Partially-massless higher-spin algebras and their finite-dimensional truncations

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Abstract: The global symmetry algebras of partially-massless (PM) higher-spin (HS) fields in (A)dS$_{d+1}$ are studied. The algebras involving PM generators up to depth $2\,(\ell-1)$ are defined as the maximal symmetries of free conformal scalar field with $2\,\ell$ order wave equation in d dimensions. We review the construction of these algebras by quotienting certain ideals in the universal enveloping algebra of $(A)dS_{d+1}$ isometries, and provide yet another description in terms of Howe duality. This enables us to explicitly calculate the bilinear form of the algebra and to reveal new properties — for non-negative integer values of $\ell-d/2$ the PM algebra develops additional ideals with corresponding finite dimensional coset algebras spanned by massless and PM generators.

1 Introduction

Higher spin (HS) symmetries play a central role in the consistency of HS theories [1–3], and serve as important guideline in the understanding of different AdS/CFT models. In the recent work [4] the authors of the current article have studied several aspects — such as minimal coadjoint orbit, minimal representation and Joseph ideal — related to symmetries of massless HS fields of symmetric index type, and found a convenient formula for computing trace in HS algebra. In this article, we continue our previous work and clarify analogous aspects of a more general type of HS symmetries — global symmetries of a theory involving partially-massless (PM) HS fields [5].

PM fields are special spectra which exist (or stay irreducible) only in the backgrounds with non-vanishing constant curvature. More precisely, they are unitary irreducible in dS background, whereas in AdS they are irreducible but not unitary. For a given spin s, there are s different PM fields labelled by their *depth* $t = 0, 1, \ldots, s-1$, where $t = 0$ corresponds to the massless field.[1] They are described by gauge potentials that transform with gauge parameters of lower rank compared to massless fields of the same spin:

$$\delta\, \varphi^{(t)}{}_{\mu_1\cdots\mu_s} = \nabla_{(\mu_1} \cdots \nabla_{\mu_{t+1}}\, \varepsilon^{(t)}{}_{\mu_{t+2}\cdots\mu_s)} + \mathcal{O}(\Lambda)\,, \tag{1.1}$$

where $\varphi^{(t)}$ is a PM field of depth t, ∇ denotes (A)dS covariant derivative and Λ is the cosmological constant. The propagating degrees of freedom of the spin s PM field with depth $t > 0$ are more than those of the massless spin s but less than those of the massive one. In the flat space limit, PM field of depth t is decomposed into a collection of massless fields of spin $s, s-1, \ldots, s-t$. The Killing tensor of a PM field of spin s and depth t are given by $O(d+2)$ Young diagram with two rows of respective length $s-1$ and $s-1-t$ [7]. Therefore, algebras involving such generators can be interpreted as global symmetry of a theory involving PM fields.

In order to construct a physically consistent algebra based on these generators, we need to require following two conditions:

- The isometry algebra $\mathfrak{so}_{d+2}$[2] is a subalgebra. This implies that the corresponding HS theory contains gravity sector.
- All the generators of different spins transform covariantly under isometries. This is to say that in the corresponding theory, HS fields couple to gravity in a diffeomorphism invariant manner, which is tantamount to the requirement of the equivalence principle to hold.

A priori, finding out an algebra satisfying the above conditions is highly non-trivial task in a bottom-up approach, as we have to solve for the Lie brackets among different PM generators (labelled by infinitely many spins and depths) satisfying Jacobi identity. Fortunately, a series of algebras involving PM HS generators and satisfying the above conditions have been already found in the literature. The first discussions are from mathematics literature

[1] Note that our definition of "depth" is different from the one used in [6].

[2] In this article, we do not consider any issue related to the reality structure.

[8–10] where certain PM HS algebras are defined as the maximal symmetries of higher order Laplace operators

$$\Box^{\ell}\,\phi = 0\,, \tag{1.2}$$

generalizing the definition of the massless HS algebra $(\ell = 1)$. In physics, they are investigated in [6, 11, 12], where the authors analyze the boundary values of PM fields, generalize Flato-Fronsdal theorem to higher-order singletons and provide arguments for a conjecture of AdS/CFT duality, relating a Vasiliev-type theory with PM symmetric tensor fields to the $O(N)$ model at a multicritical isotropic Lifshitz point.

In this paper, we revisit the PM algebras including the aforementioned one from various points of view. We first define the PM algebras as particular cosets of universal enveloping algebra (UEA) of the isometry algebra. This requires to identify proper ideals of the UEA and such ideals depend on a parameter λ. For generic value of λ, the algebra is spanned by PM generators of any even depths. When λ takes an integer value ℓ, it truncates into the algebra associated with (1.2) allowing only the generators of depth smaller than $2\,\ell$. We also show how the same algebras can be constructed making use of oscillators and Howe duality. The oscillator description allows us to perform more concrete calculations and we derive the explicit form of trace and bilinear form for these PM algebras. The explicit expression of bilinear form reveals a new structure of PM algebras. For (half-)integer values of λ greater than $d/2$, with (odd)even d, the algebras truncate again into *finite dimensional* ones.

The paper is organized as follows. In Section 2, we provide generalities on the global symmetries of free PM HS fields and the vector space of PM HS algebra. In Section 3, we define PM HS algebras as cosets of the UEA of (A)dS isometry algebra. In Section 4, we present Howe duality and oscillator realization of PM HS algebras. In Section 5, the trace for the PM HS algebras are defined and the bilinear forms are calculated. In Section 6, we show the finite-dimensional truncation of algebras for special values of λ. In Section 7, the relations of the PM HS algebras to conformal higher spin theory are discussed. Appendix A contains computational details.

2 Partially-Massless Fields and Their Global Symmetries

The global-symmetry generators, or Killing tensors, corresponding to PM fields can be identified by analyzing the relevant Killing equations. They were first identified in the frame-like formulation in [7]. Below, we shall rederive the PM Killing tensors in the metric-like ambient formulation. As it has been shown in [13, 14], the Fierz system of the PM field of spin s and depth t has the following gauge equivalence relation,

$$\delta_E\,\Phi^{(t)}(X,U) = (U\cdot\partial_X)^{t+1}\,E^{(t)}(X,U)\,, \tag{2.1}$$

where PM gauge parameters $E^{(t)}$ are traceless tensors:

$$\partial_U^2\,E(X,U) = 0\,. \tag{2.2}$$

As usual, being ambient-space fields, $\Phi^{(t)}$ and $E^{(t)}$ should satisfy the tangentiality and homogeneity conditions:

$$X\cdot\partial_U\,\Phi^{(t)}(X,U)=0\,,\qquad (X\cdot\partial_X-U\cdot\partial_U+2+t)\;\Phi^{(t)}(X,U)=0\,,$$
$$X\cdot\partial_U\,E^{(t)}(X,U)=0\,,\qquad (X\cdot\partial_X-U\cdot\partial_U-t)\;E^{(t)}(X,U)=0\,,\tag{2.3}$$

in order to be reduced to (A)dS Fierz system. The global symmetries are given by parameters, that satisfy the Killing equation, $\delta_E\,\Phi^{(t)}=0$, which combined with the tangentiality and homogeneity conditions defines the following system,[3]

$$X\cdot\partial_U\,E^{(t)}=0\,,\quad (U\cdot\partial_X)^{t+1}\,E^{(t)}=0\,,\quad (X\cdot\partial_X-U\cdot\partial_U-t)\,E^{(t)}=0\,.\tag{2.6}$$

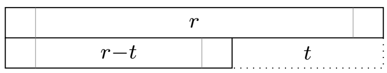
It is obvious to see from here that the PM Killing tensors of depth t corresponds to the $(t+1)$-dimensional representation of $\mathfrak{sp}_2$:

$$[h,e]=2\,e\,,\qquad [h,f]=-2\,f\,,\qquad [e,f]=h\,,$$
$$e=U\cdot\partial_X\,,\qquad f=X\cdot\partial_U\,,\qquad h=X\cdot\partial_X-U\cdot\partial_U\,.\tag{2.7}$$

We shall also see the relevance of this $\mathfrak{sp}_2$ representation later in the other construction. The conditions (2.6) and (2.2) are solved by

$$E^{(t)}(X,U)=\sum_{r=0}^{\infty}\frac{1}{r!\,(r-t)!}\,X^{M_1}\cdots X^{M_r}\,U^{N_1}\cdots U^{N_{r-t}}\,M^{(r,t)}_{M_1\cdots M_r,N_1\cdots N_{r-t}}\,,\tag{2.8}$$

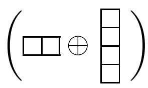
with the tensor $M^{(r,t)}$ taking values in the $\{r,r-t\}$ Young diagram of $O(d+2)$:

$$M_{M_1\cdots M_r,N_1\cdots N_{r-t}}\sim$$  . (2.9)

Therefore, any theory involving a PM field of spin s and depth t should admit global symmetry containing generator $M^{(s-1,t)}$.

