stab}(v)$. So,

$$W_v \subseteq \mathcal{F}^G_v \subseteq S^{\mathrm{Stab}(v)}\ ,$$

and by hypothesis equality holds. By Lemma 3.9, $W = \mathcal{F}^G$, which is therefore finitely generated. Therefore, to prove the theorem we must construct a finitely generated W satisfying (3.13).
Choose a group H which is the stabilizer of a point of V, and let $s_1, \dots, s_k$ be a basis of S^H. For each s_i, we shall construct an equivariant map

$$\tilde{s}_i : V \to S$$

such that $\tilde{s}_i(v)$ is a non-zero multiple of s_i at each v with stabilizer H. By equivariance, it will follow that $\tilde{s}_1(v), \dots, \tilde{s}_k(v)$ is a basis of $S^{\mathrm{Stab}\, v}$ for all v with stabilizer conjugate to H. By Lemma 3.10, it is sufficient to consider a finite number of H; thus, we obtain a finite collection of equivariant maps from V to S, which generate a module W with the required properties. It only remains to construct the $\tilde{s}_i$.

Observe first that since G is compact, the number of $K \leq G$ which have the same Lie algebra as H is finite. For every such K containing H, choose an element $g_K \in K \setminus H$. At each point v of V^H, consider the map

$$\mathfrak{g}/\mathfrak{h} \to \mathrm{End}(V)$$

induced by the infinitesimal action of G; this map is injective if and only if $\mathrm{Stab}\, v$ has the same Lie algebra as H. Therefore we can define a smooth function f on V^H satisfying

$$\begin{cases} f(v) > 0 & \text{if } \mathrm{Stab}\, v \text{ has the same Lie algebra as } H\ , \\ f(v) = 0 & \text{otherwise.} \end{cases}$$

Now consider the function on V^H

$$\tilde{f}(v) = f(v) \prod_{\substack{\mathfrak{k}=\mathfrak{h} \\ H \leq K \leq G}} \|g_K(v) - v\|^2\ ;$$

by construction, $\tilde{f}^{-1}(0)$ is the subset of V^H where the stabilizer is bigger than H. So, we can set

$$\tilde{s}_i : V^H \ni v \to \tilde{f}(v) s_i \in S\ ,$$

which extends to an equivariant map on GV^H. Since GV^H is closed, we can apply Theorem 3.11 to extend $\tilde{s}_i$ to an equivariant map defined on all of V. $\square$

When checking the criterion of Theorem 3.12, it may be useful to separate forms by degree. More precisely, we say that a form in $\mathcal{F}$ has *bidegree* (p, q) if it takes values in

$$S_{p,q} = \Lambda^p T \otimes \Lambda^q V\ ;$$

we denote $\mathcal{F}_{p,q}$ the space of forms of bidegree (p, q). We have thus made $\mathcal{F}$ into a bigraded vector space. As the $S_{p,q}$ are G-stable, Theorem 3.12 still holds if one replaces (3.13) with

$$(W \cap \mathcal{F}_{p,q})_v = (S_{p,q})^{\mathrm{Stab}\, v} \quad \forall v \in V\ .$$

3.5 The fibre $\mathfrak{so}(3)$

In this section we give a more explicit version of Theorem 3.12 for the case $V = \mathfrak{so}(3)$, where either $G = \mathrm{SO}(3)$ acting through the adjoint representation, or there exists an epimorphism

$$p : G \to \mathrm{SO}(3)$$

such that the action of G on $\mathfrak{so}(3)$ is induced by the adjoint representation. In fact, the latter case includes the former, setting p equal to the identity. We set $K = \ker p$, so that $G/K \cong \mathrm{SO}(3)$. Given a G-module Y, this identification makes Y^K into an $\mathrm{SO}(3)$-module.

Theorem 3.13. *If $V = \mathfrak{so}(3)$, the space of G-invariant forms $\mathcal{F}^G$ is a finitely generated $C^\infty(V)^G$-module. With notation from Section 2.4, write*

$$(\Lambda^p T^*)^K \cong m_0^p V^0 \oplus \cdots \oplus m_{2k}^p V^{2k} \; .$$

Let $\alpha_1, \dots, \alpha_n$ be elements of $\mathcal{F}^G_{p,q}$; then $\alpha_1, \dots, \alpha_n$ generate $\mathcal{F}^G_{p,q}$ if and only if for all v in V

$$\dim\langle \alpha_1(v), \dots, \alpha_n(v)\rangle = \begin{cases} m_0^p & v = 0,\ q = 0,3 \\ m_2^p & v = 0,\ q = 1,2 \\ m_0^p + \cdots + m_{2k}^p & v \neq 0,\ q = 0,3 \\ m_0^p + 3m_2^p + \cdots + 3m_{2k}^p & v \neq 0,\ q = 1,2 \end{cases}$$

Before proving the theorem, we remark that the m_{2k}^p can be computed explicitly with the techniques explained in Section 2.4, which makes this theorem an effective criterion to determine whether a given family of invariant forms is a basis.

Proof. We view invariant forms of bidegree (p,q) as invariant maps

$$V \to S_{p,q} = \Lambda^p T^* \otimes \Lambda^q(\mathfrak{so}(3))^* \; .$$

Since K acts trivially on $\mathfrak{so}(3)$, such maps are bound to take values in

$$Z = (\Lambda^p T^*)^K \otimes \Lambda^q(\mathfrak{so}(3))^* \; .$$

By Theorem 3.12, it suffices to compute the dimension of $Z^{\mathrm{Stab}(v)}$. Now, non-zero points of $\mathfrak{so}(3)$ have stabilizer U(1), whereas zero has stabilizer SO(3). By construction, we have

$$\begin{aligned}(\Lambda^p T^*)^K &\cong m_0^p V^0 \oplus \dots\\ (\Lambda^p T^*)^K \otimes \Lambda^1(\mathfrak{so}(3))^* &\cong m_2^p V^0 \oplus \dots\\ (\Lambda^p T^*)^K \otimes \Lambda^2(\mathfrak{so}(3))^* &\cong m_2^p V^0 \oplus \dots\\ (\Lambda^p T^*)^K \otimes \Lambda^3(\mathfrak{so}(3))^* &\cong m_0^p V^0 \oplus \dots\end{aligned}$$

where the dots represent SO(3)-modules with no fixed points; so the dimension of $Z^{\mathrm{SO}(3)}$ is either m_0^p or m_2^p, depending on q.
To compute the dimension of $Z^{\mathrm{U}(1)}$, observe first that every V^{2k} contains exactly one one-dimensional trivial U(1)-module. So every non-trivial, irreducible component of Z, on tensoring with V^2, gives rise to a three-dimensional trivial U(1)-module; on the other hand, $V^0 \otimes V^2$ only contains a one-dimensional trivial U(1)-module. □

Remark. The adjoint representation of SO(3) is polar; in particular, the space of invariant functions can be identified with the space of smooth functions on the half-line $[0, +\infty)$.

3.6 Constructing invariant forms

Theorem 3.12 does not completely solve the problem of computing the algebra of invariant forms. Even when the need to compute stabilizers poses no problem, as in the case of $G = \mathrm{SO}(3)$, one still needs to determine the action of d. We know from Section 3.3 that if ω has parallel torsion, d maps invariant forms to invariant forms, thereby making $\mathcal{F}^G$ into a differential graded algebra; the action of d depends on the geometry of M - which is precisely what makes this space of forms interesting from our point of view. In this section we illustrate a technique to produce elements of $\mathcal{F}$, and to determine how d acts on them.

Fix a G-invariant metric on V. We consider invariant $\pi^*\underline{V}$-valued forms, among which we can find some canonical elements. As an obvious extension of Theorem 3.4, we find

$$\Omega(X, \pi^*\underline{V})^{\mathcal{G}} \cong C^\infty(M) \otimes \tilde{\mathcal{F}}\ , \quad \tilde{\mathcal{F}} = \Omega(V, \Lambda^* T \otimes V)^G\ .$$

Let $i : V^* \to \Lambda^k T^*$ be a non-zero, G-equivariant map for some k. In applications, i will factor through a G-equivariant metric $V^* \to V$, and moreover V will be irreducible, forcing i to be injective; however, we shall not need these hypotheses for the moment. The corresponding element

$$c_i \in \Lambda^k T^* \otimes V$$

gives a constant element of $\tilde{\mathcal{F}}$, also denoted c_i. Geometrically, one can interpret c_i as the pullback of the section of $\Lambda^k(M, \underline{V})$ defined by i. The identity map $a : V \to V$ also lies in $\tilde{\mathcal{F}}$.

Recall that in order to view $\pi^* P$ as a G-structure, we had to fix a connection, which we are assuming to have parallel torsion; let D be the corresponding exterior covariant differential. By Proposition 3.6, $\tilde{\mathcal{F}}$ is D-closed. It is quite obvious that D satisfies the Leibnitz rule, which we state in the bilinear case:

Proposition 3.14. *Let $f : V \otimes V \to V$ be a linear, G-equivariant map. Then the induced map*

$$f_* : \tilde{\mathcal{F}} \otimes \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$$

satisfies

$$Df_*(\alpha_1 \otimes \alpha_2) = f_*(D\alpha_1 \otimes \alpha_2) + (-1)^{\deg \alpha_1} f_*(\alpha_1 \otimes D\alpha_2) \ ,$$

for every α_1, α_2 in $\tilde{\mathcal{F}}$.

Now consider the space $\mathrm{Hom}(\mathcal{T}, V)$, where

$$\mathcal{T} = \bigoplus_{r \geq 0} V^{\otimes^r}$$

We think of multilinear maps $V^{\otimes^r} \to V$ as elements of $\mathrm{Hom}(\mathcal{T}, V)$ by extending to zero. In particular, $\mathrm{Hom}(\mathcal{T}, V)$ has a canonical element given by the identity $V \to V$. Using the metric,

$$\mathrm{Hom}(V^{\otimes^r}, V) \cong V^{\otimes^{r+1}}$$

and therefore $\mathrm{Hom}(\mathcal{T}, V) = \bigoplus_{r>0} V^{\otimes^r}$ which gives $\mathrm{Hom}(\mathcal{T}, V)$ an algebra structure (without unit); explicitly, if α has degree p,

$$(\alpha \cdot \beta)(v_1, \dots, v_n) = \alpha(v_1, \dots, v_p)\beta(v_{p+1}, v_{n-1}) \cdot v_n \ .$$

However, there is a choice involved in this definition, because one could replace v_i with $v_{\sigma(i)}$, σ being a permutation of $1, \ldots, n$. We shall loosely say that some elements of $\mathrm{Hom}(\mathcal{T}, V)$ generate $\mathrm{Hom}(\mathcal{T}, V)^G$ if a basis of the latter can be obtained from the given elements through multiplications, with respect to any of this algebra structures.
If $f \in \mathrm{Hom}(\mathcal{T}, V)^G$, then f_* maps r-tuples of invariant elements to invariant elements. Now let $\tilde{\mathcal{A}}$ be the smallest subspace of $\tilde{\mathcal{F}}$ satisfying the following conditions:

- $\tilde{\mathcal{A}}$ contains $D^n a$ for all $n \geq 0$, where $D^n = D \circ \cdots \circ D$ n times;
- $\tilde{\mathcal{A}}$ contains $D^n c_i$ for all $n \geq 0$, and all equivariant $i : V^* \to \Lambda^k T^*$;
- $\tilde{\mathcal{A}}$ contains $f_*(\alpha_1 \otimes \cdots \otimes \alpha_p)$ for every p-tuple $(\alpha_1, \ldots, \alpha_p)$ of elements of $\tilde{\mathcal{A}}$ and every G-invariant multilinear $f : V^{\otimes p} \to V$.

Remark. When checking the last condition, it suffices to consider a set of generators of $\mathrm{Hom}(\mathcal{T}, V)^G$.

The metric $V \otimes V \to \mathbb{R}$ induces a map

$$\cdot : \tilde{\mathcal{F}} \times \tilde{\mathcal{F}} \to \mathcal{F} ;$$

let $\mathcal{A}$ be the subalgebra of $\mathcal{F}$ generated by the image of $\tilde{\mathcal{A}}$ under $\cdot$.

Remark. There would nothing to be gained in considering all of the multilinear maps in $\mathrm{Hom}(\mathcal{T}, \mathbb{R})^G$ rather than just $\cdot$, because of the definition of $\tilde{\mathcal{A}}$.

Proposition 3.15. *$\mathcal{A}$ is a differential graded subalgebra of $\mathcal{F}^G$.*

Proof. The generators a and c_i are clearly G-invariant. The maps f_* (with notation from Proposition 3.14) and D preserve invariance, so G acts trivially on $\tilde{\mathcal{A}}$. Since the map $\cdot$ is equivariant, it follows that $\mathcal{A}$ is contained in $\mathcal{F}^G$. The fact that $\mathcal{A}$ is stable under d follows from the Leibniz rule

$$d(\alpha_1 \cdot \alpha_2) = D\alpha_1 \cdot \alpha_2 + (-1)^{\deg \alpha_1} \alpha_1 \cdot D\alpha_2 . \qquad \square$$

Remark. In general, one cannot hope that $\mathcal{F}^G = \mathcal{A}$. For instance, if there is no equivariant i with $k = 1$, all 1-forms in $\mathcal{A}$ vanish at the zero section, whereas the same need not hold for $\mathcal{F}^G$. However, in many cases this is the only thing that goes wrong, and we can work around this problem dividing by suitable elements of $C^\infty(V)^G$.

Remark. The vector bundle $\Lambda^k(X)$ has a tautological section, induced by the map i: this section is actually $a \cdot c_i$. In fact, write c for c_i and let s be a local section of P; then a choice of an orthonormal basis on V gives a local orthonormal basis $e_1, \dots, e_n$ of X_x, so that the generic element of X has the form $(x;\, a^j e_j)$, where summation over $j = 1, \dots, n$ is implied. Then

$$a_{(x;\, a^j e_j)} = a^j \otimes e_j \quad \text{and} \quad c_{(x;\, a^j e_j)} = \underline{i}(e^j) \otimes e_j \ , \tag{3.14}$$

so that

$$(a \cdot c)_{(x;\, a^j e_j)} = a^j \otimes \underline{i}(e^j) = \underline{i}(a^j e^j) \ .$$

We can define the bidegree of an element of $\tilde{\mathcal{F}}$ just like we did for elements of $\mathcal{F}$; it is clear that c_i has bidegree $(k, 0)$. Let $b = Da$. We immediately conclude:

Lemma 3.16. *As an element of $\tilde{\mathcal{F}}$, b has bidegree $(0,1)$. At a point x in X, the linear maps*

$$(c_i)_x : \Lambda^k(T_xX) \to X_x \subset \Lambda^k(T_x^*M) \tag{3.15}$$

$$b_x : T_xX \to X_x \tag{3.16}$$

are onto.

Remark. A canonical element of $\Omega^0(X)^G$ is the function $a \cdot a$. We have

$$d(a \cdot a) = 2a \cdot b \ .$$

Example 3.16.1. Suppose that G is the trivial group; applying our technique yields:

$$\mathcal{A} = \{\alpha \in \mathcal{F}^G \mid \deg \alpha \neq 1 \text{ or } \alpha(0) = 0\} \ .$$

Indeed, we know that

$$\mathcal{F}^G = \mathcal{F} \cong \Omega(V, \Lambda^* T) = \Omega(V) \otimes \Lambda^* T \ ;$$

in particular, $\mathcal{F}^G_{0,0}$ is the space of functions on V. Take a basis $e_1, \dots, e_n$ of V and a basis $\alpha^1, \dots, \alpha^m$ of T^*. Define maps

$$w_{ij} : V^* \to T^*$$

by $w_{ij}(e^k) = \delta_{jk}\alpha_i$, which of course are equivariant with respect to the action of the trivial group. By construction, $e_{ij} = e_{w_{ij}}$ is in $\tilde{\mathcal{A}}$. Hence,

$$a \cdot e_{ij} = a^k e_k \cdot \alpha_i e_j = a^j \alpha^i$$

is in $\mathcal{A}$. On the other hand, write $b = b^k e_k, a = a^k e_k$, and define endomorphisms f_{ij} by $f_{ij}(e_k) = \delta_{ik} e_j$, . By construction, $\tilde{\mathcal{A}}$ contains $(f_{ij})_* b$ for all i, and $\mathcal{A}$ contains

$$a \cdot (f_{ij})_* b = a^k e_k \cdot b^i e_j = a^j b^i \ .$$

By Lemma 3.16, at each point the 1-forms $(b^i)_x$ span V^*.
By using Taylor series, one sees immediately that every degree 1 element of $\mathcal{F}$ which vanishes at the origin of V belongs to the $C^\infty(V)$-module generated by $\{a^j \alpha^i, a^j b^i\}$, and therefore to $\mathcal{A}$. On the other hand if one looks at, say, two-forms, one finds that

$$e_{ij} \cdot e_{kj} = \alpha^{ik} \ , \qquad e_{ij} \cdot b_{kj} = \alpha^i \wedge b^k \ , \qquad b_{ij} \cdot b_{kj} = b^{ik} \ ,$$

and a similar result holds for all p-forms, $p > 1$.
Thus, we have proved our claim that $\mathcal{A}$ is strictly contained in $\mathcal{F}^G$. However, it is clear that by allowing division by $a \cdot a$ we obtain the full algebra of invariant forms.

By the Leibnitz rule, in order to compute the action of d on $\mathcal{A}$ it is sufficient to compute Db and Dc_i. We have the following:

Lemma 3.17. *We have*

$$Db = \Omega \wedge a \ ,$$

where contraction $\mathfrak{g} \otimes V \to V$ is implied. If ω is torsion-free, then $Dc_i = 0$, and as a consequence,

$$d\,(a \cdot c_i) = b \cdot c_i \ .$$

Proof. Since Db is $D^2 a$, by a general formula $Db = \Omega \wedge a$, even though Ω might not lie in $\mathcal{F}^G$.
Now assume ω is torsion-free, and let θ be the solder form. By construction,

$$c_i = i^{-1}(\theta^k) \ ,$$

where contraction $T \otimes \cdots \otimes T \to \Lambda^k T$ is implied, and by hypothesis $D\theta = 0$; the Leibnitz rule gives $Dc_i = 0$. □

3.7 Homogeneous spaces

In this section we consider the particular case of a homogeneous space

$$M = G/H \ ;$$

we view G as a principal bundle P over M with fibre H. In this case, we can define a stronger notion of invariance, using the left action of G on itself. This has the consequence of eliminating the factor $C^\infty(M)$ appearing in Theorem 3.4, because left-invariant functions on M are constant; in other words, the space of invariant forms is simply $\mathcal{F}^H$. It is clear that d preserves left-invariance, i.e. $\mathcal{F}^H$ is d-closed. In general, we cannot assume the existence of an invariant torsion-free connection, in order to apply Lemma 3.17; on the other hand, all the calculations needed to determine the action of d reduce to simple calculations on a Lie algebra.

We begin this discussion with a general remark of topological nature.

Theorem 3.18. *If G is a connected, compact group acting on X, the inclusion $\Omega(X)^G \to \Omega(X)$ induces an isomorphism in cohomology.*

For the proof, we refer to [23]. In the present setup, Theorem 3.18 tells us that $\mathcal{F}^H$ has the same cohomology as X and therefore G/H, provided G is compact and connected. This is related to the fact that while the definition of $\mathcal{F}^H$ only depends on H and V, the action of d depends on the local geometry of G/H.

Remark. Another possibility is to fix a H-stable submanifold $j : N \to V$, and consider the manifold

$$\underline{N} = G \times_H N \ .$$

Suppose that every invariant form on N can be extended to an invariant form on V; then Theorem 3.18 implies that the cohomology of $\underline{N}$ coincides with the cohomology of $j^*(\mathcal{F}^H)$.

We shall now restate some well-known results on homogeneous spaces in a language that suits our needs; for details, we refer to [26]. We essentially retain the notation from the last sections, except that the fibre of P is now called H instead of G: so, we fix a H-module V and set $X = G \times_H V$. Let $T = \mathfrak{g}/\mathfrak{h}$.

