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Ph.D. Thesis

Asymptotic problems for some classes of dispersive PDEs

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To Piero and Paolo

Penso che la matematica sia una delle manifestazioni più significative dell'amore per la sapienza e come tale la matematica è caratterizzata, da un lato, da una grande libertà e, dall'altro, da una intuizione che il mondo, diciamo, è grandissimo, è fatto di cose visibili e invisibili, e la matematica ha forse una capacità unica tra tutte le scienze: di passare dall'osservazione delle cose visibili all'immaginazione delle cose invisibili. Questo forse è il segreto della forza della matematica.

(Ennio De Giorgi)

Contents

Introduction

This work deals with some classes of nonlinear dispersive evolution PDEs: in particular, under some non classical framework we will consider a class of nonlinear Schrödinger equation, a class of nonlinear Klein-Gordon equation and a system of PDEs called Zakharov system which couples a Schrödinger-type equation with a nonlinear wave equation. For all of these equations the associated Cauchy problem will be considered. More specifically we will examine two different asymptotic limits: for the Schrödinger and the Klein-Gordon equations we will deal with the problem of scattering: roughly speaking, we wonder weather solutions to the nonlinear Cauchy problem behave as linear solutions for large times. The second asymptotic problem is instead a singular limit result related to the solution of the Zakharov system, which depends on a physical parameter: the qualitative investigation of the solutions for large value of such parameter is meaningful from a physical point of view. We mentioned the fact that these equations are considered in a non standard setting: more precisely, the main feature in our context about the Schrödinger equation concerns the presence of a linear perturbation by means of a partially periodic time-independent potential (in the sense that it is periodic with respect to all but one direction) which furthermore does not decay towards zero along the remaining direction: this kind of potential will be denoted as steplike; about the Klein-Gordon equation, the novelty with respect to the usual literature is due to the fact that the equation is not posed on a Euclidean space but on a manifold which mixes a Euclidean part and a compact part, explicitly given, for technical reasons, by the one-dimensional torus; the Zakharov system considered in this work is the so-called vectorial Zakharov equation: in its formulation the Schrödinger equation, coupled with a wave equation, is not the usual one, in the sense that the classical Laplace operator appearing in the literature, is replaced by a second order operator involving a parameter which has got a relevant physical meaning. Let us introduce rigorously the three Cauchy problems mentioned so far, then the arguments of research about them. They are the following: the pure-power nonlinear Schrödinger equation (NLS) given by

$$\begin{cases} i\partial_t u + \Delta u - Vu = |u|^\alpha u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d \\ u(0,x) = u_0 \in H^1(\mathbb{R}^d) \end{cases}, \tag{0.0.1}$$

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ and $V : \mathbb{R}^d \to \mathbb{R}$, is a non-decaying, partially periodic potential, the pure-power nonlinear Klein-Gordon equation (NLKG) given by

$$\begin{cases} \partial_{tt} u - \Delta_{x,y} u + u = -|u|^\alpha u, \quad (t, x, y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \\ u(0, x, y) = f(x, y) \in H^1(\mathbb{R}^d \times \mathbb{T}) \\ \partial_t u(0, x, y) = g(x, y) \in L^2(\mathbb{R}^d \times \mathbb{T}) \end{cases}, \tag{0.0.2}$$

where $u : \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \to \mathbb{R}$, $\mathbb{T}$ being the one dimensional flat torus and $\Delta_{x,y}$ the Laplace operator on $\mathbb{R}^d \times \mathbb{T}$, while the 3D vectorial Zakharov system is as follows

$$\begin{cases} i\partial_t u - \omega \nabla \times \nabla \times u + \nabla(\operatorname{div} u) = nu \\ \frac{1}{c_s^2}\partial_{tt} n - \Delta n = \Delta |u|^2 \end{cases}, \tag{0.0.3}$$

considered under the initial conditions

$$(u(0), n(0), \partial_t n(0)) = (u_0, n_0, n_1) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3),$$

where $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^3$, $n : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $\nabla \times$ is the curl operator. We postpone the description of the parameters appearing in the equations above in the devoted chapters.

Let us focus on (0.0.1) and (0.0.2): both of them are nonlinear equations in a defocusing energy-subcritical regime. The energies are defined (respectively for (0.0.1) and (0.0.2)) as

$$E_{NLS}(u(t)) = \frac{1}{2} \int_{\mathbb{R}^d} \left(|\nabla u(t)|^2 + V |u(t)|^2 + \frac{2}{\alpha + 2} |u(t)|^{\alpha+2} \right) dx,$$
$$E_{NLKG}(u(t), \partial_t u(t)) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{T}} \left(|\partial_t u(t)|^2 + |u|^2 + |\nabla u(t)|^2 + \frac{2}{\alpha + 2} |u(t)|^{\alpha+2} \right) dx\, dy.$$

Both these quantities are conserved along the flows, in the sense that they do not depend on time and therefore

$$\begin{aligned} E_{NLS}(u(t)) &= E_{NLS}(u_0) \quad \forall\, t \in \mathbb{R}, \\ E_{NLKG}(u(t), \partial_t u(t)) &= E_{NLKG}(f, g) \quad \forall\, t \in \mathbb{R}. \end{aligned} \tag{0.0.4}$$

We recall that for the Schrödinger equation, the following conservation law (beside some other ones) holds true:

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad \forall\, t \in \mathbb{R},$$

and the latter one is called conservation of mass. Defocusing means, roughly speaking, that there is no competition between the quadratic part of the energy and the potential energy (i.e. the $L^{\alpha+2}$-norm in (0.0.4)), namely that the energy is nonnegative (the focusing regime corresponds to the opposite sign in front of the nonlinearity in the equations and consequently it is reflected by an opposite sign in front of the super-quadratic terms of the energies. This would lead, in general, to non-global solution). Moreover, they are investigated in the so-called energy subcritical case (and mass-supercritical case).

This sub-criticality corresponds to the fact that the nonlinear term is weaker than the linear terms of the equation, in the sense that the quadratic part of the energy, i.e. the H^1-norm of the solution, controls the super-quadratic term of the energy, by means of the Sobolev embeddings, both for NLS and NLKG. A consequence of this fact is that once the local well-posedness theory is established, then local in time solutions can be extended globally in time. A natural question is now the asymptotic behavior of this solutions. In the energy topology, is it true that the wave operator is surjective? The mathematical formulation of this question is as follows: consider the initial value problems associated to (0.0.1) and (0.0.2) respectively given by

$$\begin{cases} i\partial_t v + \Delta v - Vv = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ v(0) = h_0^{\pm} \in H^1(\mathbb{R}^d) \end{cases} \tag{0.0.5}$$

and

$$\begin{cases} \partial_{tt} v - \Delta_{x,y} v + v = 0, \quad (t,x,y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \\ v(0,x,y) = f^{\pm} \in H^1(\mathbb{R}^d \times \mathbb{T}) \\ \partial_t v(0,x,y) = g^{\pm} \in L^2(\mathbb{R}^d \times \mathbb{T}) \end{cases}; \tag{0.0.6}$$

then one wonders if there exist suitable initial data $h^{\pm} \in H^1(\mathbb{R}^d)$ and corresponding linear solutions $v^{\pm}$ to the Cauchy problem (0.0.5) such that

$$\lim_{t\to\pm\infty} \|u(t,x) - v^{\pm}(t,x)\|_{H^1(\mathbb{R}^d)} = 0$$

and suitable initial data $(f^{\pm}, g^{\pm}) \in H^1(\mathbb{R}^d \times \mathbb{T}) \times L^2(\mathbb{R}^d \times \mathbb{T})$ and corresponding linear solutions $v^{\pm}$ to the Cauchy problem (0.0.6) such that

$$\lim_{t\to\pm\infty} \left\|u(t,x) - v^{\pm}(t,x)\right\|_{H^1(\mathbb{R}^d\times\mathbb{T})} + \left\|\partial_t u(t,x) - \partial_t v^{\pm}(t,x)\right\|_{L^2(\mathbb{R}^d\times\mathbb{T})} = 0.$$

The big issue in such kind of problem is to pass from small initial data to arbitrarily large initial data. Here, again, smallness and largeness are defined by means of the conserved quantities introduced in (0.0.4). For small initial data, the tool of the Strichartz estimates in fact guarantees a positive answer to the scattering problem by using a simple perturbative argument, while the passage to the large data theory will be attacked by a powerful technique introduced by Kenig and Merle which is by now well celebrated and is known with the name of Concentration/Compactness & Rigidity method. Briefly, once it is known that a small data scattering result holds true (and this is a consequence of a basic local theory), the Kenig and Merle road map consists in the following steps: a Profile Decomposition Theorem, which allows to decompose any element of a sequence (actually, up to subsequences) of initial data belonging to the energy space as a superposition of $J \geq 1$ free solutions (linear Schrödinger or linear Klein-Gordon solutions, when dealing with (0.0.1) or (0.0.2), respectively) evaluated at zero-time plus a remainder term which is still a free solution. The decomposition of the initial data sequence is characteristic: it is a superposition of suitable free solutions on which the non-compact groups of transformation leaving invariant the equations act.

Basically, they are the group of time and space translations. Such translation sequences enjoy some properties leading to a very weak interaction of the free solutions, while the remainder term is small is some norm: the latter is not the energy one, but a Strichartz norm. The second step is the perturbation result: in some sense, the remainder in the Profile Decomposition Theorem must be absorbed, to build a minimal (with respect to the energy) global non-scattering solution. Loosely speaking, since we already know that for small initial data in the energy space scattering holds, here minimal means the smallest energy which an initial datum can have such that the corresponding nonlinear solution to the associated Cauchy problem does not scatter. Such minimal non-scattering solution enjoys some compactness property and the last step consists therefore to exclude such non-scattering solution by means of Liouville-type theorem. In literature, this last step is usually proved by using a well known ingredient, the so-called virial identities. In our work, since dealing with the defocusing regimes, we present an alternative approach for the latter step (which is the real nonlinear step in the Kenig and Merle strategy) which relies only on one dimensional a priori estimates known as Nakanishi/Morawetz estimates. We emphasize how a one dimensional tool is enough to conclude the desired results although the considered equation are posed on multidimensional frameworks.

Let us turn our attention now to the Zakharov system (0.0.3). Our aim is to provide a local well-posedness theory and to study the behavior of the solution for large values of ω. To establish a local well-posedness theory we adopt a method developed by Ozawa and Tsutsumi in the context of the scalar Zakharov system, where the second order operator in the first equation of (0.0.3) is replaced by a classical Laplace operator. Basically, this technique is established to avoid the loss of derivative occurring from the right-hand side of the wave equation. The physical model takes into account the second order operator as in (0.0.3) and the parameter ω is basically related to the temperature of the plasma for which the Zakharov equations describes the oscillation of the Langmuir waves. Direct observations shows that for large value of ω the models is prescribed to be irrotational: the second aim of our work is therefore to show that the evolution of the electric field envelope u is asymptotically constrained (for $\omega \to \infty$) onto the space of irrotational vector fields. More precisely we control separately, in the Strichartz norm, the fast and slow dynamics of the problem, showing that a solution (u^ω, n^ω) to (0.0.3) converges, for $\omega \to \infty$, to a solution to

$$\begin{cases} i\partial_t u + \Delta u = \mathbf{Q}(nu) \\ \frac{1}{c_s^2}\partial_{tt} n - \Delta n = \Delta |u|^2 \end{cases}$$

with some well-prepared initial data (i.e. irrotational), where $\mathbf{Q} := -(-\Delta)^{-1}\nabla \operatorname{div}$ is the Helmholtz projection operator onto irrotational vector fields.

For a rigorous analysis and explanation of the concepts introduced before, as well as physical motivations to study them, we refer to the specific chapters of this work. A detailed bibliographical description of the problems can be also found in the corresponding chapters.

The results contained in this thesis can be found in the papers by the author and collaborators:

[1] *The electrostatic limit for the 3D Zakharov system* (joint work with P. Antonelli), Nonlinear Analysis **163** (2017), $19-33$;

[44] *Double Scattering Channels for 1D NLS in the Energy Space and its Generalization to Higher Dimensions* (joint work with N. Visciglia), Journal of Differential Equations, **264** (2018) 929-958;

[43] *Large data scattering for the defocusing NLKG on waveguide* $\mathbb{R}^d \times \mathbb{T}$ (joint work with L. Hari), submitted, archived at https://arxiv.org/abs/1709.03101.

Notations

We collect here the notations used along this Thesis, although some of them will be recalled along the work for the reader's convenience. Other ones will be introduced in the sequel in specific contexts.

$\mathbb{R}$ and $\mathbb{R}^d$ are the one-dimensional and d-dimensional euclidean spaces, respectively, while $\mathbb{C}$ is the complex plane. $\mathcal{M}^k$ will denote a k-dimensional compact manifold and $\mathbb{T}$ the one-dimensional torus.

Given a complex number z, $\bar{z}$ will be its complex conjugate, while $\Re z$ and $\Im z$ will be its real and imaginary parts, respectively.

As usual in the mathematical literature, ∂_x will denote the partial derivative operator with respect to the variable x, ∇ will be the standard gradient operator, Δ will be the Laplace operator, *div* the divergence operator and $\nabla\times$ the curl operator in $\mathbb{R}^3$. The operator $\tau_z f(x) := f(x-z)$ is the classical translation operator. In the sequel, by $\mathcal{F}, \mathcal{F}^{-1}$ we mean the Fourier transform operator and its inverse, respectively, and often $\hat{f}$ will stand for $\mathcal{F}f$. Moreover, for $x \in \mathbb{R}^d$, the function $\langle x\rangle$ is classically defined by $\sqrt{1+|x|^2}$.

With standard notation, for $1 \leq p < \infty$, $L^p = L^p(\mathbb{R}^d)$ will be the usual Lebesgue space on $\mathbb{R}^d$ of p-summable functions with norm given by $\|f\|_{L^p} = (\int_{\mathbb{R}^d} |f|^p\,dx)^{1/p}$. $W^{k,p} = W^{k,p}(\mathbb{R}^d)$ is instead the Sobolev space of functions in L^p with weak derivatives in L^p up to the order k, endowed with the norm $\|f\|_{W^{k,p}} = \left(\sum_{|\alpha|=0}^k \|\partial^{|\alpha|} f\|_{L^p}^p\right)^{1/p}$. If $p=2$, $H^k := W^{k,2}$. If $p=\infty$, then L^∞ (and $W^{k,\infty}$, consequently) is defined with $\|f\|_{L^\infty} := \operatorname{ess\,sup}_{x\in\mathbb{R}^d} |f(x)|$. These spaces are similarly defined with $\mathbb{R}^d$ replaced by $\mathbb{T}$ or $\mathbb{R}^d \times \mathbb{T}$; if not otherwise specified, the L^p-norm ($W^{k,p}$-norm) $\|f\|_{L^p}$ ($\|f\|_{W^{k,p}}$ respectively) of a function f defined on a product space will refer to the whole set of variables; if instead subscripts with some explicit variable appear, this will mean that the norm is considered for that subset of variables only.

Given a vector-valued function $f(t)$ from $I \subseteq \mathbb{R}$ into a Banach space $(X, \|\cdot\|)$, $L^p(I;X)$ will denote the Bochner space of that functions having finite $L^p(I;X)$-norm, the last defined as $\|f\|_{L^p(I;X)} = (\int_I \|f(t)\|_X^p\,dt)^{1/p}$. If $I=\mathbb{R}$ then the notation can be contracted in L^pX.

If a vector-valued function $f: I \to X$ is continuous, we will write $f \in \mathcal{C}(I;X)$ or $f \in \mathcal{C}^m(I;X)$ if it is continuos with continuous derivatives up to the order m.

The arrow $\rightharpoonup$ will denote the convergence in a weak topology, while $\hookrightarrow$ will be for a continuous embedding between two spaces.

For $1 \leq p \leq \infty$, its conjugate p' is defined by $p' = \frac{p}{p-1}$. If there exists a constant C such that two quantities A and B are related by $A \leq CB$ or $A \geq CB$, it will be written $A \lesssim B$ or $A \gtrsim B$, respectively. If both the previous relations hold, it will be written $A \sim B$. $A << B$ or $A >> B$ will mean that the quantity A is "much smaller than" or "much bigger than" the quantity B.

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Chapter 1

Strichartz estimates

The aim of this chapter is twofold: first, to recall the fundamental machinery given by the Strichartz estimates for the analysis of the dispersive PDEs, in particular by reporting the basic facts and by now well known results in the framework of the classical (i.e. the unperturbed one) nonlinear Schrödinger equation, and then to analyze the NLS perturbed by a steplike potential and the Klein-Gordon equation posed on $\mathbb{R}^d \times \mathbb{T}$, $\mathbb{T}$ being the one-dimensional flat torus. For the Zakharov system, they arise by a simple scaling argument from the unperturbed Schrödinger equation, once the propagator associated to the linear flow of the Schrödinger-type equation is explicitly determined in term of the classical Schrödinger group.

1.1 The toy model: the unperturbed NLS on $\mathbb{R}^d$

Let us treat the unperturbed linear Schrödinger equation. Consider the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases}; \tag{1.1.1}$$

by employing the Fourier transform method, at least assuming smooth initial data, we can represent explicitly the solution to (1.1.1) as

$$u(t,x) = U(t)u_0 := \mathcal{F}^{-1}(e^{-i|\xi|^2 t}\hat{u}_0(\xi))(x) = K(t,x) * u_0 \tag{1.1.2}$$

with

$$K(t,x) := \left(\frac{1}{4i\pi t}\right)^{d/2} e^{\frac{i|x|^2}{4t}}.$$

It is straightforward that the propagator $U(t)$ is unitary in H^s for every $s \in \mathbb{R}$.

Theorem 1.1.1. *For every $p \in [2,\infty]$ and $t \neq 0$ the group $U(t)$ maps $L^{p'}$ into L^p continuously, in particular the following estimate holds:*

$$\|U(t)f\|_{L^p} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|f\|_{L^{p'}} \tag{1.1.3}$$

Proof. By the explicit expression for the Schrödinger propagator (1.1.2), using the Young's inequality for convolution, follows that

$$\|U(t)f\|_{L^\infty} \lesssim |t|^{-\frac{d}{2}}\|f\|_{L^1} \tag{1.1.4}$$

and so by using the property that $U(t)$ is unitary from L^2 into itself and the Riesz-Thorin Theorem we end up with (1.1.3). □

Definition 1.1.2. The estimate (1.1.4) is called *Dispersive estimate.*

Let us report here a definition of dispersion as given in Palais, see [105], by quoting a version for Schrödinger contained in [113]:

A definition of dispersion. Let us consider linear wave equations of the form

$$\partial_t u + P(\partial_x)u = 0.$$

where P is polynomial. Recall that a solution $u(t,x)$, which Fourier transform is of the form $e^{i(kx-\omega t)}$ is called a plane-wave solution; k is called the wave number (waves per unit of length) and ω the (angular) frequency. Rewriting this in the form $e^{ik(x-(\omega/k)t)}$ we recognize that this is a traveling wave of velocity ω/k. If we substitute this $u(t,x)$ into our wave equation, we get a formula determining a unique frequency $\omega(k)$ associated to any wave number k, which we can write in the form

$$\omega(k) = \frac{1}{ik}P(ik).$$

This is called the *dispersive relation* for this wave equation. Note that it expresses the velocity for the plane-wave solution with wave number k. For example $P(\partial_x) = c\partial_x$ gives the linear advection equation $\partial_t u + c\partial_x = 0$ which has the dispersion relation $\omega/k = c$ showing of course that all plane-wave solutions travel at the same velocity c, and we say that we have trivial dispersion in this case. On the other hand if we take $P(\partial_x) = -i(\partial_x)^2$ then our wave equation is $i\partial_t u + \partial_x^2 u = 0$, which is the linear Schrödinger equation, and we have the non-trivial dispersion relation $\omega/k = k$. In this case, plane waves of large wave-number (and hence high frequency) are traveling much faster than low-frequency waves. The effect of this is to broaden a wave packet. That is, suppose our initial condition is $u_0(x)$. We can use the Fourier transform to write u_0 in the form

$$u_0(x) = \int \hat{u}_0(k)e^{ikx}\,dk$$

and then, by superposition, the solution to our wave equation will be

$$u(t,x) = \int \hat{u}_0(k)e^{ik(x-(\omega/k)t)}\,dk.$$

Suppose for example that our initial wave form is a highly peaked Gaussian. Then in the case of the linear advection equation all the Fourier modes travel together at the same speed and the Gaussian lump remains highly peaked over time. On the other hand, for the linearized Schrödinger equation the various Fourier modes all travel at different velocities, so after time they start canceling each other by destructive interference, and the original sharp Gaussian quickly broadens.

To better understand how the Strichartz estimates that will be introduced in a while can be viewed as a regularizing effect of the free propagator $U(t)$, let us recall the following.

Proposition 1.1.3. *For any $p > 2$ and $\bar{t} \in \mathbb{R}$, there exists a function $f \in L^2$ such that $U(\bar{t})f \notin L^p$. Furthermore it can be shown that there exists a dense subset of L^p functions contained in L^2, say D_p, such that for any $f \in D_p$, $U(t)f \notin L^p$.*

Because of the group $U(t)$ does not map the space L^p into itself for a generic $p \neq 2$, the estimate (1.1.3), even thought remarkable and very useful, does not allow us to deal with the NLS equation, but it can be used to prove some space-time estimates, called *Strichartz estimates*, and the latter ones are essential to study the NLS equation. These estimates were introduced in [115] by Strichartz in the context of a very interesting and fascinating problem in harmonic analysis known as Fourier Restriction Problem (see for example Wolff's lecture notes [133]), then in the work of Ginibre and Velo [51] was given a simpler proof along with a generalization of them. For a general treatment we address to the monographs of Cazenave [20], or Linares and Ponce [90]. See also Tao's book [119].

Let us introduce the concept of admissible pair. As we already said, Strichartz estimates are space-time summability properties for the solution to (1.1.1) (and its inhomogeneous version); in particular they claim that solutions to (1.1.1) belong to some Bochner space $L^q(I; L^r)$, where $I \subseteq \mathbb{R}$. Since (1.1.1) is invariant under the scaling $u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$, it arises that some algebraic conditions for the exponents (q,r) must be satisfied. They are precisely the following:

Definition 1.1.4. A pair (q,r) of reals is said *admissible* if

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right)$$

and the following condition are satisfied:

$$\begin{array}{ll} 2 \leq r \leq \frac{2d}{d-2} & if \;\; d \geq 3, \\ 2 \leq r < \infty & if \;\; d = 2, \\ 2 \leq r \leq \infty & if \;\; d = 1. \end{array}$$

The case corresponding to $r = \frac{2d}{d-2}$ when $d \geq 3$ is known as *endpoint case*. The prove of Strichartz estimates for such exponents is given in the famous and remarkable work by Keel and Tao [76]. The following collects the set of the Strichartz estimates:

Theorem 1.1.5. *Consider a function $f \in L^2$. Then the function $t \to U(t)f$ belongs to*

$$L^q(\mathbb{R}; L^r(\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d))$$

provided (q,r) is admissible and the following estimate holds:

$$\|U(t)f\|_{L^qL^r} \le C\|f\|_{L^2} \tag{1.1.5}$$

where the constant C is independent from f, in particular $C = C(q,r)$ and thus C is also dependent from d. Moreover, if $f \in L^{\gamma'}(\mathbb{R}; L^{\rho'}(\mathbb{R}^d))$ with (γ,ρ) admissible, then for every (q,r) admissible the function

$$t \to \mathcal{D}_f(t) := \int_{\mathbb{R}} U(t-s)f(s)\, ds$$

belongs to

$$L^q(\mathbb{R}; L^r(\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d))$$

provided (q,r) is admissible and the following estimate holds:

$$\|\mathcal{D}_f\|_{L^q(\mathbb{R};L^r)} \le C\|f\|_{L^{\gamma'}(\mathbb{R};L^{\rho'}(\mathbb{R}^d))}. \tag{1.1.6}$$

where $C = C(q,r)$.

The proof for the non-endpoint cases is a consequence of the pointwise estimate (1.1.3), the TT^* method and the Hardy-Sobolev-Littlewood inequality, see [20, 90, 119] for instance.

Corollary 1.1.6. *After having proved the Strichartz estimates, it can be claimed that if $f \in L^2$ then for almost every $t \in \mathbb{R}$, $U(t)f \in L^p$. This fact does not contradict Proposition 1.1.3 and it is the smoothing effect that we mentioned before.*

Finally, thanks to the Christ and Kiselev Lemma, see [25], the *retarded Strichartz estimates* also hold, namely:

Lemma 1.1.7. *The same estimate of (1.1.6) holds if we replace $\mathcal{D}_f(t)$ with*

$$\mathcal{D}^r_f(t) := \int_{s<t} U(t-s)f(s)\, ds.$$

1.2 Strichartz estimates for NLS perturbed by a step-like potential

It is quite clear that Strichartz estimates for the Schrödinger equation arise once the dispersive estimate holds. We saw how simple is to establish it in the unperturbed case. This is no more the case when we consider a perturbation of the Laplace operator, in fact we cannot rely on the explicit representation of the solution in term of the Fourier transform. In the presence of a perturbation $V(x)$, by noting $U_V(t) := e^{it(\Delta - V)}$ the linear propagator associated to

$$i\partial_t u + \Delta u - Vu = 0, \tag{1.2.1}$$

the dispersive estimate would read

$$\|U_V(t)f\|_{L^\infty} \lesssim |t|^{-d/2}\|f\|_{L^1}, \quad \forall\, t \neq 0, \quad \forall\, f \in L^1. \tag{1.2.2}$$

If (1.2.2) holds, as first consequence we get the following Strichartz estimates

$$\|U_V(t)f\|_{L^aL^b} \lesssim \|f\|_{L^2}$$

where $a, b \in [1, \infty]$ are assumed to be Strichartz admissible in the sense of Definition 1.1.4, namely

$$\frac{2}{a} = d\left(\frac{1}{2} - \frac{1}{b}\right). \tag{1.2.3}$$

A lot of works have been written in the literature about the topic of the validity of (1.2.2), both in 1D and in higher dimensions. We briefly cite works of Artbazar and Yajima [4], Christ and Kiselev[25], D'Ancona and Fanelli [33], Goldberg and Schlag [58], Weder[131], [132] and Yajima [135] for the one dimensional case and the papers by Burq, Planchon, Stalker and Tahvildar-Zadeh [18], Goldberg, Vega and Visciglia [59], Jensen and Kato [73], Journée, Soffer, Sogge [74], Rauch [109], Rodnianski and Schlag [110] for the higher dimensional case, referring to the bibliographies contained in these papers for a more detailed list of works on the subject. It is worth mentioning that in all the papers mentioned above, the perturbation V is assumed to decay at infinity, but our aim is to deal with steplike potentials.

Let us focus in the 1D case: we define as steplike a real potential $V : \mathbb{R} \to \mathbb{R}$ such that

$$a_+ = \lim_{x\to+\infty} V(x) \neq \lim_{x\to-\infty} V(x) = a_-, \tag{1.2.4}$$

where $a_- < a_+$ and without loss of generality it can be assumed $a_- = 0$ and $a_+ = 1$.

Under suitable assumptions, beside the fact that V is steplike, (1.2.2) with $d = 1$ was proved in D'Ancona and Selberg [34] by considering at first the steplike perturbation of the Laplacian given by the Heaviside function $\chi(x) = \chi_{[0,+\infty)}(x)$, then allowing $L^1(\mathbb{R})$ perturbations of it. The strategy to prove (1.2.2) was to derive explicitly the kernel of the second order operator $-\partial_x^2 + \chi$ and the representation of the fundamental solution corresponding to the non-stationary evolution equation associated to it. If the steplike perturbation is simply the characteristic function of $[0, +\infty)$, the calculation of the kernel for $-\partial_x^2 + \chi$ follows from the standard theory for the ODEs and classical functional calculus, i.e. spectral analysis and the limiting absorption principle. We observe that the spectrum of $-\partial_x^2 + \chi$ is $[0, +\infty)$ and in particular the point spectrum is empty. This observation is essential when dealing with the problem of dispersive property for evolution equation: in fact, if the empty spectrum were not empty, then solitons would appear: consider a solution to

$$(-\Delta + V)\, Q = \mu Q, \qquad \mu \in \mathbb{R},$$

then $u(t,x) = e^{i\mu t}Q$ is a solution to $i\partial_t u + \Delta u - Vu = 0$. But clearly such u cannot disperse, since any L^p-norm is preserved, namely it is constant in time. For this reason, in [34], the dispersive estimate in its general formulation is not as in (1.2.2), but it is necessary to project the flow on the absolutely continuous component of the spectrum, yielding to the estimate

$$\|P_{ac}U_V(t)f\|_{L^\infty} \lesssim |t|^{-1/2}\|f\|_{L^1}, \qquad \forall\, t \neq 0.$$

Beside the presence of a non-empty point spectrum, another obstruction to dispersion is the presence of the so-called *resonance at zero energy.* Although it is not our purpose to investigate such problem, for which there is a huge literature (see for example [71–73,109]), we quote here a definition given in [110] which defines when $\lambda = 0$ is a *regular point* and some comments about it.

Definition 1.2.1. $\lambda = 0$ is called regular point if it is neither an eigenvalue for $-\Delta + V$ nor a resonance. Resonance means that the equation $-\Delta u + Vu = 0$ has no solutions in $S = \cap_{\gamma > 1/2} L^{2,-\gamma}$, with $L^{2,-\gamma} = \left\{ f : (1+|x|^2)^{\gamma/2} \in L^2 \right\}$.

In [110] is also remarked that in general it is not a matter of regularity of the potential or its decay if the condition given in Definition 1.2.1 is satisfied, but sometimes is automatically fulfilled; for example when the potential is nonnegative, which is a condition imposed on our steplike potential in the analysis carried in chapter 3.

When considering a multidimensional setting, we will consider as steplike a real potential $V : \mathbb{R}^d \to \mathbb{R}$ such that is steplike in one direction, say x_1, and which is periodic with respect to the remaining variables, i.e. $\bar{x} = (x_2, \dots, x_d)$. More rigorously, we consider real potentials $V : \mathbb{R}^d \to \mathbb{R}$ such that uniformly in $\bar{x} \in \mathbb{R}^{d-1}$,

$$a_- = \lim_{x_1 \to -\infty} V(x_1, \bar{x}) \neq \lim_{x_1 \to +\infty} V(x_1, \bar{x}) = a_+, \qquad \text{where} \qquad x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$$

and moreover we assume the existence of $d-1$ linear independent vectors $P_2, \dots, P_d \in \mathbb{R}^{d-1}$ such that for any fixed $x_1 \in \mathbb{R}$, the following holds:

$$V(x_1, \bar{x}) = V(x_1, \bar{x} + k_2 P_2 + \cdots + k_d P_d),$$
$$\forall\, \bar{x} = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}, \quad \forall\, (k_2, \dots, k_d) \in \mathbb{Z}^{d-1}.$$

For such kind of potentials, it is worth highlighting that the dispersive estimate (1.2.2) is assumed to be satisfied, and then the Strichartz estimates below. But some remarks about the potential must be done. In fact we note that:

Remark 1.2.2. When considering the multidimensional version of our considered equation, see (3.1.7), an alternative steplike potential could be provided: for instance we could assume the existence of the following limits

$$\lim_{r \to \infty} V(r\sigma) = l_\sigma$$

that can change according with $\sigma \in S^{d-1}$, being the last one the unit sphere in $\mathbb{R}^d$. However, since we assume as black-box the validity of the Strichartz estimates for the linear propagator, this type of potentials cannot be allowed. In fact the Strichartz estimates are forbidden for this kind of perturbations, see the work of Goldberg, Vega and Visciglia [59].

When dealing with the nonlinear counterpart of (1.2.1), namely

$$i\partial_t u + \Delta u - Vu = f(u),$$

with $f(u) = |u|^\alpha u$, we fix the following Lebesgue exponents

$$r = \alpha + 2, \qquad p = \frac{2\alpha(\alpha+2)}{4-(d-2)\alpha}, \qquad q = \frac{2\alpha(\alpha+2)}{d\alpha^2+(d-2)\alpha-4} \tag{1.2.5}$$

and we conclude this section giving the linear estimates that will be fundamental in our analysis:

$$\|e^{it(\Delta-V)}f\|_{L^{\frac{4(\alpha+2)}{d\alpha}}L^r} \lesssim \|f\|_{H^1}, \tag{1.2.6}$$

$$\|e^{it(\Delta-V)}f\|_{L^{\frac{2(d+2)}{d}}L^{\frac{2(d+2)}{d}}} \lesssim \|f\|_{H^1}, \tag{1.2.7}$$

$$\|e^{it(\Delta-V)}f\|_{L^pL^r} \lesssim \|f\|_{H^1}. \tag{1.2.8}$$

The last estimate that we need is the so-called inhomogeneous Strichartz estimate for non-admissible pairs:

$$\left\|\int_0^t e^{i(t-s)(\Delta-V)}g(s)\,ds\right\|_{L^pL^r} \lesssim \|g\|_{L^{q'}L^{r'}}, \tag{1.2.9}$$

whose proof is contained in [22].

Remark 1.2.3. In the unperturbed framework, i.e. in the absence of the potential, and for general dimensions, we refer to [42] for comments and references about Strichartz estimates (1.2.6), (1.2.7), (1.2.8) and (1.2.9).

Remark 1.2.4. About Strichartz estimates for not admissible pairs, we address the reader to the works of Foschi [45] and Vilela [129].

1.3 Strichartz estimates for NLKG on product spaces

In this section we prove some Strichartz estimates on waveguides. Let us consider for the moment the equation posed on a k-dimensional compact manifold, say $\mathcal{M}^k$: unlike the euclidean setting, the presence of periodic solutions induces a lack of (global-in-time) summability on them. The analysis of a mixed situation, in which an euclidean part guarantees at least enough decay in time for solution to the equation, goes back to the work of Tzvetkov and Visciglia [126], where the authors investigates the NLS equation posed on $\mathbb{R}^d \times \mathcal{M}^k$, proving a small data scattering result in some anisotropic Sobolev space. This section is devoted to the proof of some Strichartz estimates for NLKG on $\mathbb{R}^d \times \mathbb{T}$, where $\mathbb{T}$ is the one-dimensional flat torus. The method we apply to obtain Strichartz estimates on the whole product space is used in Hari and Visciglia [63] (which is inspired in some sense from [126]) for the energy critical NLKG posed on $\mathbb{R}^d \times \mathcal{M}^2$. The method is divided into the following steps:

1. state the estimates on $\mathbb{R}^d$, involving Besov spaces;
2. use embedding theorems to deduce some estimates that hold in Lebesgue spaces posed on $\mathbb{R}^d$;
3. use a scaling argument to handle masses different from one;
4. write (1.3.2) in the basis of eigenfunctions of $\mathbb{T}$ and prove Theorem 1.3.1 in the fashion of [126] and [63].

In [63], dealing with energy-critical nonlinearities, only critical embeddings were needed to prove small data theory. In our subcritical setting, one has to consider a wider range of Strichartz estimates to prove such results, obtained with “subcritical” embeddings. Deeper discussions about these estimates will be made along the proof of Theorem 1.3.1 which is as follows and it is the main result of this section.

Theorem 1.3.1. *Let $d \in \mathbb{N}$ and $1 \leq q, r \leq \infty$ such that (q, r) satisfies*

$$\begin{aligned} &\frac{2q}{q-4} \leq r \qquad\qquad\qquad q \geq 4 \quad \text{if } d = 1, \\ &\frac{2dq}{dq-4} \leq r \leq \frac{2q(d+1)}{q(d-1)-2}, \quad q > 2 \quad \text{if } d = 2, \quad q \geq 2 \text{ if } d \geq 3. \end{aligned} \tag{1.3.1}$$

Let $w \in \mathcal{C}(\mathbb{R}; H^1) \cap \mathcal{C}^1(\mathbb{R}; L^2)$ be the unique solution to the following nonlinear problem:

$$\begin{cases} \partial_{tt} u - \Delta u + u = F, \quad (t, x, y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \\ u(0, x, y) = f \in H^1 \\ \partial_t u(0, x, y) = g \in L^2 \end{cases} \tag{1.3.2}$$

where $F = F(t, x, y) \in L^1 L^2$. Then the estimate below holds:

$$\|w\|_{L^q L^r} \leq C \left(\|f\|_{H^1} + \|g\|_{L^2} + \|F\|_{L^1 L^2} \right).$$

In order to prove Theorem 1.3.1 we recall the definition of the Besov spaces. Given a cut-off function χ_0 such that

$$C_c^\infty(\mathbb{R}^d; \mathbb{R}) \ni \chi_0(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \leq 1 \\ 0 & \text{if} \quad |\xi| > 2 \end{cases}$$

then are defined the following dyadic functions

$$\varphi_j(\xi) = \chi_0(2^{-j}\xi) - \chi_0(2^{-j+1}\xi),$$

yielding to the partition of the unity

$$\chi_0(\xi) + \sum_{j>0} \varphi_j(\xi) = 1, \qquad \forall\, \xi \in \mathbb{R}^d.$$

By denoting with $\mathcal{S}'(\mathbb{R}^d)$ the set of all tempered distributions on $\mathbb{R}^d$, we denote by P_j, $j \in \mathbb{N} \cup \{0\}$, the following operators acting on $f \in \mathcal{S}'(\mathbb{R}^d)$:

$$\begin{aligned} P_0 f &:= \mathcal{F}^{-1}\left(\chi_0 \mathcal{F}(f)\right), \\ P_j f &:= \mathcal{F}^{-1}\left(\varphi_j \mathcal{F}(f)\right), \quad \forall j \in \mathbb{N}. \end{aligned}$$

Let $s \in \mathbb{R}$. Then, for $0 < q \leq \infty$, the Besov space $B^s_{q,2}$ is defined by

$$B^s_{q,2}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \middle| \left\{ 2^{js} \| P_j f \|_{L^q(\mathbb{R}^d)} \right\}_{j \in \mathbb{N} \cup \{0\}} \in l^2 \right\}, \tag{1.3.3}$$

where l^2 is the classical space of square-summable sequences.

We rigorously prove the steps listed above in order to prove Theorem 1.3.1.

We begin therefore with the following proposition which is given in a pure euclidean context.

Proposition 1.3.2. *Let $d \geq 1$ and $2 \leq q, \rho \leq \infty$ such that*

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{\rho}\right) \quad (\textit{with the restrictions } q > 2 \textit{ if } d = 2,\ q \geq 4 \textit{ if } d = 1). \tag{1.3.4}$$

Consider $w = w(t, x)$ satisfying

$$\begin{cases} \partial_{tt} w - \Delta_{\mathbb{R}^d} w + w = F, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d \\ w(0, x) = f \in H^1(\mathbb{R}^d) \\ \partial_t w(0, x) = g \in L^2(\mathbb{R}^d) \end{cases} \tag{1.3.5}$$

where $F = F(t, x) \in L^1(\mathbb{R}; L^2(\mathbb{R}^d))$. Then

$$\|w\|_{L^q(\mathbb{R}; B^s_{\rho,2}(\mathbb{R}^d))} \leq C \left(\|f\|_{H^1(\mathbb{R}^d)} + \|g\|_{L^2(\mathbb{R}^d)} + \|F\|_{L^1(\mathbb{R}; L^2(\mathbb{R}^d))} \right), \tag{1.3.6}$$

where $C > 0$ depends only on the choice of the pair (q, r) and on the dimension d and $s \in [0, 1]$ is defined by

$$s = 1 - \frac{1}{2}\left(\frac{d}{2} + 1\right)\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) = 1 - \frac{1}{2}\left(\frac{d}{2} + 1\right)\left(1 - \frac{2}{\rho}\right). \tag{1.3.7}$$

The proof is detailed in Nakanishi and Schlag [102], using previous results from Brenner [12, 13], Ginibre and Velo [52–54], Ibrahim, Masmoudi and Nakanishi [69], Keel and Tao [76], Nakamura and Ozawa [99], Pecher [106] (see for example [52–54, 76] for the proofs and [76] for the endpoint cases when $d \geq 3$. See also Machihara, Nakanishi and Ozawa [91, 92]).
The estimates in previous works are more general: the source term can be handled in a "dual" Besov space. We chose to handle the source term in the only homogeneous space we can work with, using a scaling method.

As second step, we state the following embedding theorem contained in [124, 125] and references therein.

Theorem 1.3.3. *Let $d \geq 1$, $s > 0$, and $1 < r, \rho < \infty$. Consider the Besov space $B^s_{\rho,2}(\mathbb{R}^d)$ and the Lebesgue space $L^r(\mathbb{R}^d)$. Then the embedding relations below hold:*

1. $B^s_{\rho,2}(\mathbb{R}^d) \hookrightarrow L^\rho(\mathbb{R}^d)$ *($\rho = 1, \infty$ allowed);*
2. *If* $\rho^* := \dfrac{d\rho}{d - s\rho}$, *when* $d > s\rho$, *then* $B^s_{\rho,2}(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$ *for* $\rho \leq r \leq \rho^*$*;*
3. *If* $d \leq s\rho$, *then* $B^s_{\rho,2}(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$ *for* $\rho \leq r < +\infty$.