3 Coset Construction from Universal Enveloping Algebra

Similarly to the massless HS algebra, the algebras involving PM HS generators can be approached from the universal enveloping algebra (UEA) of $\mathfrak{so}_{d+2}$. One of the advantages of UEA construction is that the two physical consistency conditions mentioned in Introduction are automatically satisfied. Let us remind that the massless HS algebra can be obtained as the coset,

$$hs(\mathfrak{so}_{d+2})=\mathcal{U}\Big/$$  , (3.1)

[3]One can equally describe PM fields in the Stueckelberg formulation where the gauge transformation has the standard form,

$$\delta_E\,\Phi^{(t)}(X,U)=U\cdot\partial_X\,E^{(t)}(X,U)\,,\tag{2.4}$$

whereas the tangentiality condition is modified to

$$(X\cdot\partial_U)^{t+1}\,E^{(t)}(X,U)=0\,.\tag{2.5}$$

Therefore, in the Stueckelberg formulation, we get conditions equivalent to (2.6) but X and U exchanged.

where $\mathcal{U}$ denotes the UEA of $\mathfrak{so}_{d+2}$, while (—) denotes the Joseph ideal, generated by the following elements in $\mathfrak{so}_{d+2} \odot \mathfrak{so}_{d+2}$:

$$J_{ab} := M_{(a}{}^{c} \odot M_{b)c} - \frac{\eta_{ab}}{d+2}\, M^{cd} \odot M_{cd} \sim \square\!\square\,, \qquad J_{abcd} := M_{[ab} \odot M_{cd]} \sim \begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array}. \tag{3.2}$$

This procedure determines the eigenvalues of all Casimir operators of $\mathfrak{so}_{d+2}$. In particular, the quadratic Casimir is given by

$$C_2 := \frac{1}{2}\, M_{ab} \odot M^{ba} = -\frac{(d+2)(d-2)}{4}\,. \tag{3.3}$$

Let us now consider deformations of the above construction where we take a quotient with the ideal generated by only one of J_{ab} and J_{abcd} . In such cases, the quadratic Casimir C_2 remains arbitrary while the other Casimirs are fixed as functions of C_2 as shown in [15, 16]. Hence, one can take a further quotient with $C_2 - \nu$. In this way, we have two ideals labelled by a continuous parameter ν as

$$\mathcal{I}(\nu) = \left(\begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array} \oplus (C_2 - \nu) \right), \qquad \mathcal{J}(\nu) = \left(\square\!\square \oplus (C_2 - \nu) \right). \tag{3.4}$$

Since these ideals contain fewer elements than the Joseph ideal, the corresponding coset algebras contain more generators than the original one. These one-parameter families of algebras have already been investigated in [10] and [16], respectively. In the case of $\mathcal{I}(\nu)$, we get the global symmetry algebra of a theory involving PM HS fields, whereas for $\mathcal{J}(\nu)$, we get the symmetry of a theory involving mixed symmetry HS fields. Leaving aside the discussion about the mixed symmetry case, in the current article we shall consider only the ideal $\mathcal{I}(\nu)$ and corresponding PM HS algebras.

Let us resume the discussion with the coset algebra,

$$\mathcal{A}_\lambda = \mathcal{U}/\,\mathcal{I}(\nu_\lambda)\,, \qquad \nu_\lambda = -\frac{(d-2\,\lambda)(d+2\,\lambda)}{4}\,, \tag{3.5}$$

where we have parametrized ν in terms of λ for later convenience. The ideal generated by J_{abcd} contains all $GL(d+2)$-tensors in $\mathcal{U}$ having more than two rows, hence the algebra $\mathcal{A}_\lambda$ is spanned by the generators which have the symmetry of two-row $GL(d+2)$ Young diagrams. When decomposed into the traceless $O(d+2)$ tensors, it is given by

$$\mathcal{A}_\lambda \simeq \bigoplus_{p=0}^{\infty} K_p\,, \tag{3.6}$$

where K_p are the vector space of the Killing tensors associated with the PM field of any spin and depth $t = 2p$:

$$K_p = \bigoplus_{r=0}^{\infty} \begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{r+2p} \\ \hline r & \multicolumn{1}{|c}{2p} \\ \cline{1-1} \end{array}\,. \tag{3.7}$$

Hence, the generators of $\mathcal{A}_\lambda$ are not only the massless Killing tensors K_0 but also the PM ones $K_{p\geq 1}$ corresponding to the depth $2p$. For generic values of λ , the algebra corresponds to the global symmetry of a theory involving PM HS fields of any *even* depth. One may wonder whether there exists particular values of λ leading to some special properties. As has been shown in [10], when λ takes integer values $\ell = 1, 2, \ldots$, the algebra $\mathcal{A}_\ell$ develops the ideals $\mathfrak{q}_\ell$ which consist of PM Killing tensors of even depth not smaller than $2\,\ell$:

$$\mathfrak{q}_\ell \simeq \bigoplus_{p=\ell}^{\infty} K_p\,. \tag{3.8}$$

As a result, the algebra is decomposed into

$$\mathcal{A}_\ell = \mathfrak{p}_\ell \oplus \mathfrak{q}_\ell\,, \tag{3.9}$$

where $\mathfrak{p}_\ell$ is the coset algebra:

$$\mathfrak{p}_\ell = \mathcal{A}_\ell / \mathfrak{q}_\ell \simeq \bigoplus_{p=0}^{\ell-1} K_p\,, \tag{3.10}$$

containing generators of even depth smaller than $2\,\ell$. Therefore, we obtain two algebras $\mathfrak{q}_\ell$ and $\mathfrak{p}_\ell$ for $\lambda = \ell$: the former algebra is the global symmetry of an interacting theory involving PM fields of even depths not smaller than $2\,\ell$, whereas the latter one is associated with the theory involving PM fields of even depths smaller than $2\,\ell$. The usual massless HS algebra corresponds to $\mathfrak{p}_1$.

Let us focus now on the coset algebra $\mathfrak{p}_\ell$ which contains the usual HS symmetry in the $\ell = 1$ case. In fact, it can be obtained directly from $\mathcal{U}$ as the coset,

$$\mathfrak{p}_\ell = \mathcal{U}/\mathcal{I}_\ell\,, \tag{3.11}$$

where the ideal $\mathcal{I}_\ell$ is given by [9]

$$\mathcal{I}_\ell = \left(\boxed{ 2\,\ell } \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \right). \tag{3.12}$$

The vector space (3.10) of the $\mathfrak{p}_\ell$ can be straightforwardly identified using this ideal, or equivalently from the corresponding coadjoint orbit defined by

$$W^{(a_1}{}_{b_1}\, W^{c_1}{}_{d_1} \cdots W^{a_\ell}{}_{b_\ell}\, W^{c_\ell)}{}_{d_\ell}\, \eta^{b_1 d_1} \cdots \eta^{b_\ell d_\ell} = 0\,, \qquad W^{a[b}\, W^{cd]} = 0\,. \tag{3.13}$$

Any polynomials of W^{ab} subject to the above conditions are dual to the generators of $\mathfrak{p}_\ell$ so give rise to the vector space (3.10). The values of the quadratic Casimir can be also calculated from the ideal $\mathcal{I}_\ell$ by inquiring the consistency of the ideal.

The algebra $\mathfrak{p}_\ell$ corresponds in fact to the symmetry algebra of the ℓ-th power of Laplacian operator, or in other words the symmetry of the action,

$$S[\phi] = \int d^d x\, \phi\, \Box^\ell\, \phi\,. \tag{3.14}$$

This symmetry has been studied for $\ell = 1, 2$ in [8, 17] and for arbitrary positive integers ℓ in [9]. The aspect of holographic correspondence has been also explored recently in [6].