Proposition 3.19. *Let (W, ρ_W) be a H-module. The restriction to the identity defines an isomorphism from the space of G-invariant W-valued pseudotensorial forms on P to the space*

$$(\Lambda^*(T \oplus \mathfrak{h}) \otimes W)^H \ .$$

Similarly, invariant tensorial forms are identified with

$$(\Lambda^* T \otimes W)^H \ .$$

Proof. Let ω be a G-invariant W-valued pseudotensorial p-form on P, i.e. an element of

$$\{\omega \in \Lambda^p(G, W) \mid R_h^*\omega = \rho_W(h^{-1}) \circ \omega \; \forall h \in H, \; L_g^*\omega = \omega \; \forall g \in G\} \; .$$

From $\mathrm{Ad}(g) = L_g \circ R_{g^{-1}}$, it follows that $\mathrm{Ad}(h)^*\omega = \rho_W(h) \circ \omega$ for all h in H. This means that ω_e is H-equivariant as a map from $\Lambda^p\mathfrak{g}$ to W, namely H-invariant as an element of $\Lambda^p\mathfrak{g}^* \otimes W$.
Vice versa, let $\omega_e : \Lambda^p\mathfrak{g} \to W$ be H-equivariant. Define a form ω on P by $\omega_g = L_{g^{-1}}^*\omega_e$; then ω is clearly G-invariant and pseudotensorial at the identity, and consequently pseudotensorial everywhere.
The second assertion follows from the fact that a G-invariant pseudotensorial form is tensorial if and only if it is at the identity. □

Corollary 3.20. *The space of G-invariant connections on P can be identified with*

$$\{\mathfrak{m} \mid \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{g}, [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}\} \; .$$

The corresponding operator D from invariant pseudotensorial p-forms to invariant tensorial $p+1$-forms can be expressed as

$$\begin{aligned} D : (\Lambda^p\mathfrak{g}^* \otimes W)^H &\to (\Lambda^{p+1}\mathfrak{g}^* \otimes W)^H \\ \alpha \otimes w &\to d\alpha|_{\Lambda^{p+1}m^*} \otimes w \end{aligned}$$

Proof. By Proposition 3.19, the connection form corresponds to a H-invariant element $\pi_\mathfrak{h}$ of $\mathrm{Hom}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{g}^* \otimes \mathfrak{h}$, and by the definition of a connection form $\pi_\mathfrak{h}$ must restrict to the identity on $\mathfrak{h}$. The kernel m of $\pi_\mathfrak{h}$ (which corresponds to the distribution of horizontal spaces and determines $\pi_\mathfrak{h}$) must then be H-invariant, and consequently $\mathfrak{h}$-invariant. The converse is proved in the same way. □

Remark. If α is tensorial, by Proposition 1.32 we can write

$$D(\alpha \otimes w) = d\alpha \otimes w + (-1)^p \alpha \wedge \pi_\mathfrak{h} \cdot w \; ,$$

where $\pi_\mathfrak{h}$ is the connection form.

Now fix a connection $\mathfrak{m}$, and identify T with $\mathfrak{m}$. We write

$$[X, Y] = [X, Y]_\mathfrak{h} + [X, Y]_\mathfrak{m} \; ,$$

according to the splitting $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. In the language of Proposition 3.19, the canonical form is the projection $\pi_\mathfrak{m} : \mathfrak{g} \to \mathfrak{m}$. From this and Corollary 3.20, one easily proves:

Theorem 3.21. *On G/H with the connection determined by $\mathfrak{m}$, let Θ be the torsion and R the curvature tensor. Then*

- $\Theta(X,Y) = -\frac{1}{2}[X,Y]_{\mathfrak{m}}$ *for all* X, Y *in* $\mathfrak{m}$.
- $R(X,Y)Z = -[[X,Y]_{\mathfrak{h}}, Z]$ *for all* X, Y, Z *in* $\mathfrak{m}$.
- $\nabla\underline{\Theta} = 0$
- $\nabla R = 0$

Remark. Theorem 3.21 shows that a torsion-free $\mathfrak{m}$ exists if and only if G/H is a symmetric space. It also shows that we can apply Theorem 3.8, which in this case reduces to the trivial remark that the space of G-invariant forms on X is closed under d.

We can now apply these results to the study of invariant forms on X. As usual, let $\pi : X \to M$ be the projection; then the pullback to X of P is

$$\pi^* P = G \times V \ ,$$

with principal H action given by $(R_h, \rho_V(h^{-1}))$. The tangent space of $\pi^* P$ on $\{e\} \times V$ is $\mathfrak{g} \oplus V$ and its vertical subspace is $\mathfrak{h} \times \{0\}$; the horizontal subspace with respect to the pullback of the connection induced by $\mathfrak{m}$ is $\mathfrak{m} \oplus V$, the connection form being given by the projection. The analogue of Proposition 3.19 is:

Proposition 3.22. *The restriction to $\{e\} \times V$ defines an isomorphism from the space of G-invariant W-valued tensorial forms on $\pi^* P$ to the space*

$$\{\alpha : V \to \Lambda^*(T \oplus V) \otimes W \mid \alpha \text{ is } H\text{-equivariant}\} \ ,$$

and the space of invariant tensorial forms is identified with

$$\Omega(V, \Lambda T^* \otimes W)^H \tag{3.17}$$

In particular,

$$\Omega(X)^G \cong \Omega(V, \Lambda T^*)^H$$

Proof. The first statement is completely analogous to Proposition 3.19. The second statement follows from the fact that under the action of H one has a decomposition

$$\Lambda(T \oplus V)^* = \sum_{h \geq 0, k \geq 0} \Lambda^h T^* \otimes \Lambda^k V^* \ ,$$

and every invariant α splits into invariant components accordingly. $\square$

$\Omega^0(X)^G$ consists of H-invariant functions on V, which is typically infinite dimensional as a vector space, and so is $\Omega^p(X)^G$, which we view as a free $\Omega^0(X)^G$-module.
The analogue of the formula in Corollary 3.20 is slightly more complicated: consider the maps

$$\Omega^p(V, \Lambda\mathfrak{m}^*) \ni \alpha \to d\alpha \in \Omega^{p+1}(V, \Lambda\mathfrak{m}^*)$$
$$\Omega^p(V, \Lambda\mathfrak{m}^*) \ni \alpha \to d \circ \alpha \in \Omega^p(V, \Lambda\mathfrak{g}^*)$$

the second of which is induced by $d : \Lambda^k\mathfrak{m}^* \to \Lambda^{k+1}\mathfrak{g}^*$. We can replace Lemma 3.17 with the following:

Lemma 3.23. *Let $\alpha \otimes w$ be an element of $\Omega^p(V, \Lambda\mathfrak{m}^* \otimes W)^H$, where α is a p-form taking values in $\Lambda\mathfrak{m}^*$ and w is a W-valued function. Then*

$$D(\alpha \otimes w) = d(\alpha \otimes w) + (-1)^p (d \circ \alpha) \otimes w + \pi_{\mathfrak{h}} \wedge \alpha \cdot w \ .$$

Proof. Follows from the general formula of Proposition 1.32 and the fact that the connection form is π_h. $\square$

Remark. If G/H is symmetric, Lemma 3.17 completely determines the action of d on $\mathcal{A}$: the c_i are D-closed, and Db is determined by the curvature (which can be computed using Theorem 3.21). However, in Section 4.6 we shall consider a symmetric space for which $\mathcal{A}$ is strictly contained in $\mathcal{F}^G$; in order to compute the action of d on forms not lying in $\mathcal{A}$, we shall need Lemma 3.23.

Chapter 4

Explicit invariant metrics

In this chapter we apply the results of Chapter 3 to construct invariant special geometries explicitly.

In the first section, we compute the algebra of SO(3)-invariant forms on T^*S^2. We give a simple characterization in terms of forms of hyperkähler structures in four dimensions, and use it to construct the celebrated Eguchi-Hanson metric on T^*S^2.
In the second section, we describe a technique of Apostolov and Salamon to produce G_2-holonomy metrics, starting from a hyperkähler 4-manifold. We apply this technique to the hyperkähler structure on T^*S^2 to produce a new G_2-holonomy metric on $T^*S^3 \times \mathbb{R}$.
The third section is preparatory to the study of the remaining examples, where the fibre equals $\mathfrak{so}(3)$. As opposed to the similarly named section of Chapter 3, this one is concerned with the construction of $\mathcal{A}$. In particular we show that in this case, the space of "letters" $\tilde{\mathcal{A}}$ forms a Lie superalgebra.
In the fourth section we consider another example related to holonomy G_2: the bundle $\Lambda^2_- S^4$ with the natural action of SO(5). Again, we compute the algebra of invariant forms; this time, the complete G_2-holonomy metric that we write down is well known.
In the fifth section, we look at $\mathbb{CP}^3$ as a subbundle of $\Lambda^2_- S^4$, corresponding to the twistor fibration. One can think of $\Lambda^2_- S^4$ as a blow-up of the cone over $\mathbb{CP}^3$; one of the G_2-holonomy metrics of $\Lambda^2_- S^4$ is compatible with this description, yielding a nearly-Kähler structure on $\mathbb{CP}^3$. Other non-conical G_2-holonomy metrics induce half-flat structures on $\mathbb{CP}^3$. What we prove in this section is that all invariant SU(3)-structures are half-flat, and essentially only one of them is nearly-Kähler. Having discovered that the underlying

almost complex structure is never integrable, we write down a more general family of invariant almost-hermitian structures which does contain complex (in fact, Kähler) structures.
In the sixth section we study the total space of the vector bundle

$$\frac{\mathrm{SU}(3)}{\mathrm{SO}(3)} \times_{\mathrm{SO}(3)} \mathfrak{so}(3) ,$$

motivated by quaternionic geometry.

4.1 The Eguchi-Hanson metric on T^*S^2

In this section we construct a hyperkähler structure on T^*S^2, invariant under the global action of SO(3).
In the language of Section 3.7, we set

$$G = \mathrm{SO}(3), \qquad H = \mathrm{SO}(2), \qquad V = (\mathfrak{so}(3)/\mathfrak{so}(2))^* .$$

Clearly, $G/H = S^2$ and $X = T^*S^2$. Let e_1, e_2, e_3 be a basis of $\mathfrak{so}(3)$ reflecting the splitting

$$\mathfrak{so}(3) = \mathfrak{so}(2) \oplus \mathbb{R}^2 ;$$

the dual basis e^1, e^2, e^3 of $\mathfrak{so}(3)^*$ may be assumed to satisfy

$$de^1 = e^{23}, \qquad de^2 = e^{31}, \qquad de^3 = e^{12} .$$

As an SO(2)-module, $V = (\mathbb{R}^2)^* \cong \mathbb{R}^2$. The adjoint action of e_1 defines an almost complex structure J on V with $Je_2 = -e_3$; the space of SO(2)-invariant maps $\mathcal{T} \to V$ is generated by the identity and J. The corresponding maps $V \otimes V \to V$ are the inner product $\cdot$ and the determinant σ, where $\sigma(X, Y) = JX \cdot Y$.
Let $i : V^* \to \Lambda^1(\mathbb{R}^2)^*$ be the identity, and let c be the corresponding element of $\Omega^1(X)^G$. For short, we shall write, for example, ab for $a \cdot b$. Recall the differential graded algebra $\mathcal{A}$ constructed in Section 3.6. In this case, we find that $\mathcal{A} = \mathcal{F}^H$, and more precisely:

Theorem 4.1. *The space of invariant forms $\Omega(T^*S^2)^{\mathrm{SO}(3)}$ coincides with $\mathcal{A}$. A basis over the algebra of invariant functions is*

Degree	*Rank*	*Basis*
0	1	1
1	4	$ab, ac, \sigma(a,b), \sigma(a,c)$
2	6	$bc, \sigma(b,b), \sigma(b,c), \sigma(c,c), ab\wedge ac, \sigma(a,b)\wedge\sigma(a,c)$
3	4	$ab\wedge\sigma(c,c), \sigma(a,b)\wedge\sigma(c,c), ac\wedge\sigma(b,b), \sigma(a,c)\wedge\sigma(b,b)$
4	1	bc^2

Proof. We apply Theorem 3.12. Let $W\subseteq\mathcal{F}$ be the algebra generated by the forms appearing in the table. Recall that for v in V, W_v is defined as the image of $\{v\}\times W$ in $\Lambda^*V\otimes\Lambda^*\mathbb{R}^2$ under evaluation. Now, we have to compute

$$\dim(\Lambda^p(V\oplus\mathbb{R}^2)^*)^{\mathrm{Stab}(v)}\ ,\quad v\in V\ .$$

For $v\neq 0$, $\mathrm{Stab}(v)$ is trivial and this dimension is $\binom{4}{p}$, which is also the dimension of

$$W_v^p=\{\alpha\in W_v\mid\deg\alpha=p\}\ .$$

On the other hand, $\mathrm{Stab}(0)=\mathrm{SO}(2)$, and

$$\dim\left(\Lambda^p(V\oplus\mathbb{R}^2)^*\right)^{\mathrm{SO}(2)}=\begin{cases}1\ , & p=0,4\\ 0\ , & p=1,3\\ 4\ , & p=2\ ,\end{cases}$$

which coincide with the dimension of W_0^p. In conclusion, by dimension count

$$W_v^p=\left(\Lambda^p(V\oplus\mathbb{R}^2)^*\right)^{\mathrm{Stab}(v)}\quad\forall v\in V\ . \qquad\square$$

All we need in order to compute the exterior derivative of elements of $\mathcal{A}$ is a formula for Dc and a formula for Db. First, we must fix a connection; we take $\mathbb{R}^2$ as the horizontal space, so that the connection form is $\psi=e^1\otimes e_1$. By Proposition 3.20 (or Theorem 3.21), the curvature is

$$\Psi=e^{23}\otimes e_1\ ;$$

we can write $e^{23}=-\frac{1}{2}\sigma(c,c)$, so

$$Db=\Psi\cdot a=-\frac{1}{2}\sigma(c,c)\,Ja\ . \tag{4.1}$$

On the other hand, $Dc = 0$ by Lemma 3.17.

We now use the algebra of invariant forms to construct a hyperkähler structure on T^*S^2, whose underlying metric is known as the Eguchi-Hanson metric [18]. We shall need the following:

Lemma 4.2. *Given symplectic 2-forms ω_1, ω_2, ω_3 on a 4-manifold M satisfying*

$$\omega_i \wedge \omega_j = \delta_{ij}\nu$$

for some volume form ν, define endomorphisms J_1, J_2, J_3 of TM by

$$J_3X \lrcorner\, \omega_2 = X \lrcorner\, \omega_1 \tag{4.2}$$

and its cyclic permutations. This establishes a one-to-one correspondence between triplets $(\omega_1, \omega_2, \omega_3)$ of this kind satisfying

$$\omega_3(X, J_3X) > 0$$

and hyperkähler structures on M.

Remark. From the proof of Lemma 4.2, it will follow that $\omega_3(X, J_3X) > 0$ if and only if, say, $\omega_1(X, J_1X) > 0$.

Proof. Suppose we have ω_i as in the hypothesis, and define J_i accordingly; we have to show that $J_3 = J_1J_2$ and (ω_i, J_i) defines a Kähler structure, the corresponding metric not depending on i. Observe that

$$\omega_1(X, J_3X) = \omega_2(J_3X, J_3X) = 0\ ,$$

and the same holds for ω_2. Fix a vector field X, rescaled so that

$$\omega_3(X, J_3X) = 1\ .$$

Then

$$0 = J_3X \lrcorner\, X \lrcorner\, (\omega_1 \wedge \omega_3) = -(X \lrcorner\, \omega_1) \wedge (J_3X \lrcorner\, \omega_3) + (J_3X \lrcorner\, \omega_1) \wedge (X \lrcorner\, \omega_3) + \omega_1\ ,$$

and the same holds for ω_2; consequently we have

$$\begin{aligned} \omega_1 &= (X \lrcorner\, \omega_1) \wedge (J_3X \lrcorner\, \omega_3) + (X \lrcorner\, \omega_3) \wedge (J_3X \lrcorner\, \omega_1) \\ \omega_2 &= (X \lrcorner\, \omega_2) \wedge (J_3X \lrcorner\, \omega_3) + (X \lrcorner\, \omega_3) \wedge (J_3X \lrcorner\, \omega_2) \end{aligned} \tag{4.3}$$

Applying (4.2) to a vector field Z, we conclude

$$\omega_3(X,Z)J_3^2X - \big(\omega_3(J_3X,Z) + \omega_3(X,J_3Z)\big)J_3X + \omega_3(J_3X,J_3Z)X = 0 \quad (4.4)$$

From (4.3) we also get

$$\begin{aligned} 0 = \omega_1 \wedge \omega_2 &= (X \lrcorner\, \omega_3) \wedge (J_3X \lrcorner\, \omega_1) \wedge (X \lrcorner\, \omega_2) \wedge (J_3X \lrcorner\, \omega_3) = \\ &= (X \lrcorner\, \omega_3) \wedge (J_3^2X \lrcorner\, \omega_2) \wedge (X \lrcorner\, \omega_2) \wedge (J_3X \lrcorner\, \omega_3) \quad (4.5) \end{aligned}$$

Now, since

$$0 \neq \omega_2^2 = 2(X \lrcorner\, \omega_2) \wedge (J_3X \lrcorner\, \omega_3) \wedge (X \lrcorner\, \omega_3) \wedge (J_3X \lrcorner\, \omega_2) \; ,$$

we discover that on writing J_3^2X as a linear combination of J_3X and X in accordance with (4.4), the J_3X component is zero, i.e. J_3^2X is a multiple of X. From $\omega_1^2 = \omega_2^2$, we deduce

$$\begin{aligned} &(X \lrcorner\, \omega_1) \wedge (J_3X \lrcorner\, \omega_3) \wedge (X \lrcorner\, \omega_3) \wedge (J_3X \lrcorner\, \omega_1) = \\ &\qquad = (X \lrcorner\, \omega_2) \wedge (J_3X \lrcorner\, \omega_3) \wedge (X \lrcorner\, \omega_3) \wedge (J_3X \lrcorner\, \omega_2) \end{aligned}$$

which tells us that $J_3^2X = -X$. So J_3 is an almost complex structure; moreover, by (4.4)

$$\omega_3(X,Z) = \omega_3(J_3X, J_3Z) \; .$$

The same argument applies to ω_1 and ω_2 as well; the conclusion is that (ω_i, J_i) are almost-Kähler structures. Moreover, $J_3 = J_1J_2$, because

$$J_2J_1J_3X \lrcorner\, \omega_1 = J_1J_3X \lrcorner\, \omega_3 = J_3X \lrcorner\, \omega_2 = X \lrcorner\, \omega_1 \; .$$

It remains to check that the three Kähler metrics coincide. Indeed, by (4.2)

$$\omega_1(X, J_1Y) = -\omega_1(J_1Y, X) = \omega_3(J_2J_1Y, X) = \omega_3(X, J_3Y) \; ,$$

and the same holds for ω_2. By a result mentioned in Section 1.3, (ω_i, J_i) is a hyperkähler structure on M.

Conversely, if M is hyperkähler, then to each complex structure J_i there corresponds a Kähler form ω_i; then ω_i^2 equals twice the volume form for all i. We need to check that (4.2) holds. Define J by $X \lrcorner\, \omega_1 = JX \lrcorner\, \omega_2$ and let g be the Riemannian metric; then

$$g(J_1X, Y) = g(J_2JX, Y) \; ,$$

so $J_1 = J_2 J$. Since $J_1 = J_2 J_3$, we conclude that $J = J_3$.
It remains to prove that, say, $\omega_1 \wedge \omega_2 = 0$. Observe first that

$$\omega_1(X, J_3 X) = \omega_2(J_3 X, J_3 X) = 0 \ ,$$

and the same holds for ω_2. Therefore,

$$X \lrcorner\, J_3 X \lrcorner\, (\omega_1 \wedge \omega_2) = (X \lrcorner\, \omega_1) \wedge (J_3 X \lrcorner\, \omega_2) - (J_3 X \lrcorner\, \omega_1) \wedge (X \lrcorner\, \omega_2)$$

and both summands clearly vanish, proving that $\omega_1 \wedge \omega_2$ vanishes. $\square$

At this point, we know what to look for and we know where to look; we find the following:

Theorem 4.3. *The total space of T^*S^2 has a hyperkähler structure defined as follows:*

$$\begin{cases} \omega_1 = bc \\ \omega_2 = \dfrac{1}{2}(aa+k)^{-1/2}\sigma(b,b) - \dfrac{1}{2}(aa+k)^{1/2}\sigma(c,c) \\ \omega_3 = \sigma(b,c) \end{cases} \tag{4.6}$$

where k is any positive constant.

Proof. We know that $Dc = 0$; in particular, $\sigma(c,c)$ is closed. From Proposition 3.14 and (4.1)

$$D\sigma(b,c) = -\frac{1}{2}\sigma\big(\sigma(c,c)Ja, c\big) = 0$$

because $\Lambda^3(\mathbb{R}^2) = 0$. The same argument (or $d^2 = 0$) gives $d(bc) = 0$. Now

$$d\sigma(b,b) = -\sigma\big(\sigma(c,c)\, Ja, b\big) = -\sigma(c,c)\,\sigma(Ja,b) = \sigma(c,c)\, ab \ ,$$

so it is easy to check that all the ω_i are closed. It is also easy to check that

$$\omega_i \wedge \omega_j = \delta_{ij}\, bc^2 \ . \tag{4.7}$$

Since J_2 acts as J vertically, and as $-J$ horizontally, $\omega_2(X, J_2 X) > 0$ and Lemma 4.2 concludes the proof. $\square$

4.2 $T^*S^3 \times \mathbb{R}$

We construct an integrable G_2-structure on $T^*S^3 \times \mathbb{R}$ using the algebra of invariant forms on T^*S^2, its hyperkähler structure defined in (4.6) and a technique introduced by Apostolov and Salamon in [3, p.11].
The general idea of this technique is the following. Start with a hyperkähler manifold M with defining two-forms ω_1, ω_2, ω_3; suppose that $[\omega_1]$ and $[\omega_2]$ are integral classes. Take a U(1) bundle with Chern class $[\omega_1]$, and choose a connection form ξ on this bundle such that $d\xi = \omega_1$. On the 5-manifold thus obtained, take a U(1) bundle with Chern class $[-\omega_2]$, and choose a connection form η such that $d\eta = -\omega_2$. Then take the product with $\mathbb{R}^+$, with coordinate t. Define a 3-form on the resulting 7-manifold by

$$\varphi = t\omega_1 \wedge \eta + t\omega_2 \wedge \xi + t^4\omega_3 \wedge dt + dt \wedge \xi \wedge \eta \ ;$$

setting

$$\hat{\omega}_i = t^2\omega_i$$
$$e^5 = t^{-1}\eta \qquad e^6 = t^{-1}\xi \qquad e^7 = t^2 dt$$

we can write

$$\varphi = \hat{\omega}^1 \wedge e^5 + \hat{\omega}^2 \wedge e^6 + \hat{\omega}^3 \wedge e^7 - e^{567} \ ;$$

since our convention is that the hyperkähler two-forms are self-dual (as opposed to anti-self-dual), this is equivalent to (1.4). Since the volume form relative to the metric induced by φ is $\frac{1}{2}\hat{\omega}_i \wedge \hat{\omega}_i \wedge e^{567}$,

$$*\varphi = \hat{\omega}^1 \wedge e^{67} + \hat{\omega}^2 \wedge e^{75} + \hat{\omega}^3 \wedge e^{56} - \frac{1}{2}\hat{\omega}^1 \wedge \hat{\omega}^1 \ .$$

Then both φ and $*\varphi$ are automatically closed; therefore, the G_2-structure is integrable.