Remark 1.3.4. Observe that in the statement of the Embedding Theorem, s is not assumed to be the same of (1.3.7). We kept the same station since later on they will be identified.

We notice that for s satisfying (1.3.7) and (q, ρ) as in (1.3.4), the conditions of the theorem yield

$$\begin{array}{lll} d - s\rho < 0 & \text{if} & d = 1, \\ d - s\rho = 0 & \text{if} & d = 2, \\ d - s\rho > 0 & \text{if} & d \geq 3. \end{array}$$

By [123], we have in our setting $B^s_{\rho,2}(\mathbb{R}^d) \subset W^{s,\rho}(\mathbb{R}^d)$. Thus with Sobolev embeddings $W^{s,\rho}(\mathbb{R}^d) \subset L^r(\mathbb{R}^d)$ with

$$\begin{array}{lll} r \in [\rho, \infty] & \text{if} & d = 1, \\ r \in [\rho, \infty) & \text{if} & d = 2, \\ r \in [\rho, \rho^*] & \text{if} & d \geq 3, \end{array}$$

and computing ρ, ρ^* in terms of q, we obtain

$$\frac{2dq}{dq - 4} \leq r \qquad \text{if } d = 1 \quad \text{or} \quad d = 2, \tag{1.3.8}$$

$$\frac{2dq}{dq - 4} \leq r \leq \frac{2d^2 q}{d^2 q - 2d - 2dq + 4} \qquad \text{if } d \geq 3. \tag{1.3.9}$$

Strichartz estimates involving Lebesgue spaces instead of Besov spaces follow immediately by applying the previous embedding theorem to Proposition 1.3.2, with r satisfying

(1.3.8) or (1.3.9).

The third step is the scaling argument. By defining $w_\lambda := w\left(\sqrt{\lambda}t, \sqrt{\lambda}x\right)$ with w as in (1.3.5) and noticing that it satisfies

$$\begin{cases} \partial_{tt} w_\lambda - \Delta_{\mathbb{R}^d} w_\lambda + \lambda w_\lambda = F_\lambda, & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ w_\lambda(0,\cdot) = f_\lambda \\ \partial_t w_\lambda(0,\cdot) = g_\lambda \end{cases}, \tag{1.3.10}$$

where

$$f_\lambda(x) = f\left(\sqrt{\lambda}x\right), \quad g_\lambda(x) = \sqrt{\lambda} g\left(\sqrt{\lambda}x\right), \quad F_\lambda(t,x) = \lambda F\left(\sqrt{\lambda}t, \sqrt{\lambda}x\right),$$

we can easily prove the next result, whose detailed proof can be found in [63].

Proposition 1.3.5. *Consider a pair (q,ρ) as in (1.3.4), s given by (1.3.7) and r as in Theorem 1.3.3. Consider w given by (1.3.5) for which Proposition 1.3.2 holds. Then one has for (1.3.10)*

$$\begin{aligned} \|w_\lambda\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d))} \leq C\lambda^{-\frac{1}{2}\left(\frac{d}{r}+\frac{1}{q}-\frac{d}{2}+1\right)} \Big(\sqrt{\lambda}\|f_\lambda\|_{L^2(\mathbb{R}^d)} + \|f_\lambda\|_{\dot{H}^1(\mathbb{R}^d)} \\ + \|g_\lambda\|_{L^2(\mathbb{R}^d)} + \|F_\lambda\|_{L^1(\mathbb{R};L^2(\mathbb{R}^d))}\Big). \end{aligned} \tag{1.3.11}$$

The final step is the proof of Theorem 1.3.1, by using the tools of the previous steps.

Proof of Theorem 1.3.1. Once we can rely on the ingredients above, we finally use the strategy from [126] to conclude with the desired result. We write $\{\lambda_j\}_{j\geq 0}$ for the eigenvalues of $-\Delta_{\mathbb{T}}$, sorted in ascending order and taking in account their multiplicities; we also introduce $\{\Phi_j(y)\}_{j\geq 0}$, the eigenfunctions associated with λ_j, i.e.

$$-\Delta_{\mathbb{T}}\Phi_j = \lambda_j \Phi_j, \quad \lambda_j \geq 0, \quad j \in \mathbb{N} \cup \{0\}. \tag{1.3.12}$$

This provides an orthonormal basis of $L^2(\mathbb{T})$. We now consider the solution to (1.3.2) and we write the functions in terms of (1.3.12):

$$\begin{aligned} w(t,x,y) &= \sum_{j=0}^{\infty} w_j(t,x)\Phi_j(y), \\ F(t,x,y) &= \sum_{j=0}^{\infty} F_j(t,x)\Phi_j(y), \\ f(x,y) &= \sum_{j=0}^{\infty} f_j(x)\Phi_j(y), \\ g(x,y) &= \sum_{j=0}^{\infty} g_j(x)\Phi_j(y), \end{aligned} \tag{1.3.13}$$

with $w_j = w_j(t,x)$ satisfying

$$\begin{cases} \partial_{tt} w_j - \Delta_{\mathbb{R}^d} w_j + w_j + \lambda_j w_j = F_j, & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ w_j(0,\cdot) = f_j & \\ \partial_t w_j(0,\cdot) = g_j & \end{cases}. \tag{1.3.14}$$

Taking $\lambda = 1 + \lambda_j$ in (1.3.11) it follows that

$$(\lambda_j+1)^{\frac{1}{2}\left(\frac{d}{r}+\frac{1}{q}+1-\frac{d}{2}\right)} \|w_j\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d))} \leq C\left((\lambda_j+1)^{1/2}\|f_j\|_{L^2(\mathbb{R}^d)} + \|f_j\|_{\dot{H}^1(\mathbb{R}^d)} \right.$$
$$\left. + \|g_j\|_{L^2(\mathbb{R}^d)} + \|F_j\|_{L^1(\mathbb{R};L^2(\mathbb{R}^d))}\right).$$

Then, summing in j the squares as in [126] and [63] one obtains

$$\left\|(\lambda_j+1)^{\frac{1}{2}\left(\frac{d}{r}+\frac{1}{q}+1-\frac{d}{2}\right)} w_j\right\|_{l_j^2 L^q(\mathbb{R};L^r(\mathbb{R}^d))} \leq C\left(\left\|(\lambda_j+1)^{1/2} f_j\right\|_{l_j^2 L^2(\mathbb{R}^d)} + \|f_j\|_{l_j^2 \dot{H}^1(\mathbb{R}^d)} \right.$$
$$\left. + \|g_j\|_{l_j^2 L^2(\mathbb{R}^d)} + \|F_j\|_{l_j^2 L^1(\mathbb{R};L^2(\mathbb{R}^d))}\right).$$

Since $\max\{1,2\} \leq 2 \leq \min\{q,\rho\}$, the Minkowski inequality can be applied, hence

$$\left\|(\lambda_j+1)^{\frac{1}{2}\left(\frac{d}{r}+\frac{1}{q}+1-\frac{d}{2}\right)} w_j\right\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d)) l_j^2} \leq C\left(\left\|(\lambda_j+1)^{1/2} f_j\right\|_{L^2(\mathbb{R}^d) l_j^2} + \|g_j\|_{L^2(\mathbb{R}^d) l_j^2} \right.$$
$$\left. + \|F_j\|_{L^1(\mathbb{R};L^2(\mathbb{R}^d)) l_j^2}\right),$$

and by the Plancherel identity one is able to handle the y variable getting

$$\left\|(1-\Delta_y)^{\frac{1}{2}\left(\frac{d}{r}+\frac{1}{q}+1-\frac{d}{2}\right)} w\right\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d) L^2(\mathbb{T}))} \leq C\left(\|f\|_{H^1(\mathbb{R}^d\times\mathbb{T})} + \|g\|_{L^2(\mathbb{R}^d\times\mathbb{T})} \right.$$
$$\left. + \|F\|_{L^1(\mathbb{R};L^2(\mathbb{R}^d\times\mathbb{T}))}\right),$$

which in turn implies

$$\|w\|_{L^q(\mathbb{R};L^r_x(\mathbb{R}^d) H^\gamma_y(\mathbb{T}))} \leq C\left(\|f\|_{H^1(\mathbb{R}^d\times\mathbb{T})} + \|g\|_{L^2(\mathbb{R}^d\times\mathbb{T})} + \|F\|_{L^1(\mathbb{R};L^2(\mathbb{R}^d\times\mathbb{T}))}\right)$$

where

$$\gamma = \left(\frac{d}{r} + \frac{1}{q} + 1 - \frac{d}{2}\right).$$

We easily see that $\gamma \geq 0$ when $d = 1, 2$, and by computing the condition $\gamma \geq 0$ for $d \geq 3$, we have

$$\begin{aligned} \gamma \geq 0 \iff & \frac{2dq + 2r + 2qr - dqr}{2qr} \geq 0 \\ \iff & 2dq + 2r + 2qr - dqr \geq 0 \\ \iff & r \leq \frac{2dq}{dq - 2q - 2}. \end{aligned}$$

It is easy to check that $\frac{2dq}{dq-2q-2} \geq \rho^* > r$ which establishes that under (1.3.9) γ is always nonnegative.

The proof is then completed by using a Sobolev embedding available for $\gamma \geq 0$

$$H^\gamma(\mathbb{T}) \hookrightarrow L^r(\mathbb{T}) \tag{1.3.15}$$

which holds (at least) under one of the following conditions:

$$\begin{array}{ll} 2\gamma < 1, & \frac{2}{1-2\gamma} \geq r \\ 2\gamma \geq 1, & r \geq 2 \end{array}. \tag{1.3.16}$$

Remark 1.3.6. The first one of (1.3.16) is the "usual" condition to have Sobolev embedding, while the second one ensures (1.3.15) with $H^\gamma(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ allowing to control any L^r-norm with the H^γ-norm since $\mathbb{T}$ is of finite volume.

Then by gluing together all conditions (1.3.8),(1.3.9),(1.3.16) in terms of q, we exhibit the exponent r for which the Strichartz estimates can be proved: for $d = 1$, since $\gamma > 1/2$, we have $H^\gamma(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ and so

$$\frac{2q}{q-4} \leq r.$$

For $d \geq 2$

$$\frac{2dq}{dq-4} \leq r \leq \min\left\{\frac{2d^2q}{d^2q-2d-2dq+4}, \frac{2q(d+1)}{dq-q-2}\right\},$$

that is

$$\frac{2dq}{dq-4} \leq r \leq \frac{2q(d+1)}{dq-q-2},$$

which concludes the proof of Theorem 1.3.1. □

1.4 A dispersive estimate for NLKG on flat waveguide $\mathbb{R}^d \times \mathbb{T}$

In order to prove the Profile Decomposition Theorem in chapter 4 we use a decay property from [32]. The key argument to show it on a waveguide is, again, a scaling argument. We will briefly sketch the proof given in [32, Example 1.2].

By means of the basis $\{\Phi_j(y)\}_{j\in\mathbb{N}}$ given in (1.3.12) and (1.3.13) we decompose

$$e^{it\sqrt{1-\Delta_{x,y}}} f(x,y) = \sum_{j\in\mathbb{N}} e^{it\sqrt{1+\lambda_j-\Delta_x}} f_j(x)\Phi_j(y).$$

Thus we get

$$\left\| e^{it\sqrt{1-\Delta_{x,y}}} f \right\|_{L^\infty(\mathbb{R}^d\times\mathbb{T})} \leq \sum_{j\in\mathbb{N}} \left\| e^{it\sqrt{1+\lambda_j-\Delta_x}} f_j(\cdot) \right\|_{L^\infty(\mathbb{R}^d)} \|\Phi_j(\cdot)\|_{L^\infty_y}.$$

From [32], we have

$$\left\| e^{it\sqrt{1-\Delta_x}} f \right\|_{L^\infty_x} \leq C|t|^{-d/2} \|f\|_{B^{\frac{d}{2}+1}_{1,1}}$$

with $B^{\frac{d}{2}+1}_{1,1}$ defined similarly to (1.3.3) with the obvious modifications. The function $w_m(t,x) = e^{it\sqrt{m-\Delta_x}} f$ satisfies the equation $\partial_{tt} w_m - \Delta_x w_m + m w_m = 0$ with $w(0,x) = f(\sqrt{m}x) := f_m$, with $w_m := w(\sqrt{m}t, \sqrt{m}x)$ and w satisfying $\partial_{tt} w - \Delta_x w + w = 0$ with $w(0,x) = f(x)$. We use a scaling argument to deduce an estimate for f_m, noticing that for $m \geq 1$, the Besov norm of a rescaled function can be bounded by:

$$\|f_m\|_{B^{\frac{d}{2}+1}_{1,1}} \leq m^{\frac{d+2}{4}} \|f\|_{B^{\frac{d}{2}+1}_{1,1}},$$

giving the following estimate with $m = 1 + \lambda_j > 1$

$$\begin{aligned}
\left\| e^{it\sqrt{1-\Delta_{x,y}}} f \right\|_{L^\infty(\mathbb{R}^d \times \mathbb{T})} &\leq C|t|^{-\frac{d}{2}} \sum_{j\in\mathbb{N}} \sqrt{1+\lambda_j} \|f_j\|_{B^{\frac{d}{2}+1}_{1,1}} \|\Phi_j(y)\|_{L^\infty_y} \\
&= C|t|^{-\frac{d}{2}} \sum_{j\in\mathbb{N}} (1+\lambda_j)^{d+1} (1+\lambda_j)^{-d-1/2} \|f_j\|_{B^{\frac{d}{2}+1}_{1,1}} \|\Phi_j(y)\|_{L^\infty_y} \\
&\lesssim |t|^{-\frac{d}{2}} \sum_{j\in\mathbb{N}} (1+\lambda_j)^{d+1} \|f_j\|_{B^{\frac{d}{2}+1}_{1,1}} \|\Phi_j(y)\|_{L^\infty_y}.
\end{aligned}$$

Noticing that the righthand side can be expressed a term involving derivatives in (x,y), one can find $N \in \mathbb{N}$ large enough to have

$$\left\| e^{it\sqrt{1-\Delta_{x,y}}} f \right\|_{L^\infty(\mathbb{R}^d \times \mathbb{T})} \leq C|t|^{-\frac{d}{2}} \|f\|_{W^{N,1}(\mathbb{R}^d \times \mathbb{T})}. \tag{1.4.1}$$

1.5 Strichartz estimates for the Zakharov system

About the Zakharov system

$$\begin{cases} i\partial_t u - \omega \nabla \times \nabla \times u + \nabla(\operatorname{div} u) = nu \\ \frac{1}{c_s^2} \partial_{tt} n - \Delta n = \Delta |u|^2 \end{cases}, \tag{1.5.1}$$

let us remark since now that for the subsequent analysis Strichartz estimates for the wave equation will not be used, being necessary only the energy estimates for the wave unknown. Let us therefore consider the free propagator related to the (linear) Schrödinger-type equation of (1.5.1), namely

$$i\partial_t u = \omega \nabla \times \nabla \times u - \nabla \operatorname{div} u. \tag{1.5.2}$$

Lemma 1.5.1. *Let u solve (1.5.2) with initial datum $u(0) = u_0$, then*

$$u(t) = \mathcal{Z}(t) u_0 = [U(\omega t)\mathbf{P} + U(t)\mathbf{Q}] u_0, \tag{1.5.3}$$

where $U(t) = e^{it\Delta}$ is the Schrödinger evolution operator, $\mathbf{Q} := -(-\Delta)^{-1}\nabla \operatorname{div}$ and $\mathbf{P} := \mathbf{1} - \mathbf{Q}$.

Proof. By taking the Fourier transform of (1.5.2) we have

$$\begin{aligned} i\partial_t \hat{u} &= -\omega\xi \times \xi \times \hat{u} + \xi(\xi \cdot \hat{u}) \\ &= |\xi|^2 \left(\omega \hat{\mathbf{P}}(\xi) + \hat{\mathbf{Q}}(\xi)\right) \hat{u}(\xi), \end{aligned}$$

where $\hat{\mathbf{P}}(\xi), \hat{\mathbf{Q}}(\xi)$ are two $(3 \times 3)-$matrices defined by $\hat{\mathbf{Q}}(\xi) = \frac{\xi \otimes \xi}{|\xi|^2}$, $\hat{\mathbf{P}}(\xi) = \mathbf{1} - \hat{\mathbf{Q}}(\xi)$ where $\mathbf{1}$ is the identity matrix. Hence we may write

$$\hat{u}(t) = e^{-i\omega t|\xi|^2 \hat{\mathbf{P}}(\xi) - it|\xi|^2 \hat{\mathbf{Q}}(\xi)} \hat{u}_0(\xi).$$

It is straightforward to see that $\hat{\mathbf{Q}}(\xi)$ is a projection matrix, $0 \leq \hat{\mathbf{Q}}(\xi) \leq 1$, $\hat{\mathbf{Q}}(\xi) = \hat{\mathbf{Q}}^2(\xi)$, hence $\hat{\mathbf{P}}(\xi)$ is its orthogonal projection. Consequently we have

$$\begin{aligned} \hat{u}(t) &= e^{-i\omega t|\xi|^2 \hat{\mathbf{P}}(\xi)} e^{-it|\xi|^2 \hat{\mathbf{Q}}(\xi)} \hat{u}_0(\xi) \\ &= \left(e^{-i\omega t|\xi|^2} \hat{\mathbf{P}}(\xi) + \hat{\mathbf{Q}}(\xi)\right) \left(e^{-it|\xi|^2} \hat{\mathbf{Q}}(\xi) + \hat{\mathbf{P}}(\xi)\right) \hat{u}_0(\xi) \\ &= \left(e^{-i\omega t|\xi|^2} \hat{\mathbf{P}}(\xi) + e^{-it|\xi|^2} \hat{\mathbf{Q}}(\xi)\right) \hat{u}_0(\xi). \end{aligned}$$

By taking the inverse Fourier transform we find (1.5.3). □

By the dispersive estimates for the standard Schrödinger evolution operator of the previous section, see (1.1.3), we have

$$\begin{aligned} \|U(t)\mathbf{Q}f\|_{L^p} &\lesssim |t|^{-3\left(\frac{1}{2}-\frac{1}{p}\right)} \|\mathbf{Q}f\|_{L^{p'}}, \\ \|U(\omega t)\mathbf{P}f\|_{L^p} &\lesssim |\omega t|^{-3\left(\frac{1}{2}-\frac{1}{p}\right)} \|\mathbf{P}f\|_{L^{p'}}, \end{aligned} \tag{1.5.4}$$

for any $2 \leq p \leq \infty$, $t \neq 0$. These two estimates together give

$$\|\mathcal{Z}(t)f\|_{L^p} \lesssim |t|^{-3\left(\frac{1}{2}-\frac{1}{p}\right)} \|f\|_{L^{p'}},$$

for $2 \leq p < \infty$. Let us notice that the dispersive estimate for $p = \infty$ does not hold for $\mathcal{Z}(t)$ anymore because the projection operators $\mathbf{Q}, \mathbf{P}$ are not bounded from L^1 into itself (the symbol associated to $\mathbf{Q}$ is basically the same of the one defining the Riesz Transform; it is a well-known property that the Riesz Transform does not map L^1 into itself). Nevertheless by using the dispersive estimates in (1.5.4) and the result in [76] we infer the whole set of Strichartz estimates for the irrotational and solenoidal part, separately. By summing them up we thus find the Strichartz estimates for the propagator in (1.5.3).

Lemma 1.5.2. *Let (q, r), (γ, ρ) be two arbitrary admissible pairs (in the sense of Definition 1.1.4 with $d = 3$) and let $\omega \geq 1$, then we have*

$$\begin{aligned} \|U(\omega t)\mathbf{P}f\|_{L^q(I;L^r)} &\leq C\omega^{-\frac{2}{q}} \|f\|_{L^2}, \\ \left\| \int_0^t U(\omega(t-s))\mathbf{P}F(s)\, ds \right\|_{L^q(I;L^r)} &\leq C\omega^{-\left(\frac{1}{q}+\frac{1}{\gamma}\right)} \|F\|_{L^{\gamma'}(I;L^{\rho'})}, \end{aligned} \tag{1.5.5}$$

and

$$\|U(t)\mathbf{Q}f\|_{L^q(I;L^r)} \le C\|f\|_{L^2},$$

$$\left\|\int_0^t U(t-s)\mathbf{Q}F(s)\,ds\right\|_{L^q(I;L^r)} \le C\|F\|_{L^{\gamma'}(I;L^{\rho'})}.$$

Therefore the linear propagator $\mathcal{Z}(t)$ satisfies

$$\|\mathcal{Z}(t)g\|_{L^q(I;L^r)} \le C\|f\|_{L^2}, \tag{1.5.6}$$

$$\left\|\int_0^t \mathcal{Z}(t-s)F(s)\,ds\right\|_{L^q(I;L^r)} \le C\|F\|_{L^{\gamma'}(I;L^{\rho'})}. \tag{1.5.7}$$

Remark 1.5.3. From the estimates in the Lemma above it is already straightforward that, at least in the linear evolution, we can separate the fast and slow dynamics and that the fast one is asymptotically vanishing. This is somehow similar to what happens with rapidly varying dispersion management, see for example Antonelli, Saut and Sparber [2].

Remark 1.5.4. Let us notice that the constants in (1.5.6) and (1.5.7) are uniformly bounded for $\omega \ge 1$. This is straightforward but it is a necessary remark to infer that the existence time in the local well-posedness result of Section 5.2 is uniformly bounded from below for any $\omega \ge 1$.

Chapter 2

The Concentration/Compactness & Rigidity scheme

We now introduce the Concentration/Compactness & Rigidity method, first developed by Kenig and Merle in their famous papers on energy critical radial NLS equation [77] and energy critical radial nonlinear wave (NLW) equation [78]. This method in the last ten years have had a tremendous impact on the fields of dispersive PDEs, with an enormous production of mathematical results in the study of global well-posedness and scattering for such kind of equations. See below for a (not exhaustive) list of references.

The Kenig and Merle approach can be view as a version of the *induction on the energy* developed by Bourgain in [10] to treat the large data problem for the defocusing energy critical radial NLS on $\mathbb{R}^3$. The Kenig and Merle strategy actually applies for both defocusing and focusing problems, and also for non-critical equations aside from the critical ones for which the method was developed at first instance. Refinements of the method led to the removal of the constraint of radial solutions along with the restriction to low dimensions. We illustrate now how the scheme (which is a indirect method) proceeds; since it is very general in its plan, we illustrate it in the case of the intra-critical NLS with constant coefficients in arbitrary dimension, borrowing from [111]. For a sketch of the strategy we also refer to [42, Section 2], while for a more complete (and historical) review on the maturation of the method developed by Kenig and Merle we refer to [108]. We further emphasize the main differences between this class of equations and the NLS equation perturbed with steplike potentials, which is the subject of study of the next chapter. This strategy will be also adopted for the NLKG equation on waveguide in chapter 4.

2.1 The Kenig and Merle road map

Consider the Cauchy problems

$$\begin{cases} i\partial_t u + \Delta u = \pm|u|^\alpha u, & (t,x) \in \mathbb{R}\times\mathbb{R}^d \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases} \tag{2.1.1}$$

(the sign $+$ corresponding to the defocusing case while $-$ to the focusing one) with

$$\alpha \in \begin{cases} \left(\frac{4}{d}, \frac{4}{d-2}\right) & \text{if} \quad d \geq 3 \\ \left(\frac{4}{d}, \infty\right) & \text{if} \quad d \leq 2 \end{cases}.$$

Solutions to (2.1.1) conserve the L^2-norm and the energy, defined as

$$E(u(t)) = \frac{1}{2}\left(\int_{\mathbb{R}^d} |\nabla u(t)|^2 \pm \frac{2}{\alpha+2}|u(t)|^{\alpha+2}\,dx\right). \tag{2.1.2}$$

The method we are going to illustrate allows to establish the free dynamics property for solutions to (2.1.1). We recall the definition of scattering (already introduced in the first chapter for the equations (0.0.1) and (0.0.2)) in the context of (2.1.1). For this purpose it is considered the linear problem associated to (2.1.1) which reads as follows:

$$\begin{cases} i\partial_t v + \Delta v = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^d \\ v(0) = v_0 \in H^1(\mathbb{R}^d) \end{cases}. \tag{2.1.3}$$

Definition 2.1.1. Given a global solution $u \in \mathcal{C}(\mathbb{R}; H^1)$ to (2.1.1), we say that it scatters if for large times it behaves like a solution to (2.1.3). More rigorously, u scatters if for $t \to \pm\infty$ there exist $v_0^\pm$, respectively, such that

$$\lim_{t\to\pm\infty} \|u(t) - v^\pm(t)\|_{H^1} = 0$$

where $v^\pm(t)$ solve the Cauchy problems (2.1.3) with $v_0^\pm$ as initial data, respectively.

For such kind of problems we can separate the situation in two cases:

1. small data theory: if $\|u_0\|_{H^1}$ is "small" then global well-posedness of the problems is quite simple to establish as well as the long time dynamics, which is free (in the sense that solutions scatter), no matter on the sing in front of the nonlinearity;

2. large data theory: in the defocusing case, in the energy sub-critical regime, once the local well-posedness is established, the sub-criticality conditions enable to control the energy norm by means of Sobolev embedding and since the local time of existence depends only on the size of the initial datum, then it is possible to extend globally in time the solution by time-stepping. This is no more the case for the energy critical problem since the time of existence depends on the initial

profile and not only on its size. For the focusing problems, it can be proved in a quite simple way, see the Glassey's argument in [57], that some initial data lead to blowing-up solutions in finite time. But in general, if we assume that we are dealing with global solutions, both in the defocusing and focusing regimes, the dynamics at large times is not simple to establish, since a perturbative argument is no more exploitable.

The Kenig and Merle road map therefore is a tool which enables us to attack the large data problems. It is based, as mentioned in the Introduction, on four main points. For sake of clarity, let us consider since now on the defocusing problems.

Remark 2.1.2. As established by Cazenave and Weissler (see [23]) a sufficient condition to have scattering for a global solution $u(t,x)$ to (2.1.1) is that $u \in L^pL^r$, with (p,r) defined as in (1.2.5). Such condition implies that u belongs to any $L^\gamma L^\rho$ with (γ,ρ) Strichartz admissible pair. This is enough to show that $e^{-it\Delta}u(t)$ is a Cauchy sequence in H^1, and this a necessary and sufficient condition to have the scattering property. Therefore the Kenig and Merle method aims to show that for any initial datum, a solution to (2.1.1) has finite L^pL^r-norm.

The four steps are in order:

1. *Small data theory*, which is "guaranteed" by perturbative argument, see Lemma 3.2.2 and Theorem 4.2.1 for NLS and NLKG, respectively;

2. *Construction of a "critical solution"*: from the previous point, it is known that small initial data, and consequently small energies lead to global and scattering solutions. The method goes on now by exploiting a reasoning *by the absurd:* if there exists a global non-scattering solution, then there exists a minimal energy $E_c > 0$ (boundedness away from zero is guaranteed by the *Step 1*) for which the free dynamics fails to hold true. This minimal energy is defined as follows:

$$E_c = \sup\Big\{E > 0 \textit{ such that if } u_0 \in H^1 \textit{ with } E(u_0) < E \\ \textit{then the solution of } (2.1.1) \textit{ with initial data } u_0 \textit{ is in } L^pL^r\Big\}.$$

   Then the first non-trivial step of the method is the construction of a "critical solution" u_c such that $E(u_c) = E_c$, and which is global and non-scattering. Therefore E_c is not a maximum and it is the smallest energy such that the solutions having such energy do not scatter;

3. *Compactness of the flow*: the minimality of E_c has another fundamental consequence: the critical solution built in the previous step is precompact modulo the symmetries of the equation in the energy space. This means that $\{u_c(t)\}_{t\in\mathbb{R}^+}$ (up to symmetries) is precompact in H^1;

4. *Rigidity*: the last step, which is the most nonlinear ingredient of the method, is the exclusion of solutions enjoying the compactness property of the previous point. This is done by means of Liouville-type theorems and in general relies on virial

identities. It shows that $u_c(t) \equiv 0$, then one concludes with $E_c = \infty$ and therefore scattering holds for any initial datum.

Remark 2.1.3. In the subsequent chapters, the rigidity part will be proved by the 1D Nanakishi/Morawetz estimates. It is worth highlighting since right now that for NLS perturbed with a potential, as in the situation of chapter 3, the idea to use a 1D tool allows us to work with potentials which are repulsive in only one direction, although working in a multidimensional setting. The novelty is that in the previous literature, as far as we know, only repulsive potentials with respect to the full set of variables had been considered.

Let us discuss briefly the points above. As said, the first nontrivial point is the *Step 2*, and it is based on the Concentration/Compactness method. The Concentration/Compactness procedure appears in the Calculus of Variation with the seminal works of Lions [88, 89]; there the author analyzes minimization problems on unbounded domains. Due to the invariance of such domains under the action of non-compact groups of transformation, for example dilations and translation on $\mathbb{R}^d$, some loss of compactness arises. Other early works in this direction are due to Brezis and Coron, see [14], Struwe, see [116], Lieb, see [87], to quote some of them. The precise description of the loss of compactness in the Sobolev embedding is given in the work of Gérard [46]. When dealing with critical dispersive equations, the Concentration/Compactness method appears in the works of Bahouri and Gérard for the (critical) wave equation, see [5, 47], and for the (critical) Schrödinger equation in the papers by Merle and Vega, see [96], Keraani, see [80], Hmidi and Keraani, see [66], where the authors describe the loss of compactness in the Strichartz estimates. How to concern extensions to subcritical cases, see the works by Holmer and Roudenko [67], Duykaerts, Holmer and Roudenko, [40], Fanelli and Visciglia [41], Cazenave, Fang, Xie [42], to quote some of them. The Profile Decomposition Theorem needed for (2.1.1) is given in [42] and is as follows.

Theorem 2.1.4. *Let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^d)$ be a bounded sequence. There exist $(t_n^j, x_n^j) \subset \mathbb{R}\times\mathbb{R}^d$ and sequences of profiles $\{\psi_j^n\}_{n,j\in\mathbb{N}} \subset H^1$ such that, up to subsequences, one can write*

$$u_n = \sum_{1\le j\le J} e^{it_n^j\Delta}\tau_{x_n^j}\psi^j + R_n^J, \qquad \forall\, J \in \mathbb{N}. \tag{2.1.4}$$

Furthermore:

- *(dichotomy of the parameters) for any fixed j we have:*

$$\begin{aligned} &\text{either} \quad t_n^j = 0 \quad \forall\, n\in\mathbb{N} \quad \text{or} \quad t_n^j \overset{n\to\infty}{\longrightarrow} \pm\infty, \\ &\text{either} \quad x_n^j = 0 \quad \forall\, n\in\mathbb{N} \quad \text{or} \quad |x_n^j| \overset{n\to\infty}{\longrightarrow} \pm\infty; \end{aligned}$$

- *(orthogonality of the parameters) for any $j \neq k$*

$$|x_n^j - x_n^k| + |t_n^j - t_n^k| \overset{n\to\infty}{\longrightarrow} \infty;$$

- *(smallness of the remainder)* $\forall\, \varepsilon > 0 \quad \exists\, J = J(\varepsilon)$ *such that*

$$\limsup_{n\to\infty} \|e^{it\Delta} R_n^J\|_{L^p L^r} \leq \varepsilon,$$

(see (1.2.5) *for the definition of* (p, r)*);*

- *(Pythagorean expansion of the Sobolev norms) for any* $0 \leq s \leq 1$*, as* $n \to \infty$*,*

$$\|u_n\|_{\dot H^s}^2 = \sum_{1\leq j\leq J} \|\psi^j\|_{\dot H^s}^2 + \|R_n^J\|_{\dot H^s}^2 + o(1), \quad \forall\, J \in \mathbb{N}; \tag{2.1.5}$$

- *(Pythagorean expansion of the potential energy)* $\forall\, J \in \mathbb{N}$ *and* $\forall\, 2 < q < 2^*$ *we have, as* $n \to \infty$*,*

$$\|u_n\|_{L^q}^q = \sum_{1\leq j\leq J} \|e^{it_n^j\Delta}\psi^j\|_{L^q}^q + \|R_n^J\|_{L^q}^q + o(1);$$

- *(Pythagorean expansion of the energy) with* $E(u)$ *defined as* (2.1.2) *(with the* $+$ *sign), we have, as* $n \to \infty$*,*

$$E(u_n) = \sum_{1\leq j\leq J} E(e^{it_n^j\Delta}\psi^j) + E(R_n^J) + o(1), \quad \forall\, J \in \mathbb{N}. \tag{2.1.6}$$

Once a similar theorem is proved, the key consideration in the Kenig and Merle scheme is that due to the minimality of E_c, only one nontrivial profile exists in the decomposition (2.1.4). More precisely:

Step 1. Consider a minimizing sequence of initial data $\{u_n(0)\}_{n\in\mathbb{N}}$ with $E(u_n(0)) \to E_c$ (which is strictly away from zero, since small data scattering holds) and such that the corresponding solutions $\{u_n(t)\}_{n\in\mathbb{N}}$ to (2.1.1) satisfy $\|u_n\|_{L^pL^r} \to \infty$ as $n \to \infty$, one employs the Concentration/Compactness decomposition to $\{u_n(0)\}_{n\in\mathbb{N}}$ and assumes that there exists two nontrivial profiles, namely $J \geq 2$ in (2.1.4).

Step 2. Assume for example that the first two profiles are nontrivial, namely the ones indexed by $j = 1, 2$. Due to the energy-norm orthogonal expansion both of them have energy less than E_c. Due to the dichotomy condition on the parameters, by the local well-posedness theory at $t = 0$ and $t = \pm\infty$ is it possible to associate to these two linear solutions two nonlinear profiles, say V^1, V^2 : here V^1, V^2 are solutions to (2.1.1) with suitable initial data such that

$$\|e^{-t_n^j\Delta}\psi^j - V^j(t_n^j)\| \overset{n\to\infty}{\longrightarrow} 0 \qquad j = 1, 2.$$

Furthermore, from the Pythagorean expansion of the energy (2.1.6) one can claim that both $E(V^1) < E_c$ and $E(V^2) < E_c$, hence they scatter.

Step 3. By considering a large $J > 1$ such that the remainder in (2.1.1) is small enough (this smallness given by the small data theory), a suitable perturbation result implies that the error term can be absorbed in the nonlinear profiles, leading to the uniform estimate

$$\sup_{n\in\mathbb{N}} \|u_n\|_{L^pL^r} \leq C < \infty.$$

But this is a contradiction with respect to the assumption on $\{u_n\}_{n\in\mathbb{N}}$, therefore $J = 1$, hence the compactness (up to subsequences) of $\{u_n(0)\}_{n\in\mathbb{N}}$, then $E_c = E(V^1)$ and the critical solution is given by $u_c(t) = V^1(t)$.

Step 4. By considering $\{u_c(t_n)\}_{n\in\mathbb{N}}$, repetition of *Step 3* on this sequence yields to the precompactness of the critical element modulo the symmetries of the equation.

Step 5. The rigidity part (i.e. the proof that $u_c(t) \equiv 0$) is given by means of localized virial identities, by exploiting concentration of the energy and a convexity argument. Also a priori uniform bounds like Morawetz estimates can be used to conclude the argument. It is worth mentioning that such kind of estimates are no more exploitable in the context of focusing problems and furthermore that for our next results we will use only one-dimensional version of them, see Remark 2.1.3.

2.2 Remarks

This conclusive section contains remarks on the Concentration/Compactness & Rigidity method. Moreover some differences with respect to the NLS with non-constant coefficients (3.1.7) are highlighted, along with some features arising in the NLKG (4.0.1) which do not appear in the context of the Schrödinger equation. The following are in order.

Remark 2.2.1 (Focusing case). The method illustrated above also applies to the focusing equation. But it is well-known that in this situation the dynamics of the solutions is richer: presence of possible blowing-up solutions in finite time, solitons, stationary solutions... We refer to [42, Introduction] for more comments of the focusing case.

Remark 2.2.2 (Critical problems). Aside from the defocusing/focusing regimes, also critical problem can be attacked with the method. The (radial) energy critical Schrödinger and wave equations are exactly the contents of the Kenig and Merle early papers [77, 78].

Remark 2.2.3 (Symmetries). As in the variational problems, the lack of compactness comes from the symmetries of the problems. We observe that space-time translations left the equation invariant and they do appear in the Profile Decomposition Theorem. If we consider radial problems, then only time translations appear, since the problem is spherically symmetric and the group of rotation is compact, therefore if we consider also rotations, up to subsequence they can be absorbed in the profiles. For critical problems, furthermore, there is also the scaling invariance. For such problems, in the Profile Decomposition Theorem a sequence of scaling parameters appears, satisfying, as the translation sequences, some orthogonality conditions.

Remark 2.2.4 (Alternative proof of scattering in defocusing case). Let us point out that in the energy sub-critical, defocusing regime, this method provides an alternative proof of the already known scattering results of Morawetz [97] and Morawetz and Strauss [98] in $\mathbb{R}^d$ for $d \geq 3$ and Nakanishi [100] for the low dimensional cases $d = 1, 2$. This remark also applies in the energy critical case: the method provides a different proof of the scattering in $\mathbb{R}^3$ with respect to the results of Bourgain in [10].

We now give some comments on the differences between the Schrödinger equation (2.1.1) and the one studied in chapter 3.

Remark 2.2.5. The Profile Decomposition Theorem related to (3.1.7) with steplike, partially period coefficients, takes into account the fact that the equation is no more invariant under space translations. This is reflected in the dichotomy condition on the space sequences, as well as in the the Pythagorean expansion of the energy. See Theorem 3.3.1.

Remark 2.2.6. The rigidity step is proved by using only 1D tools, despite the multi-dimensional setting where the equation is posed. These are the Nakanishi/Morawetz estimates found by Nakanishi in [100] to prove scattering in low dimensional $(d = 1, 2)$ cases for (2.1.1) (defocusing case). This kind on tool is also used fro the rigidity result about NLKG in chapter 4: it is worth mentioning that in this situation a fundamental property of the Klein-Gordon equation which does not hold for NLS is the so-called "finite propagation speed".

Remark 2.2.7. About the two previous remarks, it is worth mentioning that while the the Rigidity step is related to the invariance of the equation under the action of some non-compact groups of the Poincaré Group, the Profile Decomposition Theorem is an abstract result, somehow classic, now. Recently Banica and Visciglia, see [6], gave an abstract version of it, needed by the authors to prove by scattering for the mass supercritical NLS on the line perturbed by a repulsive δ-interaction. Since we will work with a NLS equation with non constant coefficients, we will give, borrowing from [6], a detailed version of the Concentration/Compactness Theorem in chapter 3, fitting the structure of our considered equation. The same strategy will be adopted in chapter 4 in the context of Klein-Gordon equation posed on flat waveguide.

Remark 2.2.8. In the construction of the critical element, due to the different behaviors at infinity of the potential, the association of nonlinear profiles to the linear ones is more involved. See Claim 3.5.2.

Beside the already mentioned papers (and those that will be cited in the next chapters) where the Concentration/Compactness & Rigidity method is used, it is worth mentioning that this road map found a very large range of applicability, yielding for example to solutions for the mass-critical problems about NLS in the works of Dodson, see [36–39] or in the fields of wave maps, see the work of Krieger and Schlag [86], Schrödinger maps, see Bejenaru, Ionescu, Kenig and Tataru [8] and also for energy supercritical problems, see for example Bulut [16] or Killip and Visan [83, 84].

Chapter 3

Scattering for a class of NLS with a steplike potential

This chapter is devoted to the analysis of the behavior for large times of solutions to the following 1D Cauchy problems

$$\begin{cases} i\partial_t u + \partial_x^2 u - Vu = |u|^\alpha u, & (t,x) \in \mathbb{R} \times \mathbb{R}, \quad \alpha > 4 \\ u(0) = u_0 \in H^1(\mathbb{R}) \end{cases}, \tag{3.0.1}$$

namely we treat the L^2-supercritical defocusing power nonlinearities, and $V : \mathbb{R} \to \mathbb{R}$ is a real time-independent steplike potential. More precisely we assume that $V(x)$ has two different asymptotic behaviors at $\pm\infty$:

$$a_+ = \lim_{x\to+\infty} V(x) \neq \lim_{x\to-\infty} V(x) = a_-. \tag{3.0.2}$$

In order to simplify the presentation we shall assume in our treatment

$$a_+ = 1 \quad \text{and} \quad a_- = 0,$$

but of course the arguments and the results below can be extended to the general case $a_+ \neq a_-$. Furthermore, we will give a suitable extension in a multidimensional framework, by considering potential which are of steplike type along one direction and partially periodic in the remaining ones (see (3.1.7) below).

3.1 Motivations and main results

We recall that in physics literature the steplike potentials are called *barrier potentials* and are very useful to study the interactions of particles with the boundary of a solid (see Gesztesy [48] and Gesztesy, Nowell and Pötz [49] for more details). We also mention the paper by Davies and Simon [35] where, in between other results, it is studied via the twisting trick the long time behavior of solutions to the propagator $e^{it(\partial_x^2 - V)}$, where $V(x)$ is steplike (see below for more details on the definition of the double scattering channels).

