On Higher Spin Symmetries in AdS_5

R. Manvelyan [†], K. Mkrtchyan[♭], R. Mkrtchyan [†] and S. Theisen [‡]

[†] *Yerevan Physics Institute, Alikhanian Br. Str. 2, 0036 Yerevan, Armenia*
[♭] *Scuola Normale Superiore and INFN, Piazza dei Cavalieri 7, 56126 Pisa, Italy*
[‡] *Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut,*
Am Mühlenberg 1, 14476 Golm, Germany

manvel@physik.uni-kl.de; karapet.mkrtchyan@sns.it; mrl55@list.ru;
stefan.theisen@aei.mpg.de

Abstract

A special embedding of the $SU(4)$ algebra in $SU(10)$, including both spin two and spin three symmetry generators, is constructed. A five dimensional action for massless spin two and three fields is proposed. The connection with the previously investigated higher spin theories in AdS_5 background is discussed. Generalization to the more general case of symmetries, including spins $2, 3, \dots s$, is shown.

1 Introduction

Higher Spin gauge theories have different structure in different space-time dimensions. The first example of a consistent fully nonlinear HS theory in four dimensions was given in [1]. Less is known for higher dimensions. In dimensions higher than four Higher Spin theories are getting more complicated in general, allowing fields of mixed symmetry type. At the same time, for the restricted spectra of only symmetric fields, Vasiliev equations are available for any space-time dimension [2]. They are defined unambiguously and describe totally symmetric bosonic fields of all spins.

Recent progress in three dimensional AdS higher spin gravity resulted in new relations between topological Chern-Simons theory, two-dimensional conformal field theories with higher spin symmetry, and new three-dimensional black hole solutions with higher spin charges ([3]-[8] and references therein). It also points out again the importance of an AdS background for the construction of consistent nonlinear higher spin interactions with a finite number of interacting higher spin gauge fields. These recent results are based on the embedding of the gravitational gauge group into a larger group, unifying higher spin gauge symmetry with the AdS group. In the three dimensional case it amounts to embedding $SL(2)$ into $SL(3)(SL(n))$ in the case of spin three (up to spin n) gravity, and the corresponding field theory is described by a three-dimensional Chern-Simons action with $SL(3) \times SL(3)$ $(SL(n) \times SL(n))$ gauge group. The case of three dimensions is singled out by the existence of a one-parameter family of Higher Spin algebras, that underlie the construction of Chern-Simons actions for the gauge fields [9, 10, 11, 12] and Vasiliev equations, describing the interaction of Higher Spin gauge fields with scalar matter [13].

The main goal of this paper is to generalize this approach to five dimensions, and to propose possible nonlinear interacting theories with finite number of higher spin fields in an AdS_5 background. Moreower we show in this paper the existence of a family of Lie algebras, the generators of which can be identified with the generators of Higher Spin gauge symmetries for a finite number of symmetric fields in $(A)dS_5$, analogously to the case of three dimensions.

As a realization of this idea we construct in the next section a special embedding of the spin two and spin three symmetry generators in frame formalism into a unifying $SU(10)$ Lie algebra, where the spin two generators correspond to the $SU(4)$ subalgebra and the spin three generators to the remaining part of $SU(10)$. In Section 3 we construct gauge fields and curvatures. The latter include interactions and self-interactions of the spin-2 and spin-3 fields. In the fourth section we discuss a possible action as a realization of the unified spin 2 and 3 gauge field theory. The first idea which comes to mind is a five-dimensional Chern-Simons action for the $SU(10)$ gauge field. This idea is also based on the fact that for unitary groups one can find invariant third rank symmetric tensors which provides an invariant trace for the construction of the Chern-Simons action in five dimensions. But it is well known since many years [14][15] that this action, even in the pure gravity case ($SO(6)$ gauge group) leads to Gauss-Bonnet (Lovelock) gravity with a special combination of terms quadratic and linear in curvatures and without a propagator for spin two fluctuations in an AdS_5 background. Higher Spin Chern-Simons gravity in 5d was discussed in [16], where the authors considered also the dynamics of linearized spin 3 gauge fields. However, another possible Lagrangian formulation for theories of spin 2 and higher in an AdS background in the frame formulation is known. It is the so-called MacDowell-Mansouri-Stelle-West formulation [17, 18] used by Vasiliev for perturbative analysis of interactions [19, 20, 21]. Taking into account all of this, we propose in Section 4 a generalization of the coset construction of [17, 18] and introduce a compensator field living on the coset $SU(10)/SO(10)$. The limit of pure spin two field and the free limit of the spin three field in an AdS-background are correctly captured. Generalization to any spin is discussed in the section 5.

2 Unification of spin 2 and 3 symmetries on AdS_5

Gravitational theories in frame formalism can be formulated as gauge theories. Since our construction draws some of its motivation from the three dimensional case, we will briefly recall it. There pure gravity with a negative cosmological constant can be written as a $SO(2,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons theory. The generalization to higher spin is to replace $SL(2)$ by a bigger group G with a special embedding $SL(2,\mathbb{R}) \hookrightarrow G$, the simplest case being $G = SL(3,\mathbb{R})$ with the principal embedding, leading to a unified description of a spin-three field coupled to gravity.

Five dimensional gravity in AdS_5 space is a gauge theory of $SO(2,4)$ (pure AdS) or $SO(1,5)$ (Euclidian AdS). The corresponding fünfbein and spin connection can be extracted from the gauge field, which is an algebra-valued one-form, by decomposition of the adjoint representation of $SO(2,4)$ or $SO(1,5)$ into the adjoint and vector representations of $SO(1,4)$. For simplicity and without loss of generality we can replace these non-compact groups by their compact versions. Namely we consider instead of the AdS_5 group the six dimensional rotation group $SO(6)$ and expand the gauge field with respect to the "space-time rotation" group $SO(5)$, just separating the sixth component as the vector representation and obtaining correspondingly a fünfbein and a spin-connection:

$$\begin{aligned} A^{AB}_{\mu} dx^{\mu} &= A^{AB} = -A^{BA}, \quad A, B, \dots = 1, \dots, 6, \\ A^{AB} &= \{A^{a6}, A^{ab}\} = \{e^a, \omega^{ab}\}, \quad a, b = 1, \dots, 5. \end{aligned} \tag{2.1}$$

We can then impose constraints of vanishing torsion and express the spin connection in terms of fünfbein and inverse fünfbein fields.