Let $M = T^*S^2$ with the Eguchi-Hanson hyperkähler structure; we shall write the resulting G_2-structure explicitly. Let $\mathbb{H}$ be the space of quaternions; identify S^3 with the unit quaternions, and let

$$S^2 = \operatorname{Im}\mathbb{H} \cap S^3 \ .$$

By the standard scalar product on $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$, i.e. $\langle p, q\rangle = \operatorname{Re} p\overline{q}$, one can identify $\operatorname{Im}\mathbb{H}$ with its dual, and TS^2 with T^*S^2. Define the Hopf fibration by

$$S^3 \ni q \to \pi(q) = \overline{q}iq \in S^2 \ .$$

Even though TS^2 is not trivial, π^*TS^2 is; in fact $T_{\pi(q)}S^2$ has $\{\overline{q}jq, \overline{q}kq\}$ as an oriented orthonormal basis, and so the circle subbundle has fibre

$$\left\{\overline{q}je^{i\theta}q, \theta \in \mathbb{R}\right\}$$

at $\overline{q}iq \in S^2$. Let $s : S^3 \to T^*S^2$ be defined by

$$s(q) = (\overline{q}iq; \overline{q}jq) \ ;$$

setting $B = \pi^*T^*S^2$, the following diagram commutes but for the upper triangle (i.e. $\pi \neq s \circ \pi_B$):

$$\begin{tikzcd} B \arrow[r, "\pi"] \arrow[d, "\pi_B"'] & T^*S^2 \arrow[d, "\pi_M"] \\ S^3 \arrow[r, "\pi"] \arrow[ur, "s"] & S^2 \end{tikzcd} \tag{4.8}$$

It follows that s is the standard two-to-one covering from the circle subbundle of the tautological bundle on $\mathbb{CP}^1 = S^2$ to the circle subbundle of T^*S^2. In particular, $s^*aa = 1$.
Consider the 1-form on S^3 given by

$$\beta = -2s^*\sigma(a, b) \ ,$$

where

$$d\sigma(a, b) = \sigma(b, b) - \frac{1}{2}aa\,\sigma(c, c) \ .$$

We have $s^*aa = 1$, and $s^*\sigma(b, b) = 0$ because $\sigma(b, b)$ has bidegree $(0, 2)$ (Lemma 3.16) and at a point q, the projection of $s_{*q}(T_qS^3)$ on the vertical space at $s(q)$ is one-dimensional. Since $\sigma(c, c)$ is the pullback of a form on S^2, which we denote in the same way, and the lower triangle in diagram (4.8) commutes, one obtains

$$d\beta = s^*\sigma(c, c) = \pi^*\sigma(c, c) \ . \tag{4.9}$$

Now, B is a subbundle of T^*S^3, but β is not a section of B: if it were, it would kill both horizontal (being the pullback of $\sigma(a, b)$) and vertical vectors (being a section of B). So the line bundle L generated by β gives a decomposition $T^*S^3 = B \oplus L$, meaning that as a manifold,

$$T^*S^3 = B \times \mathbb{R} \ ;$$

call y the standard coordinate on $\mathbb{R}$. More precisely, let ρ_x be an element of B_x; the identification is given by

$$B \times \mathbb{R} \ni (\rho_x, y) \xrightarrow{\cong} \rho_x + y\beta_x \in T^*S^3 \ .$$

The Apostolov-Salamon construction gives a G_2-holonomy metric on the 7-manifold $T^*S^3 \times \mathbb{R}^+$, diffeomorphic to $S^3 \times \mathbb{R}^4$. Indeed, let t be the standard coordinate on $\mathbb{R}^+$, and define

$$\xi = ac + dy$$

on $T^*S^2 \times \mathbb{R}$; then

$$d\xi = \omega_1 \ . \tag{4.10}$$

On T^*S^3, which we might view as a circle bundle over $T^*S^2 \times \mathbb{R}$, define a connection form

$$\eta = h\pi_B^*\beta + \pi^* g\, \sigma(a,b) \ , \qquad \text{where} \quad g = \frac{2h - \sqrt{2(k+aa)}}{aa} \ ,$$

and h is an arbitrary, non-zero real constant; then, using

$$ab \wedge \sigma(a,b) = \frac{1}{2} aa \wedge \sigma(b,b) \ ,$$

we find

$$d\eta = \left(h - \frac{1}{2} aa\, g\right) \sigma(c,c) + (g'\, aa + g)\sigma(b,b) = -\omega_2 \ . \tag{4.11}$$

Now define

$$\varphi = t\omega_1 \wedge \eta + t\omega_2 \wedge \xi + t^4 \omega_3 \wedge dt + dt \wedge \xi \wedge \eta \ ;$$

from the above discussion, it follows that $d\varphi = 0$ and $d * \varphi = 0$.

Remark. A straightforward calculation shows that η is singular at $aa = 0$ unless $k = h^2$, and non-vanishing at $aa = \infty$.

4.3 The fibre $\mathfrak{so}(3)$ (continued)

In Sections 4.4 and 4.6 we shall consider examples which fit in the general scheme of Section 3.5: the fibre V is $\mathfrak{so}(3)$, and there exists an epimorphism

$$p : G \to \mathrm{SO}(3)$$

such that the action of G on $\mathfrak{so}(3)$ is induced by the adjoint representation. In this section we show that in this setup the construction of $\mathcal{A}$ has a natural interpretation in terms of the Lie algebra structure of $\mathfrak{so}(3)$. The language introduced in this section will prove convenient in the sequel, when computing $\mathcal{A}$.

The SO(3)-invariant metric on $\mathfrak{so}(3)$ is

$$\langle X, Y \rangle = -\operatorname{tr}(X^T Y) \ . \tag{4.12}$$

As we pointed out in Section 3.5, $\Omega^0(X)^G$ consists of functions of aa.

Lemma 4.4. *Let* SO(3) *act on* $\mathfrak{so}(3)$ *by the adjoint representation. Then* $\mathrm{Hom}(\mathcal{T}(\mathfrak{so}(3)), \mathfrak{so}(3))^{\mathrm{SO}(3)}$ *is generated by the Lie bracket*

$$[,] : \mathfrak{so}(3) \otimes \mathfrak{so}(3) \to \mathfrak{so}(3) \ .$$

Proof. Follows from the fact that $\mathrm{Hom}\,(\mathcal{T}(\mathfrak{so}(3)), \mathbb{R})^{\mathrm{SO}(3)}$ is generated by the metric and the determinant. □

We shall write $[\alpha, \beta]$ for $[\,,]_*(\alpha \otimes \beta)$; this is a *Lie superalgebra* structure on $\tilde{\mathcal{A}}$. This means that the usual properties defining a Lie algebra hold, provided that one changes the sign whenever an odd element is moved past another odd element. For instance, if α has degree p and β has degree q then

$$[\alpha, \beta] + (-1)^{p+q}[\beta, \alpha] = 0 \ ;$$

we shall say that α and β anticommute up to *Quillen's law*. However, one can only write equations up to Quillen's law unambiguously when symbols occur exactly once in both sides of an equation.
On the other hand, for every choice of V, $\tilde{\mathcal{A}}$ is a $\mathbb{Z}$-graded differential space with differential given by D. In our case, Lemma 4.4 immediately gives:

Proposition 4.5. *For* $V = \mathfrak{so}(3)$*,* $\tilde{\mathcal{A}}$ *is the* $\mathbb{Z}$*-graded differential Lie algebra generated by* a *and the* c_i*.*

Remark. By a general property of the Lie bracket on $\mathfrak{so}(3)$, the following hold in $\tilde{\mathcal{A}}$ up to Quillen's law:

$$[[\alpha,\beta],\gamma] = \beta(\alpha\cdot\gamma) - \alpha(\beta\cdot\gamma)$$
$$[\alpha,[\beta,\gamma]] = \beta(\alpha\cdot\gamma) - (\alpha\cdot\beta)\gamma$$

Before going on to studying concrete examples, we conclude this section by establishing more notation. We define

$$\sigma(\alpha,\beta,\gamma) = [\alpha,\beta]\cdot\gamma = \alpha\cdot[\beta,\gamma]\ .$$

For $\{\alpha_1,\alpha_2,\alpha_3\}\subset\{a,b,c\}$ we write $\alpha_1\alpha_2\alpha_3$ for $\sigma(\alpha_1,\alpha_2,\alpha_3)$, and $\alpha_1\alpha_2$ for $\alpha_1\cdot\alpha_2$. Also, we shall omit the wedge symbol when only a,b,c are involved.

4.4 A G_2 metric on $\Lambda^2_- S^4$

In this section we show how the algebra of invariant forms can be used to define one of the first examples of complete G_2-holonomy metrics to be discovered [13].
By definition, $\Lambda^2_- S^4$ is the bundle of anti-self-dual forms on S^4, given by

$$P_{\mathrm{SO}(4)} \times_{\mathrm{SO}(4)} \Lambda^2_-(\mathbb{R}^4)\ .$$

Here, $P_{\mathrm{SO}(4)}$ is the standard SO(4)-structure on S^4, and $\Lambda^2_-(\mathbb{R}^4)$ is spanned by

$$\sigma_1 = e^{12}-e^{34}\ ,\qquad \sigma_2 = e^{13}-e^{42}\ ,\qquad \sigma_3 = e^{14}-e^{23}\ ,$$

where $e^1,\dots,e^4$ is the standard basis of $(\mathbb{R}^4)^*$. We identify S^4 with the homogenous space G/H, where $G=SO(5)$, $H=SO(4)$; write

$$\mathfrak{so}(4) = \mathfrak{so}(3)_+ \oplus \mathfrak{so}(3)_-\ ,$$

and let $V=\mathfrak{so}(3)_-$. As SO(4)-modules,

$$\mathfrak{so}(3)_- \cong \Lambda^2_-(\mathbb{R}^4)\ , \qquad (4.13)$$

giving

$$X \cong \Lambda^2_- S^4\ .$$

We use (4.13) to define a V-valued form c.

Define a basis on $\mathfrak{so}(5)$ using the one we gave in Section 2.2, replacing v_i with $w_i = 2\,v_i$; this basis is then orthogonal with respect to the metric (4.12). We identify $\langle w_1, w_2, w_3, w_4\rangle$ with $\mathbb{R}^4$, and consider the connection form ψ given by projection on $\mathbb{R}^4$. The curvature is a map

$$R : \Lambda^2\mathbb{R}^4 \to \mathfrak{so}(4) \ ,$$

given explicitly by $R(w_i, w_j) = -2e^i \wedge e^j$, and the curvature form is:

$$\Psi(w_i, w_j) = -e^i \wedge e^j \ .$$

This means that Ψ acts on V in the same way as $-c$. In particular,

$$Db = [-c, a] = [a, c] \ .$$

Theorem 4.6. *The space of invariant forms $\Omega(\Lambda^2_- S^4)^{\mathrm{SO}(5)}$ coincides with $\mathcal{A}$. A basis over the algebra of invariant functions is*

Degree	*Rank*	*Basis*
0	1	1
1	1	ab
2	2	ac, abb
3	4	$bc, abc, bbb, ab\ ac$
4	4	$cc, bbc, ab\ bc, ab\ abc$
5	2	$ab\ cc, ac\ bbb$
6	1	$abb\ cc$
7	1	$bbb\ cc$

Proof. In the language of Theorem 3.13, $K = \mathrm{SO}(3)_+$, so

$$(\Lambda^p(\mathbb{R}^4)^*)^K = \begin{cases} \mathbb{R} & p = 0, 4 \\ 0 & p = 1, 3 \\ \Lambda^2_- & p = 2 \end{cases}$$

Hence, the only non-zero m^p_{2k} are

$$m^0_0 = 1 \ , \qquad m^2_2 = 1 \ , \qquad m^4_0 = 1 \ .$$

Applying Theorem 3.13, we find that the list in the table is complete. $\square$

Remark. Through (4.13), the scalar product (4.12) yields

$$\alpha \wedge \beta = -2\langle \alpha, \beta \rangle e^{1234} \,.$$

So when one uses an orthonormal basis as in (3.14), one has

$$i(e_i) \wedge i(e_j) = \frac{1}{3}\delta_{ij}\, cc \tag{4.14}$$

where $-cc$ is the volume form. As a consequence, if α is in $\Omega^p(X,X)$ then

$$3c \wedge \alpha \cdot c = \alpha \wedge cc.$$

The following relations also hold:

$$\begin{aligned}
3\alpha \cdot c \wedge \beta \cdot c &= \alpha \cdot \beta \wedge cc \\
3(b,b,\alpha) \wedge (b \cdot \gamma) &= (-1)^{\deg \alpha} bbb \wedge (\alpha \cdot \gamma) \\
ac\; abb &= aa(bbc) - 2ab\; abc \\
ac\; bbc &= -\, bc\; abc = \frac{1}{3} cc\; abb
\end{aligned}$$

It is now easy to compute the action of d on the space of invariant forms (see Table 4.1). In particular, we see that $H^4(\Omega(X))^G$ has dimension 1, with generator cc. This is consistent with Theorem 3.18, which in this case implies that the cohomology of $\Omega(X)^G$ is the same as the cohomology of S^4.
The following is now straightforward:

Theorem 4.7. *There is a complete G_2 holonomy metric on $\Lambda^2_- S^4$ defined by the 3-form*

$$\phi = u\, be + \frac{1}{6} u^{-1} bbb$$

where

$$u = \left(\frac{1}{6} aa + 1\right)^{1/4} .$$

4.5 SU(3)-structures on $\mathbb{CP}^3$

Retaining notation from the last section, in this section we show that $\mathbb{CP}^3$ is a hypersurface in X, so that $\mathcal{A}$ restricts to an algebra of invariant forms on

Table 4.1: Action of d on $\Omega(\Lambda^2_- S^4)^{\mathrm{SO}(5)}$

$\deg\alpha$	α	$d\alpha$
0	aa	$2\,ab$
1	ab	0
2	ac	bc
	abb	$bbb + 2\,ab\;ac - 2\,aa\;bc$
3	bc	0
	abc	$bbc - \frac{2}{3}aa\;cc$
	bbb	$6\,ab\;bc$
	$ab\;ac$	$-ab\;bc$
4	cc	0
	bbc	$\frac{4}{3}ab\;cc$
	$ab\;bc$	0
	$ab\;abc$	$-\frac{1}{3}ac\;bbb + \frac{2}{3}aa\;ab\;cc$
5	$ab\;cc$	0
	$ac\;bbb$	0
6	$abb\;cc$	$bbb\;cc$
7	$bbb\;cc$	0

$\mathbb{CP}^3$. We use this algebra to construct a two-parameter family of invariant half-flat structures on $\mathbb{CP}^3$, proving that one of these SU(3)-structures is nearly-Kähler, but none of them is Kähler. As we shall explain, this is consistent with the existence of an invariant Kähler structure on $\mathbb{CP}^3$, but it implies that this structure does not admit an invariant reduction to SU(3).

The action of the group SO(4) on $\mathfrak{so}(3)_-$ is induced by the action of $\mathrm{Sp}(1)_-$ on $\mathfrak{sp}(1)_-$ given by $q(p) = qp\overline{q}$; so, the two-sphere in $\mathfrak{so}(3)_-$ with respect to the metric (4.12) can be identified with

$$\frac{\mathrm{SO}(4)}{\mathrm{U}(2)} \cong \frac{\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-}{\mathrm{Sp}(1)_+ \times \mathrm{U}(1)}\ ,$$

where U(2) is the stabilizer of σ_1. Hence the twistor space $\overline{\mathrm{SO}(4)/\mathrm{U}(2)}$ of the four-sphere is the sphere subbundle of X with respect to its metric (i.e. the hypersurface of X defined by $aa = 1$), and it can clearly be identified with

$$\frac{\mathrm{SO}(5)}{\mathrm{U}(2)} \cong \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)_+ \times \mathrm{U}(1)} \cong \mathbb{CP}^3\ .$$

Consider the restriction of the algebra $\mathcal{A}$ to $\mathbb{CP}^3$.

Theorem 4.8. *The space of invariant forms $\Omega(\mathbb{CP}^3)^{\mathrm{SO}(5)}$ is the restriction of $\Omega(X)^{\mathrm{SO}(5)}$. A basis over $\mathbb{R}$ is given by:*

Degree	*Dimension*	*Basis*
0	1	1
1	0	
2	2	ac, abb
3	2	bc, abc
4	2	cc, bbc
5	0	
6	1	$abb\ cc$

Proof. We show that every invariant α on $\mathbb{CP}^3$ is the restriction of an invariant form on X. Indeed, X is the cone on $\mathbb{CP}^3$; consequently, α is the restriction of a form defined on X minus the zero section, which we also denote by α. Then, $aa\,\alpha$ is an invariant form on X which coincides with α on $\mathbb{CP}^3$.
The table of generators is then obtained from the table in Theorem 4.6, observing that ab and bbb are zero on $\mathbb{CP}^3$, because

$$2\,ab = d(aa)\;, \quad bbb = 3\,ab\;abb\;.$$

These elements are generators over $C^\infty(aa)$ by construction, and therefore over $\mathbb{R}$ because $aa = 1$ on $\mathbb{CP}^3$. □

Recall from Section 2.7 that an SU(3)-structure on a 6-manifold is defined by a non-degenerate 2-form ω and a 3-form ψ^+ which is the real part of a decomposable form $\psi^+ + i\psi^-$, satisfying the compatibility relations

$$\begin{cases} \omega \wedge \psi^+ = 0 \\ \psi^+ \wedge \psi^- = \dfrac{2}{3}\omega^3 \end{cases} \tag{4.15}$$

We say that an SU(3)-structure is *nearly-Kähler* if

$$\begin{cases} d\omega = 3\,\psi_+ \\ d\psi^- = -2\,\omega^2 \end{cases} \tag{4.16}$$

This means that the underlying U(3)-structure is nearly-Kähler in the sense of Definition 1.21.

Table 4.2: Action of d on $\Omega(\mathbb{CP}^3)^{\mathrm{SO}(5)}$

$\deg\alpha$	α	$d\alpha$
0	aa	0
2	ac	bc
	abb	$-2\,bc$
3	bc	0
	abc	$bbc - \frac{2}{3}cc$
4	cc	0
	bbc	0
6	$abb\ cc$	0

Theorem 4.9. *For $\epsilon = \pm 1$, we can define a two-parameter family of half-flat structures on $\mathbb{CP}^3$ by*

$$\omega = \lambda\, ac + \epsilon\mu\, abb\,, \qquad \psi^+ = \lambda\sqrt{2\mu}\; bc\,, \qquad \psi^- = -\epsilon\lambda\sqrt{2\mu}\; abc\,.$$

where $\lambda, \mu > 0$. For $\epsilon = -1$, $\lambda = \frac{1}{2}$, $\mu = \frac{1}{8}$, this structure is nearly-Kähler.

Proof. It is easy to check that $\psi^+ + i\psi^-$ is decomposable; the fact that (ω, ψ^+, ψ^-) satisfy (4.15) for all choices of parameters is a consequence of the remark following Theorem 4.6.
The rest of the statement follows from Table 4.2. $\square$

Remark. By Theorem 4.8, this family of SU(3)-structures is essentially complete: every invariant SU(3)-structure on $\mathbb{CP}^3$ belongs to this family up to changing the sign of ω and rotating ψ^+ and ψ^-.