For a more complete list of references devoted to the analysis of steplike potentials we refer to [34]. Nevertheless, at the best of our knowledge, no results are available about the long time behavior of solutions to nonlinear Cauchy problem (3.0.1) with a steplike potential.

Roughly speaking the Cauchy problem (3.0.1) looks like the following Cauchy problems respectively for $x >> 0$ and $x << 0$:

$$\begin{cases} i\partial_t v + \partial_x^2 v = |v|^\alpha v \\ v(0) = v_0 \in H^1(\mathbb{R}) \end{cases} \tag{3.1.1}$$

and

$$\begin{cases} i\partial_t v + (\partial_x^2 - 1)v = |v|^\alpha v \\ v(0) = v_0 \in H^1(\mathbb{R}) \end{cases}. \tag{3.1.2}$$

We recall that in 1D (and also in the $2D$ case) the long time behavior of solutions to (3.1.1) (and also to (3.1.2)) has been first obtained in the work by Nakanishi (see [100]), who proved that the solutions to (3.1.1) (and also (3.1.2)) scatter to a free wave in $H^1(\mathbb{R})$ (see Definition 3.1.4 for a precise definition of scattering from nonlinear to linear solutions in a general framework). The Nakanishi argument is a combination of the induction on the energy in conjunction with a suitable version of Morawetz inequalities with time-dependent weights. Alternative proofs based on the use of the interaction Morawetz estimates, first introduced in [28], have been obtained later (see for example Colliander, Grillakis and Tzirakis [26], Colliander, Holmer, Visan and Zhang [27], Planchon and Vega [107], Visciglia [130] and the references therein).
As far as we know, there are not results available in the literature about the long time behavior of solutions to NLS perturbed by a steplike potential, and this is the main motivation of this work.

It is worth mentioning that in 1D, we can rely on the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Hence it is straightforward to show that the Cauchy problem (3.0.1) is locally well posed in the energy space $H^1(\mathbb{R})$. For higher dimensions the local well-posedness theory is still well known, see for example Cazenave's monograph [20], once the good dispersive properties of the linear flow are established. Moreover, thanks to the defocusing character of the nonlinearity, we can rely (as already explained in the Introduction) on the conservation of the mass and of the energy below, valid in any dimension: for any $t \in \mathbb{R}$

$$\|u(t)\|_{L^2(\mathbb{R}^d)} = \|u(0)\|_{L^2(\mathbb{R}^d)} \tag{3.1.3}$$

and

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}^d} \left(|\nabla u(t)|^2 + V|u(t)|^2 + \frac{2}{\alpha+2}|u(t)|^{\alpha+2} \right) dx = E(u(0)) \tag{3.1.4}$$

in order to deduce that the solutions are global. Hence for any initial datum $u_0 \in H^1(\mathbb{R}^d)$ there exists one unique global solution $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ to (3.0.1) for $d=1$ and to (3.1.7) below in higher dimension.

At the best of our knowledge, the unique paper where the dispersive properties of the corresponding 1D linear flow perturbed by a steplike potential $V(x)$ have been analyzed is [34], where the $L^1 - L^\infty$ decay estimate in 1D is proved:

$$\|e^{it(\partial_x^2 - V)} f\|_{L^\infty(\mathbb{R})} \lesssim |t|^{-1/2} \|f\|_{L^1(\mathbb{R})}, \quad \forall\, t \neq 0, \quad \forall\, f \in L^1(\mathbb{R}). \tag{3.1.5}$$

We point out again that beside the different spatial behavior of $V(x)$ on left and on right of the line, other assumptions must be satisfied by the potential, see comments contained in Section 1.2.

Nevertheless we shall not focus on them since our main point is to show how to go from (3.1.5) to the analysis of the long time behavior of solutions to (3.0.1). We will assume therefore as black-box the dispersive relation (3.1.5) and its multidimensional version

$$\|e^{it(\Delta - V)} f\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2} \|f\|_{L^1(\mathbb{R}^d)}, \quad \forall\, t \neq 0 \quad \forall\, f \in L^1(\mathbb{R}^d). \tag{3.1.6}$$

Our first aim is to provide a nonlinear version of the *double scattering channels* in 1D that has been established in the literature in the linear context, see [35].

Definition 3.1.1. Let $u_0 \in H^1(\mathbb{R})$ be given and $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}))$ be the unique global solution to (3.0.1) with $V(x)$ that satisfies (3.0.2) with $a_- = 0$ and $a_+ = 1$. Then we say that $u(t,x)$ satisfies the *double scattering channels* provided that

$$\lim_{t \to \pm\infty} \left\| u(t,x) - e^{it\partial_x^2} \eta_\pm - e^{it(\partial_x^2 - 1)} \gamma_\pm \right\|_{H^1(\mathbb{R})} = 0,$$

for suitable $\eta_\pm, \gamma_\pm \in H^1(\mathbb{R})$.

We can now state our first result in 1D.

Theorem 3.1.2. *Assume that $V : \mathbb{R} \to \mathbb{R}$ is a bounded, nonnegative potential satisfying (3.0.2) with $a_- = 0$ and $a_+ = 1$, and (3.1.5). Furthermore, suppose that:*

- $|\partial_x V(x)| \overset{|x| \to \infty}{\longrightarrow} 0$*;*
- $\lim_{x \to +\infty} |x|^{1+\varepsilon} |V(x) - 1| = 0$, $\lim_{x \to -\infty} |x|^{1+\varepsilon} |V(x)| = 0$ *for some* $\varepsilon > 0$*;*
- $x \cdot \partial_x V(x) \leq 0$.

Then for every $u_0 \in H^1(\mathbb{R})$ the corresponding unique solution $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}))$ to (3.0.1) satisfies the double scattering channels *(according to Definition 3.1.1).*

Remark 3.1.3. It is worth mentioning that the assumption (3.1.5) it may look somehow quite strong. However we emphasize that the knowledge of the estimate (3.1.5) provides for free informations on the long time behavior of nonlinear solutions for small data, but in general it is more complicated to deal with large data, as it is the case in Theorem 3.1.2. For instance consider the case of 1D NLS perturbed by a periodic potential. In this situation it has been established in the literature the validity of the dispersive estimate for the linear propagator, see the work by Cuccagna [29], and also the small data nonlinear scattering, see Cuccagna and Visciglia [31]. However, at the best of our knowledge, it is unclear how to deal with the large data scattering.

The proof of Theorem 3.1.2 goes in two steps. The first one is to show that solutions to (3.0.1) scatter to solutions of the linear problem and actually we do that for general space dimensions (see Definition 3.1.4 for a rigorous definition of scattering in a general framework); the second one is the asymptotic description of solutions to the linear problem associated with (3.0.1) (hence the 1D case) in the energy space H^1 (see Theorem 3.1.8). Concerning the first step we use the technique of Concentration/Compactness & Rigidity pioneered by Kenig and Merle in [77, 78], introduced in chapter 2. Since this argument is rather general, we shall present it in a more general higher dimensional setting.
More precisely in higher dimension we consider the following family of NLS

$$\begin{cases} i\partial_t u + \Delta u - Vu = |u|^\alpha u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases}, \tag{3.1.7}$$

where

$$\alpha \in \begin{cases} \left(\frac{4}{d}, \frac{4}{d-2}\right) & \text{if} \quad d \geq 3 \\ \left(\frac{4}{d}, \infty\right) & \text{if} \quad d \leq 2 \end{cases}.$$

The potential $V(x)$ is assumed to satisfy, uniformly in $\bar{x} \in \mathbb{R}^{d-1}$,

$$a_- = \lim_{x_1 \to -\infty} V(x_1, \bar{x}) \neq \lim_{x_1 \to +\infty} V(x_1, \bar{x}) = a_+, \qquad \text{where} \qquad x = (x_1, \bar{x}). \tag{3.1.8}$$

Moreover we assume $V(x)$ periodic with respect to the variables $\bar{x} = (x_2, \ldots, x_d)$. Namely we assume the existence of $d-1$ linear independent vectors $P_2, \ldots, P_d \in \mathbb{R}^{d-1}$ such that for any fixed $x_1 \in \mathbb{R}$, the following holds:

$$\begin{gathered} V(x_1, \bar{x}) = V(x_1, \bar{x} + k_2 P_2 + \cdots + k_d P_d), \\ \forall\, \bar{x} = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}, \quad \forall\, (k_2, \ldots, k_d) \in \mathbb{Z}^{d-1}. \end{gathered} \tag{3.1.9}$$

Some comments about this choice of assumptions on $V(x)$ are given in Remark 3.1.6 along with Remark 1.2.2.

Next we recall the classical definition of scattering from nonlinear to linear solutions in a general setting. We recall that by classical arguments we have that once (3.1.6) is granted, then the local (and also the global, since the equation is defocusing) existence and uniqueness of solutions to (3.1.7) follow by standard arguments.

Definition 3.1.4. Let $u_0 \in H^1(\mathbb{R}^d)$ be given and $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ be the unique global solution to (3.1.7). Then we say that $u(t,x)$ *scatters to a linear solution* provided that

$$\lim_{t \to \pm\infty} \|u(t,x) - e^{it(\Delta - V)}\psi^\pm\|_{H^1(\mathbb{R}^d)} = 0$$

for suitable $\psi^\pm \in H^1(\mathbb{R}^d)$.

In the sequel we will also use the following auxiliary Cauchy problems that roughly speaking represent the Cauchy problems (3.1.7) in the regions $x_1 << 0$ and $x_1 >> 0$ (provide that we assume $a_- = 0$ and $a_+ = 1$ in (3.1.8)):

$$\begin{cases} i\partial_t u + \Delta u = |u|^\alpha u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0) = \psi \in H^1(\mathbb{R}^d) \end{cases} \tag{3.1.10}$$

and

$$\begin{cases} i\partial_t u + (\Delta - 1)u = |u|^\alpha u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0) = \psi \in H^1(\mathbb{R}^d) \end{cases}. \tag{3.1.11}$$

Notice that those problems are respectively the analogue of (3.1.1) and (3.1.2) in higher dimensional setting.

We can now state our main result about scattering from nonlinear to linear solutions in general dimension $d \geq 1$.

Theorem 3.1.5. *Let $V \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R})$ be a bounded, nonnegative potential which satisfies (3.1.8) with $a_- = 0$, $a_+ = 1$, (3.1.9) and assume moreover:*

- $|\nabla V(x_1, \bar{x})| \overset{|x_1| \to \infty}{\longrightarrow} 0$ *uniformly in* $\bar{x} \in \mathbb{R}^{d-1}$;

- *the decay estimate (3.1.6) is satisfied;*

- $x_1 \cdot \partial_{x_1} V(x) \leq 0$ *for any* $x = (x_1, \dots, x_d) \in \mathbb{R}^d$.

Then for every $u_0 \in H^1(\mathbb{R}^d)$ the unique corresponding global solution $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ to (3.1.7) scatters.

Remark 3.1.6. Next we comment about the assumptions done on the potential $V(x)$ along Theorem 3.1.5. Roughly speaking we assume that the potential $V(x_1, \dots, x_d)$ is steplike and repulsive with respect to x_1 and it is periodic with respect to $(x_2, \dots, x_d)$. The main motivation of this choice is that this situation is reminiscent, according with [35], of the higher dimensional version of the 1D double scattering channels mentioned above. Moreover we highlight the fact that the repulsivity of the potential in one unique direction is sufficient to get scattering, despite to other situations considered in the literature where repulsivity is assumed with respect to the full set of variables $(x_1, \dots, x_d)$. Another point is that along the proof of Theorem 3.1.5 we show how to deal with a partially periodic potential $V(x)$, despite to the fact that, at the best of our knowledge, the large data scattering for potentials periodic with respect to the full set of variables has not been established elsewhere, either in the 1D case (see Remark 3.1.3).

Remark 3.1.7. Next we discuss about the repulsivity assumption on $V(x)$. As pointed out in the paper by Hong, see [68], this assumption on the potential plays the same role of the convexity assumption for the obstacle problem studied by Killip, Visan and Zhang in [85]. The author highlights the fact that both strict convexity of the obstacle and the repulsivity of the potential prevent wave packets to refocus once they are reflected by the obstacle or by the potential. From a technical point of view the repulsivity assumption is done in order to control the right sign in the virial identities, and hence to conclude the rigidity part of the Kenig and Merle argument. In this work, since we assume repulsivity only in one direction we use a suitable version of the Nakanishi-Morawetz time-dependent estimates in order to get the rigidity part in the Kenig and Merle road map. Of course it is a challenging mathematical question to understand whether or not the repulsivity assumption (partial or global) on $V(x)$ is a necessary condition in order to get scattering.

When we specialize in 1D we are able to complete the theory of double scattering channels in the energy space. Therefore how to concern the linear part of our work, we give the following result, that in conjunction with Theorem 3.1.5 where we fix $d = 1$, provides the proof of Theorem 3.1.2.

Theorem 3.1.8. *Assume that $V(x) \in \mathcal{C}(\mathbb{R};\mathbb{R})$ satisfies the following space decay rate:*

$$\lim_{x\to+\infty} |x|^{1+\varepsilon}|V(x)-1| = \lim_{x\to-\infty} |x|^{1+\varepsilon}|V(x)| = 0 \quad \textit{for some} \quad \varepsilon > 0. \tag{3.1.12}$$

Then for every $\psi \in H^1(\mathbb{R})$ we have

$$\lim_{t\to\pm\infty} \left\| e^{it(\partial_x^2 - V)}\psi - e^{it\partial_x^2}\eta_\pm - e^{it(\partial_x^2-1)}\gamma_\pm \right\|_{H^1(\mathbb{R})} = 0$$

for suitable $\eta_\pm, \gamma_\pm \in H^1(\mathbb{R})$.

Notice that Theorem 3.1.8 is a purely linear statement. The main point (compared with other results in the literature) is that the asymptotic convergence is stated with respect to the H^1-topology and not with respect to the weaker L^2-topology. Indeed we point out that the content of Theorem 3.1.8 is well-known and has been proved in [35] in the L^2 setting. However, it seems natural to us to understand, in view of Theorem 3.1.5, whether or not the result can be extended in the H^1 setting. In fact according with Theorem 3.1.5 the asymptotic convergence of the nonlinear dynamic to linear dynamic occurs in the energy space and not only in L^2. As far as we know the issue of H^1 linear scattering has not been previously discussed in the literature, not even in the case of a potential which decays in both directions $\pm\infty$.

For this reason we have decided to state Theorem 3.1.8 as an independent result.

3.2 Small data theory and perturbative results

Along this section we assume that the estimate (3.1.6) is satisfied by the propagator associated with the potential $V(x)$. We do not need for the moment to assume the other assumptions done on $V(x)$.

We also specify that in the sequel the Lebesgue exponents p, r, q are the ones given in (1.2.5), but for reader's convenience we recall them:

$$r = \alpha + 2, \qquad p = \frac{2\alpha(\alpha+2)}{4-(d-2)\alpha}, \qquad q = \frac{2\alpha(\alpha+2)}{d\alpha^2+(d-2)\alpha-4}$$

Lemma 3.2.1. *Let $u_0 \in H^1$ and assume that the corresponding solution to (3.1.7) satisfies $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1) \cap L^pL^r$. Then $u(t,x)$* scatters *to a linear solution in H^1.*

Proof. It is a standard consequence of Strichartz estimates. □

Lemma 3.2.2. *There exists $\varepsilon_0 > 0$ such that for any $u_0 \in H^1$ with $\|u_0\|_{H^1} \le \varepsilon_0$, the solution $u(t,x)$ to the Cauchy problem (3.1.7)* scatters *to a linear solution in H^1.*

Proof. It is enough to prove that is $\|u_0\|_{H^1}$ is small enough, then $u \in L^pL^r$, hence the thesis follows from Lemma 3.2.1. By writing the solution in the Duhamel's formulation, by the Strichartz estimates above we get that, for any $T > 0$

$$\|u\|_{L^p((-T,T);L^r)} \lesssim \|u_0\|_{H^1} + \|u\|^{\alpha+1}_{L^p((-T,T);L^r)}.$$

Therefore a continuity argument gives that if $\|u_0\|_{H^1} << 1$ then

$$\sup_{T>0} \|u\|_{L^p((-T,T);L^r)} < \infty.$$

□

Lemma 3.2.3. *For every $M > 0$ there exist $\varepsilon = \varepsilon(M) > 0$ and $C = C(M) > 0$ such that: if $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1)$ is the unique global solution to (3.1.7) and $w \in \mathcal{C}(\mathbb{R}; H^1) \cap L^pL^r$ is a global solution to the perturbed problem*

$$\begin{cases} i\partial_t w + \Delta w - Vw = |w|^\alpha w + e(t,x) \\ \qquad\qquad w(0,x) = w_0 \in H^1 \end{cases}$$

satisfying the conditions $\|w\|_{L^pL^r} \le M$, $\left\|\int_0^t e^{i(t-s)(\Delta-V)} e(s)\,ds\right\|_{L^pL^r} \le \varepsilon$ and $\|e^{it(\Delta-V)}(u_0 - w_0)\|_{L^pL^r} \le \varepsilon$, then $u \in L^pL^r$ and $\|u - w\|_{L^pL^r} \le C\varepsilon$.

Proof. The proof is contained in [42, Proposition 4.7] and it relies on (1.2.9). □

3.3 Profile Decomposition Theorem for steplike perturbations of the Laplacian

The main content of this section is the following profile decomposition theorem. From now on we will use the notation $\tau_{z'} f(z) := f(z - z')$, as usual for the translation operator.

Theorem 3.3.1. *Let $V(x) \in L^\infty$ satisfies: $V \geq 0$, (3.1.9), (3.1.8) with $a_- = 0$ and $a_+ = 1$, the dispersive relation (3.1.6) and suppose that $|\nabla V(x_1, \bar{x})| \to 0$ as $|x_1| \to \infty$ uniformly in $\bar{x} \in \mathbb{R}^{d-1}$. Given a bounded sequence $\{v_n\}_{n\in\mathbb{N}} \subset H^1$, $\forall\, J \in \mathbb{N}$ and $\forall\, 1 \leq j \leq J$ there exist two sequences $\{t_n^j\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{x_n^j\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ and $\psi^j \in H^1$ such that, up to subsequences,*

$$v_n = \sum_{1\leq j\leq J} e^{it_n^j(\Delta - V)}\tau_{x_n^j}\psi^j + R_n^J$$

with the following properties:

- *(time translation sequences' dichotomy) for any fixed j we have the following dichotomy for the time parameters t_n^j:*

$$\text{either} \quad t_n^j = 0 \quad \forall\, n \in \mathbb{N} \quad \text{or} \quad t_n^j \overset{n\to\infty}{\longrightarrow} \pm\infty;$$

- *(space translation sequences' trichotomy) for any fixed j we have the following scenarios for the space parameters $x_n^j = (x_{n,1}^j, \bar{x}_n^j) \in \mathbb{R} \times \mathbb{R}^{d-1}$:*

$$\begin{aligned} &\text{either} \quad x_n^j = 0 \quad \forall\, n \in \mathbb{N} \\ &\text{or} \quad |x_{n,1}^j| \overset{n\to\infty}{\longrightarrow} \infty \\ \text{or} \quad x_{n,1}^j = 0, \quad \bar{x}_n^j = \sum_{l=2}^d k_{n,l}^j P_l \quad &\text{with} \quad k_{n,l}^j \in \mathbb{Z} \quad \text{and} \quad \sum_{l=2}^d |k_{n,l}^j| \overset{n\to\infty}{\longrightarrow} \infty, \end{aligned}$$

  where P_l are given in (3.1.9);

- *(orthogonality condition) for any $j \neq k$*

$$|x_n^j - x_n^k| + |t_n^j - t_n^k| \overset{n\to\infty}{\longrightarrow} \infty;$$

- *(smallness of the remainder) $\forall\, \varepsilon > 0 \quad \exists\, \bar{J} = \bar{J}(\varepsilon)$ such that for any $J > \bar{J}$*

$$\limsup_{n\to\infty} \|e^{it(\Delta - V)} R_n^J\|_{L^pL^r} \leq \varepsilon;$$

- *(orthogonality of the free energy) by defining $\|v\|_H^2 = \int (|\nabla v|^2 + V|v|^2)\, dx$ we have, as $n \to \infty$,*

$$\begin{aligned} \|v_n\|_{L^2}^2 &= \sum_{1\leq j\leq J} \|\psi^j\|_{L^2}^2 + \|R_n^J\|_{L^2}^2 + o(1), \quad \forall\, J \in \mathbb{N}, \\ \|v_n\|_H^2 &= \sum_{1\leq j\leq J} \|\tau_{x_n^j}\psi^j\|_H^2 + \|R_n^J\|_H^2 + o(1), \quad \forall\, J \in \mathbb{N}; \end{aligned}$$

- *(orthogonality of the potential energy) $\forall\, J \in \mathbb{N}$ and $\forall\, 2 < q < 2^*$ we have, as $n \to \infty$,*

$$\|v_n\|_{L^q}^q = \sum_{1\leq j\leq J} \|e^{it_n^j(\Delta - V)}\tau_{x_n^j}\psi^j\|_{L^q}^q + \|R_n^J\|_{L^q}^q + o(1);$$

- *(orthogonality of the energy) with* $E(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V|v|^2 + \frac{2}{\alpha+2}|v|^{\alpha+2}\right) dx$*, we have, as* $n \to \infty$*,*

$$E(v_n) = \sum_{1\le j\le J} E(e^{it_n^j(\Delta - V)}\tau_{x_n^j}\psi^j) + E(R_n^J) + o(1), \quad \forall\, J \in \mathbb{N}. \tag{3.3.1}$$

First we prove the following lemma.

Lemma 3.3.2. *Given a bounded sequence* $\{v_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^d)$ *we define*

$$\Lambda = \left\{ w \in L^2 \quad | \quad \exists \{x_k\}_{k\in\mathbb{N}} \quad \text{and} \quad \{n_k\}_{k\in\mathbb{N}} \quad \text{such that} \quad \tau_{x_k} v_{n_k} \overset{L^2}{\rightharpoonup} w \right\}$$

and

$$\lambda = \sup_{w\in\Lambda} \|w\|_{L^2}.$$

Then for every $q \in (2, 2^*)$ *there exists a constant* $M = M(\sup_{n\in\mathbb{N}} \|v_n\|_{H^1}) > 0$ *and an exponent* $e = e(d,q) > 0$ *such that*

$$\limsup_{n\to\infty} \|v_n\|_{L^q} \le M\lambda^e.$$

Proof. We consider a Fourier multiplier ζ where ζ is defined as

$$C_c^\infty(\mathbb{R}^d;\mathbb{R}) \ni \zeta(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \le 1 \\ 0 & \text{if} \quad |\xi| > 2 \end{cases}.$$

By setting $\zeta_R(\xi) = \zeta(\xi/R)$, we define the pseudo-differential operator with symbol ζ_R, classically given by $\zeta_R(|D|)f = \mathcal{F}^{-1}(\zeta_R \mathcal{F} f)(x)$ and similarly we define the operator $\tilde{\zeta}_R(|D|)$ with associated symbol given by $\tilde{\zeta}_R(\xi) = 1 - \zeta_R(\xi)$. For any $q \in (2, 2^*)$ there exists a $\epsilon \in (0,1)$ such that $H^\epsilon \hookrightarrow L^{\frac{2d}{d-2\epsilon}} =: L^q$. Then, with the standard notation $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$, we get

$$\begin{aligned} \|\tilde{\zeta}_R(|D|)v_n\|_{L^q} &\lesssim \|\langle\xi\rangle^\epsilon \tilde{\zeta}_R(\xi)\hat{v}_n(\xi)\|_{L^2_\xi} \\ &= \|\langle\xi\rangle^{\epsilon-1}\langle\xi\rangle\tilde{\zeta}_R(\xi)\hat{v}_n(\xi)\|_{L^2_\xi} \\ &\lesssim R^{-(1-\epsilon)} \end{aligned}$$

where we have used the boundedness of $\{v_n\}_{n\in\mathbb{N}}$ in H^1 at the last step.

For the localized part we consider instead a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that

$$\|\zeta_R(|D|)v_n\|_{L^\infty} \le 2|\zeta_R(|D|)v_n(y_n)|$$

and we have that up to subsequences, by using the well-known properties $\mathcal{F}^{-1}(fg) = \mathcal{F}^{-1}f * \mathcal{F}^{-1}g$ and $\mathcal{F}^{-1}\left(f\left(\frac{\cdot}{r}\right)\right) = r^d(\mathcal{F}^{-1}f)(r\cdot)$ (at least for $f, g \in \mathcal{S}$ and for any $r > 0$),

$$\limsup_{n\to\infty} |\zeta_R(|D|)v_n(y_n)| = R^d \limsup_{n\to\infty} \left| \int \eta(Rx)v_n(x - y_n)\, dx \right| \lesssim R^{d/2}\lambda$$

where we denoted $\eta = \mathcal{F}^{-1}\zeta$ and we used Cauchy-Schwartz inequality. Given $\theta \in (0,1)$ such that $\frac{1}{q} = \frac{1-\theta}{2}$, by interpolation follows that

$$\|\zeta_R(|D|)v_n\|_{L^q} \le \|\zeta_R(|D|)v_n\|_{L^\infty}^{\theta} \|\zeta_R(|D|)v_n\|_{L^2}^{1-\theta} \lesssim R^{\frac{d\theta}{2}}\lambda^{\theta}$$

$$\limsup_{n\to\infty} \|v_n\|_{L^q} \lesssim \left(R^{\frac{d\theta}{2}}\lambda^{\theta} + R^{-1+\epsilon}\right)$$

and the proof is complete provided we select as radius $R = \lambda^{-\beta}$ with $0 < \beta = \theta\left(1-\epsilon+\frac{d\theta}{2}\right)^{-1}$ and so $e = \theta(1-\epsilon)\left(1-\epsilon+\frac{d\theta}{2}\right)^{-1}$. $\square$

Based on the previous lemma we can prove the following result.

Lemma 3.3.3. *Let $\{v_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. There exists, up to subsequences, a function $\psi \in H^1$ and two sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that*

$$\tau_{-x_n} e^{it_n(\Delta - V)} v_n = \psi + W_n, \tag{3.3.2}$$

where the following conditions are satisfied:

$$W_n \overset{H^1}{\rightharpoonup} 0,$$

$$\limsup_{n\to\infty} \|e^{it(\Delta-V)}v_n\|_{L^\infty L^q} \le C\left(\sup_n \|v_n\|_{H^1}\right)\|\psi\|_{L^2}^{e}$$

with the exponent $e > 0$ given in Lemma 3.3.2. Furthermore, as $n \to \infty$, v_n fulfills the Pythagorean expansions below:

$$\|v_n\|_{L^2}^2 = \|\psi\|_{L^2}^2 + \|W_n\|_{L^2}^2 + o(1), \tag{3.3.3}$$

$$\|v_n\|_{H}^2 = \|\tau_{x_n}\psi\|_{H}^2 + \|\tau_{x_n}W_n\|_{H}^2 + o(1), \tag{3.3.4}$$

$$\|v_n\|_{L^q}^q = \|e^{it_n(\Delta-V)}\tau_{x_n}\psi\|_{L^q}^q + \|e^{it_n(\Delta-V)}\tau_{x_n}W_n\|_{L^q}^q + o(1), \qquad q \in (2, 2^*). \tag{3.3.5}$$

Moreover we have the following dichotomy for the time parameters t_n:

$$\text{either} \quad t_n = 0 \quad \forall n \in \mathbb{N} \quad \text{or} \quad t_n \overset{n\to\infty}{\longrightarrow} \pm\infty. \tag{3.3.6}$$

Concerning the space parameters $x_n = (x_{n,1}, \bar{x}_n) \in \mathbb{R} \times \mathbb{R}^{d-1}$ we have the following scenarios:

$$\text{either} \quad x_n = 0 \quad \forall n \in \mathbb{N} \tag{3.3.7}$$

$$\text{or} \quad |x_{n,1}| \overset{n\to\infty}{\longrightarrow} \infty$$

$$\text{or} \quad x_{n,1} = 0, \quad \bar{x}_n^j = \sum_{l=2}^{d} k_{n,l}P_l \quad \text{with} \quad k_{n,l} \in \mathbb{Z} \quad \text{and} \quad \sum_{l=2}^{d} |k_{n,l}| \overset{n\to\infty}{\longrightarrow} \infty.$$

Proof. Let us choose a sequences of times $\{t_n\}_{n\in\mathbb{N}}$ such that

$$\|e^{it_n(\Delta-V)}v_n\|_{L^q} > \frac{1}{2}\|e^{it(\Delta-V)}v_n\|_{L^\infty L^q}. \tag{3.3.8}$$

According to Lemma 3.3.2 we can consider a sequence of space translations such that

$$\tau_{-x_n}(e^{it_n(\Delta-V)}v_n) \stackrel{H^1}{\rightharpoonup} \psi,$$

which yields (3.3.2). Let us remark that the choice of the time sequence in (3.3.8) is possible since the norms H^1 and H are equivalent. Then

$$\limsup_{n\to\infty} \|e^{it_n(\Delta-V)}v_n\|_{L^q} \lesssim \|\psi\|_{L^2}^{e},$$

which in turn implies by (3.3.8) that

$$\limsup_{n\to\infty} \|e^{it(\Delta-V)}v_n\|_{L^\infty L^q} \lesssim \|\psi\|_{L^2}^{e},$$

where the exponent is the one given in Lemma 3.3.2. By definition of ψ we can write

$$\tau_{-x_n}e^{it_n(\Delta-V)}v_n = \psi + W_n, \qquad W_n \stackrel{H^1}{\rightharpoonup} 0 \tag{3.3.9}$$

and the Hilbert structure of L^2 gives (3.3.3).

Next we prove (3.3.4). We have

$$v_n = e^{-it_n(\Delta-V)}\tau_{x_n}\psi + e^{-it_n(\Delta-V)}\tau_{x_n}W_n, \qquad W_n \stackrel{H^1}{\rightharpoonup} 0$$

and we conclude provided that we show

$$(e^{-it_n(\Delta-V)}\tau_{x_n}\psi, e^{-it_n(\Delta-V)}\tau_{x_n}W_n)_H \stackrel{n\to\infty}{\longrightarrow} 0. \tag{3.3.10}$$

Since we have

$$(e^{-it_n(\Delta-V)}\tau_{x_n}\psi, e^{-it_n(\Delta-V)}\tau_{x_n}W_n)_H = (\psi, W_n)_{\dot{H}^1} + \int V(x+x_n)\psi(x)\bar{W}_n(x)\,dx$$

and $W_n \stackrel{H^1}{\rightharpoonup} 0$, it is sufficient to show that

$$\int V(x+x_n)\psi(x)\bar{W}_n(x)\,dx \stackrel{n\to\infty}{\longrightarrow} 0. \tag{3.3.11}$$

If (up to subsequence) $x_n \stackrel{n\to\infty}{\longrightarrow} x^* \in \mathbb{R}^d$ or $|x_{n,1}| \stackrel{n\to\infty}{\longrightarrow} \infty$, where we have splitted $x_n = (x_{n,1}, \bar{x}_n) \in \mathbb{R}\times\mathbb{R}^{d-1}$, then we have that the sequence $\tau_{-x_n}V(x) = V(x+x_n)$ pointwise converges to the function $\tilde{V}(x) \in L^\infty$ defined by

$$\tilde{V}(x) = \begin{cases} 1 & if \quad x_{n,1} \stackrel{n\to\infty}{\longrightarrow} +\infty \\ V(x+x^*) & if \quad x_n \stackrel{n\to\infty}{\longrightarrow} x^* \in \mathbb{R}^d \\ 0 & if \quad x_{n,1} \stackrel{n\to\infty}{\longrightarrow} -\infty \end{cases}$$

and hence

$$\int V(x+x_n)\psi(x)\bar{W}_n(x)\,dx = \int [V(x+x_n)-\tilde{V}(x)]\psi(x)\bar{W}_n(x)\,dx \\ + \int \tilde{V}(x)\psi(x)\bar{W}_n(x)\,dx.$$

The function $\tilde{V}(x)\psi(x)$ belongs to L^2 since $\tilde{V}$ is bounded and $\psi \in H^1$, and since $W_n \rightharpoonup 0$ in H^1 (and then in L^2) we have that

$$\int \tilde{V}(x)\psi(x)\bar{W}_n(x)\,dx \overset{n\to\infty}{\longrightarrow} 0.$$

Moreover by using Cauchy-Schwartz inequality

$$\left|\int \left[V(x+x_n)-\tilde{V}(x)\right]\psi(x)\bar{W}_n(x)\,dx\right| \leq \sup_{n\in\mathbb{N}} \|W_n\|_{L^2}\|[V(\cdot+x_n)-\tilde{V}(\cdot)]\psi(\cdot)\|_{L^2};$$

since $\left|[V(\cdot+x_n)-\tilde{V}(\cdot)]\psi(\cdot)\right|^2 \lesssim |\psi(\cdot)|^2 \in L^1$ we claim, by dominated convergence theorem, that also

$$\int [V(x+x_n)-\tilde{V}(x)]\psi(x)\bar{W}_n(x)\,dx \overset{n\to\infty}{\longrightarrow} 0,$$

and we conclude (3.3.11) and hence (3.3.10). It remains to prove (3.3.10) in the case when, up to subsequences, $x_{n,1} \overset{n\to\infty}{\longrightarrow} x_1^*$ and $|\bar{x}_n| \overset{n\to\infty}{\longrightarrow} \infty$. Up to subsequences we can assume therefore that $\bar{x}_n = \bar{x}^* + \sum_{l=2}^d k_{n,l}P_l + o(1)$ with $\bar{x}^* \in \mathbb{R}^{d-1}$, $k_{n,l} \in \mathbb{Z}$ and $\sum_{l=2}^d |k_{n,l}| \overset{n\to\infty}{\longrightarrow} \infty$. Then by using the periodicity of the potential V with respect to the $(x_2,\dots,x_d)$ variables we get:

$$\begin{aligned}
&(e^{-it_n(\Delta-V)}\tau_{x_n}\psi, e^{-it_n(\Delta-V)}\tau_{x_n}W_n)_H \\
&= (e^{-it_n(\Delta-V)}\tau_{(x_1^*,\bar{x}_n)}\psi, e^{-it_n(\Delta-V)}\tau_{(x_1^*,\bar{x}_n)}W_n)_H + o(1) \\
&= (\tau_{(x_1^*,\bar{x}^*)}\psi, \tau_{(x_1^*,\bar{x}^*)}W_n)_H + o(1) \\
&= (\psi, W_n)_{\dot{H}^1} + \int V(x+(x_1^*,\bar{x}^*))\psi(x)\bar{W}_n\,dx = o(1)
\end{aligned}$$

where we have used the fact that $W_n \overset{H^1}{\rightharpoonup} 0$.

We now turn our attention to the orthogonality of the non quadratic term of the energy, namely (3.3.5). The proof is almost the same of the one carried out in [6], with some modification.

Case 1. Suppose $|t_n| \overset{n\to\infty}{\longrightarrow} \infty$. By (1.2.2) we have $\|e^{it(\Delta-V)}\|_{\mathcal{L}(L^1;L^\infty)} \lesssim |t|^{-d/2}$ for any $t \neq 0$. Here $\mathcal{L}(X;Y)$ stands for the operator norm of bounded linear operators from the normed space X onto the normed space Y. We recall that for the evolution operator $e^{it(\Delta-V)}$ the L^2-norm is conserved, so the estimate $\|e^{it(\Delta-V)}\|_{\mathcal{L}(L^{p'};L^p)} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}$ is guaranteed by the Riesz-Thorin Theorem, thus we have the conclusion provided that

$\psi \in L^1 \cap L^2$. If this is not the case we can conclude by a straightforward approximation argument. This implies that if $|t_n| \to \infty$ as $n \to \infty$ then for any $p \in (2, 2^*)$ and for any $\psi \in H^1$

$$\left\| e^{it_n(\Delta - V)} \tau_{x_n} \psi \right\|_{L^p} \overset{n\to\infty}{\longrightarrow} 0.$$

Thus we conclude by (3.3.9).

Case 2. Assume now that $t_n \overset{n\to\infty}{\longrightarrow} t^* \in \mathbb{R}$ and $x_n \overset{n\to\infty}{\longrightarrow} x^* \in \mathbb{R}^d$. In this case the proof relies on a combination of the Rellich-Kondrachov theorem and the Brezis-Lieb Lemma contained in [15], provided that

$$\left\| e^{it_n(\Delta - V)}(\tau_{x_n}\psi) - e^{it^*(\Delta - V)}(\tau_{x^*}\psi) \right\|_{H^1} \overset{n\to\infty}{\longrightarrow} 0, \qquad \forall\, \psi \in H^1.$$

But this is a straightforward consequence of the continuity of the linear propagator (see [6] for more details).

Case 3. It remains to consider $t_n \overset{n\to\infty}{\longrightarrow} t^* \in \mathbb{R}$ and $|x_n| \overset{n\to\infty}{\longrightarrow} \infty$. Also here we can proceed as in [6] provided that for any $\psi \in H^1$ there exists a $\psi^* \in H^1$ such that

$$\| \tau_{-x_n}(e^{it_n(\Delta - V)}(\tau_{x_n}\psi)) - \psi^* \|_{H^1} \overset{n\to\infty}{\longrightarrow} 0.$$

Since translations are isometries in H^1, it suffices to show that for some $\psi^* \in H^1$

$$\| e^{it_n(\Delta - V)} \tau_{x_n}\psi - \tau_{x_n}\psi^* \|_{H^1} \overset{n\to\infty}{\longrightarrow} 0.$$

We decompose $x_n = (x_{n,1}, \bar{x}_n) \in \mathbb{R} \times \mathbb{R}^{d-1}$ and we consider the two scenarios: $|x_{n,1}| \overset{n\to\infty}{\longrightarrow} \infty$ and $\sup_{n\in\mathbb{N}} |x_{n,1}| < \infty$.

If $x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty$, by continuity in H^1 of the flow, it is enough to prove that

$$\| e^{it^*(\Delta - V)} \tau_{x_n}\psi - e^{it^*\Delta}\tau_{x_n}\psi \|_{H^1} \overset{n\to\infty}{\longrightarrow} 0.$$

We observe that

$$e^{it^*(\Delta - V)} \tau_{x_n}\psi - e^{it^*\Delta}\tau_{x_n}\psi = \int_0^{t^*} e^{i(t^*-s)(\Delta - V)}(V e^{-is\Delta}\tau_{x_n}\psi)(s)\, ds$$

and hence,

$$\| e^{it^*(\Delta - V)} \tau_{x_n}\psi - e^{it^*\Delta}\tau_{x_n}\psi \|_{H^1} \leq \int_0^{t^*} \|(\tau_{-x_n}V) e^{is\Delta}\psi\|_{H^1}\, ds.$$

We will show that

$$\int_0^{t^*} \|(\tau_{-x_n}V) e^{is\Delta}\psi\|_{H^1}\, ds \overset{n\to\infty}{\longrightarrow} 0. \tag{3.3.12}$$

Since we are assuming $x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty$, for fixed $x \in \mathbb{R}^d$ we get $V(x + x_n) \overset{n\to\infty}{\longrightarrow} 0$, namely $(\tau_{-x_n}V)(x) \overset{n\to\infty}{\longrightarrow} 0$ pointwise; since $V \in L^\infty$, $|\tau_{-x_n}V|^2 |e^{it\Delta}\psi|^2 \leq \|V\|_{L^\infty}^2 |e^{it\Delta}\psi|^2$ and $|e^{it\Delta}\psi|^2 \in L^1$, the dominated convergence theorem yields to

$$\|(\tau_{-x_n}V) e^{it\Delta}\psi\|_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Analogously, since $|x_{n,1}| \overset{n\to\infty}{\longrightarrow} \infty$ implies $|\nabla \tau_{-x_n} V(x)| \overset{n\to\infty}{\longrightarrow} 0$, we obtain

$$\|\nabla(\tau_{-x_n} V e^{it\Delta}\psi)\|_{L^2} \leq \|(e^{it\Delta}\psi)\nabla\tau_{-x_n} V\|_{L^2} + \|(\tau_{-x_n} V)\nabla(e^{it\Delta}\psi)\|_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

We conclude (3.3.12) by using the dominated convergence theorem with respect to the measure ds. For the case $x_{n,1} \overset{n\to\infty}{\longrightarrow} \infty$ we proceed similarly.

If $\sup_{n\in\mathbb{N}} |x_{n,1}| < \infty$, then up to subsequence $x_{n,1} \overset{n\to\infty}{\longrightarrow} x_1^* \in \mathbb{R}$. The thesis follows by choosing $\psi^* = e^{it^*(\Delta-V)}\tau_{(x_1^*,\bar{x}^*)}\psi$, with $\bar{x}^* \in \mathbb{R}^{d-1}$ defined as follows (see above the proof of (3.3.4)): $\bar{x}_n = \bar{x}^* + \sum_{l=2}^d k_{n,l} P_l + o(1)$ with $k_{n,l} \in \mathbb{Z}$ and $\sum_{l=2}^d |k_{n,l}| \overset{n\to\infty}{\longrightarrow} \infty$.
Finally, it is straightforward from [6] that the conditions on the parameters (3.3.6) and (3.3.7) hold. □

Proof of Theorem 3.3.1. The proof of the profile decomposition theorem can be carried out as in [6] iterating the previous lemma. □

3.4 Nonlinear profiles

As already pointed out, the Profile Decomposition Theorem is a pure linear statement. Therefore we need to associate to every linear profile a non linear profile, which will be crucial along the construction of the minimal element.