4 Howe Duality and Oscillator Realization

The coset construction from UEA provides a concise definition of the PM symmetries, but for explicit calculations we may consider another description of PM algebra. Similarly to massless case, PM algebra $\mathcal{A}_\lambda$ can be defined using a reductive dual pair correspondence, aka Howe duality, as

$$\big(\,Sp(2)\,,\,O(d+2)\,\big)\subset Sp(2(d+2))\,. \tag{4.1}$$

The starting point is again the metaplectic representation of $\mathfrak{sp}_{2(d+2)}$ which can be realized by the star product algebra with the product,[4]

$$(f\star g)(y)=\exp\left(\frac{1}{2}\,\epsilon_{\alpha\beta}\,\partial_{y_\alpha}\cdot\partial_{z_\beta}\right)f(y)\,g(z)\Big|_{z=y}\,. \tag{4.2}$$

The mutual stabilizers $\mathfrak{sp}_2$ and $\mathfrak{so}_{d+2}$ correspond to

$$K_{\alpha\beta}=y_\alpha\cdot y_\beta\,,\qquad M_{ab}=y_{\alpha a}\,y^\alpha{}_b\,, \tag{4.3}$$

and the off-shell HS algebra $\widetilde{hs}(\mathfrak{so}_{d+2})$ is defined as the centralizer of $\mathfrak{sp}_2$: any element $f(y)$ in $\widetilde{hs}(\mathfrak{so}_{d+2})$ satisfies

$$[\,f(y)\stackrel{\star}{,}K_{\alpha\beta}\,]=0\,. \tag{4.4}$$

By solving this condition, one can show that $\widetilde{hs}(\mathfrak{so}_{d+2})$ is spanned by polynomials of M_{ab} .

The usual (on-shell) massless HS algebra is the coset of $\widetilde{hs}(\mathfrak{so}_{d+2})$ by the $\mathfrak{sp}_2$-triviality relation, $K_{\alpha\beta}\sim 0$. This relation actually defines the trivial representation of $\mathfrak{sp}_2$, which is dual to the singleton representation of $\mathfrak{so}_{d+2}$. In the following, we shall consider generalization of the last step and define the PM algebras $\mathcal{A}_\lambda$ and $\mathfrak{p}_\ell$.

PM algebra $\boldsymbol{\mathcal{A}_\lambda}$ In order to obtain $\mathcal{A}_\lambda$ from $\widetilde{hs}(\mathfrak{so}_{d+2})$, we can quotient the latter with the $\mathfrak{sp}_2$ Casimir relation,

$$C_2(\mathfrak{sp}_2)=\frac{1}{2}\,K_{\alpha\beta}\star K^{\alpha\beta}\sim(1-\lambda)(1+\lambda)\,, \tag{4.5}$$

which also fixes the quadratic Casimir of $\mathfrak{so}_{d+2}$ in terms of λ to (3.5) since in $\widetilde{hs}(\mathfrak{so}_{d+2})$ the two Casimirs are related by

$$K_{\alpha\beta}\star K^{\alpha\beta}+M_{ab}\star M^{ba}=-\frac{(d+2)(d-2)}{2}\,. \tag{4.6}$$

In the case of $\lambda=\ell$, the relation (4.5) gives an indecomposable representation consisting of the $(\ell+1)$-dimensional representation and the infinite-dimensional representation with lowest (or heighest) weight $\ell+1$ (or $-(\ell+1)$) . Hence, the algebra $\mathcal{A}_\ell$ becomes the semi-direct-sum of the coset algebra $\mathfrak{p}_\ell$ and the ideal algebra $\mathfrak{q}_\ell$ as in (3.9).

[4] As in [4], we denote $\mathfrak{sp}_2$ indices by greek letters and $\mathfrak{so}_{d+2}$ indices by roman letters. The $\mathfrak{sp}_2$ indices are rised and lowered using antisymmetric invariant tensors $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$, while $\mathfrak{so}_{d+2}$ indices are rised and lowered by symmetric metric tensor of $\mathfrak{so}_{d+2}$ (which can be taken as Kroneker delta, since we work with the complex algebra, not a particular real form) and its inverse.

PM algebra $\mathfrak{p}_\ell$ In order to obtain directly $\mathfrak{p}_\ell$ from $\widetilde{hs}(\mathfrak{so}_{d+2})$, we can quotient the latter with the following equivalence relations

$$K_{++} \sim 0\,, \quad K_{--}^{\ell} \sim 0\,, \quad (K_{+-} + \ell - 1) \sim 0\,, \tag{4.7}$$

which define the $(\ell+1)$-dimensional representation of $\mathfrak{sp}_2$. This relations have asymmetric form with respect to $\mathfrak{sp}_2$ indices, but one can show in fact that any other relations defining the $(\ell+1)$-dimensional representation such as,

$$K_{++}^{1+k} \sim 0\,, \quad K_{--}^{\ell-k} \sim 0\,, \quad (K_{+-} + \ell - k - 1) \sim 0\,, \qquad [\,k = 0, 1, \ldots, \ell\,]\,, \tag{4.8}$$

give the same algebra $\mathfrak{p}_\ell$. Instead of (4.8), one may also use the equivalence relation

$$K_{(\alpha_1\alpha_2} \star K_{\ \alpha_3\alpha_4} \star \cdots \star K_{\alpha_{2\ell-1}\alpha_{2\ell})} \sim 0\,, \tag{4.9}$$

which generalizes the relation $K_{\alpha\beta} \sim 0$ of the massless HS algebra to PM cases in a manifestly $\mathfrak{sp}_2$ covariant manner. However, this relation is weaker than (4.8) since the relations $K_{(\alpha_1\alpha_2} \star \cdots \star K_{\alpha_{2n-1}\alpha_{2n})} \sim 0$ with $n = 0, 1, \ldots, \ell - 1$ also imply (4.9). Therefore, only complemented by (4.5), the relation (4.9) becomes equivalent to (4.8) so defines the PM algebra $\mathfrak{p}_\ell$. In the following we shall use the relations (4.5) and (4.9) to derive the trace of PM algebras. Instead, in Appendix A, we make use of relations (4.7) for alternative derivation of the trace.

Decomposition of traceful generators into PM ones

For concreteness, let us look at the generators of $\mathcal{A}_\lambda$ (and $\mathfrak{p}_\ell$) more closely and show that they indeed correspond to PM Killing tensors given in (2.9). We begin with $\widetilde{hs}(\mathfrak{so}_{d+2})$ spanned by

$$\tilde{M}^{(r)}_{a_1\cdots a_r, b_1\cdots b_r} = y_{a_1\alpha_1}\, y_{b_1}{}^{\alpha_1}\, \cdots\, y_{a_r\alpha_r}\, y_{b_r}{}^{\alpha_r}\,, \tag{4.10}$$

which are traceful tensors corresponding to $GL(d+2)$ $\{r, r\}$ Young diagrams. The traceful tensor $\tilde{M}^{(r)}$ (4.10) can be decomposed into traceless tensors $M^{(r,2t)}$ with $t = 0, \ldots, [r/2]$,

$$M^{(r,2t)}_{a_1\cdots a_r, b_{2t+1}\cdots b_r} = \tilde{M}^{(r)}_{a_1\cdots a_r, b_1\cdots b_r}\, \eta^{b_1 b_2} \cdots \eta^{b_{2t-1} b_{2t}} + (\text{traces})\,, \tag{4.11}$$

which can be interpreted as PM generators of spin $r + 1$ and depth $2t$. These tensors can be conveniently handled by contracting them with

$$\tilde{W}^{ab} = \tilde{w}^a_\alpha\, \tilde{w}^{\alpha b}\,. \tag{4.12}$$

Then, the traceless generators of $\widetilde{hs}(\mathfrak{so}_{d+2})$ given by

$$M^{(r,2t)}(\tilde{W}) = M^{(r,2t)}_{a_1\cdots a_r, b_{2t+1}\cdots b_r}\, \eta_{b_1 b_2} \cdots \eta_{b_{2t-1} b_{2t}}\, \tilde{W}^{a_1 b_1} \cdots \tilde{W}^{a_r b_r}\,, \tag{4.13}$$

can be expressed solely through traces of M_{ab}'s and $\tilde{W}^{ab}$'s,

$$\langle M^p\, \tilde{W}^q\, M^r\, \cdots\, \tilde{W}^s \rangle\,, \tag{4.14}$$

as they do not have any free index. Using the following identity based on (4.3) and (4.12),

$$A\,B\,A = \frac{1}{2}\,A\,\langle A\,B\rangle\,, \qquad [\,A, B = M \text{ or } \tilde{W}\,], \tag{4.15}$$

one can show that any trace of type (4.14) will be broken down to a function of

$$\langle M^2\rangle\langle \tilde{W}^2\rangle\,, \qquad \langle M\,\tilde{W}\rangle\,, \qquad \langle (M^2\,\tilde{W}^2)^p\rangle\,. \tag{4.16}$$