Applying Theorem 2.22, we obtain:

Corollary 4.10. *There is an integrable G_2-structure on $\Lambda^2_- S^4$ minus the zero section, defined by the 3-form*

$$\varphi = aa\, ab\, ac - \frac{1}{12} aa^2\, bbb + \frac{1}{4} aa^3\, bc\,.$$

It is well known that $\mathbb{CP}^3$ has an invariant Kähler U(3)-structure. One might wonder whether this U(3)-structure can be reduced to an invariant SU(3)-structure. The answer is no; for aesthetic reasons, we prove it without directly using Theorem 4.9:

Theorem 4.11. *All* SO(5)*-invariant* SU(3)*-structures on* $\mathbb{CP}^3$ *are half-flat. Their underlying almost complex structures are not integrable.*

Proof. From table 4.2, it is clear that the 4-form ω^2 has to be closed. On the two-dimensional space of invariant 3-forms on $\mathbb{CP}^3$, d has a one-dimensional kernel. This means that every invariant SU(3)-structure, up to rotating ψ^+ and ψ^-, satisfies

$$d\psi^+ = 0 \;, \quad d\psi^- \neq 0 \;.$$

So, all invariant SU(3)-structures are half-flat.
Consider the real 3-form

$$\Psi = \psi^+ + i\psi^- \;;$$

a priori $d\Psi$ has type $(3,1)+(2,2)$, but being purely imaginary it is bound to have type $(2,2)$. Since $d\Psi \neq 0$, it follows that the almost complex structure is not integrable. □

So, in order to express the invariant Kähler metric on $\mathbb{CP}^3$ in our language, we have to consider almost-hermitian structures, i.e. replace the 3-form ψ^+ with an almost complex structure J. An almost complex structure J on $\mathbb{CP}^3$ induces endomorphisms of $\tilde{\mathcal{A}}$ and $\mathcal{A}$, both of which will be denoted J and defined as

$$J(\alpha)(X_1,\dots,X_p) = \alpha(-JX_1,\dots,-JX_p) \;.$$

Having fixed a connection on S^4, the tangent space of X splits as

$$T_xX = T_{\pi(x)}M \oplus \ker \pi_{*x} \;.$$

Consider the natural invariant map $\mathrm{SO}(4)/\mathrm{U}(2) \to \mathrm{End}(\mathbb{R}^4)$ which maps the identity to the almost complex structure on $\mathbb{R}^4$ given by

$$e_1 \to e_2 \;, \qquad\qquad e_3 \to -e_4 \;.$$

This induces a map

$$\underline{\mathrm{SO}(4)/\mathrm{U}(2)} \to \underline{\mathrm{End}(\mathbb{R}^4)} = \mathrm{End}(TS^4) \;;$$

an almost complex structure J_h is thus defined on $T_{\pi(x)}M$, which depends on x as well as $\pi(x)$. Define an almost complex structure J_v on $\ker \pi_{*x}$ by imposing the condition

$$J_v b = -[a,b] \;.$$

Proposition 4.12. *For $\epsilon = \pm 1$, we can define a two-parameter family of invariant almost-hermitian structures (ω, J) by*

$$\omega = \lambda\, ac + \epsilon\mu\, abb\ , \qquad\qquad J = J_h \oplus \epsilon J_v\ .$$

For $\epsilon = -1$, this almost-hermitian structure has a half-flat reduction to SU(3)*. For $\epsilon = 1$, $\lambda = 1 = 2\mu$, we get the standard Kähler structure of* $\mathbb{CP}^3$*.*

Proof. We start by showing that (ω, J) is an almost-hermitian structure; we have

$$Jabb = J([a,b]\cdot b) = J[a,b]\cdot Jb = -b\cdot [a,b] = abb\ .$$

Moreover, the tautological 2-form ac is J-invariant; so, ω is J-invariant. Recall that with our convention,

$$g(JX, Y) = 2\omega(X, Y)\ ;$$

positive definiteness of g implies that $\omega(X, JX)$ must be greater than zero for all X. By construction, if u is a frame at $x \in S^4$, then

$$J_{[u,\sigma_1]}[u, e_1] = [u, e_2]\ , \qquad\qquad J_{[u,\sigma_1]}[u, e_3] = -[u, e_4]\ .$$

On the other hand, $ac_{[u,\sigma_1]} = [u, \sigma_1]$, so

$$ac(X, JX) > 0$$

for all X. The same holds for $\epsilon\, abb = -Jb\cdot b$, and any linear combination of the two with non-negative coefficients. It follows that (ω, J) is an almost-hermitian structure.

Now,

$$d\omega = (\lambda - 2\epsilon\mu) bc\ ,$$

so for $\epsilon = 1$, $\lambda = 2\mu$, the almost-hermitian structure is almost-Kähler. If we normalize to $\lambda = 1$, the metric induced on the fibre SO(4)/U(2) is the standard metric on the sphere, and we obtain the standard Kähler metric on $\mathbb{CP}^3$ (see [4]).

Now set $\epsilon = -1$; we must check that $\psi^+ + i\psi^-$, as defined in Theorem 4.9, has type $(3,0)$ with respect to J. It is sufficient to do so at a generic point of $\mathbb{CP}^3$, which we write as $[u, \sigma_1]$. Write

$$b_{[u,\sigma_1]} = b_1 \otimes \sigma_1 + b_2 \otimes \sigma_2 + b_3 \otimes \sigma_3\ ,$$

and do the same for c. Then

$$(\psi^+ + i\psi^-)_{[u,\sigma_1]} = (b_2 - ib_3) \wedge (c_2 + ic_3) \ ,$$

which by construction has type $(3,0)$. $\square$

The almost-hermitian structures with $\epsilon = -1$ were introduced in [17].

Remark. For $\epsilon = \pm 1$, we can still write

$$\psi^+ + i\psi^- = \lambda\sqrt{2\mu}(bc - i\,\epsilon\,abc) = \lambda\sqrt{2\mu}(b + i\,Jb)\cdot c \ ;$$

by construction $b + i\,Jb$ has type $(1,0)$ and c has type $(2,0)+(0,2)$. When $\epsilon = -1$, the component of type $(0,2)$ gives no contribution to $\psi^+ + i\psi^-$, which therefore has type $(3,0)$. However, when $\epsilon = 1$, it is the component of type $(2,0)$ that gives no contribution, so $\psi^+ + i\psi^-$ has type $(1,2)$ and we do not have a reduction to SU(3).

4.6 SU(3)/SO(3)

So far, we have only seen examples where the algebra $\mathcal{A}$ more or less coincides with the algebra of invariant forms. In this section we consider an example for which this fact does not hold. This is because the generators we considered were *constant* equivariant maps

$$c_i : V \to \Lambda^*(T \oplus V) \otimes V \ ;$$

while this simplifies computations considerably (for instance, for symmetric spaces we were able to conclude that the c_i are D-closed), it means leaving out all non-constant linear equivariant maps. These non-constant maps turn out to be necessary in the example to follow.

We set

$$G = \mathrm{SU}(3), \qquad H = \mathrm{SO}(3), \qquad V = \mathfrak{so}(3) \ .$$

Let $e_1, \dots, e_8$ be an orthonormal basis of $\mathfrak{su}(3)$ reflecting the splitting

$$\mathfrak{su}(3) = \mathfrak{so}(3) \oplus T \ ;$$

the dual basis $e^1, \ldots, e^8$ of $\mathfrak{su}(3)^*$ may be assumed to satisfy

$$\begin{aligned}
de^1 &= -e^{23} - e^{45} + 2\,e^{67}, & de^2 &= e^{13} + e^{46} - e^{57} - \sqrt{3}\,e^{58}, \\
de^3 &= -e^{12} - e^{47} + \sqrt{3}\,e^{48} - e^{56}\ , & de^4 &= e^{15} - e^{26} + e^{37} - \sqrt{3}\,e^{38}, \\
de^5 &= -e^{14} + e^{27} + e^{36} + \sqrt{3}\,e^{28}, & de^6 &= -2\,e^{17} + e^{24} - e^{35}, \\
de^7 &= 2\,e^{16} - e^{25} - e^{34}, & de^8 &= -\sqrt{3}e^{25} + \sqrt{3}\,e^{34}\ .
\end{aligned}$$

The lack of components in $\Lambda^2\mathfrak{so}(3)^*$ in $de^4, \ldots, de^8$ shows that SU(3)/SO(3) is a symmetric space. It is known that SU(3)/SO(3) is not homeomorphic to S^5, even though it has the same real cohomology as the sphere [23].

For $k = 2, 3$, we have equivariant maps

$$i_k : V^* \to \Lambda^k T^*\ ;$$

defined as follows: i_2 is the composition

$$\mathfrak{so}(3)^* \xrightarrow{d} \Lambda^2\mathfrak{su}(3)^* \xrightarrow{\text{proj}} \Lambda^2 T^*\ ,$$

and i_3 is the composition

$$V^* \xrightarrow{i_2} \Lambda^2 T^* \cong \Lambda^3 T^*\ .$$

Let $\beta = c_{i_2}$, and $\tilde{\beta} = c_{i_3}$. It is clear that the algebra $\mathcal{A}$ contains no elements of bidegree $(1,0)$; however, by Theorem 3.13, $\mathcal{F}_{1,0}$ does contain invariant elements. We solve this problem by replacing $\tilde{\mathcal{A}}$ with a bigger subspace $\tilde{\mathcal{B}}$ of $\tilde{\mathcal{F}}$.

As a representation of SO(3), T is irreducible, i.e. $T \cong V^4$. Consider the bilinear map

$$i_2 \wedge i_2 : V \otimes V \to \Lambda^4 T^* = T^*\ ,$$

inducing an equivariant map

$$\epsilon : V \to T^* \otimes V \subset (T \oplus V)^* \otimes V\ .$$

Not being constant, ϵ is not an element of $\tilde{\mathcal{A}}$; it is, however, an element of $\tilde{\mathcal{F}}$. Define $\tilde{\mathcal{B}}$ as the differential Lie superalgebra generated by a, β, $\tilde{\beta}$ and ϵ; in other words, $\tilde{\mathcal{B}}$ is the smallest subspace of $\tilde{\mathcal{F}}$ satisfying

- $\tilde{\mathcal{B}}$ contains $D^n a$ for all $n \geq 0$, where $D^n = D \circ \cdots \circ D$ n times;

- $\tilde{\mathcal{B}}$ contains $D^n c_i$ for all $n \geq 0$, and all equivariant $i : V^* \to \Lambda^k T^*$;
- $\tilde{\mathcal{B}}$ contains $D^n \epsilon$ for all $n \geq 0$;
- $\tilde{\mathcal{B}}$ is closed under $[\,,]$.

The meaning of this definition is that we enlarged $\tilde{\mathcal{A}}$, adding the generator ϵ. Notice that ϵ vanishes at the zero section.

We set $\gamma = D\epsilon$; we shall see that γ has bidegree $(1,1)$ and is nowhere zero on all of V. By Theorem 3.13, the dimension of $\mathcal{F}_{p,q}$ at a point v is given by the following table, where e.g. $0/1$ means 0 at $v=0$ and 1 at $v \neq 0$:

$p\backslash q$	$0,3$	$1,2$
$0,5$	$1/1$	$0/1$
$1,4$	$0/1$	$0/3$
$2,3$	$0/2$	$1/6$

Theorem 4.13. *The algebra $\mathcal{B}$ generated by the image of $\tilde{\mathcal{B}}$ under $\cdot$ coincides with $\mathcal{F}^G$.*

Proof. It is clear that the forms of Table 4.3 belong to $\mathcal{B}$; we must check that they satisfy the hypotheses of Theorem 3.13. This follows from two facts: these forms are independent, and γ is a nowhere-vanishing form of bidegree $(1,1)$; both can be proved writing down ϵ and γ explicitly.
Let v_1, v_2, v_3 be a basis of V, where $v_i = e_i$ as an element of $\mathfrak{so}(3)$. Write

$$a = a_1 \otimes v_1 + a_2 \otimes v_2 + a_3 \otimes v_3 \;;$$

then

$$\begin{aligned}\epsilon = &\left(-4a_1 e^8 - 2\sqrt{3}a_2\, e^4 - 2\sqrt{3}\, a_3\, e^5\right) \otimes v_1 + \\ &+ \left(-2\sqrt{3}\, a_1\, e^4 + a_2\,(2\, e^8 - 2\sqrt{3}\, e^7) - 2\sqrt{3}\, a_3\, e^6\right) \otimes v_2 + \\ &+ \left(-2\sqrt{3}\, a_1\, e^5 - 2\sqrt{3}\, a_2\, e^6 + a_3\,(2\sqrt{3}\, e^7 + 2\, e^8)\right) \otimes v_3 \;.\end{aligned}$$

Similarly, writing

$$b = b_1 \otimes v_1 + b_2 \otimes v_2 + b_3 \otimes v_3 \;,$$

from Lemma 3.23 we obtain:

$$\gamma = \Big(4\, b_1 \wedge e^8 - 2\sqrt{3} b_2 \wedge e^4 - 2\sqrt{3}\, b_3 \wedge e^5\Big) \otimes v_1 +$$
$$+ \Big(-2\sqrt{3}\, b_1 \wedge e^4 - 2\sqrt{3} b_2 \wedge e^7 + 2 b_2 \wedge e^8 - 2\sqrt{3} b_3 \wedge e^6\Big) \otimes v_2 +$$
$$+ \Big(-2\sqrt{3}\, b_1 \wedge e^5 - 2\sqrt{3} b_2 \wedge e^6 + 2\, b_3 \wedge (e^8 + \sqrt{3} e^7)\Big) \otimes v_3$$

The rest of the proof is a straightforward computation. □

Remark. The generators appearing in Table 4.3 were chosen arbitrarily. We give a list of those forms consisting of a single "piece" which were omitted, and their equivalents in terms of the generators.

$$\begin{array}{lllll} a\gamma = b\epsilon & a\beta\epsilon = 0 & \beta\epsilon = a\tilde{\beta} & b\gamma = 0 & \\ ab\gamma = -\frac{1}{2} bb\epsilon & \beta\beta = 0 & a\beta\gamma = -b\beta\epsilon & b\tilde{\beta} = \frac{1}{8}\beta\gamma & \\ ab\tilde{\beta} = -\frac{1}{4} b\beta\epsilon & \tilde{\beta}\gamma = 0 & a\tilde{\beta}\gamma = \frac{1}{4}\beta\epsilon\gamma & \gamma\gamma = 12\, bb\beta & bb\gamma = 0 \end{array}$$

Remark. By definition $Da = b$ and $D\epsilon = \gamma$; since $\mathrm{SU}(3)/\mathrm{SO}(3)$ is symmetric, $D\beta = 0$. The curvature is β: so, $Db = -[a, \beta]$ and $D\gamma = -[\epsilon, \beta]$. This is essentially all one needs to know in order to determine the action of d on $\mathcal{B}$.

We shall not give applications of Theorem 4.13. However, one could in theory try to repeat what we did in Section 4.5: restrict to the hypersurface $\{aa = 1\}$, and classify invariant G_2-structures on the resulting 7-manifold. In Table 4.5 we give the restricted action of d on the hypersurface $\{aa = 1\}$. By looking at the table, it is easy to see that cocalibrated structures exist; by Theorem 2.25, they can be used to produce integrable invariant Spin(7)-structures on an open set of X. One might also try to construct such a structure directly, by producing a closed 4-form with stabilizer Spin(7). This would be more difficult: while the space of closed 4-forms is easy to identify (see Table 4.4), the form defining Spin(7) is not stable, unlike the G_2 case.

Table 4.3: Invariant forms on X

p	q	Rank	Basis
0	0	1	1
1	0	1	$a\epsilon$
0	1	1	ab
2	0	2	$a\beta, a\epsilon\epsilon$
1	1	3	$b\epsilon, ab\epsilon, ab\, a\epsilon$
0	2	1	abb
3	0	2	$\beta\epsilon, \epsilon\epsilon\epsilon$
2	1	6	$b\beta, ab\beta, b\epsilon\epsilon, a\gamma\epsilon, a\epsilon\, b\epsilon, a\epsilon\, ab\epsilon$
1	2	3	$bb\epsilon, abb\, a\epsilon, ab\, b\epsilon$
0	3	1	bbb
4	0	1	$\beta\epsilon\epsilon$
3	1	6	$\beta\gamma, b\beta\epsilon, \gamma\epsilon\epsilon, ab\, \beta\epsilon, ab\, \epsilon\epsilon\epsilon, a\epsilon\, ab\beta$
2	2	6	$bb\beta, b\epsilon\gamma, a\epsilon\, bb\epsilon, b\epsilon\, b\epsilon, a\beta\, abb, a\epsilon\epsilon\, abb$
1	3	1	$bbb\, a\epsilon$
5	0	1	$\beta\tilde{\beta}$
4	1	3	$\beta\epsilon\gamma, ab\, \beta\epsilon\epsilon, a\epsilon\, \beta\gamma$
3	2	6	$bb\tilde{\beta}, ab\, \beta\gamma, ab\, b\beta\epsilon, ab\, \gamma\epsilon\epsilon, a\epsilon\, bb\beta, abb\, \epsilon\epsilon\epsilon$
2	3	2	$a\beta\, bbb, a\epsilon\epsilon\, bbb$
5	1	1	$ab\, \beta\tilde{\beta}$
4	2	3	$abb\, \beta\epsilon\epsilon, \beta\epsilon\, bb\epsilon, b\epsilon\, \beta\gamma$
3	3	2	$bbb\, \epsilon\epsilon\epsilon, \beta\epsilon\, bbb$
5	2	1	$\beta\tilde{\beta}\, abb$
4	3	1	$bbb\, \beta\epsilon\epsilon$
5	3	1	$bbb\, \beta\tilde{\beta}$

Table 4.4: Action of d on 4-forms

α	$d\alpha$
$\beta\epsilon\epsilon$	$-2\beta\epsilon\gamma$
$\beta\gamma$	0
$b\beta\epsilon$	$-\frac{8}{3}aa\,\beta\tilde{\beta} - 4bb\tilde{\beta}$
$\gamma\epsilon\epsilon$	0
$ab\,\beta\epsilon$	$-ab\,\beta\gamma$
$ab\,\epsilon\epsilon\epsilon$	$-3ab\,\gamma\epsilon\epsilon$
$a\epsilon\,ab\beta$	$-2a\epsilon\,bb\beta + 8aa\,bb\tilde{\beta} + \frac{16}{15}aa^2\,\beta\tilde{\beta} + 4ab\,b\beta\epsilon$
$bb\beta$	$\frac{1}{6}\beta\epsilon\gamma$
$b\epsilon\gamma$	0
$a\epsilon\,bb\epsilon$	0
$b\epsilon\,b\epsilon$	0
$a\beta\,abb$	$\frac{1}{6}aa\,\beta\epsilon\gamma + \frac{1}{6}ab\,\beta\epsilon\epsilon + \frac{4}{3}a\beta\,bbb$
$a\epsilon\epsilon\,abb$	$2a\epsilon\epsilon\,bbb$
$bbb\,a\epsilon$	$-\frac{1}{4}ab\,\gamma\epsilon\epsilon$

Table 4.5: Action of d on $\{aa = 1\}$

p	q	α	$d\alpha$
3	0	$\beta\epsilon$	$\beta\gamma$
		$\epsilon\epsilon\epsilon$	$3\,\gamma\epsilon\epsilon$
2	1	$b\beta$	0
		$ab\beta$	$bb\beta + \frac{1}{12}\beta\epsilon\epsilon$
		$b\epsilon\epsilon$	0
		$a\epsilon\,b\epsilon$	$2\,b\epsilon\,b\epsilon$
1	2	$bb\epsilon$	$\frac{1}{12}\gamma\epsilon\epsilon$
4	0	$\beta\epsilon\epsilon$	$-2\,\beta\epsilon\gamma$
3	1	$\beta\gamma$	0
		$b\beta\epsilon$	$-\frac{8}{3}\,\beta\tilde{\beta} - 4bb\tilde{\beta}$
		$\gamma\epsilon\epsilon$	0
		$a\epsilon\,ab\beta$	$-2\,a\epsilon\,bb\beta + 8\,bb\tilde{\beta} + \frac{16}{15}\,\beta\tilde{\beta}$
2	2	$b\epsilon\,b\epsilon$	0
		$bb\beta$	$\frac{1}{6}\,\beta\epsilon\gamma$

Chapter 5

Hypo manifolds

In this chapter, we continue the study of SU(2)-structures on 5-manifolds we began in Chapter 2, using a language closer to that of Chapters 3 and 4.

Indeed, in the first section we show how SU(2)-structures on a 5-manifold can be defined using differential forms, and how these forms are related to the differential forms defining SU(3)-structures on a 6-manifold.
Recall that a hypo structure is an SU(2)-structure on a 5-manifold defined by a generalized Killing spinor, generalizing the notion of Einstein-Sasaki structure. In the second section, we express the intrinsic torsion of an SU(2)-structure in terms of the defining forms, and we use this result to characterize hypo and Einstein-Sasaki structures in terms of differential forms.

After that, we temporarily turn our attention to $SL(2,\mathbb{C})$-structures: the third section is centred on algebraic preliminaries for the discussion of the embedding property, whereas the fourth section is devoted to the study of the intrinsic torsion of $SL(2,\mathbb{C})$-structures.
In the fifth section, we discuss the embedding property for hypo manifolds. We introduce the space of deformations of a hypo structure, and define regularly hypo structures. We then prove the embedding theorem for regularly hypo structures.
In the sixth section we provide examples of hypo structures; indeed, we give a complete list of the 5-dimensional nilmanifolds which admit an invariant hypo structure. Notice that the analogous problem in dimension 6 is still open, although of the 34 isomorphism classes of 6-dimensional nilmanifolds, 12 are known to admit invariant half-flat structures ([16, 14]), and we have been able to produce 11 more.
In the seventh section, we give an example of a regularly hypo manifold

which does not satisfy the hypothesis of the Bär-Gauduchon-Moroianu theorem (Theorem 2.23), showing that our embedding theorem is independent of that result.
In the appendix, we state some results in analysis which were used in the proof of the embedding theorem.