Lemma 3.4.1. *Let $\psi \in H^1$ and $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be such that $|x_{n,1}| \overset{n\to\infty}{\longrightarrow} \infty$. Up to subsequences we have the following estimates:*

$$x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty \implies \|e^{it\Delta}\psi_n - e^{it(\Delta-V)}\psi_n\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0, \tag{3.4.1}$$

$$x_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty \implies \|e^{it(\Delta-1)}\psi_n - e^{it(\Delta-V)}\psi_n\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0, \tag{3.4.2}$$

where $\psi_n := \tau_{x_n}\psi$.

Proof. Assume $x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty$ (the case $x_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty$ can be treated similarly). We first prove that

$$\sup_{n\in\mathbb{N}} \|e^{it(\Delta-V)}\psi_n\|_{L^p((T,\infty);L^r)} \overset{T\to\infty}{\longrightarrow} 0. \tag{3.4.3}$$

Let $\varepsilon > 0$. By density there exists $\tilde{\psi} \in C_c^\infty$ such that $\|\tilde{\psi} - \psi\|_{H^1} \leq \varepsilon$, then by the estimate (1.2.8)

$$\|e^{it(\Delta-V)}(\tilde{\psi}_n - \psi_n)\|_{L^pL^r} \lesssim \|\tilde{\psi}_n - \psi_n\|_{H^1} = \|\tilde{\psi} - \psi\|_{H^1} \lesssim \varepsilon.$$

Since $\tilde{\psi} \in L^{r'}$, by interpolation between the dispersive estimate (1.2.2) and the conservation of the mass along the linear flow, we have

$$\|e^{it(\Delta-V)}\tilde{\psi}_n\|_{L^r} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)}\|\tilde{\psi}\|_{L^{r'}},$$

and since $f(t) = |t|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)} \in L^p(|t|>1)$, there exists $T>0$ such that

$$\sup_n \|e^{it(\Delta-V)}\tilde{\psi}_n\|_{L^p((|t|\geq T);L^r)} \leq \varepsilon,$$

hence we get (3.4.3). In order to obtain (3.4.1), we are reduced to show that for a fixed $T>0$

$$\|e^{it\Delta}\psi_n - e^{it(\Delta-V)}\psi_n\|_{L^p((0,T);L^r)} \overset{n\to\infty}{\longrightarrow} 0.$$

Since $w_n = e^{it\Delta}\psi_n - e^{it(\Delta-V)}\psi_n$ is the solution of the following linear Schrödinger equation

$$\begin{cases} i\partial_t w_n + \Delta w_n - V w_n = -V e^{it\Delta}\psi_n \\ w_n(0) = 0 \end{cases},$$

by combining (1.2.8) with the Duhamel formula we get

$$\|e^{it\Delta}\psi_n - e^{it(\Delta-V)}\psi_n\|_{L^p((0,T);L^r)} \lesssim \|(\tau_{-x_n}V)e^{it\Delta}\psi\|_{L^1((0,T);H^1)}.$$

The thesis follows from the dominated convergence theorem. □

Lemma 3.4.2. *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence such that $x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty$, (resp. $x_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty$) and $v \in \mathcal{C}(\mathbb{R};H^1)$ be the unique solution to (3.1.10) (resp. (3.1.11)). Define $v_n(t,x) := v(t, x-x_n)$. Then, up to a subsequence, the followings hold:*

$$\left\|\int_0^t \left[e^{i(t-s)\Delta}\left(|v_n|^\alpha v_n\right)(s) - e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)(s)\right] ds\right\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0 \tag{3.4.4}$$

$$\left(\textit{resp.} \quad \left\|\int_0^t \left[e^{i(t-s)(\Delta-1)}\left(|v_n|^\alpha v_n\right)(s) - e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)(s)\right] ds\right\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0\right). \tag{3.4.5}$$

Proof. Assume $x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty$ (the case $x_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty$ can be treated similarly). Our proof starts with the observation that

$$\lim_{T\to\infty}\left(\sup_{n\in\mathbb{N}}\left\|\int_0^t e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)(s)\,ds\right\|_{L^p((T,\infty);L^r)}\right) = 0. \tag{3.4.6}$$

By Minkowski inequality and the interpolation of the dispersive estimate (1.2.2) with the conservation of the mass, we have

$$\begin{aligned}\left\|\int_0^t e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)(s)\,ds\right\|_{L^r_x} &\lesssim \int_0^t |t-s|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)}\||v_n|^\alpha v_n(s)\|_{L^{r'}_x}\,ds \\ &\lesssim \int_{\mathbb{R}} |t-s|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)}\||v|^\alpha v(s)\|_{L^{r'}_x}\,ds = |t|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)} * g\end{aligned}$$

with $g(s) = C\| |v|^\alpha v(s)\|_{L_x^{r'}}$. We conclude (3.4.6) provided that we show $|t|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)} * g(t)$ belongs to $L^p(\mathbb{R})$. By using the Hardy-Littlewood-Sobolev inequality (see [114, Theorem 1, page 119]) we assert

$$\left\| |t|^{-1+\frac{(2-d)\alpha+4}{2(\alpha+2)}} * g(t) \right\|_{L^p} \lesssim \| |v|^\alpha v\|_{L^{\frac{2\alpha(\alpha+2)}{((2-d)\alpha+4)(\alpha+1)}} L^{r'}} = \|v\|_{L^pL^r}^{\alpha+1}.$$

Since v scatters, then it belongs to L^pL^r, and so we can deduce the validity of (3.4.6).

Consider now T fixed: we are reduced to show that

$$\left\| \int_0^t \left[e^{i(t-s)\Delta} \left(|v_n|^\alpha v_n\right)(s) - e^{i(t-s)(\Delta - V)} \left(|v_n|^\alpha v_n\right)(s) \right] ds \right\|_{L^p((0,T);L^r)} \overset{n\to\infty}{\longrightarrow} 0.$$

As usual we observe that

$$w_n(t,x) = \int_0^t e^{i(t-s)\Delta} \left(|v_n|^\alpha v_n\right)(s)\, ds - \int_0^t e^{i(t-s)(\Delta-V)} \left(|v_n|^\alpha v_n\right)(s)\, ds$$

is the solution of the following linear Schrödinger equation

$$\begin{cases} i\partial_t w_n + \Delta w_n - V w_n = -V \displaystyle\int_0^t e^{i(t-s)\Delta} \left(|v_n|^\alpha v_n\right)(s)\, ds \\ w_n(0) = 0 \end{cases},$$

and likely for Lemma 3.4.1 we estimate

$$\begin{aligned} &\left\| \int_0^t e^{i(t-s)\Delta} \left(|v_n|^\alpha v_n\right)(s)\, ds - \int_0^t e^{i(t-s)(\Delta-V)} \left(|v_n|^\alpha v_n\right)(s)\, ds \right\|_{L^p((0,T);L^r)} \\ &\qquad \lesssim \|(\tau_{-x_n} V)|v|^\alpha v\|_{L^1((0,T);H^1)}. \end{aligned}$$

By using the dominated convergence theorem we conclude the proof.

□

The previous results imply the following useful corollaries.

Corollary 3.4.3. *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence such that $x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty$, and let $v \in \mathcal{C}(\mathbb{R}; H^1)$ be the unique solution to (3.1.10) with initial datum $v_0 \in H^1$. Then*

$$v_n(t,x) = e^{it(\Delta-V)} v_{0,n} - i \int_0^t e^{i(t-s)(\Delta-V)} \left(|v_n|^\alpha v_n\right)(s)\, ds + e_n(t,x)$$

where $v_{0,n}(x) := \tau_{x_n} v_0(x)$, $v_n(t,x) := v(t, x - x_n)$ and $\|e_n\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0$.

Proof. It is a consequence of (3.4.1) and (3.4.4). □

Corollary 3.4.4. *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence such that $x_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty$, and let $v \in \mathcal{C}(\mathbb{R}; H^1)$ be the unique solution to (3.1.11) with initial datum $v_0 \in H^1$. Then*

$$v_n(t,x) = e^{it(\Delta-V)} v_{0,n} - i \int_0^t e^{i(t-s)(\Delta-V)} \left(|v_n|^\alpha v_n\right)(s)\, ds + e_n(t,x)$$

where $v_{0,n}(x) := \tau_{x_n} v_0(x)$, $v_n(t,x) := v(t, x - x_n)$ and $\|e_n\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0$.

Proof. It is a consequence of (3.4.2) and (3.4.5). □

Lemma 3.4.5. *Let $v(t,x) \in \mathcal{C}(\mathbb{R}; H^1)$ be a solution to (3.1.10) (resp. (3.1.11)) and let $\psi_\pm \in H^1$ (resp. $\varphi_\pm \in H^1$) be such that*

$$\|v(t,x) - e^{it\Delta}\psi_\pm\|_{H^1} \overset{t\to\pm\infty}{\longrightarrow} 0$$

$$\left(\textit{resp.} \quad \|v(t,x) - e^{it(\Delta-1)}\varphi_\pm\|_{H^1} \overset{t\to\pm\infty}{\longrightarrow} 0 \right).$$

Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$, $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be two sequences such that $x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty$ (resp. $x_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty$) and $|t_n| \overset{n\to\infty}{\longrightarrow} \infty$. Let us define moreover $v_n(t,x) := v(t-t_n, x-x_n)$ and $\psi_n^\pm(x) := \tau_{x_n}\psi_\pm(x)$ (resp. $\varphi_n^\pm(x) = \tau_{x_n}\varphi_\pm(x)$). Then, up to subsequence, we get

$$\begin{gathered} t_n \to \pm\infty \Longrightarrow \|e^{i(t-t_n)\Delta}\psi_n^\pm - e^{i(t-t_n)(\Delta-V)}\psi_n^\pm\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \\ \left\| \int_0^t \left[e^{i(t-s)\Delta} \left(|v_n|^\alpha v_n\right)(s) - e^{i(t-s)(\Delta-V)} \left(|v_n|^\alpha v_n\right)(s) \right] ds \right\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0 \end{gathered} \tag{3.4.7}$$

$$\left(\textit{resp. } t_n \to \pm\infty \Longrightarrow \|e^{i(t-t_n)(\Delta-1)}\varphi_n^\pm - e^{i(t-t_n)(\Delta-V)}\varphi_n^\pm\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \right.$$

$$\left. \left\| \int_0^t \left[e^{i(t-s)(\Delta-1)} \left(|v_n|^\alpha v_n\right)(s) - e^{i(t-s)(\Delta-V)} \left(|v_n|^\alpha v_n\right)(s) \right] ds \right\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0 \right).$$

Proof. It is a multidimensional suitable version of [6, Proposition 3.6]. Nevertheless, since in [6] the details of the proof are not given, we expose below the proof of the most delicate estimate, namely the second estimate in (3.4.7). After a change of variable in time, proving (3.4.7) is clearly equivalent to prove

$$\left\| \int_{-t_n}^t e^{i(t-s)\Delta} \tau_{x_n}(|v|^\alpha v)(s) ds - \int_{-t_n}^t e^{i(t-s)(\Delta-V)} \tau_{x_n}(|v|^\alpha v)(s)\, ds \right\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0.$$

We can focus on the case $t_n \to \infty$ and $x_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty$, being the other cases similar. The idea of the proof is to split the estimate above in three different regions, i.e. $(-\infty, -T) \times \mathbb{R}^d$, $(-T,T) \times \mathbb{R}^d$, $(T,\infty) \times \mathbb{R}^d$ for some fixed T which will be chosen in an appropriate way below. The strategy is to use translation in the space variable to gain smallness in the strip $(-T,T) \times \mathbb{R}^d$ while we use smallness of Strichartz estimate in half spaces $(-T,T)^c \times \mathbb{R}^d$. Actually in (T,∞) the situation is more delicate and we will also use the dispersive relation.

Let us define $g(t) = \|v(t)\|_{L^{(\alpha+1)r'}}^{\alpha+1}$ and for fixed $\varepsilon > 0$ let us consider $T = T(\varepsilon) > 0$ such that:

$$\begin{cases} \||v|^\alpha v\|_{L^{q'}((-\infty,-T);L^{r'})} < \varepsilon \\ \||v|^\alpha v\|_{L^{q'}((T,+\infty);L^{r'})} < \varepsilon \\ \||v|^\alpha v\|_{L^1((-\infty,-T);H^1)} < \varepsilon \\ \left\| |t|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)} * g(t) \right\|_{L^p(T,+\infty)} < \varepsilon \end{cases} . \tag{3.4.8}$$

The existence of such a T is guaranteed by the integrability properties of v and its decay at infinity (in time). We can assume without loss of generality that $|t_n| > T$.
We split the term to be estimated as follows:

$$\begin{aligned}&\int_{-t_n}^{t} e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)\,ds - \int_{-t_n}^{t} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)\,ds\\ = e^{it\Delta}&\int_{-t_n}^{-T} e^{-is\Delta}\tau_{x_n}(|v|^\alpha v)(s)\,ds - e^{it(\Delta-V)}\int_{-t_n}^{-T} e^{-is(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)\,ds\\ &+\int_{-T}^{t} e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)\,ds - \int_{-T}^{t} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)\,ds.\end{aligned}$$

By Strichartz estimate (1.2.8) and the third one of (3.4.8), we have, uniformly in n,

$$\left\| e^{it\Delta}\int_{-t_n}^{-T} e^{-is\Delta}\tau_{x_n}(|v|^\alpha v)(s)\,ds\right\|_{L^pL^r} \lesssim \varepsilon,$$
$$\left\| e^{it(\Delta-V)}\int_{-t_n}^{-T} e^{-is(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)\,ds\right\|_{L^pL^r} \lesssim \varepsilon.$$

Thus, it remains to prove

$$\left\|\int_{-T}^{t} e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)\,ds - \int_{-T}^{t} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)\,ds\right\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0$$

and we split it by estimating it in the regions mentioned above. By using (1.2.9) and the first one of (3.4.8) we get uniformly in n the following estimates:

$$\left\|\int_{-T}^{t} e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)\,ds\right\|_{L^p((-\infty,-T);L^r)} \lesssim \||v|^\alpha v\|_{L^{q'}((-\infty,-T);L^{r'})} \lesssim \varepsilon,$$
$$\left\|\int_{-T}^{t} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)\,ds\right\|_{L^p((-\infty,-T);L^r)} \lesssim \||v|^\alpha v\|_{L^{q'}((-\infty,-T);L^{r'})} \lesssim \varepsilon.$$

The difference $w_n = \int_{-T}^{t} e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)\,ds - \int_{-T}^{t} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)\,ds$ satisfies the following Cauchy problem:

$$\begin{cases} i\partial_t w_n + (\Delta - V)w_n = -V\displaystyle\int_{-T}^{t} e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)\,ds \\ w_n(-T) = 0\end{cases}.$$

Then w_n satisfies the integral equation

$$w_n(t) = \int_{-T}^{t} e^{i(t-s)(\Delta-V)}\left(-V\int_{-T}^{s} e^{i(s-\sigma)\Delta}\tau_{x_n}(|v|^\alpha v)(\sigma)\,d\sigma\right)ds$$

which we estimate in the region $(-T,T)\times\mathbb{R}^d$. By Sobolev embedding $H^1 \hookrightarrow L^r$, Hölder and Minkowski inequalities we have therefore

$$\begin{aligned}\left\|\int_{-T}^{t} e^{i(t-s)(\Delta-V)}\left(-V\int_{-T}^{s} e^{i(s-\sigma)\Delta}\tau_{x_n}(|v|^\alpha v)(\sigma)\,d\sigma\right)ds\right\|_{L^p((-T,T);L^r)} &\lesssim\\ \lesssim T^{1/p}\int_{-T}^{T}\left\|(\tau_{-x_n}V)\int_{-T}^{s} e^{i(s-\sigma)\Delta}|v|^\alpha v(\sigma)\,d\sigma\right\|_{H^1} ds \lesssim \varepsilon&\end{aligned}$$

by means of Lebesgue's theorem. What is left is to estimate in the region $(T, \infty) \times \mathbb{R}^d$ the terms

$$\int_{-T}^{t} e^{i(t-s)(\Delta - V)} \tau_{x_n}(|v|^\alpha v)(s)\, ds \qquad \text{and} \qquad \int_{-T}^{t} e^{i(t-s)\Delta} \tau_{x_n}(|v|^\alpha v)(s)\, ds.$$

We consider only one term being the same for the other. Let us split the estimate as follows:

$$\begin{aligned} \left\| \int_{-T}^{t} e^{i(t-s)(\Delta-V)} \tau_{x_n}(|v|^\alpha v)(s)\, ds \right\|_{L^p((T,\infty);L^r)} &\leq \\ \leq \left\| \int_{-T}^{T} e^{i(t-s)(\Delta-V)} \tau_{x_n}(|v|^\alpha v)(s)\, ds \right\|_{L^p((T,\infty);L^r)} & \\ + \left\| \int_{T}^{t} e^{i(t-s)(\Delta-V)} \tau_{x_n}(|v|^\alpha v)(s)\, ds \right\|_{L^p((T,\infty);L^r)} &. \end{aligned}$$

The second term is controlled by Strichartz estimates, and it is $\lesssim \varepsilon$ since we are integrating in the region where $\||v|^\alpha v\|_{L^{q'}((T,\infty);L^{r'})} < \varepsilon$ (by using the second of (3.4.8)), while the first term is estimated by using the dispersive relation. More precisely

$$\begin{aligned} \left\| \int_{-T}^{T} e^{i(t-s)(\Delta-V)} \tau_{x_n}(|v|^\alpha v)(s)\, ds \right\|_{L^p((T,\infty);L^r)} &\lesssim \\ \lesssim \left\| \int_{-T}^{T} |t-s|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)} \|v(s)\|^{\alpha+1}_{L_x^{(\alpha+1)r'}}\, ds \right\|_{L^p(T,\infty)} & \\ \lesssim \left\| \int_{\mathbb{R}} |t-s|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)} \|v(s)\|^{\alpha+1}_{L_x^{(\alpha+1)r'}}\, ds \right\|_{L^p(T,\infty)} &\lesssim \varepsilon \end{aligned}$$

where in the last step we used Hardy-Sobolev-Littlewood inequality and the fourth of (3.4.8). $\square$

As consequences of the previous lemma we obtain the following corollaries.

Corollary 3.4.6. *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence such that $x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty$ and let $v \in \mathcal{C}(\mathbb{R}; H^1)$ be a solution to (3.1.10) with initial datum $\psi \in H^1$. Then for a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $|t_n| \overset{n\to\infty}{\longrightarrow} \infty$*

$$v_n(t,x) = e^{it(\Delta-V)}\psi_n - i\int_0^t e^{i(t-s)(\Delta-V)} \left(|v_n|^\alpha v_n\right)(s)\, ds + e_n(t,x)$$

where $\psi_n := e^{-it_n(\Delta-V)}\tau_{x_n}\psi$, $v_n := v(t-t_n, x-x_n)$ and $\|e_n\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0$.

Corollary 3.4.7. *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence such that $x_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty$ and let $v \in \mathcal{C}(\mathbb{R}; H^1)$ be a solution to (3.1.11) with initial datum $\psi \in H^1$. Then for a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $|t_n| \overset{n\to\infty}{\longrightarrow} \infty$*

$$v_n(t,x) = e^{it(\Delta-V)}\psi_n - i\int_0^t e^{i(t-s)(\Delta-V)} \left(|v_n|^\alpha v_n\right)(s)\, ds + e_n(t,x)$$

where $\psi_n := e^{-it_n(\Delta-V)}\tau_{x_n}\psi$, $v_n := v(t-t_n, x-x_n)$ and $\|e_n\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0$.

We shall also need the following results, for whose proof we refer to [6].

Proposition 3.4.8. *Let $\psi \in H^1$. There exists $\hat{U}_\pm \in \mathcal{C}(\mathbb{R}^\pm; H^1) \cap L^p(\mathbb{R}^\pm; L^r)$ solution to (3.1.7) such that*

$$\|\hat{U}_\pm(t, \cdot) - e^{-it(\Delta - V)}\psi\|_{H^1} \overset{t\to\pm\infty}{\longrightarrow} 0.$$

Moreover, if $t_n \to \mp\infty$, then

$$\hat{U}_{\pm,n} = e^{it(\Delta-V)}\psi_n - i\int_0^t e^{i(t-s)(\Delta-V)}\left(|\hat{U}_{\pm,n}|^\alpha \hat{U}_{\pm,n}\right)(s)\,ds + h_{\pm,n}(t,x)$$

where $\psi_n := e^{-it_n(\Delta-V)}\psi$, $\hat{U}_{\pm,n}(t,\cdot) := \hat{U}_\pm(t - t_n, \cdot)$ and $\|h_{\pm,n}(t,x)\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0$.

3.5 Construction of the minimal element

In view of the results stated in Section 3.2, we define the following quantity belonging to $(0, \infty]$:

$$E_c = \sup\Big\{E > 0 \text{ such that if } \varphi \in H^1 \text{ with } E(\varphi) < E$$
$$\text{then the solution of (3.1.7) with initial data } \varphi \text{ is in } L^pL^r\Big\}.$$

Our aim is to show that $E_c = \infty$ and hence we get the large data scattering.

Proposition 3.5.1. *Suppose $E_c < \infty$. Then there exists $\varphi_c \in H^1$, $\varphi_c \not\equiv 0$, such that the corresponding global solution $u_c(t,x)$ to (3.1.7) does not scatter. Moreover, there exists $\bar{x}(t) \in \mathbb{R}^{d-1}$ such that $\{u_c(t, x_1, \bar{x} - \bar{x}(t))\}_{t\in\mathbb{R}^+}$ is a relatively compact subset in H^1.*

Proof. If $E_c < \infty$, there exists a sequence φ_n of elements of H^1 such that

$$E(\varphi_n) \overset{n\to\infty}{\longrightarrow} E_c,$$

and by denoting with $u_n \in \mathcal{C}(\mathbb{R}; H^1)$ the corresponding solution to (3.0.1) with initial datum φ_n then

$$u_n \notin L^pL^r.$$

We apply the profile decomposition to φ_n :

$$\varphi_n = \sum_{j=1}^{J} e^{-it_n^j(-\Delta+V)}\tau_{x_n^j}\psi^j + R_n^J. \tag{3.5.1}$$

Claim 3.5.2. *There exists only one non-trivial profile, that is $J = 1$.*

Assume $J > 1$. For $j \in \{1, \ldots, J\}$ to each profile ψ^j we associate a nonlinear profile U_n^j. We can have one of the following situations, where we have reordered without loss of generality the cases in these way:

1. $(t_n^j, x_n^j) = (0,0) \in \mathbb{R} \times \mathbb{R}^d$,
2. $t_n^j = 0$ and $x_{n,1}^j \overset{n\to\infty}{\longrightarrow} -\infty$,
3. $t_n^j = 0$, and $x_{n,1}^j \overset{n\to\infty}{\longrightarrow} +\infty$,
4. $t_n^j = 0$, $x_{n,1}^j = 0$ and $|\bar{x}_n^j| \overset{n\to\infty}{\longrightarrow} \infty$,
5. $x_n^j = \vec{0}$ and $t_n^j \overset{n\to\infty}{\longrightarrow} -\infty$,
6. $x_n^j = \vec{0}$ and $t_n^j \overset{n\to\infty}{\longrightarrow} +\infty$,
7. $x_{n,1}^j \overset{n\to\infty}{\longrightarrow} -\infty$ and $t_n^j \overset{n\to\infty}{\longrightarrow} -\infty$,
8. $x_{n,1}^j \overset{n\to\infty}{\longrightarrow} -\infty$ and $t_n^j \overset{n\to\infty}{\longrightarrow} +\infty$,
9. $x_{n,1}^j \overset{n\to\infty}{\longrightarrow} +\infty$ and $t_n^j \overset{n\to\infty}{\longrightarrow} -\infty$,
10. $x_{n,1}^j \overset{n\to\infty}{\longrightarrow} +\infty$ and $t_n^j \overset{n\to\infty}{\longrightarrow} +\infty$,
11. $x_{n,1}^j = 0$, $t_n^j \overset{n\to\infty}{\longrightarrow} -\infty$ and $|\bar{x}_n^j| \overset{n\to\infty}{\longrightarrow} \infty$,
12. $x_{n,1}^j = 0$, $t_n^j \overset{n\to\infty}{\longrightarrow} +\infty$ and $|\bar{x}_n^j| \overset{n\to\infty}{\longrightarrow} \infty$.

Notice that despite to [6] we have twelve cases to consider and not six (this is because we have to consider a different behavior of $V(x)$ as $|x| \to \infty$). Since the argument to deal with the cases above is similar to the ones considered in [6] we skip the details. The main point is that for instance in dealing with the cases (2) and (3) above we have to use respectively Corollary 3.4.3 and Corollary 3.4.4.

When instead $|\bar{x}_n^j| \overset{n\to\infty}{\longrightarrow} \infty$ and $x_{1,n}^j = 0$ we use the fact that this sequences can be assumed, according with the profile decomposition Theorem 3.3.1 to have components which are integer multiples of the periods, so the translations and the nonlinear equation commute and if $|t_n| \overset{n\to\infty}{\longrightarrow} \infty$ we use moreover Proposition 3.4.8. We skip the details. Once it is proved that $J = 1$ and

$$\varphi_n = e^{it_n(\Delta - V)}\psi + R_n$$

with $\psi \in H^1$ and $\limsup_{n\to\infty} \|e^{it(\Delta-V)} R_n\|_{L^p L^r} = 0$, then the existence of the critical element follows now by [42], ensuring that, up to subsequence, φ_n converges to ψ in H^1 and so $\varphi_c = \psi$. We define by u_c the solution to (3.1.7) with Cauchy datum φ_c, and we call it critical element (or soliton-like solution). This is the minimal (with respect to the energy) non-scattering solution to (3.1.7). We can assume therefore with no loss of generality that $\|u_c\|_{L^p(\mathbb{R}^+;L^r)} = \infty$. The precompactenss of the trajectory up to translation by a path $\bar{x}(t)$ follows again by [42]. □

3.6 Death of the soliton-like solution

Next we show that the unique solution that satisfies the compactness properties of the critical element $u_c(t,x)$ (see Proposition 3.5.1) is the trivial solution. Hence we get a contradiction and we deduce that necessarily $E_c = \infty$.

The tool that we shall use is the following Nakanishi-Morawetz type estimate.

Lemma 3.6.1. *Let $u(t,x)$ be the solution to (3.1.7), where $V(x)$ satisfies $x_1 \cdot \partial_{x_1} V(x) \le 0$ for any $x \in \mathbb{R}^d$, then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \frac{t^2 |u|^{\alpha+2}}{(t^2 + x_1^2)^{3/2}} \, dx_1 \, d\bar{x} \, dt < \infty. \tag{3.6.1}$$

Proof. The proof follows the ideas of Nakanishi, see [100]; we shall recall it shortly, with the obvious modifications of our context. Let us introduce

$$m(u) = a\partial_{x_1} u + gu$$

with

$$a = -\frac{2x_1}{\lambda}, \quad g = -\frac{t^2}{\lambda^3} - \frac{it}{\lambda}, \quad \lambda = (t^2 + x_1^2)^{1/2}$$

and by using the equation solved by $u(t,x)$ we get

$$\begin{aligned} 0 &= \Re\left\{(i\partial_t u + \Delta u - Vu - |u|^\alpha u)\bar{m})\right\} \\ &= \frac{1}{2}\partial_t \left(-\frac{2x_1}{\lambda}\Im\{\bar{u}\partial_{x_1} u\} - \frac{t|u|^2}{\lambda}\right) \\ &\quad + \partial_{x_1}\Re\left\{\partial_{x_1} u\bar{m} - a l_V(u) - \partial_{x_1} g\frac{|u|^2}{2}\right\} \\ &\quad + \frac{t^2 G(u)}{\lambda^3} + \frac{|u|^2}{2}\Re\{\partial_{x_1}^2 g\} \\ &\quad + \frac{|2it\partial_{x_1} u + x_1 u|^2}{2\lambda^3} - x_1\partial_{x_1} V\frac{|u|^2}{\lambda} \\ &\quad + div_{\bar{x}}\Re\{\bar{m}\nabla_{\bar{x}} u\}. \end{aligned} \tag{3.6.2}$$

with $G(u) = \frac{\alpha}{\alpha+2}|u|^{\alpha+2}$, $l_V(u) = \frac{1}{2}\left(-\Re\{i\bar{u}\partial_t u\} + |\partial_{x_1} u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2} + V|u|^2\right)$ and $div_{\bar{x}}$ is the divergence operator with respect to the $(x_2, \dots, x_d)$ variables. Making use of the repulsivity assumption in the x_1 direction, we get (3.6.1) by integrating (3.6.2) on $\{1 < |t| < T\} \times \mathbb{R}^d$, obtaining

$$\int_1^T \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \frac{t^2 |u|^{\alpha+2}}{(t^2 + x_1^2)^{3/2}} \, dx_1 \, d\bar{x} \, dt \le C,$$

where $C = C(M,E)$ depends on mass and energy and then letting $T \to \infty$. □

Lemma 3.6.2. *Let $u(t,x)$ be a nontrivial solution to (3.1.7) such that for a suitable choice $\bar{x}(t) \in \mathbb{R}^{d-1}$ we have that $\{u(t, x_1, \bar{x} - \bar{x}(t))\} \subset H^1$ is a precompact set. If $\bar{u} \in H^1$ is one of its limit points, then $\bar{u} \ne 0$.*

Proof. This property simply follows from the conservation of the energy. $\square$

Lemma 3.6.3. *If $u(t,x)$ is an in Lemma 3.6.2 then for any $\varepsilon > 0$ there exists $R > 0$ such that*

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}^{d-1}} \int_{|x_1|>R} \left(|u|^2 + |\nabla_x u|^2 + |u|^{\alpha+2}\right) d\bar{x}\, dx_1 < \varepsilon.$$

Proof. This is a well-known property implied by the precompactness of the sequence. $\square$

Lemma 3.6.4. *If $u(t,x)$ is an in Lemma 3.6.2 then there exist $R_0 > 0$ and $\varepsilon_0 > 0$ such that*

$$\int_{\mathbb{R}^{d-1}} \int_{|x_1|<R_0} |u(t, x_1, \bar{x} - \bar{x}(t))|^{\alpha+2}\, d\bar{x}\, dx_1 > \varepsilon_0 \qquad \forall\, t \in \mathbb{R}^+.$$

Proof. It is sufficient to prove that $\inf_{t\in\mathbb{R}^+} \|u(t, x_1, \bar{x} - \bar{x}(t))\|_{L^{\alpha+2}} > 0$, then the result follows by combining this fact with Lemma 3.6.3. If by the absurd it is not true then there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ such that $u(t_n, x_1, \bar{x} - \bar{x}(t_n)) \overset{n\to\infty}{\longrightarrow} 0$ in $L^{\alpha+2}$. On the other hand by the compactness assumption, it implies that $u(t_n, x_1, \bar{x} - \bar{x}(t_n)) \overset{n\to\infty}{\longrightarrow} 0$ in H^1, and it is in contradiction with Lemma 3.6.2. $\square$

We now conclude the proof of scattering for large data, by showing the extinction of the minimal element. Let $R_0 > 0$ and $\varepsilon_0 > 0$ be given by Lemma 3.6.4, then

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \frac{|u|^{\alpha+2} t^2}{(t^2 + x_1^2)^{3/2}}\, dx_1\, d\bar{x}\, dt &\geq \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \int_{|x_1|<R_0} \frac{t^2 |u(t, x_1, \bar{x} - \bar{x}(t))|^{\alpha+2}}{(t^2 + x_1^2)^{3/2}}\, dx_1\, d\bar{x}\, dt \\ &\geq \varepsilon_0 \int_1^T \frac{t^2}{(t^2 + R_0^2)^{3/2}}\, dt \to \infty \qquad \text{if} \quad T \to \infty.\end{aligned}$$

Hence we contradict (3.6.1) and we get that the critical element cannot exist.

3.7 Double scattering channels in 1D

This last section is devoted to prove Theorem 3.1.8. Following [35, example 1, page 283] we have the following property:

$$\begin{aligned}&\forall\, \psi \in L^2 \quad \exists\, \eta_\pm, \gamma_\pm \in L^2 \text{ such that} \\ &\|e^{it(\partial_x^2 - V)}\psi - e^{it\partial_x^2}\eta_\pm - e^{it(\partial_x^2 - 1)}\gamma_\pm\|_{L^2} \overset{t\to\pm\infty}{\longrightarrow} 0.\end{aligned} \tag{3.7.1}$$

Our aim is now to show that (3.7.1) actually holds in H^1 provided that $\psi \in H^1$. We shall prove this property for $t \to +\infty$ (the case $t \to -\infty$ is similar). We divide the proof in two steps.

Step 1. Convergence (3.7.1) occurs in H^1 provided that $\psi \in H^2$

In order to do that it is sufficient to show that

$$\psi \in H^2 \Longrightarrow \eta_+, \gamma_+ \in H^2. \tag{3.7.2}$$

Once it is proved then we conclude the proof of this first step by using the following interpolation inequality

$$\|f\|_{H^1} \leq \|f\|_{L^2}^{1/2} \|f\|_{H^2}^{1/2}$$

in conjunction with (3.7.1) and with the bound

$$\sup_{t \in \mathbb{R}} \|e^{it(\partial_x^2 - V)}\psi - e^{it\partial_x^2}\eta_+ - e^{it(\partial_x^2-1)}\gamma_+\|_{H^2} < \infty$$

(in fact this last property follows by the fact that the domain of the operator $\partial_x^2 - V(x)$ is $\mathcal{D}(\partial_x^2 - V(x)) = H^2$ is preserved along the linear flow and by (3.7.2)). Thus we show (3.7.2). Notice that by (3.7.1) we get

$$\|e^{-it\partial_x^2} e^{it(\partial_x^2 - V)}\psi - \eta_+ - e^{-it}\gamma_+\|_{L^2} \overset{t\to\infty}{\longrightarrow} 0,$$

and by choosing as subsequence $t_n = 2\pi n$ we get

$$\|e^{-it_n\partial_x^2} e^{it_n(\partial_x^2 - V)}\psi - \eta_+ - \gamma_+\|_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

By combining this fact with the bound $\sup_n \|e^{-it_n\partial_x^2} e^{it_n(\partial_x^2 - V)}\psi\|_{H^2} < \infty$ we get $\eta_+ + \gamma_+ \in H^2$. Arguing as above but by choosing $t_n = (2n+1)\pi$ we also get $\eta_+ - \gamma_+ \in H^2$ and hence necessarily $\eta_+, \gamma_+ \in H^2$.

Step 2. The map $H^2 \ni \psi \mapsto (\eta_+, \gamma_+) \in H^2 \times H^2$ satisfies $\|\gamma_+\|_{H^1} + \|\eta_+\|_{H^1} \lesssim \|\psi\|_{H^1}$

Once this step is proved then we conclude by a straightforward density argument. By a linear version of the conservation laws (3.1.3), (3.1.4) we get

$$\|e^{it(\partial_x^2 - V)}\psi\|_{H^1_V} = \|\psi\|_{H^1_V} \tag{3.7.3}$$

where

$$\|w\|_{H^1_V}^2 = \|w\|_H^2 + \|w\|_{L^2}^2 = \int |\partial_x w|^2 \, dx + \int V|w|^2 \, dx + \int |w|^2 \, dx.$$

Notice that this norm is clearly equivalent to the usual norm of H^1.
Next notice that by using the conservation of the mass we get

$$\|\eta_+ + \gamma_+\|_{L^2}^2 = \|\eta_+ + e^{-2n\pi i}\gamma_+\|_{L^2}^2 = \|e^{i2\pi n\partial_x^2}\eta_+ + e^{i2\pi n(\partial_x^2 - 1)}\gamma_+\|_{L^2}^2$$

and by using (3.7.1) we get

$$\|\eta_+ + \gamma_+\|_{L^2}^2 = \lim_{t\to\infty} \|e^{it(\partial_x^2 - V)}\psi\|_{L^2}^2 = \|\psi\|_{L^2}^2$$

Moreover we have

$$\begin{aligned} \|\partial_x(\eta_+ + \gamma_+)\|_{L^2}^2 &= \|\partial_x(\eta_+ + e^{-2n\pi i}\gamma_+)\|_{L^2}^2 = \|\partial_x(e^{i2\pi n\partial_x^2}(\eta_+ + e^{-i2\pi n}\gamma_+))\|_{L^2}^2 \\ &= \|\partial_x(e^{i2\pi n\partial_x^2}\eta_+ + e^{i2\pi n(\partial_x^2-1)}\gamma_+)\|_{L^2}^2 \end{aligned}$$

and by using the previous step and (3.7.3) we get

$$\begin{aligned}\|\partial_x(\eta_+ + \gamma_+)\|_{L^2}^2 &= \lim_{t\to+\infty} \|\partial_x(e^{it(\partial_x^2 - V)}\psi)\|_{L^2}^2 \\ &\leq \lim_{t\to+\infty} \|e^{it(\partial_x^2 - V)}\psi\|_{H_V^1}^2 = \|\psi\|_{H_V^1}^2 \lesssim \|\psi\|_{H^1}^2.\end{aligned}$$

Summarizing we get

$$\|\eta_+ + \gamma_+\|_{H^1} \lesssim \|\psi\|_{H^1}.$$

By a similar argument and by replacing the sequence $t_n = 2\pi n$ by $t_n = (2n+1)\pi$ we get

$$\|\eta_+ - \gamma_+\|_{H^1} \lesssim \|\psi\|_{H^1}.$$

The conclusion follows.

Chapter 4

Scattering for a class of NLKG on waveguides

We consider the following Cauchy problem for the pure-power defocusing nonlinear Klein-Gordon equation posed on the waveguide $\mathbb{R}^d \times \mathbb{T}$, with $1 \leq d \leq 4$

$$\begin{cases} \partial_{tt}u - \Delta_{x,y}u + u = -|u|^\alpha u, & (t,x,y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \\ u(0,x,y) = f(x,y) \in H^1(\mathbb{R}^d \times \mathbb{T}) \\ \partial_t u(0,x,y) = g(x,y) \in L^2(\mathbb{R}^d \times \mathbb{T}) \end{cases}, \tag{4.0.1}$$

where $\mathbb{T}$ is the one-dimensional flat torus and $\Delta_{x,y} = \Delta_x + \Delta_y$ is the usual Laplace operator $\sum_{i=1}^d \partial_{x_i}^2 + \partial_y^2$. We consider nonlinearities that are energy subcritical on $\mathbb{R}^{d+1}$ and mass supercritical on $\mathbb{R}^d$, namely we restrict our attention to $\frac{4}{d} < \alpha < \frac{4}{d-1}$ for $2 \leq d \leq 4$ while $\alpha > 4$ for $d = 1$. For some particular choices of nonlinearities, aside from the natural question of existence of solutions, it is of interest to try to relate the long-time behavior of nonlinear solutions to linear solutions in appropriate functional spaces. We wish to investigate the energy scattering for (4.0.1).

4.1 Motivations and main results

About the pure euclidean case $\mathbb{R}^d$, there is a huge mathematical literature, not only for the Klein-Gordon equation but in general for other dispersive PDEs such as the NLS equation and the NLW equation. We recall that Strichartz estimates play an essential role for the local well-posedness and for the large time analysis of the solutions - once Strichartz estimates have been proved to hold globally in time. The nonlinear Klein-Gordon equation has been deeply studied in the euclidean framework, producing a huge literature. We only give here some references amongst others about the scattering results: in high dimension cases $d \geq 3$, we mention the early works by Morawetz [97] and Morawetz and Strauss [98], the works by Brenner [12, 13], Ginibre and Velo [52–54], while for the low dimensional case $\mathbb{R}^d$ with $d = 1$ and $d = 2$ the question of scattering has been solved by Nakanishi in [100]. The focusing case have been investigated in [69, 70] by Ibrahim, Masmoudi and Nakanishi both in the energy subcritical and critical cases. For a more complete picture of the known results, we refer the reader to the references contained in the previously cited papers.

Unlike the euclidean setting, the compact one does not exhibit the same phenomenon. This is due to the presence of periodic solutions inducing a lack of (global-in-time) summability on them. Nevertheless, it is worth pointing out that existence properties on compact manifolds have been investigated for NLS by Bourgain in [9] and later by Burq, Gerard and Tzvetkov in [17]. For existence results for NLKG, valid on more general manifolds, we refer the reader to [75].

The question of "mixing" both configurations, to understand the competition of induced phenomena is natural. The study of scattering properties for solutions to NLS posed on a product space was proved for small data on $\mathbb{R}^d \times \mathcal{M}^k$ - $\mathcal{M}^k$ being a compact Riemannian manifold - by Tzvetkov and Visciglia [126] followed by a theorem of large data scattering by the previous authors in [127]. We also mention the existence of several related results in mixed settings, among which the papers by Cheng, Guo, Yang and Zhao [24], Grébert, Paturel and Thomann [60], Hani and Pausader [61], Hani, Pausader, Tzvetkov and Visciglia [62], Tarulli, [120], Vilaça da Rocha [128] and references therein.