Then we propose the following extension to include spin 3 fields (and higher). The $SO(6)$ representation of the gravitational fields (2.1) is via the antisymmetric two cell Young tableau

$$A^{AB} \Rightarrow Y^{SO(6)}_{A^{AB}} = \begin{array}{|c|}\hline ~ \\ \hline ~ \\ \hline\end{array}\,, \quad \dim(Y^{SO(6)}_{A^{AB}}) = 15\ . \tag{2.2}$$

In terms of Young tableaux, the expansion (2.1) is

$$\begin{array}{|c|}\hline ~ \\ \hline ~ \\ \hline\end{array}_{\,SO(6)} = \left(\begin{array}{|c|}\hline ~ \\ \hline\end{array} + \begin{array}{|c|}\hline ~ \\ \hline ~ \\ \hline\end{array} \right)_{SO(5)}\,, \tag{2.3}$$

or in terms of dimensions:

$$\underline{\mathbf{15}}_{SO(6)} = (\underline{\mathbf{5}} + \underline{\mathbf{10}})_{SO(5)}\ . \tag{2.4}$$

From this point of view the spin 3 field corresponds to the $SO(6)$ window diagram

$$A^{AB,CD} \Rightarrow Y^{SO(6)}_{A^{AB,CD}} = \begin{array}{|c|c|}\hline ~ & ~ \\ \hline ~ & ~ \\ \hline\end{array}\,, \qquad \dim\big(Y^{SO(6)}_{A^{AB,CD}}\big) = 84\ , \tag{2.5}$$

and the corresponding $SO(5)$ expansion to a spin 3 tetrad and connections looks like

$$\begin{array}{ccccccccc} A^{AB,CD} & & & e^{ab} & & \omega^{ab,c} & & \omega^{ab,cd} & \\ \begin{array}{|c|c|}\hline ~ & ~ \\ \hline ~ & ~ \\ \hline\end{array}_{\,SO(6)} & = & \Big(& \begin{array}{|c|c|}\hline ~ & ~ \\ \hline\end{array} & + & \begin{array}{|c|c|}\hline ~ & ~ \\ \hline ~ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & + & \begin{array}{|c|c|}\hline ~ & ~ \\ \hline ~ & ~ \\ \hline\end{array} & \Big)_{SO(5)}\,, \\ \underline{\mathbf{84}}_{SO(6)} & = & (& \underline{\mathbf{14}} & + & \underline{\mathbf{35}} & + & \underline{\mathbf{35}} &)_{SO(5)}\,, \end{array} \tag{2.6}$$

where we have identified

$$\{A^{a6,b6}, A^{ab,c6}, A^{ab,cd}\} = \{e^{ab}, \omega^{ab,c}, \omega^{ab,cd}\}. \tag{2.7}$$

The $\omega^{ab,cd}$ are so-called extra fields (which are absent in $d = 3$).

For the unification of the spin 2 and spin 3 degrees of freedom into one field, we should first of all find a Lie group G with dimension

$$\underline{\mathbf{15}}_{SO(6)} + \underline{\mathbf{84}}_{SO(6)} = \underline{\mathbf{99}}_{\mathbf{G}}\ . \tag{2.8}$$

Taking into account that $SO(6)$ is equivalent* to $SU(4)$ we see that the natural choice for G is $SU(10)$†. The 15 generators of spin 2 gauge symmetry and 84 generators of spin 3 gauge symmetry can be combined into the 99 generators of $SU(10)$.

*See the appendix for details on the isomorphism $so(6) \simeq su(4)$ and other relevant formulae.

†For other signatures of the initial space-time isometry algebra, we have, of course, different real forms of $SL(10, \mathbb{C})$.

To proceed, we have to find an embedding of $SU(4)$ into $SU(10)$ such that the adjoint of the latter decomposes w.r.t. the former as in (2.8). That amounts to finding a representation of $SU(4)$ of dimension 10. Such representation of $SU(4)$ exists in the space of symmetric second-rank tensors. We arrive at the following embedding procedure:[‡]

- Denote the 99 generators of the $SU(10)$ algebra by

$$U^I_J, \quad U^I_I = 0, \quad I, J, \dots \in \{1, 2, \dots, 10\}. \tag{2.9}$$

- We can present the $SU(10)$ vector indices $I, J, \dots$ as symmetric pairs of vector indices of $SU(4)$

$$\begin{aligned} I, J, \dots \ &\rightarrow \ (\alpha\beta), (\gamma\delta), \dots, \quad \alpha, \beta, \dots \in \{1, 2, 3, 4\}, \\ U^I_J \ &\rightarrow \ U^{\alpha\beta}_{\gamma\delta} = U^{\beta\alpha}_{\gamma\delta} = U^{\alpha\beta}_{\gamma\delta}, \quad U^{\alpha\beta}_{\alpha\beta} = 0. \end{aligned} \tag{2.10}$$

- The $SU(4) \hookrightarrow SU(10)$ embedding can then be realized as the decomposition into single and double traceless parts of $U^{\alpha\beta}_{\gamma\delta}$

$$\begin{aligned} U^{\alpha\beta}_{\gamma\delta} &= W^{\alpha\beta}_{\gamma\delta} + \frac{1}{6}\delta^{(\alpha}_{(\gamma} L^{\beta)}_{\delta)}, \\ L^{\beta}_{\delta} &= U^{\alpha\beta}_{\alpha\delta}, \\ W^{\alpha\beta}_{\alpha\delta} &= L^{\beta}_{\beta} = 0, \end{aligned} \tag{2.11}$$

  where L^{β}_{δ} are the 15 generators of $SU(4)$.

This shows that (2.11) is a realization of the embedding:

$$\underline{\mathbf{99}}_{SU(10)} = (\underline{\mathbf{15}} + \underline{\mathbf{84}})_{SO(6)} \ . \tag{2.12}$$

Using the explicit form of the $SU(10)$ generators, it is straightforward to work out the commutation relations of L and W. The result is given in the appendix.

To summarize, we constructed a Lie algebra of spin 3 and spin 2 transformations in AdS_5 using a special embedding $SO(6) \simeq SU(4) \hookrightarrow SU(10)$. From (A.6) one sees that the difference between $SU(10)$ and $SU(4)$ is precisely the tensor representation of $SU(4)$ corresponding to the window tableau of $SO(6)$.

In the subsequent sections we attempt to construct a non-linearly interacting gauge field theory corresponding to the above unified algebra, and show a connection to Vasiliev's free higher spin action in AdS background [19].