5.1 SU(2)-structures and hypo hypersurfaces

In this section we show how SU(2)-structures, and hypo structures in particular, can be defined using differential forms; in this language we repeat the construction described in Section 2.1. Namely, we use forms to show that SU(2)-structures arise naturally on hypersurfaces of 6-manifolds endowed with an SU(3)-structure (see Proposition 2.2), and that if the SU(3)-structure is integrable, the corresponding SU(2)-structure is hypo (see Proposition 2.5). A particular case is when the 6-manifold is a cone on the 5-manifold; in that case, the metric induced on the latter is Einstein-Sasaki. In fact, hypo structures generalize Einstein-Sasaki structures: the former are defined by a generalized Killing spinor, and the latter by a Killing spinor.
In the proof of Lemma 4.2, we established conditions which a triplet $(\omega_1, \omega_2, \omega_3)$ of 2-forms on a 4-manifold must satisfy in order to define an SU(2)-structure. A similar result holds in dimension 5:

Proposition 5.1. SU(2)*-structures on a 5-manifold are in one-to-one correspondence with quadruplets* $(\alpha, \omega_1, \omega_2, \omega_3)$*, where* α *is a 1-form and* ω_i *are 2-forms, satisfying:*

$$\omega_i \wedge \omega_j = \delta_{ij} \upsilon \tag{5.1}$$

for some 4-form υ *with* $\upsilon \wedge \alpha \neq 0$*, and*

$$X \lrcorner \omega_1 = Y \lrcorner \omega_2 \implies \omega_3(X, Y) \geq 0 \, . \tag{5.2}$$

Equivalently, an SU(2)*-structure can be defined by a 1-form* α*, a 2-form* ω_1 *and a complex 2-form* Φ*, corresponding to* $\omega_2 + i\omega_3$*, such that*

$$\alpha \wedge \omega_1^2 \neq 0 \qquad \omega_1 \wedge \Phi = 0 \tag{5.3}$$

$$\Phi^2 = 0 \qquad 2\omega_1^2 = \Phi \wedge \overline{\Phi} \tag{5.4}$$

and Φ *is* $(2, 0)$ *with respect to* ω_1*.*

Remark. If we start with an SO(5)-structure, we can understand a reduction to SU(2) as follows. The form α defines a splitting $\mathbb{R}^5 = \mathbb{R} \oplus \mathbb{R}^4$; a metric and an orientation is induced on $\mathbb{R}^4$. The eigenspace decomposition relative to the Hodge star gives $\Lambda^2\mathbb{R}^4 = \Lambda^2_+ \oplus \Lambda^2_-$, which corresponds to writing SO(4) as $\mathrm{SU}(2)_+\mathrm{SU}(2)_-$. The choice of a basis $\omega_1, \omega_2, \omega_3$ of Λ^2_+, corresponding to (5.1), reduces then SO(4) to $\mathrm{SU}(2)_-$. Since Λ^2_+ has a natural orientation, one can always assume that $\omega_1, \omega_2, \omega_3$ be a positively oriented, orthogonal basis of this space, so that the map $\mathbb{R}^3 \ni (a^i) \to (a^i\omega_i) \in \Lambda^2_+$ is $\mathrm{SU}(2)_-$-equivariant; this assumption corresponds to (5.2).

By (α, ω_i) we shall mean a quadruplet $(\alpha, \omega_1, \omega_2, \omega_3)$ satisfying Proposition 5.1 and the orientation condition, and for a form ω on a manifold M, we define

$$\omega^0 = \{X \in TM \mid X \lrcorner\, \omega = 0\} \ .$$

Before proving Proposition 5.1, it is convenient to prove the following.

Proposition 5.2. *For (α, ω_i) as above, we have $\omega_i^0 = \upsilon^0$, $i = 1, 2, 3$.*

Proof. Define β_i and γ_i by the condition that $\omega_i = \alpha \wedge \beta_i + \gamma_i$, with β_i^0 and γ_i^0 containing υ^0. Then by (5.1),

$$\alpha \wedge \beta_i \wedge \gamma_i + \gamma_i^2 = \upsilon \ .$$

Take a non-zero X in υ^0; then

$$0 = X \lrcorner\, (\alpha \wedge \beta_i \wedge \gamma_i) = (X \lrcorner\, \alpha)\beta_i \wedge \gamma_i \ ,$$

and so $\beta_i \wedge \gamma_i = 0$. On the other hand, $0 \neq \alpha \wedge \upsilon = \alpha \wedge \gamma_i^2$ shows that γ_i is non-degenerate on α^0, and therefore $\beta_i = 0$. Thus $\omega_i^0 \supseteq \upsilon^0$; the opposite inclusion follows from $\upsilon = \omega_i^2$. □

Corollary 5.3. *Given (α, ω_i) as above, one can always find a basis $e^1, \dots, e^5$ of forms on M such that*

$$\begin{cases} \alpha = e^5 & \omega_1 = e^{12} + e^{34} \\ \omega_2 = e^{13} + e^{42} & \omega_3 = e^{14} + e^{23} \end{cases} \tag{5.5}$$

Moreover, one may require that (for example) e^1 equal a fixed unit form orthogonal to α.

A consequence of the corollary itself is that a global nowhere vanishing 1-form in $\alpha^{\perp}$ exists only if M is parallelizable; in general, (5.5) can only be used locally.

Proof of Proposition 5.1. It is sufficient to show that if $e^1, \dots, e^5$ is the standard basis of $(\mathbb{R}^5)^*$ and (α, ω_i) are as in (5.5), the stabilizers of α, ω_1, ω_2 and ω_3 have intersection SU(2). In fact, if $A \in GL(5, \mathbb{R})$ preserves these forms, it must preserve the splitting $\mathbb{R}^5 = \alpha^0 \oplus \upsilon^0$, so that

$$A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}, \; B \in GL(4, \mathbb{R}) \; .$$

On the other hand, such an A preserves $\omega_1, \omega_2, \omega_3$ if and only if B preserves the standard hyperkähler structure on $\mathbb{R}^4$, i.e. B lies in $\mathrm{Sp}(1) = \mathrm{SU}(2)$. $\square$

Remark. One has to fix a choice of reference forms on $\mathbb{R}^5$ in order to actually identify an SU(2)-structure with a quadruplet of forms (α, ω_i), or a triplet (α, ω_1, Φ); we shall henceforth use (5.5) to do so, and associate to a frame u forms (α, ω_i) such that $u^*\alpha = e^5$, and so on.

Remark. In the proof of Proposition 5.1 we have used the fact that an SU(2)-structure on M defines an orthogonal splitting $TM = \upsilon^0 \oplus \alpha^0$.

Proposition 5.4. *Let $i : M \to N$ be an immersion of an oriented 5-manifold into a 6-manifold. Then an* SU(3)*-structure on N defines an* SU(2)*-structure on M in a natural way. Conversely, an* SU(2)*-structure on M defines an* SU(3)*-structure on $M \times \mathbb{R}$ in a natural way.*

Proof. The SU(3)-structure on N defines a non-degenerate 2-form ω and a complex 3-form Ψ with stabilizer $\mathrm{SL}(3, \mathbb{C})$. Since both M and N are oriented, the normal bundle to M has a canonical unit section, which using the metric lifts to a section V of i^*TN. Define forms on M by $\alpha = V \lrcorner\, \omega$, $\Phi = iV \lrcorner\, \Psi$, $\omega_1 = i^*\omega$. Choose a local basis of 1-forms on N such that V is dual to $-e^6$ and

$$\omega = e^{12} + e^{34} + e^{56} \; , \qquad \Psi = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) \; ; \qquad (5.6)$$

then $i^*e^1, \dots, i^*e^5$ satisfy (5.5).
Vice versa, given an SU(2)-structure on M, an SU(3)-structure on $M \times \mathbb{R}$ is defined by (ω, Ψ) given by

$$\omega = \omega_1 + \alpha \wedge dt \; , \qquad\qquad \Psi = \Phi \wedge (\alpha + idt) \; , \qquad (5.7)$$

where t is a coordinate on $\mathbb{R}$. $\square$

In this chapter we study SU(2)-structures satisfying

$$d\omega_1 = 0\ , \qquad d(\alpha\wedge\omega_2) = 0\ , \qquad d(\alpha\wedge\omega_3) = 0\ . \tag{5.8}$$

Remark. If (α, ω_i) satisfies (5.8), then the SU(2)-structures obtained rotating ω_2 and ω_3 also satisfy (5.8); moreover, they induce the same metric. This is akin to the case of integrable (i.e. Calabi-Yau) SU(n)-structures on $2n$-dimensional manifolds, where multiplying the holomorphic n-form by $e^{i\theta}$ gives a different integrable structure corresponding to the same metric.

We shall show in the next section that these SU(2)-structures coincide with hypo structures are defined Section 2.2. For the moment, we state their most important property:

Proposition 5.5. *Let $i\colon M\to \mathbf{M}$ be an immersion of an oriented 5-manifold in a 6-manifold with an integrable* SU(3)*-structure. Then the* SU(2)*-structure induced on M satisfies (5.8).*

Proof. From (5.5) and (5.6) it follows that $i^*\Psi = i\Phi\wedge\alpha$; recall also that $\omega_1 = i^*\omega$. Since Ψ and ω are closed, and i^* commutes with d, (5.8) holds. □

By construction, Equations 5.8 are exactly the conditions one obtains on the SU(2)-structure induced on a hypersurface in a 6-dimensional manifold with a parallel spinor; it is therefore not surprising that these structures are characterized by the existence of a generalized Killing spinor, as we shall prove in the next section.

5.2 Differential forms versus spinors

Fix a 5-manifold M. By Proposition 1.9, SU(2)-structures on M are in one-to-one correspondence with pairs $(P_{\mathrm{Spin}(5)}, \psi)$, where $P_{\mathrm{Spin}(5)}$ is a spin structure on M and ψ is a unit spinor; explicitly, one has

$$\psi = [s, u_0]$$

for every local section s of $P_{\mathrm{SU}(2)}$. On the other hand, SU(2)-structures are in one-to-one correspondence with quadruplets (α, ω_i), as stated in Proposition 5.1. We shall compare properties of ψ with properties of (α, ω_i).
Fix an SU(2)-structure $P_{\mathrm{SU}(2)}$ on M; let ψ be the defining spinor and (α, ω_i)

the defining forms. In Section 2.4, we saw that the intrinsic torsion of an SU(2)-structure takes values in

$$7\mathbb{R} \oplus 4\Lambda^1 \oplus 4\Lambda^2_- \ . \tag{5.9}$$

Clearly, if $P_{\mathrm{SU}}(2)$ is defined by (α, ω_i), its intrinsic torsion can be expressed in terms of $\nabla\alpha$, $\nabla\omega_i$. It turns out that it is sufficient to consider the exterior derivative, rather than the covariant derivative:

Proposition 5.6. *Write*

$$\begin{aligned} d\alpha &= \alpha \wedge \beta + \sum f^i \omega_i + \omega^- \ , \\ d\omega_i &= \gamma_i \wedge \omega_i + \sum f_i^j \alpha \wedge \omega_j + \alpha \wedge \sigma_i^- . \end{aligned}$$

then $f_i^j = \lambda\delta_{ij} + g_i^j$ *where* $g_i^j = -g_j^i$ *and according to the splitting (5.9) the intrinsic torsion can be written*

$$[\Theta](u) = \big((f^i, \lambda, g_i^j), (u^*\beta, u^*\gamma_i), (u^*\omega^-, u^*\sigma_i^-)\big) \ .$$

We can now prove that the two definitions of hypo structures we have given are equivalent.

Proposition 5.7. *Equation 5.8 holds if and only if* ψ *is a generalized Killing spinor.*

Proof. By Proposition 5.6, each condition means that the intrinsic torsion is forced to lie in a SU(2)-module isomorphic to

$$2\mathbb{R} \oplus \Lambda^1 \oplus 3\Lambda^2_- \ .$$

A priori these isomorphic modules need not coincide; however, they have to coincide if one of them is contained in the other one. It is therefore sufficient to prove the "if" part of the statement.

Let ψ be a generalized Killing spinor. By (2.6), for the Levi-Civita connection we have

$$\begin{aligned} \nabla\alpha &= A^1 \otimes e^2 - A^2 \otimes e^1 + A^3 \otimes e^4 - A^4 \otimes e^3 \\ \nabla\omega_1 &= -A^1 \otimes v^1 - A^2 \otimes v^2 - A^3 \otimes v^3 - A^4 \otimes v^4 \\ \nabla\omega_2 &= -A^5 \otimes \omega_3 - A^1 \otimes v^4 - A^2 \otimes v^3 + A^3 \otimes v^2 + A^4 \otimes v^1 \\ \nabla\omega_3 &= A^5 \otimes \omega_2 + A^1 \otimes v^3 - A^2 \otimes v^4 - A^3 \otimes v^1 + A^4 \otimes v^2 \end{aligned}$$

Using the symmetry of A, we conclude that ω_1, $\omega_2 \wedge \alpha$ and $\omega_3 \wedge \alpha$ are closed. $\square$

A special case is when ψ is a Killing spinor, i.e. the underlying U(2)-structure is Einstein-Sasaki. The results on Einstein-Sasaki manifolds that were proved with spinors in Chapter 1 can also be derived using differential forms:

Lemma 5.8. *The conical* SU(3)*-structure on* $N = M \times \mathbb{R}_+$ *induced by* $P_{\mathrm{SU}(2)}$*, defined by*

$$\omega = t^2\omega_1 + t\alpha \wedge dt \ , \qquad \Psi = t^2\Phi \wedge (t\alpha + i dt) \ , \tag{5.10}$$

is integrable if and only if

$$d\alpha = -2\omega_1 \ , \qquad d\Phi = -3i\alpha \wedge \Phi \ . \tag{5.11}$$

Proof. The SU(3)-structure is integrable if and only if ω and Ψ are closed; a straightforward calculation completes the proof. □

From the proof of Proposition 5.7, we see that (5.11) is equivalent to

$$A = -\frac{1}{2}\mathrm{Id} \ .$$

As a consequence, we have the following characterization:

Proposition 5.9. *The following are equivalent:*

1. *The almost contact structure underlying* $P_{\mathrm{SU}(2)}$ *is Einstein-Sasaki;*
2. ψ *is a Killing spinor with Killing constant* $-1/2$*;*
3. *(5.11) holds;*
4. *the* SU(3)*-structure on* $M \times \mathbb{R}_+$ *given by (5.10) is integrable.*

Remark. If (M, g) is a Riemannian 5-manifold, and the cone $M \times_r \mathbb{R}_+$ has holonomy SU(3), then M has an Einstein-Sasaki structure: indeed, one can choose an integrable SU(3)-structure on $M \times \mathbb{R}_+$; this will be conical in the sense of (5.10), as it is preserved by parallel transport with respect to the Levi-Civita connection, and an SU(2)-structure satisfying (5.11) is thus defined on M (not uniquely).

Remark. Proposition 5.9 shows that Einstein-Sasaki 5-manifolds are the analogue of nearly-Kähler 6-manifolds; this analogy will be carried further in Section 5.5.

Having proved Proposition 5.7, we shall not need to consider spinors anymore. In particular, we drop the convention that the letter ψ represents a spinor and the letter A represents a symmetric endomorphism of the tangent bundle.

5.3 Stable forms and $\mathrm{SL}(2,\mathbb{C})$ structures

One can think of SU(2)-structures on 5-manifolds as the 5-dimensional analogue of the more popular SU(3)-structures on 6-manifolds, or as the sort of structures which are defined by spinors in dimension 5. However, one could also introduce SU(2) structures from a theory of stable forms. From this point of view, it is natural to see SU(2) as the intersection of two copies of $\mathrm{SL}(2,\mathbb{C})$; it turns out that some of the structure coming from the SU(2)-structure is already present in the $\mathrm{SL}(2,\mathbb{C})$ case, which is why we devote this section to the study of $\mathrm{SL}(2,\mathbb{C})$-structures, in preparation for the proof of the evolution theorem.

Let T be a real vector space of dimension 5. The set

$$\{\omega \in \Lambda^2 T^* \mid \omega^2 \neq 0\}$$

is clearly open in $\Lambda^2 T^*$, as well as stable under $\mathrm{GL}(5,\mathbb{R})$. Its elements can be interpreted as pairs (L,σ) where L is a line in T and σ is a non-degenerate two-form on T/L. Since the action of $\mathrm{GL}(5,\mathbb{R})$ on such pairs (L,σ) is clearly transitive, the space defined above consists of a single $\mathrm{GL}(5,\mathbb{R})$ orbit. In the terminology of Hitchin, we shall call these elements *stable* 2-forms.
Using the non-degenerate pairing $\Lambda^2 T \otimes \Lambda^3 T \to \mathbb{R}$ defined by any volume form, we see that $\Lambda^3 T \cong \Lambda^2 T^*$ as $\mathrm{GL}(5,\mathbb{R})$-modules. Hence, $\Lambda^3 T^*$ also has a unique open orbit, whose elements will be called stable 3-forms.

Now fix an orientation on T and consider the isomorphism

$$A : \Lambda^n T^* \to \Lambda^{5-n} T \otimes \Lambda^5 T^* \ , \quad n = 0, \dots, 5 \ .$$

Although we shall not fix an isomorphism $\Lambda^5 T^* \cong \mathbb{R}$, we think of elements of $T \otimes \Lambda^5 T^*$, for instance, as vectors whose length depends on the choice of a volume form, and write formulas accordingly. We have a natural map $X : \Lambda^2 T^* \to T \otimes \Lambda^5 T^*$ defined by

$$X(\omega) = A(\omega^2) \ ;$$

interchanging T with T^* and composing with A, a map $\alpha : \Lambda^3T^* \to T^*\otimes\Lambda^5T^*$ is induced; explicitly,

$$\alpha(\psi) = A((A\psi)^2) .$$

Note that $X(\omega)$ spans $(\omega^2)^0$, and therefore ω^0, by Proposition 5.2. By duality, this shows that

$$\alpha(\psi) \wedge \psi = 0 . \tag{5.12}$$

Now define a map $V^2 : \Lambda^2T^* \times \Lambda^3T^* \to (\Lambda^5T^*)^2$ by

$$V^2(\omega, \psi) = \alpha(\psi)(X(\omega)) ; \tag{5.13}$$

then $V^2(\omega, \psi) = \omega^2(A\psi \wedge A\psi)$, showing that $V^2 \neq 0$ forces ω and ψ to be stable. If $V^2(\omega, \psi)$ admits a "square root" in Λ^5T^*, we write $V^2(\omega, \psi) > 0$ and we define $V(\omega, \psi)$ as the positively oriented square root of $V^2(\omega, \psi)$.

Wherever $V^2 > 0$, we have a splitting

$$T = \langle X(\omega)\rangle \oplus \alpha(\psi)^0 . \tag{5.14}$$

More, the pair (ω, ψ) defines a one-form α and a vector X with $\alpha(X) = 1$ by

$$\alpha = V(\omega, \psi)^{-1}\alpha(\psi) , \qquad X = V(\omega, \psi)^{-1}X .$$

We can then restate Proposition 5.1 in the following way:

Proposition 5.10. *An* $\mathrm{SL}(2, \mathbb{C})$ *structure is a pair* (ω, ψ) *with*

$$V^2(\omega, \psi) > 0 , \qquad \omega \wedge \psi = 0 .$$

An $\mathrm{SU}(2)$*-structure is a triple* (ω, ψ_2, ψ_3) *such that*

$$\alpha(\psi_2) = \alpha(\psi_3) , \qquad \omega \wedge \psi_2 = 0 = \omega \wedge \psi_3 , \qquad (X \lrcorner \psi_2) \wedge \psi_3 = 0 .$$

and $V(\omega, \psi_2)$*, which in this hypothesis equals* $V(\omega, \psi_3)$*, is non-zero.*

Proof. Identify $\mathbb{R}^5$ with $\mathbb{C}^2 \oplus \mathbb{R}$; this determines an action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{R}^5$. An $\mathrm{SL}(2, \mathbb{C})$ structure on T defines, uniquely up to $\mathrm{SL}(2, \mathbb{C})$ action, a basis of 1-forms $e^1, \dots, e^5$, which determine a pair (ω, ψ) by

$$\omega = e^{12} + e^{34} , \qquad \psi = e^{135} + e^{425} .$$

Conversely, given (ω, ψ), the splitting (5.14) reduces the structure group to $\mathrm{GL}(4, \mathbb{R})$; to reduce to $\mathrm{SL}(2, \mathbb{C})$, we need a simple, non-degenerate complex

two-form.
Observe that

$$V^2(\omega,\psi)=\omega^2\wedge\alpha(\psi)\ , \tag{5.15}$$

and using $A\psi\wedge X(\omega)=A(X(\omega)\lrcorner\,\psi)$, we see that the dual equation

$$V^2(\omega,\psi)=X(\omega)\lrcorner\,\psi\wedge\psi \tag{5.16}$$

also holds.
By construction, the restriction of ω to α^0 is non-degenerate, and so is $X\lrcorner\,\psi$. By (5.12), $\psi=\alpha\wedge(X\lrcorner\,\psi)$; consequently, $\omega\wedge\psi=0$ implies $(X\lrcorner\,\psi)\wedge\omega=0$.

We show that

$$\omega^2=(X\lrcorner\,\psi)^2\ . \tag{5.17}$$

Indeed, using (5.16):

$$(X\lrcorner\,\psi)^2=V^{-1}(\omega,\psi)X(\omega,\psi)\lrcorner\,(X(\omega)\lrcorner\,\psi\wedge\psi)=X(\omega,\psi)\lrcorner\,V(\omega,\psi)\ .$$

Hence, $\omega+i(X\lrcorner\,\psi)$ is a non-degenerate complex 2-form whose square vanishes, and therefore simple.