Our purpose is to carry on with the investigation of the second author and Visciglia started in [63]. In that paper the authors proved scattering for small energy data for the pure-power nonlinear energy-critical Klein-Gordon equation in the framework of $\mathbb{R}^d \times \mathcal{M}^2$, in both defocusing and focusing regimes (the latter corresponding to an opposite sign in front of the nonlinear term in (4.0.1) and where $\mathcal{M}^2$ is a bidimensional compact manifold. For small initial data, once Strichartz estimates have been proved to hold globally in time, the global well-posedness and scattering can be proved by a perturbative argument.

This is no more the case when dealing with initial data without smallness assumption. We apply the Kenig and Merle scheme. To this aim, after having studied small data scattering on $\mathbb{R}^d \times \mathbb{T}$, our first step is to prove a profile decomposition theorem on the considered product space. Then thanks to a perturbative argument, we construct a minimal energy solution which is global in time but does not enjoy a finite Strichartz bound which would lead to the scattering property. Moreover, we prove that the trajectory of this solution is precompact in the energy space, and this will give a contradiction to its existence once combined with Nakanishi/Morawetz estimates. The choice of the strategy *à la Kenig and Merle* seems to be the best adapted to our setting, to deal with either defocusing or focusing nonlinearities and may be revisited for critical cases. We however recall that, combined with the mixed geometry, the "bad sign" of the energy in the focusing case is usually an obstruction to prove a priori bounds such as the Morawetz or the Nakanishi/Morawetz estimates, whereas for the energy-critical cases, it is expected that the lack of scaling invariance of NLKG will bring a delicate technical issue. Therefore, we restrict our attention on the defocusing (energy) subcritical cases, and especially on the adjustment of the euclidean arguments and tools in our setting, whereas the other cases are objects of future investigation.

We briefly recall what we intend as scattering property, as defined in the Introduction: we investigate the completeness of the wave operator by showing that, a global solution $u(t,x,t)$ to (4.0.1) behaves, as time t tends to $\pm\infty$, like a solution to the following linear equation

$$\begin{cases} \partial_{tt}v - \Delta_{x,y}v + v = 0, \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{T} \\ v(0,x,y) = f^{\pm} \in H^1(\mathbb{R}^d\times\mathbb{T}) \\ \partial_t v(0,x,y) = g^{\pm} \in L^2(\mathbb{R}^d\times\mathbb{T}) \end{cases} \tag{4.1.1}$$

for some initial data $(f^{\pm}, g^{\pm}) \in H^1(\mathbb{R}^d\times\mathbb{T})\times L^2(\mathbb{R}^d\times\mathbb{T})$. The main result of this Chapter is stated as follows.

Theorem 4.1.1. *Assume that $d = 1$ and $\alpha > 4$ or $2 \le d \le 4$ and $\frac{4}{d} < \alpha < \frac{4}{d-1}$. Let*

$$u \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d\times\mathbb{T})) \cap \mathcal{C}^1(\mathbb{R}; L^2(\mathbb{R}^d\times\mathbb{T})) \cap L^{\alpha+1}(\mathbb{R}; L^{2(\alpha+1)}(\mathbb{R}^d\times\mathbb{T})). \tag{4.1.2}$$

be the unique global solution to (4.0.1): then for $t\to+\infty$ (respectively $t\to-\infty$) there exists $(f^+, g^+) \in H^1(\mathbb{R}^d\times\mathbb{T})\times L^2(\mathbb{R}^d\times\mathbb{T})$ (respectively $(f^-, g^-) \in H^1(\mathbb{R}^d\times\mathbb{T})\times L^2(\mathbb{R}^d\times\mathbb{T})$) such that

$$\lim_{t\to+\infty} \left\|u(t,x) - u^+(t,x)\right\|_{H^1(\mathbb{R}^d\times\mathbb{T})} + \left\|\partial_t u(t,x) - \partial_t u^+(t,x)\right\|_{L^2(\mathbb{R}^d\times\mathbb{T})} = 0, \tag{4.1.3}$$

$$\left(\textit{respectively} \quad \lim_{t\to-\infty} \left\|u(t,x) - u^-(t,x)\right\|_{H^1(\mathbb{R}^d\times\mathbb{T})} + \left\|\partial_t u(t,x) - \partial_t u^-(t,x)\right\|_{L^2(\mathbb{R}^d\times\mathbb{T})} = 0\right),$$

where $u^+(t,x,y), u^-(t,x,y) \in H^1(\mathbb{R}^d\times\mathbb{T})\times L^2(\mathbb{R}^d\times\mathbb{T})$ are the corresponding solutions to (4.1.1) with initial data (f^+, g^+) and (f^-, g^-), respectively.

4.2 Small data theory and perturbative results

Once Strichartz estimates are available (see Section 1.3), they enable to deal with small data scattering problem. As for the Schrödinger equation examined in chapter 3, this is the first step we need to proceed with the Concentration/Compactness scheme.

Theorem 4.2.1. *Let $d = 1$ and $\alpha \ge 4$ or $2 \le d \le 5$ and α be such that $\frac{4}{d} \le \alpha \le \frac{4}{d-1}$. Then there exists $\varepsilon > 0$ such that for all $(f,g) \in H^1\times L^2$ satisfying $\|f\|_{H^1} + \|g\|_{L^2} < \varepsilon$, the global nonlinear solution u to the Cauchy problem (4.0.1)*

$$u \in \mathcal{C}(\mathbb{R}; H^1) \cap \mathcal{C}^1(\mathbb{R}; L^2) \cap L^{\alpha+1}L^{2(\alpha+1)}$$

scatters in the sense of (4.1.3).

Proof. We recall that in the framework of Theorem 4.2.1, we consider $\frac{4}{d} \le \alpha \le \frac{4}{d-1}$ for $2 \le d \le 5$ and $\alpha \ge 4$ if $d = 1$. We handle both focusing and defocusing nonlinearities arguing as in [63].

We rewrite (4.0.1) in an Hamiltonian form, as a first order system. More precisely if u is a solution to (4.0.1) then the vector $(u, \partial_t u)^T$ satisfies

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Delta + 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 \\ |u|^\alpha u \end{pmatrix}.$$

We have that the following exponential matrix operator

$$e^{tH} = \begin{pmatrix} \cos\left(t \cdot \sqrt{1-\Delta}\right) & \dfrac{\sin\left(t \cdot \sqrt{1-\Delta}\right)}{\sqrt{1-\Delta}} \\ -\sin\left(t \cdot \sqrt{1-\Delta}\right) \cdot \left(\sqrt{1-\Delta}\right) & \cos\left(t \cdot \sqrt{1-\Delta}\right) \end{pmatrix}, \tag{4.2.1}$$

is unitary on the energy space $H^1 \times L^2$ (see [102]). Moreover

$$\begin{pmatrix} u \\ \partial_t u \end{pmatrix} = e^{tH} \begin{pmatrix} f \\ g \end{pmatrix} + \int_0^t e^{(t-s)H} \begin{pmatrix} 0 \\ |u|^\alpha u \end{pmatrix} (s)\, ds$$

and then, since e^{tH} is skew self-adjoint

$$e^{-tH} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} + \int_0^t e^{-sH} \begin{pmatrix} 0 \\ |u|^\alpha u \end{pmatrix} (s)\, ds.$$

We now write $\vec{V}(t) = e^{-tH} \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$, and consider $0 < \tau < t$. Then

$$\|\vec{V}(t) - \vec{V}(\tau)\|_{H^1 \times L^2} \leq C \int_\tau^t \||u|^\alpha u(s)\|_{L^2}\, ds \leq C \|u\|^{\alpha+1}_{L^{\alpha+1}((\tau,t);L^{2(\alpha+1)})},$$

and it is obvious that $\|u\|^{\alpha+1}_{L^{\alpha+1}((\tau,t);L^{2(\alpha+1)})}$ tends to zero as t, τ tend towards infinity, since the solution belongs to $L^{\alpha+1}(\mathbb{R}; L^{2(\alpha+1)})$.

Therefore, there exist $(f^\pm, g^\pm) \in H^1 \times L^2$ such that $\vec{V}(t) \to \begin{pmatrix} f^\pm \\ g^\pm \end{pmatrix}$ in $H^1 \times L^2$ as $t \to \pm\infty$. □

Remark 4.2.2. It is worth mentioning that the analysis for small initial data can be stated without any further restriction in the focusing case, namely replacing in (4.0.1) the sign in front of the nonlinear term with a plus sign. Furthermore, observe that the result of the theorem above is valid also in the critical cases. The main restriction on α is carried by the fact that $(\alpha + 1, 2\alpha + 2)$ should satisfy (1.3.1). It is easy to check that $d = 5, \alpha = 1$ is the only case that can be handled for $d > 4$ and it is critical.

The second tool will be a suitable Profile Decomposition Theorem and will be proved in the next section. As already mentioned along this work, the latter one is a linear result, hence in order to deal with the non linear equation and associate a non linear profile to every linear Klein-Gordon solution given by the Profile Decomposition Theorem, we give the following perturbation lemma which will enable us to absorb the nonlinear terms in the remainders of the Profile Decomposition Theorem, in a proper way. The following long time perturbation theorem is contained in [102] where it is proved for cubic focusing NLKG on $\mathbb{R}^3$, namely for $\alpha = 2$, with the opposite sign in front of the nonlinearity and in an euclidean framework. We report here the statement modified to fit our setting, for the sake of completeness.

Lemma 4.2.3. *For any $M > 0$ there exist $\varepsilon = \varepsilon(M) > 0$ (possibly very small) and $c = c(M) > 0$ (possibly very large) such that the following fact holds. Fix $t_0 \in \mathbb{R}$ and suppose that*

$$\|v\|_{L^{\alpha+1}L^{2(\alpha+1)}} \leq M$$
$$\|e_u\|_{L^1L^2} + \|e_v\|_{L^1L^2} + \|w_0\|_{L^{\alpha+1}L^{2(\alpha+1)}} \leq \varepsilon' \leq \varepsilon(M)$$

where $u, v \in \bigcap_{h=0}^{1} \mathcal{C}^h(\mathbb{R}; H^{1-h})$, $e_z = \partial_{tt}z - \Delta_{x,y}z + z + |z|^\alpha z$ and $\vec{w}_0(t) = e^{(t-t_0)H}(\vec{u}(t_0) - \vec{v}(t_0))$. Then

$$\|u\|_{L^{\alpha+1}L^{2(\alpha+1)}} < \infty,$$
$$\|\vec{u} - \vec{v} - \vec{w}_0\|_{L^\infty\mathcal{H}} + \|u - v\|_{L^{\alpha+1}L^{2(\alpha+1)}} \leq c(M)\varepsilon'.$$

Proof. This retraces the same proof as in [102] (where the nonlinearity is u^2u), but with the following inequality to estimate the nonlinear part:

$$||u+v|^\alpha(u+v) - |u|^\alpha u| \leq C(|u|^\alpha + |v|^\alpha)|v| = C(|u|^\alpha|v| + |v|^{\alpha+1}).$$

□

4.3 Profile Decomposition Theorem on flat waveguide $\mathbb{R}^d \times \mathbb{T}$

In this section we follow the arguments of [6, 102] to provide a profile decomposition theorem which is the main ingredient in the proof of scattering properties in the whole energy space.
We start with the following preliminary lemma. We use the following convention

$$2^* = \begin{cases} \frac{2(d+1)}{d-1}, & \text{if} \quad d \geq 2 \\ +\infty, & \text{if} \quad d = 1 \end{cases}. \tag{4.3.1}$$

Lemma 4.3.1. *Let $\{v_n(x,y)\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^d \times \mathbb{T})$, with $1 \leq d \leq 4$, be a bounded sequence. Define the set*

$$\begin{aligned}\Lambda(v_n) = \{w(x,y) \in L^2 \,\Big|\, &\exists\, \{(x_n, y_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^d \times \mathbb{T} \quad \textit{such that,} \\ &\textit{up to subsequence,} \quad v_n(x - x_n, y - y_n) \overset{L^2}{\rightharpoonup} w(x,y)\}\end{aligned} \tag{4.3.2}$$

and let

$$\lambda(v_n) = \sup_{w \in \Lambda(v_n)} \|w\|_{L^2}. \tag{4.3.3}$$

Then, for any q such that $2q \in (2, 2^*)$ we have

$$\limsup_{n\to\infty} \|v_n\|_{L^{2q}} \lesssim \lambda^e(v_n)$$

where

$$e = e(q,d) = \frac{q-1}{3-5q}\left(\frac{q(d-1)-(d+1)}{q}\right) > 0.$$

Proof. By the Sobolev embedding theorem (see [64]), the energy space embeds continuously in the Lebesgue space L^{2^*}. In particular $H^1(\mathbb{R}^d \times \mathbb{T}) \hookrightarrow L^{2q}(\mathbb{R}^d \times \mathbb{T})$ for any $q \in [1, 2^*/2]$ if $d \geq 2$ while $q \geq 1$ if $d = 1$, where 2^* is defined in (4.3.1). Similarly to the proof of Lemma 3.3.2, we consider, as Fourier multiplier, a cut-off function in the flat-frequencies space $\mathbb{R}^d_\xi$ where the cut-off is given by

$$C_c^\infty(\mathbb{R}^d;\mathbb{R}) \ni \chi(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \leq 1 \\ 0 & \text{if} \quad |\xi| > 2 \end{cases}.$$

By setting $\chi_R(\xi) = \chi(\xi/R)$, $R > 0$, we define the pseudo-differential operator with symbol χ_R. It is given by $\chi_R(|D|)f = \mathcal{F}^{-1}(\chi_R \mathcal{F} f)(x)$ and similarly we define the operator $\tilde{\chi}_R(|D|)$ with the associated symbol given by $\tilde{\chi}_R(\xi) = 1 - \chi_R(\xi)$. Later on we will also use the well-known properties

$$\begin{aligned} \mathcal{F}(fg) &= \mathcal{F}(f) * \mathcal{F}(g) \\ \mathcal{F}(f(\sigma\cdot)) &= \sigma^{-d}\mathcal{F}f(\cdot/\sigma) \end{aligned} \tag{4.3.4}$$

which hold for any smooth functions $f, g : \mathbb{R}^d \to \mathbb{R}$ and any nonnegative real number σ. In order to apply the Hausdorff-Young inequality $\mathcal{F} : L^p \to L^{p'}$ for any $p \in [1, 2]$, we set $2q = p'$ and $p = \frac{p'}{p'-1} = \frac{2q}{2q-1} \in (1, 2)$. We then use Hölder inequality with $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$ and exploiting the precise structure of $\mathbb{T}$ we can write, for every $n \in \mathbb{N}$,

$$v_n(x,y) = \sum_{k\in\mathbb{Z}} v_n^k(x) e^{iky}, \tag{4.3.5}$$

where the functions v_n^k are the Fourier coefficients, and similarly

$$\tilde{\chi}_R(|D|)v_n(x,y) = \sum_{k\in\mathbb{Z}} \tilde{\chi}_R(|D|)v_n^k(x)e^{iky}.$$

We first notice the embedding $H^{\frac{1}{2}-\frac{1}{2q}}(\mathbb{T}) \hookrightarrow L^{2q}(\mathbb{T})$, allowing us to write

$$\begin{aligned} \|\tilde{\chi}_R(|D|)v_n\|^2_{L^{2q}} &\lesssim \|\tilde{\chi}_R(|D|)v_n\|^2_{L^{2q}_x H_y^{\frac{1}{2}-\frac{1}{2q}}} = \left\| \sum_{k\in\mathbb{Z}} \langle k\rangle^{1-\frac{1}{q}} |\tilde{\chi}_R(|D|)v_n^k|^2 \right\|_{L^q_x} \\ &\lesssim \sum_{k\in\mathbb{Z}} \langle k\rangle^{1-\frac{1}{q}} \|\tilde{\chi}_R(|D|)v_n^k\|^2_{L^{2q}_x} \lesssim \sum_{k\in\mathbb{Z}} \langle k\rangle^{1-\frac{1}{q}} \|\mathcal{F}^{-1}(\tilde{\chi}_R(|\xi|)\hat{v}_n^k)(x)\|^2_{L^{2q}_x} \\ &\lesssim \sum_{k\in\mathbb{Z}} \langle k\rangle^{1-\frac{1}{q}} \|\tilde{\chi}_R(|\xi|)\hat{v}_n^k(\xi)\|^2_{L_\xi^{2q/(2q-1)}} \\ &\lesssim \sum_{k\in\mathbb{Z}} \langle k\rangle^{1-\frac{1}{q}} \|\langle\xi\rangle^{\frac{1}{2}+\frac{1}{2q}}\hat{v}_n^k(\xi)\|^2_{L^2_\xi} \|\tilde{\chi}_R(|\xi|)\langle\xi\rangle^{-\frac{1}{2}-\frac{1}{2q}}\|^2_{L_\xi^{2q/(q-1)}} \end{aligned} \tag{4.3.6}$$

where an Hölder inequality was used in the last step. We notice that the last factor in the RHS term is easily controlled as follows:

$$
\begin{aligned}
\|\tilde{\chi}_R(|\xi|)\langle\xi\rangle^{-\frac{1}{2}-\frac{1}{2q}}\|^2_{L^{2q/(q-1)}_\xi} &\lesssim \left(\int_{|\xi|\geq R} \frac{d\xi}{(1+|\xi|^2)^{\left(\frac{1}{4}+\frac{1}{4q}\right)\left(\frac{2q}{q-1}\right)}}\right)^{\frac{q-1}{q}} \\
&\lesssim \left(\int_{|\xi|\geq R} |\xi|^{-\frac{q+1}{q-1}}\, d\xi\right)^{\frac{q-1}{q}} \\
&\lesssim \left(\int_R^\infty \rho^{-\frac{q+1}{q-1}+d-1}\, d\rho\right)^{\frac{q-1}{q}} \\
&\lesssim \left(R^{d-\frac{q+1}{q-1}}\right)^{\frac{q-1}{q}} = R^{\frac{d(q-1)}{q}-\frac{q+1}{q}}
\end{aligned}
$$

where the integrability of the term has been checked and $\frac{d(q-1)}{q}-(1+\frac{1}{q})<0$. Thus, by the Plancherel identity, it may be concluded that the estimate (4.3.6) satisfies:

$$
\begin{aligned}
\|\tilde{\chi}_R(|D|)v_n\|^2_{L^{2q}} &\lesssim R^{\frac{d(q-1)}{q}-(1+\frac{1}{q})} \sum_{k\in\mathbb{Z}} \langle k\rangle^{1-\frac{1}{q}} \int \langle\xi\rangle^{1+\frac{1}{q}} |\hat{v}_n^k(\xi)|^2\, d\xi \\
&\lesssim R^{\frac{d(q-1)}{q}-(1+\frac{1}{q})} \|v_n\|^2_{H^1}.
\end{aligned}
$$

Recalling that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in H^1, we summarize with

$$
\|\tilde{\chi}_R(|D|)v_n\|_{L^{2q}_{x,y}} \lesssim R^{\frac{d(q-1)}{2q}-\frac{q+1}{2q}} = R^{\frac{q(d-1)-(d+1)}{2q}}.
$$

We now use (4.3.5) and we define the localized part of v_n as

$$
\begin{aligned}
\chi_R(|D|)v_n(x,y) &= \sum_{|k|\leq M} \chi_R(|D|)v_n^k(x)e^{iky} + \sum_{|k|>M} \chi_R(|D|)v_n^k(x)e^{iky} \\
&:= \chi_R^{\leq M}(|D|)v_n + \chi_R^{>M}(|D|)v_n.
\end{aligned}
$$

We estimate the tail $\chi_R^{>M}(|D|)v_n$ as follows. By means of Minkowski and Cauchy-Schwartz inequalities we get

$$
\begin{aligned}
\|\chi_R^{>M}(|D|)v_n\|_{L^2} &\leq C(Vol(\mathbb{T})) \sum_{|k|>M} \|\chi_R(|D|)v_n^k(x)\|_{L^2_x} \\
&\lesssim \left(\sum_{|k|>M} \frac{1}{k^2}\right)^{1/2} \left(\sum_{|k|>M} k^2 \|\chi_R(|\xi|)\hat{v}_n^k(\xi)\|^2_{L^2_\xi}\right)^{1/2} \\
&\lesssim \left(\sum_{|k|>M} \frac{1}{k^2}\right)^{1/2} \left(\sum_{|k|>M} k^2 \|\hat{v}_n^k(\xi)\|^2_{L^2_\xi}\right)^{1/2} \\
&\lesssim \left(\sum_{|k|>M} \frac{1}{k^2}\right)^{1/2} \|v_n\|_{L^2_x H^1_y} \lesssim \left(\sum_{|k|>M} \frac{1}{k^2}\right)^{1/2}.
\end{aligned}
$$

Since

$$\sum_{k=M+1}^{\infty} a_k \leq \int_M^{\infty} f(x)\,dx$$

where $f:\mathbb{R}\to\mathbb{R}^+$ is a decreasing function such that $a_k = f(k)$ (it is assumed here that $f(x) = x^{-2}$), then

$$\left(\sum_{|k|>M+1} k^{-2}\right)^{1/2} \lesssim M^{-1/2}$$

and so

$$\|\chi_R^{>M}(|D|)v_n\|_{L^2} \lesssim M^{-1/2}.$$

The following interpolation result is a straightforward classical application of the Hölder inequality.

Lemma 4.3.2. *Let $p_1 \leq p \leq p_2$ and $f \in L^{p_1} \cap L^{p_2}$. Then $f \in L^p$ and given $\theta \in [0,1]$ such that $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ the following estimates holds:*

$$\|f\|_{L^p} \leq \|f\|_{L^{p_1}}^{\theta} \|f\|_{L^{p_2}}^{1-\theta}.$$

Therefore by Lemma 4.3.2 we have

$$\begin{aligned}\|\chi_R^{>M}(|D|)v_n\|_{L^{2q}} &\leq \|\chi_R^{>M}(|D|)v_n\|_{L^2}^{\theta}\|\chi_R^{>M}(|D|)v_n\|_{L^{2^*}}^{1-\theta} \\ &\lesssim \|\chi_R^{>M}(|D|)v_n\|_{L^2}^{\frac{d+1}{2q}-\frac{d-1}{2}} \lesssim M^{-\frac{1}{2}\left(\frac{d+1}{2q}-\frac{d-1}{2}\right)}.\end{aligned} \tag{4.3.7}$$

It remains to estimate the term $\sum_{|k|\leq M} \chi_R(|D|)v_n^k(x)e^{iky}$. Denoting by D_M the Dirichlet Kernel

$$D_M(y) = \sum_{k=-M}^{M} e^{iky},$$

we can write

$$\chi_R^{\leq M}(|D|)v_n(x,y) = \sum_{|k|\leq M} \chi_R(|D|)v_n^k(x)e^{iky} = \int_{\mathbb{T}} \chi_R(|D|)v_n(x,z)D_M(y-z)\,dz,$$

and we choose a sequence $(x_n, y_n) \in \mathbb{R}^d \times \mathbb{T}$ such that

$$\begin{aligned}\|\chi_R^{\leq M}(|D|)v_n\|_{L^\infty_{x,y}} &\leq 2\left|\chi_R^{\leq M}(|D|)v_n(x_n,y_n)\right| \\ &= 2R^d\left|\int_{\mathbb{R}^d\times\mathbb{T}} \eta(Rx)D_M(y)v_n(x-x_n, y-y_n)\,dx\,dy\right|,\end{aligned}$$

where $R^d\eta(Rx) = \mathcal{F}^{-1}(\chi_R(|\xi|))$. Observe that $\eta(Rx)D_M(y)$ is a function in L^2 and that $\|\eta(Rx)D_M(y)\|_{L^2} \lesssim R^{-d/2}M\|\eta\|_{L^2}$. Up to subsequences, from (4.3.2) and (4.3.3) we get

$$\begin{aligned}\limsup_{n\to\infty} \|\chi_R^{\leq M}(|D|)v_n\|_{L^\infty} &\leq \limsup_{n\to\infty} 2R^d\left|\int_{\mathbb{R}^d\times\mathbb{T}} \eta(Rx)D_M(y)v_n(x-x_n,y-y_n)\,dx\,dy\right| \\ &= 2R^d\left|\int_{\mathbb{R}^d\times\mathbb{T}} \eta(Rx)D_M(y)w(x,y)\,dx\,dy\right| \\ &\leq 2R^{d/2}M\|\eta\|_{L^2}\lambda \lesssim R^{d/2}M\lambda\end{aligned}$$

thus, again by interpolation, we infer that

$$\|\chi_R^{\leq M}(|D|)v_n\|_{L^{2q}} \lesssim \|\chi_R^{\leq M}(|D|)v_n\|_{L^\infty}^{1-1/q}\|\chi_R^{\leq M}(|D|)v_n\|_{L^2}^{1/q} \lesssim \|\chi_R^{\leq M}(|D|)v_n\|_{L^\infty}^{1-1/q},$$

and then

$$\limsup_{n\to\infty} \|\chi_R^{\leq M}(|D|)v_n\|_{L^{2q}} \lesssim R^{\frac{d}{2}\left(\frac{q-1}{q}\right)} M^{\frac{q-1}{q}} \lambda^{\frac{q-1}{q}}. \tag{4.3.8}$$

Combining (4.3.6),(4.3.7) and (4.3.8), we obtain

$$\limsup_{n\to\infty} \|v_n\|_{L^{2q}} \lesssim R^{\frac{q(d-1)-(d+1)}{2q}} + M^{\frac{1}{2}\left(\frac{q(d-1)-(d+1)}{2q}\right)} + R^{\frac{d}{2}\left(\frac{q-1}{q}\right)} M^{\frac{q-1}{q}} \lambda^{\frac{q-1}{q}},$$

and by choosing $M \sim R^2$ we end up with

$$\limsup_{n\to\infty} \|v_n\|_{L^{2q}} \lesssim R^{\frac{q(d-1)-(d+1)}{2q}} + \left(R^{\frac{d+4}{2}}\lambda\right)^{\frac{q-1}{q}}.$$

We now consider as radius $R = \lambda^\beta$, and so

$$\limsup_{n\to\infty} \|v_n\|_{L^{2q}} \lesssim \lambda^{\beta\left(\frac{q(d-1)-(d+1)}{2q}\right)} + \left(\lambda^{\beta\left(\frac{d+4}{2}\right)+1}\right)^{\frac{q-1}{q}}.$$

Defining now β in this way:

$$\begin{aligned}
&\beta\left(\frac{q(d-1)-(d+1)}{2q}\right) = \frac{q-1}{q}\left(\beta\left(\frac{d+4}{2}\right)+1\right)\\
\iff\ &\beta\left(\frac{q(d-1)-(d+1)}{2q} - \frac{q-1}{q}\frac{d+4}{2}\right) = \frac{q-1}{q}\\
\iff\ &\beta(q(d-1)-(d+1)-(q-1)(d+4)) = 2(q-1)\\
\iff\ &\beta(3-5q) = 2(q-1) \iff \beta = \frac{2(q-1)}{3-5q},
\end{aligned}$$

we observe that $\beta = \frac{2(q-1)}{3-5q} < 0$, and we conclude with

$$\limsup_{n\to\infty} \|v_n\|_{L^{2q}} \lesssim \lambda^{\frac{q-1}{3-5q}\left(\frac{q(d-1)-(d+1)}{q}\right)}.$$

□

Remark 4.3.3. We notice that w actually belongs to H^1 since the weak limit clearly enjoys this regularity.

We now fix some notations used in the following part. We define with $v(t,x,y)$ or simply $v(t)$ the free evolution with respect to the linear Klein-Gordon equation, with Cauchy datum $\vec{v}^0 = (v_0, v_1)$ and we define by $\vec{v}(t) = e^{tH}\vec{v}^0 = (v(t), \partial_t v(t))^T$, where e^{tH} has been introduced in (4.2.1). Then, we give the following decomposition for a time-independent bounded sequence in $H^1 \times L^2$. We first introduce the following lemma which will be useful after.

To shorten notation, we write from now on $\mathcal{H} = \mathcal{H}(\mathbb{R}^d\times\mathbb{T}) = H^1(\mathbb{R}^d\times\mathbb{T})\times L^2(\mathbb{R}^d\times\mathbb{T})$.

Lemma 4.3.4. *Let $\vec{f}_n \rightharpoonup 0$ in $\mathcal{H}$. Then we have:*

- $t_n \to \bar{t} \in \mathbb{R} \implies e^{t_n H}\vec{f}_n(x,y) \rightharpoonup 0$ *in* $\mathcal{H}$,
- $e^{(t_n^2 - t_n^1)H}\vec{f}_n(x-(x_n^1 - x_n^2), y) \rightharpoonup \vec{g} \neq 0 \implies |t_n^2 - t_n^1| + |x_n^2 - x_n^1| \to +\infty.$

Proof. For the first point, we make use of the continuity property of the propagator: by denoting by $(\cdot,\cdot)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$, for any $\vec{\psi} \in \mathcal{H}$ we have

$$\begin{aligned}(e^{t_n H}\vec{f}_n, \vec{\psi})_{\mathcal{H}} &= (e^{t_n H}\vec{f}_n - e^{\bar{t}H}\vec{f}_n, \vec{\psi}) + (\vec{f}_n, e^{-\bar{t}H}\vec{\psi})_{\mathcal{H}} \\ &= (\vec{f}_n, e^{-t_n H}\vec{\psi} - e^{\bar{t}H}\vec{\psi})_{\mathcal{H}} + (\vec{f}_n(x,y), e^{-\bar{t}H}\vec{\psi})_{\mathcal{H}} \\ &= (\vec{f}_n, e^{-t_n H}\vec{\psi} - e^{\bar{t}H}\vec{\psi})_{\mathcal{H}} + o(1).\end{aligned}$$

The conclusion follows, up to subsequences, since it holds in the $L^2 \times L^2$-topology by exploiting the continuity of the flow.

The second point is proved in its contrapositive form. Suppose that $s_n := (t_n^2 - t_n^1)$ and $\xi_n := (x_n^2 - x_n^1)$ are bounded. Then, up to subsequences, $s_n \to s \in \mathbb{R}$ and $\xi_n \to \bar{\xi} \in \mathbb{R}^d$. We prove that $e^{s_n H}\vec{f}_n(x-\xi_n, y) \rightharpoonup 0$ in $\mathcal{H}$. But as before

$$\begin{aligned}(e^{s_n H}\vec{f}_n(x-\xi_n,y), \vec{\psi})_{\mathcal{H}} &= (e^{sH}\vec{f}_n(x,y), \vec{\psi}(x+\xi,y))_{\mathcal{H}} + o(1) \\ &= (\vec{f}_n(x,y), e^{-sH}\vec{\psi}(x+\xi,y))_{\mathcal{H}} + o(1) \rightharpoonup 0.\end{aligned}$$

□

We can now state the following result, whose iteration will give the Profile Decomposition Theorem.

Proposition 4.3.5. *Let $\{\vec{v}_n^0\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}$ and $1 \le d \le 4$. Then, for suitable sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$, possibly after extractions of subsequences (still denoted with the subscript n), we can write, for every $n \in \mathbb{N}$*

$$\vec{v}_n(-t_n, x - x_n, y) = \vec{\psi}(x,y) + \vec{W}_n(x,y)$$

where $\vec{v}_n(t,x,y) = e^{tH}\vec{v}_n^0$ and where the components of $\vec{\psi}$ are denoted by $(\psi, \partial\psi)$. Moreover, the following properties hold:

$$\vec{W}_n \overset{n\to\infty}{\rightharpoonup} 0 \quad \textit{in} \quad \mathcal{H},$$

$$\limsup_{n\to\infty} \|v_n(t,x,y)\|_{L^\infty L^q} \lesssim \|\psi\|_{L^2}^e \quad \textit{for any} \quad q \in (2, 2^*), \tag{4.3.9}$$

where $e > 0$ is given in Lemma 4.3.1 and as $n \to \infty$

$$\|\vec{v}_n^0\|_{\mathcal{H}}^2 = \|\vec{\psi}\|_{\mathcal{H}}^2 + \|\vec{W}_n\|_{\mathcal{H}}^2 + o(1). \tag{4.3.10}$$

Similarly, for the $L^{\alpha+2}$-norm, as $n \to \infty$, we have

$$\|v_n^0\|_{L^{\alpha+2}}^{\alpha+2} = \|\psi\|_{L^{\alpha+2}}^{\alpha+2} + \|W_n\|_{L^{\alpha+2}}^{\alpha+2} + o(1). \tag{4.3.11}$$

Furthermore, the translation sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ satisfy the dichotomies below:

$$\begin{aligned}&\textit{either} \quad t_n = 0 \quad \forall n \in \mathbb{N} \qquad \textit{or} \quad t_n \overset{n\to\infty}{\longrightarrow} \pm\infty; \\ &\textit{either} \quad x_n = 0 \quad \forall n \in \mathbb{N} \qquad \textit{or} \quad |x_n| \overset{n\to\infty}{\longrightarrow} \infty.\end{aligned} \tag{4.3.12}$$

Proof. Define $\vec{v}_n(t,x,y) := e^{tH}\vec{v}_n^0$, namely $\vec{v}_n(t)$ is the linear evolution of $\vec{v}_n^0$ by the linear Klein-Gordon flow. Since the energy is preserved along the flow, the sequence $\vec{v}_n(t)$ is bounded in $L^\infty\mathcal{H}$ and by Sobolev embedding the sequence $\{v_n(t)\}_{n\in\mathbb{N}}$ is bounded in $L^\infty L^q$-norm, for any $q\in(2,2^*)$. Thus, let us now choose a sequence of times $\{t_n\}_{n\in\mathbb{N}}$ such that

$$\|v_n(-t_n)\|_{L^q} > \frac{1}{2}\|v_n(\cdot)\|_{L^\infty L^q}. \tag{4.3.13}$$

In the spirit of previous lemma, we consider $\Lambda\left(v_n(-t_n,x,y)\right)$ and $\lambda\left(v_n(-t_n,x,y)\right)$. Let $\{(x_n,y_n)\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d\times\mathbb{T}$ and $\vec{\psi}(x,y)=(\psi,\partial\psi)(x,y)\in\mathcal{H}$ be such that, up to subsequences,

$$\vec{v}_n(-t_n,x-x_n,y-y_n)\rightharpoonup\vec{\psi}$$

in $\mathcal{H}$ as $n\to\infty$. Then we get

$$\vec{v}_n(-t_n,x-x_n,y-y_n)=\vec{\psi}+\vec{W}_n,\qquad \vec{W}_n\rightharpoonup 0, \tag{4.3.14}$$

the latter weak convergence occurring in $\mathcal{H}$ and in addition

$$\lambda\left(v_n(-t_n,x,y)\right)\lesssim\|\psi\|_{L^2}. \tag{4.3.15}$$

The relation (4.3.15) along with Lemma 4.3.1 implies that

$$\limsup_{n\to\infty}\|v_n(-t_n)\|_{L^q}\lesssim\|\psi\|_{L^2}^e \quad \text{for any} \quad q\in(2,2^*),$$

and then (4.3.9) follows by (4.3.13).

By definition, from (4.3.14) we can write

$$\vec{v}_n^0(x,y)=e^{t_nH}\vec{\psi}(x+x_n,y+y_n)+e^{t_nH}\vec{W}_n(x+x_n,y+y_n), \tag{4.3.16}$$

and since e^{tH} is an isometry on $\mathcal{H}$ and its adjoint is given by e^{-tH}, together with the fact that $\vec{W}_n\rightharpoonup 0$, we get, as $n\to\infty$,

$$\|\vec{v}_n^0\|_{\mathcal{H}}^2=\|\vec{\psi}\|_{\mathcal{H}}^2+\|\vec{W}_n\|_{\mathcal{H}}^2+o(1).$$

We pursue the proof by showing the orthogonality property of the potential energy, by distinguishing three cases. In the following, the Lebesgue exponent $\alpha+2$, is defined by the same α appearing in the nonlinearity of (4.0.1).

Case 1: $|t_n|\to\infty$. From (4.3.16) we see that (4.3.11) holds, observing that W_n is uniformly bounded and using the dispersive estimate (1.4.1) and a density argument; hence the orthogonality in $L^{\alpha+2}$.

Since $\{y_n\}_{n\in\mathbb{N}}\subset\mathbb{T}$ which is compact, in the next two cases we can assume that up to subsequence $y_n\to\bar{y}\in\mathbb{T}$.

Case 2: $t_n \to \bar{t}$ *&* $x_n \to \bar{x}$. We claim the following:

$$\vec{v}_n^0(x,y) - e^{t_n H}\vec{\psi}(x+x_n, y+y_n) = e^{t_n H}\vec{W}_n(x+x_n, y+y_n) \to 0,$$

for almost every $(x,y) \in \mathbb{R}^d \times \mathbb{T}$. In fact

$$(e^{t_n H}\vec{W}_n(x+x_n, y+y_n), \vec{\psi})_{\mathcal{H}} = (\vec{W}_n, e^{-\bar{t}H}\vec{\psi}(x-\bar{x}, y-\bar{y}))_{\mathcal{H}} + o(1) = o(1),$$

if we localize in the euclidean part, i.e. if we consider the restriction of $e^{t_n H}\vec{W}_n(x + x_n, y+y_n)$ on a compact set $K \subset \mathbb{R}^d$. The compactness of $K \times \mathbb{T}$ gives, by the Rellich-Kondrakhov theorem, that $W_n(t_n, x+x_n, y+y_n)$ strongly converges towards zero in $L^p(K \times \mathbb{T})$ for any $p \in (2, 2^*)$, see [64]. Therefore we have (x,y)-almost everywhere convergence towards zero of $W_n(t_n, x+x_n, y+y_n)$. We recall that the Brezis-Lieb Lemma (see [15]) holds on a general measure space, therefore the same argument given in [6] yields to the $L^{\alpha+2}$-orthogonality in the case $t_n \to \bar{t}$ and $x_n \to \bar{x}$.

Case 3: if $t_n \to \bar{t}$ *&* $|x_n| \to \infty$. Similar arguments apply to the remaining situation $t_n \to \bar{t}$ and $|x_n| \to \infty$.

It remains to prove that we can rearrange the sequences of translation parameters $\{t_n\}_{n\in\mathbb{N}}$, $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$. Namely, we wish to have that for any $n \in \mathbb{N}$, $t_n = 0$ or $t_n \to \pm\infty$, and similarly for $\{x_n\}_{n\in\mathbb{N}}$, while y_n can be assumed to be trivial. In the following, by $t_n \to \bar{t}$ and $x_n \to \bar{x}$ we will implicitly assume that this possibly holds after extraction of subsequences from bounded sequences.

Case 1: $t_n \to \bar{t}$ *&* $|x_n| \to \infty$. By continuity of the linear flow

$$e^{t_n H}\vec{\psi} \xrightarrow{\mathcal{H}} \vec{\phi}, \qquad \vec{\phi}(x,y) := e^{\bar{t}H}\vec{\psi}(x,y).$$

We rewrite $\vec{v}_n^0$ as

$$\vec{v}_n^0(x-x_n, y-y_n) = \vec{\phi}(x,y) + e^{t_n H}\vec{W}_n(x,y) + \vec{r}_n(x,y) = \vec{\phi}(x,y) + \vec{\rho}_n(x,y),$$

where $\vec{r}_n \to 0$ strongly in $\mathcal{H}$ and $\vec{\rho}_n = e^{t_n H}\vec{W}_n(x,y) + \vec{r}_n(x,y)$. From Lemma 4.3.4 it follows that if $\vec{h}_n \rightharpoonup 0$ in $\mathcal{H}$ and $t_n \to \bar{t}$ then $e^{t_n H}\vec{h}_n \rightharpoonup 0$. Therefore $\vec{\rho}_n \rightharpoonup 0$ in $\mathcal{H}$. It is true whether $x_n \to x_0 \in \mathbb{R}^d$ or $|x_n| \to \infty$. Translating the profiles by $\bar{y}$, namely by choosing $\vec{\phi}(x,y) := e^{\bar{t}H}\vec{\psi}(x, y-\bar{y})$ we can also assume that $y_n = 0$.

Case 2: $t_n \to \bar{t}$ *&* $x_n \to \bar{x}$. If $t_n \to \bar{t} \in \mathbb{R}$ and also $x_n \to \bar{x} \in \mathbb{R}^d$ we proceed similarly by adding a space translation: namely as before but considering $\vec{\phi} := e^{\bar{t}H}\vec{\psi}(x-\bar{x}, y-\bar{y})$.

Case 3: $t_n \to \pm\infty$ *&* $x_n \to \bar{x}$. If $t_n \to \pm\infty$ and $x_n \to \bar{x} \in \mathbb{R}$ then we change the function by translating in the space variables only, i.e. we consider $\vec{\phi} := \vec{\psi}(x-\bar{x}, y-\bar{y})$.

Case 4: $|t_n| \to \infty$ *&* $|x_n| \to \infty$. By extracting subsequences we have the desired property, again by translating the profiles in the y variable only.

$\square$

We can now state the Profile Decomposition Theorem for a bounded sequence of linear solutions in the energy space.