Here again, one can show that the last object with $p \geq 2$ is not independent and can be expressed in terms of the rest. Hence, $M^{(r,2t)}(\tilde{W})$ in (4.13) are polynomials of $\langle M^2\rangle\langle \tilde{W}^2\rangle$, $\langle M\,\tilde{W}\rangle$ and $\langle M^2\,\tilde{W}^2\rangle$. Rewriting them as $\star$ polynomials, we can replace $\langle M^2\rangle$ by $2(1-\lambda^2)$ in $\mathcal{A}_\lambda$. In this way, any element of $\mathcal{A}_\lambda$, namely PM generators, can be written as $\star$ polynomials of $\langle M\,\tilde{W}\rangle$ and $\langle M^2\,\tilde{W}^2\rangle$ with $\langle \tilde{W}^2\rangle$-dependent coefficients:

$$\begin{aligned} M^{(r,2t)}(\tilde{W}) \sim\ & \langle M^2\,\tilde{W}^2\rangle^{\star t} \star \langle M\,\tilde{W}\rangle^{\star(r-2t)} + \langle \tilde{W}^2\rangle \Big(a\,\langle M^2\,\tilde{W}^2\rangle^{\star(t-1)} \star \langle M\,\tilde{W}\rangle^{\star(r-2t-2)} \\ & + b\,\langle M^2\,\tilde{W}^2\rangle^{\star(t-2)} \star \langle M\,\tilde{W}\rangle^{\star(r-2t-4)} + \cdots \Big) + \cdots\,, \end{aligned} \tag{4.17}$$

where a and b are some constants. Here, one can notice that the $\star$ polynomial corresponding to $M^{(r,2t)}$ has maximal orders $r-2t$ and t in $\langle M\,\tilde{W}\rangle$ and $\langle M^2\,\tilde{W}^2\rangle$, respectively.

In $\mathcal{A}_\lambda$, any $\star$ powers of $\langle M^2\,\tilde{W}^2\rangle$ can appear so PM generators of any even depth are present there. However in $\mathfrak{p}_\ell$, one can show using (4.9) that the ℓ-th $\star$ power of $\langle M^2\,\tilde{W}^2\rangle$ reduces to lower order ones:

$$\langle M^2\,\tilde{W}^2\rangle^{\star \ell} \sim a\,\langle \tilde{W}^2\rangle\,\langle M^2\,\tilde{W}^2\rangle^{\star(\ell-1)} + \cdots + \langle \tilde{W}^2\rangle^{\ell}\,, \tag{4.18}$$

with some constant a. Therefore, the generators of depth not smaller than $2\,\ell$ become linear combinations of generators of lower depth. In order to remove this redundancy, one can impose on $\tilde{W}^{ab}$ the orbit condition (3.13), which is equivalent to

$$W^{ab} = w^{a\alpha}\,w^{b}{}_{\alpha}\,, \qquad w_{(\alpha_1}\cdot w_{\alpha_2}\,\cdots\,w_{\alpha_{2\ell-1}}\cdot w_{\alpha_{2\ell})} = 0\,. \tag{4.19}$$

Here, W^{ab} denotes $\tilde{W}^{ab}$ satisfying the above condition. With this, any higher than $\ell-1$ $\star$-powers of $\langle M^2\,W^2\rangle$ vanish in $\mathfrak{p}_\ell$:

$$\langle M^2\,W^2\rangle^{\star n} = 0 \qquad [\,n = \ell, \ell+1, \ldots\,]\,. \tag{4.20}$$

It will be convenient to require another condition on W^{ab},

$$\langle W^2\rangle = 0 \qquad \Leftrightarrow \qquad w_\alpha\cdot w_\gamma\,w^\gamma\cdot w_\beta = 0\,, \tag{4.21}$$

to hold. Note that this requirement does not project any component of generators in (4.13) so we do not loose any generality by imposing it. To summarize, with (4.19) and (4.21), any elements of $\widetilde{hs}(\mathfrak{so}_{d+2})$ are projected to the generators of $\mathcal{A}_\lambda$ (or $\mathfrak{p}_\ell$) in a simple manner as

$$M^{(r,2t)}(W) = \begin{cases} \langle M^2\,W^2\rangle^{\star t} \star \langle M\,W\rangle^{\star(r-2t)}\,, & t = 0,1,2\ldots \\ 0\,, & t = \ell, \ell+1, \ldots \end{cases}\,, \tag{4.22}$$

where the second case only holds for $\mathfrak{p}_\ell$.

5 Trace and Bilinear Form

Analogously to the case of massless HS algebra, it will be useful to define a trace in PM HS algebras, from which the bi-/tri-linear forms and the structure constants can be derived. We define the trace as the coefficient of identity, similarly to the massless HS algebra,

$$\mathrm{Tr}\,[c_0 + c_a T^a] = c_0\,, \tag{5.1}$$

where T^a denotes a generator of PM HS algebra. The expression of trace for any element in the algebra can be obtained in principle by only using the definition (5.1) as in the case of massless HS algebra [4], and the trace formula for PM HS algebra $\mathfrak{p}_\ell$ is obtained in that way for a few lower ℓ's in Appendix A.

In the following, we will derive the bilinear form of $\mathcal{A}_\lambda$, that should manifest the appearance of the ideal $\mathfrak{q}_\ell$ and the corresponding coset $\mathfrak{p}_\ell$ when $\lambda = \ell$.

Non-Gaussian projector

In the massless case, it turns out [4] that the trace obtained from the definition (5.1) admits a simple representation,

$$\mathrm{Tr}_{\mathfrak{p}_1}[f] = (\Delta \star f)(0)\,, \tag{5.2}$$

where $\Delta(y)$ is the *projector*[5] introduced in [18, 19]. The function $\Delta(y)$ giving the trace is not unique as shown in [4] even though the definition (5.1) is unambiguous.

Generalizing this massless result to PM one, we assume that the trace of the PM algebra $\mathcal{A}_\lambda$ admits also an analogous expression,

$$\mathrm{Tr}_{\mathcal{A}_\lambda}[f] = (\Delta_\lambda \star f)(0)\,, \tag{5.3}$$

with the conditions

$$[\,\Delta_\lambda \overset{\star}{,} K_{\alpha\beta}\,] = 0 = [\,\Delta_\lambda \overset{\star}{,} M_{ab}\,]\,. \tag{5.4}$$

The above conditions are satisfied by the ansatz,

$$\Delta_\lambda = \Delta_\lambda(z)\,, \qquad z = K_{\alpha\beta}\,K^{\alpha\beta}\,. \tag{5.5}$$

In the massless case, we also impose the condition that $K_{\alpha\beta} \star \Delta_1(z) = 0$. In the case of $\mathcal{A}_\lambda$, we impose instead

$$(C_2(\mathfrak{sp}_2) - (1-\lambda)(1+\lambda)) \star \Delta_\lambda(z) = 0\,. \tag{5.6}$$

In order to determine such Δ_λ, we assume

$$\Delta_\lambda(z) = \sum_{n=1}^{\infty} d_n(\lambda)\,\Pi_n(z)\,, \tag{5.7}$$

with $\Pi_n(z)$ satisfying

$$K_{(\alpha_1\alpha_2} \star K_{\alpha_3\alpha_4} \star \cdots \star K_{\alpha_{2n-1}\alpha_{2n})} \star \Pi_n(z) = 0\,. \tag{5.8}$$

[5]Strictly speaking it is not a projector, since $\Delta \star \Delta$ is not well defined as shown in [18].

We shall show below that (5.6) fixes uniquely the coefficients $d_n(\lambda)$ and determines the projector. Since the condition (4.9) should hold for $\lambda = \ell$, the coefficients in (5.7) will satisfy $d_{n\geq\ell}(\ell) = 0$. The identification of $\Pi_n(z)$ is in parallel with Δ_1 in [18], and the equation (5.8) for Π_n takes the following form,

$$\left(2\,z\,\partial_z^2 + (d+1)\,\partial_z + 1\right)^n \Pi_n(z) = 0\,. \tag{5.9}$$

Clearly $\Pi_m(z)$ with $m < n$ also solves the equation for Π_n but since we are considering a linear combination of them, without a loss of generality we can take a particular solution of (5.9),

$$\Pi_n(z) = \int_{-1}^{1} ds\,(1-s^2)^{\frac{d}{2}-n}\,e^{i\,s\,\sqrt{2\,z}}\,, \tag{5.10}$$

for which $\left(2\,z\,\partial_z^2 + (d+1)\,\partial_z + 1\right)^{n-1} \Pi_n(z) \neq 0$. With the above $\Pi_n(z)$'s, the condition (5.6) fixes the coefficients $d_n(\lambda)$ in the formula (5.7) as

$$d_n(\lambda) = c\,\frac{(1-\lambda)_{n-1}\,(1+\lambda)_{n-1}}{(\frac{3}{2})_{n-1}\,(n-1)!}\,\frac{1}{d-2n}\,, \tag{5.11}$$

up to an overall factor c which can in turn be fixed from the normalization condition $\mathrm{Tr}(1) = 1$ (or, equivalently $\Delta_\lambda(0) = 1$). Notice that the coefficient $d_n(\lambda)$ is ill-defined for $d = 2n$. This problem is in fact an artifact of non-Gaussian projector. In terms of Gaussian projector, this factor will disappear and in the end all formulas are well-defined. Details of computation for (5.11) can be found in Appendix A. Remark also that the dependence of λ and d in (5.11) is such that there exist special values of λ where the series (5.7) acquire special properties. These special cases deserve more careful analysis since they may imply the appearance of an ideal in the algebra $\mathcal{A}_\lambda$. For concrete treatment, we shall explicitly calculate the bilinear form using the trace formula. However, as in the massless case, the non-Gaussian dependence on oscillators in the projector complicates the computations. Fortunately, we can find an alternative projector which gives the same trace but admits a treatable expression.