To prove the second assertion, set $\omega_i=X(\omega)\lrcorner\,\psi_i$ for $i=2,3$, and set $\alpha=V(\omega,\psi_i)^{-1}\alpha(\psi_i)$, not depending on $i=2,3$. Then by (5.17) Proposition 5.1 is satisfied. $\square$

The differential of V at a point is an element of

$$(\Lambda^2T^*\oplus\Lambda^3T^*)^*\otimes\Lambda^5T^*\cong(\Lambda^2T\oplus\Lambda^3T)\otimes\Lambda^5T^*=\Lambda^3T^*\oplus\Lambda^2T^*\ .$$

Note also that $X(\omega)$ only depends on ω^2, so it makes sense to define X on 4-forms; in order to make subsequent formulae simpler, we set $X(\upsilon)=2\,A(\upsilon)$. Therefore, given a 3-form ψ and a 4-form υ one can define $V(\psi,\upsilon)$, $X(\psi,\upsilon)$ and $\alpha(\psi,\upsilon)$ as above; the differential of V in this sense is an element of $T^*\oplus\Lambda^2T^*$. The following Lemma will be needed in Section 5.5:

Lemma 5.11. *Set $\hat\omega=\alpha(\omega,\psi)\wedge\omega$, $\hat\psi=X(\omega,\psi)\lrcorner\,\psi$; then*

$$dV_{(\omega,\psi)}(\sigma,\phi)=\hat\omega\wedge\sigma+\hat\psi\wedge\phi\ . \tag{5.18}$$

Now set $\hat\psi=X(\psi,\upsilon)\lrcorner\,\psi$; then

$$dV_{(\psi,\upsilon)}(\phi,\sigma)=\hat\psi\wedge\phi+\alpha(\psi,\upsilon)\wedge\sigma\ . \tag{5.19}$$

Proof. Differentiating (5.15) with respect to ω,

$$2V(\omega,\psi)dV_{(\omega,\psi)}(\sigma,0) = 2\,\omega\wedge\sigma\wedge\alpha(\psi)\ ,$$

so

$$dV_{(\omega,\psi)}(\sigma,0) = \omega\wedge\sigma\wedge\alpha(\omega,\psi)\ .$$

Similarly, from (5.16) we obtain

$$dV_{(\omega,\psi)}(0,\phi) = X(\psi,v)\lrcorner\,\psi\wedge\phi\ ,$$

and (5.18) is obtained by summing the two equations.
The proof of (5.19) is completely analogous. $\square$

5.4 Geometry of $\mathrm{SL}(2,\mathbb{C})$-structures

In this section we study the intrinsic torsion of $\mathrm{SL}(2,\mathbb{C})$-structures; in particular, we characterize the intrinsic torsion of $\mathrm{SL}(2,\mathbb{C})$-structures on M such that the underlying almost contact structure is normal, i.e. the product complex structure on $M\times S^1$ is integrable. The results of this section will not be used in the rest of this chapter.
Let T be a 5-dimensional, oriented real vector space with an $\mathrm{SL}(2,\mathbb{C})$ structure (ω,ψ).

Proposition 5.12. *Let $i : M\to N$ be an immersion of an oriented 5-manifold into a 6-manifold; with a section V of i^*TN which is transverse to TM at each point. Then the choice of V defines a bijection between the $\mathrm{SL}(3,\mathbb{C})$-structures on i^*TN and the $\mathrm{SL}(2,\mathbb{C})$-structures on M compatible with its orientation.*

Proof. An $\mathrm{SL}(3,\mathbb{C})$-structure on i^*TN is identified by a real positive stable 3-form ψ^+, whereas an $\mathrm{SL}(2,\mathbb{C})$-structure on TN is identified by a pair (ω,ϕ) as above plus an orientation. Let e^6 be the form dual to V, i.e. the section of $i^*\Lambda^1N$ defined by $e^6(V)=1$, $e^6|_{TM}=0$. Given ψ^+, set

$$\omega = -V\lrcorner\,\psi^+\ ,\qquad\qquad \phi = i^*\psi^+\ .$$

Conversely, given (ω,ϕ), write $\psi^+ = \phi-\omega\wedge e^6$. It is easy to check that this establishes a bijection. $\square$

Write T as in (5.14). Algebraically, the choice of V in Proposition 5.12 corresponds to the choice of a complex structure on $X(\omega) \oplus \mathbb{R}$; using the complex structure on $\alpha(\psi)^0$, this choice determines a complex structure on $T \oplus \mathbb{R}$. We shall show that there is a well-defined notion of forms of type (p,q) on T which does not depend on this choice. To this effect, let (W, J) be a vector space with a complex structure.

Lemma 5.13. *If $\Theta \in \Lambda^{p,q}W$, then Θ^0 is J-stable.*

Proof. Take $V \in \Theta^0$; then by definition $V \lrcorner \Theta = 0$. The forms $(V + iJV) \lrcorner \Theta$ and $(V - iJV) \lrcorner \Theta$ are respectively of type $(p, q-1)$ and $(p-1, q)$; since their sum is $2\,V \lrcorner \Theta = 0$, they must both vanish, implying $JV \lrcorner \Theta = 0$. □

As a consequence, a form on $T \subset T \oplus \mathbb{R}$ is of pure type if and only if it is zero on $\langle X(\omega) \rangle$ and it is of pure type on $\alpha(\psi)^0$. We shall therefore declare a complex form on T to have type (p,q) if it is zero on $\langle X(\omega) \rangle$ and it has type (p,q) on $\alpha(\psi)^0$.

Proposition 5.14. *Let M have an $\mathrm{SL}(2,\mathbb{C})$ structure $(\alpha, \omega_1, \omega_2)$. Then the almost complex structure of the induced $\mathrm{SL}(3,\mathbb{C})$-structure on $M \times S^1$ is integrable if and only if*

$$d\alpha \wedge \omega_1 = 0, \qquad d\alpha \wedge \omega_2 = 0, \qquad d\omega_2 + i d\omega_1 \in \Lambda^{2,1} .$$

Proof. By Proposition 5.12, the defining $(3,0)$-form on $M \times S^1$ is

$$\Psi = (\omega_2 + i\omega_1) \wedge (\alpha + i d\theta) .$$

Thus, integrability is equivalent to $d\Psi$ being of type $(3,1)$. This implies

$$0 = (\alpha + i d\theta) \wedge d\Psi = (\alpha + i d\theta) \wedge (\omega_2 + i\omega_1) \wedge d\alpha ,$$

so $(\omega_2 + i\omega_1) \wedge d\alpha = 0$ and

$$d\Psi = (d\omega_2 + i d\omega_1) \wedge (\alpha + i d\theta) . \tag{5.20}$$

Hence $d\Psi$ being of type $(3,1)$ implies that $(d\omega_2 + i d\omega_1)$ must be of type $(2,1)$. Conversely, if (5.20) holds and $(d\omega_2 + i d\omega_1)$ is of type $(2,1)$, then $d\Psi$ is of type $(3,1)$. □

Remark. An $\mathrm{SL}(2,\mathbb{C})$-structure induces an almost-contact structure by extension. The integrability of the $\mathrm{SL}(3,\mathbb{C})$-structure on $M \times S^1$ is equivalent to this almost contact structure being normal (see Section 1.5).

Now consider the skew-symmetrization map ∂:

$$\mathfrak{sl}(2,\mathbb{C}) \otimes T^* \to T \otimes T^* \otimes T^* \to T \otimes \Lambda^2 T^* \ .$$

An exact sequence is induced:

$$0 \to \mathfrak{sl}(2,\mathbb{C})^{(1)} \to \mathfrak{sl}(2,\mathbb{C}) \otimes T^* \to T \otimes \Lambda^2 T^* \to \operatorname{Coker} \partial \to 0 \ .$$

Since $\mathfrak{sl}(2,\mathbb{C})$ is a vector space over $\mathbb{C}$, one has

$$\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{R}} T^* \cong \mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} T^*_{\mathbb{C}} \ ;$$

it is therefore equivalent to consider the complexification of ∂, which is a map

$$\partial_{\mathbb{C}} : \mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} T^*_{\mathbb{C}} \to T_{\mathbb{C}} \otimes \Lambda^2 T^*_{\mathbb{C}} \ .$$

Lemma 5.15. *The kernel of $\partial_{\mathbb{C}}$ is the subspace of $\mathfrak{sl}(2,\mathbb{C}) \otimes \Lambda^{1,0} T^*$ given by*

$$\left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} b & d \\ -a & -b \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} \mid a,b,c,d \in \mathbb{C} \right\} \ .$$

Proof. Consider $T \oplus \mathbb{R} \cong \mathbb{C}^3$; the action of $\mathfrak{sl}(2,\mathbb{C})$ on $T \oplus \mathbb{R}$ is $\mathbb{C}$-linear. Relative to the action on $(T \oplus \mathbb{R})_{\mathbb{C}}$, we have

$$\mathfrak{sl}(2,\mathbb{C}) \otimes T^*_{\mathbb{C}} \subset \mathfrak{sl}(2,\mathbb{C}) \otimes \Lambda^{1,0}(T \oplus \mathbb{R})^* \oplus \mathfrak{sl}(2,\mathbb{C}) \otimes \Lambda^{0,1}(T \oplus \mathbb{R})^* \ .$$

Since the restriction of $\partial_{\mathbb{C}}$ to the second component is injective, and by Lemma 5.13 $\Lambda^{1,0}(T \oplus \mathbb{R})^* = \Lambda^{1,0} T^*$, the first assertion is proved.
Now write the generic element of $\mathfrak{sl}(2,\mathbb{C}) \otimes \Lambda^{1,0} T^*$ as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & -b_{11} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ ;$$

evaluating on $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \wedge \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ proves the lemma. $\square$

We conclude that $\operatorname{Coker} \partial$ has dimension $8 - 30 + 50 = 28$, i.e. the intrinsic torsion of an $\mathrm{SL}(2,\mathbb{C})$-structure takes values in a 28-dimensional space. We can express the intrinsic torsion in terms of the defining forms, like in the case of $\mathrm{SU}(2)$. Let $\mathrm{SL}(2,\mathbb{C})$ act on $\mathbb{R}^5 = \mathbb{C}^2 \oplus \mathbb{R}$; we are viewing $\mathbb{C}^2$ as a real vector space with an $\mathrm{SL}(2,\mathbb{C})$ action.

Proposition 5.16. *The intrinsic torsion of an* $\mathrm{SL}(2,\mathbb{C})$*-structure takes values in*

$$4\mathbb{R} \oplus 3\mathbb{C}^2 \oplus 3[\Lambda^{1,1}] .$$

For any $\mathrm{SL}(2,\mathbb{C})$*-structure* $(\alpha, \omega_1, \omega_2)$ *we can write*

$$\begin{aligned} d\alpha &= \alpha \wedge \beta + f^1\omega_1 + f^2\omega_2 + \tau , \\ d\omega_1 &= \gamma_1 \wedge \omega_1 + \lambda\alpha \wedge \omega_1 + \mu\alpha \wedge \omega_2 + \alpha \wedge \rho_1. , \\ d\omega_2 &= \gamma_2 \wedge \omega_2 - \mu\alpha \wedge \omega_1 + \lambda\alpha \wedge \omega_2 + \alpha \wedge \rho_2. , \end{aligned}$$

and the intrinsic torsion can be written

$$[\Theta](u) = \big((f^1, f^2, \lambda, \mu), (u^*\beta, u^*\gamma_1, u^*\gamma_2), (u^*\tau, u^*\rho_1, u^*\rho_2)\big) .$$

The condition of Proposition 5.14 kills all components but τ, γ_1 and γ_2; moreover, it forces $\gamma_1 + \gamma_2 = 0$. Indeed, $(\omega_1 + i\omega_2) \wedge \gamma_1$ has type $(2,1)$, so

$$\gamma_2 \wedge \omega_2 + i\gamma_1 \wedge \omega_1 \in \Lambda^{2,1} \iff \gamma_2 \wedge \omega_2 + \gamma_1 \wedge \omega_2 \in \Lambda^{2,1} ;$$

since the latter form is real, it follows that $\gamma_1 + \gamma_2 = 0$. We can then restate Proposition 5.14 as follows:

Theorem 5.17. *An* $\mathrm{SL}(2,\mathbb{C})$ *structure is normal if and only if its intrinsic torsion lies in* $\mathbb{C}^2 \oplus [\Lambda^{1,1}]$.

5.5 Evolution theory

We have seen that a hypersurface M inside a holonomy SU(3) 6-manifold $\mathbf{M}$ is naturally endowed with a hypo structure $P_{\mathrm{SU}(2)}$; hypo manifolds $(M, P_{\mathrm{SU}(2)})$ which occur this way are said to have the embedding property. In this section we prove that a large class of compact hypo manifolds, called regularly hypo, have the embedding property. We also show that the geometry of $\mathbf{M}$ can be recovered (at least locally) from the geometry of M.

Fix a compact 5-dimensional manifold M; let B^p be the space of exact p-forms, Z^p the space of closed p-forms, Ω^p the space of p-forms, and Λ^p the bundle of p-forms. We are now thinking of an SU(2)-structure as defined by a quadruplet $(\omega, \psi_2, \psi_3, \upsilon)$, where (ω, ψ_2, ψ_3) play the same rôle as in Proposition 5.10, and $\upsilon = \omega^2/2$. This is not unnatural, because an SU(3)-structure on a 6-manifold is determined by a 3-form ψ^+ and a 4-form ω^2, i.e. a section of $\Lambda^3 \oplus \Lambda^4$, and the pullback of this bundle to a hypersurface splits up as $\Lambda^2 \oplus \Lambda^3 \oplus \Lambda^3 \oplus \Lambda^4$.

Let (ω, ψ_2, ψ_3) define a hypo SU(2)-structure on M, and set

$$\mathcal{H} = (\omega + B^2) \times (\psi_2 + B^3) \times (\psi_3 + B^3) \times \left(\frac{1}{2}\omega^2 + B^4\right) .$$

Let $\tilde{\mathcal{V}}$ be the space of SU(2) structures on M, composed of quadruples $(\omega_1, \psi_2, \psi_3, \upsilon)$ in $\Omega^2 \oplus \Omega^3 \oplus \Omega^3 \oplus \Omega^4$ satisfying

$$\begin{cases} \alpha(\psi_2) = \alpha(\psi_3) & X(\omega_1) = X(\upsilon) \\ \omega_1 \wedge \psi_2 = 0 = \omega_1 \wedge \psi_3 & 0 = (X(\omega_1) \lrcorner \psi_2) \wedge \psi_3 \end{cases} \tag{5.21}$$

and $V^2(\omega_1, \psi_2) > 0$. We can view all the Ω^p as Banach spaces, in such a way that $d : \Omega^p \to \Omega^{p+1}$ is continuous (see the Appendix to this chapter). In particular, $\tilde{\mathcal{V}}$ is a Banach space.

We view $\tilde{\mathcal{V}}$ as the preimage of the zero section under the smooth, continuous map on sections $\underline{B}$ determined by the bundle map

$$\begin{aligned} B : \Lambda^2 \oplus \Lambda^3 \oplus \Lambda^3 \oplus \Lambda^4 &\to (\Lambda^1 \otimes \Lambda^5) \oplus (T \otimes \Lambda^5) \oplus \Lambda^5 \oplus \Lambda^5 \oplus (\Lambda^5)^2 \\ (\omega_1, \psi_2, \psi_3, \upsilon) &\to (\alpha(\psi_2) - \alpha(\psi_3), X(\omega_1) - X(\upsilon), \\ &\qquad\qquad \omega_1 \wedge \psi_2, \omega_1 \wedge \psi_3, (X(\omega_1) \lrcorner \psi_2) \wedge \psi_3) \end{aligned}$$

Lemma 5.18. *The zero section is a regular value for $\underline{B}$. In particular, $\tilde{\mathcal{V}}$ is smooth and*

$$T_x \tilde{\mathcal{V}} = \ker \underline{B}_{*x} .$$

Proof. Since $\underline{B}$ is a bundle map, it is sufficient to show that zero is a regular value for B (Proposition 5.31). We can therefore work on $\mathbb{R}^5$. Let $x = (\omega_1, \psi_2, \psi_3, \upsilon)$ with $B(x) = 0$; define

$$\alpha = \alpha(\omega_1, \psi_2) , \qquad \omega_2 = X(\omega_1, \psi_2) \lrcorner \psi_2 , \qquad \omega_3 = X(\omega_1, \psi_3) \lrcorner \psi_3 .$$

We must show that B_{*x} is surjective. In the tangent space at x, consider the 13-dimensional subspace

$$\begin{aligned} W = \Big\langle &\left\{(0, 0, \beta \wedge \omega_3, 0) \mid \beta \in (\mathbb{R}^5)^*\right\}, \left\{(0, 0, 0, v \lrcorner (\alpha \wedge \omega_3^2)) \mid v \in \mathbb{R}^5\right\}, \\ &(\omega_2, 0, 0, 0), (\omega_3, 0, 0, 0), (0, 0, \alpha \wedge \omega_2, 0) \Big\rangle ; \end{aligned}$$

we claim that B_{*x} is injective on W. We have

$$\alpha(\omega_3 \wedge \beta) = c\beta \otimes (\beta \wedge \omega_3^2) ,$$

where c is a constant which depends on the conventions. Now, replace β with a curve $\beta(t)$ with $\beta(0) = \alpha$. At $t = 0$, we have

$$\partial_t \alpha(\omega_3 \wedge \beta) = c\,\beta' \otimes (\alpha \wedge \omega_3^2) + c\,\alpha \otimes (\beta' \wedge \omega_3^2) .$$

For β' ranging in $\alpha^\perp$, the second summand vanishes and $\partial_t \alpha(\omega_3 \wedge \beta)$ spans $\alpha^\perp \otimes \Lambda^5$. On the other hand, setting $\beta' = \alpha$, we find

$$\partial_t \alpha(\omega_3 \wedge \beta) = 2\,c\,\alpha \otimes (\alpha \wedge \omega_3^2) ,$$

so B_{*x} is injective on

$$\left\langle \left\{ (0, 0, \beta \wedge \omega_3, 0) \mid \beta \in (\mathbb{R}^5)^* \right\} \right\rangle ;$$

it is now easy to see that B_{*x} is injective on all of W. By a dimension count, it is also surjective. $\square$

Remark. It is certainly not surprising that $\tilde{\mathcal{V}}$ is smooth, but the identification of its tangent bundle will be a crucial point in the proof of Theorem 5.20.

We consider the *space of deformations* of (ω, ψ_2, ψ_3)

$$\mathcal{V} = \tilde{\mathcal{V}} \cap \mathcal{H} .$$

This space of deformations of a hypo structure consists in the space of SU(2)-structures whose defining forms are in the same cohomology class as those defining the given structure; by construction, these deformed structures are also hypo.

Definition 5.19. An SU(2)-structure P in $\tilde{\mathcal{V}}$ is *regularly hypo* if in a neighbourhood of P, $\mathcal{V}$ is a smooth immersed submanifold of $\Omega^2 \oplus \Omega^3 \oplus \Omega^3 \oplus \Omega^4$.

Remark. Since $\tilde{\mathcal{V}}$ is the zero locus of $\underline{B}$, to determine whether a given hypo structure P is regularly hypo, one has to look at the restriction of $\underline{B}_*$ to $\mathcal{H}$; incidentally, the υ component plays a tautological role and may be neglected in this calculations. Since the map B is polynomial, we expect most points of $\mathcal{H}$ to be regularly hypo. We shall make this argument rigorous when treating the case of nilmanifolds.

Note that from (5.13), it is obvious that $V^2(\omega_1, \psi_2) = V^2(\psi_3, \upsilon)$ on $\mathcal{V}$. We recall that $X(\omega_1) = X(\upsilon)$ is equivalent to $\upsilon = \frac{1}{2}\omega_1^2$.
In order to apply Hitchin's technique, we consider the skew-symmetric form on $B^2 \times B^3 \times B^3 \times B^4$ defined by

$$\Big\langle (\dot{\omega}, \dot{\psi}_2, \dot{\psi}_3, \dot{\upsilon}), (d\beta, d\tau_2, d\tau_3, d\gamma) \Big\rangle = \int_M \Big(\dot{\omega} \wedge \gamma - \dot{\upsilon} \wedge \beta - \dot{\psi}_2 \wedge \tau_3 - \dot{\psi}_3 \wedge \tau_2 \Big) .$$

Then $\langle \, , \rangle$ makes $\mathcal{H}$ into a symplectic manifold.
A point of $\mathcal{V}$ defines a hypo structure on M; in particular, it defines forms α, ω_2 and ω_3.