[‡]We do not distinguish between the components of a tensor in the adjoint representation and the generators of $SU(10)$.

3 Gauge fields and Curvatures

In this section we apply the $SU(4) \hookrightarrow SU(10)$ embedding to gauge fields and curvatures. First of all we can equip a general one-form gauge field and zero-form gauge parameter with $SU(10)$ indices expressed as symmetric pairs of $SU(4)$ indices

$$\mathbf{A} = A^{\alpha\beta}_{\gamma\delta}\, U^{\gamma\delta}_{\alpha\beta}, \quad \epsilon = \epsilon^{\alpha\beta}_{\gamma\delta}\, U^{\gamma\delta}_{\alpha\beta}\ , \tag{3.1}$$
$$\delta\mathbf{A} \ = \ D\epsilon \quad \Rightarrow \quad \delta A^{\alpha\beta}_{\gamma\delta} = d\epsilon^{\alpha\beta}_{\gamma\delta} + A^{\alpha\beta}_{\lambda\rho}\epsilon^{\lambda\rho}_{\gamma\delta} - A^{\lambda\rho}_{\gamma\delta}\epsilon^{\alpha\beta}_{\lambda\rho}\ .$$

From now on we use for algebra valued objects a component formalism, i.e. stripping off the generators. In this notation the $SU(10)$ Yang-Mills field strength is

$$F^{\alpha\beta}_{\gamma\delta} = dA^{\alpha\beta}_{\gamma\delta} + A^{\alpha\beta}_{\lambda\rho} \wedge A^{\lambda\rho}_{\gamma\delta}, \quad F^{\alpha\beta}_{\alpha\beta} = 0. \tag{3.2}$$

Using the embedding (2.11) we can extract from the $SU(10)$ gauge field and field strength the spin 2 and spin 3 gauge fields and curvatures:

$$\begin{aligned} A^{\alpha\beta}_{\gamma\delta} &= W^{\alpha\beta}_{\gamma\delta} + \frac{1}{6}\delta^{(\alpha}_{(\gamma}\omega^{\beta)}_{\delta)}, \quad W^{\alpha\beta}_{\alpha\delta} = \omega^{\beta}_{\beta} = 0, \\ F^{\alpha\beta}_{\gamma\delta} &= R^{\alpha\beta}_{\gamma\delta} + \frac{1}{6}\delta^{(\alpha}_{(\gamma}r^{\beta)}_{\delta)}, \quad R^{\alpha\beta}_{\alpha\delta} = r^{\beta}_{\beta} = 0. \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} R^{\alpha\beta}_{\gamma\delta} &= D_\omega W^{\alpha\beta}_{\gamma\delta} + W^{\alpha\beta}_{\lambda\rho} \wedge W^{\lambda\rho}_{\gamma\delta} - \frac{1}{6}\delta^{(\alpha}_{(\gamma} W^{\beta)\sigma}_{|\lambda\rho|} \wedge W^{\lambda\rho}_{\delta)\sigma}, \\ D_\omega W^{\alpha\beta}_{\gamma\delta} &= dW^{\alpha\beta}_{\gamma\delta} + \frac{1}{3}\omega^{(\alpha}_{\lambda} \wedge W^{\beta)\lambda}_{\gamma\delta} - \frac{1}{3}\omega^{\lambda}_{(\gamma} \wedge W^{\alpha\beta}_{\delta)\lambda}, \\ r^{\alpha}_{\beta} &= d\omega^{\alpha}_{\beta} + \frac{1}{3}\omega^{\alpha}_{\lambda} \wedge \omega^{\lambda}_{\beta} + W^{\alpha\sigma}_{\lambda\rho} \wedge W^{\lambda\rho}_{\beta\sigma}. \end{aligned} \tag{3.4}$$

Structure and couplings of fields in the curvatures reflect the structure of the commutators (A.6)[§].

4 Topological Actions and Coset Construction

We begin with a brief review of the Macdowell-Mansouri-Stelle-West action principle for the case of usual spin two gravity in five dimensions. The task could be formulated in the following way: we have to write a topological action for five dimensional gauge theory with $SO(6)$ gauge group. This means that we should

[§]After rescaling of the spin two field $\omega \to 3\omega$ the curvature transforms to the usual Riemann form.

construct a five-form enabling us to integrate over a general five dimensional manifold M_5 in a metric independent way. Introduce a field strength

$$F^{AB} = dA^{AB} + A^A{}_C \wedge A^{CB}, \quad A, B, \dots = 1, 2 \dots 6\ . \tag{4.1}$$

The natural choice for the action is

$$S_{SO(6)} \sim \int_{M_5} \epsilon_{ABCDEF}\, B^{AB} \wedge F^{CD} \wedge F^{EF}\ , \tag{4.2}$$

where $B^{AB} = -B^{BA}$ is an $SO(6)$ algebra valued gauge covariant one-form constructed from some compensator field. The compensator field should be introduced in a way that does not lead to equations of motion purely quadratic in the field strength

$$\epsilon_{ABCDEF} F^{CD} \wedge F^{EF} = 0, \tag{4.3}$$

as happens in the Chern-Simons case and which leads to a vanishing propagator in an AdS background $F^{AB} = F^{AB}_{AdS} = 0$. A possible solution is to take the compensator as an element of the coset G/H where G in this case is $SO(6)$ and the stabilizer H should be taken in a way to keep "Lorentz" covariance as the remaining symmetry after gauge fixing. The natural choice in this case is $H = SO(5)$. This construction leads to consistent gravity action, which is equivalent to the Einstein-Hilbert action in the linearized limit. In summary, we define the compensator field as an element of a five dimensional sphere

$$S^5 = SO(6)/SO(5)\ . \tag{4.4}$$

The sphere can be realized, in a manifestly $SO(6)$ invariant way, as a unit vector in $\mathbb{R}^6$:

$$V^A, \quad V^A V_A = 1. \tag{4.5}$$

The $SO(6)$ covariant one-form and the corresponding action can then be constructed from (4.5) uniquely:

$$B^{AB} = V^{[A} D V^{B]}, \quad DV^B = dV^B + A^B{}_C V^C, \tag{4.6}$$

$$S_{SO(6)} \sim \int_{M_5} \epsilon_{ABCDMN}\, V^A D V^B \wedge F^{CD} \wedge F^{MN}\ . \tag{4.7}$$

A detailed analysis of the equations of motions and symmetries of this action can be found in [19]. Here we only note that using local $SO(6)$ invariance of the theory, we can bring the vector field $V^A(x)$ to the constant unit vector in the sixth direction, and the remaining $SO(5)$ invariance will still be sufficient for covariance in the language of fünfbein and spin connection (2.1). Another important aspect of this construction is that the remaining $SO(5)$ invariance, combined with diffeomorphism invariance will still be sufficient for full AdS invariance of the theory [19].