Gaussian projector

The integral form (5.10) of Π_n is simple but not very useful for actual computations as it involves

$$\exp\left(i\,s\sqrt{\frac{y_\alpha\cdot y_\beta\,y^\alpha\cdot y^\beta}{2}}\right). \tag{5.12}$$

However, using the same method as the one described in the Appendix of [4], one can show that the expression

$$P_n = \int_0^1 dx\,x^{\frac{1}{2}}\,(1-x)^{\frac{d-2}{2}-n}\,e^{-2\,\sqrt{x}\,y_+\cdot y_-} \tag{5.13}$$

is equivalent to Π_n in the following sense:

$$(\Pi_n \star f)(0) = (d-2n)\,(P_n \star f)(0)\,. \tag{5.14}$$

Here, one can see that the problematic factor $d-2n$ cancels out when working with the Gaussian projector. Since Δ_λ will be used only within the setting of (5.3), we can replace Π_n by $(d-2n)P_n$ in the formula of the projector. Evaluating the sum in (5.7), we obtain new projector as

$$D_\lambda = N_\lambda \int_0^1 dx\, x^{\frac{1}{2}}\,(1-x)^{\frac{d-4}{2}}\,{}_2F_1\left(1+\lambda\,,\,1-\lambda\,;\,\frac{3}{2}\,;\,\frac{1}{1-x}\right) e^{-2\sqrt{x}\,y_+\cdot y_-}\,, \tag{5.15}$$

with

$$N_\lambda = \frac{(-1)^{\lambda-1}\,\Gamma(d+1)}{2^{d-1}\,\Gamma(\frac{d}{2}-\lambda)\,\Gamma(\frac{d}{2}+\lambda)}\,. \tag{5.16}$$

Let us emphasize again that this projector is equivalent to Δ_λ since

$$\mathrm{Tr}(f) = (\Delta_\lambda \star f)\,(0) = (D_\lambda \star f)\,(0)\,, \qquad \forall\, f \in \widetilde{hs}(\mathfrak{so}_{d+2})\,. \tag{5.17}$$

The apparent advantage of this projector is that it involves a Gaussian function of oscillators and therefore is convenient for actual calculations. Hence, we will use the expression (5.15) in the computation of invariant bilinear form of the PM HS algebra. The trace defined with (5.15) can be equally obtained starting from the definition (5.1) as we have demonstrated for $\mathfrak{p}_\ell$ in Appendix A.

Bilinear form

Analogously to massless case [4], we consider now the invariant bilinear form of the algebra given as a trace from star-product of generating functions of all PM generators:

$$B(W_1,W_2) = \mathrm{Tr}(M(W_1)\star M(W_2)) = \mathrm{Tr}\left(e^{y_+\cdot W_1\cdot y_-}\star e^{y_+\cdot W_2\cdot y_-}\right). \tag{5.18}$$

We use (5.17) for the trace, hence consider first

$$\frac{1}{G^{(2)}(\rho,W)} = e^{\rho\, y_+\cdot y_-}\star e^{y_+\cdot W_1\cdot y_-}\star e^{y_+\cdot W_2\cdot y_-}\,\big|_{y_\alpha=0}\,, \tag{5.19}$$

which can be evaluated using the star product composition formula derived in [20] and used in [4] as

$$G^{(2)}(\rho,W) = \frac{\det_{N\times N}\left[\frac{1}{2}\,\frac{1+\rho}{1-\rho}\,\frac{2+W_1}{2-W_1}\,\frac{2+W_2}{2-W_2}+\frac{1}{2}\right]}{\det_{N\times N}\left[\frac{1}{2}\,\frac{1+\rho}{1-\rho}+\frac{1}{2}\right]\det_{N\times N}\left[\frac{1}{2}\,\frac{2+W_1}{2-W_1}+\frac{1}{2}\right]\det_{N\times N}\left[\frac{1}{2}\,\frac{2+W_2}{2-W_2}+\frac{1}{2}\right]}\,. \tag{5.20}$$

This expression involves determinants of $N\times N$ matrices, which can be simplified to

$$G^{(2)}(\rho,W) = \det_{N\times N}\left[1+\rho\,\frac{W_1+W_2}{2}+\frac{W_1\,W_2}{4}\right]. \tag{5.21}$$

Further simplification can be made by defining the *dual* matrices V_{ij} as

$$(V_{ij})_{\alpha\beta} = (w_i)_\alpha{}^A\,(w_j)_{\beta A} \qquad \left[\,(W_i)^{AB} = (w_i)^{\alpha A}(w_i)_\alpha{}^B\,\right], \tag{5.22}$$

and this enables us to rewrite (using Sylvester's determinant theorem) the $N\times N$ determinant as 4×4 one as

$$G^{(2)}(\rho,W)=\det{}_{4\times4}\begin{bmatrix}1+\frac{\rho}{2}\,V_{11}+\frac{1}{4}\,V_{12}\,V_{21} & \frac{\rho}{2}\,V_{12}+\frac{1}{4}\,V_{12}\,V_{22}\\ \frac{\rho}{2}\,V_{21} & 1+\frac{\rho}{2}\,V_{22}\end{bmatrix}. \tag{5.23}$$

Using the parameterization (4.19) and the condition (4.21), the above reduces to

$$G^{(2)}(\rho,W)=\det{}_{2\times2}\left[1+\frac{1-\rho^2}{4}\,V_{12}\,V_{21}+\frac{\rho}{2}\,V_{11}-\frac{\rho\,(1-\rho^2)}{8}\,V_{12}\,V_{22}\,V_{21}\right], \tag{5.24}$$

which can be straightforwardly evaluated to give

$$G^{(2)}(\rho,W)=\left(1+\frac{1-\rho^2}{8}\,\langle W_1\,W_2\rangle\right)^2+\frac{\rho^2\,(1-\rho^2)}{16}\,\langle W_1^2\,W_2^2\rangle\,. \tag{5.25}$$

Finally, using (5.15), (5.19) and (5.25), the invariant bilinear form can be expressed as

$$B(W_1,W_2)=N_\lambda\int_0^1 dx\,\frac{x^{\frac{1}{2}}\,(1-x)^{\frac{d-4}{2}}\,{}_2F_1(1+\lambda,\ 1-\lambda;\ \frac{3}{2};\ \frac{1}{1-x})}{\left(1+\frac{1-x}{8}\,\langle W_1\,W_2\rangle\right)^2+\frac{x\,(1-x)}{16}\,\langle W_1^2\,W_2^2\rangle}\,. \tag{5.26}$$

Let us make a few observations on the bilinear form (5.26). First of all, it is manifestly symmetric in $\lambda\to-\lambda$, which is a straightforward consequence of the symmetry of the quadratic Casimir. Moreover, when $\lambda=\ell$ the bilinear form become degenerate for the generators $M^{(r,2t)}$ with $t\geq\ell$. This degeneracy is less manifest since $\langle W_1\,W_2\rangle^{r-2t}\,\langle W_1^2\,W_2^2\rangle^t$ does not exactly correspond to the contribution of $M^{(r,2t)}$ — in other words, the bilinear form is not diagonal. Taking the simplest example of $\lambda=\ell=1$ case, the W^2-order term of the bilinear form is given by

$$\begin{aligned}a\left(\langle W_1\,W_2\rangle^2-\frac{4}{d}\,\langle W_1^2\,W_2^2\rangle\right)&=W_1^{a_1b_1}\,W_1^{a_2b_2}\,W_2^{c_1d_1}\,W_2^{c_2d_2}\,\times\\ &\times\left[\mathrm{Tr}\left(M^{(2,0)}_{a_1a_2,b_1b_2}\,M^{(2,0)}_{c_1c_2,d_1d_2}\right)+\mathrm{Tr}\left(M^{(2,2)}_{a_1a_2}\,M^{(2,2)}_{c_1c_2}\right)\eta_{b_1b_2}\,\eta_{d_1d_2}\right],\end{aligned} \tag{5.27}$$

with some constant a. By decomposing the LHS of the equation into traceless tensors, one can verify that the PM generators of the second depth have vanishing bilinear from:

$$\mathrm{Tr}\left(M^{(2,2)}_{a_1a_2}\,M^{(2,2)}_{c_1c_2}\right)=0\,, \tag{5.28}$$

hence form an ideal. Likewise, the appearance of the ideal $\mathfrak{q}_\ell$ and the corresponding coset algebra $\mathfrak{p}_\ell$ in $\mathcal{A}_\ell$ can be checked with diagonalization. And as we discussed before, the ideal part — having vanishing bilinear form — can be conveniently discarded by contracting with W^{ab} satisfying the orbit condition (3.13). We shall keep this condition in the following section.