Theorem 5.20. *Let P be a regularly hypo* $\mathrm{SU}(2)$*-structure on a compact manifold M, and define $\mathcal{V}$ as above, so that $P \in \mathcal{V}$. Define $H : \mathcal{H} \to \mathbb{R}$ by*

$$H(\omega, \psi_2, \psi_3, \upsilon) = \int_M \left(V(\omega, \psi_2) - V(\psi_3, \upsilon) \right) .$$

Then the integral curve through P of the Hamiltonian flow of H is contained in $\mathcal{V}$, when restricted to an interval (a, b). This curve gives rise to a one-parameter family of hypo structures $(\omega(t), \psi_2(t), \psi_3(t))$ satisfying

$$\begin{cases} \partial_t \omega = -d\alpha \\ \partial_t \psi_2 = -d\omega_3 \\ \partial_t \psi_3 = d\omega_2 \end{cases} \tag{5.22}$$

for all $t \in (a, b)$; moreover

$$\Omega = \alpha \wedge dt + \omega$$
$$\Psi = (\omega_2 + i\omega_3) \wedge (\alpha + idt)$$

defines an integrable $\mathrm{SU}(3)$*-structure on $M \times (a, b)$.*
Conversely, if $\mathbf{M}$ is a 6-manifold with an integrable $\mathrm{SU}(3)$*-structure and $i : M \to \mathbf{M}$ is a compact embedded hypersurface with the induced hypo structure, then i can be extended to an embedding $i : M \times (a, b) \to \mathbf{M}$ such that the integrable* $\mathrm{SU}(3)$*-structure defined above coincides with the pullback of the* $\mathrm{SU}(3)$*-structure on $\mathbf{M}$.*

Proof. Suppose that P is regularly hypo. For brevity's sake, on $\mathcal{H}$ we define

$$\alpha_2 = \alpha(\omega_1, \psi_2), \quad \alpha_3 = \alpha(\psi_3, \upsilon), \quad \omega_2 = X(\omega_1, \psi_2) \lrcorner \, \psi_2, \quad \omega_3 = X(\psi_3, \upsilon) \lrcorner \, \psi_3 .$$

Then by Lemma 5.11,

$$dH_{(\omega_1,\psi_2,\psi_3,\upsilon)}(d\beta,d\tau_2,d\tau_3,d\gamma) =$$
$$= \int_M (\alpha_2 \wedge \omega_1 \wedge d\beta + \omega_2 \wedge d\tau_2 - \omega_3 \wedge d\tau_3 - \alpha_3 \wedge d\gamma) =$$
$$= \int_M (d\alpha_2 \wedge \omega_1 \wedge \beta - d\omega_2 \wedge \tau_2 + d\omega_3 \wedge \tau_3 - d\alpha_3 \wedge \gamma) \text{ by Stokes' theorem.} \tag{5.23}$$

Using the non-degeneracy of $\langle\,,\rangle$, one can define the skew gradient X_H by

$$\langle X_H, \cdot\rangle = dH \ ;$$

then

$$(X_H)_{(\omega,\psi_2,\psi_3,\upsilon)} = (-d\alpha_3, -d\omega_3, d\omega_2, -\omega_1 \wedge d\alpha_2) \ . \tag{5.24}$$

Let $(\omega(t),\psi_2(t),\psi_3(t),\upsilon(t))$ be an integral curve through P for X_H in $\mathcal{H}$. The condition $V^2 > 0$ is open, and therefore holds on the curve when restricting to some interval (a,b). We shall prove that X_H is tangent to $\mathcal{V}$; by the smoothness of $\mathcal{V}$ and the uniqueness of integral curves for a vector field on a smooth manifold, it will follow that the curve lies in $\mathcal{V}$.
For any vector field X on M, consider the functional on $\mathcal{V}$ defined by

$$\mu_X(\omega,\psi_2,\psi_3,\upsilon) = \int_M ((X \lrcorner\, \omega) \wedge \upsilon + (X \lrcorner\, \psi_2) \wedge \psi_3) \ .$$

It is easy to see that μ_X vanishes at $t = 0$ for all X. Since the group of diffeomorphisms homotopic to the identity $\mathrm{Diff}^0(M)$ acts on $\mathcal{H}$, X induces a vector field $\tilde{X}$ on $\mathcal{H}$ such that

$$\tilde{X}_{(\omega,\psi_2,\psi_3,\upsilon)} = (\mathcal{L}_X\omega, \mathcal{L}_X\psi_2, \mathcal{L}_X\psi_3, \mathcal{L}_X\upsilon) \ .$$

We claim that μ is the moment map with respect to the action of $\mathrm{Diff}^0(M)$, i.e.

$$d\mu_X(Y) = \langle \tilde{X}, Y\rangle \quad \forall\, Y \in T\mathcal{H} \ . \tag{5.25}$$

Since H is $\mathrm{Diff}^0(M)$-invariant, it will follow that

$$0 = \tilde{X}H = dH(\tilde{X}) = \langle X_H, \tilde{X}\rangle = -d\mu_X(X_H) = -X_H\mu_X \ .$$

Therefore, μ_X is identically zero on our curve. Take $X = f\,X(\omega, \psi_2)$, where f is a function on M; then

$$0 = \mu_X = \int_M f\,\omega_2 \wedge \psi_3 \ ,$$

which holds for all f; therefore $\omega_2 \wedge \psi_3 = 0$ for all t.

To prove our claim that X_H is tangent to $T\mathcal{V}$, we assume that $(\omega_1, \psi_2, \psi_3, \upsilon)$ lies in $\mathcal{V}$ at t and we show that the derivative of

$$t \to B(\omega_1(t), \psi_2(t), \psi_3(t), \upsilon(t))$$

vanishes at t. By construction, Hamilton's equations hold:

$$\begin{cases} \partial_t\omega_1 = -d\alpha_3 & \partial_t\psi_2 = -d\omega_3 \\ \partial_t\upsilon = -\omega_1 \wedge d\alpha_2 & \partial_t\psi_3 = d\omega_2 \end{cases}$$

implying

$$\partial_t(\omega_1 \wedge \psi_2) = -d\alpha_3 \wedge \psi_2 - \omega_1 \wedge d\omega_3 = -\alpha_3 \wedge d\psi_2 + d\omega_1 \wedge \omega_3 = 0$$

because $\alpha_3 \wedge \psi_2$ and $\omega_1 \wedge \omega_3$ vanish identically on $\mathcal{V}$. Similar arguments show that $\partial_t(\omega_1 \wedge \psi_3) = 0$ and $\partial_t\omega_1^2 = 2\,\partial_t\upsilon$, which is equivalent to $\partial_t X(\omega_1) = \partial_t X(\upsilon)$. It remains to prove

$$\partial_t\alpha_2 = \partial_t\alpha_3 \ . \tag{5.26}$$

We shall make this follow from

$$\partial_t(\alpha(\psi_2) \wedge \psi_3) = 0 \ , \qquad \partial_t V^2(\omega, \psi_2) = \partial_t V^2(\psi_3, \upsilon) \ ;$$

indeed, at points of $\mathcal{V}$, the first equation gives

$$\begin{aligned} &0 = \partial_t\big(\alpha(\psi_2) \wedge \alpha(\psi_3) \wedge \omega_3\big) = \big(\partial_t\big(\alpha(\psi_2) - \alpha(\psi_3)\big)\big) \wedge \alpha(\psi_3) \wedge \omega_3 \implies \\ &0 = \big(\partial_t\big(\alpha(\psi_2) - \alpha(\psi_3)\big)\big) \wedge \alpha(\psi_3) \end{aligned}$$

by the non-degeneracy of ψ_3. On the other hand, the second equation gives

$$\begin{aligned} 0 = \partial_t\big(\alpha(\psi_2) - \alpha(\psi_3)\big)\big(X(\omega_1)\big) + \alpha(\psi_2)\,\big(\partial_t\big(X(\omega_1) - X(\upsilon)\big)\big) = \\ = \partial_t\big(\alpha(\psi_2) - \alpha(\psi_3)\big)\big(X(\omega_1)\big) \end{aligned}$$

because $\partial_t X_2 = \partial_t X_3$; Equation 5.26 clearly follows. Moreover, since we are working at points of $\mathcal{V}$, it is sufficient to prove

$$\partial_t(\alpha_2 \wedge \psi_3) = 0 \ , \qquad\qquad \partial_t V(\omega, \psi_2) = \partial_t V(\psi_3, \upsilon) \ .$$

Observe that at points of $\mathcal{V}$, along the curve

$$\partial_t \int_M (X \lrcorner\, \omega_1) \wedge \upsilon = \int_M ((X \lrcorner\, \partial_t \omega_1) \wedge \upsilon + (X \lrcorner\, \omega_1) \wedge \partial_t \omega_1 \wedge \omega_1) =$$

$$= \int_M X \lrcorner\, (\partial_t \omega_1 \wedge \upsilon) = 0 \ . \quad (5.27)$$

Now choose any 1-form β; by the non-degeneracy of ω_2, one has

$$\beta = X \lrcorner\, \omega_2 + f\alpha_2$$

for some vector field X and some function f, and we can assume that $\alpha_2(X) = 0$. Using

$$(X \lrcorner\, \psi_2) \wedge \psi_3 = (X \lrcorner\, \alpha_2) \wedge \omega_2 \wedge \psi_3 - \alpha_2 \wedge (X \lrcorner\, \omega_2) \wedge \psi_3 \ ,$$

we find

$$0 = \partial_t \mu_X = \int_M \beta \wedge \partial_t(\alpha_2 \wedge \psi_3)$$

for all β; therefore $\partial_t(\alpha_2 \wedge \psi_3) = 0$. By (5.18) and (5.19),

$$\partial_t V(\omega_1, \psi_2) = \alpha_2 \wedge \omega_1 \wedge \partial_t \omega_1 + \omega_2 \wedge \partial \psi_2 = -\alpha_2 \wedge \omega_1 \wedge d\alpha_3 - \omega_2 \wedge d\omega_3$$
$$\partial_t V(\psi_3, \upsilon) = \alpha_3 \wedge \partial_t \upsilon + \omega_3 \wedge \partial \psi_3 = -\alpha_3 \wedge \omega_1 \wedge d\alpha_2 + \omega_3 \wedge d\omega_2$$

At points of $\mathcal{V}$, $\alpha_2 \wedge \alpha_3 = 0$ and $\omega_2 \wedge \omega_3 = 0$, proving that

$$\partial_t V(\omega, \psi_2) = \partial_t V(\psi_3, \upsilon) \ .$$

As we observed, (5.26) must then hold.

To conclude the proof that the curve is contained in $\mathcal{V}$, there only remains to prove (5.25). Let $Y = (d\beta, d\tau_2, d\tau_3, d\gamma)$; then

$$d\mu_X(Y) = \int_M ((X \lrcorner\, \omega) \wedge d\gamma - (X \lrcorner\, \upsilon) \wedge d\beta + (X \lrcorner\, \psi_2) \wedge d\tau_3 + (X \lrcorner\, \psi_3) \wedge d\tau_2) =$$

$$= \int_M (d(X \lrcorner\, \omega) \wedge \gamma - d(X \lrcorner\, \upsilon) \wedge \beta - d(X \lrcorner\, \psi_2) \wedge \tau_3 - d(X \lrcorner\, \psi_3) \wedge \tau_2) =$$

$$= \int_M (\mathcal{L}_X \omega \wedge \gamma - \mathcal{L}_X \upsilon \wedge \beta - \mathcal{L}_X \psi_2 \wedge \tau_3 - \mathcal{L}_X \psi_3 \wedge \tau_2) = \langle \tilde{X}, Y \rangle \quad (5.28)$$

Since a point of $\mathcal{V}$ defines a hypo structure on M, the curve defines a one-parameter family of hypo structures. Moreover $\upsilon(t) = \omega^2(t)/2$ and Hamilton's equations are equivalent to (5.22). To prove that the SU(3)-structure on $M \times (a, b)$ is integrable, set $\Phi = \omega_2 + i\omega_3$. Then $d\Phi = i\partial_t(\Phi \wedge \alpha)$, and

$$d\Omega = d\alpha \wedge dt + dt \wedge \partial_t \omega = 0$$
$$d\Psi = d\Phi \wedge (\alpha + idt) + \Phi \wedge d\alpha + idt \wedge \partial_t(\Phi \wedge \alpha) = 0$$

proving the first part of the theorem.

On the other hand, the right-hand sides are zero if and only if (5.22) hold. Therefore, if M arises as a hypersurface in $\mathbf{M}$, we can use the exponential mapping to embed $M \times (a, b)$ in $\mathbf{M}$ so that ∂_t is a unit vector orthogonal to M, and the hypo structures induced on $M \times \{t\}$ will evolve according to (5.22). This proves the second part of the theorem. □

Remark. A hypo structure is Einstein-Sasaki if and only if the components (5.10) satisfy (5.22); similarly, nearly-Kähler half-flat structures are characterized by the evolution being conical [25]. In this sense, evolution theory is a generalization of the construction of a manifold with a parallel spinor as the cone on a manifold with a Killing spinor, where the cone is replaced by more complicated evolution equations and the Killing spinor by a generalized Killing spinor.

5.6 Hypo nilmanifolds: The classification

It is well known that nilmanifolds do not admit Einstein-Sasaki structures; in fact, Einstein-Sasaki manifolds have finite fundamental group, and therefore $b_1 = 0$, which cannot occur for nilmanifolds. Not surprisingly, most 5-nilmanifolds do admit (invariant) hypo structures. Indeed, consider

$$M = \Gamma \backslash G ,$$

where G is a 5-dimensional nilpotent group, Γ a discrete subgroup of G and M is compact; an invariant structure on the nilmanifold M is a structure which pulls back to a left-invariant structure on G. With this setting in mind, we define:

Definition 5.21. A hypo structure on $\mathfrak{g}$ is a quadruplet (α, ω_i) of forms on $\mathfrak{g}$ satisfying Proposition 5.1 and Equation 5.8.

Borrowing notation from [32], we represent Lie algebras using symbolic expressions such as $(0,0,0,0,12)$, which represents a Lie algebra with a basis $e^1,\dots,e^5$ such that $de^i=0$ for $i=1,\dots,4$, and $de^5=e^{12}$.

Theorem 5.22. *The nilpotent 5-dimensional Lie algebras not admitting a hypo structure are* $(0,0,12,13,23)$*,* $(0,0,0,12,14)$ *and* $(0,0,12,13,14+23)$*.*

Using the classification of five-dimensional nilpotent Lie algebras, the theorem is equivalent to the following table:

$\mathfrak{g}$	step	b_2	Admits hypo
$0,0,12,13,14+23$	3	3	no
$0,0,12,13,14$	3	3	yes
$0,0,12,13,23$	2	3	no
$0,0,0,12,14$	2	4	no
$0,0,0,12,13+24$	2	4	yes
$0,0,0,12,13$	2	6	yes
$0,0,0,0,12+34$	1	5	yes
$0,0,0,0,12$	1	7	yes
$0,0,0,0,0$	0	10	yes

Remark. Any hypo nilmanifold $\Gamma\backslash G$ provides an example of a compact hypo manifold with $b_1>0$. Moreover the pullback hypo structure on the nilpotent group G gives an example of a non-compact complete hypo manifold.

We start with a list of examples of hypo structures on nilpotent Lie algebras which do admit such structures; we shall then prove that any nilpotent Lie algebra with a hypo structure must be one of these. Theorem 5.22 will then follow from the classification of nilpotent Lie algebras, which is not otherwise used.

- $(0,0,12,13,14)$ has a hypo structure given by

$$\alpha=e^1\qquad \omega_1=e^{25}+e^{43}\qquad \omega_2=e^{24}+e^{35}\qquad \omega_3=e^{23}+e^{54}$$

- $(0,0,0,12,13+24)$ has a one-parameter family of hypo structures given by

$$\begin{aligned}\alpha&=e^1+e^5 & \omega_1&=e^4\wedge(-c\,e^2-e^3)+e^{25}\\ \omega_2&=e^{42}+e^5\wedge(-c\,e^2-e^3) & \omega_3&=e^{45}+(-c\,e^2-e^3)\wedge e^2\end{aligned}$$

- $(0,0,0,12,13)$ has a hypo structure given by

$$\alpha = e^1 \qquad \omega_1 = e^{35} + e^{24} \qquad \omega_2 = e^{32} + e^{45} \qquad \omega_3 = e^{34} + e^{52}$$

  Taking the product of this nilmanifold with a circle, one obtains the half-flat symplectic structure studied by Giovannini in [21].

- $(0,0,0,0,12+34)$ has hypo structures given by

$$\begin{array}{llll} \alpha = e^5 & \omega_1 = e^{12} + e^{34} & \omega_2 = e^{13} + e^{42} & \omega_3 = e^{14} + e^{23} \\ \alpha = e^5 & \omega_1 = e^{12} - e^{34} & \omega_2 = e^{13} - e^{42} & \omega_3 = e^{14} - e^{23} \end{array}$$

  These structures arise as circle bundles over the hyperkähler torus.

- $(0,0,0,0,12)$ has hypo structures given by

$$\begin{array}{llll} \alpha = e^1 & \omega_1 = e^{25} + e^{34} & \omega_2 = e^{23} + e^{45} & \omega_3 = e^{24} + e^{53} \\ \alpha = e^5 & \omega_1 = e^{12} + e^{34} & \omega_2 = e^{13} + e^{42} & \omega_3 = e^{14} + e^{23} \\ \alpha = e^2 - e^5 & \omega_1 = e^{34} + e^{15} & \omega_2 = e^{31} + e^{54} & \omega_3 = e^{35} + e^{41} \end{array}$$

- Every SU(2)-structure on $(0,0,0,0,0)$ is hypo.

The rest of this section consists of the proof of the theorem. Suppose $\mathfrak{g}$ is a non-trivial nilpotent Lie algebra carrying a hypo structure. Since $\mathfrak{g}$ is nilpotent, one can fix a filtration of vector spaces V^i, $\dim V^i = i$, such that

$$V^1 \subset V^2 \subset \cdots \subset V^5 = \mathfrak{g}^*, \quad d(V^i) \subset \Lambda^2 V^{i-1} .$$

This filtration can be chosen so that $V^i = \ker d$ for some i; in particular, one has $V^2 \subset \ker d \subset V^4$. Note that the first Betti number b_1 is the dimension of $\ker d$.

It is convenient to distinguish three cases, according to whether α lies in V^4, $(V^4)^\perp$ or neither.

5.6.1 First case

We first consider the case when α is in V^4.

Theorem 5.23. *If α is in V^4, then $\mathfrak{g}$ is either $(0,0,0,0,12)$, $(0,0,0,12,13)$, or $(0,0,12,13,14)$.*

Proof. Fix a unit e^5 in $(V^4)^\perp$ and apply Corollary 5.3 to obtain a coframe $e^1, \dots, e^5$ such that

$$\alpha = e^1 \qquad \omega_1 = e^{25} + e^{34} \qquad \omega_2 = e^{23} + e^{45} \qquad \omega_3 = e^{24} + e^{53}$$

From $d\omega_1 = 0$, it follows that

$$e^2 \wedge de^5 - de^{34} = de^2 \wedge e^5 \; ; \tag{5.29}$$

since the left-hand side lies in $\Lambda^3 V^4$ and the right-hand side lies in $e^5 \wedge \Lambda^2 V^4$, both must vanish and $de^2 = 0$. Using the fact that $\alpha \wedge \omega_2$ and $\alpha \wedge \omega_3$ are closed, we obtain:

$$\begin{aligned} 0 &= de^{123} + de^{14} \wedge e^5 + e^{14} \wedge de^5 \\ 0 &= de^{124} - de^{13} \wedge e^5 - e^{13} \wedge de^5 \end{aligned}$$

Therefore e^{13} and e^{14} are closed; since e^2 is closed as well, we get

$$0 = e^{14} \wedge de^5 \; , \qquad\qquad 0 = e^{13} \wedge de^5 \; . \tag{5.30}$$

Suppose first that e^1 is not in V^3. By dimension count, $\langle e^3, e^4 \rangle$ has nonzero intersection with V^3, and we can therefore rotate e^3 and e^4 to get a different hypo structure with $e^3 \in V^3$; then $de^{13} = 0$ implies that e^3 is closed and $e^3 \wedge de^1 = 0$. Similarly, de^4 is a multiple of de^1, for if one were to rotate e^4 and e^1 in order to have $e^4 \in V^3$, then e^4 would become closed. It follows that e^{34} is closed; then (5.29) becomes

$$e^2 \wedge de^5 = 0 \; ,$$

showing that, because of (5.30), up to a constant de^5 equals e^{12}, which is therefore closed. Thus, all 2-forms on V^4 are closed, implying that V^4 is trivial as a Lie algebra; consequently, $\mathfrak{g} = (0, 0, 0, 0, 12)$.

Assume now that e^1 is in V^3; we can rotate e^3 and e^4 to get e^3 in V^3, e^4 in $(V^3)^\perp$. From $de^{14} = 0$ we find that e^1 is closed and $de^4 \wedge e^1 = 0$. Wedging the left-hand side of (5.29) with e^1 and using $de^{13} = 0$, we see that

$$e^{12} \wedge de^5 = e^1 \wedge de^{34} = e^{14} \wedge de^3 - e^{13} \wedge de^4 = e^{34} \wedge de^1 = 0 \; .$$

Together with (5.30), this implies that de^5 is in $\langle e^{12}, e^{13}, e^{14} \rangle$. Now consider the endomorphism f of $\alpha^\perp$ defined by $e^1 \wedge f(\eta) = d\eta$; its matrix with respect to $\{e^2, e^3, e^4, e^5\}$ is strictly upper triangular. Its Jordan canonical form is therefore one of

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

giving $\mathfrak{g} = (0, 0, 0, 0, 12)$, $(0, 0, 0, 12, 13)$, or $(0, 0, 12, 13, 14)$ respectively. □

5.6.2 Second case

The key tool to classify the remaining hypo structures is the following Lemma, which shows that α is orthogonal to V^4 if and only if ω_2, ω_3 are closed and $b_1 = 4$.