Theorem 4.3.6. *Let $\{\vec{u}_n(t,x,y)\}_{n\in\mathbb{N}}$ be a sequence of solutions to the linear Klein-Gordon equation, bounded in $H^1(\mathbb{R}^d\times\mathbb{T})\times L^2(\mathbb{R}^d\times\mathbb{T})$ for $1\le d\le 4$. Recall that $\|\vec{u}_n(t,x,y)\|_{\mathcal{H}}=\|\vec{u}_n(0,x,y)\|_{\mathcal{H}}$, thus we are assuming that $\sup_n\|\vec{u}_n(0)\|_{\mathcal{H}}<\infty$. For any integer $J\ge 1$ the decomposition below, up to subsequences, holds:*

$$\vec{u}_n(t,x,y)=\sum_{1\le j\le J}\vec{v}^j(t-t_n^j,x-x_n^j,y)+\vec{R}_n^J(t,x,y),$$

where $\vec{v}^j$ are solutions to linear Klein-Gordon with suitable initial data and the translation sequences satisfy

$$\lim_{n\to\infty}\left(|t_n^k-t_n^j|+|x_n^k-x_n^j|\right)=\infty,\qquad \forall\, j\neq k,$$

along with the same dichotomy property of (4.3.12). Moreover, for $q\in(2,2^*)$

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|R_n^J\|_{L^\infty L^q}=0$$

which in turn implies that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|R_n^J\|_{L^{\alpha+1}L^{2(\alpha+1)}}=0.$$

Furthermore as $n\to\infty$,

$$\|\vec{u}_n(0,x,y)\|_{\mathcal{H}}^2=\sum_{1\le j\le J}\|\vec{v}_n^j\|_{\mathcal{H}}^2+\|\vec{R}_n^J\|_{\mathcal{H}}^2+o(1),$$

and

$$\|u_n(0,x,y)\|_{L^{\alpha+2}}^{\alpha+2}=\sum_{1\le j\le J}\|v_n^j\|_{L^{\alpha+2}}^{\alpha+2}+\|R_n^J\|_{L^{\alpha+2}}^{\alpha+2}+o(1).$$

Proof. We iterate several times the result of Proposition 4.3.5. We consider $\{\vec{v}_n\}_{n\in\mathbb{N}}$ as the sequence of initial data of the linear solution $\{\vec{u}_n(t,x,y)\}_{n\in\mathbb{N}}$; namely we consider the sequence $\{\vec{u}_n(0,x,y)\}_{n\in\mathbb{N}}$ as a bounded sequence in $\mathcal{H}$. Let $\{t_n^1\}_{n\in\mathbb{N}}$ be the sequence given in the proposition above and $\{x_n^1\}_{n\in\mathbb{N}}\subset\mathbb{R}^d$ be such that, up to subsequences,

$$\vec{u}_n(-t_n^1,x-x_n^1,y)\rightharpoonup\vec{\psi}^1(x,y)$$

in $\mathcal{H}$. Then

$$\vec{u}_n(-t_n^1,x-x_n^1,y)=\vec{\psi}^1(x,y)+\vec{W}_n^1(x,y),$$

with $\vec{W}_n^1\rightharpoonup 0$ in $\mathcal{H}$. It follows, as $n\to\infty$, that

$$\vec{u}_n(0,x,y)=e^{t_n^1H}\vec{\psi}^1(x+x_n^1,y)+e^{t_n^1H}\vec{W}_n^1(x+x_n^1,y):=e^{t_n^1H}\vec{\psi}^1(x+x_n^1,y)+\vec{R}_n^1(x,y),$$

where

$$e^{-t_n^1H}\vec{R}_n^1(x-x_n^1,y)=\vec{W}_n^1(x,y)\rightharpoonup 0\tag{4.3.17}$$

in $\mathcal{H}$, and that

$$\|\vec{u}_n(0)\|^2_{\mathcal{H}} = \|\vec{\psi}^1\|^2_{\mathcal{H}} + \|\vec{R}^1_n\|^2_{\mathcal{H}} + o(1) = \|\vec{\psi}^1\|^2_{\mathcal{H}} + \|\vec{W}^1_n\|^2_{\mathcal{H}} + o(1).$$

Similar claim can be proved for the $L^{\alpha+2}$-norm. We now consider the functions $\vec{R}^1_n(x,y) = e^{t^1_n H}\vec{W}^1_n(x+x^1_n, y)$ as bounded sequence in $\mathcal{H}$. As before, we can write

$$\vec{R}^1_n(x,y) = e^{t^2_n H}\vec{\psi}^2(x+x^2_n, y) + e^{t^2_n H}\vec{W}^2_n(x+x^2_n, y) := e^{t^2_n H}\vec{\psi}^2(x+x^2_n, y) + \vec{R}^2_n(x,y),$$

where $\vec{W}^2_n \rightharpoonup 0$ in $\mathcal{H}$ and

$$\|\vec{R}^1_n\|^2_{\mathcal{H}} = \|\vec{\psi}^2\|^2_{\mathcal{H}} + \|\vec{R}^2_n\|^2_{\mathcal{H}} + o(1) = \|\vec{\psi}^2\|^2_{\mathcal{H}} + \|\vec{W}^2_n\|^2_{\mathcal{H}} + o(1).$$

It implies that at the second step we have

$$\vec{u}_n(0,x,y) = e^{t^1_n H}\vec{\psi}^1(x+x^1_n, y) + e^{t^2_n H}\vec{\psi}^2(x+x^2_n, y) + \vec{R}^2_n(x,y),$$

and by acting with the linear propagator on both sides we get

$$\vec{u}_n(t,x,y) = e^{(t+t^1_n)H}\vec{\psi}^1(x+x^1_n, y) + e^{(t+t^2_n)H}\vec{\psi}^2(x+x^2_n, y) + e^{tH}\vec{R}^2_n(x,y).$$

Moreover, as $n \to \infty$,

$$\|\vec{u}(t,x,y)_n\|^2_{\mathcal{H}} = \|\vec{\psi}^1\|^2_{\mathcal{H}} + \|\vec{\psi}^2\|^2_{\mathcal{H}} + \|\vec{R}^2_n\|^2_{\mathcal{H}} + o(1) = \|\vec{\psi}^1\|^2_{\mathcal{H}} + \|\vec{\psi}^2\|^2_{\mathcal{H}} + \|\vec{W}^2_n\|^2_{\mathcal{H}} + o(1),$$

and the orthogonality for the $L^{\alpha+2}$-norm can be similarly proved. Recall that

$$e^{t^1_n H}\vec{W}^1_n(x+x^1_n, y) = \vec{R}^1_n(x,y) = e^{t^2_n H}\vec{\psi}^2(x+x^2_n, y) + e^{t^2_n H}\vec{W}^2_n(x+x^2_n, y),$$

and so

$$e^{(t^1_n - t^2_n)H}\vec{W}^1_n(x+(x^1_n - x^2_n), y) = \vec{\psi}^2(x,y) + \vec{W}^2_n(x,y),$$

with $\vec{W}^2_n \rightharpoonup 0$ in $\mathcal{H}$, and this implies the weak convergence in $\mathcal{H}$

$$e^{(t^1_n - t^2_n)H}\vec{W}^1_n(x+(x^1_n - x^2_n), y) \rightharpoonup \vec{\psi}^2(x,y).$$

Lemma 4.3.4, which is the equivalent of [6, Lemma 2.1] in our context, allows us to conclude with the orthogonality condition

$$|t^1_n - t^2_n| + |x^1_n - x^2_n| \to \infty.$$

Iterating this construction we end up, at the J^{th} step, with

$$\vec{u}_n(t,x,y) = e^{(t+t^1_n)H}\vec{\psi}^1(x+x^1_n, y) + \cdots + e^{(t+t^{J-1}_n)H}\vec{\psi}^{J-1}(x+x^{J-1}_n, y) + e^{tH}\vec{R}^J_n(x,y),$$

where

$$\vec{R}^J_n(x,y) = e^{t^J_n H}\vec{W}^J_n(x+x^k_n, y), \qquad W^J_n \rightharpoonup 0.$$

Moreover the free energy orthogonality holds:

$$\|\vec{u}_n(t,x,y)\|^2_{\mathcal{H}} = \|\vec{\psi}^1\|^2_{\mathcal{H}} + \cdots + \|\vec{\psi}^{J-1}\|^2_{\mathcal{H}} + \|\vec{R}^J_n\|^2_{\mathcal{H}},$$

and by the fact that the LHS is uniformly bounded in $L^\infty\mathcal{H}$ we get

$$\lim_{J\to\infty} \|\psi^J\|_{L^2} \le \lim_{J\to\infty} \|\psi^J\|_{H^1} \le \lim_{J\to\infty} \|\vec{\psi}^J\|_{\mathcal{H}} = 0.$$

Using (4.3.15) we obtain the smallness of the remainders in the sense of

$$\limsup_{J\to\infty} \limsup_{n\to\infty} \|R^J_n\|_{L^\infty L^q} = 0.$$

The proof of the smallness in the Strichartz norm

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|R^J_n\|_{L^{\alpha+1}L^{2(\alpha+1)}} = 0,$$

is done by interpolation:

Lemma 4.3.7. *Let $\alpha \in \left(\frac{4}{d}, \frac{4}{d-1}\right)$ for $2 \le d \le 4$ or $\alpha > 4$ if $d = 1$. Consider $\{u_n\}_{n\in\mathbb{N}}$ a sequence of solutions to*

$$\begin{cases} \partial_{tt}u_n - \Delta_{x,y}u_n + u_n = 0, & (t,x,y) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{T} \\ u_n(0,x,y) = f_n(x,y) \in H^1(\mathbb{R}^d\times\mathbb{T}) & \\ \partial_t u_n(0,x,y) = g_n(x,y) \in L^2(\mathbb{R}^d\times\mathbb{T}) & \end{cases},$$

with $\sup_{n\in\mathbb{N}} \|\,(f_n,g_n)\,\|_{\mathcal{H}} \le C < \infty$. Suppose that for any $q \in (2, 2^*)$, with 2^* defined in (4.3.1)

$$\lim_{n\to\infty} \|u_n\|_{L^\infty L^q} = 0.$$

Then

$$\lim_{n\to\infty} \|u_n\|_{L^{\alpha+1}L^{2(\alpha+1)}} = 0.$$

Proof. We drop the subscript n to lighten the notations. We first make a formal computation (from Hölder inequality) without adjusting the parameters:

$$\begin{aligned}
\|u\|_{L^{\alpha+1}L^{2(\alpha+1)}} &= \left(\int\left(\int |u|^a|u|^b\,dx\,dy\right)^{1/2} dt\right)^{1/(\alpha+1)} \\
&\le \left(\int\left(\int |u|^{ar}\,dx\,dy\right)^{1/(2r)}\left(\int |u|^{bs}\,dx\,dy\right)^{1/(2s)} dt\right)^{1/(\alpha+1)} \\
&\le \left(\int \|u\|^{a/2}_{L^{ar}}\,\|u\|^{b/2}_{L^{bs}}\,dt\right)^{1/(\alpha+1)} \\
&\le \|u\|^{a/(2\alpha+2)}_{L^\infty L^{ar}}\|u\|^{b/(2\alpha+2)}_{L^{b/2}L^{bs}}. \qquad (4.3.18)
\end{aligned}$$

The claim of Lemma 4.3.7 is satisfied if the following conditions are fulfilled in (4.3.18):

$$a + b = 2\alpha + 2, \qquad a, b > 0, \tag{4.3.19}$$
$$r = q/a > 1, \tag{4.3.20}$$
$$s = r/(r-1) = q/(q-a), \tag{4.3.21}$$
$$(b/2, bs) \quad \text{is a Strichartz pair as in } \textit{Proposition } 1.3.2. \tag{4.3.22}$$

Under these conditions, we may have by hypothesis along with the energy conservation

$$\|u_n\|_{L^{\alpha+1}L^{2\alpha+2}} \leq \|u_n\|^{\gamma}_{L^\infty L^q} E^{1-\gamma} \to 0,$$

where $\gamma \in (0, 1)$. Note that it is enough to have the convergence to zero in only one $L^\infty L^q$.

Let us now check that all conditions are non-empty.

Case $d = 1$. Let $b = 2\alpha + \epsilon$ and $a = 2 - \epsilon$. We impose that $\epsilon \in (0, 2)$ in order to satisfy (4.3.19). Strichartz admissibility conditions read $b \geq 8$ and $s \geq 2/(b - 8)$. We strengthen the first requirement to $b > 8$. By definition of s we have

$$\frac{q}{q - 2 + \epsilon} \geq \frac{2}{2\alpha + \epsilon - 8} \iff q(2\alpha - 10 + \epsilon) \geq 2\epsilon - 4.$$

For $\alpha \geq 5$ and for any $q \in (2, 2^*)$ the LHS of the last inequality is positive for any choice of $\epsilon \in (0, 2)$, since the RHS is always negative for such values of ϵ. Then any $q \in (2, 2^*)$ yields to Strichartz admissibility condition.

If $\alpha \in (4, 5)$ we further impose on ϵ the condition $\epsilon > 10 - 2\alpha$ beside the upper bound $\epsilon < 2$ so that the LHS is still positive and hence any $0 < q \in (2, 2^*)$ is good for our purpose.

Case $d = 2$. Recall that in this dimension $2^* = 4$. We chose $q = q(\alpha) = 2\alpha - 2$. We observe that $q(\alpha) \in (2, 2^*)$ for any $\alpha \in (2, 4)$ which is the range where α is allowed in dimension $d = 2$.

Strichartz admissibility reads $b/2 > 2 \iff b > 4$ and $\frac{2}{b-4} \leq s \leq \frac{6}{b-4}$ which is equivalent to

$$\begin{aligned}
\frac{2}{b-4} \leq \frac{q}{q-a} \leq \frac{6}{b-4} &\iff \frac{2}{b-4} \leq \frac{2\alpha - 2}{2\alpha - 2 - (2\alpha + 2 - b)} \leq \frac{6}{b-4} \\
&\iff \frac{2}{b-4} \leq \frac{2\alpha - 2}{b-4} \leq \frac{6}{b-4} \\
&\iff 2 \leq \alpha \leq 4
\end{aligned}$$

which is satisfied for any intra-critical $\alpha \in (2, 4)$.

Case $d = 3$. In this case $2^* = 4$ and $\alpha \in (4/3, 2)$. To satisfy the admissibility condition, at first we impose $b \geq 4$. The second Strichartz condition reads

$$\frac{6}{3b - 8} \leq \frac{q}{q-a} \leq \frac{4}{b-2}.$$

Let us focus on the LHS condition.

$$\frac{6}{3b-8} \le \frac{q}{q-a} \iff 6(q-2\alpha-2+b) \le q(3b-8)$$
$$\iff b(6-3q) \le 12\alpha+12-14q$$
$$\iff b \ge \frac{12\alpha+12-14q}{6-3q} := c_1(\alpha,q)$$

If we impose $c_1 < 4$ we are done. But $c_1 < 4 \iff q < 6(\alpha-1)$. So we restrict the upper bound for the choice of q as

$$q < \min\{4, 6(\alpha-1)\}.$$

Let us now focus on the RHS condition.

$$\frac{q}{q-a} \le \frac{4}{b-2} \iff \frac{q}{(q-2\alpha-2+b)} \le \frac{4}{b-2}$$
$$\iff q(b-2) \le 4(q-2\alpha-2+b)$$
$$\iff b(4-q) \ge 8\alpha+8-6q$$
$$\iff b \ge \frac{8\alpha+8-6q}{4-q} := c_2(\alpha,q)$$

If we impose $c_2 < 4$ we are done. But this last condition is equivalent to $q > 4(\alpha-1)$ and then by considering

$$q > \max\{2, 4(\alpha-1)\}$$

we are able to conclude summarizing with

$$\max\{2, 4(\alpha-1)\} < q < \min\{4, 6(\alpha-1)\}.$$

Case $d = 4$. In this case $2^* = 10/3$ and $\alpha \in (1, 4/3)$. To satisfy the admissibility condition, at first we impose $b \ge 4$. The second Strichartz condition reads

$$\frac{2}{b-2} \le \frac{q}{q-a} \le \frac{10}{3b-4}.$$

Let us focus on the LHS condition.

$$\frac{2}{b-2} \le \frac{q}{q-a} \iff \frac{2}{b-2} \le \frac{q}{q-2\alpha-2+b}$$
$$\iff b(q-2) \ge 4q-4\alpha-4$$
$$\iff b \ge \frac{4q-4\alpha-4}{q-2} := c_3(\alpha,q)$$

If we impose $c_3 < 4$ we are done. But $c_3 < 4 \iff \alpha > 1$ which is always satisfied under the intra-criticality condition.

Let us now focus on the RHS condition.

$$\begin{aligned}
\frac{q}{q-a} \leq \frac{10}{3b-4} &\iff \frac{q}{(q-2\alpha-2+b)} \leq \frac{10}{3b-4} \\
&\iff q(3b-4) \leq 10(q-2\alpha-2+b) \\
&\iff b(10-3q) \geq 20\alpha+20-14q \\
&\iff b \geq \frac{20\alpha+20-14q}{10-3q} := c_4(\alpha,q)
\end{aligned}$$

If we impose $c_4 < 4$ we are done. But this last condition is equivalent to $q > 10(\alpha-1)$ and then by considering

$$q > \max\{2, 10(\alpha-1)\}$$

we are able to conclude summarizing with

$$\max\{2, 4(\alpha-1)\} < q < \frac{10}{3}.$$

□

The proof of Theorem 4.3.6 is complete. □

4.4 Construction of the minimal element

Theorem 4.3.6 is the key tool for the construction of a minimal (with respect to the energy) non-scattering solution to (4.0.1) with some compactness property. We define the following critical energy:

$$\begin{aligned}
E_c = \sup\{E > 0 \,|\, (f,g) \in \mathcal{H} \quad &and \quad E(f,g) < E \\
&\implies u_{(f,g)}(t) \in L^{\alpha+1}L^{2(\alpha+1)} < \infty\}
\end{aligned}$$

where $u_{(f,g)}(t)$ denotes here the global solution to (4.0.1) with Cauchy data f and g. Our final aim, at the end of the chapter, is to exclude that E_c is finite.

The result stated in Theorem 4.2.1 ensures that $E_c > 0$. The strategy consists in a contradiction argument. If we suppose that E_c is finite, we will show that there exists a critical solution $u_c(t)$ to (4.0.1), with energy E_c, such that it does not belong to the Strichartz space $L^{\alpha+1}L^{2(\alpha+1)}$. It will moreover enjoy some compactness properties. The latter will imply that such critical solution must be the trivial one, hence a contradiction.

We first proceed with the construction of the critical solution, based on Theorem 4.3.6 and Lemma 4.2.3.

Once every ingredient is given, we continue with the extraction of the critical solution. We therefore assume that $E_c < \infty$. Let $(f_n, g_n) \in \mathcal{H}$ be a sequence of Cauchy data such that $E(f_n, g_n) \to E_c$ as $n \to +\infty$ and let $u_n(t) := u_{(f_n,g_n)}(t)$ be the corresponding solutions to (4.0.1) which exist globally in time but do not belong to $L^{\alpha+1}L^{2(\alpha+1)}$, i.e. $\|u_n\|_{L^{\alpha+1}L^{2(\alpha+1)}} = \infty$. The last condition means that we are considering a maximizing sequence $(f_n, g_n) \in \mathcal{H}$ whose corresponding solutions do not satisfy the scattering property.

Since $E(f_n, g_n) \to E_c$ and the energy is a conserved quantity, we can state that $\vec{u}_n^0 := (f_n, g_n)$ is uniformly bounded in $\mathcal{H}$. And since the Klein-Gordon linear flow preserves the $\mathcal{H}$-norm, the sequence $e^{tH}\vec{u}_n^0$ is uniformly bounded in $L^\infty\mathcal{H}$. Thus we can apply the linear profile decomposition to this sequence of free solutions and we can write

$$e^{tH}\vec{u}_n^0 = \sum_{1\le j\le J} \vec{v}_n^j(t) + \vec{R}_n^J(t,x,y), \tag{4.4.1}$$

where $\vec{v}_n^j(t) = \vec{v}^j(t - t_n^j, x - x_n^j, y) = e^{(t-t_n^j)H}\vec{\psi}^j(x - x_n^j, y)$ for suitable $\vec{\psi}^j \in \mathcal{H}$. We recall that the profile decomposition theorem given above ensures the orthogonality of the translation sequences in the sense of

$$\lim_{n\to+\infty} \left(|t_n^h - t_n^j| + |x_n^h - x_n^j|\right) = +\infty, \tag{4.4.2}$$

for all $j \neq h$, the smallness of the remainders in the sense of

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|R_n^J(t)\|_{L^\infty L^q \cap L^{\alpha+1}L^{2(\alpha+1)}} = 0, \tag{4.4.3}$$

as well as the pythagorean expansions of the quadratic and super quadratic terms of the energy. More precisely, for $n \to \infty$,

$$\|(f_n, g_n)\|_{\mathcal{H}}^2 = \|\vec{u}_n(0,x,y)\|_{\mathcal{H}}^2 = \|\vec{u}_n(t,x,y)\|_{\mathcal{H}}^2 = \sum_{1\le j\le J} \|\vec{v}_n^j\|_{\mathcal{H}}^2 + \|\vec{R}_n^J\|_{\mathcal{H}}^2 + o(1), \tag{4.4.4}$$

and

$$\|u_n(0,x,y)\|_{L^{\alpha+2}}^{\alpha+2} = \sum_{1\le j\le J} \|v_n^j\|_{L^{\alpha+2}}^{\alpha+2} + \|R_n^k\|_{L^{\alpha+2}}^{\alpha+2} + o(1). \tag{4.4.5}$$

We suppose that $J > 1$ and we follow the same strategy as in [6, 102]. We have that, at most, in one case we can have that both space and time translations sequences are trivial, due to (4.4.2). Without loss of generality we can suppose that this case happens when $j = 1$, and since we are assuming $J > 1$ we have, by orthogonality of the energy expressed by summing up (4.4.4) and (4.4.5), that $\vec{\psi}^1$ is such that the corresponding solution $z^1 := u_{\vec{\psi}^1}$ to (4.0.1) scatters, as it belongs to $L^{\alpha+1}L^{2(\alpha+1)}$ by definition. In the other cases $j \geq 2$, we associate to a linear profile $\vec{\psi}^j$, a nonlinear profile in a proper way. We associate a nonlinear profile V^j to each linear profile v^j thanks to the following procedure: V^j is a nonlinear solution to (4.0.1) such that

$$\lim_{n\to\infty} \|\vec{v}^j(t_n^j) - \vec{V}^j(t_n^j)\|_{\mathcal{H}} = 0.$$

Recall that by the dichotomy property of the parameters, for every j, $\lim_{n\to\infty} t_n^j = 0$ or $\lim_{n\to\infty} |t_n^j| = \infty$. Then V^j is locally defined both in a neighborhood of $t = 0$ or $|t| = \infty$:

the first property follows by the local well-posedness theory, while the second one by the existence of the wave operators. Due to the defocusing nature of the equation, V^j is actually globally defined. Orthogonality of the energy given by (4.4.4) together with (4.4.5) implies that any nonlinear profile V^j has an energy less than the minimal one E_c. Let us define

$$V(t) = \sum_{j=1}^{J} V^j(t - t_n^j, x - x_n^j, y);$$

we use the perturbation lemma with V instead of v in Lemma 4.2.3 and u_n in the role of u of the Lemma 4.2.3. As in [102] this would imply that

$$\limsup_{n\to\infty} \| \sum_{j=1}^{J} V^j(t - t_n^j, x - x_n^j, y)\|_{L^{\alpha+1}L^{2(\alpha+1)}} < C < \infty, \qquad \text{uniformly in } \; J,$$

and Lemma 4.2.3 gives

$$\limsup_{n\to\infty} \|u_n\|_{L^{\alpha+1}L^{2(\alpha+1)}} < C,$$

which is a contradiction. Therefore $J = 1$, and the precompactness of the trajectory up to a translation also follows by [102]. We can summarize the core result of this section in the following theorem.

Theorem 4.4.1. *There exists an initial datum $(f_c, g_c) \in H^1(\mathbb{R}^d \times \mathbb{T}) \times L^2(\mathbb{R}^d \times \mathbb{T})$ such that the corresponding solution $u_c(t)$ to (4.0.1) is global and $\|u_c\|_{L^{\alpha+1}L^{2(\alpha+1)}} = \infty$. Moreover there exists a path $x(t) \in \mathbb{R}^d$ such that $\{u_c(t, x - x(t), y), \partial_t u_c(t, x - x(t), y)\}_{t\in\mathbb{R}^+}$ is precompact in $H^1(\mathbb{R}^d \times \mathbb{T}) \times L^2(\mathbb{R}^d \times \mathbb{T})$.*

4.5 Death of the soliton-like solution

This section establishes that the critical (or soliton-like) solution $u_c(t)$ built in the previous section cannot exist. The first step is to prove the validity of the finite propagation speed in our framework. It will be useful to control the growth of the translation path $x(t) \in \mathbb{R}^d$ given in Theorem 4.4.1. Let us first recall this simple result.

Lemma 4.5.1. *Let f be smooth and $B(x_0, r) \subset \mathbb{R}^d$ the ball centered in x_0 with radius r. The following equality holds:*

$$\frac{d}{dr} \int_{B(x_0,r)} f(x)\, dx = \int_{\partial B(x_0,r)} f(\sigma)\, d\sigma$$

where $\partial B(x_0, r)$ is the boundary of $B(x_0, r)$ and $d\sigma$ is the surface measure on $\partial B(x_0, r)$.

Proof. The proof is straightforward once switched in radial coordinates. □

We then state the following, which is the finite time propagation speed mentioned above. The notation $B(x_0, r)^c$ stands for $\mathbb{R}^d \setminus B(x_0, r)$.

Proposition 4.5.2. *Let u be the solution to (4.0.1) with Cauchy datum (u_0, u_1) vanishing on $B(x_0, r)^c \times \mathbb{T}$, for some $r > 0$. Then $\vec{u}(t) = (u, \partial_t u)(t)$ vanishes on $K(x_0, r) := \{t \geq 0,\ x \in B(x_0, r+t)^c,\ y \in \mathbb{T}\}$.*

Proof. Fix $r > 0$, $x_0 \in \mathbb{R}^d$, consider the balls $B(x_0, r+t) := B(t+r)$ and define the local energy $E_r(t)$ as

$$E_r(t) = \frac{1}{2} \int_{\mathbb{T}} \int_{B(r+t)} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2} |u|^{\alpha+2} \right)(t)\, dx\, dy.$$

Assume that $u(t, x, y)$ is smooth enough (by a classical regularization argument, the following then extends to rougher solutions), and let us calculate the first time derivative of the local energy:

$$\begin{aligned}
\frac{d}{dt} E_r(t) = & \int_{\mathbb{T}} \int_{B(r+t)} \partial_t u \partial_{tt} u + \sum_{i \in \{1,\dots,d\}} \partial_{x_i} u\, \partial_{x_i} \partial_t u + \partial_y u\, \partial_y \partial_t u \ dx\, dy \\
& + \int_{\mathbb{T}} \int_{B(r+t)} u \partial_t u + \frac{1}{\alpha+2} |u|^{\alpha} u \partial_t u \ dx\, dy \\
& + \frac{1}{2} \int_{\mathbb{T}} \int_{\partial B(r+t)} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2} |u|^{\alpha+2} \right)(t)\, d\sigma\, dy \\
= & \int_{\mathbb{T}} \int_{B(r+t)} \partial_t u \partial_{tt} u + div_x(\partial_t u \nabla_x u) - \partial_t u \Delta_x u + \partial_y(\partial_t u \partial_y u) - \partial_t u \partial_{yy} u \ dx\, dy \\
& + \int_{\mathbb{T}} \int_{B(r+t)} u \partial_t u + \frac{1}{\alpha+2} |u|^{\alpha} u \partial_t u\, dx\, dy \\
& + \frac{1}{2} \int_{\mathbb{T}} \int_{\partial B(r+t)} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2} |u|^{\alpha+2} \right)(t)\, d\sigma\, dy \\
= & \int_{\mathbb{T}} \int_{B(r+t)} div_x(\partial_t u \nabla_x u)\, dx\, dy + \int_{B(r+t)} \int_{\mathbb{T}} \partial_y(\partial_t u \partial_y u)\, dy\, dx \\
& + \frac{1}{2} \int_{\mathbb{T}} \int_{\partial B(r+t)} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2} |u|^{\alpha+2} \right)(t)\, d\sigma\, dy \\
= & - \int_{\mathbb{T}} \int_{\partial B(r+t)} \partial_t u \nabla u \cdot n_i\, d\sigma\, dy \\
& + \frac{1}{2} \int_{\mathbb{T}} \int_{\partial B(r+t)} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2} |u|^{\alpha+2} \right)(t)\, d\sigma\, dy.
\end{aligned}$$

where $n_i = n_i(x)$, $x \in \partial B$, denotes the inner normal vector to the boundary of B. Recall that the energy on the whole space in conserved, and so by using Cauchy-Schwartz inequality

$$
\begin{aligned}
\frac{d}{dt}(E - E_r(t)) &= \frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{T}}\int_{B(r+t)^c}\left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t)\,dx\,dy\right\} \\
&= \int_{\mathbb{T}}\int_{\partial B(r+t)} \partial_t u \nabla u \cdot n_i \,d\sigma\,dy \\
&\quad - \frac{1}{2}\int_{\mathbb{T}}\int_{\partial B(r+t)}\left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t)\,d\sigma\,dy \\
&\leq \frac{1}{2}\int_{\mathbb{T}}\int_{\partial B(r+t)} |\partial_t u|^2 + |\nabla u|^2 \,d\sigma\,dy \\
&\quad - \frac{1}{2}\int_{\mathbb{T}}\int_{\partial B(r+t)}\left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t)\,d\sigma\,dy \leq 0,
\end{aligned}
$$

and we obtain

$$
\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{T}}\int_{B(r+t)^c}\left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t)\,dx\,dy\right\} \leq 0,
$$

namely the energy on $B(x_0, r+t)^c \times \mathbb{T}$ is decreasing. The conclusion follows. □

We now give an estimate from above of a portion away from zero of the potential energy. This will be essential in the last section dealing with the rigidity part in the Kenig and Merle scheme. We follows the ideas of Bulut in [16], who in turn was inspired by Killip and Visan [82–84]. We point out that in [16] the situation is much more involved, since the author is considering energy supercritical NLW.

Lemma 4.5.3. *Let $u(t,x,y)$ be a solution to (4.0.1). If $\{\vec{u}(t)\}_{t\in\mathbb{R}} \subset \mathcal{H}$ is a relatively compact set and $\vec{u}^* \in \mathcal{H}$ is one of its limit points, then $\vec{u}^* \neq 0$.*

Proof. This property simply follows from the conservation of the energy. □

At this point we can give the following lemma, essentially based on the well-posedness of (4.0.1), in particular its continuous dependence on the initial data.

Lemma 4.5.4. *Let $u(t)$ be a nontrivial solution to (4.0.1) such that*

$$
\{u(t, x - x(t), y), \partial_t u(t, x - x(t), y)\}_{t\in\mathbb{R}} \quad \textit{is relatively compact in} \quad \mathcal{H}.
$$

Then for any $A > 0$, there exists $C(A) > 0$ such that for any $t \in \mathbb{R}$

$$
\int_t^{t+A}\int_{\mathbb{T}}\int_{|x - x(s)|\leq R} |u|^{\alpha+2}(s,x,y)\,dx\,dy\,ds \geq C(A), \tag{4.5.1}
$$

for $R = R(A)$ large enough.

Proof. We argue by contradiction, supposing that there exists a sequence of times $\{t_n\}_{n\in\mathbb{N}}$ such that

$$\int_{t_n}^{t_n+A}\int_{\mathbb{R}^d\times\mathbb{T}}|u|^{\alpha+2}(s,x,y)\,dx\,dy\,ds<\frac{1}{n}.$$

By compactness, up to subsequence still denoted with the subscript n,

$$(u(t_n,x-x(t_n),y),\partial_t u(t_n,x-x(t_n),y))\to(f,g)\in\mathcal{H}.$$

Let $(w(0),\partial_t w(0))=(f,g)$ be an initial datum and $w(t)$ be the corresponding solution to (4.0.1): then we have, by the fact that $u\neq 0$,

$$\begin{aligned}E(w,\partial_t w)&=E(f,g)\\&=\lim_{n\to\infty}E(u(t_n,x-x(t_n),y),\partial_t u(t_n,x-x(t_n),y))\\&=E(u_0,u_1)\neq 0.\end{aligned}\tag{4.5.2}$$

Local well-posedness and Strichartz estimates imply

$$\begin{aligned}0&=\lim_{n\to\infty}\int_{t_n}^{t_n+A}\int_{\mathbb{R}^d\times\mathbb{T}}|u|^{\alpha+2}(s,x,y)\,dx\,dy\,ds\\&=\lim_{n\to\infty}\int_0^A\int_{\mathbb{R}^d\times\mathbb{T}}|u|^{\alpha+2}(t_n+s,x,y)\,dx\,dy\,ds\\&=\lim_{n\to\infty}\int_0^A\int_{\mathbb{R}^d\times\mathbb{T}}|u|^{\alpha+2}(t_n+s,x-x(t_n),y)\,dx\,dy\,ds\\&=\int_0^A\int_{\mathbb{R}^d\times\mathbb{T}}|w|^{\alpha+2}(s,x,y)\,dx\,dy\,ds,\end{aligned}$$

which in turn gives that $w(t)=0$ almost everywhere in $(0,A)$. This contradicts (4.5.2), then

$$\int_t^{t+A}\int_{\mathbb{T}\times\mathbb{R}^d}|u|^{\alpha+2}\,dx\,dy\,ds\geq C'(A).$$

By exploiting again the precompactness property of the solution

$$\begin{aligned}&\int_t^{t+A}\int_{\mathbb{T}}\int_{|x-x(s)|\leq R}|u|^{\alpha+2}\,dx\,dy\,ds\\&=\int_t^{t+A}\left\{\int_{\mathbb{T}\times\mathbb{R}^d}|u|^{\alpha+2}\,dx\,dy-\int_{\mathbb{T}}\int_{|x-x(s)|\geq R}|u|^{\alpha+2}\,dx\,dy\right\}ds\\&\geq C'(A)-\frac{C'(A)}{2}=\frac{C'(A)}{2}=:C(A).\end{aligned}$$

□

Corollary 4.5.5. *By interpolation the same property can be claimed for the localized L^2-norm of u. More precisely, under the same assumption of Lemma 4.5.4 on u, for any $A>0$ there exists $C(A)>0$ such that for any $t\in\mathbb{R}$*

$$\int_t^{t+A}\int_{\mathbb{T}}\int_{|x-x(s)|\leq R}|u|^2(s,x,y)\,dx\,dy\,ds\geq C(A)\tag{4.5.3}$$

for $R=R(A)$ large enough.

The last ingredient to derive a contradiction to the existence of a such precompact solution is an a priori bound for the super-quadratic term of the energy which is due to Nakanishi, see [100]. The latter is a remarkable extension in the euclidean framework $\mathbb{R}^m$ with $m = 1, 2$ of the well-known Morawetz estimate proved by Morawetz and Strauss, see [97, 98], for higher dimensions. This a priori bounds lead to the scattering in energy space both for the nonlinear Klein-Gordon equation and the nonlinear Schrödinger equation posed in the euclidean space.

4.5.1 Nakanishi/Morawetz-type estimate

We begin this section by giving the analogue in our domain of the decay result due to Nakanishi, [100]. Our approach is to simply use a multiplier that does not consider all the variables: neither the compact factor of the product space we work on (the y variable), nor a set of $d-1$ euclidean variables ($\{x_2, \ldots, x_d\}$, for instance) will be "seen" by the multiplier. Consequently, we will show how the Nakanishi/Morawetz type estimate in one dimension is enough for a contradiction argument which will exclude soliton-like solutions, i.e. the u_c built in Theorem 4.4.1.

We report verbatim the proof contained in [100, Lemma 5.1, equation (5.1)], then we analyze the extra term given by the remaining part of the second order space operator involved in the equation. First, Nakanishi introduces the following term with relative notations (recall that in the following we are in a pure euclidean framework, with $x \in \mathbb{R}^m$ and $m = 1, 2$):

$$r = |x|, \quad \theta = \frac{x}{r}, \quad \lambda = \sqrt{t^2 + r^2}, \quad \Theta = \frac{(-t, x)}{\lambda}$$

$$u_r = \theta \cdot \nabla_x u, \quad u_\theta = \nabla_x u - \theta u_r$$

$$l(u) = \frac{1}{2}\left(-|\partial_t u|^2 + |\nabla_x u|^2 + |u|^2 + \frac{2}{\alpha + 2}|u|^{\alpha+2}\right)$$

$$(\partial_0, \partial_1, \partial_2) = (-\partial^0, \partial^1, \partial^2) = (\partial_t, \nabla_x)$$

$$g = \frac{m-1}{2\lambda} + \frac{t^2 - r^2}{2\lambda^3}, \quad M = \Theta \cdot (\partial_t u, \nabla_x u) + ug$$

$$(\partial_t^2 - \Delta_x) g = -\frac{5}{2\lambda^3} + 3\frac{t^2 - r^2}{\lambda^5} + 15\frac{(t^2 - r^2)^2}{2\lambda^7}.$$

Then by multiplying the equation $\partial_t^2 u - \Delta_x u + u + |u|^\alpha u = 0$ by M, with $u = u(t, x)$, we obtain the relation

$$\begin{aligned} 0 = (\partial_t^2 u - \Delta_x u + u + |u|^\alpha u) M = \sum_{\beta=0}^{m} \partial_\beta \left(-M \partial^\beta u + l(u)\Theta_\beta + \frac{|u|^2}{2}\partial^\beta g\right) \\ + \frac{|u_\omega|^2}{\lambda} + \frac{|u|^2}{2}(\partial_t^2 - \Delta_x) g + \frac{\alpha}{\alpha+2}|u|^{\alpha+2} g, \end{aligned} \tag{4.5.4}$$

where u_ω is the projection of $(\partial_t u, \nabla_x u)$ on the tangent space of $t^2 - |x|^2 = c$, c being a constant.