One can rewrite this action equivalently in $SU(4)$ form. This can be done in two ways, leading to the same result, of course. The first one is a direct transformation to chiral spinor indices $\alpha, \beta, \dots \in \{1,2,3,4\}$ using standard identities for chiral Dirac matrices in six dimensions[¶]

$$
\begin{aligned}
V^{\alpha\beta} = i(\Sigma^A)^{\alpha\beta} V_A \qquad &\longleftrightarrow \qquad V^A = \tfrac{i}{4}\Sigma^A_{\alpha\beta} V^{\alpha\beta}, \qquad V^{\alpha\beta} = -V^{\beta\alpha}, \\
F_\alpha^\beta = (\Sigma_{AB})_\alpha^\beta F^{AB} \quad &\longleftrightarrow \quad F^{AB} = -\tfrac{1}{2}(\Sigma^{AB})_\beta^\alpha F_\alpha^\beta, \qquad F_\alpha^\alpha = 0.
\end{aligned} \tag{4.8}
$$

The constraint on $V^{\alpha\beta}$ which follows from (4.5) is

$$
V^{\alpha\gamma} V_{\beta\gamma} = \delta_\beta^\alpha, \quad V_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta} V^{\gamma\delta}. \tag{4.9}
$$

With the help of the identity (A.11) one obtains from (4.7)

$$
S_{SU(4)} \sim i \int_{M_5} V^{\alpha\lambda} DV_{\beta\lambda} \wedge F_\rho^\beta \wedge F_\alpha^\rho. \tag{4.10}
$$

So we recognize the $SU(4)$ covariant algebra-valued one-form

$$
\begin{aligned}
&B_\beta^\alpha = iV^{\alpha\lambda}(DV)_{\beta\lambda}, \quad B_\alpha^\alpha = 0, \\
&(DV)_{\beta\lambda} = dV_{\beta\lambda} + A^\rho_{[\beta} V_{\lambda]\rho}.
\end{aligned} \tag{4.11}
$$

The second way is to observe that the integrand in (4.7) is just the $SO(6)$ invariant trace of three elements of the $SO(6)$ algebra or, equivalently, that ϵ_{ABCDEF} is the d-symbol of $SO(6) \simeq SU(4)$. With this observation it is immediate how to generalize the topological action for any Lie group G:

$$
S_G \sim \int_{M_5} d_{\Omega\Theta\Lambda} B^\Omega \wedge F^\Theta \wedge F^\Lambda \ , \tag{4.12}
$$

where capital Greek indices $\Gamma, \Theta, \Lambda \dots \in \{1, \dots, \dim(G)\}$. The crucial point of this construction is the choice of the coset G/H whose element will be used for the construction of the G covariant one-form B^Ω. In the case of $G = SO(6)$ we have $H = SO(5)$ and the compensator field is an element of the five-sphere. Equivalently for the same system, if $G = SU(4)$ we identify the stabilizer group $H = Sp(4) \simeq SO(5)$ and the compensator $V^{\alpha\beta}$ is an element of the coset

$$
SU(4)/Sp(4) \ , \tag{4.13}
$$

and is expressed as an antisymmetric $SU(4)$ tensor constrained by (4.9). Then the $SU(4)$ algebra valued one-form can be constructed as (4.11) and the general

[¶]Further details are given in the appendix.

action (4.12) transforms into (4.10). Note also that in the same fashion as we fixed the gauge using local $SO(6)$ rotations,

$$\begin{aligned} V^A &= (V^a, V^6), \quad (a = 1, \dots, 5), \\ V^{(0)A} &= (0, 1), \end{aligned} \tag{4.14}$$

in the $SU(4)$ formulation, we can bring the compensator field $V_{\alpha\beta}(x)$ to the constant symplectic form $V^{(0)}_{\alpha\beta}$, leaving an unbroken symmetry $Sp(4)$. The relation corresponding to (4.14) is

$$V_{\alpha\beta}(x) = V^{(0)}_{\alpha\beta} = i\,\Sigma^6_{\alpha\beta}. \tag{4.15}$$

We now turn to our proposal for unifying spin 2 and spin 3 invariance. To this end we consider an action with gauge group $SU(10)$ with the special embedding of $SU(4)$ discussed above. This means that we identify in (4.12) the field strength F^Λ with the $SU(10)$ field strength (3.2). In other words we replace the indices $\Gamma, \Theta, \Lambda, \dots$ by two symmetrised pairs of $SU(4)$ indices ${}^{\alpha\beta}_{\gamma\delta}$ with the corresponding $SU(10)$ rule for taking the trace, e.g. using the d-symbol (A.3)

$$S_{SU(10)} = \int_{M_5} B^{\alpha\beta}_{\mu\nu} \wedge F^{\mu\nu}_{\lambda\rho} \wedge F^{\lambda\rho}_{\alpha\beta}, \tag{4.16}$$

$F^{\alpha\beta}_{\gamma\delta}$ was defined in (3.2). It remains to define the correct coset space and compensator, and to construct an $SU(10)$ covariant one-form

$$B^{\alpha\beta}_{\gamma\delta}\,, \qquad B^{\alpha\beta}_{\alpha\beta} = 0, \tag{4.17}$$

$$\delta B^{\alpha\beta}_{\gamma\delta} = B^{\alpha\beta}_{\lambda\rho}\epsilon^{\lambda\rho}_{\gamma\delta} - B^{\lambda\rho}_{\gamma\delta}\epsilon^{\alpha\beta}_{\lambda\rho}.$$