Lemma 5.24. *Let $\alpha \notin V^4$. Then*

1. *If all ω_i are closed, α is orthogonal to V^4.*
2. *If α is orthogonal to V^4, $V^4 = \ker d$. In particular, all ω_i are closed.*

Proof. Let e^5 be a unit form in $(V^4)^\perp$. By hypothesis, $\alpha = a\, e^5 + \eta$ where η is in V^4 and $a \neq 0$. To prove the first statement, suppose η is non-zero and let e^4 be a unit form in $\langle e^5, \eta\rangle$ orthogonal to α. Using Corollary 5.3 we can write

$$\omega_1 = e^{12} + e^{34} \qquad \omega_2 = e^{13} + e^{42} \qquad \omega_3 = e^{14} + e^{23}$$

The space $\langle e^1, e^2, e^3\rangle$ is orthogonal to e^5, and is therefore contained in V^4, whereas e^4 is not. Since ω_1 is closed,

$$e^3 \wedge de^4 - de^{12} = de^3 \wedge e^4\ ; \tag{5.31}$$

both sides must then vanish, and so e^3 is closed; applying the same argument to ω_2 and ω_3 (which are closed by hypothesis) one finds that e^1 and e^2 are also closed. From (5.31) and its analogues obtained using ω_2 and ω_3, it follows that $de^4 \wedge \langle e^1, e^2, e^3\rangle$ is trivial. Hence e^4 is closed and is therefore in V^4, which is absurd.

To prove the second assertion, let $\eta = 0$, i.e. $\alpha = e^5$ (up to sign). From (5.8), it follows that ω_2 and ω_3 are closed. Pick a unit e^4 in $(V^3)^\perp \cap V^4$, and define e^1, e^2, e^3 so as to obtain (5.5); then $\langle e^1, e^2, e^3\rangle = V^3$. The same argument as above gives $V^4 = \ker d$. □

It is now easy to prove:

Theorem 5.25. *If α is orthogonal to V^4, then $\mathfrak{g}$ is either* $(0,0,0,0,12)$ *or* $(0,0,0,0,12+34)$.

Proof. By Lemma 5.24, $V^4 = \ker d$. Then either $d\alpha$ is simple, and one can choose a basis $e^1,\dots,e^4$ of V^4 such that $d\alpha = e^{12}$, or it is not simple, and one can choose a basis such that $d\alpha = e^{12}+e^{34}$. □

5.6.3 Third case

The last case is the one with α neither in V^4 nor in $(V^4)^\perp$. Lemma 5.24 suggests that the span of $d\omega_2$ and $d\omega_3$ is relevant to the classification of hypo structures; we shall use the dimension of $\langle d\omega_2, d\omega_3\rangle \cap \Lambda^3 V^4$ to distinguish two subcases. In fact, we shall prove that this dimension can only be 1 or 2.

Theorem 5.26. *If α is neither in V^4 nor in $(V^4)^\perp$ and $d\omega_2$, $d\omega_3$ are in $\Lambda^3 V^4$, then* $\mathfrak{g} = (0,0,0,0,12)$.

Proof. Let $\alpha+\gamma$ be a generator of $(V^4)^\perp$ with γ in $\alpha^\perp$, and let k be the norm of γ; then $\alpha - k^{-2}\gamma$ lies in V^4. Consider now the hypo structure obtained multiplying α by k. Let $e^4 = -k^{-1}\gamma$ and define

$$\eta = \frac{1}{2}\left(\alpha + e^4\right)\ , \qquad\qquad \xi = \frac{1}{2}\left(\alpha - e^4\right)\ ;$$

then ξ generates $(V^4)^\perp$ and η is in V^4.
Using Corollary 5.3 we can write

$$\omega_1 = e^{12}+e^{34} \qquad \omega_2 = e^{13}+e^{42} \qquad \omega_3 = e^{14}+e^{23}$$

The space $\langle e^1, e^2, e^3\rangle$, being orthogonal to both e^4 and α, is orthogonal to ξ, and is therefore contained in V^4; on the other hand e^4 is not in V^4. Since ω_1 is closed,

$$e^3\wedge de^4 - de^{12} = de^3 \wedge e^4\ ; \tag{5.32}$$

both sides must then vanish, and so e^3 is closed. Similarly, write

$$\begin{aligned} d\omega_2 &= de^{13} + de^4\wedge e^2 - \eta\wedge de^2 + \xi\wedge de^2 \\ d\omega_3 &= de^{23} - de^4\wedge e^1 + \eta\wedge de^1 - \xi\wedge de^1 \end{aligned}$$

So far we have only used the fact that α is orthogonal to V^4. Writing

$$\Lambda^3\mathfrak{g}^* = \Lambda^3 V^4 \oplus \xi \wedge \Lambda^2 V^4 \ ,$$

the hypotheses on $d\omega_2$, $d\omega_3$ imply that e^2 and e^1 are closed. By (5.8),

$$0 = d(\omega_2\alpha) = d(e^{13} \wedge (\eta + \xi) - 2\, e^2 \wedge \eta \wedge \xi) \tag{5.33}$$
$$0 = d(\omega_3\alpha) = d(e^{23} \wedge (\eta + \xi) + 2\, e^1 \wedge \eta \wedge \xi) \tag{5.34}$$

Relative to $\Lambda^4\mathfrak{g}^* = \xi \wedge \Lambda^3 V^4 \oplus \Lambda^4 V^4$, the first components give $e^1 \wedge d\eta = 0$, $e^2 \wedge d\eta = 0$; using this, the second components give

$$(e^{13} - 2\, e^2 \wedge \eta) \wedge d\xi = 0$$
$$(e^{23} + 2\, e^1 \wedge \eta) \wedge d\xi = 0$$

The left-hand side of (5.32) gives

$$e^3 \wedge d\eta = e^3 \wedge d\xi \ . \tag{5.35}$$

Wedging by e^1, e^2 we see that $d\xi \wedge e^{13}$ and $d\xi \wedge e^{23}$ are zero, so our equations reduce to

$$(e^2 \wedge \eta) \wedge d\xi = 0 \ , \qquad\qquad (e^1 \wedge \eta) \wedge d\xi = 0 \ .$$

Therefore, $d\xi$ lies in $\langle e^{12}, e^3 \wedge \eta\rangle$. Suppose that η is not closed; then $d\eta$ is a non-zero multiple of e^{12}, so $d^2 = 0$ implies that $d\xi$ must also be a multiple of e^{12}. Then (5.35) implies that e^4 is closed, which is absurd because e^4 cannot be in V^4.
Thus, η is closed and from (5.35) we must have $d\xi = e^3 \wedge \eta$ up to a constant, which we can take to be 1 by introducing a global scale factor. The basis $\{e^3, \eta, e^1, e^2, e^4\}$ reveals $\mathfrak{g}$ to be $(0, 0, 0, 0, 12)$. □

Suppose now that the dimension of $\langle d\omega_2, d\omega_3\rangle \cap \Lambda^3 V^4$ is one; then up to rotating ω_2 and ω_3 we can assume that, say, $d\omega_2$ is in $\Lambda^3 V^4$. The following Lemma shows that this can always be done.

Lemma 5.27. *If α is neither in V^4 nor in $(V^4)^\perp$, up to rotating ω_2 and ω_3 we can always assume that $d\omega_2$ is in $\Lambda^3 V^4$.*

Proof. Suppose $d\omega_2$ is not in $\Lambda^3 V^4$; then α cannot be orthogonal to V^4, because otherwise ω_2 would be in $\Lambda^2 V^4$. We can then proceed as in the proof of Theorem 5.26. If de^1 and de^2 are linearly dependent, by rotating ω_2 and ω_3 one can construct a hypo structure with $d\omega_2$ in $\Lambda^3 V^4$; assume that they are independent.
Consider the bilinear form B on V^4 defined by

$$B(\alpha, \beta) e^{123} \wedge \eta = \alpha \wedge \beta \;;$$

its signature is $(+,+,+,-,-,-)$. By the classification of four-dimensional Lie algebras, $V^4 = (0,0,12,13)$; an explicit computation then shows that space Z_2 of closed 2-forms has dimension 4 and the signature of B on Z_2 is $(0,0,+,-)$.
On the other hand, the components of (5.33), (5.34) containing ξ give

$$d(e^{13} + 2\,\eta \wedge e^2) = 0 \;, \qquad\qquad d(2\,e^1 \wedge \eta + e^{23}) = 0 \;,$$

giving a two-dimensional subspace in Z_2 on which B is positive definite, which is absurd. $\square$

Theorem 5.28. *Let $\alpha \notin V^4$. If $d\omega_3$ is not in $\Lambda^3 V^4$, then*

$$\mathfrak{g} = (0,0,0,12,13+24) \;.$$

Proof. Since $d\omega_3$ is not in $\Lambda^3 V^4$, α cannot be orthogonal to V^4, because otherwise ω_3 would be in $\Lambda^2 V^4$. We can then proceed as in the proof of Theorem 5.26; in fact, everything applies verbatim until the conclusion that e^1 and e^2 are closed, as in the present case e^2 is closed but e^1 is not. Equations 5.33 and 5.34 also hold; the vanishing of the $\xi \wedge \Lambda^3 V^4$ component of (5.34) shows that $d(1 \wedge \eta)$=0. This implies that $d\eta = k\, de^1$ for some constant k, for if one were to rotate η and e^1 in order to have $\eta \in V^3$, then η would become closed. Rewrite (5.32), (5.33) and (5.34) as

$$(k\, e^3 - e^2) \wedge de^1 - e^3 \wedge d\xi = 0 \tag{5.36}$$
$$de^1 \wedge (2\, k\, e^2 + e^3) = 0 \tag{5.37}$$
$$(e^{13} - 2\, e^2 \wedge \eta) \wedge d\xi = 0 \tag{5.38}$$
$$k\, e^{23} \wedge de^1 + (e^{23} + 2\, e^1 \wedge \eta) \wedge d\xi = 0 \tag{5.39}$$

Wedging (5.36) with e^3 shows that $de^1 \wedge e^{23} = 0$; wedging with e^2 shows that $d\xi \wedge e^{23} = 0$; (5.39) is therefore equivalent to $d\xi \wedge e^1 \wedge \eta = 0$. From $d(e^1 \wedge \eta) = 0$ and (5.37), we find that up to a non-zero multiple

$$de^1 = (2\, k\, e^2 + e^3) \wedge (\eta - k\, e^1) \;.$$

Then from (5.36) we get

$$d\xi \in \langle e^2 \wedge (\eta - k\, e^1)\rangle \oplus e^3 \wedge V^4 \; ; \tag{5.40}$$

write $d\xi = \sigma_1 + \sigma_2$ accordingly and note that σ_1 cannot be zero, as in that case (5.36) would imply $(2k^2+1)\, e^{23} = 0$. From (5.40) and (5.39), we know that σ_2 must be in $e^3 \wedge \langle e^1, \eta\rangle$; let

$$\sigma_2 = c_1\, e^3 \wedge (\eta - k\, e^1) + c_2\, e^{31} \; .$$

If c_2 is zero, (5.38) implies that $d\xi = de^1$ up to a multiple. So $\langle e^1, \xi\rangle$ has non-zero intersection with $\ker d \subset V^4$, contradicting the fact that $\langle e^1, \xi\rangle$ intersects V^4 in $\langle e^1\rangle$ and $de^1 \neq 0$.
Hence c_2 is not zero, and since σ_1 is closed, σ_2 must be closed as well; therefore $de^1 \wedge e^3 = 0$, implying $k = 0$. We can then rescale everything so that, using (5.36) and (5.38),

$$d\eta = 0 \; , \qquad de^1 = 2\, e^3 \wedge \eta \; , \qquad d\xi = 2\, e^2 \eta + e^{13} + c_1\, e^3 \wedge \eta \; .$$

Relative to the basis $\{-2\,\eta, e^3, -e^2 - \frac{c_1}{2}\, e^3, e^1, e^4\}$ we see that

$$\mathfrak{g} = (0,0,0,12,13+24) \; . \qquad \square$$

5.7 Hypo nilmanifolds: Examples

The constructions of Section 5.5 also work for hypo structures on a Lie algebra $\mathfrak{g}$, if one replaces $\Omega(M)$ with $\Lambda^*\mathfrak{g}$, $\mathrm{Diff}(M)$ with the space of inner automorphisms of $\mathfrak{g}$, and so on. The space of deformations is then a finite-dimensional, algebraic variety; therefore, it is smooth on a Zariski-open set. Since Zariski-open sets are dense, we conclude:

Proposition 5.29. *The space of deformations of a hypo structure on* $\mathfrak{g}$ *is almost everywhere smooth.*

In other words, almost every hypo manifolds is regularly hypo.

Remark. Lacking an estimate on the interval where evolution equations can be solved, being able to approximate a hypo structure with regularly hypo structures does not help to extend the evolution theorem.

We now give an example of a regularly hypo structure not satisfying the hypotheses of Theorem 2.23, proving that Theorem 5.20 is in some sense more general. Consider $\mathfrak{g} = (0, 0, 0, 12, 13 + 24)$, with

$$\alpha = e^1 + e^5\ , \qquad \omega_1 = e^{34} + e^{25}\ , \qquad \omega_2 = e^{42} + e^{35}\ , \qquad \omega_3 = e^{45} + e^{23}\ .$$

We have

$$B^2 = \langle e^{12}, e^{13} + e^{24}\rangle\ , \qquad\qquad B^3 = \langle e^{123}, e^{124}, e^{234}, e^{125} - e^{134}\rangle\ .$$

Write

$$\begin{aligned}
\omega_1 &= e^{34} + e^{25} + x_1 e^{12} + x_2(e^{13} + e^{24})\\
\psi_2 &= e^{142} + e^{135} + e^{542} + p_1\, e^{123} + p_2\, e^{124} + p_3\, e^{234} + p_4(e^{125} - e^{134})\\
\psi_3 &= e^{145} + e^{123} + e^{523} + q_1\, e^{123} + q_2\, e^{124} + q_3\, e^{234} + q_4(e^{125} - e^{134})\\
\upsilon &= \frac{1}{2}\omega_1 \wedge \omega_1
\end{aligned}$$

A straightforward calculation gives

$$\mathcal{V} = \left\{q_1 = -{p_4}^2 - p_2 + {q_4}^2,\ q_2 = p_1 - 2\,p_4 q_4,\ q_3 = -p_4,\ q_4 = p_3\right\}\ ,$$

which is clearly smooth. So, all of the hypo structures in $\mathcal{V}$ are regularly hypo. Moreover, observe that the dimension of $\mathcal{V}$ is 6, which is half the dimension of $\mathcal{H}$; it is not hard to check that $\mathcal{V}$ is Lagrangian in $\mathcal{H}$.
Lastly, we compute the intrinsic torsion $\overline{A} : P_{\mathrm{SU}(2)} \to \mathrm{Sym}(\mathbb{R}^5)$. For convenience, we work on a left-invariant reduction of $P_{\mathrm{SU}(2)}$ to the trivial group $\{e\}$, i.e. with a global frame; then the connection form can be written

$$\begin{pmatrix}
0 & 0 & \theta^4 & 0 & \theta^3\\
0 & 0 & 0 & \theta^4 & -\theta^5\\
-\theta^4 & 0 & 0 & -\theta^1 & 0\\
0 & -\theta^4 & \theta^1 & 0 & -\theta^2\\
-\theta^3 & \theta^5 & 0 & \theta^2 & 0
\end{pmatrix},$$

where θ^k is the k-th component of the solder form. Using the formulae appearing in the proof of Proposition 5.7, the restriction of $\overline{A}$ to $P_{\{e\}}$ is

$$\overline{A} = \begin{pmatrix}
0 & 0 & 0 & 0 & -1\\
0 & 0 & -1 & 0 & 0\\
0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and the tensorial 1-form given by its exterior covariant differential is

$$D\overline{A} = \begin{pmatrix} -2\theta^3 & \theta^5 - \theta^4 & 0 & \theta^2 & 0 \\ \theta^5 - \theta^4 & 0 & 0 & -\theta^1 & 0 \\ 0 & 0 & 0 & \theta^4 & \theta^4 - \theta^5 \\ \theta^2 & -\theta^1 & \theta^4 & 0 & 0 \\ 0 & 0 & \theta^4 - \theta^5 & 0 & 2\theta^3 \end{pmatrix},$$

so $\nabla \underline{A}$ as a tensor taking values in $T^* \otimes T^* \otimes T^*$ is not totally symmetric.

We conclude this section by solving the evolution equations in one example.

Example 5.29.1. Let $\mathfrak{g} = (0, 0, 0, 0, 12 + 34)$, with

$$\alpha = e^5\,, \qquad \omega_1 = e^{12} + e^{34}\,, \qquad \omega_2 = e^{13} + e^{42}\,, \qquad \omega_3 = e^{14} + e^{23}\,.$$

The evolution equations are solved by

$$\begin{aligned} \alpha &= (1-2t)^{-1/2} e^5 & \qquad \omega_1 &= (1-2t)^{1/2}(e^{12} + e^{34}) \\ \omega_2 &= (1-2t)^{1/2}(e^{13} + e^{42}) & \qquad \omega_3 &= (1-2t)^{1/2}(e^{14} + e^{23}) \end{aligned}$$

and the Calabi-Yau structure is

$$\begin{aligned} \omega &= (1-2t)^{-1/2} e^5 \wedge dt + (1-2t)^{1/2}(e^{12} + e^{34}) \\ \Psi &= (e^5 + (1-2t)^{1/2} i dt) \wedge (e^1 + ie^2) \wedge (e^3 + ie^4) \end{aligned}$$

Appendix

In this appendix we give some details on the theory of infinite-dimensional manifolds which were used in the proof of Theorem 5.20. Our discussion is based on [27].

Infinite-dimensional manifolds are defined using atlases, like in the finite-dimensional case; however, charts are required to take values in a topological vector space admitting a Banach structure. In particular, the tangent space at a point can be viewed as a Banach space. A closed subspace E of a Banach space F is said to *split* if it admits a closed complement $E' \subset F$, such that the natural linear isomorphism $E \oplus E' \cong F$ is a homeomorphism. A smooth map $f : M \to N$ is a *submersion* at $x \in M$ if $(f_*)_x$ is surjective and $\ker(f_*)_x$ splits. We say that $y \in N$ is a *regular value* for f if f is a submersion at all x in $f^{-1}(y)$. As a consequence of the implicit mapping theorem, we get:

Theorem 5.30. *Let $f : M \to N$ be smooth. If y is a regular value for f, then $f^{-1}(y)$ is a submanifold of M.*

Remark. By a submanifold of M we mean a subset $X \subset M$ such that each $x \in X$ belongs to a chart $\phi : U \to V_1 \times V_2$, where $V_i \subset E_i$ is an open set in a Banach space, and

$$\phi(X \cap U) = V_1 \times \{a_2\} \ ,$$

where a_2 is some point of V_2.

Now consider a finite-dimensional, compact manifold M, and fix a Riemannian structure $P_{O(n)}$. For every $O(n)$-module V, we can define a countable family of norms on $\Gamma(\underline{V})$, by taking the L^∞ norms of

$$\alpha, \nabla\alpha, \nabla\nabla\alpha, \dots \ ,$$

and combine them into a single norm

$$\|\alpha\| = \sum_{n\in\mathbb{N}} 2^{-n} \left\|\nabla^n \alpha\right\|_\infty \ .$$

This gives a Banach structure to $\Gamma(\underline{V})$. In particular, the spaces $\Omega^p(M)$ are Banach spaces and

$$d : \Omega^p(M) \to \Omega^{p+1}(M)$$

is continuous.
The following elementary result was used in the proof of Lemma 5.18:

Proposition 5.31. *If $f : V \to W$ is smooth and $O(n)$-equivariant, then $\underline{f} : \Gamma(\underline{V}) \to \Gamma(\underline{W})$ is smooth. Moreover, if 0 is a regular value for f, then the zero section is a regular value for $\underline{f}$.*

Proof. Continuity and smoothness of $\underline{f}$ are obvious. Fix a section α of $\underline{V}$ with $f(\alpha(x)) = 0$ for all x in M; define vector bundles on M by

$$K_x = \ker(f_*)_{\alpha(x)} \ , \qquad\qquad I = \mathrm{Im}\,(f_*)_{\alpha(x)} \ .$$

By construction,

$$\Gamma(K) = \ker(\underline{f})_{*\alpha} \ , \qquad\qquad \Gamma(I) = \mathrm{Im}\,(\underline{f})_{*\alpha} \ .$$

We must show that $\Gamma(K) \subset \Gamma(\underline{V})$ splits and $\Gamma(I) = \Gamma(\underline{W})$. Since 0 is a regular value for f, we can choose a linear map $s : \underline{W} \to \underline{V}$ such that $(f_*)_x \circ s_x$ is the identity for all x. Then,

$$(\underline{f})_{*\alpha} \circ s = \mathrm{Id} : \Gamma(\underline{W}) \to \Gamma(\underline{W}) ,$$

so the sequence

$$0 \to \Gamma(K) \to \Gamma(\underline{V}) \to \Gamma(\underline{W}) \to 0$$

splits, which is what we had to prove. $\square$

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