We focus on $m = 1$ and we go back to (4.0.1). We introduce the compact notation

$$\mathbb{R}^{d-1} \times \mathbb{T} =: \mathcal{Y} \ni z := (\bar{x}, y) = (x_2, \dots, x_d, y).$$

Then the analogous of (4.5.4) is the following:

$$\begin{aligned} 0 = (\partial_t^2 u - \Delta u + u + |u|^\alpha u)M = &\sum_{\beta \in \{0,1\}} \partial_\beta \left(-M\partial^\beta u + l(u)\Theta_\beta + \frac{|u|^2}{2}\partial^\beta g \right) \\ &+ \frac{|u_\omega|^2}{\lambda} + \frac{|u|^2}{2}(\partial_t^2 - \Delta_x)g + \frac{\alpha}{\alpha+2}|u|^{\alpha+2} g \\ &- M\Delta_z u. \end{aligned} \tag{4.5.5}$$

Observe that the term g is nonnegative only in the region where $r < t$. Then after integrating (4.5.5) (now $u = u(t, x_1, z)$) on $\mathcal{C} := \{(t, x_1) \,|\, 2 < t < T, |x_1| = r < t\} \times \mathcal{Y}$, using the divergence theorem, the last relation we obtain is:

$$\begin{aligned} &\left\{ \int_{\mathcal{Y}} \int_{r<t} -\partial_t u M + l(u)\frac{t}{\lambda} + \frac{|u|^2}{2}\partial_t g \, dx_1 \, dz \right\} \Bigg|_{t=2}^{t=T} \\ &= \int_{\mathcal{C}} \frac{|u_\omega|^2}{\lambda} + \frac{|u|^2}{2}(\partial_t^2 - \partial_{x_1}^2)g + \frac{\alpha}{\alpha+2}|u|^{\alpha+2} g \, dx_1 \, dz \, dt \\ &\quad + \frac{\sqrt{2}}{2} \int_{\mathcal{Y}} \int_{2<r=t<T} |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2} \, dx_1 \, dz \\ &\quad - \int_{\mathcal{C}} M\Delta_z u \, dx_1 \, dz \, dt, \end{aligned}$$

noticing that $|u_\theta|^2 = 0$ if $m = 1$. The LHS of the above identity is bounded by the energy, as well as the middle term in the first integral in the RHS thanks to the estimate

$$\left| \int_{\mathcal{C}} \frac{|u|^2}{2}(\partial_t^2 - \partial_{x_1}^2)g \right| \lesssim \int_2^T \int_{\mathbb{T}\times\mathbb{R}^d} \frac{|u|^2}{t^3} \, dx_1 \, dz \, dt \lesssim E.$$

The energy flux through the curved surface, i.e. the second integral in the RHS is estimated by the energy. In fact we have the following:

Lemma 4.5.6. *Any smooth solution u to (4.0.1) satisfies:*

$$\int_{\mathcal{Y}} \int_{2<|x_1|=t<T} |\partial_t u - \theta\partial_{x_1} u|^2 + |\nabla_z u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2} \, d\sigma \, dz \lesssim E. \tag{4.5.6}$$

Proof. The proof repeats the same analysis performed to prove the finite propagation speed property. Define

$$e(t) := \frac{1}{2} \int_{\mathcal{Y}} \int_{|x_1|<t} \left(|\partial_t u|^2 + |u|^2 + |\partial_{x_1} u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2} \right)(t, x_1, z) \, dx_1 \, dz.$$

Differentiating $e(t)$ with respect to t, we obtain

$$\begin{aligned}
\frac{d}{dt}e(t) &= \int_{\mathcal{Y}}\int_{|x_1|<t} \left(\partial_t u \partial_t^2 u + \partial_{x_1} u \partial_{x_1}\partial_t u + \partial_t u u + |u|^\alpha u \partial_t u\right)\, dx_1\, dz \\
&\quad + \frac{1}{2}\int_{\mathcal{Y}}\int_{|x_1|=t} \left(|\partial_t u|^2 + |u|^2 + |\partial_{x_1} u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)\, d\sigma\, dz \\
&= \int_{\mathcal{Y}}\int_{|x_1|<t} \partial_t u \left(\partial_t^2 u - \partial_{x_1}^2 u + u + |u|^\alpha u\right) + \partial_{x_1}(\partial_{x_1} u \cdot \partial_t u)\, dx_1\, dz \\
&\quad + \frac{1}{2}\int_{\mathcal{Y}}\int_{|x_1|=t} \left(|\partial_t u|^2 + |u|^2 + |\partial_{x_1} u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)\, d\sigma\, dz \\
&= \int_{\mathcal{Y}}\int_{|x_1|<t} \partial_t u \left(\partial_t^2 u - \Delta u + u + |u|^\alpha u\right) + \partial_t u \Delta_z u + \partial_{x_1}(\partial_{x_1} u \cdot \partial_t u)\, dx_1\, dz \\
&\quad + \frac{1}{2}\int_{\mathcal{Y}}\int_{|x_1|=t} \left(|\partial_t u|^2 + |u|^2 + |\partial_{x_1} u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)\, d\sigma\, dz \\
&= \int_{\mathcal{Y}}\int_{|x_1|<t} \partial_t u \Delta_z u + \partial_{x_1}(\partial_{x_1} u \cdot \partial_t u)\, dx_1\, dz \\
&\quad + \frac{1}{2}\int_{\mathcal{Y}}\int_{|x_1|=t} \left(|\partial_t u|^2 + |u|^2 + |\partial_{x_1} u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)\, d\sigma\, dz \\
&= -\frac{1}{2}\int_{\mathcal{Y}}\int_{|x_1|<t} \partial_t |\nabla_z u|^2\, dx_1\, dz \\
&\quad + \frac{1}{2}\int_{\mathcal{Y}}\int_{|x_1|=t} \left(|\partial_t u|^2 + |u|^2 + |\partial_{x_1} u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2} - 2\theta\partial_{x_1} u \cdot \partial_t u\right)\, d\sigma\, dz \\
&= -\frac{1}{2}\frac{d}{dt}\int_{\mathcal{Y}}\int_{|x_1|<t} |\nabla_z u|^2\, dx\, dz + \frac{1}{2}\int_{\mathcal{Y}}\int_{|x_1|=t} |\nabla_z u|^2\, d\sigma\, dz \\
&\quad + \frac{1}{2}\int_{\mathcal{Y}}\int_{|x_1|=t} \left(|\partial_t u - \theta\partial_{x_1} u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)\, d\sigma\, dz
\end{aligned}$$

therefore, integrating with respect to the time variable from 2 to T we obtain (4.5.6). $\square$

Moreover, the energy estimate on the surface of the light cone gives

$$\sup_t \int_{\mathbb{R}^{d-1}\times\mathbb{T}}\int_{\mathbb{R}} |u(|x_1|+t, x_1, z)|^2\, dx_1\, dz \lesssim E.$$

We now analyze the term $-\int_{\mathcal{C}} M\Delta_z u\, dx_1\, dz\, dt$ in (4.5.5). We rewrite explicitly the term to be integrated as

$$-M\Delta_z u = -div_z\left(M\nabla_z u\right) + \nabla_z u \cdot \nabla_z M := \mathcal{A} + \mathcal{B}.$$

The second term is explicitly given by

$$
\begin{aligned}
\mathcal{B} &= -\frac{t}{2\lambda}\partial_t|\nabla_z u|^2 + \frac{1}{2\lambda}x_1\cdot\partial_{x_1}|\nabla_z u|^2 + g|\nabla_z u|^2 \\
&= -\frac{1}{2}\partial_t\left(\frac{t}{\lambda}|\nabla_z u|^2\right) + \frac{1}{2}|\nabla_z u|^2\partial_t\left(\frac{t}{\lambda}\right) + \frac{1}{2\lambda}\left(\partial_{x_1}(x_1|\nabla_z u|^2) - |\nabla_z u|^2\right) + g|\nabla_z u|^2 \\
&= -\frac{1}{2}\partial_t\left(\frac{t}{\lambda}|\nabla_z u|^2\right) + \frac{|x_1|^2}{2\lambda^3}|\nabla_z u|^2 + \frac{1}{2\lambda}\partial_{x_1}(x_1|\nabla_z u|^2) - \frac{|\nabla_z u|^2}{2\lambda} + g|\nabla_z u|^2 \\
&= -\frac{1}{2}\partial_t\left(\frac{t}{\lambda}|\nabla_z u|^2\right) + \frac{|x_1|^2}{2\lambda^3}|\nabla_z u|^2 + \partial_{x_1}\left(\frac{x_1}{2\lambda}|\nabla_z u|^2\right) \\
&\quad - \partial_{x_1}\left(\frac{1}{2\lambda}\right)x_1|\nabla_z u|^2 - \frac{|\nabla_z u|^2}{2\lambda} + g|\nabla_z u|^2 \\
&= -\frac{1}{2}\partial_t\left(\frac{t}{\lambda}|\nabla_z u|^2\right) + \frac{|x_1|^2}{2\lambda^3}|\nabla_z u|^2 + \partial_{x_1}\left(\frac{x_1}{2\lambda}|\nabla_z u|^2\right) \\
&\quad + \frac{|x_1|^2}{2\lambda^3}|\nabla_z u|^2 - \frac{|\nabla_z u|^2}{2\lambda} + g|\nabla_z u|^2 \\
&= -\frac{1}{2}\partial_t\left(\frac{t}{\lambda}|\nabla_z u|^2\right) + \partial_{x_1}\left(\frac{x_1}{2\lambda}|\nabla_z u|^2\right)
\end{aligned}
$$

and then, after integration, it can be estimated by the energy on the whole space, while the divergence term $\mathcal{B}$ disappears using the Gauss-Green theorem.

In conclusion

$$
\int_2^\infty \int_{\mathcal{Y}\times\{|x_1|<t\}} \frac{|u_\omega|^2}{\lambda} + \frac{\alpha}{\alpha+2}|u|^{\alpha+2}g\,dx_1\,dz\,dt \lesssim E.
$$

The Nakanishi/Morawetz-type estimate follows as in [100]:

$$
\int_{\mathbb{R}}\int_{\mathbb{R}^d\times\mathbb{T}} \frac{\min\{|u|^2,|u|^{\alpha+2}\}}{\langle t\rangle\log(|t|+2)\log(\max\{|x_1|-t,2\})}\,dx\,dy\,dt \lesssim E. \tag{4.5.7}
$$

We have now all the elements allowing the exclusion of the soliton-like solution.

4.5.2 Extinction of the minimal element

With the aforementioned tool, we are in position to obtain a contradiction with respect to the our hypothesis on the finiteness of the critical energy E_c. Consider the upper bound $C = C(E(u))$ appearing in (4.5.7), then for any $T > 2$ we can write

$$
\begin{aligned}
C &\geq \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}\times\mathbb{T}}\int_{\mathbb{R}} \frac{\min\{|u|^2,|u|^{\alpha+2}\}}{\langle t\rangle\log(|t|+2)\log(\max\{|x_1|-t,2\})}\,dx_1\,dz\,dt \\
&\geq \int_2^T\int_{\mathbb{R}^{d-1}\times\mathbb{T}}\int_{\mathbb{R}} \frac{\min\{|u|^2,|u|^{\alpha+2}\}}{\langle t\rangle\log(|t|+2)\log(\max\{|x_1|-t,2\})}\,dx_1\,dz\,dt \\
&\geq \int_2^T\int_{\mathbb{T}}\int_{|x-x(t)|\leq R} \frac{\min\{|u|^2,|u|^{\alpha+2}\}}{\langle t\rangle\log(|t|+2)\log(\max\{|x_1|-t,2\})}\,dx_1\,dz\,dt.
\end{aligned} \tag{4.5.8}
$$

The finite propagation speed implies that $|x(t) - x(0)| \leq t + c_0$ for $t > 0$, then

$$|x| \leq |x - x(t)| + |x(t) - x(0)| + |x(0)| \leq R + t + c_0 + c_1,$$

so that $|x_1| - t \leq c + R$. With $[T]$ being the usual floor function of T, we are able to carry on with the chain above with

$$\begin{aligned}
(4.5.8) &\gtrsim \int_2^T \frac{1}{\langle t \rangle \log(|t| + 2)} \int_{\mathbb{T}} \int_{|x - x(t)| \leq R} \min\{|u|^2, |u|^{\alpha+2}\}\, dx\, dy\, dt \\
&\gtrsim \int_2^{[T]} \frac{1}{\langle t \rangle \log(|t| + 2)} \int_{\mathbb{T}} \int_{|x - x(t)| \leq R} \min\{|u|^2, |u|^{\alpha+2}\}\, dx\, dy\, dt \\
&= \sum_{j=3}^{[T]} \int_{j-1}^{j} \frac{1}{\langle t \rangle \log(|t| + 2)} \int_{\mathbb{T}} \int_{|x - x(t)| \leq R} \min\{|u|^2, |u|^{\alpha+2}\}\, dx\, dy\, dt \\
&\gtrsim \sum_{j=3}^{[T]} \frac{1}{\langle j \rangle \log(j + 2)} \int_{j-1}^{j} \int_{\mathbb{T}} \int_{|x - x(t)| \leq R} \min\{|u|^2, |u|^{\alpha+2}\}\, dx\, dy\, dt \\
&\gtrsim C(1) \sum_{j=3}^{[T]} \frac{1}{\langle j \rangle \log(j + 2)} \sim \int_2^T \frac{1}{\langle t \rangle \log(t + 2)}\, dt.
\end{aligned}$$

In the last step we used the property stated in Lemma 4.5.4 and Corollary 4.5.5 above (more precisely (4.5.1) and (4.5.3)) for a suitable choice of the radius R. This suffices to establish a contradiction by taking T large enough, since for $T \to +\infty$

$$\int_2^T \frac{1}{\langle t \rangle \log t}\, dt \sim \int_2^\infty \frac{1}{t \log t}\, dt$$

and the latter diverges, while the chain of inequalities above should imply a uniform bound.

Chapter 5

A singular limit result for the Zakharov system in 3D

This chapter is devoted to the local well-posedness theory and the qualitative behavior of solutions to the initial value problem for the vectorial Zakharov system first derived by Zakharov in [136] in order to describe the so-called Langmuir waves in a weakly magnetized plasma. From a Euler-Maxwell two-fluids model, after rescaling of variables, see [118], vectorial Zakharov equations are given by

$$\begin{cases} i\partial_t u - \omega\nabla\times\nabla\times u + \nabla(\operatorname{div} u) = nu \\ \frac{1}{c_s^2}\partial_{tt} n - \Delta n = \Delta|u|^2 \end{cases} . \tag{5.0.1}$$

Here $u : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}^3$ describes the slowly varying envelope of the highly oscillating electric field, whereas $n : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ is the ion density fluctuation. The initial value problem associated to (5.0.1) is subject to initial conditions

$$u(0) = u_0, \quad n(0) = n_0, \quad \partial_t n(0) = n_1.$$

The rescaled constants in (5.0.1) are $\omega = \frac{c^2}{3v_e^2}$, c being the speed of light and $v_e = \sqrt{\frac{T_e}{m_e}}$ the electron thermal velocity, while c_s is proportional to the ion acoustic speed.

5.1 Motivations and main results

In many physical situations the parameter ω is relatively large, see for example [122, table 1, p. 47], hence hereafter we will only consider $\omega \geq 1$. In the large ω regime, the electric field is almost irrotational and in the electrostatic limit $\omega \to \infty$ the dynamics is asymptotically described by

$$\begin{cases} i\partial_t u + \Delta u = \mathbf{Q}(nu) \\ \frac{1}{c_s^2}\partial_{tt} n - \Delta n = \Delta|u|^2 \end{cases} , \tag{5.1.1}$$

where $\mathbf{Q} = -(-\Delta)^{-1}\nabla \operatorname{div}$ is the Helmholtz projection operator onto irrotational vector fields. Later on we will also use the orthogonal projector to $\mathbf{Q}$ given by $\mathbf{P} := \mathbf{1} - \mathbf{Q}$, which is the projector onto solenoidal vector fields. By further simplifying (5.0.1) it is possible to consider the so called scalar Zakharov system

$$\begin{cases} i\partial_t u + \Delta u = nu \\ \frac{1}{c_s^2}\partial_{tt} n - \Delta n = \Delta |u|^2 \end{cases}, \tag{5.1.2}$$

which retains the main features of (5.1.1). For a rigorous derivation of (5.1.2) we refer to the work of Texier [121], or the one by Masmoudi and Nakanishi [93], who derived the model from a Klein-Gordon-Zakharov system. In the subsonic limit $c_s \to \infty$ we find the cubic focusing nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + |u|^2 u = 0.$$

The Cauchy problem for the Zakharov system has been extensively studied in the mathematical literature. For the local and global well-posedness, see for example Bourgain and Colliander [11], Kenig, Ponce and Vega [79], Ozawa and Tsutsumi [103, 104], Sulem and Sulem [117], and the recent results by Bejenaru and Herr [7] and Ginibre, Tsutsumi and Velo [50] concerning low regularity solutions. In Merle, see [95], formation of blow-up solutions is studied by means of virial identities, see also the papers by Glangetas and Merle [55, 56] where self-similar solutions are constructed in two space dimensions. The subsonic limit $c_s \to \infty$ for (5.1.2) is investigated by Schochet and Weinstein in [112]. Here we do not consider such limits, hence without loss of generalities we can set $c_s = 1$. Furthermore, some related singular limits are also studied by Masmoudi and Nakanishi, see [93], considering the Klein-Gordon-Zakharov system. The aim of our research is to rigorously study the electrostatic limit for the vectorial Zakharov equation, namely we show that mild solutions to (5.0.1) converge towards solutions to (5.1.1) as $\omega \to \infty$.

As we will see later on, we will investigate this limit by exploiting two auxiliary systems associated to (5.0.1), (5.1.1), namely the following below:

$$\begin{cases} i\partial_t v - \omega\nabla \times \nabla \times v + \nabla \operatorname{div} v = nv + \partial_t nu \\ \partial_{tt} n - \Delta n = \Delta |u|^2 \\ iv - \omega\nabla \times \nabla \times u + \nabla \operatorname{div} u = nu \end{cases} \tag{5.1.3}$$

and

$$\begin{cases} i\partial_t v^\infty + \Delta v^\infty = \mathbf{Q}(n^\infty v^\infty + \partial_t n^\infty u^\infty) \\ \partial_{tt} n^\infty - \Delta n^\infty = \Delta |u^\infty|^2 \\ iv^\infty + \Delta u^\infty = \mathbf{Q}(n^\infty u^\infty) \end{cases}. \tag{5.1.4}$$

Those are obtained by considering $v = \partial_t u$ as a new variable and by studying the Cauchy problem for the auxiliary system describing the dynamics for (v, n) and a state equation for u. This approach was introduced in [103, 104] to study local and global well-posedness for the Zakharov system (5.1.2), and overcomes the problem generated by the loss of derivatives on the term $|u|^2$ in the wave equation, but in our context it introduces a new difficulty. Indeed the initial data $v(0)$ for (5.2.1) below is not uniformly bounded for $\omega \geq 1$.

For this reason we will need to consider a family of well-prepared initial data; more precisely we will take a set u_0^ω of initial states for the Schrödinger part in (5.0.1) which converges to an irrotational initial datum for (5.1.1).

We consider initial data $(u_0^\omega, n_0^\omega, n_1^\omega) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) =: \mathcal{H}_2$ for (5.0.1), converging in the same space to a set of initial data $(u_0^\infty, n_0^\infty, n_1^\infty) \in \mathcal{H}_2$, with u_0^∞ an irrotational vector field, and we show the convergence in the space

$$\begin{aligned}\mathcal{X}_T := \Big\{(u,n) \ : u \in L^q((0,T); W^{2,r}(\mathbb{R}^3)),\ \forall\ (q,r) \text{ admissible pair}, \\ n \in L^\infty((0,T); H^1(\mathbb{R}^3)) \cap W^{1,\infty}((0,T); L^2(\mathbb{R}^3))\Big\}.\end{aligned}$$

Here "admissible" has the same meaning of Definition 1.1.4. Before stating our main result we first recall the local well-posedness result in $\mathcal{H}_2$ for system (5.1.1).

Theorem 5.1.1 ([103])**.** *Let $(u_0, n_0, n_1) \in \mathcal{H}_2$, then there exist a maximal time $0 < T_{max} \leq \infty$ and a unique solution (u, n) to (5.1.1) such that $u \in \mathcal{C}([0, T_{max}); H^2) \cap \mathcal{C}^1([0, T_{max}); L^2)$, $n \in \mathcal{C}([0, T_{max}); H^1) \cap \mathcal{C}^1([0, T_{max}); L^2)$. Furthermore the solution depends continuously on the initial data and the standard blow-up alternative holds true: either $T_{max} = \infty$ and the solution is global or $T_{max} < \infty$ and we have*

$$\lim_{t\to T_{max}} \|(u, n, \partial_t n)(t)\|_{\mathcal{H}_2} = \infty.$$

Analogously we are going to prove the same local well-posedness result for system (5.0.1). Furthermore, although the initial datum for (5.2.1) is not uniformly bounded for $\omega \geq 1$ (see the discussion at the beginning of Section 5.2), we can anyway infer some a priori bounds in ω for the solution (u^ω, n^ω) to (5.0.1).

Theorem 5.1.2. *Let $(u_0^\omega, n_0^\omega, n_1^\omega) \in \mathcal{H}_2$, then there exist a maximal time $T_{max}^\omega > 0$ and a unique solution (u^ω, n^ω) to (5.0.1) such that $u^\omega \in \mathcal{C}([0, T_{max}^\omega); H^2) \cap \mathcal{C}^1([0, T_{max}^\omega); L^2)$ and $n^\omega \in \mathcal{C}([0, T_{max}^\omega); H^1) \cap \mathcal{C}^1([0, T_{max}^\omega); L^2)$. Furthermore the existence times T_{max}^ω are uniformly bounded from below, $0 < T^* \leq T_{max}^\omega$ for any $\omega \geq 1$, and we have*

$$\|(u^\omega, n^\omega, \partial_t n^\omega)\|_{L^\infty((0,T);\mathcal{H}_2)} + \|\partial_t u^\omega\|_{L^2((0,T);L^6)} \leq C(T, \|u_0^\omega, n_0^\omega, n_1^\omega\|_{\mathcal{H}_2}),$$

for any $0 < T < T_{max}^\omega$, where the constant above does not depend on $\omega \geq 1$.

Our main result in this chapter is the following one.

Theorem 5.1.3. *Let $(u_0^\omega, n_0^\omega, n_1^\omega) \in \mathcal{H}_2$ and let (u^ω, n^ω) be the maximal solution to (5.0.1) defined on the time interval $[0, T_{max}^\omega)$. Let us assume that*

$$\lim_{\omega\to\infty} \|(u_0^\omega, n_0^\omega, n_1^\omega) - (u_0^\infty, n_0^\infty, n_1^\infty)\|_{\mathcal{H}_2} = 0,$$

for some $(u_0^\infty, n_0^\infty, n_1^\infty) \in \mathcal{H}_2$ such that $u_0^\infty = \mathbf{Q} u_0^\infty$, and let (u^∞, n^∞) be the maximal solutions to (5.1.1) in the interval $[0, T_{max}^\infty)$ with such initial data. Then

$$\liminf_{\omega\to\infty} T_{max}^\omega \geq T_{max}^\infty$$

and we have the following convergence

$$\lim_{\omega\to\infty} \|(u^\omega, n^\omega) - (u^\infty, n^\infty)\|_{\mathcal{X}_T} = 0,$$

for any $0 < T < T^\infty_{max}$.

The proof of the theorem above relies on a bootstrap argument, beside other considerations on the well prepared initial data.

5.2 Local existence theory

In this Section we study the local well-posedness of (5.0.1) in the space $\mathcal{H}_2$. We are going to perform a fixed point argument in order to find a unique local solution in the time interval $[0, T]$, for some $0 < T < \infty$. By standard arguments it is then possible to extend the solution up to a maximal time T_{max} for which the blow-up alternative holds. However, due to the loss of derivatives on the term $\Delta|u|^2$, we cannot proceed in a straightforward way, thus we follow the approach in [103] where the authors use an auxiliary system to overcome this difficulty. More precisely, let us define $v := \partial_t u$, then by differentiating the Schrödinger equation in (5.0.1) with respect to time variable, we write the following system

$$\begin{cases} i\partial_t v - \omega\nabla\times\nabla\times v + \nabla \operatorname{div} v = nv + \partial_t n u \\ \partial_{tt} n - \Delta n = \Delta|u|^2 \\ iv - \omega\nabla\times\nabla\times u + \nabla \operatorname{div} u = nu \end{cases}. \tag{5.2.1}$$

Differently from [103], here we encounter a further difficulty. Indeed we have that the initial datum for v is given by

$$v(0) = -i\omega\nabla\times\nabla\times u_0 + i\nabla \operatorname{div} u_0 - in_0u_0, \tag{5.2.2}$$

which in general is not uniformly bounded in L^2 for $\omega \geq 1$. Hence the standard fixed point argument applied to the integral formulation of (5.2.1) would give a local solution on a time interval $[0, T^\omega]$, where T^ω goes to zero as ω goes to infinity. For this reason we introduce the alternative variable

$$\tilde{v}(t) := v(t) - U(\omega t)\mathbf{P}(i\omega\Delta u_0), \tag{5.2.3}$$

for which we prove that the existence time T^ω is uniformly bounded from below for $\omega \geq 1$. The main result of this Section concerns the local well-posedness for (5.2.1).

Proposition 5.2.1. *Let* $(u_0, n_0, n_1) \in \mathcal{H}_2$ *be such that*

$$M := \|(u_0, n_0, n_1)\|_{\mathcal{H}_2}.$$

Then, for any $\omega \geq 1$ *there exists* $\tau = \tau(M)$ *and a unique local solution* $(u, n) \in \mathcal{C}([0, \tau]; \mathcal{H}_2)$ *to* (5.0.1) *such that*

$$\sup_{[0,\tau]} \|(u, n, \partial_t n)(t)\|_{\mathcal{H}_2} \leq 2M$$

and

$$\|v\|_{L^2L^6} \leq CM,$$

where C does not depend on $\omega \geq 1$.

We fix now some notation. Given a time interval $I \subset \mathbb{R}$ we denote the Strichartz space $S^0(\mathrm{I})$ to be the closure of the Schwartz space with the norm

$$\|u\|_{S^0(I)} := \sup_{(q,r)} \|u\|_{L^q(I;L^r(\mathbb{R}^3))},$$

where the supremum is taken over all admissible pairs; furthermore we write

$$S^2(I) = \{u \in S^0(I) \ : \ \nabla^2 u \in S^0(I)\}.$$

We define moreover the space

$$\mathcal{W}^1(I) = \{n \ : \ n \in L^\infty(I; H^1) \cap W^{1,\infty}(I; L^2)\}$$

endowed with the norm

$$\|n\|_{\mathcal{W}^1(I)} = \|n\|_{L^\infty(I;H^1)} + \|\partial_t n(t)\|_{L^\infty(I;L^2)}.$$

The space of solutions we consider in this chapter is given by

$$\mathcal{X}_T = \{(u,n) \ : \ u \in S^2([0,T]), \ n \in \mathcal{W}^1([0,T])\}.$$

We will also use the following notation:

$$\begin{aligned} \mathcal{C}([0,T); \mathcal{H}_2) = \Big\{(u,n) \ &: u \in \mathcal{C}([0,T); H^2) \cap \mathcal{C}^1([0,T); L^2), \\ &n \in \mathcal{C}([0,T); H^1) \cap \mathcal{C}^1([0,T); L^2)\Big\}. \end{aligned}$$

By standard arguments we then extend the local solution in Proposition 5.2.1 to a maximal existence interval where the standard blow-up alternative holds true.

Theorem 5.2.2. *Let $(u_0, n_0, n_1) \in \mathcal{H}_2$, then for any $\omega \geq 1$ there exists a unique maximal solution $(u^\omega, v^\omega, n^\omega)$ to (5.2.1) with initial data $(u_0, v(0), n_0, n_1)$, $v(0)$ given by (5.2.2), on the maximal existence interval $I_\omega := [0, T^\omega_{max})$, for some $T^\omega_{max} > 0$. The solution satisfies the following regularity properties:*

- $u^\omega \in \mathcal{C}(I_\omega; H^2)$, $u^\omega \in S^2([0,T])$, $\forall\, 0 < T < T^\omega_{max}$,
- $v^\omega \in \mathcal{C}(I_\omega; L^2)$, $v^\omega \in S^0([0,T])$, $\forall\, 0 < T < T^\omega_{max}$,
- $n^\omega \in \mathcal{C}(I_\omega; H^1) \cap \mathcal{C}^1(I_\omega; L^2)$.

Moreover, the following blow-up alternative holds true: $T^\omega_{max} < \infty$ if and only if

$$\lim_{t \to T^\omega} \|(u^\omega, n^\omega, \partial_t n^\omega)(t)\|_{\mathcal{H}_2} = \infty.$$

Finally, the map $\mathcal{H}_2 \to \mathcal{C}([0, T_{max}); \mathcal{H}_2)$ associating any initial datum to its solution is a continuous operator.

Remark 5.2.3. The blow-up alternative above also implies in particular that the family of maximal existence times T^ω is strictly bounded from below by a positive constant, i.e. there exists a $T^* > 0$ such that $T^* \le T^\omega$ for any $\omega \ge 1$.

Theorem 5.1.2 yields in a straightforward way from Theorem 5.2.2 above.

Proof of Theorem 5.1.2. Let $(u^\omega, v^\omega, n^\omega)$ be the solution to (5.2.1) constructed in Theorem 5.2.2, then to prove the Theorem 5.1.2 we only need to show that we identify $\partial_t u^\omega = v^\omega$ in the distribution sense. Let us differentiate with respect to t the equation

$$(1 - \omega\Delta\mathbf{P} - \Delta\mathbf{Q})\, u = iv - (n-1)\left(u_0 + \int_0^t v(s)\, ds\right)$$

obtaining

$$(1 - \omega\Delta\mathbf{P} - \Delta\mathbf{Q})\, \partial_t u = i\partial_t v - (n-1)v - \partial_t n\left(u_0 + \int_0^t v(s)\, ds\right), \tag{5.2.4}$$

this equation holding in H^{-2}, while the first equation of (5.2.1) gives us

$$(1 - \omega\Delta\mathbf{P} - \Delta\mathbf{Q})\, v = i\partial_t v - (n-1)v - \partial_t n\left(u_0 + \int_0^t v(s)\, ds\right).$$

Also the equation above is satisfied in H^{-2} and therefore in the same distributional sense we have $\partial_t u = v$. Moreover from (5.2.4) we get

$$\partial_t u = (1 - \omega\Delta\mathbf{P} - \Delta\mathbf{Q})^{-1}\left(i\partial_t v - (n-1)v - \partial_t n\left(u_0 + \int_0^t v(s)\, ds\right)\right) \in \mathcal{C}(I; L^2)$$

therefore $u \in \mathcal{C}^1(I; L^2)$. It is straightforward that $u^\omega(0, x) = u_0$ and so the proof is complete. □

Proof of Theorem 5.2.2. As discussed above, we are going to prove the result by means of a fixed point argument. Let us define the function

$$\tilde{v}(t) := v(t) - U(\omega t)\mathbf{P}(i\omega\Delta u_0).$$

We look at the integral formulation for (5.2.1), namely

$$v(t) = \mathcal{Z}(t)v(0) - i\int_0^t \mathcal{Z}(t-s)\,(nv + \partial_t n u)\,(s)\, ds \tag{5.2.5}$$

$$n(t) = \cos(t|\nabla|)n_0 + \frac{\sin(t|\nabla|)}{|\nabla|}n_1 + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}\left(\Delta|u|^2\right)(s)\, ds,$$

with u determined by the following elliptic equation

$$-\omega\nabla\times\nabla\times u + \nabla \operatorname{div} u = n\left(u_0 + \int_0^t v(s)\, ds\right) - iv,$$

and $v(0)$ is given by (5.2.2). This implies that $\tilde{v}$ must satisfy the following integral equation

$$\begin{aligned}\tilde{v}(t) = U(\omega t)\mathbf{P}(-in_0u_0) + U(t)\mathbf{Q}(i\Delta u_0 - in_0u_0) \\ - i\int_0^t \mathcal{Z}(t-s)\left(\tilde{v}n + nU(\omega\cdot)\mathbf{P}(i\omega\Delta u_0) + \partial_t nu\right)(s)\, ds.\end{aligned}$$

Let us consider the space

$$\begin{aligned}X = \Big\{(\tilde{v}, n) : \tilde{v} \in S^2([0,T]), n \in \mathcal{W}^1([0,T]), \\ \|\tilde{v}\|_{S^2(I)} \leq M, \|n\|_{\mathcal{W}^1(I)} \leq M\Big\},\end{aligned}$$

endowed with the norm

$$\|(\tilde{v}, n)\|_X := \|\tilde{v}\|_{S^2(I)} + \|n\|_{\mathcal{W}^1(I)}.$$

Here $0 < T \leq 1, M > 0$ will be chosen subsequently and $I := [0,T]$. From the third equation in (5.2.1) and the definition of $\tilde{v}$ we have

$$\begin{aligned}-\omega\nabla\times\nabla\times u + \nabla\operatorname{div} u = -i\tilde{v} - iU(\omega t)(i\omega\Delta\mathbf{P}u_0) \\ - in\left(u_0 + \int_0^t \tilde{v}(s) + U(\omega s)(i\omega\Delta\mathbf{P}u_0)\, ds\right),\end{aligned} \tag{5.2.6}$$

thus it is straightforward to see that given $n, \tilde{v}$, then u is uniquely determined. Furthermore, by applying the projection operators $\mathbf{P}, \mathbf{Q}$, respectively, to (5.2.6) we obtain

$$\omega\Delta\mathbf{P}u = -i\mathbf{P}[\tilde{v} + U(\omega t)\mathbf{P}(i\omega\Delta u_0)] + \mathbf{P}\left[n\left(u_0 + \int_0^t \tilde{v}(s) + U(\omega s)\mathbf{P}(i\omega\Delta u_0)\, ds\right)\right]$$

and

$$\Delta\mathbf{Q}u = -i\mathbf{Q}\tilde{v} + \mathbf{Q}\left[n\left(u_0 + \int_0^t \tilde{v}(s) + U(\omega s)\mathbf{P}(i\omega\Delta u_0)\, ds\right)\right].$$

We now estimate the irrotational and solenoidal parts of Δu separately. Let us start with $\mathbf{Q}\Delta u$: by Hölder inequality and Sobolev embedding we obtain

$$\begin{aligned}\|\Delta\mathbf{Q}u\|_{L^\infty L^2} \lesssim \|\tilde{v}\|_{L^\infty L^2} + \|n\|_{L^\infty H^1}\|u_0\|_{H^2} + T^{1/2}\|n\|_{L^\infty H^1}\|\tilde{v}\|_{L^2L^6} \\ + T^{1/2}\|n\|_{L^\infty H^1}\|U(\omega t)\mathbf{P}(i\omega\Delta u_0)\|_{L^2L^6}.\end{aligned}$$

To estimate the last term, we use the Strichartz estimate in (1.5.5); let us notice that by choosing the admissible exponents $(q, r) = (2, 6)$ we obtain a factor ω^{-1} in the estimate, which balances the term ω appearing above. We thus have

$$\|\Delta\mathbf{Q}u\|_{L^\infty L^2} \lesssim (\|u_0\|_{H^2} + 1)M + M^2.$$

By similar calculations, we also obtain an estimate for $\mathbf{P}\Delta u$,

$$\|\mathbf{P}\Delta u\|_{L^\infty L^2} \lesssim \|u_0\|_{H^2}^2 + \|u_0\|_{H^2}M + M^2.$$

We then sum up the contributions given by the irrotational and solenoidal parts to get

$$\|u\|_{L^\infty H^2} \lesssim \|u_0\|_{H^2}^2 + \|u_0\|_{H^2} M + M^2 \le C(\|u_0\|_{H^2})\big(1+M^2\big). \tag{5.2.7}$$

Similar calculations also give

$$\begin{aligned}\|u-u'\|_{L^\infty(I;H^2)} &\lesssim \|\tilde{v}-\tilde{v}'\|_{L^\infty L^2} + \|n-n'\|_{L^\infty H^1} \\ &\quad + M(\|n-n'\|_{L^\infty H^1} + \|\tilde{v}-\tilde{v}'\|_{L^2 L^6}) \\ &\le C(1+M)\|(\tilde{v},n)-(\tilde{v}',n')\|_X.\end{aligned}$$

Given $(\tilde{v},n)\in X$ we define the map $\Phi : X \to X$, $\Phi(\tilde{v},n) = (\Phi_S,\Phi_W)(\tilde{v},n)$ by

$$\begin{aligned}\Phi_S &= U(\omega t)\mathbf{P}(-in_0u_0) + U(t)\mathbf{Q}(i\Delta u_0 - in_0u_0) \\ &\quad - i\int_0^t U(\omega(t-s))\mathbf{P}(\tilde{v}n + nU(\omega\cdot)\mathbf{P}(i\omega\Delta u_0) + \partial_t nu)(s)\,ds \\ &\quad - i\int_0^t U(t-s)\mathbf{Q}\left(n\tilde{v} + nU(\omega\cdot)(i\omega\Delta u_0) + \partial_t nu\right)(s)\,ds\end{aligned} \tag{5.2.8}$$

$$\Phi_W = \cos(t|\nabla|)n_0 + \frac{\sin(t|\nabla|)}{|\nabla|}n_1 + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}\left(\Delta|u|^2\right)(s)\,ds \tag{5.2.9}$$

where u in the formulas above is given by (5.2.6) and its $L^\infty H^2$-norm is bounded in (5.2.7). Let us first prove that, by choosing T and M properly, Φ maps X into itself.

Let us first analyze the Schrödinger part (5.2.8), by the Strichartz estimates in Lemma 1.5.2, Hölder inequality and Sobolev embedding we have

$$\|U(\omega t)\mathbf{P}(-in_0u_0) + U(t)\mathbf{Q}(i\Delta u_0 - in_0u_0)\|_{L^qL^r} \lesssim \|u_0\|_{H^2} + \|n_0\|_{H^1}\|u_0\|_{H^2}$$

We treat the inhomogeneous part similarly:

$$\begin{aligned}&\left\|\int_0^t \mathcal{Z}(t-s)\left(n\tilde{v} + nU(\omega\cdot)(i\omega\mathbf{P}\Delta u_0)\right)(s)\,ds\right\|_{L^qL^r} \lesssim \|n\tilde{v} + nU(\omega\cdot)(i\omega\Delta\mathbf{P}u_0)\|_{L^1L^2} \\ &\lesssim T^{1/2}\|n\|_{L^\infty H^1}(\|\tilde{v}\|_{L^2L^6} + \|U(\omega\cdot)\mathbf{P}(i\omega\Delta u_0)\|_{L^2L^6}) \lesssim T^{1/2}M(M+\|u_0\|_{H^2})\end{aligned}$$

where in the last inequality we again used (1.5.5) with $(2,6)$ as admissible pair. Similarly,

$$\begin{aligned}\left\|\int_0^t \mathcal{Z}(t-s)\left(\partial_t nu\right)(s)\,ds\right\|_{L^qL^r} &\lesssim T\|\partial_t n\|_{L^\infty L^2}\|u\|_{L^\infty H^2} \\ &\lesssim C(\|u_0\|_{H^2})TM\big(1+M^2\big),\end{aligned}$$

where in the last line we use the bound (5.2.7). Collecting these estimates we get

$$\|\Phi_S(\tilde{v},n)\|_{L^qL^r} \le C(\|u_0\|_{H^2},\|n_0\|_{L^2}) + CT^{1/2}M(1+M). \tag{5.2.10}$$

For the wave component we use formula (5.2.9) and Hölder inequality to obtain

$$\begin{aligned}\|\Phi_W(v,n)\|_{\mathcal{W}^1(I)} &\le C(1+T)\|n_0\|_{H^1} + \|n_1\|_{L^2} + \|\Delta|u|^2\|_{L^1L^2} \\ &\le C\left(\|n_0\|_{H^1} + \|n_1\|_{L^2}\right) + T\|u\|_{L^\infty H^2}^2,\end{aligned}$$

where we used the fact that $H^2(\mathbb{R}^3)$ is an algebra. From (5.2.7) we infer

$$\|\Phi_W(v,n)\|_{\mathcal{W}^1(I)} \leq C(\|n_0\|_{H^1}, \|n_1\|_{L^2}) + T\Big(M + M^4\Big). \tag{5.2.11}$$

The bounds (5.2.10) and (5.2.11) together yield

$$\|\Phi(\tilde{v},n)\|_X \leq C(\|(u_0,n_0,n_1)\|_{\mathcal{H}_2}) + CT^{1/2}M(1+M^3).$$

Let us choose M such that

$$\frac{M}{2} = C(\|(u_0,n_0,n_1)\|_{\mathcal{H}_2})$$

and T such that

$$CT^{1/2}(1+M^3) < \frac{1}{2},$$

we then obtain $\|\Phi(\tilde{v},n)\|_X \leq M$. Hence Φ maps X into itself. It thus remains to prove that Φ is a contraction. Arguing similarly to what we did before we obtain

$$\begin{aligned}\|\Phi_S(\tilde{v},n) - \Phi_S(\tilde{v}',n')\|_{L^qL^r} &\leq CT^{1/2}(1+M)\|(\tilde{v},n) - (\tilde{v}',n')\|_{L^qL^r}\\ \|\Phi_W(\tilde{v},n) - \Phi_W(\tilde{v}',n')\|_{\mathcal{W}^1(I)} &\leq CT\Big(1+M^3\Big)\|(\tilde{v},n) - (\tilde{v}',n')\|_{\mathcal{W}^1(I)}.\end{aligned}$$

By possibly choosing a smaller $T>0$ such that $CT^{1/2}(1+M^3) < 1$ then we see that $\Phi : X \to X$ is a contraction and consequently there exists a unique $(\tilde{v},n) \in X$ which is a fixed point for X. Let us notice that the time T depends only on M, hence $T = T(\|(u_0,n_0,n_1)\|_{\mathcal{H}_2})$. Furthermore from the definition of $\tilde{v}$ it follows that (u,v,n) is a solution to (5.2.1), where $v(t) = \tilde{v}(t) + U(\omega t)\mathbf{P}(i\omega\Delta u_0)$. From (5.2.7) we also see that the $L^\infty H^2$-norm of u is uniformly bounded in ω.

Finally, from standard arguments we extend the solution on a maximal time interval, on which the standard blow-up alternative holds true and we can also infer the continuous dependence on the initial data. □

5.3 Convergence of solutions

Given the well-posedness results of the previous Section, we are now ready to study the electrostatic limit for the vectorial Zakharov system (5.0.1). In order to understand the effective dynamics we consider the system (5.0.1) in its integral formulation, by splitting the Schrödinger linear propagator in its fast and slow dynamics, that is $\mathcal{Z}(t) = U(\omega t)\mathbf{P} + U(t)\mathbf{Q}$. In particular for u^ω we have

$$u^\omega(t) = U(\omega t)\mathbf{P}u_0 + U(t)\mathbf{Q}u_0 - i\int_0^t U(\omega(t-s))\mathbf{P}(nu)(s)\,ds - i\int_0^t U(t-s)\mathbf{Q}(nu)(s)\,ds.$$

Due to fast oscillations, we expect that the terms of the form $U(\omega t)f$ go weakly to zero as $\omega \to 0$. This fact can be quantitatively seen by using the Strichartz estimates in (1.5.5). However, while for the third term we can choose (γ,ρ) in a suitable way such that it converges to zero in every Strichartz space, by the unitarity of $U(\omega t)$ we see that $\|U(\omega t)\mathbf{P}u_0\|_{L^\infty L^2}$ cannot converge to zero, while $\|U(\omega t)\mathbf{P}u_0\|_{L^qL^r} \to 0$ for any admissible pair $(q,r) \neq (\infty, 2)$.