Searching for a suitable stabilizer for the coset G/H constructed from $G = SU(10)$, we arrive at $H = SO(10)$. This choice of compensator allows the background value described by the SU(4)/Sp(4) coset construction. This property we use below in the analysis of the linearized limit. From

$$\begin{aligned} G/H &= SU(10)/SO(10)\,, \\ \dim(G/H) &= \dim(SU(10)) - \dim(SO(10)) = 54\,. \end{aligned} \tag{4.18}$$

we conclude that the compensator should appear as a 54-dimensional representation of $SO(10)$. For $SU(10)$ covariance of B or, equivalently, for $SU(10)$ invariance of the action (4.16), this representation should be expressed as a constrained representation of $SU(10)$. From an $SO(10)$ point of view it is a second rank *symmetric* traceless tensor with 54 independent real components, which we can express as an $SU(10)$ object in the following way. Consider the space of complex tensors symmetric in a pair of lower indices and its complex conjugate tensor with upper indices

$$V_{IJ} = V_{JI}, \quad \bar{V}^{IJ} = \bar{V}^{JI} = (V_{IJ})^*, \quad I, J, \dots \in \{1, \dots 10\}. \tag{4.19}$$

It has 55 independent complex components. The natural $SU(10)$ invariant (real) constraint

$$\bar{V}^{IK}V_{KJ} = \delta^I_J \qquad \text{or} \quad V^*V = \mathbb{1}, \tag{4.20}$$

reduces the number of independent real components to 55. If we construct an $SU(10)$ covariant one-form in the usual way

$$\begin{aligned} B^I_J &= i\bar{V}^{IK}DV_{KJ}, \\ DV_{KJ} &= dV_{KJ} - A^L_{(K}V_{J)L}, \end{aligned} \tag{4.21}$$

we see that in this case the constraint (4.20) is not sufficient for rendering (4.21) traceless and therefore $SU(10)$ algebra valued. In any case, we need one more real constraint on (4.19) to reduce the number of independent components to 54 in order to identify this tensor with an element of the symmetric space (4.18). The following $SU(10)$ invariant constraint

$$\det(V) = 1, \tag{4.22}$$

solves both problems and completes the construction of a covariant one-form in the $SU(10)$ case. Replacing capital Latin indices with symmetrized pairs of $SU(4)$ indices as before, we arrive at the following expression for $B^{\alpha\beta}_{\mu\nu}$ in (4.16)

$$\begin{aligned} B^{\alpha\beta}_{\gamma\delta} &= i\bar{V}^{\alpha\beta,\lambda\rho}DV_{\gamma\delta,\lambda\rho}, \\ B^{\alpha\beta}_{\alpha\beta} &= 0, \end{aligned} \tag{4.23}$$

where the $SU(10)/SO(10)$ compensator field is defined as

$$\begin{aligned} &V_{\alpha\beta,\lambda\rho} = V_{\lambda\rho,\alpha\beta}, \\ &\bar{V}^{\alpha\beta,\lambda\rho} = (V_{\alpha\beta,\lambda\rho})^*, \\ &\bar{V}^{\alpha\beta,\lambda\rho}V_{\lambda\rho,\gamma\delta} = \delta^{\alpha\beta}_{\gamma\delta}, \\ &\det(V_{(\alpha\beta),(\gamma\delta)}) = 1. \end{aligned} \tag{4.24}$$

In this case we can also use local $SU(10)$ transformations of the compensator field and set

$$V^{(0)}_{\alpha\beta,\lambda\rho} = \delta_{(\alpha\beta),(\lambda\rho)}. \tag{4.25}$$

The unbroken symmetry is $SO(10)$, because the r.h.s. of (4.25) remains invariant under $SO(10)$ rotations.

We now address the embedding of the $SU(4)/Sp(4)$ compensator $V_{\alpha\beta}$ into the $SU(10)/SO(10)$ element (4.24). It is easy to see that the restrictions imposed by the ansatz

$$\begin{aligned} V_{\alpha\beta,\sigma\delta} &= \frac{1}{2}(V_{\alpha\sigma}V_{\beta\delta} + V_{\beta\sigma}V_{\alpha\delta}), \\ \bar{V}^{\alpha\beta,\sigma\delta} &= \frac{1}{2}(V^{\alpha\sigma}V^{\beta\delta} + V^{\beta\sigma}V^{\alpha\delta}), \end{aligned} \tag{4.26}$$

supplemented with

$$A^{\alpha\beta}_{\mu\nu} \sim \delta^{(\alpha}_{(\mu}\omega^{\beta)}_{\nu)}, \tag{4.27}$$

lead to a reduction of the one-forms

$$\begin{aligned} B^{\alpha\beta}_{\gamma\delta} =& i\bar{V}^{\alpha\beta,\lambda\rho}DV_{\lambda\rho,\gamma\delta} = \frac{1}{2}\delta^{(\alpha}_{(\gamma}B^{\beta)}_{\delta)}, \\ B^{\beta}_{\delta} =& iV^{\alpha\beta}DV_{\alpha\delta}. \end{aligned} \tag{4.28}$$

This means that putting the spin three gauge field to zero and using the ansatz (4.26), we obtain the purely gravitational action (4.10) from the $SU(10)$ invariant action. This immediately shows that the equations of motion have AdS_5 background solutions. Note also that the restriction (4.26) leading to the correct $SU(4)/Sp(4)$ coset construction can be realized only for an $SO(10)$ stabilizer.

We now analyze the part of the quadratic action in AdS_5 background that depends only on the spin three field. We require this part to coincide with the free action of [19] for the spin three case. However, one immediately realizes that the $SU(10)$ invariant action (4.16) does not suffice. Indeed, the free action for spin three consist of two parts [19]

$$\begin{aligned} S^{s=3}_{SO(6)} \sim & \int_{M_5} \epsilon_{ABCDMN}\, V^A D_0 V^B \wedge \big(R_1^{CC_1,DD_1} \wedge R_1^{M}{}_{C_1}{}^{,N}{}_{D_1} \\ & + 4\, R_1{}^{CC_1,DD_1} \wedge R_1^{M}{}_{C_1}{}^{,ND_2} V_{D_1} V_{D_2}\big), \end{aligned} \tag{4.29}$$

where $D_0 = d + \omega_0$ is background covariant derivative, $R_1 = D_0\omega$ the linearized curvature and the relative coefficient between the two terms is fixed such that the equation of motion for the unwanted "extra" fields corresponding to the $SO(5)$ window like Young tableau in (2.6) trivializes. Using results from the appendix we can transform this action to $SU(4)$ invariant form:

$$\begin{aligned} S^{s=3}_{SU(4)} \sim & \; i\int_{M^5} V^{\alpha\lambda} D_0 V_{\mu\lambda} \wedge \big(R_1^{\mu\sigma}{}_{\delta_1\delta_2} \wedge R_1^{\delta_1\delta_2}{}_{\alpha\sigma} \\ & + \frac{1}{2} R_1^{\mu\rho_1}{}_{\sigma\delta_1} \wedge R_1^{\sigma\rho_2}{}_{\alpha\delta_2} V_{\rho_1\rho_2} V^{\delta_1\delta_2}\big). \end{aligned} \tag{4.30}$$