6 Finite dimensional truncations

The normalization factor N_λ of the bilinear form contains

$$\frac{1}{\Gamma(\frac{d}{2}-\lambda)}\,, \tag{6.1}$$

which vanishes for $\lambda - \frac{d}{2} = 0, 1, \ldots$. Remind that N_λ is fixed with the condition $\mathrm{Tr}(1) = 1$ which means the bilinear form of the identity is normalized to one. When N_λ itself vanishes, the only way to keep the normalization of the identity is that the integral (5.26) should diverge for the identity part. In this case, if there exists generators which give finite integral, then their bilinear form will vanish and the algebra develops a new ideal corresponding to such generators.

In order to study this new ideal and the corresponding coset algebra, it is useful to use the transformation of hypergeometric function,

$$ {}_2F_1(1+\lambda,\, 1-\lambda;\, \tfrac{3}{2};\, \tfrac{1}{1-x}) = (\tfrac{x}{x-1})^{\lambda-1}\, {}_2F_1(\tfrac{1}{2}-\lambda,\, 1-\lambda;\, \tfrac{3}{2};\, \tfrac{1}{x})\,, \tag{6.2} $$

to rewrite the bilinear form as

$$ B(W_1, W_2) = N_\lambda\, (-1)^{\lambda-1} \int_0^1 dx\, \frac{x^{\lambda-\frac{1}{2}}\, (1-x)^{\frac{d-2}{2}-\lambda}\, {}_2F_1\left(\frac{1}{2}-\lambda\,,\, 1-\lambda\,;\, \frac{3}{2}\,;\, \frac{1}{x}\right)}{\left(1+\frac{1-x}{8}\, \langle W_1\, W_2\rangle\right)^2 + \frac{x\,(1-x)}{16}\, \langle W_1^2\, W_2^2\rangle}\,. \tag{6.3} $$

Focusing on the $M^{(r,2t)}$ part, which corresponds to

$$ \langle W_1\, W_2\rangle^{r-2t}\, \langle W_1^2\, W_2^2\rangle^t\,, \tag{6.4} $$

the integral is proportional, up to finite coefficient, to

$$ \begin{aligned} &\int_0^1 dx\, x^{\lambda-\frac{1}{2}+t}\, (1-x)^{\frac{d-2}{2}-\lambda+r-t}\, {}_2F_1\left(\tfrac{1}{2}-\lambda\,,\, 1-\lambda\,;\, \tfrac{3}{2}\,;\, \tfrac{1}{x}\right) \\ &= \Gamma\left(\frac{d}{2}-\lambda+r-t\right) \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-\lambda)_n\, (1-\lambda)_n\, \Gamma(\lambda+t+\frac{1}{2}-n)}{n!\, (\frac{3}{2})_n\, \Gamma(\frac{d+1}{2}+r-n)}\,. \end{aligned} \tag{6.5} $$

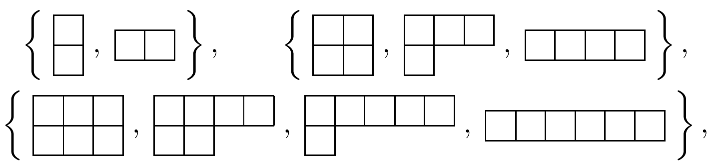
This integral diverges when the first factor $\Gamma(\frac{d}{2}-\lambda+r-t)$ diverges while the summation part is always finite. Hence, for a fixed value of λ with vanishing N_λ:

$$ \lambda = \frac{d}{2} + k\,, \qquad k = 0, 1, \ldots \tag{6.6} $$

the algebra develops an infinite-dimensional ideal corresponding to the generators $M^{(r,2t)}$ with $r - t > k$. The coset algebra, denoted henceforth by $\mathfrak{f}_k$, is finite dimensional one with the generators $M^{(r,2t)}$ satisfying

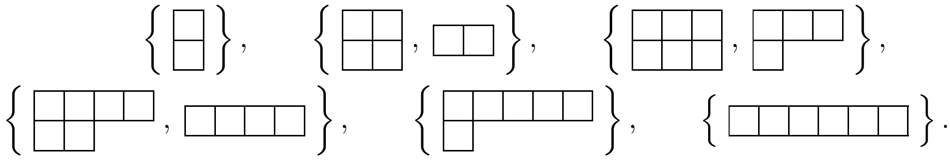
$$ r - t \le k\,. \tag{6.7} $$

The Young diagram of $M^{(r,2t)}$ has r and $r-2t$ boxes in the first and second row respectively, so the total number is $2(r-t)$. Therefore, the algebra $\mathfrak{f}_k$ consists of the generators whose Young diagram contains no more than $2k$ boxes. For example, $\mathfrak{f}_k$ with $k = 3$ have (we omit the identity generator, corresponding to Young diagram with no boxes)



(6.8)

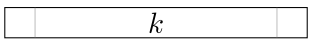
where we have organized the generators in terms of the number of boxes. Regrouping the same set according to spins, we get

 (6.9)

The dimension of $\mathfrak{f}_3$ is $\left(\frac{(d+1)(d+2)(d+6)}{6}\right)^2$. For generic integer value of k, the dimension of $\mathfrak{f}_k$ is given again by a perfect square M_k^2 with

$$M_k = \frac{(d+1)_{k-1}(d+2k)}{k!}\,. \tag{6.10}$$

This suggests that the algebra $\mathfrak{f}_k$ is isomorphic to $\mathfrak{gl}_{M_k}$, an endomorphism of a $\mathfrak{so}_{d+2}$ representation with dimension M_k. It turns out that this representation corresponds to the finite-dimensional one-row $\mathfrak{so}_{d+2}$ Young diagram with length k:

 . (6.11)

This can be verified by comparing the quadratic Casimir of (6.11) — given by $k(k+d)$ — with that of (6.6): the latter gives

$$C_2(so(d+2)) = -\frac{(d-2\,\lambda)(d+2\,\lambda)}{4} = k(k+d)\,, \tag{6.12}$$

hence they coincide as expected. This proves that the finite-dimensional coset algebra $\mathfrak{f}_k$ corresponds to the symmetries of the finite-dimensional representation (6.11). The latter representation can be also understood from the boundary point of view: they should correspond to the solution (sub)space of higher-order Laplace equation for scalar field $\Box^{d/2+k}\phi = 0$. Since any higher-order Laplace equation, $\Box^{\ell}\phi = 0$, is equivalent to the set of equations in the $(d+2)$-dimensional ambient space:

$$\partial_X^2\,\Phi(X) = 0\,, \qquad \left(X\cdot\partial_X + \frac{d}{2} - \ell\right)\Phi(X) = 0\,, \tag{6.13}$$

one can study its solutions using the above system (see e.g. [6])[6]. For $\ell = \frac{d}{2} + k$, the ambient field Φ has homogeneity degree k, any k-th order polynomials in X^a dual to traceless tensors (6.11) become solutions of the system (6.13).

[6] In (6.13), we can consider the transformation

$$\Phi(X) \quad \to \quad (X^2)^{\ell-d}\,\Phi(X)\,, \tag{6.14}$$

to flip the sign of ℓ. Higher order singletons are usually described with the positive sign — that is, with homogeneity $-(d+2\ell)/2$ — together with $(X^2)^{\ell}\,\Phi(X) = 0$.