This is indeed due to the presence of an initial layer for the electrostatic limit for (5.0.1) when dealing with "ill-prepared" initial data. In general, for arbitrary initial data, the right convergence, as $\omega \to \infty$, should be given by

$$\tilde{u}^\omega(t) := u^\omega(t) - U(\omega t)\mathbf{P}u_0 \to u^\infty$$

in all Strichartz spaces, where u^∞ is the solution to (5.1.1). Let us notice that $\tilde{u}^\omega$ is related to the auxiliary variable $\tilde{v}^\omega$ defined in (5.2.3) and used to prove the local well-posedness results in Section 5.2, since we have $\tilde{v}^\omega = \partial_t \tilde{u}^\omega$.

Our strategy to prove the electrostatic limit goes through studying the convergence of $(v^\omega, n^\omega, u^\omega)$, studied in the previous Section, towards solutions to

$$\begin{cases} i\partial_t v^\infty + \Delta v^\infty = \mathbf{Q}(n^\infty v^\infty + \partial_t n^\infty u^\infty) \\ \partial_{tt} n^\infty - \Delta n^\infty = \Delta |u^\infty|^2 \\ iv^\infty + \Delta u^\infty = \mathbf{Q}(n^\infty u^\infty) \end{cases}, \tag{5.3.1}$$

which is the auxiliary system associated to (5.1.1). Again, we exploit such auxiliary formulations in order to overcome the difficulty generated by the loss of derivatives on the terms $|u^\omega|^2$ and $|u^\infty|^2$.

Unfortunately our strategy is not suitable to study the limit in the presence of an initial layer. Indeed for ill-prepared data we should consider $\tilde{u}^\omega$ and consequently $\tilde{v}^\omega$ defined in (5.2.3) for the auxiliary system. This means that when studying the auxiliary variable v^ω the initial layer itself becomes singular. For this reason here we restrict ourselves to study the limit with well-prepared data. More specifically, we consider $(u_0^\omega, n_0^\omega, n_1^\omega) \in \mathcal{H}_2$ such that

$$\|(u_0^\omega, n_0^\omega, n_1^\omega) - (u_0^\infty, n_0^\infty, n_1^\infty)\|_{\mathcal{H}_2} \to 0 \tag{5.3.2}$$

for some $(u_0^\infty, n_0^\infty, n_1^\infty) \in \mathcal{H}_2$ and

$$\|\mathbf{P}u_0^\omega\|_{H^2} \to 0 \tag{5.3.3}$$

as $\omega \to \infty$. This clearly implies that the initial datum for the limit equation (5.1.1) is irrotational, i.e. $\mathbf{P}u_0^\infty = 0$.

Remark 5.3.1. In view of the above discussion, it is reasonable to think about studying the initial layer by considering the Cauchy problem for the Zakharov system in low regularity spaces, by exploiting recent results in [7, 11, 50]. However this goes beyond the scope of our work and it could be the subject of some future investigations.

To prove the convergence result stated in Theorem 5.1.3 we will study the convergence from (5.2.1) to (5.3.1). The main result of this Section is the following.

Theorem 5.3.2. *Let $\omega \geq 1$ and let $(u_0^\omega, n_0^\omega, n_1^\omega)$, $(u_0^\infty, n_0^\infty, n_1^\infty) \in \mathcal{H}_2$ be initial data such that (5.3.2) and (5.3.3) hold true. Let $(u^\omega, v^\omega, n^\omega)$ be the maximal solution to (5.2.1) with Cauchy data $(u_0^\omega, n_0^\omega, n_1^\omega)$ given by Theorem 5.2.2 and analogously let $(u^\infty, v^\infty, n^\infty)$ be the maximal solution to (5.3.1) in the interval $[0, T_{max}^\infty)$ accordingly to Theorem 5.1.1. Then for any $0 < T < T_{max}^\infty$ we have*

$$\lim_{\omega\to\infty} \|(u^\omega, v^\omega, n^\omega) - (u^\infty, v^\infty, n^\infty)\|_{L^\infty((0,T);\mathcal{H}_2)} = 0.$$

The proof of the Theorem above is divided in two main steps. First of all we prove in Lemma 5.3.3 that, as long as the $\mathcal{H}_2$-norm of $(u^\omega(T), n^\omega(T), \partial_t n^\omega(T))$ is bounded, then the convergence holds true in $[0, T]$. The second one consists in proving that the $\mathcal{H}_2$ bound on $(u^\omega(T), n^\omega(T), \partial_t n^\omega(T))$ holds true for any $0 < T < T_{max}^\infty$. A similar strategy of proof is already exploited in the literature to study the asymptotic behavior of time oscillating nonlinearities, see for example the works by Cazenave and Scialom [21] where the authors consider a time oscillating nonlinearity or Antonelli and Weishäupl [3] where in a system of two nonlinear Schrödinger equations a rapidly varying linear coupling term is averaging out the effect of nonlinearities. We also mention the paper by Carvajal, Panthee and Scialom [19], where a similar strategy is also used to study a time oscillating critical Korteweg-de Vries equation.

Lemma 5.3.3. *Let $(u^\omega, v^\omega, n^\omega)$, $(u^\infty, v^\infty, n^\infty)$ be defined as in the statement of Theorem 5.3.2 and let us assume that for some $0 < T_1 < T_{max}^\infty$ we have*

$$\sup_{\omega\geq 1} \|(u^\omega, n^\omega, \partial_t n^\omega)\|_{L^\infty((0,T_1);\mathcal{H}_2)} < \infty.$$

It follows that

$$\lim_{\omega\to\infty} (\|u^\omega - u^\infty\|_{L^\infty H^2} + \|v^\omega - v^\infty\|_{L^2 L^6} + \|n^\omega - n^\infty\|_{\mathcal{W}^1}) = 0,$$

where all the norms are taken in the space-time slab $[0, T_1] \times \mathbb{R}^3$. In particular we have

$$\lim_{\omega\to\infty} \|(u^\omega, n^\omega, \partial_t n^\omega) - (u^\infty, n^\infty, \partial_t n^\infty)\|_{L^\infty((0,T_1);\mathcal{H}_2)} = 0.$$

We assume for the moment that Lemma 5.3.3 holds true, then we first show how this implies Theorem 5.3.2.

Proof of Theorem 5.3.2. Let $0 < T < T_{max}^\infty$ be fixed and let us define

$$N := 2\|(u^\infty, n^\infty, \partial_t n^\infty)\|_{L^\infty((0,T);\mathcal{H}_2)}.$$

From the local well-posedness theory, see Proposition 5.2.1, there exists $\tau = \tau(N)$ such that the solution $(u^\omega, n^\omega, \partial_t n^\omega)$ to (5.2.1) exists on $[0, \tau]$ and we have

$$\|(u^\omega, n^\omega, \partial_t n^\omega)\|_{L^\infty((0,T_1);\mathcal{H}_2)} < \infty.$$

We observe that, because of what we said before, the choice $T_1 = \tau$ is always possible. By the Lemma 5.3.3 we infer that

$$\lim_{\omega\to\infty} \|(u^\omega, n^\omega, \partial_t n^\omega) - (u^\infty, n^\infty, \partial_t n^\infty)\|_{L^\infty((0,T_1);\mathcal{H}_2)} = 0.$$

On the other hand by the definition of N we have that, for $\omega \geq 1$ large enough,

$$\begin{aligned}\|(u^\omega, n^\omega, \partial_t n^\omega)(T_1)\|_{\mathcal{H}_2} \leq \|&(u^\omega, n^\omega, \partial_t n^\omega)(T_1) - (u^\infty, n^\infty, \partial_t n^\infty)(T_1)\|_{\mathcal{H}_2} \\ &+ \|(u^\infty, n^\infty, \partial_t n^\infty)(T_1)\|_{\mathcal{H}_2} \leq N.\end{aligned}$$

Consequently we can apply Proposition 5.2.1 to infer that (u^ω, n^ω) exists on a larger time interval $[0, T_1 + \tau]$, provided $T_1 + \tau \leq T$, and again

$$\|(u^\omega, n^\omega, \partial_t n^\omega)\|_{L^\infty((0,T_1+\tau);\mathcal{H}_2)} \leq 2N.$$

We can repeat the argument iteratively on the whole interval $[0, T]$ to infer

$$\|(u^\omega, n^\omega, \partial_t n^\omega)\|_{L^\infty((0,T);\mathcal{H}_2)} \leq 2N.$$

By using Lemma 5.3.3 this proves the Theorem. □

It only remains now to prove Lemma 5.3.3.

Proof of Lemma 5.3.3. Let us fix

$$M := \sup_\omega \sup_{[0,T_1]} \|(u^\omega, n^\omega, \partial_t n^\omega)(t)\|_{\mathcal{H}_2}.$$

By using the integral formulation for (5.2.1) and (5.3.1) we have

$$\begin{aligned}v^\omega(t) - v^\infty(t) = U(\omega t)&\mathbf{P}(\omega\Delta u_0^\omega - i u_0^\omega n_0^\omega) + U(t)\mathbf{Q}(v_0^\omega - v_0^\infty) \\ &- i\int_0^t U(\omega(t-s))[\mathbf{P}(\partial_t(n^\omega u^\omega))](s)\,ds \\ &- i\int_0^t U(t-s)[\mathbf{Q}(\partial_t(n^\omega u^\omega) - \partial_t(n^\infty u^\infty))](s)\,ds.\end{aligned}$$

Now we use the Strichartz estimates in Lemma 1.5.2 to get

$$\begin{aligned}\|v^\omega - v^\infty\|_{L^2L^6} \lesssim \|&\mathbf{P}u_0^\omega\|_{H^2} + \omega^{-1}\|n_0^\omega\|_{H^1}\|u_0^\omega\|_{H^2} + \|v_0^\omega - v_0^\infty\|_{L^2} \\ &+ \omega^{-1/2}\|n^\omega v^\omega + \partial_t n^\omega u^\omega\|_{L^1L^2} \\ &+ \|n^\omega v^\omega - n^\infty v^\infty\|_{L^1L^2} + \|\partial_t n^\omega u^\omega - \partial_t n^\infty u^\infty\|_{L^1L^2}.\end{aligned}$$

It is straightforward to check that, by Hölder inequality and Sobolev embedding,

$$\begin{aligned}\|n^\omega v^\omega + \partial_t n^\omega u^\omega\|_{L^1L^2} &\leq C(T, M), \\ \|n^\omega v^\omega - n^\infty v^\infty\|_{L^1L^2} &\leq CT^{1/2}(\|n^\omega - n^\infty\|_{L^\infty H^1} + \|v^\omega - v^\infty\|_{L^2L^6}), \\ \|\partial_t n^\omega u^\omega - \partial_t n^\infty u^\infty\|_{L^1L^2} &\leq CT\left(\|\partial_t n^\omega - \partial_t n^\infty\|_{L^\infty L^2} + \|u^\omega - u^\infty\|_{L^\infty H^2}\right).\end{aligned}$$

By putting al the estimates together we obtain

$$\|v^\omega - v^\infty\|_{L^2L^6} \lesssim \|\mathbf{P}u_0^\omega\|_{H^2} + \omega^{-1}\|n_0^\omega\|_{H^1}\|u_0^\omega\|_{H^2}\|u_0^\omega - u_0^\infty\|_{H^2} + \omega^{-1/2} + \|n_0^\omega - n_0^\infty\|_{H^1}$$
$$+ T^{1/2}(\|u^\omega - u^\infty\|_{L^\infty H^2} + \|v^\omega - v^\infty\|_{L^2L^6} + \|n^\omega - n^\infty\|_{\mathcal{W}^1}).$$

To estimate the wave part in (5.2.1) and (5.3.1), we write

$$n^\omega - n^\infty = \cos(t|\nabla|)(n_0^\omega - n_0^\infty) - \frac{\sin(t|\nabla|)}{|\nabla|}(n_1^\omega - n_1^\infty)$$
$$+ \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}\Delta(|u^\omega|^2 - |u^\infty|^2)(s)\, ds,$$

whence, by using again that $H^2(\mathbb{R}^3)$ is an algebra,

$$\|n^\omega - n^\infty\|_{\mathcal{W}^1} \lesssim \|n_0^\omega - n_0^\infty\|_{H^1} + \|n_1^\omega - n_1^\infty\|_{L^2} + T\|u^\omega - u^\infty\|_{L^\infty H^2}.$$

The estimate for the difference $u^\omega - u^\infty$ is more delicate. From the third equations in (5.2.1) and (5.3.1) we have

$$-\omega\nabla \times \nabla \times u^\omega + \nabla \operatorname{div}(u^\omega - u^\infty) = i(v^\omega - v^\infty) - n^\omega u^\omega + \mathbf{Q}(n^\infty u^\infty).$$

Again, here we estimate separately the irrotational and solenoidal parts of the difference. For the solenoidal part we obtain

$$\omega\|\mathbf{P}\Delta u^\omega\|_{L^\infty L^2} \lesssim \|v^\omega\|_{L^\infty L^2} + \|n^\omega u^\omega\|_{L^\infty L^2}.$$

To estimate the $L^\infty L^2$-norm of v^ω on the right hand side we use (5.2.5) and Strichartz estimates to infer

$$\|v^\omega\|_{L^\infty L^2} \lesssim \omega\|\mathbf{P}u_0^\omega\|_{H^2}\|u_0^\omega\|_{H^2}\|n_0^\omega\|_{H^1} + 1.$$

Hence

$$\omega\|\mathbf{P}\Delta u^\omega\|_{L^\infty L^2} \lesssim \omega\|\mathbf{P}u_0^\omega\|_{H^2} + \|u_0^\omega\|_{H^2}\|n_0\|_{H^1} + 1.$$

For the irrotational part

$$\|\mathbf{Q}\Delta(u^\omega - u^\infty)\|_{L^\infty L^2} \lesssim \|\mathbf{Q}(v^\omega - v^\infty)\|_{L^\infty L^2} + \|n^\omega - u^\omega - n^\infty u^\infty\|_{L^\infty L^2}. \tag{5.3.4}$$

By using (5.2.5), the analogue integral formulation for v^∞ and by applying the Helmholtz projection operator $\mathbf{Q}$ to their difference we have that the first term on the right hand side is bounded by

$$\|\mathbf{Q}(v^\omega - v^\infty)\|_{L^\infty L^2} \lesssim \|u_0^\omega - u_0^\infty\|_{H^2} + \|n_0^\omega - n_0^\infty\|_{H^1}$$
$$+ T^{1/2}\left(\|u^\omega - u^\infty\|_{L^\infty H^2} + \|v^\omega - v^\infty\|_{L^2L^6} + \|n^\omega - n^\infty\|_{\mathcal{W}^1}\right).$$

The second term on the right hand side of (5.3.4) is estimated by

$$\begin{aligned}\|n^\omega u^\omega - n^\infty u^\infty\|_{L^\infty L^2} &\lesssim \|n^\omega - n^\infty\|_{L^\infty L^2}\|u^\omega\|_{L^\infty H^2} \\ &\quad + \|n^\infty(u_0^\omega - u_0^\infty)\|_{L^\infty L^2} + \left\|n^\infty \int_0^t (v^\omega - v^\infty)(s)\, ds\right\|_{L^\infty L^2} \\ &\lesssim \left(\|n_0^\omega - n_0^\infty\|_{L^2} + T\|\partial_t n^\omega - \partial_t n^\infty\|_{L^\infty L^2}\right) M \\ &\quad + \|n^\infty\|_{L^\infty L^2}\|u_0^\omega - u_0^\infty\|_{H^2} \\ &\quad + T^{1/2}\|n^\infty\|_{L^\infty H^1}\|v^\omega - v^\infty\|_{L^2L^6}.\end{aligned}$$

By summing up the two contribution in (5.3.4) we then get

$$\begin{aligned}\|\mathbf{Q}\Delta(u^\omega - u^\infty)\|_{L^\infty L^2} \lesssim & \|u_0^\omega - u_0^\infty\|_{H^2} + \|n_0^\omega - n_0^\infty\|_{H^1} \\ & + T^{1/2}\left(\|u^\omega - u^\infty\|_{L^\infty H^2} + \|v^\omega - v^\infty\|_{L^2 L^6} + \|n^\omega - n^\infty\|_{\mathcal{W}^1}\right).\end{aligned}$$

Finally, we notice that, by using the Schrödinger equations in (5.0.1) and (5.1.1), we have

$$\|u^\omega - u^\infty\|_{L^\infty L^2} \lesssim T\left(\|n^\omega - n^\infty\|_{L^\infty H^1} + \|u^\omega - u^\infty\|_{L^\infty H^2}\right),$$

so that

$$\begin{aligned}\|u^\omega - u^\infty\|_{L^\infty H^2} \lesssim & \|u_0^\omega - u_0^\infty\|_{H^2} + \|n_0^\omega - n_0^\infty\|_{H^1} + \|\mathbf{P}u_0^\omega\|_{H^2} + \omega^{-1} \\ & + T^{1/2}\left(\|u^\omega - u^\infty\|_{L^\infty H^2} + \|v^\omega - v^\infty\|_{L^2 L^6} + \|n^\omega - n^\infty\|_{\mathcal{W}^1}\right).\end{aligned}$$

Now we put everything together, finally obtaining

$$\begin{aligned}&\|v^\omega - v^\infty\|_{L^2 L^6} + \|n^\omega - n^\infty\|_{\mathcal{W}^1} + \|u^\omega - u^\infty\|_{L^\infty H^2} \lesssim \\ &\quad \lesssim \|\mathbf{P}u_0^\omega\|_{H^2} + \omega^{-1} + \|u_0^\omega - u_0^\infty\|_{H^2} + \|n_0^\omega - n_0^\infty\|_{H^1} + \|n_1^\omega - n_1^\infty\|_{L^2} \\ &\quad\quad + T^{1/2}\left(\|u^\omega - u^\infty\|_{L^\infty H^2} + \|v^\omega - v^\infty\|_{L^2 L^6} + \|n^\omega - n^\infty\|_{\mathcal{W}^1}\right).\end{aligned}$$

By choosing T small enough depending on M we can infer

$$\begin{aligned}&\|v^\omega - v^\infty\|_{L^2 L^6} + \|n^\omega - n^\infty\|_{\mathcal{W}^1} + \|u^\omega - u^\infty\|_{L^\infty H^2} \lesssim \\ &\quad \lesssim \|\mathbf{P}u_0^\omega\|_{H^2} + \omega^{-1} + \|u_0^\omega - u_0^\infty\|_{H^2} + \|n_0^\omega - n_0^\infty\|_{H^1} + \|n_1^\omega - n_1^\infty\|_{L^2}.\end{aligned}$$

This proves the convergence in the time interval $[0, T]$, for $T > 0$ small enough. Let now $0 < T_1$ be as in the statement of Lemma, we can divide $[0, T_1]$ into many subintervals of length T such that the convergence holds in any small interval. By gluing them together we prove the Lemma. □

Bibliography

[1] P. Antonelli and L. Forcella, *The electrostatic limit for the 3D Zakharov system*, Nonlinear Anal. **163** (2017), 19–33.

[2] P. Antonelli, J.-C. Saut, and C. Sparber, *Well-posedness and averaging of NLS with time-periodic dispersion management*, Adv. Differential Equations **18** (2013), no. 1-2, 49–68.

[3] P. Antonelli and R. M. Weishäupl, *Asymptotic behavior of nonlinear Schrödinger systems with linear coupling*, J. Hyperbolic Differ. Equ. **11** (2014), no. 1, 159–183.

[4] G. Artbazar and K. Yajima, *The L^p-continuity of wave operators for one dimensional Schrödinger operators*, J. Math. Sci. Univ. Tokyo **7** (2000), no. 2, 221–240.

[5] H. Bahouri and P. Gérard, *High frequency approximation of solutions to critical nonlinear wave equations*, Amer. J. Math. **121** (1999), no. 1, 131–175.

[6] V. Banica and N. Visciglia, *Scattering for NLS with a delta potential*, J. Differential Equations **260** (2016), no. 5, 4410–4439.

[7] I. Bejenaru and S. Herr, *Convolutions of singular measures and applications to the Zakharov system*, J. Funct. Anal. **261** (2011), no. 2, 478–506.

[8] I. Bejenaru, A. Ionescu, C. Kenig, and D. Tataru, *Equivariant Schrödinger maps in two spatial dimensions*, Duke Math. J. **162** (2013), no. 11, 1967–2025.

[9] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations*, Geom. Funct. Anal. **3** (1993), no. 2, 107–156.

[10] ———, *Global well-posedness of defocusing critical nonlinear Schrödinger equation in the radial case*, J. Amer. Math. Soc. **12** (1999), no. 1, 145–171.

[11] J. Bourgain and J. Colliander, *On well-posedness of the Zakharov system*, Internat. Math. Res. Notices **11** (1996), 515–546.

[12] P. Brenner, *On space-time means and everywhere defined scattering operators for nonlinear Klein-Gordon equations*, Math. Z. **186** (1984), no. 3, 383–391.

[13] ———, *On scattering and everywhere defined scattering operators for nonlinear Klein-Gordon equations*, J. Differential Equations **56** (1985), no. 3, 310–344.

[14] H. Brezis and J.-M. Coron, *Convergence of solutions of H-systems or how to blow bubbles*, Arch. Rational Mech. Anal. **89** (1985), no. 1, 21–56.

[15] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490.

[16] A. Bulut, *The defocusing energy-supercritical cubic nonlinear wave equation in dimension five*, Trans. Amer. Math. Soc. **367** (2015), no. 9, 6017–6061.

[17] N. Burq, P. Gérard, and N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, Amer. J. Math. **126** (2004), no. 3, 569–605.

[18] N. Burq, F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential*, J. Funct. Anal. **203** (2003), no. 2, 519–549.

[19] X. Carvajal, M. Panthee, and M. Scialom, *On the critical KdV equation with time-oscillating nonlinearity*, Differential Integral Equations **24** (2011), no. 5-6, 541–567.

[20] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[21] T. Cazenave and M. Scialom, *A Schrödinger equation with time-oscillating nonlinearity*, Rev. Mat. Complut. **23** (2010), no. 2, 321–339.

[22] T. Cazenave and F. B. Weissler, *The Cauchy problem for the nonlinear Schrödinger equation in* H^1, Manuscripta Math. **61** (1988), no. 4, 477–494.

[23] ———, *Rapidly decaying solutions of the nonlinear Schrödinger equation*, Comm. Math. Phys. **147** (1992), no. 1, 75–100.

[24] X. Cheng, Z. Guo, K. Yang, and L. Zhao, *On scattering for the cubic defocusing nonlinear Schrödinger equation on waveguide* $\mathbb{R}^2 \times \mathbb{T}$. Preprint, archived at https://arxiv.org/abs/1705.00954.

[25] M. Christ and A. Kiselev, *Scattering and wave operators for one-dimensional Schrödinger operators with slowly decaying nonsmooth potentials*, Geom. Funct. Anal. **12** (2002), no. 6, 1174–1234.

[26] J. Colliander, M. Grillakis, and N. Tzirakis, *Tensor products and correlation estimates with applications to nonlinear Schrödinger equations*, Comm. Pure Appl. Math. **62** (2009), no. 7, 920–968.

[27] J. Colliander, J. Holmer, M. Visan, and X. Zhang, *Global existence and scattering for rough solutions to generalized nonlinear Schrödinger equations on* $\mathbb{R}$, Commun. Pure Appl. Anal. **7** (2008), no. 3, 467–489.

[28] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in* $\mathbb{R}^3$, Ann. of Math. (2) **167** (2008), no. 3, 767–865.

[29] S. Cuccagna, *Dispersion for Schrödinger equation with periodic potential in 1D*, Comm. Partial Differential Equations **33** (2008), no. 10-12, 2064–2095.

[30] S. Cuccagna, V. Georgiev, and N. Visciglia, *Decay and scattering of small solutions of pure power NLS in* $\mathbb{R}$ *with* $p > 3$ *and with a potential*, Comm. Pure Appl. Math. **67** (2014), no. 6, 957–981.

[31] S. Cuccagna and N. Visciglia, *Scattering for small energy solutions of NLS with periodic potential in 1D*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 5-6, 243–247 (English, with English and French summaries).

[32] P. D'Ancona, *Smoothing and dispersive properties of evolution equations with potential perturbations*, Hokkaido Math. J. **37** (2008), no. 4, 715–734.

[33] P. D'Ancona and L. Fanelli, L^p*-boundedness of the wave operator for the one dimensional Schrödinger operator*, Comm. Math. Phys. **268** (2006), no. 2, 415–438.

[34] P. D'Ancona and S. Selberg, *Dispersive estimate for the 1D Schrödinger equation with a steplike potential*, J. Differential Equations **252** (2012), no. 2, 1603–1634.

[35] E. B. Davies and B. Simon, *Scattering theory for systems with different spatial asymptotics on the left and right*, Comm. Math. Phys. **63** (1978), no. 3, 277–301.

[36] B. Dodson, *Global well-posedness and scattering for the defocusing, L^2-critical nonlinear Schrödinger equation when $d \geq 3$*, J. Amer. Math. Soc. **25** (2012), no. 2, 429–463.

[37] ______, *Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d = 2$*, Duke Math. J. **165** (2016), no. 18, 3435–3516.

[38] ______, *Global well-posedness and scattering for the defocusing, L^2 critical, nonlinear Schrödinger equation when $d = 1$*, Amer. J. Math. **138** (2016), no. 2, 531–569.

[39] ______, *Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state*, Adv. Math. **285** (2015), 1589–1618.

[40] T. Duyckaerts, J. Holmer, and S. Roudenko, *Scattering for the non-radial 3D cubic nonlinear Schrödinger equation*, Math. Res. Lett. **15** (2008), no. 6, 1233–1250.

[41] L. Fanelli and N. Visciglia, *The lack of compactness in the Sobolev-Strichartz inequalities*, J. Math. Pures Appl. (9) **99** (2013), no. 3, 309–320 (English, with English and French summaries).

[42] D. Fang, J. Xie, and T. Cazenave, *Scattering for the focusing energy-subcritical nonlinear Schrödinger equation*, Sci. China Math. **54** (2011), no. 10, 2037–2062.

[43] L. Forcella and L. Hari, *Large data scattering for the defocusing NLKG on waveguide $\mathbb{R}^d \times \mathbb{T}$*, preprint, archived at https://arxiv.org/abs/1709.03101.

[44] L. Forcella and N. Visciglia, *Double Scattering Channels for 1D NLS in the Energy Space and its Generalization to Higher Dimensions*, Journal of Differential Equations **264** (2018), 929–958.

[45] D. Foschi, *Inhomogeneous Strichartz estimates*, J. Hyperbolic Differ. Equ. **2** (2005), no. 1, 1–24.

[46] P. Gérard, *Description du défaut de compacité de l'injection de Sobolev*, ESAIM Control Optim. Calc. Var. **3** (1998), 213–233 (French, with French summary).

[47] ______, *Oscillations and concentration effects in semilinear dispersive wave equations*, J. Funct. Anal. **141** (1996), no. 1, 60–98.

[48] F. Gesztesy, *Scattering theory for one-dimensional systems with nontrivial spatial asymptotics*, Schrödinger operators, Aarhus 1985, Lecture Notes in Math., vol. 1218, Springer, Berlin, 1986, pp. 93–122.

[49] F. Gesztesy, R. Nowell, and W. Pötz, *One-dimensional scattering theory for quantum systems with nontrivial spatial asymptotics*, Differential Integral Equations **10** (1997), no. 3, 521–546.

[50] J. Ginibre, Y. Tsutsumi, and G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal. **151** (1997), no. 2, 384–436.

[51] J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Schrödinger equation revisited*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), no. 4, 309–327 (English, with French summary).

[52] ______, *The global Cauchy problem for the nonlinear Klein-Gordon equation*, Math. Z. **189** (1985), no. 4, 487–505.

[53] ______, *Time decay of finite energy solutions of the nonlinear Klein-Gordon and Schrödinger equations*, Ann. Inst. H. Poincaré Phys. Théor. **43** (1985), no. 4, 399–422.

[54] ______, *The global Cauchy problem for the nonlinear Klein-Gordon equation. II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **6** (1989), no. 1, 15–35 (English, with French summary).

[55] L. Glangetas and F. Merle, *Existence of self-similar blow-up solutions for Zakharov equation in dimension two. I*, Comm. Math. Phys. **160** (1994), no. 1, 173–215.

[56] ______, *Concentration properties of blow-up solutions and instability results for Zakharov equation in dimension two. II*, Comm. Math. Phys. **160** (1994), no. 2, 349–389.

[57] R. T. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. **18** (1977), no. 9, 1794–1797.

[58] M. Goldberg and W. Schlag, *Dispersive estimates for Schrödinger operators in dimensions one and three*, Comm. Math. Phys. **251** (2004), no. 1, 157–178.

[59] M. Goldberg, L. Vega, and N. Visciglia, *Counterexamples of Strichartz inequalities for Schrödinger equations with repulsive potentials*, Int. Math. Res. Not. (2006), Art. ID 13927, 16.

[60] B. Grébert, É. Paturel, and L. Thomann, *Modified scattering for the cubic Schrödinger equation on product spaces: the nonresonant case*, Math. Res. Lett. **23** (2016), no. 3, 841–861.

[61] Z. Hani and B. Pausader, *On scattering for the quintic defocusing nonlinear Schrödinger equation on* $\mathbb{R} \times \mathbb{T}^2$, Comm. Pure Appl. Math. **67** (2014), no. 9, 1466–1542.

[62] Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia, *Modified scattering for the cubic Schrödinger equation on product spaces and applications*, Forum Math. Pi **3** (2015), e4, 63.

[63] L. Hari and N. Visciglia, *Small data scattering for energy critical NLKG on product spaces* $\mathbb{R}^d \times \mathcal{M}^2$, Commun. Contemp. Math. **20** (2018), no. 2, 1750036, 11.

[64] E. Hebey, *Sobolev spaces on Riemannian manifolds*, Lecture Notes in Mathematics, vol. 1635, Springer-Verlag, Berlin, 1996.

[65] S. Herr, D. Tataru, and N. Tzvetkov, *Strichartz estimates for partially periodic solutions to Schrödinger equations in* 4*d and applications*, J. Reine Angew. Math. **690** (2014), 65–78.

[66] T. Hmidi and S. Keraani, *Remarks on the blowup for the* L^2*-critical nonlinear Schrödinger equations*, SIAM J. Math. Anal. **38** (2006), no. 4, 1035–1047.

[67] J. Holmer and S. Roudenko, *A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation*, Comm. Math. Phys. **282** (2008), no. 2, 435–467.

[68] Y. Hong, *Scattering for a nonlinear Schrödinger equation with a potential*, Commun. Pure Appl. Anal. **15** (2016), no. 5, 1571–1601.

[69] S. Ibrahim, N. Masmoudi, and K. Nakanishi, *Scattering threshold for the focusing nonlinear Klein-Gordon equation*, Anal. PDE **4** (2011), no. 3, 405–460.

[70] ———, *Correction to the article Scattering threshold for the focusing nonlinear Klein-Gordon equation*, Anal. PDE **9** (2016), no. 2, 503–514.

[71] A. Jensen, *Spectral properties of Schrödinger operators and time-decay of the wave functions results in* $L^2(\mathbf{R}^m)$*,* $m \geq 5$, Duke Math. J. **47** (1980), no. 1, 57–80.

[72] ———, *Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in* $L^2(\mathbf{R}^4)$, J. Math. Anal. Appl. **101** (1984), no. 2, 397–422.

[73] A. Jensen and T. Kato, *Spectral properties of Schrödinger operators and time-decay of the wave functions*, Duke Math. J. **46** (1979), no. 3, 583–611.

[74] J.-L. Journé, A. Soffer, and C. D. Sogge, *Decay estimates for Schrödinger operators*, Comm. Pure Appl. Math. **44** (1991), no. 5, 573–604.

[75] L. V. Kapitanskiĭ, *The Cauchy problem for the semilinear wave equation. III, translated in* J. Soviet. Math **62** (1990), no. 2, 2619–2645.

[76] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), no. 5, 955–980.

[77] C. E. Kenig and F. Merle, *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math. **166** (2006), no. 3, 645–675.

[78] ———, *Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation*, Acta Math. **201** (2008), no. 2, 147–212.

[79] C. E. Kenig, G. Ponce, and L. Vega, *On the Zakharov and Zakharov-Schulman systems*, J. Funct. Anal. **127** (1995), no. 1, 204–234.

[80] S. Keraani, *On the defect of compactness for the Strichartz estimates of the Schrödinger equations*, J. Differential Equations **175** (2001), no. 2, 353–392.

[81] R. Killip, B. Stovall, and M. Visan, *Scattering for the cubic Klein-Gordon equation in two space dimensions*, Trans. Amer. Math. Soc. **364** (2012), no. 3, 1571–1631.

[82] R. Killip and M. Visan, *Energy-supercritical NLS: critical $\dot{H}^s$-bounds imply scattering*, Comm. Partial Differential Equations **35** (2010), no. 6, 945–987.

[83] ———, *The defocusing energy-supercritical nonlinear wave equation in three space dimensions*, Trans. Amer. Math. Soc. **363** (2011), no. 7, 3893–3934.

[84] ———, *The radial defocusing energy-supercritical nonlinear wave equation in all space dimensions*, Proc. Amer. Math. Soc. **139** (2011), no. 5, 1805–1817.

[85] R. Killip, M. Visan, and X. Zhang, *Quintic NLS in the exterior of a strictly convex obstacle*, Amer. J. Math. **138** (2016), no. 5, 1193–1346.

[86] J. Krieger and W. Schlag, *Concentration compactness for critical wave maps*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2012.

[87] E. H. Lieb, *On the lowest eigenvalue of the Laplacian for the intersection of two domains*, Invent. Math. **74** (1983), no. 3, 441–448.

[88] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana **1** (1985), no. 1, 145–201.

[89] ———, *The concentration-compactness principle in the calculus of variations. The limit case. II*, Rev. Mat. Iberoamericana **1** (1985), no. 2, 45–121.

[90] F. Linares and G. Ponce, *Introduction to nonlinear dispersive equations*, 2nd ed., Universitext, Springer, New York, 2015.

[91] S. Machihara, K. Nakanishi, and T. Ozawa, *Nonrelativistic limit in the energy space for nonlinear Klein-Gordon equations*, Math. Ann. **322** (2002), no. 3, 603–621.

[92] ———, *Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation*, Rev. Mat. Iberoamericana **19** (2003), no. 1, 179–194.

[93] N. Masmoudi and K. Nakanishi, *Energy convergence for singular limits of Zakharov type systems*, Invent. Math. **172** (2008), no. 3, 535–583.

[94] F. Merle, *On uniqueness and continuation properties after blow-up time of self-similar solutions of nonlinear Schrödinger equation with critical exponent and critical mass*, Comm. Pure Appl. Math. **45** (1992), no. 2, 203–254.

[95] ———, *Blow-up results of virial type for Zakharov equations*, Comm. Math. Phys. **175** (1996), no. 2, 433–455.

[96] F. Merle and L. Vega, *Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D*, Internat. Math. Res. Notices **8** (1998), 399–425.

[97] C. S. Morawetz, *Time decay for the nonlinear Klein-Gordon equations*, Proc. Roy. Soc. Ser. A **306** (1968), 291–296.

[98] C. S. Morawetz and W. A. Strauss, *Decay and scattering of solutions of a nonlinear relativistic wave equation*, Comm. Pure Appl. Math. **25** (1972), 1–31.

[99] M. Nakamura and T. Ozawa, *Small data scattering for nonlinear Schrödinger wave and Klein-Gordon equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1** (2002), no. 2, 435–460.

[100] K. Nakanishi, *Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions* 1 *and* 2, J. Funct. Anal. **169** (1999), no. 1, 201–225.

[101] ______, *Scattering theory for the nonlinear Klein-Gordon equation with Sobolev critical power*, Internat. Math. Res. Notices **1** (1999), 31–60.

[102] K. Nakanishi and W. Schlag, *Invariant manifolds and dispersive Hamiltonian evolution equations*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2011.

[103] T. Ozawa and Y. Tsutsumi, *Existence and smoothing effect of solutions for the Zakharov equations*, Publ. Res. Inst. Math. Sci. **28** (1992), no. 3, 329–361.

[104] ______, *Global existence and asymptotic behavior of solutions for the Zakharov equations in three space dimensions*, Adv. Math. Sci. Appl. **3** (1993/94), no. Special Issue, 301–334.

[105] R. S. Palais, *The symmetries of solitons*, Bull. Amer. Math. Soc. (N.S.) **34** (1997), no. 4, 339–403.

[106] H. Pecher, *Low energy scattering for nonlinear Klein-Gordon equations*, J. Funct. Anal. **63** (1985), no. 1, 101–122.

[107] F. Planchon and L. Vega, *Bilinear virial identities and applications*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 2, 261–290 (English, with English and French summaries).

[108] P. Raphaël, *Concentration compacité à la Kenig-Merle*, Astérisque **352** (2013), Exp. No. 1046, vii, 121–146 (French, with French summary). Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058.

[109] J. Rauch, *Local decay of scattering solutions to Schrödinger's equation*, Comm. Math. Phys. **61** (1978), no. 2, 149–168.

[110] I. Rodnianski and W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math. **155** (2004), no. 3, 451–513.

[111] W. Schlag, *The method of concentration compactness and dispersive Hamiltonian evolution equations*, XVIIth International Congress on Mathematical Physics, World Sci. Publ., Hackensack, NJ, 2014, pp. 174–196.

[112] S. H. Schochet and M. I. Weinstein, *The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence*, Comm. Math. Phys. **106** (1986), no. 4, 569–580.

[113] G. Staffilani, *Periodic Schrödinger equations in Hamiltonian form*, HCDTE lecture notes. Part II. Nonlinear hyperbolic PDEs, dispersive and transport equations, AIMS Ser. Appl. Math., vol. 7, Am. Inst. Math. Sci. (AIMS), Springfield, MO, 2013, pp. 66.

[114] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[115] R. S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), no. 3, 705–714.

[116] M. Struwe, *Variational methods*, Springer-Verlag, Berlin, 1990. Applications to nonlinear partial differential equations and Hamiltonian systems.

[117] C. Sulem and P.-L. Sulem, *Quelques résultats de régularité pour les équations de la turbulence de Langmuir*, C. R. Acad. Sci. Paris Sér. A-B **289** (1979), no. 3, A173–A176 (French, with English summary).

[118] ______, *The nonlinear Schrödinger equation*, Applied Mathematical Sciences, vol. 139, Springer-Verlag, New York, 1999. Self-focusing and wave collapse.

[119] T. Tao, *Nonlinear dispersive equations*, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.

[120] M. Tarulli, *Well-posedness and scattering for the mass-energy NLS on* $\mathbb{R}^n \times \mathcal{M}^k$, Analysis (Berlin) **37** (2017), no. 3, 117–131.

[121] B. Texier, *Derivation of the Zakharov equations*, Arch. Ration. Mech. Anal. **184** (2007), no. 1, 121–183.

[122] S. G. Thornhill and D. ter Harr, *Langmuir Turbulence and Modulational Instability*, Phys. Reports **43** (1978), 43–99.

[123] H. Triebel, *Spaces of distributions with weights. Multipliers in* L_p*-spaces with weights*, Math. Nachr. **78** (1977), 339–355.

[124] ______, *Theory of function spaces. II*, Monographs in Mathematics, vol. 84, Birkhäuser Verlag, Basel, 1992.

[125] ______, *Theory of function spaces. III*, Monographs in Mathematics, vol. 100, Birkhäuser Verlag, Basel, 2006.

[126] N. Tzvetkov and N. Visciglia, *Small data scattering for the nonlinear Schrödinger equation on product spaces*, Comm. Partial Differential Equations **37** (2012), no. 1, 125–135.

[127] ______, *Well-posedness and scattering for nonlinear Schrödinger equations on* $\mathbb{R}^d \times \mathbb{T}$ *in the energy space*, Rev. Mat. Iberoam. **32** (2016), no. 4, 1163–1188.

[128] V. Vilaça da Rocha, *Modified scattering and beating effect for coupled Schrödinger systems on product spaces with small initial data*, Trans. Amer. Math. Soc., to appear.

[129] M. C. Vilela, *Inhomogeneous Strichartz estimates for the Schrödinger equation*, Trans. Amer. Math. Soc. **359** (2007), no. 5, 2123–2136.

[130] N. Visciglia, *On the decay of solutions to a class of defocusing NLS*, Math. Res. Lett. **16** (2009), no. 5, 919–926.

[131] R. Weder, *The* $W_{k,p}$*-continuity of the Schrödinger wave operators on the line*, Comm. Math. Phys. **208** (1999), no. 2, 507–520.

[132] ______, L^p*-*$L^{\dot{p}}$ *estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential*, J. Funct. Anal. **170** (2000), no. 1, 37–68.

[133] T. H. Wolff, *Lectures on harmonic analysis*, University Lecture Series, vol. 29, American Mathematical Society, Providence, RI, 2003. With a foreword by Charles Fefferman and preface by Izabella Łaba; Edited by Łaba and Carol Shubin.

[134] K. Yajima, *Existence of solutions for Schrödinger evolution equations*, Comm. Math. Phys. **110** (1987), no. 3, 415–426.

[135] ______, *The* $W^{k,p}$*-continuity of wave operators for Schrödinger operators*, J. Math. Soc. Japan **47** (1995), no. 3, 551–581.

[136] V. E. Zakharov, *Collapse of Langmuir waves*, Sov. Phys. JETP **35** (1972), 908–914.