However, from (4.16) after linearization of the spin three field in an AdS_5 background, i.e. with the restriction (4.26), we obtain only the first term in (4.30). To get the second one, we introduce another term in the action. Such a term can be constructed with the rank four d symbol of $SU(10)$, defined as the completely symmetrized trace of four $SU(10)$ generators:

$$S_G \sim \int_{M_5} d_{\Omega\Xi\Theta\Lambda} B^{\Omega\Xi} \wedge F^{\Theta} \wedge F^{\Lambda} \ . \tag{4.31}$$

As before, capital Greek indices refer to the adjoint representation of $SU(10)$ and we can replace them by an upper and a lower index refering to the fundamental representation of $SU(10)$ and its complex conjugate, respectively, e.g. $F^{\Lambda} \to F^I_J$ with $F^I_I = 0$ or by two pairs of symmetriced $SU(4)$ indices, i.e. $F^{\alpha\beta}_{\gamma\delta}$ with $F^{\alpha\beta}_{\alpha\beta} = 0$. The tensor B can be realized using the $SU(10)/SO(10)$ compensator field (cf. (4.19),(4.20) and (4.22))[‖]:

$$B^{IK}_{JL} = \frac{i}{2}(\bar{V}^{IK} DV_{JL} - D\bar{V}^{IK} V_{JL}) - \text{traces}. \tag{4.32}$$

In $SU(4)$ covariant notation the second part of the spin three action then becomes

$$\tilde{S}_{SU(10)} = \int_{M_5} B^{\alpha\beta,\sigma\delta}_{\mu\nu,\lambda\rho} \wedge F^{\mu\nu}_{\alpha\beta} \wedge F^{\lambda\rho}_{\sigma\delta}, \tag{4.33}$$

where

$$B^{\alpha\beta,\sigma\delta}_{\mu\nu,\lambda\rho} = \frac{i}{2}(\bar{V}^{\alpha\beta,\sigma\delta} DV_{\mu\nu,\lambda\rho} - D\bar{V}^{\alpha\beta,\sigma\delta} V_{\mu\nu,\lambda\rho}). \tag{4.34}$$

The general action should be a linear combination

$$S_{SU(10)} + \kappa\, \tilde{S}_{SU(10)}, \tag{4.35}$$

where the relative coefficient κ is fixed by comparison with the free spin three action of Vasiliev (4.30). To fix it we replace in (4.33) and (4.16) F with linearized curvatures R_1 (keeping only spin-3 fluctuation in AdS_5 background), use the $SU(4)$ restriction (4.26) for the $SU(10)$ compensator field and replace the covariant derivative by D_0. Straightforward calculation gives

$$S_{SU(10)} \to 2i \int_{M^5} V^{\alpha\lambda} D_0 V_{\mu\lambda} \wedge R_1^{\mu\sigma}{}_{\delta_1\delta_2} \wedge R_1^{\delta_1\delta_2}{}_{\alpha\sigma}, \tag{4.36}$$

$$\tilde{S}_{SU(10)} \to -2i \int_{M^5} V^{\alpha\lambda} D_0 V_{\mu\lambda} \wedge R_1^{\mu\rho_1}{}_{\sigma\delta_1} \wedge R_1^{\sigma\rho_2}{}_{\alpha\delta_2} V_{\rho_1\rho_2} V^{\delta_1\delta_2}. \tag{4.37}$$

Comparison with (4.30) fixes $\kappa = -\frac{1}{2}$. To close this section we note that keeping the spin two fluctuation we obtain also a mixed term in the linearized action, which is proportional to the torsion of spin three field. This term vanishes if the torsion vanishes, a condition which we might impose by hand.

5 Outlook

Two obvious generalizations can be envisioned: including spins higher than three and other dimensions. The first one, at least as far as the identification of G and the embedding $SO(6) \hookrightarrow G$ are concerned, is straightforward. Consider e.g. spin

[‖]The traces would give the same contribution as (4.16).

2, spin 3 and spin 4. The fields and their $SO(5)$ representations are

$$
\begin{array}{lll}
e^a \ \yng(1) \ \underline{\mathbf{5}} & e^{ab} \ \yng(2) \ \underline{\mathbf{14}} & e^{abc} \ \yng(3) \ \underline{\mathbf{30}} \\
\omega^{ab} \ \yng(1,1) \ \underline{\mathbf{10}} & \omega^{ab,c} \ \yng(2,1) \ \underline{\mathbf{35}} & \omega^{abc,d} \ \yng(3,1) \ \underline{\mathbf{81}} \\
 & \omega^{ab,cd} \ \yng(2,2) \ \underline{\mathbf{35}} & \omega^{abc,de} \ \yng(3,2) \ \underline{\mathbf{105}} \\
 & & \omega^{abc,def} \ \yng(3,3) \ \underline{\mathbf{84}}
\end{array}
\tag{5.1}
$$

The fields in each column combine into representations of $SO(6)$ whose Young tableau coincides with the last one in each column. The total of 399 fields nicely combine into the adjoint representation of $SU(20)$. The pattern repeats if we add higher spins such that for spin $2, \ldots, s$ we find $SU\big(\binom{s+2}{3}\big)$. All of the fields, that correspond to spins from 2 to s now combine into one $SU\big(\binom{s+2}{3}\big)$-valued one-form master field. We can introduce $s-1$ symmetrized $su(4)$ indices for each of the $SU\big(\binom{s+2}{3}\big)$ indices (the number of components matches exactly). The trace decomposition of the master one-form field gives all the fields, corresponding to different spins.

We expect that this result hints on the existence of one parameter family of algebras for symmetric Higher Spin fields in five dimensions, in full analogy with the three dimensional case. For the critical values of the parameter, this algebra should acquire infinite-dimensional ideals, with the remaining generators forming finite dimensional subalgebras $SU\big(\binom{s+2}{3}\big)$. If true, this family of algebras should include the known infinite dimensional Higher Spin algebras, discussed in [22, 19, 23, 2, 16]. In order to check this idea, one has to implement the more general construction of Higher Spin algebra, along the lines of [24, 25, 26].