7 PM Algebras and CHS Symmetries

Conformal Higher Spin (CHS) theory is an interacting theory of conformal higher-spin fields — that is, defined by Fradkin-Tseytlin free action [21] — of all spins [22, 23] and its symmetry coincides with the symmetry of massless HS theory in one higher dimensions. It has been shown that $(d+1)$-dimensional free conformal spin-s field can be decomposed, around $(A)dS_{d+1}$, into the set of spin-s PM fields with all depths [24–26]:

$$\mathrm{CHS}_s = \bigoplus_{t=0}^{s+\frac{d-5}{2}} \mathrm{PM}_{(s,t)}\,, \tag{7.1}$$

where $\mathrm{PM}_{(s,t)}$ with $t \geq s$ are massive fields. Therefore, CHS symmetry itself can be regarded as the symmetry of PM fields of any depths. Then a natural question arises: whether the even depth PM symmetries discussed in this paper ($\mathcal{A}_\lambda, \mathfrak{p}_\ell$ or $\mathfrak{f}_k$) can be embedded as a subalgebra in CHS symmetry. First of all, it is rather straightforward to see that the even depth PM part of CHS symmetry form a subalgebra. CHS symmetry in $(A)dS_{d+1}$ is isomorphic to the massless HS algebra in $(A)dS_{d+2}$, so can be realized by $d+2$ sets of oscillators, $Y_{\alpha A} = (y_{\alpha a}, z_\alpha)$. Now considering the map,

$$\rho\,:\;\left(\,y_{\alpha a}\;,\,z_\alpha\,\right) \quad\to\quad \left(\,y_{\alpha a}\;,\,-z_\alpha\,\right), \tag{7.2}$$

which is an automorphism for $hs(\mathfrak{so}_{d+3})$ the ρ invariant space of $hs(\mathfrak{so}_{d+3})$ forms a subalgebra and such space is generated by even depth PM generators.

The next question is to what this symmetry corresponds. One of the simplest way to answer this question is by examining the Howe duality in the oscillator construction. The quotient relation of massless HS algebra in AdS_{d+2} can be translated in AdS_{d+1} to

$$K_{\alpha\beta} + z_\alpha\, z_\beta \sim 0\,, \tag{7.3}$$

where $K_{\alpha\beta}$ is the $\mathfrak{sp}_2$ generators dual to $\mathfrak{so}_{d+2}$ — not $\mathfrak{so}_{d+3}$. The above relation simply means that the dual $\mathfrak{sp}_2$ carries the representation realized by $z_\alpha\, z_\beta$. By calculating the quadratic Casimir, we get

$$C_2(\mathfrak{sp}_2) \sim \frac{1}{2}\, z_\alpha\, z_\beta \star z^\alpha\, z^\beta = \frac{3}{4}\,, \tag{7.4}$$

and can recognize that it coincides with the C_2 of $\mathcal{A}_{\frac{1}{2}}$. This proves that $\mathcal{A}_{\frac{1}{2}}$ is the even depth subalgebra of CHS symmetry.

The existence of even depth PM subalgebra $\mathcal{A}_{\frac{1}{2}}$ inside of CHS symmetry has an interesting implication towards a unitary truncation of CHS theory. CHS theory is considered to be non-unitary since its linearized spectrum is described by a higher derivative action. The non-unitarity of the latter become clear when it is decomposed into the actions of PM fields around $(A)dS_{d+1}$:

$$S^{\mathrm{CHS}}_s = \sum_{t=0}^{s+\frac{d-5}{2}} (-1)^t\, S^{\mathrm{PM}}_{(s,t)}\,, \tag{7.5}$$

where even depth PM fields and odd depth PM fields have relatively negative sign for kinetic terms. One may wonder whether the above CHS action can be truncated into a unitary one by simply selecting positive sign part. Although it is not certain whether this truncation might be consistent, one can immediately identify one necessary condition for the consistency: the symmetry of the truncated spectrum should form a subalgebra of the original symmetry. For example, the spectrum of Conformal Gravity (CG) decomposes into massless spin two and PM spin two, and their kinetic terms have relatively negative sign. The symmetry of CG — that is the conformal symmetry — contains the symmetry of Einstein Gravity — that is the isometry — as a subalgebra, and the action of CG written as (7.5) can be consistently truncated to Einstein Hilbert one [27–29]. Now coming back to CHS theory, if CHS action can be truncated into a theory of even depth PM fields, we can escape from the non-unitarity problem at least at the linear level. The fact that $\mathcal{A}_{\frac{1}{2}}$ corresponds to a subalgebra of CHS symmetry suggests that such a truncation might be viable.

Another interesting question is that whether other even depth PM symmetries we have encountered in this paper can be still considered as a truncation of (a variant of) CHS theory. Although we do not have a proof, we do not see how the other PM algebras can be embedded in CHS symmetry. Moreover, we do not see even the massless HS symmetry, that is $\mathfrak{p}_1$, can be so: interpreted differently, we do not see how usual HS symmetry can be embedded inside of the same symmetry in one higher dimensions. Considering the embedding of $\mathfrak{p}_\ell$ within conformal-like symmetry, there is a chance that $\mathfrak{p}_\ell$ is supplemented with odd depth generators to form a new symmetry. Corresponding theories may coinside with Weyl-like theories of partially-massless fields of both even and odd depth lower than a given number that were considered in [24]. Conversely, Weyl-like theories of [24] may have even-depth unitary truncations that make use of the algebras $\mathfrak{p}_\ell$.

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A Computational Details

Gaussian Projector

We derive here the trace of PM HS algebra without initial assumption on the existence of a trace projector. We follow the same route as the derivation of the trace formula for massless HS algebras in [4]. First, we compute

$$\mathrm{Tr}\left[\exp\left(y_-\cdot\tilde{W}\cdot y_+\right)\right]=t_\ell\left(\langle\tilde{W}^2\rangle\right). \tag{A.1}$$

By taking the maximal trace of the above, we get

$$t_\ell(z) = \sum_{n=0}^{\infty} \tau_{\ell,n} \, \frac{z^n}{\left(\frac{d+2}{2}\right)_n \left(\frac{d+1}{2}\right)_n} \,, \qquad \tau_{\ell,n} = \left[\!\!\left[\left(\frac{y_+ \cdot y_{[-}\, y_{+]} \cdot y_-}{4}\right)^n \right]\!\!\right]. \tag{A.2}$$

In order to compute the coefficients τ_n , we use (4.8) and obtain the following recurrence relations,

$$\begin{aligned}
&\tau_{\ell,n+1} - \frac{1}{8}\left(n+\frac{3}{2}\right)\left(n+\frac{d+2}{2}\right)\tau_{\ell,n} + \frac{\ell+1}{4}\sigma_{\ell,n} = 0\,,\\
&\sigma_{\ell,n+1} - \frac{1}{8}\,(n+1)\left(n+\frac{d+1}{2}\right)\sigma_{\ell,n} + \frac{\ell-1}{2}\,\tau_{\ell,n+1} = 0\,,
\end{aligned} \tag{A.3}$$

where

$$\sigma_{\ell,n} := \left[\!\!\left[\frac{y_+ \cdot y_-}{2} \left(\frac{y_+ \cdot y_{[-}\, y_{+]} \cdot y_-}{4}\right)^n \right]\!\!\right]. \tag{A.4}$$

Defining $\tau_{\ell,n}$ as

$$\tau_{\ell,n} = p_\ell(n)\,\tau_{1,n}\,, \qquad \tau_{1,n} = \frac{\left(\frac{3}{2}\right)_n \left(\frac{d+2}{2}\right)_n}{8^n}\,, \tag{A.5}$$

the solutions for $p_\ell(n)$ read for a few low ℓ's

$$\begin{aligned}
&p_2(x) = \frac{d+2+4\,x}{d+2}\,, \qquad p_3(x) = \frac{3(d+2)(d+4) + 32\,(d+2)\,x + 64\,x^2}{3\,(d+2)(d+4)}\,,\\
&p_4(x) = \frac{(d+2+4\,x)[(d+4)(d+6) + 16(d+2)x + 32\,x^2]}{(d+2)(d+4)(d+6)}\,.
\end{aligned} \tag{A.6}$$

Using all these results, one can show that

$$t_\ell(z) = p_\ell(z\,\partial_z)\,t_1(z)\,, \tag{A.7}$$

where $t_1(z)$ has been determined in [4] as

$$t_1(z) = {}_2F_1\left(1\,,\,\tfrac{3}{2}\,;\,\tfrac{d+1}{2}\,;\,\tfrac{z}{8}\right) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{d-2}{2}\right)} \int_0^1 dx\, \frac{x^{\frac{1}{2}}\,(1-x)^{\frac{d-4}{2}}}{1 - x\,\frac{z}{8}}\,. \tag{A.8}$$

From (A.7) and performing the integration by part, we get

$$t_\ell(z) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\left(\frac{d+2}{2}\right)_{\ell-1}\Gamma\left(\frac{d}{2}-\ell\right)} \int_0^1 dx\, x^{\frac{1}{2}}\,(1-x)^{\frac{d}{2}-\ell}\, \frac{\bar{p}_\ell(x)}{1 - x\,\frac{z}{8}}\,, \tag{A.9}$$

where $\bar{p}_\ell$ are given for a few lower ℓ's by

$$\bar{p}_2(x) = 1 + x\,, \quad \bar{p}_3(x) = \left(1 + \tfrac{1}{3}\,x\right)(1+3\,x)\,, \quad \bar{p}_4(x) = (1+x)\left(1 + 6\,x + x^2\right). \tag{A.10}$$

Finally, the formula for the trace can be put as

$$\mathrm{Tr}\big[\,f(y)\,\big] = (f \star D_\ell)(0)\,, \tag{A.11}$$

with D_ℓ given by

$$D_\ell(y) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\left(\frac{d+2}{2}\right)_{\ell-1}\Gamma\left(\frac{d}{2}-\ell\right)} \int_0^1 dx\, x^{\frac{1}{2}}\,(1-x)^{\frac{d-2}{2}-\ell}\,\bar{p}_\ell(x)\,e^{-2\sqrt{x}\,y_+\cdot y_-}\,. \tag{A.12}$$

One can check that (A.12) coincides with (5.15) for lower ℓ examples provided.