Another observation is that for odd s the stabilizer group should be $SO\big(\binom{s+2}{3}\big)$ while for even s it is $Sp\big(\binom{s+2}{3}\big)$, the reason being that the generalization of the restriction (4.26), which is schematically $V^{(s)} \sim (V^{s=2})^{s-1}$, exists only if

$$
V_{(\alpha_1 \ldots \alpha_{s-1}),(\beta_1 \ldots \beta_{s-1})} = (-1)^{s-1} V_{(\beta_1 \ldots \beta_{s-1}),(\alpha_1 \ldots \alpha_{s-1})}. \tag{5.2}
$$

We have nothing to say about the generalization to other (odd) dimensions, except that e.g. in $d = 7$ and $s = 3$ there is no simple G which generalizes the discussion presented here.

While we have demonstrated that the actions which we have proposed have the correct limits in the cases where we either switch off the spin-3 field or formulate them in an AdS gravity background, we leave the discussion of the interacting theory to future work.

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Appendix

In this appendix we give some of the details about the Lie-algebras which were used in the main body of the paper.

The generators of $SU(n)$ in the fundamental representation can be chosen as a basis of real traceless matrices as follows:

$$(U^I_J)^i_j = \delta^{Ii}\delta_{Jj} - \frac{1}{n}\delta^I_J\delta^i_j, \tag{A.1}$$

where the range of all indices is $1,\dots,n$. These generators satisfy

$$[U^I_J, U^K_L] = \delta^K_J\, U^I_L - \delta^I_L\, U^K_J\,. \tag{A.2}$$

Using the explicit representation (A.1), one easily works out the rank three d-symbol of $SU(n)$:

$$\begin{aligned} d^{IKM}_{JLN} &= \frac{1}{2}\mathrm{tr}(U^I_J\{U^K_L, U^M_N\}) \qquad\qquad (A.3)\\ &= \frac{1}{2}\Big(\delta^I_N\delta^M_L\delta^K_J + \delta^I_L\delta^M_J\delta^K_N - \frac{2}{n}\delta^I_N\delta^K_L\delta^M_J - \frac{2}{n}\delta^M_L\delta^K_N\delta^I_J - \frac{2}{n}\delta^I_L\delta^K_J\delta^M_N + \frac{4}{n^2}\delta^I_J\delta^K_L\delta^M_N\Big). \end{aligned}$$

Considering the special embedding $SU(4)\hookrightarrow SU(10)$, we represent the $SU(10)$ indices $I, J,\dots$ by a symmetriced pair of $SU(4)$ indices, i.e. $I=(\alpha\beta)$, etc. with $\alpha,\beta,\dots = 1,\dots,4$ and rewrite (A.2) as

$$[U^{\alpha\beta}_{\gamma\delta}, U^{\mu\nu}_{\rho\sigma}] = \delta^{\mu\nu}_{\gamma\delta}\, U^{\alpha\beta}_{\rho\sigma} - \delta^{\alpha\beta}_{\rho\sigma}\, U^{\mu\nu}_{\gamma\delta}\,, \qquad \delta^{\alpha\beta}_{\gamma\delta} = \delta^\alpha_\gamma\delta^\beta_\delta + \delta^\alpha_\delta\delta^\beta_\gamma. \tag{A.4}$$

Given the decomposition**

$$U^I_J = U^{\alpha\beta}_{\gamma\delta} = W^{\alpha\beta}_{\gamma\delta} + \frac{1}{6}\delta^{(\alpha}_{(\gamma}L^{\beta)}_{\delta)}\,, \qquad W^{\alpha\beta}_{\alpha\gamma} = L^\alpha_\alpha = 0, \tag{A.5}$$

**Our conventions are $\delta^{(\alpha}_{(\gamma}L^{\beta)}_{\delta)} = \delta^\alpha_\gamma L^\beta_\delta + \delta^\beta_\gamma L^\alpha_\delta + \delta^\alpha_\delta L^\beta_\gamma + \delta^\beta_\delta L^\alpha_\gamma$.

and the algebra (A.4), it is straightforward to derive

$$\begin{aligned}
&[L^\alpha_\beta, L^\gamma_\delta] = \delta^\gamma_\beta L^\alpha_\delta - \delta^\alpha_\delta L^\gamma_\beta, \\
&[L^\alpha_\beta, W^{\mu\nu}_{\rho\sigma}] = \delta^\alpha_{(\rho} W^{\mu\nu}_{\sigma)\beta} - \delta^{(\mu}_\beta W^{\nu)\alpha}_{\rho\sigma}, \qquad \text{(A.6)} \\
&[W^{\alpha\beta}_{\gamma\delta}, W^{\mu\nu}_{\rho\sigma}] = \delta^{\mu\nu}_{\gamma\delta} W^{\alpha\beta}_{\rho\sigma} - \delta^{\alpha\beta}_{\rho\sigma} W^{\mu\nu}_{\gamma\delta} \\
&\qquad + \frac{1}{6}\Big(\delta^{\alpha\beta}_{\langle\gamma(\rho} W^{\mu\nu}_{\sigma)\delta\rangle} - \delta^{\mu\nu}_{\langle\gamma(\rho} W^{\alpha\beta}_{\sigma)\delta\rangle} - \delta^{\langle\alpha(\mu}_{\gamma\delta} W^{\nu)\beta\rangle}_{\rho\sigma} + \delta^{\langle\alpha(\mu}_{\rho\sigma} W^{\nu)\beta\rangle}_{\gamma\delta}\Big) \\
&\qquad\quad + \frac{1}{6}\Big(\delta^{\mu\nu}_{\gamma\delta}\delta^{(\alpha}_{(\rho} L^{\beta)}_{\sigma)} - \delta^{\alpha\beta}_{\rho\sigma}\delta^{(\mu}_{(\gamma} L^{\nu)}_{\delta)}\Big) \\
&\qquad\qquad + \frac{1}{72}\Big(\delta^{\alpha\beta}_{\langle\gamma(\rho}\delta^{(\mu}_{\sigma)} L^{\nu)}_{\delta\rangle} - \delta^{\mu\nu}_{\langle\rho(\gamma}\delta^{(\alpha}_{\delta)} L^{\beta)}_{\sigma\rangle} - \delta^{\langle\alpha(\mu}_{\gamma\delta}\delta^{\nu)}_{(\rho} L^{\beta\rangle}_{\sigma)} + \delta^{\langle\mu(\alpha}_{\rho\sigma}\delta^{\beta)}_{(\gamma} L^{\nu\rangle}_{\delta)}\Big),
\end{aligned}$$

where $\langle\alpha(\beta\gamma)\delta\rangle$ denotes symmetrization in (α, δ) and in (β, γ) and $\delta^{\alpha\beta}_{\gamma\delta} = \delta^\alpha_\gamma \delta^\beta_\delta + \delta^\alpha_\delta \delta^\beta_\gamma$.

The isomorphism between the vector respresentation of $SO(6)$ and the anti-symmetric second rank tensor representation of $SU(4)$ is made explicit with the help of the chiral Dirac matrices, some of whose properties are[††]

$$\begin{aligned}
&\Sigma^A_{\alpha\beta} = -\Sigma^A_{\beta\alpha}, \\
&(\Sigma^A)^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\Sigma^A_{\gamma\delta}, \qquad \text{(A.7)} \\
&(\Sigma^A)^{\alpha\gamma}\Sigma^B_{\gamma\beta} + (\Sigma^B)^{\alpha\gamma}\Sigma^A_{\gamma\beta} = 2\delta^{AB}\delta^\alpha_\beta.
\end{aligned}$$

A convenient basis for the $\Sigma^A_{\alpha\beta}$ is $\Sigma^1 = i\sigma_3 \otimes \sigma_1$, $\Sigma^2 = \mathbb{1} \otimes \sigma_2$, $\Sigma^3 = i\sigma_2 \otimes \mathbb{1}$, $\Sigma^4 = \sigma_2 \otimes \sigma_3$, $\Sigma^5 = i\sigma_1 \otimes \sigma_2$, $\Sigma^6 = \sigma_2 \otimes \sigma_1$ where σ_i are the three Pauli matrices. Then $SO(6)$ algebra generators can be constructed as

$$(\Sigma^{AB})^\gamma_\alpha = -\frac{1}{4}\big(\Sigma^A_{\alpha\beta}\Sigma^{B\beta\gamma} - \Sigma^B_{\alpha\beta}\Sigma^{A\beta\gamma}\big), \qquad \text{(A.8)}$$

Defining

$$V_{\alpha\beta} = i\,\Sigma^A_{\alpha\beta} V_A\,, \qquad V^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} V_{\gamma\delta}, \qquad \text{(A.9)}$$

one finds that (4.5) implies the constraint

$$V^{\alpha\gamma} V_{\beta\gamma} = \delta^\alpha_\beta. \qquad \text{(A.10)}$$

Using the symmetries of the l.h.s. and the fact that Σ^{AB} is traceless, leads to the identity

$$\begin{aligned}
&\epsilon_{ABCDMN}\,\Sigma^A_{\alpha\beta}\Sigma^B_{\gamma\delta}(\Sigma^{CD})^\rho_\lambda(\Sigma^{MN})^\nu_\mu \\
&\quad = 4i\big[\epsilon_{\alpha\beta\lambda\mu}\delta^{\rho\nu}_{\gamma\delta} - \epsilon_{\gamma\delta\lambda\mu}\delta^{\rho\nu}_{\alpha\beta} - \frac{1}{2}\epsilon_{\alpha\beta\gamma\lambda}\delta^\nu_\delta\delta^\rho_\mu + \frac{1}{2}\epsilon_{\alpha\beta\delta\lambda}\delta^\nu_\gamma\delta^\rho_\mu + \frac{1}{2}\epsilon_{\gamma\delta\alpha\lambda}\delta^\nu_\beta\delta^\rho_\mu \qquad \text{(A.11)} \\
&\qquad - \frac{1}{2}\epsilon_{\gamma\delta\beta\lambda}\delta^\nu_\alpha\delta^\rho_\mu - \frac{1}{2}\epsilon_{\alpha\beta\gamma\mu}\delta^\rho_\delta\delta^\nu_\lambda + \frac{1}{2}\epsilon_{\alpha\beta\delta\mu}\delta^\rho_\gamma\delta^\nu_\lambda + \frac{1}{2}\epsilon_{\gamma\delta\alpha\mu}\delta^\rho_\beta\delta^\nu_\lambda - \frac{1}{2}\epsilon_{\gamma\delta\beta\mu}\delta^\rho_\alpha\delta^\nu_\lambda\big].
\end{aligned}$$

[††]The indices $\dot{\alpha}$ referring to the other chirality are not needed here. By raising and lowering them with the charge conjugation matrix we can always convert them to un-dotted indices.

In this identity we use the notation $\delta^{\alpha\beta}_{\gamma\delta} = \delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} - \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}$. Other useful identities are

$$\epsilon_{ABCDMN}(\Sigma^{AB})^{\alpha}_{\beta}(\Sigma^{CD})^{\gamma}_{\delta}(\Sigma^{MN})^{\mu}_{\nu} = -16\ i\ d^{\alpha\gamma\mu}_{\beta\delta\nu}, \tag{A.12}$$

$$(\Sigma^{A})_{\alpha\beta}(\Sigma_{A})_{\gamma\delta} = -2\ \epsilon_{\alpha\beta\gamma\delta}, \tag{A.13}$$

and

$$\big(V^{\alpha\beta}h_{\mu\nu} - h^{\alpha\beta}V_{\mu\nu}\big) \wedge f^{\mu}_{\alpha} \wedge f^{\nu}_{\beta} = -2V^{\alpha\nu}h_{\mu\nu} \wedge f^{\mu}_{\rho} \wedge f^{\rho}_{\alpha}, \tag{A.14}$$

$$h_{\alpha\beta} \wedge h_{\gamma\delta} = -\frac{1}{2}\big(h_{\alpha\mu}V^{\mu\nu}h_{\nu\gamma}V_{\beta\delta} - (\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) + (\alpha,\gamma) \leftrightarrow (\beta,\delta)\big), \tag{A.15}$$

for a traceless two-form $f^{\alpha}_{\alpha} = 0$, h an antisymmetric one-form with $V^{\alpha\beta}h_{\alpha\beta} = 0$. This is e.g. the case for $h_{\alpha\beta} = DV_{\alpha\beta}$. We will also use

$$V_{\alpha\beta}V_{\gamma\delta} + V_{\alpha\gamma}V_{\delta\beta} + V_{\alpha\delta}V_{\beta\gamma} = \epsilon_{\alpha\beta\gamma\delta}, \tag{A.16}$$

to derive some of the formulas of Section 4.

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