Non-Gaussian Projector

We sketch the derivation of the solution to (5.9). First we perform a change of variables $z = u^2/2$ to rewrite the equation as

$$\left[\tfrac{1}{u}\,\mathcal{L}_u\right]^n \Pi_n(u) = 0, \qquad \mathcal{L}_u = u\,\partial_u^2 + d\,\partial_u + u\,, \tag{A.13}$$

which is equivalent to the recursive differential equation:

$$\mathcal{L}_u\,\Pi_n(u) = u\,\Pi_{n-1}(u)\,. \tag{A.14}$$

We move to the Fourier space where the differential equation becomes first-order one as

$$\Pi_n(u) = \int_{-1}^{1} ds\,\tilde{\Pi}_\ell(s)\,e^{i\,s\,u}\,, \qquad \left[(1-s^2)\partial_s + (d-2)s\right]\tilde{\Pi}_n(s) = \partial_s\,\tilde{\Pi}_{n-1}(s)\,, \tag{A.15}$$

and its solution can be identified with arbitrary integration constants a_k as

$$\tilde{\Pi}_n(s) = \sum_{k=1}^{n} a_k\,(1-s^2)^{\frac{d}{2}-k}\,. \tag{A.16}$$

Since we will consider the linear combination (5.7), we can take $a_k = \delta_{k,n}$ without a loss of generality.

The next point we will detail here is the determination of relative constant $d_n(\lambda)$ in (5.7) from the condition (5.6). For that, we need to compute

$$K^{\alpha\beta} \star K_{\alpha\beta} \star \Pi_n = \frac{1}{2}\left[u\,\mathcal{L}_u\,\frac{1}{u}\,\mathcal{L}_u + 6\left(\partial_u + \frac{d}{2\,u}\right)\mathcal{L}_u\right]\Pi_n\,. \tag{A.17}$$

With the identities,

$$u\,\Pi_n = -(d-2n)\,\partial_u\,\Pi_{n+1}\,, \qquad \partial_u^2\,\Pi_n = \Pi_{n-1} - \Pi_n\,, \tag{A.18}$$

one can express the action of the quadratic Casimir on Π_n as

$$C_2(\mathfrak{sp}_2) \star \Pi_n = (1-n)(1+n)\left(\Pi_n - \frac{(2n-1)(d-2n)}{2(n+1)(d-2n+2)}\,\Pi_{n-1}\right). \tag{A.19}$$

With this, the condition (5.6) defines a recurrence relation for $d_n(\lambda)$ which can be straightforwardly solved to give (5.11).

References

[1] E. Fradkin and M. A. Vasiliev, *Candidate to the Role of Higher Spin Symmetry*, *Annals Phys.* **177** (1987) 63.

[2] E. Fradkin and M. A. Vasiliev, *Cubic Interaction in Extended Theories of Massless Higher Spin Fields*, *Nucl. Phys.* **B291** (1987) 141.

[3] M. A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions*, *Phys. Lett.* **B243** (1990) 378–382.

[4] E. Joung and K. Mkrtchyan, *Notes on higher-spin algebras: minimal representations and structure constants*, *JHEP* **1405** (2014) 103, [arXiv:1401.7977].

[5] S. Deser and A. Waldron, *Partial masslessness of higher spins in (A)dS*, *Nucl. Phys.* **B607** (2001) 577–604, [hep-th/0103198].

[6] X. Bekaert and M. Grigoriev, *Higher order singletons, partially massless fields and their boundary values in the ambient approach*, *Nucl. Phys.* **B876** (2013) 667–714, [arXiv:1305.0162].

[7] E. Skvortsov and M. Vasiliev, *Geometric formulation for partially massless fields*, *Nucl. Phys.* **B756** (2006) 117–147, [hep-th/0601095].

[8] M. Eastwood and T. Leistner, *Higher Symmetries of the Square of the Laplacian*, math/0610610.

[9] A. R. Gover and J. Silhan, *Higher symmetries of the conformal powers of the Laplacian on conformally flat manifolds*, arXiv:0911.5265.

[10] J. P. Michel, *Higher Symmetries of the Laplacian via Quantization*, arXiv:1107.5840.

[11] X. Bekaert and M. Grigoriev, *Notes on the ambient approach to boundary values of AdS gauge fields*, *J. Phys.* **A46** (2013) 214008, [arXiv:1207.3439].

[12] T. Basile, X. Bekaert, and N. Boulanger, *Flato-Fronsdal theorem for higher-order singletons*, *JHEP* **1411** (2014) 131, [arXiv:1410.7668].

[13] K. Alkalaev and M. Grigoriev, *Unified BRST approach to (partially) massless and massive AdS fields of arbitrary symmetry type*, *Nucl. Phys.* **B853** (2011) 663–687, [arXiv:1105.6111].

[14] E. Joung, L. Lopez, and M. Taronna, *On the cubic interactions of massive and partially-massless higher spins in (A)dS*, *JHEP* **1207** (2012) 041, [arXiv:1203.6578].

[15] C. Iazeolla and P. Sundell, *A Fiber Approach to Harmonic Analysis of Unfolded Higher-Spin Field Equations*, *JHEP* **0810** (2008) 022, [arXiv:0806.1942].

[16] N. Boulanger and E. Skvortsov, *Higher-spin algebras and cubic interactions for simple mixed-symmetry fields in AdS spacetime*, *JHEP* **1109** (2011) 063, [arXiv:1107.5028].

[17] M. G. Eastwood, *Higher symmetries of the Laplacian*, *Annals Math.* **161** (2005) 1645–1665, [hep-th/0206233].

[18] M. Vasiliev, *Higher spin superalgebras in any dimension and their representations*, *JHEP* **0412** (2004) 046, [hep-th/0404124].

[19] M. A. Vasiliev, *Cubic interactions of bosonic higher spin gauge fields in* AdS_5, *Nucl. Phys.* **B616** (2001) 106–162, [hep-th/0106200].

[20] V. Didenko and M. Vasiliev, *Free field dynamics in the generalized AdS (super)space*, *J. Math. Phys.* **45** (2004) 197–215, [hep-th/0301054].

[21] E. S. Fradkin and A. A. Tseytlin, *Conformal Supergravity*, *Phys. Rept.* **119** (1985) 233–362.

[22] A. Y. Segal, *Conformal higher spin theory*, *Nucl. Phys.* **B664** (2003) 59–130, [hep-th/0207212].

[23] X. Bekaert, E. Joung, and J. Mourad, *Effective action in a higher-spin background*, *JHEP* **1102** (2011) 048, [arXiv:1012.2103].

[24] E. Joung and K. Mkrtchyan, *A note on higher-derivative actions for free higher-spin fields*, *JHEP* **11** (2012) 153, [arXiv:1209.4864].

[25] R. R. Metsaev, *Arbitrary spin conformal fields in (A)dS*, *Nucl. Phys.* **B885** (2014) 734–771, [arXiv:1404.3712].

[26] T. Nutma and M. Taronna, *On conformal higher spin wave operators*, *JHEP* **06** (2014) 066, [arXiv:1404.7452].

[27] R. Metsaev, *Ordinary-derivative formulation of conformal low spin fields*, *JHEP* **1201** (2012) 064, [arXiv:0707.4437].

[28] J. Maldacena, *Einstein Gravity from Conformal Gravity*, arXiv:1105.5632.

[29] S. Deser, E. Joung, and A. Waldron, *Partial Masslessness and Conformal Gravity*, *J. Phys.* **A46** (2013) 214019, [arXiv:1208.1307].